Explicit computation of symmetric differentials
and its application to quasihyperbolicity

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We develop explicit techniques to investigate algebraic quasihyperbolicity of singular surfaces through the constraints imposed by symmetric differentials. We apply these methods to prove that rational curves on Barth's sextic surface, apart from some well-known ones, must pass through at least four singularities, and that genus 1 curves must pass through at least two. On the surface classifying perfect cuboids, our methods show that rational curves, again apart from some well-known ones, must pass through at least seven singularities, and that genus 1 curves must pass through at least two.

We also improve lower bounds on the dimension of the space of symmetric differentials on surfaces with A_1-singularities, and use our work to show that Barth's decic, Sarti's surface, and the surface parametrizing 3×3 magic squares of squares are all algebraically quasihyperbolic.

1. Introduction

A complex projective surface Y is *algebraically hyperbolic* if there is an $\epsilon > 0$ such that for every curve $C \subset Y$ of geometric genus $g(C)$, we have $2g(C) - 2 \geq \epsilon \deg(C)$; in particular, such a surface does not contain curves of geometric genus 0 or 1. In this article we call a complex projective surface Y *algebraically quasihyperbolic* if it contains only finitely many curves of geometric genus 0 or 1.

Algebraically quasihyperbolic surfaces abound: for example, if $Y/\mathbb{C}$ is a smooth proper surface of general type whose Chern classes satisfy $c_1(Y)^2 > c_2(Y)$, then Y is algebraically quasihyperbolic, by work of Bogomolov [1977, Corollary 5]. Surfaces of general type with an ample cotangent bundle satisfy the requisite Chern class inequality [Fulton and Lazarsfeld 1983]. On the other hand, for a smooth complex surface $Y \subset \mathbb{P}^3$ of degree $d \geq 5$ the inequality $c_1(Y)^2 > c_2(Y)$ does not hold; nevertheless, genus bounds of Xu [1994] show that a very general such surface is also algebraically quasihyperbolic, a statement that had been conjectured by Harris. Recently, Coskun and Riedl [2019] improved Xu's bounds to show that a very general complex surface $Y \subset \mathbb{P}^3$ of degree $d \geq 5$ is in fact algebraically hyperbolic.[1]

Simple abelian surfaces are also algebraically quasihyperbolic in the sense above: it is well known that every map from $\mathbb{P}^1$ to any abelian variety is constant (e.g., such a map necessarily factors through the

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[1]See [Demailly 2020, Remark 3.4] for a history of progress on genus bounds for curves on a very general surface $Y \subset \mathbb{P}^3$.

Albanese variety of $\mathbb{P}^1$, which is a point). Alternatively, and in the spirit of our work, a regular differential on an abelian surface Y would pull back to the zero differential on any genus 0 curve $C \subset Y$. Since the cotangent sheaf Ω_Y^1 is globally generated, no such C can exist.

Not all is lost if Y is a surface for which Ω_Y^1 has no global sections, as one can mimic the above argument with a different sheaf, for instance the symmetric powers $S^m\Omega_Y^1$. This idea is central to Bogomolov's result [1977] that if the Chern classes of Y satisfy the inequality $c_1(Y)^2 > c_2(Y)$, then $S^m\Omega_Y^1$ has global sections for large enough m, and as a consequence Y is algebraically quasihyperbolic.

Bogomolov and de Oliveira [2006] observed that if $\tau : Y \to X$ is a minimal resolution of a surface X with A_1 singularities, then it is possible that $S^m\Omega_Y^1$ has global sections for large enough m even when $c_1(Y)^2 \leq c_2(Y)$.

In principle, symmetric differentials on Y, or more generally elements of $H^0(X, (\tau_*S^m\Omega_Y^1)^{\vee\vee})$, can be used to constrain the locus of genus 0 or 1 curves on X. To wit, an element $\omega \in H^0(Y, S^m\Omega_Y^1) \simeq H^0(\mathbb{P}(\Omega_Y^1), \mathscr{O}_{\mathbb{P}(\Omega_Y^1)}(m))$ defines a surface $Y' \subset \mathbb{P}(\Omega_Y^1)$ in the cotangent bundle of Y that is a multisection of degree m for the projection $\mathbb{P}(\Omega_Y^1) \to Y$. If, for example, Ω_Y^1 is big then any curve of genus 0 or 1 on Y lifts to a leaf of the foliation on Y' induced by ω. This idea is already present in Bogomolov's work [Bogomolov 1977; Deschamps 1979], and was amplified by Green and Griffiths [1980]. However, computing the integral curves defined by the degree m first-order differential equation that ω describes on Y' is in general a very difficult problem, though it has been successfully carried out in a few cases: On surfaces related to Büchi's problem, Vojta [2000] determined an explicit symmetric differential, as well as a description of the solution curves to the corresponding differential equation to determine the genus 0 and 1 curves. This line of research was significantly expanded by García-Fritz [2018a; 2018b] and by García-Fritz and Urzúa [2020] for other surfaces, including the surface parametrizing perfect cuboids, for which they show that every curve of genus 0 or 1 must pass through at least 2 nodes. The calculations in [Bogomolov and de Oliveira 2006] are asymptotic in m and thus cannot be used to explicitly determine the locus in Y containing the genus 0 and 1 curves.

1A. *Contributions to the study of symmetric differentials.* This article contributes to the study of algebraic quasihyperbolicity in two ways. First, we lay out *explicit methods* for the calculation of the restrictions imposed by symmetric differentials on curves of genus 0 or 1 on nodal surfaces, and showcase our methods on specific surfaces (e.g., Barth's sextic and the surface parametrizing perfect cuboids). Second, we give new, nonasymptotic lower bounds for the dimensions of spaces of symmetric differentials on resolutions of complete intersection surfaces with du Val (ADE) singularities. These bounds allow us to increase the range of surfaces covered by [Bogomolov and de Oliveira 2006; Roulleau and Rousseau 2014] that are known to be algebraically quasihyperbolic, which now includes, for example, Barth's decic [1996] and Sarti's surface [2001]. See Remark 4.3 for a full discussion. They also allow us to show that the surface parametrizing 3×3 magic squares of squares is algebraically quasihyperbolic.

1B. *Set-up.* For the rest of this section, we use the following notation: $X \subset \mathbb{P}^n$ denotes a complex projective surface that is a complete intersection of multidegree $(d_1, \ldots, d_{n-2})$, with a singular locus S

consisting of ℓ isolated du Val singularities $\{s_1, \ldots, s_\ell\}$. We let $\tau : Y \to X$ be a minimal resolution of X, with exceptional locus E. We write $\hat{S}^m \Omega_X^1$ for the reflexive hull $(S^m \tau_* \Omega_Y^1)^{\vee\vee}$.

1C. *Explicit methods.* Using graded modules over the coordinate ring of X, we explain how to *explicitly compute* a basis for the vector space $\mathrm{H}^0(X, \hat{S}^m \Omega_X^1)$. Current technology suffices to execute these ideas for small values of m. Using the identifications

$$\mathrm{H}^0(X, \hat{S}^m \Omega_X^1) \simeq \mathrm{H}^0(X - S, S^m \Omega_X^1) \simeq \mathrm{H}^0(Y - E, S^m \Omega_Y^1),$$

we study subspaces of sections of $\mathrm{H}^0(X, \hat{S}^m \Omega_X^1)$ that can be extended to (at least part of) the exceptional divisor E on Y. See Section 5 for details.

As mentioned already, a section $\omega \in \mathrm{H}^0(Y, S^m \Omega_Y^1)$ yields a degree m first-order differential equation that any genus 0 curve $C \subset Y$ must satisfy, reflecting that ω must pull back to 0 on C. Two linearly independent sections ω_1 and ω_2 force C to *simultaneously* satisfy two differential equations. Whether this is possible can be expressed in terms of the vanishing of a resultant variety $\mathrm{res}(\omega_1, \omega_2)$; a precise definition of this variety is given in Section 6. If this resultant does not vanish identically, it furnishes a closed proper sublocus of Y to which C must belong. An analysis of the irreducible components of $\mathrm{res}(\omega_1, \omega_2)$ then yields a complete list of genus 0 curves on Y.

In concrete examples, it is possible that $\mathrm{H}^0(Y, S^m \Omega_Y^1) = 0$ for small values of m. Not all is lost. Already two linearly independent sections $\omega_1, \omega_2 \in \mathrm{H}^0(X, \hat{S}^m \Omega_X^1)$ give rise to a closed sublocus $\mathrm{res}(\omega_1, \omega_2) \subset X$; if this closed set is strictly contained in X, then among its irreducible components one finds all *complete* curves $C \subset X$ of genus 0 that do not pass through any of the nodes of X.

Crucially, intermediate subspaces of $\mathrm{H}^0(Y, S^m \Omega_Y^1) \subset \mathrm{H}^0(X, \hat{S}^m \Omega_X^1)$ can at once strengthen the above claims on genus 0 curves on X (or Y), as well as give constraints on the locus of genus 1 curves on X. For example, we show that if $X \subseteq \mathbb{P}^3$ is a surface with ℓ isolated A_1 singularities, and if $H \subset \mathbb{P}^3$ is a plane, then two linearly independent elements ω_1 and ω_2 in the intermediate subspace

$$\mathrm{H}^0(Y, S^2 \Omega_Y^1) \subset \mathrm{H}^0\big(X - S, (S^2 \Omega_X^1)(-H)\big) \subset \mathrm{H}^0(X, \hat{S}^2 \Omega_X^1)$$

can be extended to regular differentials over the nodes $S \cap H$ (see Corollary 3.4), and therefore their pullbacks $\tau^* \omega_1$ and $\tau^* \omega_2$ extend to regular differentials on the part of the exceptional component E lying over $S \cap H$. Let $S' = S - H$; we show that any complete genus 0 curve $C \subset X - S'$ must lie in the closed proper subset $\mathrm{res}(\omega_1, \omega_2) \subset X$. In particular examples, we explicitly determine this locus and decompose it into irreducible components. Any genus 0 curve on X that passes only through nodes in $X \cap H$ must be among these components.

By varying the plane H across the set of planes in $\mathbb{P}^3$ spanned by any three A_1 singularities of X we may conclude that any curve $C \subset X$ of genus 0 must be among the curves we encountered or must pass through at least four noncoplanar nodes of X. In addition, for any complete curve $C \subset X - S$ of genus 1, the pullback of the differentials ω_1 and ω_2 to C each vanish on $C \cap H$, and thus they must both be identically zero on C. This forces the curve to be contained in the closed proper sublocus $\mathrm{res}(\omega_1, \omega_2)$. If, on the other hand, the intersection $C \cap S$ is not empty, then C must be contained in a linear space

whose dimension is one more than that of the span of $C \cap S$ (see Corollary 6.5), and this forces C to pass through at least two singularities in $C \cap S$.

As an example of this circle of ideas, we prove restrictions on the locus of genus 0 or 1 curves on Barth's sextic surface $X_6 \subset \mathbb{P}^3_{\mathbb{Q}(\sqrt{5})}$, a surface whose singular locus consists of 65 isolated A_1 singularities.

Theorem 1.1. *Let* $\phi = \frac{1}{2}(\sqrt{5}+1)$, *and let* $X_6 \subset \mathbb{P}^3_{\mathbb{Q}(\sqrt{5})}$ *be Barth's sextic surface, defined by*

$$X_6 : 4(\phi^2 x^2 - y^2)(\phi^2 y^2 - z^2)(\phi^2 z^2 - x^2) - (1 + 2\phi)(x^2 + y^2 + z^2 - w^2)^2 = 0.$$

Any genus 0 *curve on* X_6 *must pass through at least four singularities. Furthermore, there are exactly* 27 *genus* 0 *curves on* X_6 *lying on planes spanned by singularities* (they are listed in Section 8A); *any genus* 0 *curve on* X_6 *not among these* 27 *curves must pass through at least four singularities that span* $\mathbb{P}^3$. *Any genus* 1 *must pass through at least two singularities and lie in a plane or pass through at least three noncollinear singularities.*

We also obtain restrictions on the locus of genus 0 and 1 curves on the surface X_{pc} parametrizing *perfect cuboids*, i.e., cuboids with all sides x_1, x_2, x_3, diagonals y_1, y_2, y_3 and body diagonal z rational. It is a complete intersection in $\mathbb{P}^6$ of four quadrics:
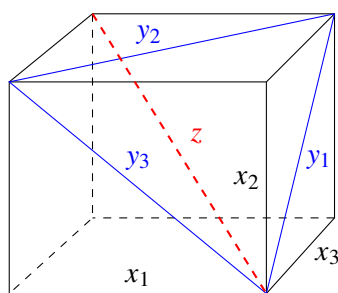
$$X_{\mathrm{pc}} : \begin{cases} y_1^2 = x_2^2 + x_3^2, \\ y_2^2 = x_3^2 + x_1^2, \\ y_3^2 = x_1^2 + x_2^2, \\ z^2 = x_1^2 + x_2^2 + x_3^2. \end{cases}$$

The surface X_{pc} has 48 singularities of type A_1; it contains 32 plane conic curves and 60 genus 1 curves identified by van Luijk in [2000] (see Section 7). We prove:

Theorem 1.2. *Let* X_{pc} *be the perfect cuboid surface. Any genus* 0 *curve on* X_{pc} *must pass through at least six distinct singularities, Any genus* 0 *curve on* X_{pc} *other than van Luijk's* 32 *plane conics must pass through at least seven singularities that span* $\mathbb{P}^6$.

