FIBREWISE STRATIFICATION OF GROUP REPRESENTATIONS

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Abstract. Given a nite cocommutative Hopf algebra A over a commutative regular ring R, the lattice of localising tensor ideals of the stable category of Gorenstein projective A-modules is described in terms of the corresponding lattices for the bres of A over the spectrum of R. Under certain natural conditions on the cohomology of A over R, this yields a stratication of the stable category. These results apply when A is the group algebra over R of a nite group, and also when A is the exterior algebra on a nite free R-module.

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1. Introduction

Following the seminal work of Hopkins [24] and Neeman [34] in stable homotopy theory and commutative algebra, much attention has been paid in the past few decades to the problem of classifying the thick subcategories of nite dimensional representations over various families of algebras, and also of the localising subcategories of all representations. In terms of the language and machinery developed in [11, 12], the goal is to prove stratication theorems. For example, in the case of modular representations of a nite group, the thick tensor ideal subcategories of the small stable module category were classied in [8], while the tensor ideal localising subcategories of the large stable module category were classied in [13]. These results were generalised to cover all nite group schemes over elds in [16].

In this paper we address the problem of change of coecients, with a focus on representations of group algebras of nite group schemes, and in particular, of nite groups. Let A be the group algebra of a nite group scheme over a commutative noetherian ring R; in other words, A is a cocommutative Hopf algebra that is nitely generated and projective over R. For example, A could be the group algebra RG of a nite group G. The appropriate analogue of the stable category of nite dimensional

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representations over a nite group, or group scheme, is the singularity category of A, in the sense of Buchweitz [18] and Orlov [37]. By a result of Buchweitz, this is equivalent to the stable category of Gorenstein projective A-modules:

$$\underline{\mathrm{Gproj}}\, A \xrightarrow{\ \sim\ } D_{sg}(A) := D^{b}(\mathrm{mod}\, A)/D^{perf}(A)\,.$$

An A-module is said to be Gorenstein projective if it occurs as a syzygy in an acyclic complex of projective A-modules. The Gorenstein projective modules in mod A form a Frobenius category, with projective-injective objects the projective A-modules; $\underline{\mathrm{Gproj}}\, A$ is the corresponding stable category. We also have to consider all Gorenstein projectives, not only the nitely generated ones, and the corresponding stable category $\underline{\mathrm{GProj}}\, A$. The category $\underline{\mathrm{GProj}}\, A$ is triangulated and compactly generated; the subcategory of compact objects is equivalent to $\underline{\mathrm{Gproj}}\, A$.

From now on assume that R is regular, for example, $R = \mathbb{Z}$. In this case, a nitely generated A-module is Gorenstein projective precisely when it is projective when viewed as an R-module. The same holds also for innitely generated A-modules when in addition dim R is nite, that is to say, when R has nite global dimension; see Lemma 3.10. Since A is a Hopf algebra over R, the tensor product over R induces on $\underline{\mathrm{Gproj}}\, A$ the structure of a tensor-triangulated category and also on $\underline{\mathrm{GProj}}\, A$. Moreover $\underline{\mathrm{GProj}}\, A$ is rigidly compactly generated. Our goal is to classify the thick tensor ideals of $\underline{\mathrm{Gproj}}\, A$ and the localising tensor ideals of $\underline{\mathrm{GProj}}\, A$.

One approach would be to extend methods developed in the case where R is a eld to cover more general coecient rings. In this work, we take a dierent tack, by viewing A as a family of Hopf algebras parameterised by Spec R, the Zariski spectrum of R. The bre over each point p in Spec R is the nite dimensional Hopf algebra $A_{k(p)} := A \otimes_R k(p)$, where k(p) is the residue eld at p. The results of [16] apply to yield a stratication of $\underline{\mathrm{GProj}}\, A_{k(p)}$ in terms of the projective spectrum of the cohomology ring of $A_{k(p)}$. Then the task becomes one of 'patching' these local stratications to obtain a global stratication of $\underline{\mathrm{GProj}}\, A$ in terms of the projective spectrum of the cohomology ring of A.

There are two aspects to this task: one representation theoretic and the other purely cohomological. The former is completely solved by the result below that can be viewed as a brewise criterion for detecting membership in localising tensor ideals. We deduce it from Theorem 4.6 that deals with the full homotopy category of projective A-modules.

1.1. Theorem. Let R be a regular ring, A a nite cocommutative Hopf R-algebra, and M; N Gorenstein projective A-modules. The conditions below are equivalent.

(1) $M \in \mathrm{Loc}^{\otimes}(N)$ in $\underline{\mathrm{GProj}}\, A$;

(2) $M_{k(p)} \in \mathrm{Loc}^{\otimes}(N_{k(p)})$ in $\underline{\mathrm{GProj}}\, A_{k(p)}$ for each $p \in \mathrm{Spec}\, R$.

The cohomological aspect concerns the relationship between the bres of the cohomology algebra $S := \mathrm{Ext}^{*}_{A}(R, R)$ of A and the cohomology algebra of the bres of the R-algebra A. Namely, for each p in Spec R there is natural map

$$\phi_{p} \colon S \otimes_{R} k(p) \longrightarrow \mathrm{Ext}^{*}_{A_{k(p)}}(k(p), k(p))$$

of k(p)-algebras. The question is when this map induces a bijection on spectra. Its import is clear from the next result.

1.2. Theorem. Let R be a regular ring and A a nite cocommutative Hopf algebra over R. If the map ϕ_{p} induces a bijection on spectra for each p in Spec R, then the tensor-triangulated category $\underline{\mathrm{GProj}}\,A$ is stratied by the action of S, and the support of $\underline{\mathrm{GProj}}\,A$ is $\mathrm{Proj}\,S$.

This result is proved at the end of Section 4, where we recall what it means for a tensor-triangulated category to be stratied by an action of a ring. For now it suces to record that the result above yields the sought after classication of localising tensor ideals of $\underline{\mathrm{GProj}}\,A$ and of the thick tensor ideals of $\underline{\mathrm{Gproj}}\,A$.

The hypothesis on $_p$ holds for the group algebra $A = RG$ over a nite group.[1] This is a result of Benson and Habegger [10]. The proof there is lacking detail, so we provide a full proof here in greater generality; see Theorem 5.5, and also the recent work of Lau [33, Section 7]. Putting this together with Theorem 1.2 yields the following stratication result.

1.3. Theorem. With G a nite group and R a regular ring, the tensor-triangulated category $\underline{\mathrm{GProj}}\,RG$ is stratied by the action of the group cohomology ring $H(G;R)$.

In fact we prove this stratication result for the slightly bigger compactly generated tensor-triangulated category $K(\mathrm{Proj}\,RG)$ consisting of complexes of projective RG-modules up to homotopy. The subcategory of compact objects identies with the bounded derived category $D^b(\mathrm{mod}\,RG)$. Another case where Theorem 1.2 ap-plies is when A is an exterior algebra, over a nite free R-module, regarded as a $Z{=}2$-graded Hopf algebra; see Example 4.14.

So far we have focused on Hopf algebras, but in fact an appropriate version of Theorem 1.1 holds for any nite projective R-algebras A; see Theorem 2.1. In such contexts the natural cohomology ring to consider, vis-avis stratication, is the Hochschild cohomology of A over R. It is plausible that the analogue of the map $_p$ in that context is the key to patching brewise stratication, when available, to get a global stratication result for A.

Related works. For G a nite group and R a regular ring, the classication of thick tensor ideals of $D^b(\mathrm{mod}\,RG)$ (which follows from Theorem 1.3) has been obtained by Lau [33]. Building on Lau's work in conjunction with developing novel homotopy theoretic methods Barthel [3, 4] established a classication of localising tensor ideals of RG-modules that are projective as R-modules, which is closely related to Theorem 1.3 (see 5.9 for more detail).

Our interest in the subject was propelled by asking a simple minded question: \Can one deduce the stratication of R-linear representations of G from that of representations over elds, where it is known?" | which echoes the approach taken by Lau for the bounded derived category. As already indicated, in this work we argue that such a reduction is indeed possible and consist of two simple steps. The rst is the brewise detection of Theorem 1.1 which works in a very general algebraic setting; and the second is the behaviour of the cohomological bres which we handle in the case of nite groups as in 5.6 and in the case of exterior algebras.

Hence, for us the main aspects of this work are the general brewise criterion Theorem 1.1 and the general stratication Theorem 1.2 with the stratication for nite groups being an application of these general techniques.

Subsequent developments. We are happy to report that the conclusion of Theorem 1.2 holds unconditionally. Building on recent work of van der Kallen [39],

[1] See the paragraph on subsequent developments further below.

in joint work with Barthel [5] we prove that the map ${}_{p}$ is always a homeomorphism, and in particular bijective. One complication that arises is that the tensor-triangulated category in question is no longer the stable category of Gorenstein projective modules but a suitable category of lattices, and the arguments needed to deal with this, and other aspects, are rather more involved.

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2. A fibrewise criterion for localising subcategories

Throughout R is a commutative noetherian ring and A a finite projective R-algebra; that is to say A is an R-algebra that is finitely generated and projective as an R-module. In particular, as a ring A is noetherian on the left and on the right. Given an A-complex X and a point p in $\operatorname{Spec} R$, the Zariski spectrum of R, set

$$X_{k(p)} := X \otimes_R k(p)$$

viewed as a complex of $A_{k(p)}$-modules. The assignment $X \mapsto X_{k(p)}$ is exact on the homotopy category of A-complexes, and hence also on any of its subcategories, in particular, on $\mathbf{K}(\operatorname{Proj} A)$, the homotopy category of projective A-modules. The latter has a natural structure of a triangulated category, with arbitrary coproducts. Given an A-complex Y in $\mathbf{K}(\operatorname{Proj} A)$ we write $\operatorname{Loc}(Y)$ for the localising subcategory of the homotopy category generated by Y. The main result in this section is the following fibrewise criterion for detecting objects in $\operatorname{Loc}(Y)$.

2.1. Theorem. *Suppose that R is regular. Let A be a finite projective R-algebra, and let X, Y be objects in $\mathbf{K}(\operatorname{Proj} A)$. The following conditions are equivalent.*

1. *$X \in \operatorname{Loc}(Y)$ in $\mathbf{K}(\operatorname{Proj} A)$;*
2. *$X_{k(p)} \in \operatorname{Loc}(Y_{k(p)})$ in $\mathbf{K}(\operatorname{Proj} A_{k(p)})$ for each $p \in \operatorname{Spec} R$.*

The ring R is by definition regular if every finitely generated R-module has finite projective dimension. An equivalent condition is that the ring R_p has finite global dimension for each p in $\operatorname{Spec} R$. The global dimension of R is then equal to $\dim R$, its Krull dimension.

The statement above is inspired by, and extends, an analogous statement for the derived category of A, established in [23]. It yields also a statement about the stable category of Gorenstein projective A-modules; see Theorem 3.7.

The proof of the theorem above takes some preparation and is given towards the end of this section. We start by recalling some properties of the homotopy category of projective modules.

The homotopy category of projectives. For the moment, A can be any ring that is noetherian on both sides; that is to say, A is noetherian as a left and as a right A-module. For us, A-modules mean left A-modules, and A^{op}-modules are identified with right A-modules. When A is an R-algebra for some commutative ring R, then it is convenient to write for any A-module the R-action on the opposite side. We denote by $\operatorname{Mod} A$ the (abelian) category of A-modules and by $\operatorname{mod} A$ its full subcategory consisting of finitely generated modules. The full subcategory of $\operatorname{Mod} A$ consisting of projective modules is denoted $\operatorname{Proj} A$.