Any genus 1 *curve on* X_{pc} *lies in a linear space of at most one dimension higher than the linear space spanned by the singularities it passes through. In particular, any genus* 1 *curve on* X_{pc} *passes through at least two singularities and is a component of a hyperplane section or passes through at least six singularities spanning a hyperplane.*

Remark 1.3. García-Fritz and Urzúa [2020] study a natural composition of cyclic covers $X_{\mathrm{pc}} \to X_0 := \mathbb{P}^1 \times \mathbb{P}^1$. By investigating integral curves for the pullback of a section $\omega \in \mathrm{H}^0\big(X_0, (S^2\Omega^1_{X_0})(2,2)\big)$, they show that every curve of genus 0 or 1 on X_{pc} must pass through at least 2 nodes. Theorem 1.2 gives stronger results vis-à-vis genus 0 curves, but similar results for genus 1 curves. It would be interesting to see if their approach can be combined with our methods to produce even stronger results.

1D. *Dimensions of spaces of symmetric differentials.* Keep the notation of Section 1B. We obtain further results on quasihyperbolicity of surfaces from another angle, by leveraging a pair of computable lower bounds for the dimensions $h^0(Y, S^m\Omega_Y^1)$ in terms of the Chern classes $c_1(Y)^2$ and $c_2(Y)$, as well as ℓ, and m. The general result is best phrased using the constituent terms χ^0 and χ^1 of Wahl's local Euler characteristic [1993] (see Section 3). For a singularity $s \in X$, we denote by X° an analytic neighborhood of s, and let Y° be its inverse image under τ; we write E_s for the component of E in Y° above s. The quantity $\chi^0(s, S^m\Omega_Y^1)$ is the codimension of the subspace of $\mathrm{H}^0(Y^\circ - E_s, S^m\Omega_{Y^\circ}^1) \simeq \mathrm{H}^0(X^\circ, \hat{S}^m\Omega_{X^\circ}^1)$ of sections that extend over s; the quantity $\chi^1(s, S^m\Omega_Y^1)$ is the dimension of $\mathrm{H}^1(Y^\circ, S^m\Omega_{Y^\circ}^1)$ around a node.

Theorem 1.4. *Let X be an irreducible complex projective surface whose singular locus S is a finite set of isolated du Val singularities. Let $\tau : Y \to X$ be a minimal resolution. Then, for $m \geq 3$, we have*

$$h^0(Y, S^m\Omega_Y^1) \geq \chi(Y, S^m\Omega_Y^1) + \sum_{s \in S} \chi^1(s, S^m\Omega_Y^1). \tag{1-1}$$

Moreover, the inequality

$$h^0(Y, S^m\Omega_Y^1) \geq h^0(X, \hat{S}^m\Omega_X^1) - \sum_{s \in S} \chi^0(s, S^m\Omega_Y^1) \tag{1-2}$$

holds for $m \geq 1$.

Conceptually, the lower bound (1-2) records that the conditions imposed by individual singularities to extend a section of $\mathrm{H}^0(X, \hat{S}^m\Omega_X^1)$ to one in $\mathrm{H}^0(Y, S^m\Omega_Y^1)$ are at worst linearly independent.

The term $\chi(Y, S^m\Omega_Y^1)$ in the right-hand side of (1-1) can be computed by combining a result of Atiyah [1958] (see Lemma 2.1) with (A-5), which is a standard Chern class computation included in the Appendix. We get

$$\chi(Y, S^m\Omega_Y^1) = \tfrac{1}{12}\big(2(K^2 - \chi)m^3 - 6\chi m^2 - (K^2 + 3\chi)m + K^2 + \chi\big),$$

where $K^2 = c_1(Y)$ and $\chi = c_2(Y)$.

In Propositions 3.3 and 3.7, we determine *exact* expressions for $\chi^0(s, S^m\Omega_Y^1)$ and $\chi^1(s, S^m\Omega_Y^1)$ in the case where s is an A_1 singularity:

$$\chi^0(s, S^m\Omega_Y^1) = \begin{cases} \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{1}{6}m & \text{if } m \equiv 0 \pmod 6, \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 - \frac{1}{12}m - \frac{35}{108} & \text{if } m \equiv 1 \pmod 6, \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{7}{18}m + \frac{5}{27} & \text{if } m \equiv 2 \pmod 6, \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 - \frac{1}{12}m - \frac{1}{4} & \text{if } m \equiv 3 \pmod 6, \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{1}{6}m - \frac{2}{27} & \text{if } m \equiv 4 \pmod 6, \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{5}{36}m - \frac{7}{108} & \text{if } m \equiv 5 \pmod 6, \end{cases}$$

$$\chi^1(s, S^m\Omega_Y^1) = \begin{cases} \frac{4}{27}m^3 + \frac{4}{9}m^2 + \frac{1}{3}m & \text{if } m \equiv 0 \pmod 3, \\ \frac{4}{27}m^3 + \frac{4}{9}m^2 + \frac{1}{3}m + \frac{2}{27} & \text{if } m \equiv 1 \pmod 3, \\ \frac{4}{27}m^3 + \frac{4}{9}m^2 + \frac{1}{9}m - \frac{5}{27} & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Putting this all together, we obtain several algebraic quasihyperbolicity results, illustrated by the following examples.

1D1. *Nodal surfaces in* $\mathbb{P}^3$. If X is a surface of degree d in $\mathbb{P}^3$, with

$$\ell > \tfrac{9}{4}(2d^2 - 5d) \tag{1-3}$$

A_1 singularities, then the lower bound (1-1) shows that $h^0(Y, S^m \Omega^1_Y) > 0$ for all $m \gg 0$ and grows cubically with m, i.e., Y has a big cotangent bundle. On the other hand, Miyaoka's bounds [1984, §2.3] on the number of quotient singularities on surfaces imply the inequality

$$\ell \leq \tfrac{4}{9}d(d-1)^2.$$

Thus, $d = 10$ is the smallest degree ≥ 5 for which there can exist a surface with sufficiently many nodes on which our results apply. Happily, such surfaces do exist: Barth's decic surface [1996], for which $(d, \ell) = (10, 345)$, has a minimal resolution with a big cotangent bundle and is therefore algebraically quasihyperbolic. In this case, we can guarantee $h^0(Y, S^m \Omega^1_Y) > 0$ once $m \geq 160$; in fact, we show $h^0(Y, S^{160} \Omega^1_Y) \geq 15755$. Sarti's surface [2001] satisfies $(d, \ell) = (12, 600)$, and is also algebraically quasihyperbolic; in this case, we can guarantee that $h^0(Y, S^m \Omega^1_Y) > 0$ once $m \geq 28$, and that $h^0(Y, S^{28} \Omega^1_Y) \geq 7646$.

We note that the results in [Roulleau and Rousseau 2014], as well as a corrected version of those in [Bogomolov and de Oliveira 2006] (henceforth abbreviated as [BdO 2006]), do not suffice to prove algebraic quasihyperbolicity of Barth's decic or Sarti's surface. Indeed, the orbifold methods in [Roulleau and Rousseau 2014] yield the slightly weaker lower bound $\ell > \tfrac{8}{3}(2d^2 - 5d)$ in place of (1-3), while a corrected version of the calculations in [BdO 2006] yield the lower bound $\ell > \tfrac{36}{11}(2d^2 - 5d)$ in place of (1-3). See Remark 4.3 for a more thorough comparison of results.

Segre [1947] constructed hypersurfaces of even degree d in $\mathbb{P}^3$ with $\ell = \tfrac{1}{4}d^2(d-1)$ nodal singularities by taking an equation of the form

$$G^2 + \lambda \prod_{i=1}^{d} L_i,$$

where G is a form of degree $\tfrac{1}{2}d$ and L_i are linear forms, and λ is a scalar (see also [Beauville 1980, p. 208]). For $d \geq 18$, this satisfies the bound (1-3).

1D2. *Nodal complete intersections of quadrics.* If X is a complete intersection of $n - 2$ quadrics in $\mathbb{P}^n$ with ℓ isolated A_1 singularities, then we use the lower bound (1-1) to show that the resolution Y has big cotangent bundle and is algebraically quasihyperbolic for $\ell \geq \ell_{\min}(n)$, where $\ell_{\min}(n)$ is defined by

n	6	7	8	9	≥ 10
$\ell_{\min}(n)$	73	145	217	145	0

The fact that a 2-dimensional complete intersection of quadrics with isolated du Val singularities in a sufficiently high-dimensional projective space has big cotangent bundle follows already from work of Roulleau and Rousseau [2014], as such surfaces have positive second Segre class [Miyaoka 1983].

As an application, we deduce that a certain surface related to magic squares is algebraically quasihyperbolic. Recall that an $n \times n$ *magic square* is an $n \times n$ grid, filled with distinct positive integers, whose rows, columns, and diagonals add up to the same number. It is unknown if there exists a 3×3 magic square

x_1^2	x_2^2	x_3^2
x_4^2	x_5^2	x_6^2
x_7^2	x_8^2	x_9^2

whose entries are distinct nonzero squares. Such a square gives rise to a rational point with nonzero coordinates on the complete intersection surface $X_{\mathrm{ms}} \subset \mathbb{P}^8$ defined by the relations

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2 + x_6^2 = x_7^2 + x_8^2 + x_9^2 = x_1^2 + x_4^2 + x_7^2$$
$$= x_2^2 + x_5^2 + x_8^2 = x_1^2 + x_5^2 + x_9^2 = x_3^2 + x_5^2 + x_7^2.$$

This surface is smooth except for 256 isolated ordinary double points. This exceeds $\ell_{\min}(8) = 217$, so we obtain the following result.

Theorem 1.5. *The complex projective surface $X_{\mathrm{ms}} \subset \mathbb{P}^8$ that parametrizes 3×3 magic squares of squares is algebraically quasihyperbolic.* $\qquad\square$

In fact, using (1-1) we find that for $m \geq 47$ there are global sections and that $H^0(Y_{\mathrm{ms}}, S^{47}\Omega_{Y_{\mathrm{ms}}}^1) \geq 8448$.

1D3. *Partial information.* Even in cases where Theorem 1.4 cannot quite prove quasihyperbolicity of a surface, we can use the ideas behind its proof to determine restrictions on the properties of genus 0 and 1 curves on Y; see Propositions 3.1 and 6.2. For instance, if the set S consists of ℓ isolated A_1 singularities, and if for some $0 \leq r < \ell$ there is a constant $C > 0$ such that

$$\chi(Y, S^m\Omega_Y^1) + \ell\chi^1(s, S^m\Omega_Y^1) + r\chi^0(s, S^m\Omega_Y^1) \sim Cm^3,$$

then X contains only finitely many genus 0 or 1 curves that pass through fewer than r singularities of type A_1.

2. Global symmetric differentials, I

Throughout this section we keep the notation of Section 1B. In particular, $\tau : Y \to X$ is a minimal resolution of a complex complete intersection surface $X \subset \mathbb{P}^n$ of multidegree $(d_1, \ldots, d_{n-2})$ with at worst a finite set S of isolated du Val singularities.

Lemma 2.1. *Let $Z \subset \mathbb{P}^n$ be a complex nonsingular complete intersection of multidegree $(d_1, \ldots, d_{n-2})$. Then*

$$\chi(Y, S^m\Omega_Y^1) = \chi(Z, S^m\Omega_Z^1).$$

Proof. This is a direct consequence of a beautiful result [Atiyah 1958] in the diffeomorphic category. We can take X as the central member of a family X_t of complete intersections, with the general member nonsingular. Then [Atiyah 1958, Theorem 3] gives that minimal resolutions of fibers are pairwise

diffeomorphic. The result now follows from comparing the central fiber X with a general member Z, because Euler characteristics are invariant under diffeomorphisms. □

Henceforth in this section we assume that $d_1 + \cdots + d_{n-2} > n + 1$.