For any additive category $\mathcal{A} \subseteq \operatorname{Mod} A$, like the ones in the last paragraph, $\mathbf{K}(\mathcal{A})$ will denote the associated homotopy category, with its natural structure as a triangulated category. Morphisms in this category are denoted $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-,-)$. An object X in $\mathbf{K}(\mathcal{A})$ is acyclic if $H^*(X) = 0$, and the full subcategory of acyclic objects in $\mathbf{K}(\mathcal{A})$ is denoted $\mathbf{K}_{\mathrm{ac}}(\mathcal{A})$.

Let $\mathbf{D}(\operatorname{Mod} A)$ denote the derived category of A-modules and $\mathbf{D}^b(\operatorname{mod} A)$ the bounded derived category of $\operatorname{mod} A$. Let $q\colon \mathbf{K}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ be the localisation functor; its kernel is $\mathbf{K}_{\mathrm{ac}}(\operatorname{Mod} A)$. We write q also for its restriction to the homotopy category of projective modules. This functor has an adjoint:

$$\mathbf{K}(\operatorname{Proj} A) \underset{q}{\overset{p}{\leftrightarrows}} \mathbf{D}(\operatorname{Mod} A). \tag{2.2}$$

Our convention is to write the left adjoint above the corresponding right one. In what follows it is convenient to conflate p with $p \circ q$. The image of p, denoted $\mathbf{K}_{\mathrm{proj}}(A)$, consists precisely of the K-projective complexes, namely, those complexes P such that $\operatorname{Hom}_{\mathbf{K}(A)}(P,-) = 0$ on acyclic complexes in $\mathbf{K}(\operatorname{Mod} A)$.

Compact objects. The category $\mathbf{K}(\operatorname{Proj} A)$ is triangulated, admits arbitrary direct sums, and is compactly generated. As in any triangulated category with arbitrary direct sums, an object X in $\mathbf{K}(\operatorname{Proj} A)$ is compact if $\operatorname{Hom}_{\mathbf{K}(A)}(X,-)$ commutes with direct sums. The compact objects in $\mathbf{K}(\operatorname{Proj} A)$ form a thick subcategory, denoted $\mathbf{K}^c(\operatorname{Proj} A)$. The assignment $M \mapsto \operatorname{Hom}_{A^{\mathrm{op}}}(pM, A)$ induces an equivalence

$$\kappa_A\colon \mathbf{D}^b(\operatorname{mod} A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{K}^c(\operatorname{Proj} A). \tag{2.3}$$

This result is due to Jørgensen [30, Theorem 3.2]; see also [27].

Regular rings. Recall that the ring R is regular if every finitely generated R-module has finite projective dimension, that is, every complex in $\mathbf{D}^b(\operatorname{mod} R)$ is perfect. We record a couple of basic facts for later use.

2.4. Lemma. *Let R be a commutative noetherian ring.*

1. *The ring R is regular if and only if every acyclic complex in $\mathbf{K}(\operatorname{Proj} R)$ is null-homotopic.*
2. *Suppose that R is regular and local of Krull dimension d with residue field k. Then $\operatorname{RHom}_R(k,R) \xrightarrow{\sim} \Sigma^{-d} k$.*

Proof. (1) The functor p in (2.2) restricted to compact objects embeds the perfect complexes over R into $\mathbf{D}^b(\operatorname{mod} R)$, via (2.3). This embedding is an equivalence if and only p is an equivalence. Clearly, p is an equivalence if and only if q is an equivalence. See also [26].

(2) The Koszul complex K on a minimal generating set for the maximal ideal of R provides a projective resolution $K \xrightarrow{\sim} k$. Since $\operatorname{Hom}_R(K,R) \cong \Sigma^{-d} K$, by [17, Proposition 1.6.10], the stated assertion follows. □

Fibres and the Koszul complex. We return to the context of finite projective algebras over a commutative ring R and wish to describe the functor $-\otimes_R k(\mathfrak{p})$ for each prime $\mathfrak{p}$ in $\operatorname{Spec} R$ via the Koszul complex for $\mathfrak{p}$. We reduce to the local case. Let $(R,\mathfrak{m},k)$ be a local ring, with maximal ideal $\mathfrak{m}$ and residue field k. In this paragraph, we write A_k instead of $A_{k(\mathfrak{m})}$. Let ϕ be the functor from $\mathbf{K}(\operatorname{Proj} A)$

to $K(\operatorname{Proj} A_k)$ given by the assignment $X \to X_k := X \otimes_R k$. It is clear that preserves coproducts and so has a right adjoint by Brown representability, say $_r$. Thus there is the adjoint pair

$$K(\operatorname{Proj} A) \rightleftarrows_{r} K(\operatorname{Proj} A_k)\,. \tag{2.5}$$

The following observation is useful in the sequel.

2.6. Lemma. Consider an adjoint pair of functors $T \rightleftarrows U$ between compactly generated triangulated categories. Then $_r$ preserves all coproducts if and only if preserves compact objects. In that case the restriction $^c\colon T^c \to U^c$ admits a right adjoint if and only if $_r$ preserves compact objects.

Proof. The first claim is [36, Theorem 5.1]; the second one is [2, Lemma 2.6(a)].

We need to understand the unit $\mathrm{id} \to {}_r$ and counit $_r \to \mathrm{id}$ of the adjunction (2.5). To that end, consider the Koszul complex, K, on a minimal generating set for the ideal m. The map $R \to K$ of R-complexes induces a natural map

$$X = X \otimes_R R \to X \otimes_R K \quad \text{for } X \text{ in } K(\operatorname{Proj} A). \tag{2.7}$$

On the other hand, the augmentation $R \to k$ factors through $R \to K$ via a map $K \to k$, which induces $V := K \otimes_R k \to k \otimes_R k \to k$ and therefore a natural map

$$V \otimes_k Y \to k \otimes_k Y = Y \quad \text{for } Y \text{ in } K(\operatorname{Proj} A_k). \tag{2.8}$$

The result below is the key to the proof of Theorem 2.1 and also in other computations that follow.

2.9. Lemma. When R is regular, the following statements hold.

(1) Both and $_r$ preserve compact objects and arbitrary direct sums.

(2) Restricting the adjunction (2.5) to compact objects gives the adjunction

$$D^b(\operatorname{mod} A^{op}) \underset{k\otimes^{L}_{R}}{\overset{d}{\rightleftarrows}} D^b(\operatorname{mod} A_k^{op})$$

where corresponds to $_r$ and is induced by restriction along the map $A \to A_k$ and $d := \dim R$. The right adjoint $k \otimes^{L}_{R}$ corresponds to .

(3) The maps (2.7) and (2.8) are the unit and the counit, respectively, of the adjunction (2.5).

(4) For $X \in K(\operatorname{Proj} A)$ and $Y \in K(\operatorname{Proj} A_k)$, there are natural isomorphisms

$$_r(X) = X \otimes_R K \qquad \text{and} \qquad _r(Y) = V \otimes_k Y\,.$$

In particular, $_r(Y)$ is isomorphic to a finite direct sum of shifts of Y.

Proof. (1) and (2) We have already observed that preserves arbitrary direct sums. We verify that preserves compacts, equivalently that its right adjoint, $_r$, preserves arbitrary direct sums. Any compact object in $K(\operatorname{Proj} A)$ is of the form $_A M = \operatorname{Hom}_A(p_{A^{op}} M, A)$ for some $M \in D^b(\operatorname{mod} A^{op})$. A direct computation gives

$$\operatorname{Hom}_A(p_{A^{op}} M, A) \otimes_R k = \operatorname{Hom}_A(p_{A^{op}} M, A) \otimes_R k$$
$$= \operatorname{Hom}_{A_k}((p_{A^{op}} M)$$

$_R$ $k; A_k)$ $\mathrm{Hom}_{A_k}(p_{A^{op}}(k$

^{L} $M); A_k)$:

Since R is regular, k is in $\mathrm{Thick}(R)$ in $D(\mathrm{Mod}\, R)$ and hence $k \,{}_R^{L}\, M$ is in $\mathrm{Thick}(M)$ in $D^b(\mathrm{mod}\, A^{op})$. Therefore the A_k-module $H(k \,{}_R^{L}\, M)$ is nitely generated, that is to say, $k \,{}^{L}_{R}\, M$ is in $D^b(\mathrm{mod}\, A_k^{op})$. Thus the complex

$$\mathrm{Hom}_{A_k}(p_{A^{op}}(k \,{}_R\, M); A_k) = {}_{A_k}(k \,{}_R\, M)$$

in $K(\mathrm{Proj}\, A_k)$ is compact. This justies the claim that preserves compacts. Along the way we have established that

$$_A(M) = {}_{A_k}(k \,{}^{L}_{R}\, M)$$

for M in $D^b(\mathrm{mod}\, A^{op})$. In other words, restricted to compacts is the functor k

$${}^{L}_{R} : D^b(\mathrm{mod}\, A^{op}) \to D^b(\mathrm{mod}\, A^{op})_k$$

via the identication in (2.3). The functor ^{d} is left adjoint to the functor above:

$$\begin{aligned}\mathrm{Hom}_{D(A)}({}^{d}N; M) = {} & \mathrm{Hom}_{D(A_k)}({}^{d}N; \mathrm{RHom}_R(k; M)) \\ = {} & \mathrm{Hom}_{D(A_k)}({}^{d}N; \mathrm{RHom}_R(k; R) \,{}^{L}_{R}\, M) \\ = {} & \mathrm{Hom}_{D(A_k)}({}^{d}N; {}^{d}k \,{}^{L}_{R}\, M) \\ = {} & \mathrm{Hom}_{D(A_k)}(N; k \,{}^{L}_{R}\, M)\end{aligned}$$

The rst isomorphism is standard adjunction, the second uses the fact that k is perfect as an R-complex, since R is regular, and the third one follows from the fact that $\mathrm{RHom}_R(k; R) \to {}^{d}k$; see Lemma 2.4.

At this point we can apply Lemma 2.6 to deduce that ${}_r$ also preserves compacts and is isomorphic to the functor ^{d} when restricted to compact objects. This settles both (1) and (2).

(3) The canonical map $K \,{}_R\, k \to k \,{}_R\, k \to k$ induces the map

$$X \,{}_R\, K \,{}_R\, k \to X \,{}_R\, k\,;$$

and this corresponds under the adjunction isomorphism

$$\mathrm{Hom}_{K(A_k)}(X \,{}_R\, K \,{}_R\, k; X \,{}_R\, k) = \mathrm{Hom}_{K(A)}(X \,{}_R\, K; {}_r(X \,{}_R\, k))$$

to a natural map

$$X \,{}_R\, K \to {}_r(X): \tag{2.10}$$

It suces to prove that this is an isomorphism. We verify this when X is compact; the general case then follows as and ${}_r$ preserve arbitrary direct sums, by (1).

Since K is a finite free R-complex, the assignment $M \mapsto \operatorname{Hom}_R(K; M)$ is an endofunctor on $D^b(\operatorname{mod} A^{op})$. Moreover, it is a simple computation to verify that under the equivalence (2.3), if X corresponds to M in $D^b(\operatorname{mod} A^{op})$, then $X \otimes_R K$ corresponds to $\operatorname{Hom}_R(K; M)$, and that the map (2.10) corresponds to the map

$$\Sigma^d(k \otimes^{L}_{R} M) \to \operatorname{Hom}_R(K; M):$$

This latter map is obtained by applying $- \otimes^{L}_{R} M$ to the isomorphism $\Sigma^d k \simeq \operatorname{RHom}_R(k; R)$

$$\simeq \operatorname{Hom}_R(K; R)$$

and is hence an isomorphism. Here again we are using the hypothesis that R is regular. This completes the proof that (2.7) is the unit of the adjunction. The claim about (2.8) can be verified along the same lines.