Lemma 2.2. *The surface Y is of general type, i.e., its canonical class K_Y is big.*

Proof. Since the singularities of X are du Val, it follows that $K_Y = \tau^* K_X$; see, e.g., the proof of [Kollár and Mori 1998, Theorem 4.20]. The hypothesis $d_1 + \cdots + d_{n-2} > n + 1$ ensures that K_X is big, and hence so is K_Y by [Lazarsfeld 2004, Lemma 2.2.43]. □

In what follows, we begin our systematic study of lower bounds for the space of global sections of $\mathcal{A} := S^m \Omega_Y^1$, in terms of m. Consider $\tau_* \mathcal{A}$ and its reflexive hull $\widehat{\mathcal{A}} = (\tau_* \mathcal{A})^{\vee\vee}$. The Leray spectral sequence

$$E_2^{p,q} := \mathrm{H}^p(X, R^q \tau_* \mathcal{A}) \Longrightarrow \mathrm{H}^{p+q}(Y, \mathcal{A})$$

gives rise to the 6-term exact sequence of low-degree terms

$$0 \to \mathrm{H}^1(X, \tau_* \mathcal{A}) \to \mathrm{H}^1(Y, \mathcal{A}) \to \mathrm{H}^0(X, R^1 \tau_* \mathcal{A}) \to \mathrm{H}^2(X, \tau_* \mathcal{A}) \to \ker\!\big(\mathrm{H}^2(Y, \mathcal{A}) \to \mathrm{H}^0(X, R^2 \tau_* \mathcal{A})\big)$$
$$\to \mathrm{H}^1(X, R^1 \tau_* \mathcal{A}).$$

The sheaf $R^1 \tau_* \mathcal{A}$ is supported on the 0-dimensional scheme S, since τ is an isomorphism outside of S. Thus

$$\mathrm{H}^1(X, R^1 \tau_* \mathcal{A}) = \mathrm{H}^2(X, R^1 \tau_* \mathcal{A}) = 0.$$

Inspecting page 2 of the spectral sequence, this last equality shows that $\mathrm{H}^0(X, R^2 \tau_* \mathcal{A}) = 0$ as well. Furthermore, since τ is an isomorphism outside S and $\mathcal{A}$ is reflexive, we see that $\widehat{\mathcal{A}}/\tau_* \mathcal{A}$ and the kernel of $\tau_* \mathcal{A} \to \widehat{\mathcal{A}}$ are both supported on S, which is 0-dimensional, so $\mathrm{H}^2(X, \tau_* \mathcal{A}) = \mathrm{H}^2(X, \widehat{\mathcal{A}})$. We simplify our sequence to

$$0 \to \mathrm{H}^1(X, \tau_* \mathcal{A}) \to \mathrm{H}^1(Y, \mathcal{A}) \to \mathrm{H}^0(X, R^1 \tau_* \mathcal{A}) \to \mathrm{H}^2(X, \widehat{\mathcal{A}}) \to \mathrm{H}^2(Y, \mathcal{A}) \to 0. \qquad (2\text{-}1)$$

Lemma 2.3. *With notation as above, for $m \geq 3$ we have $h^2(Y, S^m \Omega_Y^1) = 0$ and*

$$h^1(Y, S^m \Omega_Y^1) = h^1(X, \tau_* S^m \Omega_Y^1) + h^0(X, R^1 \tau_* S^m \Omega_Y^1).$$

Proof. By [BdO 2006, Proposition 2.3] (or [Deschamps 1979, Lemme 3.3.2]) and Lemma 2.2, we have that $h^2(X, \widehat{\mathcal{A}}) = 0$ for $m \geq 3$. The lemma now follows by looking at dimensions on (2-1). □

Corollary 2.4. *The inequality*

$$h^0(Y, S^m \Omega_Y^1) \geq \chi(Y, S^m \Omega_Y^1) + h^0(X, R^1 \tau_* S^m \Omega_Y^1)$$

holds for $m \geq 3$. □

Proof. Since $h^2(Y, S^m \Omega_Y^1) = 0$ for $m \geq 3$, we have

$$h^0(Y, S^m \Omega_Y^1) = \chi(Y, S^m \Omega_Y^1) + h^1(Y, S^m \Omega_Y^1).$$

The conclusion now follows from Lemma 2.3 and the crude estimate $h^1(X, \tau_* S^m \Omega_Y^1) \geq 0$. □

Remark 2.5. We expect improving the coarse estimate $h^1(X, \tau_* S^m \Omega_Y^1) \geq 0$ in the proof of Corollary 2.4 would significantly strengthen our results.

In the following sections, we compute the right-hand side of the inequality in Corollary 2.4 exactly in the case where S consists only of A_1 singularities. The Euler characteristic $\chi(Y, S^m\Omega^1_Y)$ is easily computed using Lemma 2.1 and (A-5), taking into account that for a nonsingular multidegree $(d_1, \ldots, d_{n-2})$ complete intersection Z, we have

$$K_Z^2 = (n+1-\sigma_1)^2 d \quad \text{and} \quad \chi_Z = \left(\binom{n+1}{2} - (n+1-\sigma_1)\sigma_1 - \sigma_2\right)d,$$

where $d = \prod_i d_i$, $\sigma_1 = \sum_i d_i$, and $\sigma_2 = \sum_{i<j} d_i d_j$. Since $R^1\tau_* S^m\Omega^1_Y$ is supported on the 0-dimensional scheme S, we compute $h^0(X, R^1\tau_* S^m\Omega^1_Y)$ point by point, restricting to sufficiently small neighborhoods around them. This requires a detailed study of local Euler characteristics, which we address in Section 3.

3. Local Euler characteristics

Let (X°, s) be an isolated normal analytic complex surface singularity, and let (Y°, E_s) be a *good resolution* of X°, by which we mean a resolution with a simple normal crossings divisor E. For a locally free coherent sheaf $\mathcal{F}$ on Y^0, following Wahl [1993], define the *local Euler characteristic* at $s \in X^0$ by

$$\chi^0(s, \mathcal{F}) := \dim\big(\mathrm{H}^0(Y^\circ - E_s, \mathcal{F})/\mathrm{H}^0(Y^\circ, \mathcal{F})\big),$$
$$\chi^1(s, \mathcal{F}) := h^1(Y^\circ, \mathcal{F}), \tag{3-1}$$
$$\chi(s, \mathcal{F}) := \chi^0(s, \mathcal{F}) + \chi^1(s, \mathcal{F}).$$

3A. *Proof of Theorem 1.4.* We now have all the necessary ingredients and notation to prove Theorem 1.4. Recall that in the statement of the theorem, the morphism $\tau : Y \to X$ follows the conventions of Section 1B. By Corollary 2.4, we know that

$$h^0(Y, S^m\Omega^1_Y) \geq \chi(Y, S^m\Omega^1_Y) + h^0(X, R^1\tau_* S^m(\Omega^1_Y)).$$

The sheaf $R^1\tau_* S^m(\Omega^1_Y)$ is supported on a 0-dimensional scheme S, so $h^0(X, R^1\tau_* S^m(\Omega^1_Y))$ is simply the sum of the contributions at each $s \in S$. We get that this contribution is $\chi^1(s, S^m\Omega^1_Y)$ by restricting to a sufficiently small affine neighborhood X° of s. This proves (1-1).

Since $\tau : (Y - E) \to (X - S)$ is an isomorphism and S is of codimension 2, we see that

$$\mathrm{H}^0(X, \hat{S}^m(\Omega^1_X)) \simeq \mathrm{H}^0(X - S, S^m\Omega^1_{X-S}) \simeq \mathrm{H}^0(Y - E, S^m\Omega^1_Y).$$

By definition, $\chi^0(s, S^m\Omega^1_Y)$ measures exactly the codimension for each singularity separately. At the worst, each of these singularities imposes independent linear conditions on sections in $\mathrm{H}^0(Y - E, S^m\Omega^1_Y)$ to extend into each component of E, giving (1-2). $\qquad\square$

Requiring regularity on only *some* components of the exceptional divisor yields stronger lower bounds. We illustrate this in the case that S consists entirely of A_1 singularities.

Proposition 3.1. *With notation as in Theorem 1.4, assume further that S consists of ℓ isolated A_1 singularities. Let $E_1, \ldots, E_r$ be exceptional components on Y above $r \leq \ell$ of the elements of S. Then, for $m \geq 3$, we have*

$$h^0\big(Y - (E_1 \cup \cdots \cup E_r), S^m\Omega^1_Y\big) \geq \chi(Y, S^m\Omega^1_Y) + \ell\chi^1(s, S^m\Omega^1_Y) + r\chi^0(s, S^m\Omega^1_Y) \tag{3-2}$$

and in fact for all $m \geq 1$ that

$$h^0\big(Y - (E_1 \cup \cdots \cup E_r), S^m \Omega^1 Y\big) \geq h^0\big(X, (S^m(\Omega^1_X))^{\vee\vee}\big) - (\ell - r)\chi^0(s, S^m \Omega^1_Y). \tag{3-3}$$

3B. Singularities of type A_1. In this section, we compute local Euler characteristics for sheaves associated to symmetric differentials in the case where s is an A_1 singularity. A model for X° is a quadric cone $x_1 x_3 = x_2^2$ in $\mathbb{A}^3$ with $s = (0, 0, 0)$, and Y° is the blow-up at the vertex, so that $E_s \simeq \mathbb{P}^1$. The assignment $(t, u) \mapsto (t, tu, tu^2)$ is an affine chart of $\tau : Y^\circ \to X^\circ$, where the exceptional fiber E_s is given by $t = 0$.

We consider the sheaves $\mathcal{A} = S^m \Omega^1_{Y^\circ}$ and $\mathcal{B}_h = \big(S^m(\Omega^1_{Y^\circ}(\log E_s))\big)(-h E_s)$. They agree on $Y^\circ - E_s$, so $\tau_* \mathcal{A}|_{X^\circ - s} = \tau_* \mathcal{B}_h|_{X^\circ - s}$. Since s has codimension 2 in X°, it follows that the reflexive hulls agree, i.e.,

$$\widehat{\mathcal{A}} := (\tau_* \mathcal{A})^{\vee\vee} = (\tau_* \mathcal{B}_h)^{\vee\vee}.$$

Ultimately, we will compute $\chi(s, S^m \Omega^1_{Y^\circ})$ by understanding for which values of h we have $\chi(s, \mathcal{B}_h) = 0$.

The singularity (X°, s) can also be viewed as a quotient singularity arising from the degree 2 finite cover $f : X' \to X^\circ$, where $X' = \mathbb{A}^2$ and f is given by $(z_1, z_2) \mapsto (z_1^2, z_1 z_2, z_2^2)$ with automorphism $\iota : (z_1, z_2) \mapsto (-z_1, -z_2)$. The unique fixed point and preimage of s is $s' = (0, 0)$.

Since $X' \to X^\circ$ is a finite quotient map with automorphism group $\langle \iota \rangle$, sections in $H^0(X^\circ - s, S^m \Omega^1_{X^\circ})$ pull back to sections in $H^0(X' - s', S^m \Omega^1_{X'})^\iota$, which by purity extend into the nonsingular point s'. Hence the vector space $H^0(Y^\circ - E_s, S^m \Omega^1_{Y^\circ}) \simeq H^0(X^\circ - s, S^m \Omega^1_{X^\circ})$ is naturally isomorphic to $H^0(X', S^m \Omega^1_{X'})^\iota$; we identify these spaces from now on.