(4) The isomorphisms were verified along the way to verifying (3). The last assertion holds because V is a finite graded k-vector space with zero differential.

Local cohomology and support. In the proof of Theorem 2.1, and later on in the sequel, we require the theory of local cohomology and support from [11], with respect to the action of the ring R on the homotopy category of projective modules. The analogue for the homotopy category of injective modules is described in [28, §7]. One could invoke that theory, for the two homotopy categories are equivalent, at least under certain minor additional constraints on A, but to keep this manuscript self-contained we develop the needed results for $\mathbf{K}(\operatorname{Proj} A)$ directly.

For any pair of objects X, Y in $\mathbf{K}(\operatorname{Proj} A)$ there is a natural R-module structure on $\operatorname{Hom}_{\mathbf{K}(A)}(X, Y)$, so that $\mathbf{K}(\operatorname{Proj} A)$ is an R-linear triangulated category, in the sense of [11, §4]. In particular, for each specialisation closed subset $\mathcal{V}$ of $\operatorname{Spec} R$ there is an exact triangle

$$(2.11) \qquad \Gamma_{\mathcal{V}} X \longrightarrow X \longrightarrow L_{\mathcal{V}} X \longrightarrow$$

such that the object $\Gamma_{\mathcal{V}} X$ is $\mathcal{V}$-torsion and $\operatorname{Hom}_{\mathbf{K}(A)}(-, L_{\mathcal{V}} X) = 0$ on $\mathcal{V}$-torsion objects. Here an object Y is by definition $\mathcal{V}$-torsion if for each compact object C in $\mathbf{K}(\operatorname{Proj} A)$ the R-module $\operatorname{Hom}_{\mathbf{K}(A)}(C, Y)$ is $\mathcal{V}$-torsion.

For any ideal $I \subseteq R$ we consider the closed set

$$\mathcal{V}(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}.$$

2.12. Lemma. *Let $I \subseteq R$ be an ideal and K the Koszul complex on a finite generating set for the ideal I. For each X in $\mathbf{K}(\operatorname{Proj} A)$ one has*

$$\operatorname{Loc}(\Gamma_{\mathcal{V}(I)} X) = \operatorname{Loc}(X \otimes_R K).$$

Proof. If K is the Koszul complex on a single element r in R, the complex $X \otimes_R K$ is isomorphic to the mapping cone of the morphism $X \xrightarrow{r} X$; in other words, $X \otimes_R K$ is the complex denoted $X/\!\!/r$ in [12, §2.5]. The Koszul complex K on a sequence of elements $r_1, \dots, r_n$ generating the ideal I can be constructed as an iterated mapping cone, so $X \otimes_R K$ represents $X/\!\!/I$. Thus the stated result is a special case of [12, Proposition 2.11(2)]. □

Fix a point $\mathfrak{p}$ in $\operatorname{Spec} R$ and consider the specialisation closed subset

$$\mathcal{Z}(\mathfrak{p}) := \operatorname{Spec} R \setminus \operatorname{Spec} R_{\mathfrak{p}} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

The localisation functor $X \to L_{\mathcal{Z}(\mathfrak{p})} X$ models localisation at $\mathfrak{p}$ in the sense that for each compact object C in $\mathbf{K}(\operatorname{Proj} A)$ the map

$$\operatorname{Hom}_{\mathbf{K}(A)}(C, X) \longrightarrow \operatorname{Hom}_{\mathbf{K}(A)}(C, L_{\mathcal{Z}(\mathfrak{p})} X)$$

of R-modules induces an isomorphism of $R_{\mathfrak{p}}$-modules

$$\operatorname{Hom}_{\mathbf{K}(A)}(C, X)_{\mathfrak{p}} \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{K}(A)}(C, L_{\mathcal{Z}(\mathfrak{p})} X).$$

See [11, Theorem 4.7], and also [12, Proposition 2.3].

The localisation functor $L_{\mathcal{Z}(\mathfrak{p})}$ admits an alternative description which will be useful. For a complex X in $\mathbf{K}(\operatorname{Proj} A)$ set

$$X_{\mathfrak{p}} := X \otimes_R R_{\mathfrak{p}}$$

viewed as a complex of $A_{\mathfrak{p}}$-modules, where $A_{\mathfrak{p}}$ denotes the $R_{\mathfrak{p}}$-algebra $A \otimes_R R_{\mathfrak{p}}$. The assignment $X \mapsto X_{\mathfrak{p}}$ yields an adjoint pair of functors

$$(2.13) \qquad \mathbf{K}(\operatorname{Proj} A) \underset{\rho_r}{\overset{\rho}{\rightleftarrows}} \mathbf{K}(\operatorname{Proj} A_{\mathfrak{p}}).$$

The right adjoint ρ_r exists as localisation preserves coproducts.

2.14. Lemma. *The right adjoint λ_r preserves coproducts. Moreover, for each X in $K(\operatorname{Proj} A)$, the unit $X \to \lambda_r\lambda(X)$ of the adjunction (2.13) is naturally isomorphic to the localisation $X \to L_{Z(\mathfrak{p})}(X)$, so that*
$$L_{Z(\mathfrak{p})}(X) \cong \lambda_r\lambda(X).$$

Proof. Observe that the functor λ also preserves compact objects, hence its right adjoint λ_r preserves coproducts. As to the second claim, it suffices to verify that for any compact object C in $K(\operatorname{Proj} A)$ the map
$$\operatorname{Hom}_{K(A)}(C, X) \to \operatorname{Hom}_{K(A)}(C, \lambda_r\lambda(X))$$
induced by the unit is localisation at $\mathfrak{p}$. Adjunction gives an isomorphism
$$\operatorname{Hom}_{K(A)}(C, \lambda_r\lambda(X)) \cong \operatorname{Hom}_{K(A_\mathfrak{p})}(\lambda C, \lambda(X)) \cong \operatorname{Hom}_{K(A_\mathfrak{p})}(C_\mathfrak{p}, X_\mathfrak{p})$$
of R-modules, so the module on the left is $\mathfrak{p}$-local. Thus one gets an induced map
$$\operatorname{Hom}_{K(A)}(C, X)_\mathfrak{p} \to \operatorname{Hom}_{K(A)}(C, \lambda_r\lambda(X))$$
and the desired result is that this is an isomorphism. Since C is compact, and localisation at $\mathfrak{p}$, and the functors λ_r and λ preserve coproducts, it suffices to verify the map above is an isomorphism when X is also compact. Consider again the adjunction isomorphism
$$\operatorname{Hom}_{K(A)}(C, \lambda_r\lambda X) \cong \operatorname{Hom}_{K(A_\mathfrak{p})}(C_\mathfrak{p}, X_\mathfrak{p}).$$
Since $C_\mathfrak{p}$ and $X_\mathfrak{p}$ are compact in $K(\operatorname{Proj} A_\mathfrak{p})$, by the description of compact objects in $K(\operatorname{Proj} A)$, the desired result is that for M, N in $D^b(\operatorname{mod} A^{\mathrm{op}})$, the map
$$\operatorname{Hom}_{D(A^{\mathrm{op}})}(M, N)_\mathfrak{p} \to \operatorname{Hom}_{D(A_\mathfrak{p}^{\mathrm{op}})}(M_\mathfrak{p}, N_\mathfrak{p})$$
is an isomorphism. But this is clear. $\square$

The *local cohomology functor* at $\mathfrak{p}$ is the functor $\Gamma_\mathfrak{p}$ on $K(\operatorname{Proj} A)$ given by

$$\Gamma_\mathfrak{p}(X) := \Gamma_{\mathcal{V}(\mathfrak{p})}L_{Z(\mathfrak{p})}(X) \cong \lambda_r\Gamma_{\mathcal{V}(\mathfrak{p}R_\mathfrak{p})}(X_\mathfrak{p}), \tag{2.15}$$

where the isomorphism is the one from Lemma 2.14.

The local cohomology functors reduce the description of localising subcategories to a local problem, because the local-to-global theorem says that

$$\operatorname{Loc}(X) = \operatorname{Loc}(\{\Gamma_\mathfrak{p}X \mid \mathfrak{p} \in \operatorname{Spec} R\}) \quad \text{for } X \in K(\operatorname{Proj} A); \tag{2.16}$$

see [12, §3] and also [38, Theorem 6.9].

Proof of Theorem 2.1. The implication (1)⇒(2) is clear since for each $\mathfrak{p}$ in $\operatorname{Spec} R$ the functor given by $X \mapsto X_{k(\mathfrak{p})}$ is exact and preserves all coproducts.

(2)⇒(1) Let X, Y be objects in $K(\operatorname{Proj} A)$. By the local-to-global theorem (2.16) it suffices to verify for each $\mathfrak{p} \in \operatorname{Spec} R$ that $X_{k(\mathfrak{p})} \in \operatorname{Loc}(Y_{k(\mathfrak{p})})$ in $K(\operatorname{Proj} A_{k(\mathfrak{p})})$ implies $\Gamma_\mathfrak{p}X \in \operatorname{Loc}(\Gamma_\mathfrak{p}Y)$ in $K(\operatorname{Proj} A)$.

We denote by K the Koszul complex on a minimal generating set for the maximal ideal $\mathfrak{p}R_\mathfrak{p}$ of $R_\mathfrak{p}$. We have
$$X_{k(\mathfrak{p})} \cong X_\mathfrak{p} \otimes_{R_\mathfrak{p}} k(\mathfrak{p}R_\mathfrak{p})$$
and therefore $X_{k(\mathfrak{p})} \in \operatorname{Loc}(Y_{k(\mathfrak{p})})$ implies
$$X_\mathfrak{p} \otimes_{R_\mathfrak{p}} K \in \operatorname{Loc}(Y_\mathfrak{p} \otimes_{R_\mathfrak{p}} K) \quad \text{in} \quad K(\operatorname{Proj} A_\mathfrak{p})$$
by Lemma 2.9. This means
$$\Gamma_{\mathcal{V}(\mathfrak{p}R_\mathfrak{p})}(X_\mathfrak{p}) \in \operatorname{Loc}(\Gamma_{\mathcal{V}(\mathfrak{p}R_\mathfrak{p})}(Y_\mathfrak{p})) \quad \text{in} \quad K(\operatorname{Proj} A_\mathfrak{p})$$

by Lemma 2.12. It remains to apply the functor $_r$. Thus

$$_r \; _{V(pR_p)}(X_p) \in \mathrm{Loc}(_r \; _{V(pR_p)}(Y_p)) \quad \text{in} \quad K(\mathrm{Proj}\, A) :$$

2.17. The derived category $D(\mathrm{Mod}\, A)$ identies with a localising subcategory of $K(\mathrm{Proj}\, A)$ via the left adjoint of the canonical functor $K(\mathrm{Proj}\, A) \to D(\mathrm{Mod}\, A)$. Thus Theorem 2.1 implies the analogous description of localising subcategories of $D(\mathrm{Mod}\, A)$ from [23]. Here is another noteworthy consequence.

2.18. Corollary. Let R be a regular ring, A a nite projective R-algebra, and $M; N$ in $D^b(\mathrm{mod}\, A)$. The following conditions are equivalent.

(1) $M \in \mathrm{Thick}(N)$ in $D^b(\mathrm{mod}\, A)$;
(2) $M_{k(p)} \in \mathrm{Thick}(N_{k(p)})$ in $D^b(\mathrm{mod}\, A_{k(p)})$ for each $p \in \mathrm{Spec}\, R$.

Proof. For compact objects $X; Y$ in any compactly generated triangulated category, one has $X \in \mathrm{Thick}(Y)$ if and only if $X \in \mathrm{Loc}(Y)$; see, for instance, [35, Lemma 2.2]. Thus the desired result is an immediate consequence of Theorem 2.1 and equivalence 2.3, applied to A^{op}.

The preceding result applied with $N = A$ implies that M is perfect if and only if it is brewise perfect. Here is a more precise result, for later use.

2.19. Lemma. Let R be a commutative noetherian ring, A a nite projective R-algebra, and M a nitely generated A-module. When M is projective as an R-module, there is an equality

$$\mathrm{proj.dim}_A M = \sup\{\mathrm{proj.dim}_{A_{k(p)}} M_{k(p)} \mid p \in \mathrm{Spec}\, R\} :$$

Moreover, it suces to take the supremum over the maximal ideals in R.

Proof. Even without the hypothesis that M is nite projective over R, one has

$$\mathrm{proj.dim}_A M = \sup\{\mathrm{proj.dim}_{A_p} M_p \mid p \in \mathrm{Spec}\, R\}$$

by [6, Corollary III.6.6]. Thus replacing R, A, and M by their localisations at p we can assume R is a local ring, say with maximal ideal m and residue eld k. Then the desired result is that

$$\mathrm{proj.dim}_A M = \mathrm{proj.dim}_{A_k} M_k :$$

Since A is semi-local, [1, Proposition A.1.5] yields

$$\mathrm{proj.dim}_A M = \max_{1 \le j \le r} \{i \in \mathbb{N} \mid \mathrm{Ext}^i_A(M; L_j) = 0\}$$

where $L_1; \dots; L_r$ are the simple A-modules. The L_j are modules over $A_k := A/mA$ and M is projective over R, so adjunction yields isomorphisms

$$\mathrm{Ext}^i_A(M; L_j) = \mathrm{Ext}^i_{A_k}(M_k; L_j) :$$

Since the L_j are the simple modules over A_k the desired result follows.

3. Gorenstein algebras

Let R be a commutative noetherian ring. Following [23, 28], we say A is a Gorenstein R-algebra when it is a nite projective R-algebra such that for each p in $\mathrm{supp}_R A$ the ring A_p has nite injective dimension on the left and on the right; that is to say, it is Iwanaga{Gorenstein. When this holds R_p is Gorenstein for each p in $\mathrm{supp}_R A$. Here is a characterisation of the Gorenstein property that is in the spirit of this work; see also [23, Theorem 6.8].

3.1. Proposition. Let R be a commutative Gorenstein ring and A a nite projective R-algebra. Then A is Gorenstein if and only the nite dimensional algebra $A_{k(\mathfrak{p})}$ is Iwanaga–Gorenstein for each $\mathfrak{p}$ in $\operatorname{Spec} R$.

Proof. By [28, Theorem 4.6], the R-algebra A is Gorenstein if and only if the A-bimodule $\operatorname{Hom}_R(A,R)$ is perfect on both sides. Since the A-module $\operatorname{Hom}_R(A,R)$ is nitely generated on both sides, it is perfect if and only if the $A_{k(\mathfrak{p})}$-module

$$\operatorname{Hom}_R(A,R)_{k(\mathfrak{p})} \cong \operatorname{Hom}_{k(\mathfrak{p})}(A_{k(\mathfrak{p})}, k(\mathfrak{p}))$$

is perfect on both sides, for each $\mathfrak{p}$ in $\operatorname{Spec} R$; this follows from Corollary 2.18 applied with $M := \operatorname{Hom}_R(A,R)$ and $N := A$. It remains to observe that this latter condition is equivalent to $A_{k(\mathfrak{p})}$ being Iwanaga–Gorenstein.

3.2. One consequence of the Gorenstein condition is that complexes in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ are totally acyclic, namely each complex $X \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ satises

$$\operatorname{Hom}_{\mathbf{K}(A)}(X, P) = 0 \quad \text{for any projective } A\text{-module } P.$$

See [28, Theorem 5.6] for a proof. The functors

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \xrightarrow{\mathrm{incl}} \mathbf{K}(\operatorname{Proj} A) \xrightarrow{q} \mathbf{D}(\operatorname{Mod} A)$$

induce a recollement of triangulated categories

$$\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \underset{\longleftarrow}{\overset{\overset{t}{\longleftarrow}}{\xrightarrow{\mathrm{incl}}}} \mathbf{K}(\operatorname{Proj} A) \underset{\longleftarrow}{\overset{\overset{p}{\longleftarrow}}{\xrightarrow{q}}} \mathbf{D}(\operatorname{Mod} A) \tag{3.3}$$

The functor t, left adjoint to the inclusion of the acyclic complexes of projectives, associates to each complex its complete resolution.

From [28, Theorem 4.6] one gets an equivalence:

$$\operatorname{RHom}_A(-, A)\colon \mathbf{D}^b(\operatorname{mod} A)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} A^{\mathrm{op}}).$$

Composing this with the equivalence (2.3) yields a canonical equivalence

$$\mathbf{D}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathbf{K}^c(\operatorname{Proj} A). \tag{3.4}$$

Gorenstein projective modules. An A-module M is Gorenstein projective if it occurs as a syzygy module in an acyclic complex of projective A-modules. Thus, there is some $X \in \mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ such that $M \cong \operatorname{Coker}(d_X^{-1})$. We write $\operatorname{GProj} A$ for the full subcategory of $\operatorname{Mod} A$ consisting of the Gorenstein projective modules, and $\operatorname{Gproj} A$ for its subcategory of nitely generated modules. Both these are Frobe-nius categories, with projective and injective objects the projective modules in the corresponding categories; see for example [32, Proposition 7.2]. The corresponding stable categories are denoted $\underline{\operatorname{GProj}} A$ and $\underline{\operatorname{Gproj}} A$, respectively. The rst part of the result below was proved by Buchweitz [18, Theorem 4.4.1] when A is Iwanaga–Gorenstein, but the same argument carries over to this context.

3.5. Theorem. The assignment $X \mapsto \operatorname{Coker}(d_X^{-1})$ induces an equivalence of R-linear triangulated categories $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} A) \xrightarrow{\sim} \underline{\operatorname{GProj}} A$. Moreover, these categories are compactly generated, and $\underline{\operatorname{Gproj}} A$ identies with the full subcategory of compact objects of $\underline{\operatorname{GProj}} A$.

Proof. In the dual setting of Gorenstein injectives the rst assertion is [32, Proposition 7.2]. In fact, we have an equivalence $\mathbf{K}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{K}(\operatorname{Proj} A)$ by [28, Theorem 5.6] and then the second assertion follows from [28, Theorem 6.5].

The Gorenstein projectivity of a module is inherited by its fibres. Without further restrictions, the converse need not hold.

3.6. Lemma. *Let R be a regular ring and A a Gorenstein R-algebra. If an A-module M is Gorenstein projective, then so is the $A_{k(\mathfrak{p})}$-module $M_{k(\mathfrak{p})}$ for each $\mathfrak{p}$ in $\operatorname{supp}_R A$.*

Proof. Let X be an acyclic complex of projective A-modules in which M is a syzygy. Since R is regular and X is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Proj} R)$, it is null-homotopic as an R-complex by Lemma 2.4. Thus $X_{k(\mathfrak{p})} = X \otimes_R k(\mathfrak{p})$ is also null-homotopic, and in particular acyclic. It consists of projective $A_{k(\mathfrak{p})}$-modules and $M_{k(\mathfrak{p})}$ is a syzygy module in it, so the latter is Gorenstein projective.

In view of Theorem 3.5 and the preceding result, one gets an analogue of Theorem 2.1 for Gorenstein projective modules.

3.7. Theorem. *Let R be a regular ring and A a Gorenstein R-algebra. For Gorenstein projective R-modules X, Y we have in $\underline{\operatorname{GProj}} A$*

$$X \in \operatorname{Loc}(Y) \iff X_{k(\mathfrak{p})} \in \operatorname{Loc}(Y_{k(\mathfrak{p})}) \text{ for each } \mathfrak{p} \in \operatorname{Spec} R.$$

In the remainder of this section, we focus on a class of Gorenstein algebras for which it is easy to describe the Gorenstein projective modules.

Fibrewise self-injective algebras. The dualising bimodule of a finite projective R-algebra A is the A-bimodule

$$\omega_{A/R} := \operatorname{Hom}_R(A, R).$$

As noted in the proof of Proposition 3.1, when A is a Gorenstein R-algebra, $\omega_{A/R}$ is perfect on either side, though not necessarily as a bimodule. Moreover, this property characterises the Gorenstein property of A when R is Gorenstein; see [28, Theorem 4.6]. In the sequel, Gorenstein algebras for which the dualising bimodule is projective on either side play a prominent role. The result below characterises these algebras in terms of their fibres.

An R-algebra A is *fibrewise self-injective* if it is a finite projective R-algebra such that the finite dimensional algebra $A_{k(\mathfrak{p})}$ is self-injective for each $\mathfrak{p}$ in $\operatorname{supp}_R A$. Our primary example is the group algebra of a finite group scheme over R.

For a module M we denote by $\operatorname{add}(M)$ the full subcategory of finite direct sums of copies of M plus their direct summands.

3.8. Lemma. *Let R be a Gorenstein ring and A a finite projective R-algebra. The conditions below are equivalent.*

(1) *The R-algebra A is fibrewise self-injective;*
(2) *$\operatorname{add}(\omega_{A/R}) = \operatorname{add}(A)$ and $\operatorname{add}(\omega_{A^{\mathrm{op}}/R}) = \operatorname{add}(A^{\mathrm{op}})$;*
(3) *The dualising bimodule $\omega_{A/R}$ is projective on the left and on the right.*

If they hold the R-algebra A is Gorenstein, and one has an equivalences of categories

$$\omega_{A/R} \otimes_A (-) \colon \operatorname{Proj} A \xrightarrow{\ \sim\ } \operatorname{Proj} A$$

with inverse $\operatorname{Hom}_A(\omega_{A/R}, -)$, and similarly for $\operatorname{Proj} A^{\mathrm{op}}$.

Proof. In what follows we use the observation that for each $\mathfrak{p}$ in $\operatorname{Spec} R$ one has an isomorphism of $A_{k(\mathfrak{p})}$-bimodules:

$$\omega_{A/R} \otimes_R k(\mathfrak{p}) \cong \omega_{A_{k(\mathfrak{p})}/k(\mathfrak{p})}.$$

Hence the A-bimodule $!_{A=R}$ is projective on either side if and only if the $A_{k(p)}$-bimodule $!_{A_{k(p)}=k(p)}$ is projective on either side for each p in Spec R; see Lemma 2.19.

(1))(2) It suces to prove that A is in $\mathrm{add}(!_{A=R})$, equivalently that the map

$$!_{A=R} \; {}_{A}\, \mathrm{Hom}_A(!_{A=R}; A) \quad ! \quad A$$

given by evaluation, is surjective. Since an R-module M is zero if and only if $M_{k(p)}$ is zero for each p in Spec R, it suces to check the surjectivity of the map on the bres, that is to say, the map

$$!_{A_{k(p)}=k(p)} \; {}_{A_{k(p)}}\, \mathrm{Hom}_{A_{k(p)}}(!_{A_{k(p)}=k(p)}; A_{k(p)}) \quad ! \quad A_{k(p)}$$

is surjective. This holds as the k(p)-algebra $A_{k(p)}$ is self-injective.

(2))(3) is clear.

(3))(1) Given the isomorphism above, condition (3) yields that the dualising bimodule of $A_{k(p)}$ over k(p) is projective, that is to say, $A_{k(p)}$ is self-injective.

It remains to verify the last part of the statement. The Gorenstein property follows from Proposition 3.1. The equivalence follows from the fact that $!_{A=R}$ is projective on both sides and the isomorphism

$$A \quad ! \quad \mathrm{Hom}_A(!_{A=R}; !_{A=R}) :$$

See also [28, Theorem 4.5].