The ring

$$\bigoplus_{m \geq 0} H^0(X', S^m \Omega^1_{X'})$$

is isomorphic to the polynomial ring $k[z_1, z_2, dz_1, dz_2]$, bigraded by the total degrees in z_1, z_2 and dz_1, dz_2 respectively, with graded parts

$$V_{m,n} = \langle z_1^j z_2^{n-j} \, dz_1^i \, dz_2^{m-i} : i = 0, \ldots, m \text{ and } j = 0, \ldots, n \rangle.$$

For the ι-invariant subring,

$$\bigoplus_m H^0(X', S^m \Omega^1_{X'})^\iota = \bigoplus_{n \equiv m \,(\mathrm{mod}\, 2)} V_{m,n}.$$

The identification $H^0(Y^\circ - E_s, S^m \Omega^1_{Y^\circ}) \simeq H^0(X', S^m \Omega^1_{X'})^\iota$ induces a valuation ord_E on the latter, which extends to all of $H^0(X', S^m \Omega^1_{X'})$ as a valuation taking values in $\frac{1}{2}\mathbb{Z}$. We describe the valuation on $k[z_1, z_2, dz_1, dz_2]$ by introducing a square root of t, denoted by $t^{1/2}$. The relations $z_1^2 = t$, $z_1 z_2 = tu$, $z_2^2 = tu^2$ give rise to relations between their derivatives as well, which can be expressed as a ring homomorphism

$$k[z_1, z_2, dz_1, dz_2] \to k(t^{1/2})[u, dt, du]$$

defined by

$$z_1 \mapsto t^{1/2}, \quad z_2 \mapsto t^{1/2}u, \quad dz_1 \mapsto \tfrac{1}{2}t^{-1/2}\, dt, \quad dz_2 \mapsto \tfrac{1}{2}(t^{-1/2}u \, dt + 2t^{1/2}\, du).$$

The valuation is the obvious one with respect to t on $k(t^{1/2})[u, dt, du]$, pulled back along this homomorphism. In particular, we have

$$\operatorname{ord}_E(z_1) = \operatorname{ord}_E(z_2) = \tfrac{1}{2} \quad \text{and} \quad \operatorname{ord}_E(dz_1) = \operatorname{ord}_E(dz_2) = -\tfrac{1}{2}.$$

Lemma 3.2. $\qquad\qquad \chi^0(s, \mathcal{B}_h) = 0 \quad \text{if and only if} \quad h < \tfrac{1}{2}(m+1).$

Proof. We observe that $\mathrm{H}^0(Y^\circ - E_s, \mathcal{B}_h) \subset \mathrm{H}^0(Y^\circ - E_s, S^m\Omega^1_{Y^\circ}) \subset k[z_1, z_2, dz_1, dz_2]$, where the last inclusion comes from the identification explained above. On the affine patch Y' of Y° with coordinates (t, u), we see that $\mathrm{H}^0(Y', S^m(\Omega^1_{Y'}(\log E_s)))$ is a free $k[t, u]$-module with basis

$$(dt/t)^m, \quad (dt/t)^{m-1}du, \quad \ldots, \quad (dt/t)(du)^{m-1}, \quad (du)^m.$$

If we also consider the complementary patch $(s, v) = (tu^2, 1/u)$ of Y°, we see that

$$(dt/t) = d(sv^2)/(sv^2) = (v^2\, ds + 2vs\, dv)/(sv^2) = ds/s + 2\, dv/v,$$

which is not a log-differential: we would need to multiply it by t. Note that

$$t(dt/t)^2 = v^2 s(ds/s)^2 + 4vs\, dv(ds/s) + 4s(dv)^2$$

is a log-differential on all of Y°. Inside $\mathrm{H}^0(Y', S^m(\Omega^1_{Y'}(\log E_s)))$ we can characterize the elements of $\mathrm{H}^0(Y^\circ, S^m(\Omega^1_{Y^\circ}(\log E_s)))$ as those forms for which the coefficient of $(dt/t)^i(ds)^{m-i}$ is divisible by $t^{\lceil i/2 \rceil}$. This coincides with $\mathrm{H}^0(X', S^m\Omega^1_{X'})^t$. We see that for $h \le m - \lceil \tfrac{1}{2}m \rceil < \tfrac{1}{2}(m+1)$ we have $\mathrm{H}^0(Y^\circ, \mathcal{B}_h) = \mathrm{H}^0(Y^\circ - E_s, \mathcal{B}_h)$. Furthermore, $(dz_1)^m$ for even m and $z_1(dz_1)^m$ for odd m, show that for larger h, equality does not hold. $\qquad\square$

We use the valuation ord_E to determine $\chi^0(s, \mathcal{A}) = \chi^0(s, S^m\Omega^1_{Y^\circ})$ as a function of m in the following proposition. From the leading coefficient one can read off the corrected asymptotics for [BdO 2006, Lemma 2.2] as well.

Proposition 3.3. *For an A_1 singularity (X°, s) and a minimal resolution $\tau : Y^\circ \to X^\circ$ we have*

$$\chi^0(s, S^m\Omega^1_{Y^\circ}) = \begin{cases} \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{1}{6}m & \text{if } m \equiv 0 \ (\mathrm{mod}\ 6), \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 - \frac{1}{12}m - \frac{35}{108} & \text{if } m \equiv 1 \ (\mathrm{mod}\ 6), \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{7}{18}m + \frac{5}{27} & \text{if } m \equiv 2 \ (\mathrm{mod}\ 6), \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 - \frac{1}{12}m - \frac{1}{4} & \text{if } m \equiv 3 \ (\mathrm{mod}\ 6), \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{1}{6}m - \frac{2}{27} & \text{if } m \equiv 4 \ (\mathrm{mod}\ 6), \\ \frac{11}{108}m^3 + \frac{11}{36}m^2 + \frac{5}{36}m - \frac{7}{108} & \text{if } m \equiv 5 \ (\mathrm{mod}\ 6). \end{cases} \tag{3-4}$$

In particular, the first few values we get are:

m	1	2	3	4	5	6	7	8	9	10	11	12
codim	0	3	5	12	21	34	49	75	98	134	174	222

Proof. We write $W_{m,n} := V_{m,n} \cap H^0(Y^\circ, \mathcal{A})$. It follows immediately that $W_{m,n} = V_{m,n}$ if $n \geq m$. In addition, we see that $\text{ord}_E(z_1 \, dz_2 - z_2 \, dz_1) = \text{ord}_E(t \, du) = 1$. In fact, by looking at leading terms with respect to u and dt, we see that $\omega \in V_{m,n}$ has $\text{ord}_E(\omega) > \frac{1}{2}(n-m)$ if and only if ω is divisible by $z_1 \, dz_2 - z_2 \, dz_1$. By applying this criterion iteratively we find that

$$W_{m,n} = \begin{cases} V_{m,n} & \text{if } n \geq m, \\ V_{(m+n)/2,(3n-m)/2}(z_1 \, dz_2 - z_2 \, dz_1)^{(m-n)/2} & \text{if } \frac{1}{3}m \leq n < m, \\ 0 & \text{if } n < \frac{1}{3}m. \end{cases}$$

Since $z_1 \, dz_2 - z_2 \, dz_1$ is bihomogeneous, it follows that

$$H^0(Y^\circ - E_s, \mathcal{A}) / H^0(Y^\circ, \mathcal{A}) = \bigoplus_{n \equiv m \pmod 2} V_{m,n} / W_{m,n}.$$

Using that $\dim V_{m,n} = (m+1)(n+1)$ and hence $\dim W_{m,n} = (n+m+2)(3n-m+2)/4$ for $m/3 \leq n \leq m$, we can find the formulas by straightforward summation. $\square$

Corollary 3.4. *If* $\omega \in H^0\big(X^\circ - s, S^m \Omega^1_{X^\circ}\big(-\lfloor \frac{1}{2}m \rfloor H\big)\big)$, *where H is a hyperplane section containing s, then $\tau^*\omega$ extends to a regular differential on the component E_s of the exceptional divisor of Y lying over s.*

Proof. The form ω pulls back to $\bigoplus_{n \geq m} V_{m,n}$ and therefore lies in $\bigoplus W_{m,n}$. $\square$

Lemma 3.5. $\chi^1(s, \mathcal{B}_h) = 0$ *if and only if* $h > \frac{1}{2}(m-2)$.

Proof. The 4-term exact sequence associated to the Leray spectral sequence for $\tau : Y^\circ \to X^\circ$ and the sheaf $\mathcal{B}_h$ is

$$0 \to H^1(X^\circ, \tau_* \mathcal{B}_h) \to H^1(Y^\circ, \mathcal{B}_h) \to H^0(X^\circ, R^1 \tau_* \mathcal{B}_h) \to H^2(X^\circ, \tau_* \mathcal{B}_h).$$

The morphism τ being proper to a locally Noetherian base, the sheaves $R^i \tau_* \mathcal{B}_h$ are coherent [EGA III$_1$ 1961, III.1 Théorème 3.2.1]. Since without loss of generality we can take X° to be affine, we have

$$H^1(X^\circ, \tau_* \mathcal{B}_h) = H^2(X^\circ, \tau_* \mathcal{B}_h) = 0;$$

see [Hartshorne 1977, III.3.5]. This shows that

$$H^1(Y^\circ, \mathcal{B}_h) \xrightarrow{\sim} H^0(X^\circ, R^1 \tau_* \mathcal{B}_h).$$

To complete the proof, we show that $R^1 \tau_* \mathcal{B}_h = 0$ precisely when $h > \frac{1}{2}(m-2)$. The sheaf $R^1 \tau_* \mathcal{B}_h$ is supported on s, so it is enough to understand its stalk $(R^1 \tau_* \mathcal{B}_h)_s$. By the theorem on formal functions [Hartshorne 1977, III.11.1], we have

$$(R^1 \tau_* \mathcal{B}_h)^\wedge_s \xrightarrow{\sim} \varprojlim_n H^1(nE, \mathcal{B}_h), \tag{3-5}$$

where $nE = Y^\circ \times_{X^\circ} \text{Spec}(\mathcal{O}_s / \mathfrak{m}_s^n)$ and by abuse of notation the sheaf $\mathcal{B}_h$ on the right-hand side is the pullback of $\mathcal{B}_h$ via the projection $nE \to Y^\circ$. Tensoring the exact sequence of sheaves

$$0 \to \mathcal{O}_E(-nE) \to \mathcal{O}_{nE} \to \mathcal{O}_{(n+1)E} \to 0$$

with the locally free sheaf $\mathcal{B}_h$ and taking cohomology we obtain the exact sequence

$$\mathrm{H}^1(E, \mathcal{O}_E(-nE) \otimes \mathcal{B}_h) \to \mathrm{H}^1(nE, \mathcal{O}_{nE} \otimes \mathcal{B}_h) \to \mathrm{H}^1((n+1)E, \mathcal{O}_{(n+1)E} \otimes \mathcal{B}_h) \to 0.$$

If $\mathrm{H}^1(E, \mathcal{O}_E(-nE) \otimes \mathcal{B}_h) = 0$ for all $n \geq 0$, then

$$\mathrm{H}^1(nE, \mathcal{O}_{nE} \otimes \mathcal{B}_h) \xrightarrow{\sim} \mathrm{H}^1((n+1)E, \mathcal{O}_{(n+1)E} \otimes \mathcal{B}_h)$$

for all $n \geq 0$, which implies in turn that the projective limit in (3-5) is isomorphic to $\mathrm{H}^1(nE, \mathcal{O}_{nE} \otimes \mathcal{B}_h)$ for all n, and in particular, it is isomorphic to $\mathrm{H}^1(E, \mathcal{O}_E \otimes \mathcal{B}_h)$.

To understand the cohomology groups $\mathrm{H}^1(E, \mathcal{O}_E(-nE) \otimes \mathcal{B}_h)$, we use the residue exact sequence

$$0 \to \Omega_E \to \Omega_{Y^\circ}(\log E)|_E \to \mathcal{O}_E \to 0.$$

This sequence does not split [Wahl 1976, 3.3], so $\Omega_{Y^\circ}(\log E)|_E$ is isomorphic to the nontrivial class in $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}(-2), \mathcal{O})$, i.e.,

$$\Omega_{Y^\circ}(\log E)|_E \simeq \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1). \tag{3-6}$$

Taking into account that $\mathcal{O}_Y(-hE)|_E \simeq \mathcal{O}_E(2h)$ because $E^2 = -2$, (3-6) shows that the restriction of $\mathcal{B}_h$ to $E \simeq \mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(-m+2h)^{\oplus(m+1)}$. We have

$$\mathrm{H}^1(E, \mathcal{O}_E(-nE) \otimes \mathcal{B}_h) \simeq \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2n-m+2h)^{\oplus(m+1)})$$
$$\simeq \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2n+m-2h-2)^{\oplus(m+1)}),$$

where the last isomorphism follows from Serre duality. This cohomology group vanishes for $n \geq 0$ precisely when $h > \frac{1}{2}(m-2)$, in which case the $n=0$ vanishing shows that $\mathrm{H}^1(E, \mathcal{O}_E \otimes \mathcal{B}_h) = 0$, and thus the projective limit (3-5) vanishes as well. $\qquad\square$

Lemmas 3.2 and 3.5 combine to the following result.

Corollary 3.6. $\qquad\qquad \chi(s, \mathcal{B}_h) = 0 \quad$ *precisely for* $\quad h = \lceil \frac{1}{2}m \rceil$. $\qquad\qquad\square$

The vanishing result in Corollary 3.6 allows us to compute $\chi(s, S^m\Omega^1_{Y^\circ})$ and $\chi^1(s, S^m\Omega^1_{Y^\circ})$ for an A_1 singularity.