In what follows $\dim_R A$ denotes the Krull dimension of A viewed as an R-module.

3.9. Proposition. Let R be a Gorenstein ring and A a brewise self-injective R-algebra. Then

$$\mathrm{inj{:}dim}_A A = \dim_R A = \mathrm{inj{:}dim}_{A^{op}} A^{op} :$$

In particular, A is Iwanaga{Gorenstein if and only if $\dim_R A$ is nite.

Proof. It suces to verify the equality for A; the one for A^{op} holds, by symmetry. Since the injective dimension of A is detected by the vanishing of $\mathrm{Ext}^i_A(\ ; A)$ on nitely generated A-modules, it suces to verify that

$$\mathrm{inj{:}dim}_{A_p} A_p = \dim_{R_p} A_p \quad \text{for each p in } \mathrm{supp}_R A.$$

We can replace R and A by their localisations at p so that R is local, and hence A is semi-local, and $\dim_R A = \dim R$. For any simple A-module L one has

$$\mathrm{Ext}^i_A(L; !_{A=R}) = \mathrm{Ext}^i_A(L; \mathrm{Hom}_R(A; R)) = \mathrm{Ext}^i_R(A \;{}_R\, L; R) :$$

Since the ring R is Gorenstein, hence of injective dimension dim R, we deduce that $\mathrm{Ext}^i_A(L; !_{A=R}) = 0$ for $i > \dim R$. Thus Lemma 3.8(3) yields

$$\mathrm{Ext}^i_A(L; !_{A=R}) = 0 \quad \text{for } i > \dim R.$$

Hence $\mathrm{inj{:}dim}_A A \;\; \dim R$ by [18, Lemma B.3.1]. For the converse equality, with k the residue eld of R, one has $\mathrm{Ext}^d_R(k; R) = k$; see Lemma 2.4. Since A is a non-zero nite free R-module, one gets

$$\mathrm{Ext}^d_A(A \;{}_R\, k; A) = \mathrm{Ext}^d_R(k; A) = 0:$$

Thus $\mathrm{inj{:}dim}_A A \;\; \dim R$. This justies the stated equalities.

In the result below, the converse statement need not hold for general Gorenstein algebras, as can be seen by contemplating the case when R is a eld.

3.10. **Lemma.** Let A be a Gorenstein R-algebra and M an A-module. When M is Gorenstein projective, it is Gorenstein projective also as an R-module. The converse holds when A is brewise self-injective and either M is nitely generated or $\dim_R A$ is nite.

Proof. Since A is nite projective as an R-module, any projective A-module is also projective as an R-module, and hence any acyclic complex of projective A-modules is an acyclic complex of projective R-modules. It follows that any Gorenstein projective A-module is Gorenstein projective also as an R-module.

Suppose that A is brewise self-injective and that M is Gorenstein projective as an R-module. We verify that $\operatorname{Ext}^i_A(M; \ ) = 0$ for i 1 and on $\operatorname{Proj} A$. Given this, if M is nitely generated one can apply [28, Lemma 6.3] to conclude that it is Gorenstein projective also as an A-module. We can draw the same conclusion from [19, Corollary 11.5.3] for a general M when we also know $\dim_R A$ is nite, for then A is Iwanaga{Gorenstein, by Proposition 3.9.

As to the vanishing of Ext, any projective A-module is a direct summand of a free A-module, and any free A-module is of the form A $_R$ F for some free R-module F. Therefore it suces to verify that

$$\operatorname{Ext}^i_A(M; A \ _R F) = 0 \quad \text{for } i \ 1.$$

Since A is in $\operatorname{add}(!_{A=R})$, by Lemma 3.8, the A-module A $_R$ F is in additive subcategory generated by

$$!_{A=R} \ _R F = \operatorname{Hom}_R(A; R) \ _R F = \operatorname{Hom}_R(A; F):$$

Thus it suces to verify that $\operatorname{Ext}^i_A(M; \operatorname{Hom}_R(A; F)) = 0$ for i 1. This follows from the adjunction isomorphism

$$\operatorname{Ext}^i_A(M; \operatorname{Hom}_R(A; F)) = \operatorname{Ext}^i_R(M; F)$$

and the hypothesis that M is Gorenstein projective as an R-module.

3.11. Let R be a regular ring of nite Krull dimension, G a nite group, and RG the group algebra. As noted earlier the R-algebra RG is brewise self-injective. Let M be an RG-module that is projective as an R-module. It follows from Lemma 3.10 and Theorem 3.7 that M is projective as an RG-module if and only if for each p in $\operatorname{Spec} R$ the $k(p)G$-module $k(p)$ $_R$ M is projective. This result appears to be in conict with the example constructed by Benson and Goodearl in [9, Section 8]. However there is no conict because the claim in [9] that the module k $_R$ M is kG-projective is not correct: the image of multiplication by 1 g is strictly contained in the kernel of the multiplication by 1 g.

Indeed, in the example in question one has $R := k[[t]]$, the ring of power series in the variable t, over a eld k of characteristic two, and $G = Z=2$. Thus $RG = R[x]=(x^2)$, with $1 + x$ representing the generator of G. The module M in question can be realised as the free R-module $R[u]$ R $R[s]$ with the action of x given by

$$x \ (f(u); r; g(u)) = (0; 0; f(u) + r)$$

for $f(u); g(u)$ in $R[u]$, and r 2 R. Evidently the element $(1; 1; 0)$ is in the kernel of multiplication by x but not in its image.

The brewise test for projectivity stated above holds over any noetherian commutative ring R; see [5, Proposition 3.5].

4. Cocommutative Hopf algebras

Throughout this section R will be a regular commutative noetherian ring and A a finite cocommutative Hopf algebra over R; this includes the condition that A is projective as an R-module. Then $\mathbf{K}(\operatorname{Proj} A)$ has a natural structure of a tensor-triangulated category, and one has an analogue of the fibrewise criterion from Section 2 that takes into account this additional structure. With some further assumption on the cohomology of A we are then able to stratify the tensor-triangulated category $\mathbf{K}(\operatorname{Proj} A)$ via the action of the cohomology ring of A.

Tensor structure. Given A-modules X and Y, there is a natural diagonal A-module structure on $X \otimes_R Y$, obtained by restricting its $A \otimes_R A$-module structure along the coalgebra map $\Delta \colon A \to A \otimes_R A$.

4.1. Lemma. *Let P, Q be A-modules. If P is projective over A and Q is projective over R, then the A-module $P \otimes_R Q$, with the usual diagonal action, is projective.*

Proof. This follows from the standard adjunction isomorphism

$$\operatorname{Hom}_A(P \otimes_R Q, -) \cong \operatorname{Hom}_A(P, \operatorname{Hom}_R(Q, -)) .$$

The preceding result implies that $\otimes_R$ induces a tensor product on $\mathbf{K}(\operatorname{Proj} A)$.

4.2. Lemma. *The triangulated category $\mathbf{K}(\operatorname{Proj} A)$ is tensor-triangulated, with product $\otimes_R$ and unit the A-complex*

$$\mathbb{1} := \operatorname{Hom}_R(\mathbf{p}_{A^{\mathrm{op}}} R, R) .$$

Proof. We have already seen that $\otimes_R$ provides a tensor product on $\mathbf{K}(\operatorname{Proj} A)$, and it remains to verify the assertion about the unit. Since A^{op} is noetherian, one can assume that for any M in $\mathbf{D}^b(\operatorname{mod} A^{\mathrm{op}})$, its projective resolution $\mathbf{p}_{A^{\mathrm{op}}} M$ is finitely generated in each degree and that $(\mathbf{p}_{M^{\mathrm{op}}})_i = 0$ for $i \ll 0$. This fact will be used multiple times in what follows.

The augmentation $\varepsilon \colon \mathbf{p}_{A^{\mathrm{op}}} R \to R$ induces the A-linear map

$$\bar{\varepsilon} \colon R \longrightarrow \operatorname{Hom}_R(\mathbf{p}_{A^{\mathrm{op}}} R, R) .$$

For each X in $\mathbf{K}(\operatorname{Proj} A)$ this induces the map

$$X \cong X \otimes_R R \xrightarrow{X \otimes \bar{\varepsilon}} X \otimes_R \operatorname{Hom}_R(\mathbf{p}_{A^{\mathrm{op}}} R, R)$$

and the desired result is that this map is an isomorphism.

Since $\mathbf{K}(\operatorname{Proj} A)$ is compactly generated, and the functors involved preserves coproducts, it suffices to verify the claim when X is compact, that is to say, of the form $\operatorname{Hom}_A(\mathbf{p}_{A^{\mathrm{op}}} M, A)$ for some M in $\operatorname{mod} A^{\mathrm{op}}$; see (2.3). Consider the diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(\mathbf{p}_{A^{\mathrm{op}}} M, A) \arrow[r, "\operatorname{Hom}_A(\mathbf{p}_{A^{\mathrm{op}}} M, A) \otimes \bar{\varepsilon}"] \arrow[dr, "\theta"'] & \operatorname{Hom}_A(\mathbf{p}_{A^{\mathrm{op}}} M, A) \otimes_R \operatorname{Hom}_R(\mathbf{p}_{A^{\mathrm{op}}} R, R) \arrow[d, "\cong"] \\
 & \operatorname{Hom}_A(\mathbf{p}_{A^{\mathrm{op}}} M \otimes_R \mathbf{p}_{A^{\mathrm{op}}} R, A)
\end{tikzcd}$$

where the vertical map is the natural one; it is an isomorphism because $\mathbf{p}_{A^{\mathrm{op}}} A$ and $\mathbf{p}_{A^{\mathrm{op}}} M$ are degreewise finitely generated. The map θ is the obvious composition. It suffices to check that θ is an isomorphism.

One can verify that the map is obtained from the map

p A o p M   p A o p

R p A o p R   !^{M}

"

p A o p M

R R = p A o p M

by applying $\mathrm{Hom}_A(\ ;A)$. Since " is a quasi-isomorphism so is the map above. Since the source and target consist of projective A^{op}-modules, and are bounded to the right, they are semi-projective complexes over A^{op}. Thus the map above is a homotopy equivalence in $K(\mathrm{Proj}\,A^{op})$. Therefore applying $\mathrm{Hom}_A(\ ;A)$ induces an isomorphism in $K(\mathrm{Proj}\,A)$. This is the desired result.

Rigidity. The tensor-triangulated category $K(\mathrm{Proj}\,A)$, which is always compactly generated, is also rigid when R is regular. We recall briey the notion of rigidity in tensor-triangulated categories and refer to [25, A.2] for details.

Let T be a compactly generated tensor-triangulated category, with product and unit 1. We assume that 1 is compact and that preserves coproducts. Being a compactly generated tensor-triangulated category, T has an internal function object, $\mathrm{Hom}(\ ; \ )$, dened by the property that

$\mathrm{Hom}_T(X$

$Y;Z) = \mathrm{Hom}_T(X;\mathrm{Hom}(Y;Z))$ for $X;Y$ and Z in T. There is

a natural map

(4.3) $\mathrm{Hom}(X;1)$
$Y \ ! \ \mathrm{Hom}(X;Y)$

and X is rigid if this map is an isomorphism for all Y. Since 1 is compact, every rigid object is compact, and one says that T is rigidly-compactly generated when the converse holds: compact objects and rigid objects coincide. It is straightforward to verify that this property holds if and only if the conditions below hold:

(1) the subcategory of compact objects is closed under
; (2) (4.3) is an isomorphism when $X;Y$ are compact.

Back to the tensor-triangulated category $K(\mathrm{Proj}\,A)$ with product
$_R$, where A is a cocommutative Hopf algebra over R. In this case, the internal function object can be described quite concretely as

$$\mathrm{Hom}(Y;Z) = j\,\mathrm{Hom}_R(Y;Z)$$

where j is the right adjoint to the inclusion of $K(\mathrm{Proj}\,A)$ into the homotopy category of at A-modules; see [27, Proposition 2.4]. Here is the pertinent result; one can also prove that $K(\mathrm{Proj}\,A)$ is not rigid when R is not regular.

4.4. Lemma. If R is regular, the tensor-triangulated category $K(\mathrm{Proj}\,A)$ is rigid.

Proof. As noted above, it suces to verify that for any $X;Y$ in $K^c(\mathrm{Proj}\,A)$, the complex X
$_R$ Y is also in $K^c(\mathrm{Proj}\,A)$ and that (4.3) is an isomorphism. Given the identication (3.4) of compact objects in $K(\mathrm{Proj}\,A)$, this is tantamount to verifying that for $M;N$ in $D^b(\mathrm{mod}\,A)$, the complex M ^L_R N is in $D^b(\mathrm{mod}\,A)$ and the map

$$\mathrm{RHom}_R(M;R) \ {}^L_R\, N \ ! \ \mathrm{RHom}_R(M;N)$$

of A-complexes is an isomorphism. Both properties are clear, since R is regular.

4.5. Remark. The functor : $K(\mathrm{Proj}\,A)$! $K(\mathrm{Proj}\,A_k)$ from (2.5) is a tensor functor which ts { together with its adjoints { into the framework discussed in [2].

Fibrewise criterion. Here is the analogue of the brewise criterion for detecting membership in localising subcategories, Theorem 2.1, in the presence of the tensor product. In the statement Loc
(Y) denotes the tensor ideal localising subcategory generated by Y.

4.6. Theorem. Let R be a regular ring and A a finite cocommutative Hopf R-algebra. Let X, Y be objects in $K(\operatorname{Proj} A)$. The following conditions are equivalent.

(1) $X \in \operatorname{Loc}(Y)$ in $K(\operatorname{Proj} A)$;

(2) $X_{k(\mathfrak{p})} \in \operatorname{Loc}(Y_{k(\mathfrak{p})})$ in $K(\operatorname{Proj} A_{k(\mathfrak{p})})$ for each $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. The argument follows the same lines as that for Theorem 2.1, using in addition that $K(\operatorname{Proj} A)$ is a rigidly generated tensor-triangulated category by Lemma 4.4. Again it is straightforward to verify (1)$\Rightarrow$(2), once one observes that the functor from (2.5) respects the tensor products: For X, Y in $K(\operatorname{Proj} A)$, there is a natural isomorphism

$$(X \otimes_R Y) \cong (X) \otimes_{k(\mathfrak{p})} (Y) \quad \text{in } K(\operatorname{Proj} A_{k(\mathfrak{p})}).$$

For the implication (2)$\Rightarrow$(1) we use the version of the local-to-global theorem (2.16) for tensor-triangulated categories in [12, Theorem 7.2]. Then the task reduces to proving for X, Y in $K(\operatorname{Proj} A)$ that when $(R, \mathfrak{m}, k)$ is a local ring and X_k is in $\operatorname{Loc}(Y_k)$, the complex $_r(X)$ is in $\operatorname{Loc}(_r(Y))$. When reducing to the local case, one uses that for each $\mathfrak{p}$ in $\operatorname{Spec} R$ the category $K(\operatorname{Proj} A_{\mathfrak{p}})$ identifies with a localising tensor ideal of $K(\operatorname{Proj} A)$ via (2.13).

Whilst the functor $_r$ need not respect tensor products, the following projection formula holds. For U in $K(\operatorname{Proj} A)$ and V in $K(\operatorname{Proj} A_k)$, there is a natural isomorphism

$$U \otimes_R {}_rV \cong {}_r((U) \otimes_k V).$$

One can verify this directly, but this is a general fact about tensor functors and their right adjoints between rigidly-compactly generated tensor-triangulated categories; see [2, Theorem 1.3]. This formula and the fact that, up to direct summands, is surjective on objects – see Lemma 2.9 – yield the desired result.

Finite generation. Let R be a commutative noetherian ring and A a finite cocommutative Hopf algebra over R. Set

$$S := \operatorname{Ext}_A(R, R).$$

This is a graded-commutative R-algebra. Van der Kallen [39] has proved that S is finitely generated as an R-algebra; equivalently, that it is noetherian. This generalises earlier work of Friedlander and Suslin [22] that dealt with the case where R is a field. The ring S can be realised as the graded-ring of morphisms

$$S = \operatorname{Hom}_{K(A)}(1, 1).$$

Since 1 is the unit of the tensor product on $K(\operatorname{Proj} A)$, it has a natural S-linear action on it. For each $\mathfrak{p}$ in $\operatorname{Spec} R$ one has that $A_{k(\mathfrak{p})}$ is a finite dimensional cocommutative Hopf algebra over $k(\mathfrak{p})$. Set

$$S(\mathfrak{p}) := \operatorname{Ext}_{A_{k(\mathfrak{p})}}(k(\mathfrak{p}), k(\mathfrak{p})). \tag{4.7}$$

Cohomological support. Let R be a commutative noetherian ring, A a finite cocommutative Hopf algebra over R, and S the cohomology ring introduced above. We write $\operatorname{Spec} S$ for the homogenous prime ideals in S. Following [11, §5], the action of S on $K(\operatorname{Proj} A)$ gives rise to a notion of support for objects in $K(\operatorname{Proj} A)$. Namely for each $\mathfrak{q}$ in $\operatorname{Spec} S$ there is a local cohomology functor functor $_{\mathfrak{q}}$, defined akin to (2.15). The support of an object $X \in K(\operatorname{Proj} A)$ is by definition the set

$$\operatorname{supp}_S X := \{\mathfrak{q} \in \operatorname{Spec} S \mid {}_{\mathfrak{q}}X \neq 0\};$$

and for any class of objects $\mathsf X$ we set

$$\operatorname{supp}_S \mathsf X := \bigcup_{X\in\mathsf X} \operatorname{supp}_S X.$$

Support and fibres. Theorem 4.6 yields a stratification – see the discussion further below – of $\mathsf K(\operatorname{Proj} A)$ in terms of subsets of the space

$$\bigsqcup_{\mathfrak p\in\operatorname{Spec} R}\operatorname{Spec} S(\mathfrak p),$$

where to each X in $\mathsf K(\operatorname{Proj} A)$ we associate the subset

$$\bigsqcup_{\mathfrak p\in\operatorname{Spec} R}\operatorname{supp}_{S(\mathfrak p)} X_{k(\mathfrak p)}.$$

The task is to relate this to $\operatorname{supp}_S X$, viewed as a subset of $\operatorname{Spec} S$.

To that end consider the structure map $\phi\colon R\to S$, which induces a map

$$\phi^a\colon \operatorname{Spec} S\to\operatorname{Spec} R.$$

The fibre of this map over $\mathfrak p$ is

$$(\phi^a)^{-1}(\mathfrak p)=\operatorname{Spec}(S\otimes_R k(\mathfrak p)),$$

which we identify with a subset of $\operatorname{Spec} S$ in the usual way.

The functor $-\otimes^{\mathbf L}_R k(\mathfrak p)$ induces a map of R-algebras

$$S=\operatorname{Ext}_A(R,R)\to\operatorname{Ext}_{A_{k(\mathfrak p)}}(k(\mathfrak p),k(\mathfrak p))=S(\mathfrak p).$$

This induces the map of graded $k(\mathfrak p)$-algebras

$$\phi_{\mathfrak p}\colon S\otimes_R k(\mathfrak p)\to S(\mathfrak p).$$

Even in the best of cases, one does not expect this to be an isomorphism. Consider the induced map on spectra:

$$\phi_{\mathfrak p}^a\colon\operatorname{Spec} S(\mathfrak p)\to\operatorname{Spec}(S\otimes_R k(\mathfrak p))\subseteq\operatorname{Spec} S. \tag{4.8}$$

The result below tracks the behavior of supports as we pass to the fibres.

4.9. Lemma. *Let R be a regular ring and A a finite cocommutative Hopf R-algebra. Fix $\mathfrak p$ in $\operatorname{Spec} R$ and let $\phi^a_{\mathfrak p}$ be the map in (4.8). For each X in $\mathsf K(\operatorname{Proj} A)$ there is an equality*

$$\phi^a_{\mathfrak p}(\operatorname{supp}_{S(\mathfrak p)} X_{k(\mathfrak p)})=\operatorname{supp}_S X\cap(\phi^a)^{-1}(\mathfrak p).$$

Proof. We can reduce to the case where $(R,\mathfrak m,k)$ is a local ring and $\mathfrak p=\mathfrak m$, the maximal ideal of R. Set $S_k:=S\otimes_R k=S/\mathfrak m S$. Via the map $\phi_{\mathfrak m}\colon S_k\to S(\mathfrak m)$ the $S(\mathfrak m)$-action on $\mathsf K(\operatorname{Proj} A_k)$ induces an S_k-action. Applying [14, Corollary 7.8(1)] to the identity functor on $\mathsf K(\operatorname{Proj} A_k)$ yields an equality

$$\phi^a_{\mathfrak m}(\operatorname{supp}_{S(\mathfrak m)} X_k)=\operatorname{supp}_{S_k} X_k.$$

It thus suffices to work with the subset on the right.

Consider the adjunction (2.5). For any compact object C in $\mathsf K(\operatorname{Proj} A)$ one has isomorphisms of graded S_k-modules

$$\begin{aligned}\operatorname{Hom}_{\mathsf K(A_k)}(C,X_k) &= \operatorname{Hom}_{\mathsf K(A_k)}(C,X)\\ &\cong \operatorname{Hom}_{\mathsf K(A)}(C,r_*X)\\ &\cong \operatorname{Hom}_{\mathsf K(A)}(C,X\otimes_R K)\end{aligned}$$

where the last isomorphism is by Lemma 2.9. Any compact object in $K(\mathrm{Proj}\,A_k)$ is a direct summand of C for some compact object C in $K(\mathrm{Proj}\,A)$ by Lemma 2.9. By [11, Theorem 5.2], one can compute $\mathrm{supp}_{S_k}(X_k)$ from the support of the S_k-modules $\mathrm{Hom}_{K(A_k)}(C; X_k)$. This gives the rst equality below

$$\mathrm{supp}_{S_k} X_k = \mathrm{supp}_S(X \otimes_R K) = \mathrm{supp}_S X \cap V(\mathfrak{m}S) = \mathrm{supp}_S X \cap (^a)^{-1}(\mathfrak{m}):$$

The second one holds because $X \otimes_R K$ represents $X=\mathfrak{m}S$; see [12, Lemma 2.6].

Stratication. Let S be a graded commutative noetherian ring and T a rigidly-compactly generated tensor-triangulated category. We denote by T^c the full subcategory of compact objects. We say that T is S-linear to mean that S acts on T via a map of graded rings

$$S \to \mathrm{End}_T(1);$$

see [12, x7]. In the context above, one says that the tensor-triangulated category T is stratied by S if for each q in $\mathrm{Spec}\,S$ the category $_qT$, consisting of the q-local and q-torsion objects in T, is either zero or minimal, in that, it has no proper localising tensor ideals; see [12, x7.2]. When this holds one has a bijection

$$\text{(4.10)} \qquad \text{Localising tensor ideals of } T \xrightarrow{\mathrm{supp}_S(\ )} \{\text{Subsets of } \mathrm{supp}_S 1\}:$$

When in addition the graded S-module $\mathrm{Ext}_T(X; X)$ is nitely generated for each $X \in T^c$, the subset $\mathrm{supp}_S X$ of $\mathrm{Spec}\,S$ is closed and, by [12, Theorem 6.1], the bijection above restricts to a bijection

$$\text{(4.11)} \qquad \text{Thick tensor ideals of } T^c \xrightarrow{\mathrm{supp}_S(\ )} \left\{\begin{matrix}\text{Specialisation closed} \\ \text{subsets of } \mathrm{supp}_S 1\end{matrix}\right\}:$$

Here is a slightly dierent perspective on the stratication property.