Proposition 3.7. *For an A_1 singularity (X°, s) and a minimal resolution $\tau : Y^\circ \to X^\circ$ we have*

$$\chi(s, S^m\Omega^1_{Y^\circ}) = \begin{cases} \frac{1}{4}m(m+1)(m+2) & \text{if } m \equiv 0 \pmod 2, \\ \frac{1}{4}(m+1)(m^2+2m-1) & \text{if } m \equiv 1 \pmod 2, \end{cases} \tag{3-7}$$

$$\chi^1(s, S^m\Omega^1_{Y^\circ}) = \begin{cases} \frac{4}{27}m^3 + \frac{4}{9}m^2 + \frac{1}{3}m & \text{if } m \equiv 0 \pmod 3, \\ \frac{4}{27}m^3 + \frac{4}{9}m^2 + \frac{1}{3}m + \frac{2}{27} & \text{if } m \equiv 1 \pmod 3, \\ \frac{4}{27}m^3 + \frac{4}{9}m^2 + \frac{1}{9}m - \frac{5}{27} & \text{if } m \equiv 2 \pmod 3. \end{cases} \tag{3-8}$$

Proof. We choose a completion X of X° such that the singular locus of X consists of just $s \in X^\circ \subset X$. We take $\tau : Y \to X$ a minimal resolution. Then Y° is isomorphic to the inverse image of X° in Y and $S^m(\Omega^1_{Y^\circ})$ is the restriction of $S^m(\Omega^1_Y)$ to Y°.

We consider the sheaves $\mathcal{A} = S^m\Omega^1_Y$ and $\mathcal{B} = \left(S^m(\Omega^1_Y(\log E_s))\right)(-hE_s)$, with $h = \lceil \frac{1}{2}m \rceil$.

The sheaf $\mathcal{B}$ is locally free and therefore reflexive, so by [Blache 1996, Lemma on p. 30], for $\widehat{\mathcal{B}} := (\tau_*\mathcal{B})^{\vee\vee}$ we know that

$$\chi(X, \widehat{\mathcal{B}}) = \chi(Y, \mathcal{B}) + \chi(s, \mathcal{B}).$$

Corollary 3.6 says $\chi(s, \mathcal{B}) = 0$, so $\chi(X, \widehat{\mathcal{B}}) = \chi(Y, \mathcal{B})$. Since $\widehat{\mathcal{A}} := (\tau_*\mathcal{A})^{\vee\vee} = (\tau_*\mathcal{B})^{\vee\vee} = \widehat{\mathcal{B}}$, we obtain

$$\chi(s, \mathcal{A}) = \chi(X, \widehat{\mathcal{A}}) - \chi(Y, \mathcal{A}) = \chi(X, \widehat{\mathcal{B}}) - \chi(Y, \mathcal{A}) = \chi(Y, \mathcal{B}) - \chi(Y, \mathcal{A}).$$

The first result now follows from (A-6) and the second from subtracting (3-4) from (3-7). $\square$

4. Global symmetric differentials, II

In this section we combine Theorem 1.4 with the calculation in Section 3 of the local Euler characteristic of symmetric differentials for A_1 singularities to obtain quasihyperbolicity results for surfaces of general type that are smooth except for finitely many isolated A_1 singularities.

Throughout this section, $X \subset \mathbb{P}^n$ denotes a complex projective surface that is a complete intersection of multidegree $(d_1, \ldots, d_{n-2})$, with a singular locus S consisting of ℓ isolated A_1 singularities $\{s_1, \ldots, s_\ell\}$. We let $\tau : Y \to X$ be a minimal resolution of X, with exceptional locus E, which consists of ℓ disjoint (-2)-curves $E_1, \ldots, E_\ell$, each isomorphic to $\mathbb{P}^1$. We assume that $d_1 + \cdots + d_{n-2} > n + 1$, so that Y is of general type.

Example 4.1. Write $c_1(Y)^2 = K^2$ and $c_2(Y) = \chi$. Using (A-5) and (3-7), inequality (1-1) gives

$$h^0(Y, S^m\Omega_Y^1) \geq \tfrac{1}{54}(9K^2 - 9\chi + 8\ell)m^3 - \left(\tfrac{1}{2}\chi - \tfrac{4}{9}\ell\right)m^2 + O(m).$$

In particular, if $\ell > \tfrac{9}{8}(\chi - K^2)$ then the surface will have regular symmetric differentials for large enough m; in fact, $\liminf_{m\to\infty} h^0(Y, S^m\Omega_Y^1)/m^3 > 0$.

Example 4.2. Let $X \subset \mathbb{P}^3$ be a hypersurface of degree $d \geq 5$ with ℓ singularities of type A_1. By (A-5), we have

$$\chi(Y, S^m\Omega_Y^1) = -\tfrac{1}{3}(2d^2 - 5d)m^3 - \tfrac{1}{2}(d^3 - 4d^2 + 6d)m^2 - \tfrac{1}{6}(2d^3 - 10d^2 + 17d)m + \tfrac{1}{6}(d^3 - 6d^2 + 11d),$$

so (1-1), together with

$$\ell > \tfrac{9}{4}(2d^2 - 5d),$$

implies $\liminf_{m\to\infty} h^0(Y, S^m\Omega_Y^1)/m^3 > 0$.

Remark 4.3. We document here where our results differ from those stated in [BdO 2006]. It has been previously noted (see [Roulleau and Rousseau 2014, Remark 12]) that [BdO 2006, Lemma 2.2] is flawed. In particular, $\chi^0(s, S^m\Omega_Y^1)$ is overestimated in [BdO 2006, (2.11)], yielding an estimate for $\chi^1(s, S^m\Omega_Y^1)$ that is only quadratic in m. With our approach (1-1), no fixed value of ℓ would be sufficient to overcome the negative coefficient of m^3 in $\chi(Y, S^m\Omega_Y^1)$ that a nodal hypersurface X would give rise to. Instead, [BdO 2006, Theorem 2.6] uses a different approach where, via Serre duality, the authors establish the inequality

$$h^0(Y, S^m\Omega_Y^1) \geq \chi(Y, S^m\Omega_Y^1) + \ell\chi^0(s, S^m\Omega_Y^1). \tag{4-1}$$

Note the difference between this inequality and (1-1): the latter uses $\chi^1(s, S^m\Omega^1_Y)$ in place of $\chi^0(s, S^m\Omega^1_Y)$. With the corrected asymptotic of $\chi^0(Y, S^m\Omega^1_Y) = \frac{11}{108}m^3 + O(m^2)$, this gives that Ω^1_Y is big if

$$\ell \geq \tfrac{36}{11}(2d^2 - 5d).$$

This result is weaker than the one in Example 4.2.

Roulleau and Rousseau [2014, Theorem 9] establish analogues of (1-1) and (4-1) for arbitrary A_k singularities without proving an analogue of Proposition 3.3 for arbitrary A_k singularities: instead, they get an asymptotic bound by cleverly taking the break-even point of the two approaches. For hypersurfaces with A_1 singularities they find the intermediate bound of

$$\ell > \tfrac{8}{3}(2d^2 - 5d).$$

It would be interesting to see which of (1-1) and (4-1) gives better results for varying k.

5. Computing regular differentials

In this section we describe how for $X \subset \mathbb{P}^n$ explicitly given as a complete intersection $f_1 = \cdots = f_{n-2} = 0$, we can compute an explicit representation of $H^0(X, (\tau_* S^m\Omega^1_Y)^{\vee\vee})$ and determine $H^0(Y, S^m\Omega^1_Y)$ as a subspace. We write $R = R_X = k[x_0, \ldots, x_n]/(f_1, \ldots, f_{n-2})$ for the projective coordinate ring and write $R(d)$ for the R-module obtained by shifting the grading so that $R(d)_i = R_{d+i}$.

An algebraic sheaf $\mathscr{F}$ on X determines a graded R-module

$$\Gamma_*(\mathscr{F}) = \bigoplus_{d \in \mathbb{Z}} H^0(X, \mathscr{F} \otimes \mathcal{O}_X(d)).$$

In turn, the sheaf $\mathscr{F}$ is determined by this graded module. Any graded R-module M also determines an algebraic sheaf $\mathscr{F}_M$ on X, and $\Gamma_*(\mathscr{F}_M)$ is the *saturation* of M. Since X is a complete intersection that is nonsingular in codimension 1, it is normal and projectively normal. This means that R_X is saturated and hence that $\Gamma_*(\mathcal{O}_X(d)) = R(d)$.

We construct a module representing Ω^1_X in the following way. Let $R_{\mathbb{P}^n} = k[x_0, \ldots, x_n]$ be the projective coordinate ring of $\mathbb{P}^n$. We have that

$$M_{\mathbb{P}^n} = \Gamma_*(\Omega^1_{\mathbb{P}^n}) \subset R_{\mathbb{P}^n}(-1)^{n+1} = \bigoplus_{i=0}^n R_{\mathbb{P}^n}\, dx_i$$

fits in an exact sequence

$$R_{\mathbb{P}^n}(-3)^{\binom{n+1}{3}} \to R_{\mathbb{P}^n}(-2)^{\binom{n+1}{2}} \to M_{\mathbb{P}^n} \to 0,$$

where the second module has an R-basis $\{\omega_{ij} : 0 \leq i < j \leq n\}$, the second map is given by $\omega_{ij} = x_i\, dx_j - x_j\, dx_i$, and the relations are generated by the obvious

$$x_i\omega_{jk} - x_j\omega_{ik} + x_k\omega_{ij} \quad \text{for } 0 \leq i < j < k \leq n.$$

In order to compute a module for Ω^1_X, we consider the submodule

$$\partial I_X := \langle \partial_{x_i}(f_j)\, dx_i : j = 1, \ldots, n-2 \text{ and } i = 0, \ldots, n \rangle \subset \bigoplus_{i=0}^n R_{\mathbb{P}^n}\, dx_i.$$

Then $(\partial I_X \cap M_{\mathbb{P}^n}) \otimes R_X$ yields the conormal sheaf on $X - S$, so the module

$$M = M_X = M_{\mathbb{P}^n}/(\partial I_X \cap M_{\mathbb{P}^n}) \otimes R_X$$

gives Ω^1_X. We then construct $S^m M_X$ as an appropriate quotient of $M_X^{\otimes m}$.

Given a graded R-module M, we consider its dual $M^\vee = \text{Hom}(M, R)$. On the level of sheaves, this corresponds (up to shift) to taking the sheaf hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{O}_X)$. By applying this operation twice, we get a graded R-module with a homomorphism $M \to M^{\vee\vee}$. We write $\hat{S}^m \Omega^1_X = \mathcal{F}_{(S^m M_X)^{\vee\vee}}$.

We emphasize here that all module operations used here can be performed by appropriate commutative algebra software such as Magma [Bosma et al. 1997] and [Macaulay2], using Gröbner bases. We have made code available that implements the above ideas in the case of Barth's sextic and the perfect cuboid surface to interested readers in the online supplement. See Sections 7–8 for more details.

5A. *Computing an abstract presentation of $\hat{S}^m \Omega^1_X$.* The following lemma collects the results that relate a graded module to regular differentials.

Lemma 5.1. *With the notation above, we have the following properties*:

(a) $\hat{S}^m \Omega^1_X$ *is reflexive.*

(b) $\hat{S}^m \Omega^1_X|_{X-S} = S^m \Omega^1_{X-S}$.

(c) $H^0(Y - E, S^m \Omega^1_Y) \simeq H^0(X - S, S^m \Omega^1_X) \simeq H^0(X, \hat{S}^m \Omega^1_X) = (S^m M_X)^{\vee\vee}_0$.

Let H_X be a hyperplane section of X and let H_Y be the proper transform of H_X on Y. Then

(d) $H^0\big(Y - E, (S^m \Omega^1_Y)(-H_Y)\big) \simeq H^0\big(X, (\hat{S}^m \Omega^1_X)(-H_X)\big) \simeq (S^m M_X)^{\vee\vee}_{-1}$.

Proof. We have (a) because the dual of a coherent sheaf on a normal variety is reflexive. Furthermore, $S^m \Omega^1_{X-S}$ is already reflexive, giving (b).

Since X is a normal variety, sections of a reflexive sheaf on $X - S$ extend uniquely to X. Since R_X is saturated, we have that $\text{Hom}(M, R_X)$ is also saturated, so $\Gamma_*(\hat{S}^m \Omega^1_X) = (S^m M_X)^{\vee\vee}$, which proves (c).

Statement (d) is most easily argued with a hyperplane H_X disjoint from S. Then τ induces

$$H^0\big(Y - E, (S^m \Omega^1_Y)(-H_Y)\big) \simeq H^0(X - S, S^m \Omega^1_X(-H_X)),$$

so the first isomorphism follows from the same argument as for (c). The second holds because for any sheaf $\mathcal{F}$ on X we have $\mathcal{F}(-H_X) \simeq \mathcal{F} \otimes \mathcal{O}_X(-1)$. $\qquad\square$

5B. *Representing global sections with Kähler differentials on $k(X)$.* We explain how an abstract representation of an element in $H^0(X, \hat{S}^m \Omega^1_X)$ can be turned into a recognizable representation of an element of $H^0(X - S, S^m \Omega^1_X)$.