4.12. **Lemma.** Let T be an S-linear tensor-triangulated category as above. Then the tensor-triangulated category T is stratied by S if and only if for any $X; Y$ in T there are equivalences

$$\mathrm{supp}_S X \subseteq \mathrm{supp}_S Y \iff \mathrm{Loc}^{\otimes}(X) \subseteq \mathrm{Loc}^{\otimes}(Y) \iff X \in \mathrm{Loc}^{\otimes}(Y):$$

Proof. We use the fact that for each q in $\mathrm{Spec}\,S$ we have

$$_qT = \{X \in T \mid \mathrm{supp}_S X \subseteq \{q\}\}$$

by [11, Corollary 5.9]. Evidently when the stated property holds the localising tensor ideal $_qT$ of T is minimal for each q in $\mathrm{Spec}\,S$, so T is stratied by S as a tensor-triangulated category. The converse is equally clear.

We focus on the case

$$T := K(\mathrm{Proj}\,A) \qquad \text{with} \qquad T^c = D^b(\mathrm{mod}\,A)$$

for a cocommutative Hopf algebra A over R, and S its cohomology algebra. Here is one of the main results of our work. When it applies[2], one gets a classication

[2] it always does; see the last paragraph of the introduction.

of the localising tensor ideals of $K(\operatorname{Proj} A)$ and also the thick tensor ideals of its subcategory of compact objects, which identies with $D^b(\operatorname{mod} A)$.

4.13. Theorem. Let R be a regular ring and A a nite cocommutative Hopf algebra over R. Let S denote the nitely generated R-algebra $\operatorname{Ext}_A(R;R)$. If the map $_p$ in (4.8) is bijective for each p in $\operatorname{Spec} R$, then the tensor-triangulated category $K(\operatorname{Proj} A)$ is stratied by the action of S, and $\operatorname{supp}_S K(\operatorname{Proj} A) = \operatorname{Spec} S$.

Proof. The main task is to verify that when $X; Y$ are objects in $K(\operatorname{Proj} A)$ with $\operatorname{supp}_S X \quad _S \operatorname{supp} Y$, the complex X is in $_p \operatorname{Loc}(Y)$; see Lemma 4.12. Since ^{a} is a homeomorphism for each p in $\operatorname{Spec} R$, Lemma 4.9 yields an inclusion

$$\operatorname{supp}_{S(p)} X_{k(p)} \quad \operatorname{supp}_{S(p)} Y_{k(p)} :$$

Since $A_{k(p)}$ is a nite dimensional cocommutative Hopf algebra over $k(p)$, the triangulated category $K(\operatorname{Proj} A_{k(p)})$ is stratied by the action of it cohomology algebra, $S(p)$; this is the main result of [16]. Thus the inclusion above implies

$$X_{k(p)} \; 2 \; \operatorname{Loc} (Y_{k(p)}) :$$

This holds for each p in $\operatorname{Spec} R$, so we can apply Theorem 4.6 to deduce that X is in $\operatorname{Loc}(Y)$ as desired.

It remains to observe that $\operatorname{supp}_S 1 = \operatorname{Spec} S$, as follows, from example, from Lemma 4.9, for $1_{k(p)}$ is the unit of $K(\operatorname{Proj} A_{k(p)})$ and its support is $\operatorname{Spec} S(p)$.

In Section 5 we prove that group algebras of nite groups satisfy the hypotheses of the preceding result. Here is one more family of examples to which it applies.

4.14. Example. Let R be a regular ring. Set $A := \wedge_R F$, the exterior algebra on a nite free R-module F. We view it as a $Z{=}2$-graded Hopf algebra, with coalgebra structure dened by $(x) = x \; 1 + 1 \; x$ for $x \; 2 \; F$. In this case $\operatorname{Ext}_A(R;R)$ is the symmetric algebra on $\operatorname{Hom}_R(F;R)$. Given this it is clear that the hypotheses of Theorem 4.13 are satised in this case.

Here is another family of examples: Suppose k is eld, R a k-algebra, and that the Hopf algebra A is of the form $R \; _k \; A^0$ where A^0 is a nite dimensional cocommutative Hopf algebra over k. Then $\operatorname{Ext}_A(R;R) = R \; _k \operatorname{Ext}_{A^0}(k;k)$ as graded R-algebras. With this, it is easy to verify that A falls under the purview of Theorem 4.13.

Next we prepare to prove Theorem 1.2 stated in the introduction.

Gorenstein projective modules. Let R be a regular ring and A a nite cocommutative Hopf R-algebra. The bres $A_{k(p)}$ are nite dimensional cocommutative Hopf algebras over $k(p)$, hence self-injective [29, Lemma I.8.7]. Thus the R-algebra A is brewise self-injective and therefore Gorenstein, by Proposition 3.1. As R is regular, Gorenstein projective R-modules are projective by Lemma 2.4. Hence a Gorenstein projective A-module is projective as an R-module, and the converse holds if the module is nitely generated or $\dim R$ is nite; see Lemma 3.10.

4.15. Lemma. Let R be a regular ring and A a nite cocommutative Hopf algebra. The tensor product $_R$ with diagonal A-action endows $\operatorname{GProj} A$ with a struc-ture of a rigidly-compactly generated tensor-triangulated category. This structure is compatible with the equivalence in Theorem 3.5.

Proof. If X is an acyclic complex of projective A-modules and N is a Gorenstein projective A-module, then the complex $X \otimes_R N$ of projective modules is also acyclic, for N is projective as an R-module. It follows that if M is a Gorenstein projective A-module, so is $M \otimes_R N$. Thus the category of Gorenstein projective A-modules is closed under $\otimes_R$, and R, viewed as an A-module via the augmentation $A \to R$ is the unit of this product. Observe that as an A-module R is Gorenstein projective, for it is finitely generated, and projective as an R-module. Since the A-module $P \otimes_R N$ is projective when P is projective, this tensor product induces one on the stable category, $\underline{\mathrm{GProj}}\,A$, making it a tensor-triangulated category, with unit R. The function object on $\underline{\mathrm{GProj}}\,A$ is $\underline{\mathrm{Hom}}_R(-,-)$, and given this it is easy to verify the rigid objects in it are precisely the compact objects, that is to say, isomorphism classes of the finitely generated Gorenstein projective modules. In summary, $\underline{\mathrm{GProj}}\,A$ is rigidly compactly generated.

A straightforward computation shows that the assignment in Theorem 3.5 is compatible with the tensor structures. □

Let $S := \mathrm{Ext}^*_A(R, R)$ be the cohomology algebra as before. We write $\mathrm{Proj}\,S$ for the projective spectrum of S, namely, those prime ideals in $\mathrm{Spec}\,S$ that do not contain the ideal $S^{\geqslant 1}$ of positive degree elements.

4.16. Lemma. *With the assumptions from Theorem 4.13, the full subcategory $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A)$ of $\mathbf{K}(\mathrm{Proj}\,A)$ is a localising tensor ideal with support $\mathrm{Proj}\,S$.*

Proof. For each $\mathfrak{p}$ in $\mathrm{Spec}\,R$ the functor $X \mapsto X_{k(\mathfrak{p})}$ maps the recollement (3.3) for A to the corresponding recollement for $A_{k(\mathfrak{p})}$. To see this, observe that the recollement (3.3) is determined by functorial exact triangles $\mathbf{p}X \to X \to \mathbf{t}X \to$ for each X in $\mathbf{K}(\mathrm{Proj}\,A)$ such $\mathbf{p}X$ is K-projective and $\mathbf{t}X$ is acyclic. These properties are preserved by $(-)_{k(\mathfrak{p})}$ since the functor is exact and preserves all coproducts. For K-projectives this is clear, since they are generated by perfect complexes which are preserved by $(-)_{k(\mathfrak{p})}$. For acyclic complexes, see Lemma 3.6.

The assertion of the lemma now follows since $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A_{k(\mathfrak{p})})$ is a localising tensor ideal of $\mathbf{K}(\mathrm{Proj}\,A_{k(\mathfrak{p})})$ with support $\mathrm{Proj}\,S(\mathfrak{p})$; see [16, §10]. □

We are now ready to prove our stratification result for representations of finite cocommutative Hopf algebras.

Proof of Theorem 1.2. We use the triangle equivalence $\underline{\mathrm{GProj}}\,A \xrightarrow{\sim} \mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A)$ from Theorem 3.5, which preserves the tensor structure and the S-action thanks to Lemma 4.15. Now the stratification of $\underline{\mathrm{GProj}}\,A$ via S follows from Theorem 4.13, since $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\,A)$ is a localising tensor ideal of $\mathbf{K}(\mathrm{Proj}\,A)$ by Lemma 4.16. In particular, the support of $\underline{\mathrm{GProj}}\,A$ is precisely $\mathrm{Proj}\,S$. □

5. Finite groups

Let G be a finite group. The main result of this section is that for $A = RG$, the group algebra of G over any commutative noetherian ring R, the map (4.8) is a homeomorphism for all primes in the spectrum of R. As a consequence we get a stratification theorem when R is regular; see Theorem 5.7. In this case the cohomology algebra $\mathrm{Ext}^*_{RG}(R, R)$ is the group cohomology algebra. This is usually denoted $H^*(G, R)$, and we follow suit.

Cup products. Let R be a commutative ring, not necessarily noetherian, and M an R-module, both viewed as G-modules with trivial action. The cup product makes $H(G;R)$ an R-algebra and $H(G;M)$ a module over $H(G;R)$. These are dened as follows: Let P be a projective resolution of the trivial ZG-module Z and $: P \to P$ ${}_{Z}\, P$ a diagonal approximation. Given classes $x \in H(G;R)$ and $y \in H(G;M)$, represented by cocycles $\tilde{x} \in \mathrm{Hom}_{ZG}(P;R)$ and $\tilde{y} \in \mathrm{Hom}_{ZG}(P;M)$ the cup product $x \cup y$ is represented by the class of the composition of maps

$$P \to P {}_{Z} P \xrightarrow{\tilde{x}\ \tilde{y}} R {}_{Z} M \to M:$$

If $I \subseteq R$ is an ideal, $\cup$ denes a product on $H(G;I)$. It is clear from the denition that if I is nilpotent of order n, then so is $H(G;I)$.

Innitesimal deformations of coecients. Let $R \to R^0$ be a surjective map of commutative rings whose kernel, say I, satises $I^2 = 0$; thus I is an R^0-module. One thinks of R as an innitesimal deformation of R^0. The exact sequence

$$0 \to I \to R \to R^0 \to 0 \tag{5.1}$$

induces a connecting homomorphism

$$: H(G;R^0) \to H(G;I):$$

Since I is an R^0-module $H(G;I)$ is a module over $H(G;R^0)$, via the cup product. The statement of the result below, and its proof, are a variation on [7, Lemma 4.3.3].