Remember that we have a representation of $M = S^m M_X$ as a quotient of $R(-2m)^{r_M}$, our generators being the monomials of degree m in $\omega_{ij} = x_i\, dx_j - x_j\, dx_i$.

We, or rather a computer algebra system, compute $M^\vee = \mathrm{Hom}_R(M, R)$ as a quotient of free modules, defined by an exact sequence

$$K_{M^\vee} \to \bigoplus_{i=1}^{r_{M^\vee}} R(d_i') \to M^\vee \to 0.$$

The bilinear pairing $M \times M^\vee \to R$ is given by an $r_M \times r_{M^\vee}$ matrix A over R. The hard work, accomplished by Gröbner basis computations, consists of determining the correct one.

Similarly, we obtain a description of $M^{\vee\vee} = \mathrm{Hom}_R(M^\vee, R)$ as a quotient defined by

$$K_{M^{\vee\vee}} \to \bigoplus_{i=1}^{r_{M^{\vee\vee}}} R(d_i) \to M^{\vee\vee} \to 0,$$

together with an $r_{M^\vee} \times r_{M^{\vee\vee}}$ matrix B over R, describing the pairing $M^\vee \times M^{\vee\vee} \to R$.

In order to get a recognizable representation of our symmetric differentials, we evaluate them at the generic point. Say, we take the affine open $X - \{x_0 = 0\}$. The dehomogenization map $R \to k[X - \{x_0 = 0\}]$ corresponding to $(x_0 : \cdots : x_n) = (1 : x_1 : \cdots : x_n)$ gives us a module M^{aff}, and we know that $M^{\mathrm{aff}} \otimes k(X)$ gives us an $(m+1)$-dimensional $k(X)$-vector space with for instance the basis $\mathcal{B} = \{dx_1^{m-i}\, dx_2^i : i = 0, \ldots, m\}$. Note that $\omega_{10} = x_1\, dx_0 - x_0\, dx_1$ equals $-dx_1$ if $x_0 = 1$, so we can readily recognize this basis from the generators we have chosen for M.

We know that

$$(M^{\vee\vee})^{\mathrm{aff}} \otimes k(X)$$

is isomorphic to this vector space. We take the submatrix A' of A consisting of the $m + 1$ rows that correspond to the basis $\mathcal{B}$. We know that A' has rank $m + 1$ over $k(X)$, so we select $m + 1$ columns of A' to get a square submatrix A'' that is invertible over $k(X)$.

Let B' be the submatrix of B obtained by taking the $m + 1$ rows matching the columns chosen for A''. Then

$$(dx_1^m, dx_1^{m-1}\, dx_2, \ldots, dx_2^m)(A'')^{-1}(B')^T$$

gives expressions for the generators of $M^{\vee\vee}$ as Kähler differentials on $k(X)$.

5C. *Determining the conditions to extend into the exceptional locus on Y.* Let $\omega \in \mathrm{H}^0(X, \hat{S}^m \Omega_X^1)$ and let $s \in S$ be an A_1 singularity. Without loss of generality, we may assume that $x_1, \ldots, x_n$ provide an affine chart around s, and that the tangent space of X at s is $x_4 = \cdots = x_n = 0$, and that the tangent cone of X at s inside the tangent space is defined by $x_1 x_3 = x_2^2$. Let E_s be the exceptional curve on Y above s. Then the completed local ring of Y at E_S is isomorphic to $k(u)[\![t]\!]$ and we have

$$(x_1, x_2, x_3, x_4, \ldots, x_n) = (t, tu, tu^2, 0, \ldots, 0) \pmod{t^2}.$$

This allows us to compute an expansion

$$\omega = \sum_{i=0}^{n} a_i(u, t)\, dt^i\, du^{n-i},$$

where $a_i(u, t) \in k[u]((t))$. Over k, there will be only finitely monomials $t^a u^b \, dt^i \, du^{n-i}$ occurring with $a < 0$, so we get a finite system of equations on $\mathrm{H}^0(X - S, S^m \Omega_X^1)$ to be satisfied for an element to extend to a regular form along E_s on Y. In fact, Proposition 3.3 gives us an upper bound on the number of equations we get.

6. Concluding quasihyperbolicity and explicitly computing the locus of special curves

In this section we consider the situation of Proposition 3.1. We take X to be a complete intersection with singular locus $S = \{s_1, \ldots, s_\ell\}$ consisting of A_1 singularities, with $s_1, \ldots, s_r$ removed. We take Y to be a minimal resolution of X, so Y has the exceptional curves $E_1, \ldots, E_r$ removed.

A regular symmetric differential on Y restricts *complete* curves on Y, so we can obtain information on curves of genus 0 and 1 on X that avoid the singularities $s_1, \ldots, s_r$.

The existence of regular symmetric differentials is usually concluded by observing that the lower bounds in Proposition 3.1 are cubic in m. The lemma below implies that regular differentials that vanish along a divisor similarly exist in that case.

Lemma 6.1. *Suppose that Y is a quasiprojective nonsingular surface of general type and suppose that there is a constant $c > 0$ such that*

$$h^0(Y, S^m \Omega_Y^1) = cm^3 + O(m^2).$$

Suppose that H is a divisor on Y (for instance, if Y is a minimal resolution of an X as above, we can take H to be the inverse image of a general hyperplane section of X). Then

$$h^0\big(Y, (S^m \Omega_Y^1)(-H)\big) > 0 \quad \text{for large enough } m$$

as well.

Proof. The condition amounts to the assertion that Ω_Y^1 is *big*. It is a standard result that for a *big* bundle $\mathcal{E}$ and a line bundle $\mathcal{L}$, we have that $h^0(Y, S^m \mathcal{E} \otimes \mathcal{L}) > 0$ for large enough m. $\square$

We use that on a complete curve $C \subset Y$ of genus 0 we have $\mathrm{H}^0(C, S^m \Omega_C^1) = 0$, and on a curve of genus 1, any section that vanishes somewhere must be identically 0. If we have $\omega \in \mathrm{H}^0(Y, S^m \Omega_Y^1)$ then we see that ω pulled back to C must be identically 0 if C is of genus 0. Similarly, if for an effective divisor H intersecting C we have $\omega \in \mathrm{H}^0\big(Y, (S^m \Omega_Y^1)(-H)\big)$ then ω restricts to a regular differential on C that vanishes somewhere, so if C is of genus 0 or 1, then ω must restrict to 0 on C. The proposition below recalls how the foliation determined by ω can be used to establish quasihyperbolicity.

Proposition 6.2. *Suppose Y is as above and that $\omega \in \mathrm{H}^0(Y, S^m \Omega_Y^1)$. Then there are only finitely many complete curves C on Y of genus at most 1 on which ω restricts to 0.*

Sketch of proof (see [Debarre 2004] for more details). The form ω defines a degree m form on the projective bundle $\mathbb{P}^1(\Omega_Y^1)$, and therefore gives rise on a surface Y' covering Y. On a desingularization $\widetilde{Y}$ of Y', the form ω induces a foliation (formed by the integral curves defined by the degree m, first-order differential equation that ω describes on Y). By the observation above, any curve C as above would be a

leaf of this foliation. By Jouanolou [1978], such a foliation either contains only finitely many algebraic leaves, or the foliation is in fact an algebraic fibration of $\widetilde{Y}$ over a curve. Since $\widetilde{Y}$ is still of general type, the general member of such a fibration must be of genus larger than 1, and therefore contain only finitely many fibers of genus 0 or 1. In either case, the result follows. $\square$

In special cases, the foliation induced by ω can be determined explicitly, but in general this seems to be hard, since it essentially requires solving a first-order differential equation of degree m on Y. We sketch another computational method here, which uses two sections ω_1, ω_2, and determines a closed locus in Y that contains all curves to which ω_1, ω_2 pull back to 0. We determine conditions for points $P \in Y$ such that there can be a curve C through P on Y on which both ω_1, ω_2 pull back to 0.

At a point $P \in Y$, a form ω defines a homogeneous degree m form on the tangent space $T_P(Y)$. If C is nonsingular at P, then the kernel of $T_P(Y)^* \to T_P(C)^*$ is generated by a single element, and $\omega(P)$ must be divisible by it. More specifically, if x, y are affine coordinates on Y such that $dx(P), dy(P)$ span $T_P(Y)^*$ then we have that $\omega(P) = \sum_{i=0}^{m} a_i \, dx^i \, dy^{m-1}$, for $a_i \in k(P)$, and that the kernel of $T_P(Y)^* \to T_P(C)^*$ is spanned by an element $\alpha_0 \, dx - \alpha_1 \, dy$. We need that $\sum_{i=0}^{m} a_i \alpha_1^i \alpha_0^{m-1} = 0$, i.e., that $(\alpha_0 : \alpha_1)$ is a root of ω as a form on $\mathbb{P}^1(k(P))$.

Suppose now that we have two such forms ω_1, ω_2; say $\omega_1(P) = \sum_{i=0}^{m-i} a_i(P) \, dx^i \, dy^{m-1}$ and $\omega_2(P) = \sum_{i=0}^{m-i} b_i(P) \, dx^i \, dy^{m-1}$. Then for P to be a nonsingular point on C, we need that $\omega_1(P)$ and $\omega_2(P)$, as forms on $\mathbb{P}^1(k(P))$, have a root in common, i.e., have a vanishing resultant. Provided that this resultant is not identically 0 on Y, we get a proper closed subset that contains any such C (since all the points of C lie in the closure of its nonsingular points).

We define the locus $\mathrm{res}_{x,y}(\omega_1, \omega_2)$ to be the locus where dx, dy do not span $T_P(Y)^*$ or where the following Sylvester determinant vanishes:

$$\mathrm{res}_{x,y}(\omega_1, \omega_2) = \det \begin{pmatrix} a_0(P) & \cdots & a_m(P) & & & \\ & \ddots & & \ddots & & \\ & & a_0(P) & \cdots & a_m(P) \\ b_0(P) & \cdots & b_m(P) & & & \\ & \ddots & & \ddots & & \\ & & b_0(P) & \cdots & b_m(P) \end{pmatrix}.$$

Let $\mathrm{res}(\omega_1, \omega_2)$ be the intersection of the vanishing of $\mathrm{res}_{x,y}(\omega_1, \omega_2)$ for all possible choices of x, y (it is sufficient to use all standard affine coordinate pairs derived from a nonsingular quasiprojective model of Y).

Proposition 6.3. *Let Y be as above and suppose that $\omega_1, \omega_2 \in \mathrm{H}^0(Y, S^m \Omega_Y^1)$. Then any complete genus 0 curve $C \subset Y$ is contained in* $\mathrm{res}(\omega_1, \omega_2)$.

Proof. As explained above, ω_1, ω_2 pull back to regular symmetric differentials on C. So, if C is a complete curve of genus 0, they pull back to 0. By the discussion above, this implies that $C \subset \mathrm{res}(\omega_1, \omega_2)$. $\square$

Proposition 6.4. *Let Y be as above and suppose $H \subset Y$ is an effective divisor on Y. If $\omega_1, \omega_2 \in \mathrm{H}^0(Y, S^m \Omega_Y^1(-H))$, then any complete genus 1 curve C on Y that intersects H is contained in* $\mathrm{res}(\omega_1, \omega_2)$.

Proof. As explained above, ω_1, ω_2 pull back to regular symmetric differentials on C. Furthermore, $C \cap H$ yields zeros of ω_1, ω_2, so they pull back to 0. $\square$

Corollary 6.5. *Let $X \subset \mathbb{P}^n$ be a complete intersection surface with a singular locus $S = \{s_1, \ldots, s_\ell\}$ consisting of A_1 singularities. Suppose that for a hyperplane section H_X of X we have $\omega_1, \omega_2 \in \mathrm{H}^0\bigl(X, \hat{S}^m \Omega_X^1 \bigl(-\lfloor \frac{1}{2} m \rfloor H_X\bigr)\bigr)$. Then any genus 0 curve C on X*

(1) *is contained in $\mathrm{res}(\omega_1, \omega_2)$, or*

(2) *is contained in one of the finitely many explicitly determinable hyperplanes, or*

(3) *passes through at least $n + 1$ distinct singularities, because $C \cap S$ spans $\mathbb{P}^n$.*

Furthermore, any genus 1 curve C on X

(4) *is contained in $\mathrm{res}(\omega_1, \omega_2)$, or*

(5) *is contained in a linear subspace of dimension at most one more than the span of $C \cap S$. In particular, C passes through at least two singularities and if $C \cap S$ spans a linear space of dimension at most $n - 2$, then the degree of C is at most the degree of X.*

Proof. We first note that hyperplane sections are linearly equivalent, so if H' is another hyperplane section then there is a function $f \in \mathrm{H}^0\bigl(X, \mathcal{O}_X\bigl(\lfloor \frac{m}{s} \rfloor (H_X - H')\bigr)\bigr)$ such that $f\omega_1, f\omega_2 \in \mathrm{H}^0\bigl(X, \hat{S}^m \Omega_X^1 \bigl(-\lfloor \frac{1}{2} m \rfloor H'\bigr)\bigr)$. Then $\mathrm{res}(\omega_1, \omega_2)$ and $\mathrm{res}(f\omega_1, f\omega_2)$ will only differ by components contained in H_X and H'.

Let us first deal with C of genus 0. Suppose we are not in case (3), so $C \cap S$ does not span $\mathbb{P}^n$. We can choose a hyperplane H' that contains the span. Since there are only finitely many possibilities for $C \cap S$, we can choose H' from a finite collection. For instance, if S as a whole spans $\mathbb{P}^n$, it is sufficient to consider all hyperplanes spanned by singularities.

Setting $S' = S - (S \cap H')$, we see that C is a complete curve in $X - S'$, so applying [Proposition 6.3](#) yields that the proper transform of C to Y lies in $\mathrm{res}(\pi^* f\omega_1, \pi^* f\omega_2)$, and therefore $C \in \mathrm{res}(f\omega_1, f\omega_2)$. Since $\mathrm{res}(\omega_1, \omega_2)$ and $\mathrm{res}(f\omega_1, f\omega_2)$ differ by a predetermined set of hyperplane sections, we see that we are in case (1) or (2).

For a genus 1 curve C for which $C \cap S$ spans a linear space of dimension at most $n - 2$, we then can choose a point P on C that is not in S and consider a hyperplane H' that contains P and $C \cap S$. By the same argument as above, we see that C must lie in $\mathrm{res}(f\omega_1, f\omega_2)$.

If the codimension of the linear span of $C \cap S$ inside the linear span of C is at least 2, we can choose our hyperplane so that C is not contained in it, forcing C to lie in $\mathrm{res}(\omega_1, \omega_2)$. If $C \cap S$ consists of just one point, then this is surely the case, since a genus 1 curve is not a line.

If the span of $C \cap S$ is of dimension at most $n - 2$ and C spans a space that is at most one dimension more, then C is a component of a hyperplane section of X. This bounds its degree. $\square$

Remark 6.6. [Corollary 6.5](#), case (5) is perhaps a little disappointing, but it still accomplishes a significant reduction: any genus 1 curve not in $\mathrm{res}(\omega_1, \omega_2)$ that passes through at most $n - 1$ singularities must be a component of a hyperplane section. The space of hyperplane sections is finite-dimensional and for a section

to have a genus 1 component it must be highly singular or reducible. These conditions define 0-dimensional loci in the space of hyperplane sections of X, which can, at least in principle, be determined explicitly.

7. Applications to complete intersections of quadrics

In this section we consider surfaces $X \subset \mathbb{P}^n$ that are complete intersections of $n - 2$ quadratic equations, with ℓ isolated A_1 singularities. In this case we have

$$K^2 = c_1(Y)^2 = (n - 5)^2 2^{n-2} \quad \text{and} \quad \chi = c_2(Y) = (n^2 - 7n + 16)2^{n-3}$$

and that Y is of general type if $n \geq 6$ by Lemma 2.2. By (A-5) we have

$$\chi(Y, S^m \Omega_Y^1) = \tfrac{1}{3} 2^{n-5} \big(2(n^2 - 13n + 34)m^3 - 6(n^2 - 7n + 16)m^2 - (5n^2 - 41n + 98)m + 3n^2 - 27 + 66 \big).$$

Theorem 1.4 gives that for $\ell \geq \ell_{\min}(n)$, we have regular symmetric differentials on Y for sufficiently large m, where $\ell_{\min}(n)$ is defined by:

n	6	7	8	9	≥ 10
$\ell_{\min}(n)$	73	145	217	145	0

In fact, using Lemma 6.1, we also have differentials vanishing along an ample divisor. Hence, using Proposition 6.2, we see that such Y are algebraically quasihyperbolic.

One concrete example is Theorem 1.5 on the surface $X_{\mathrm{ms}} \subset \mathbb{P}^8$ with $\ell = 256$ singularities of type A_1. Since $\ell > \ell_{\min}(8) = 217$, the surface is algebraically quasihyperbolic. In fact, using (1-1) we find that for $m \geq 47$ there are global sections and that $\mathrm{H}^0(Y_{\mathrm{ms}}, \Omega_{Y_{\mathrm{ms}}}^1) \geq 8448$. Unfortunately, X_{ms} is out of range of current computational techniques to explicitly determine $\hat{S}^m \Omega_{X_{\mathrm{ms}}}^1$, so we cannot apply the methods from Corollary 6.5 get an explicit description of the locus of special curves.

As a computationally more accessible example, let us consider the projective surface $X = X_{\mathrm{pc}}$ that parametrizes *perfect cuboids*, i.e., bricks with all sides x_1, x_2, x_3, diagonals y_1, y_2, y_3, and body diagonal z rational. The surface is a complete intersection in $\mathbb{P}^6$, described by the quadratic equations

$$X : \begin{cases} y_1^2 = x_2^2 + x_3^2, \\ y_2^2 = x_3^2 + x_1^2, \\ y_3^2 = x_1^2 + x_2^2, \\ z^2 = x_1^2 + x_2^2 + x_3^2. \end{cases}$$

Its singular locus S consists of $\ell = 48$ singularities of type A_1, so algebraic hyperbolicity does not follow immediately for its minimal desingularization Y. However, applying Proposition 3.1 with $r = 48 - 13$, we find that

$$h^0\big(Y - (E_1 \cup \cdots \cup E_r), S^m \Omega_Y^1\big) = \tfrac{1}{108} m^3 + O(m^2)$$

and hence that there are only finitely many curves of genus 0 or 1 on X that pass through at most 13 singularities. The lower bound based on Euler characteristics only turns positive for $m \geq 862$; a value not within range for explicit computation.

For $m = 2$, we find via explicit computation (see the online supplement) that $h^0(X, \hat{S}^2\Omega^1_X) = 13$, with generators as listed in Table 1. An indication that the ω_i are regular on $X - S$ is that the denominators listed are supported on $y_1 y_2 y_3 z = 0$, which is the branch locus of the projection on $(x_1 : x_2 : x_3)$.

Note that $\chi(X, \hat{S}^2\Omega^1_X) = 7$, so even with the assumption that $h^2(X, \hat{S}^2\Omega^1_X) = 0$, the Euler characteristic underestimates the dimension of the space of global sections. Furthermore, ω_7 vanishes along $H : x_1 = 0$ and $\langle \omega_7 \rangle = H^0(X, (\hat{S}^2\Omega^1_X)(-H))$. This means that $\hat{S}^2\Omega^1_X$ admits sections vanishing along any hyperplane.

As we show below, the foliation determined by ω_7 can be described sufficiently explicitly to obtain stronger results, but first we sketch how Proposition 6.3 can be used to obtain information on the genus 0 curves on X without solving differential equations. One can check via the approach in Section 5C that $\pi^*\omega_1$ is regular along the exceptional curves on Y over the singularities with $y_1 = 0$. We can then compute $\mathrm{res}(\omega_1, y_1\omega_7)$ to conclude that X contains no genus 0 curves that pass only through singularities for which $y_1 = 0$. By symmetry, the same holds for $y_2 = 0$ and $y_3 = 0$, and since every node on X satisfies $y_1 y_2 y_3 = 0$, we see that any genus 0 curve has to pass through at least two distinct nodes.

For the proof of Theorem 1.2 we need some information on the curves that do lie on X. The list of curves in the lemma below already appears in [van Luijk 2000].

Lemma 7.1. *Suppose $L \subset X$ is a curve of genus at most 1, contained in a hyperplane H spanned by nodes of X. Then L is one of the following curves:*

- *8 genus 0 curves satisfying $x_1^2 + x_2^2 + x_3^2 = 0$, defined over $\mathbb{Q}(i)$.*

- *24 genus 0 curves satisfying $x_1 x_2 x_3 = 0$, defined over $\mathbb{Q}(i)$.*

- *24 genus 1 curves satisfying one of three equations of the form $2x_1^2 + x_2^2 + x_3^2 = 0$, defined over $\mathbb{Q}(i, \sqrt{2})$, each through three noncollinear singularities of X.*

- *36 genus 1 curves satisfying one of three equations of the form $x_1^4 - x_2^4$, defined over $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{2})$, each through three or four noncollinear singularities of X.*

Proof. The singular locus S consists of 48 points, so there are at most $\binom{48}{6}$ hyperplanes H to be considered. As it turns out, there are somewhat less than 60,000 of them, forming 2442 orbits under the 384 obvious linear automorphisms generated by the simultaneous permutation action on x_1, x_2, x_3 and y_1, y_2, y_3 and the sign changes on each variable. We establish the lemma by considering representatives of each orbit, decomposing $H \cap X$, and checking which components are curves of genus at most 1. See the online supplement for a transcript of the computations. We find the list stated. Note that all the curves are nonsingular, that the genus 0 curves we find are plane conics and that the genus 1 curves we find are complete intersections of quadrics in $\mathbb{P}^3$, each through at least three, noncollinear singularities.   □

Lemma 7.2. *Let η be the degree two symmetric differential form on $\mathbb{P}^2$ that on the affine patch $(1 : x_2 : x_3)$ is given by $(x_3^2 + 1)(dx_2)^2 - 2x_2 x_3 \, dx_2 \, dx_3 + (x_2^2 + 1)(dx_3)^2$. The integral curves of η (that is to say, curves in $\mathbb{P}^2$ onto which η pulls back to an identically vanishing symmetric differential form) are the conic $x_1^2 + x_2^2 + x_3^2 = 0$ and the tangent lines to it, given by $Ax_1 + Bx_2 + Cx_3 = 0$ with $A^2 + B^2 + C^2 = 0$.*

$$\omega_1 = \frac{x_2 x_3}{y_3^2 z^2}(dx_2)^2 - \frac{2}{z^2}\,dx_2\,dx_3 + \frac{x_2 x_3}{y_2^2 z^2}(dx_3)^2,$$

$$\omega_2 = \frac{x_3(y_1^2 + y_3^2)}{y_1^2 y_3^2 z^2}(dx_2)^2 - \frac{2x_2}{y_1^2 z^2}\,dx_2\,dx_3 - \frac{x_3}{y_1^2 z^2}(dx_3)^2,$$

$$\omega_3 = \frac{x_2(x_3^2 - 1)}{y_1^2 y_3^2 z}(dx_2)^2 - \frac{2x_3}{y_1^2 z}\,dx_2\,dx_3 + \frac{x_2}{y_1^2 z}(dx_3)^2,$$

$$\omega_4 = \frac{1}{y_3^2 z}(dx_2)^2 - \frac{1}{y_2^2 z}(dx_3)^2,$$

$$\omega_5 = \frac{x_2}{y_1^2 z^2}(dx_2)^2 + \frac{2x_3}{y_1^2 z^2}\,dx_2\,dx_3 - \frac{x_2(y_1^2 + y_2^2)}{y_1^2 y_2^2 z^2}(dx_3)^3,$$

$$\omega_6 = \frac{x_3}{y_1^2 z}(dx_2)^2 - \frac{2x_2}{y_1^2 z}\,dx_2\,dx_3 + \frac{x_3(x_2^2 - 1)}{y_1^2 y_2^2 z}(dx_3)^2,$$

$$\omega_7 = \frac{1}{y_1 y_2 y_3 z^2}\big((x_3^2 + 1)(dx_2)^2 - 2x_2 x_3\,dx_2\,dx_3 + (x_2^2 + 1)(dx_3)^2\big),$$

$$x_2\omega_7, \quad x_3\omega_7, \quad y_1\omega_7, \quad y_2\omega_7, \quad y_3\omega_7, \quad \text{and} \quad z\omega_7.$$

Table 1. Generators for $H^0(X, \hat{S}^2\Omega_X^1)$, given on affine patch $x_1 = 1$.

Proof. It is straightforward to check that the given curves are indeed integral curves for η.

For instance, for $C \neq 0$ we use the parametrization $(x_1 : x_2 : x_3) = (1 : t : (-Bt - A)/C)$. On that line we have $dx_2 = dt$ and $dx_3 = -(B/C)\,dt$. Substitution into η yields $(A^2 + B^2 + C^2)\,dt^2/C^2$. For the conic we check similarly through the parametrization $(x_1 : x_2 : x_3) = \big(\sqrt{-1} : (1 - t^2)/(t^2 + 1) : -2t/(t^2 + 1)\big)$.