5.2. Lemma. In the context above, for $x;y \in H(G;R^0)$ one has

$$(x \cup y) = (x) \cup y + (-1)^{|x|} x \cup (y):$$

Proof. As in the proof of [7, Lemma 4.3.3], let P be the projective resolution of the trivial ZG-module Z, and $: P \to P$ ${}_{Z}\, P$ a diagonal approximation. The exact sequence (5.1) induces the exact sequence of complexes

$$0 \to \mathrm{Hom}_{ZG}(P;I) \to \mathrm{Hom}_{ZG}(P;R) \to \mathrm{Hom}_{ZG}(P;R^0) \to 0:$$

Represent x and y by cocycles $\tilde{x}$ and $\tilde{y}$ on the right hand side of this sequence. Then $x \cup y$ is represented by the composite

$$\tilde{x} \cup \tilde{y} : P \to P {}_{Z} P \xrightarrow{\tilde{x}\ \tilde{y}} R^0 {}_{Z} R^0 \to R^0$$

where is the multiplication map. To compute the eect of the connecting homomorphism, we rst lift $\tilde{x}$ and $\tilde{y}$ to cochains $\hat{x}$ and $\hat{y}$ in $\mathrm{Hom}_{ZG}(P;R)$. Since $d\tilde{x} = 0 = d\tilde{y}$, the elements $d\hat{x}$ and $d\hat{y}$ lie in I. The element $\tilde{x} \cup \tilde{y}$ lifts to $\hat{x} \cup \hat{y}$, and

$$\begin{aligned} d(\hat{x} \cup \hat{y}) &= d\hat{x} \cup \hat{y} + (-1)^{|x|} \hat{x} \cup d\hat{y} \\ &= d\hat{x} \cup \tilde{y} + (-1)^{|x|} \tilde{x} \cup d\hat{y} \end{aligned}$$

The second equality holds as $d\hat{x}$ and $d\hat{y}$ lie in I. This gives the stated equality.

In what follows, we say that an abelian group M is p-local, for a prime number p, if the natural map $M \to M_{(p)}$ is an isomorphism.

5.3. Lemma. Let p be a prime dividing $|G|$ and $: R \to R^0$ a map of p-local rings, with $\mathrm{Ker}()$ nilpotent. Then the map $H(G;)$ has nilpotent kernel, and there exists an integer n such that for any element $x \in H^{\geqslant 1}(G;R^0)$, the element x^{p^n} is in the image of the map $H(G;)$.

Proof. Set $I := \mathrm{Ker}()$. Since this ideal is nilpotent, so is $H(G;I)$, under cup products. The claim about nilpotence is clear because, by the exact sequence in cohomology arising from (5.1), the kernel of $H(G;\ )$ is the image of the map

$$H(G;I) \to H(G;R)$$

which respects cup products.

As to the second part of the statement, it suces to consider the case where $I^2 = 0$. Let n be the largest integer such that p^n divides $|G|$. Since $|G|$ annihilates $H^{>1}(G;R^0)$, and the ring R^0 and hence also $H(G;R^0)$ is p-local, one gets that

$$p^n \cdot H^{>1}(G;R^0) = 0.$$

If $|x|$ is odd, then $2 \cdot x^2 = 0$, since $H(G;R^0)$ is graded-commutative. Thus if also p is odd, then $x^2 = 0$, since we are in the p-local situation. Thus we can suppose either $|x|$ is even or $p = 2$. In either case, a repeated application of Lemma 5.2 yields $(x^i) = i x^{i-1}(x)$ for each $i \geq 1$. In particular $(x^{p^n}) = 0$. It then follows from the exact sequence in group cohomology arising from (5.1) that x^{p^n} is in the image of the map $H(G;R) \to H(G;R^0)$.

Modules with bounded torsion. Let M is an abelian group such that its torsion-subgroup, denoted $\mathrm{tors}(M)$ is bounded; that is to say, there exists an integer n such that $n \cdot \mathrm{tors}(M) = 0$. Fomin [21] proved that inclusion $\mathrm{tors}(M) \subseteq M$ splits; see also [31, Corollary pp. 134]. This result will be used below.

5.4. Lemma. *Let p be a prime dividing $|G|$ and M a p-local abelian group such that $\mathrm{tors}(M)$ is bounded. For all integers $s \geq 0$ the map*

$$H^{>1}(G;M) \to H^{>1}(G;M/p^sM)$$

induced by the surjection $M \to M/p^sM$, is one-to-one.

Proof. Since M is p-local, the only torsion is p-torsion. Choose $s \geq 0$ such that p^s

$$\mathrm{tors}(M) = 0 = p^s H^{>1}(G;M).$$

The equality on the left means that the sequence below, where the map $M \to p^sM$ is given by $m \mapsto p^sm$, is exact:

$$0 \to \mathrm{tors}(M) \to M \to p^sM \to 0.$$

This is split-exact, by Fomin's result recalled above, so the induced map

$$H(G;M) \to H(G;p^sM)$$

is surjective. The map $M \xrightarrow{p^s} M$ factors as $M \to p^sM \to M$ where the one on the right is inclusion. By the choice of p^s, the composition of the induced maps

$$H(G;M) \to H(G;p^sM) \to H(G;M)$$

is zero in degrees ≥ 1. Since the map on the left is surjective, it follows that the one on the right is zero in degrees ≥ 1. Then the cohomology exact sequence arising from the exact sequence

$$0 \to p^sM \to M \to M/p^sM \to 0$$

yields the desired statement.

Noetherian ring of coecients. The result below was proved by Benson and Habegger [10] when $R = \mathbb{Z}$; the argument given here is modeled on their proof. A general result, allowing non-trivial G-action on R, was proved by Lau [33, Section 7].

Recall that a map of rings $f : S \to T$ containing a eld of positive characteristic p is an F-isomorphism if $\ker(f)$ is nilpotent, and for each $t \in T$ there exists an n such that t^{p^n} is the image of f.

5.5. Theorem. Let G be a nite group and R a commutative noetherian ring. For each prime number p, the map

$$H^*(G;R) \otimes_R R/pR \to H^*(G;R/pR)$$ is an F-isomorphism.

Proof. Set $R_{(p)} := \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} R$. As R/pR is p-local, the map $R \to R/pR$ factors through $R_{(p)}$. As localisation is an exact functor, there are natural isomorphisms

$$\begin{aligned} H^*(G;R) \otimes_R R/pR &\cong H^*(G;R_{(p)}) \otimes_{R_{(p)}} R/pR \\ &\cong H^*(G;R_{(p)}/pR_{(p)}) . \end{aligned}$$

Thus replacing R by $R_{(p)}$ we can assume R is p-local.

For any nitely generated R-module M, the (additive) torsion submodule $\mathrm{tors}(M)$ is an R-submodule of M, and hence nitely generated as an R-module, as R is noetherian. It follows that $\mathrm{tors}(M)$ is bounded, so Lemma 5.4 applies.

Choose an ~~integer s~~ large ~~enough that~~ the ~~conclusion~~ of ~~op. cit.~~ applies to the R-modules R and pR. Consider the commutative diagram of coecients

$$\begin{tikzcd} 0 \arrow[r] & pR \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/pR \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & pR/p^{s+1}R \arrow[r] & R/p^{s+1}R \arrow[r] & R/pR \arrow[r] & 0 \end{tikzcd}$$

This induces a commutative diagram

$$\begin{tikzcd} H^*(G;R) \arrow[r] \arrow[d, "\alpha"] & H^*(G;R/pR) \arrow[r, "\delta_1"] \arrow[d, equal] & H^{*+1}(G;pR) \arrow[d, "\beta"] \\ H^*(G;R/p^{s+1}R) \arrow[r] & H^*(G;R/pR) \arrow[r, "\delta_2"] & H^{*+1}(G;pR/p^{s+1}R) \end{tikzcd}$$

where δ_1 and δ_2 are the connecting maps. The choice of s ensures that α and β are injective in positive degrees; see Lemma 5.4. Since $\beta^{\geqslant 1}$ is injective and the kernel of δ_2 is nilpotent, by Lemma 5.3, so is the kernel of δ_1 in positive degrees. This map factors through the map

$$H^*(G;R) \otimes_R R/pR \to H^*(G;R/pR)$$

so the latter is one-to-one up to nilpotence.

Fix $x \in H^{\geqslant 1}(G;R/pR)$. Applying Lemma 5.3 to the map $R/p^sR \to R/pR$ yields that for some $n \geqslant 1$ the element x^{p^n} is in the image of γ. So in the diagram above, we have $\delta_2(x^{p^n}) = 0$. Since $\beta^{\geqslant 1}$ is injective we have $\delta_1(x^{p^n}) = 0$. It follows from the exactness of the top row of the diagram that x^{p^n} is in the image of γ_1. This implies that the map in the statement of the theorem is F-onto.

Here is a consequence of the preceding theorem.

5.6. Corollary. Let G be a finite group and R a commutative noetherian ring. For any map of rings $R \to k$ with k a field of positive characteristic, the natural map

$$H(G;R) \otimes_R k \to H(G;k)$$

is an F-isomorphism, and hence the induced map on spectra

$$\operatorname{Spec} H(G;k) \to \operatorname{Spec}(H(G;R) \otimes_R k)$$

is a homeomorphism.

Proof. Let p be the characteristic of k. The map $R \to k$ factors through R/pR. Applying $-\otimes_R k$ to the F-isomorphism in Theorem 5.5 yields the F-isomorphism

$$H(G;R) \otimes_R k \to H(G;R/pR) \otimes_{R/pR} k.$$

It remains to observe that the right hand side is isomorphic to $H(G;k)$.

5.7. Theorem. With G a finite group and R a regular ring, the tensor-triangulated category $K(\operatorname{Proj} RG)$ is stratified by the action of $H(G;R)$.

Proof. The R-algebra $H(G;R)$ is finitely generated by a result of Evens [20] and Venkov [40]. Thus the result follows from Theorem 4.13 and Corollary 5.6.

An immediate, and standard, consequence of the stratification is a classification of thick tensor ideals in the bounded derived category (and also the small stable module category) of RG-modules. An analogous classification with Proj holds for Gproj.

5.8. Corollary. Let G be a finite group and R be a regular local ring. There is a one-to-one correspondence between thick tensor ideal subcategories in $D^b(\operatorname{mod} RG)$ and specialisation closed subsets in $\operatorname{Spec}(H(G;R))$, the spectrum of homogeneous prime ideals in $H(G;R)$.

5.9. Theorem 5.7 yields classifications of thick and localising tensor ideals which have recent predecessors. Corollary 5.8 is established as a main theorem in Lau's work [33] where algebraic techniques similar to ours are used. Lau only works with finitely generated modules computing the Balmer spectrum of the category of perfect complexes on the Deligne Mumford stack $[(\operatorname{Spec} R)/G]$, that is the derived category of bounded complexes of G-equivariant finitely generated projective R-modules. In the case of a regular ring R this is precisely the derived category $D^b(\operatorname{mod} RG)$. Though Lau stays in the realms of small categories, he allows for non-trivial action of G on the ring R so his result is more general in that setting.

In the series of papers [3, 4], Barthel takes a very different approach to the stratification of representations of a finite group G over a ring R, by developing powerful homotopy theory machinery, with an eye towards applications to many other topological situations. In particular, in contrast to our fibrewise approach, Quillen's philosophy of reducing the question to elementary abelian groups enters into Barthel's homotopy theoretic methods. To make a direct comparison of our specific applications, Barthel's [4, Theorem C], which classifies localising tensor ideals in the (exact) category of R-linear representations of a finite group G whose underlying R-module is projective, can be seen as a direct consequence of Theorem 5.7, under the assumption that R is regular. We arrive to this application though through entirely different routes.

5.10. For a group algebra RG there are other possible versions of a stable module category. For instance, one can endow the category of RG-modules with the exact structure given by those short exact sequences which are split exact when restricted to the trivial subgroup. Then the category $\operatorname{mod} RG$ of nitely presented RG-modules is a Frobenius category, and the corresponding stable category $\operatorname{stmod} RG$ is in fact a tensor triangulated category [15]. For general R this category is dierent from $\underline{\operatorname{Gproj}} RG$, as can be seen from the discussion in [15, x7] for $R = \mathbb{Z}$.

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