Any point not on $x_1^2 + x_2^2 + x_3^2 = 0$ has exactly two tangent lines to the conic passing through it. Since η is of degree 2, an integral curve passing through a point P must have one of at most two tangent directions. It follows that an integral curve to η that is nonsingular at a point P outside $x_1^2 + x_2^2 + x_3^2 = 0$ must be one of the tangent lines locally and therefore globally. This is sufficient to establish the lemma. $\square$

Proof of Theorem 1.2. Suppose that $L \subset X$ is a genus 0 curve such that the singularities of X it passes through are contained in a hyperplane H. Let h be the linear form defining H. Then $\omega = h\omega_7 \in H^0(X, \hat{S}^2\Omega_X^1)$ vanishes along H. By Corollary 3.4, for any singularity s of X in H, we have that $\pi^*\omega$ is regular on the exceptional curve $E_s \subset Y$. Hence, we see that ω pulls back to 0 on L.

We observe that $\phi : X \to \mathbb{P}^2$ given by $(x_1 : x_2 : x_3)$ expresses X as a finite, multiquadratic cover of $\mathbb{P}^2$ of degree 16, ramified over $(x_1^2 + x_2^2)(x_1^2 + x_3^2)(x_2^2 + x_3^2)(x_1^2 + x_2^2 + x_3^2) = 0$. For

$$\eta = (x_3^2 + 1)(dx_2)^2 - 2x_2 x_3\,dx_2\,dx_3 + (x_2^2 + 1)(dx_3)^2$$

and $h = x_1$, we see that $\omega = \phi^*(\eta)/(y_1 y_2 y_3 z^2)$. It follows that L must lie in H or that $\phi(L)$ is a solution curve to η. Lemma 7.1 lists the curves contained a hyperplane H spanned by singularities.

The alternative is that $\phi : L \to \phi(L)$ expresses L as a cover of one of the curves classified by Lemma 7.2. The curves that cover $x_1^2 + x_2^2 + x_3^2 = 0$ are contained in the hyperplane $z = 0$ and are included in Lemma 7.1. Therefore, let us assume that $\phi(L)$ is given by $Ax_1 + Bx_2 + Cx_3 = 0$, with $A^2 + B^2 + C^2 = 0$. Note that X is a compositum
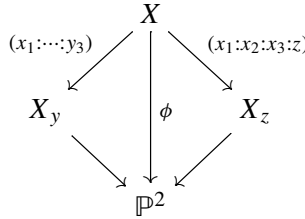
and that $X_z \to \mathbb{P}^2$ is ramified over $x_1^2 + x_2^2 + x_3^2 = 0$. Since $\phi(L)$ is tangent to this locus, we see that $\phi(L)$ pulls back to two components on X_z. On X_y, generically $\phi(L)$ pulls back to a nonsingular complete intersection of quadrics in $\mathbb{P}^4 \subset \mathbb{P}^5$, so is a canonical genus 5 curve. It follows in those cases that L itself is isomorphic to this genus 5 curve, contradicting that L has genus 0. Riemann–Hurwitz shows that this can only be avoided if $\phi(L)$ passes through a singular point of the branch locus of $X_y \to \mathbb{P}^2$. However, that branch locus consists of tangent lines to $x_1^2 + x_2^2 + x_3^2 = 0$, so this only happens if $\phi(L)$ is one of the components of $(x_1^2 + x_2^2)(x_1^2 + x_3^2)(x_2^2 + x_3^2) = 0$. But such curves L are contained in a hyperplane (such as $x_1 + ix_2 = 0$). These are included in the list of curves in Lemma 7.1, and these give genus 1 curves.

Next we show that X does not contain genus 1 curves C for which $X \cap S$ generates a linear space of codimension at least 2 in the linear space generated by C. We argue by contradiction and assume C is such a curve. Then we can choose a point P on C outside of the singular locus on X such that C is not contained in the linear span on $C \cap S$ and P. That means we can choose a hyperplane H that contains $C \cap S$ and P, but not C entirely. Let h be the linear form defining H. By Corollary 3.4, the differential $h\omega_7$ pulls back to a regular one with a zero at P on C, so it must pull back to 0. However, we have constructed H to not contain all of C, so $\phi : C \to \phi(C)$ expresses C as a cover of one of the curves classified by Lemma 7.2. As mentioned above, we do find some genus 1 curves, but these are of degree 4 and $C \cap S$ is easily checked to be of codimension at most 1 in the linear space generated by C.

As special cases, note that a genus 1 curve cannot be contained in a 1-dimensional linear space, so any genus 1 curve on X must pass through at least two singularities.

Furthermore. if $C \cap S$ consists of at most 5 points, we see that C must be contained in a hyperplane section, limiting the degree of C to 16. □

8. Applications to nodal surfaces in $\mathbb{P}^3$

Let $X \subset \mathbb{P}^3$ be a hypersurface of degree $d \geq 5$ with ℓ singularities of type A_1 and let Y be its minimal resolution. We saw in Example 4.2 that for $\ell > \frac{9}{4}(2d^2 - 5d)$ and sufficiently large m we have $h^0(Y, S^m \Omega_Y^1) > 0$. There are a few well-known surfaces of low degree d and with many A_1 singularities; see Table 2.

Lemma 6.1 allows us to conclude that Barth's decic surface and Sarti's surface are algebraically quasihyperbolic. In fact, from our lower bounds we find the lowest m for which there are guaranteed to

	d	ℓ
Barth's sextic surface [1996]	6	65
Barth's decic surface [1996]	10	345
Sarti's surface [2001]	12	600

Table 2. Well-known surfaces of low degree d and with many A_1 singularities.

be global sections. For example, for Barth's decic surface we find that

$$h^0(Y, S^m \Omega_Y^1) \geq \begin{cases} \frac{2}{9}m^3 - \frac{538}{3}m^2 - 82m + 85 & \text{for } m \equiv 0 \pmod 3, \\ \frac{2}{9}m^3 - \frac{538}{3}m^2 - 82m + \frac{991}{9} & \text{for } m \equiv 1 \pmod 3, \\ \frac{2}{9}m^3 - \frac{538}{3}m^2 - \frac{472}{3}m + \frac{200}{9} & \text{for } m \equiv 2 \pmod 3. \end{cases}$$

These bounds turn positive when $m \geq 160$, where we find $h^0(Y, S^{160}\Omega_Y^1) \geq 15755$. For Sarti's surface, a similar computation shows that the bounds turn positive when $m \geq 28$ and that $h^0(Y, S^{28}\Omega_Y^1) \geq 7646$. Neither of these values is within the range of practical computation to explicitly determine the locus of rational and genus 1 curves.

For Barth's decic surface $X = X_{10}$, Magma 2.24-6 [Bosma et al. 1997] is just about capable of computing the graded module representing $S^2(\Omega_X^1)^{\vee\vee}$ over a finite field. We did so over $\mathbb{F}_{10009}$ and $\mathbb{F}_{50021}$. In both cases this took about 8 hours of computations on an Intel Xeon CPU E5-2660, 2.20GHz, using most of the 64GB main memory. We find in both cases that $h^0(X, \hat{S}^2\Omega_X^1(-H)) = 7$ and that $h^0(X, \hat{S}^2\Omega_X^1) = 7 \cdot 4 + 27 = 55$. We expect that these results are representative of what happens in characteristic 0, which one could confirm by rational reconstruction (see below for a case where we actually executed this procedure). With that in place, the following results would be within reach:

- By Propositions 3.3 and 3.1, we can choose any 17 singularities and find $55 - 3 \cdot 17 = 4$ differentials that extend to the exceptional components above them, by the method explained in Section 5C (and we find we cannot extend them into all 345 components). Hence, Proposition 6.3 would give an approach to determining all genus 0 curves on X that pass through at most 17 singularities, at the cost of some significant combinatorics.

- We can apply Corollary 6.5 to find an explicit description of the genus 0 curves C on X for which $C \cap X$ does not span all of $\mathbb{P}^3$, in particular, all curves passing through at most 3 singularities, as well as all genus 1 curves that pass through at most 2 singularities.

We do not pursue these particular results, but instead demonstrate similar results on the sextic surface X_6, as described below.

8A. *Barth's sextic surface.* As an illustration, we perform similar computations for Barth's sextic surface X_6, defined over $\mathbb{Q}(\sqrt{5})$, with $\phi = \frac{1}{2}(\sqrt{5} + 1)$, by

$$X_6 : 4(\phi^2 x^2 - y^2)(\phi^2 y^2 - z^2)(\phi^2 z^2 - x^2) - (1 + 2\phi)(x^2 + y^2 + z^2 - w^2)^2 = 0.$$

We find the following genus 0 curves on X_6:

- 6 degree 1 curves, each through 5 singularities.
- 6 degree 2 curves, each through 10 singularities.
- 15 degree 6 plane curves, each through 10 singularities.

In addition we find the following genus 1 curves on X_6:

- 20 degree 3 plane curves, each through 15 singularities.
- 10 pairs of degree 3 plane curves through 9 singularities, defined over $\mathbb{Q}(\sqrt{5}, i)$.
- 48 degree 5 plane curves, each through 10 singularities.
- 15 degree 4, nonplanar curves, each through 16 singularities.

This list includes all genus 0 or 1 curves on X_6 that lie in planes spanned by nodes.

Proof of Theorem 1.1. As it turns out, computing the graded module representing $S^2(\Omega_X^1)^{\vee\vee}$ directly in characteristic 0 is not quite feasible with Magma 2.24-6 [Bosma et al. 1997]. Over a finite field, however, it can do so in only a matter of minutes, even for a 50-digit characteristic for which 5 is not a square in the prime field. We then use rational reconstruction to compute the trace and norm of each coefficient, and choose the conjugate in $\mathbb{Q}(\sqrt{5})$ that reduces to the coefficient. This allows us to lift the representations of the modules M, $M^\vee$, $M^{\vee\vee}$, together with the pairing matrices A, B, as in Section 5.

In order to verify that these reconstructed modules indeed have the right properties, we check that the matrices A, B define well-defined pairings, i.e., that $A K_{M^\vee} \subset K_M$ etc. This establishes that $M_0^{\vee\vee}$ indeed encodes sections of $S^2 \Omega_X^1$ that are regular outside of the locus where an appropriate 3×3 submatrix A' is singular. Thus, if we establish that the base locus of the appropriate 3×3 minors of A is supported on the singular locus of X, then we establish that $M_0^{\vee\vee}$ determines global sections of $\hat{S}^2 \Omega_X^1$. Magma is capable of directly verifying in characteristic 0 that the reconstructed module $M^{\vee\vee}$ equals its double dual, establishing that it is reflexive. See the online supplement for a transcript of the computations verifying these claims. We find that $h^0(X_6, \hat{S}^2 \Omega^1(-H)) = 3$ and $h^0(X_6, \hat{S}^2 \Omega^1) = 15$. The forms themselves are a little unappetizing to display here.

Let ω_1, ω_2, ω_3 span $H^0(X, \hat{S}^2 \Omega^1(-H))$. We apply Corollary 6.5 to get information on genus 0 curves L for which $L \cap S$ is contained in a hyperplane and on genus 1 curves C for which $C \cap S$ spans a space of codimension at most one in the linear space spanned by C. We find that outside of the vanishing locus of some of the 3×3 minors $\det(A')$ of A, the locus $\mathrm{res}(\omega_1, \omega_2) \cap \mathrm{res}(\omega_1, \omega_3) \cap \mathrm{res}(\omega_2, \omega_3)$ is 0-dimensional. That shows that any such genus 0 curve L needs to lie in a plane spanned by singularities from S, or in the locus defined by $\det(A') = 0$. This leaves us with analyzing finitely many loci. We can use the automorphism group of X to significantly reduce the amount of computation required. We find the curves listed. See the online supplement for a transcript of the computations. Interestingly, the loci $\det(A') = 0$ also yield some nonplanar genus 1 curves passing through singularities that span $\mathbb{P}^3$. □

Remark 8.1. We mention here that in the master's thesis [Alaei 2015] completed under the supervision of the first author, the same 27 genus 0 curves are already mentioned, and a similar argument to the one here is used to prove the slightly weaker result that any genus 0 curve on X_6 has to pass through at least one node.

Appendix: Hirzebruch–Riemann–Roch for twists of symmetric powers

We record the result of standard calculations of Chern classes of certain sheaves needed in the body of the article.

Let Y be a smooth projective surface over a field. By a *vector sheaf of rank r* we mean a locally free sheaf of rank r on Y; when $r = 1$ we call such a sheaf a *line bundle*. Let $\mathcal{E}$ be a vector sheaf of rank 2 on Y. Let $c_1(\mathcal{E})$, $c_2(\mathcal{E})$ be the usual Chern classes of $\mathcal{E}$. Using the splitting principle [Fulton 1998, Remark 3.2.3], we compute the Chern classes for the symmetric power $\mathcal{A} = S^m\mathcal{E}$, a rank $m + 1$ locally free sheaf on Y:

$$
\begin{aligned}
c_1(\mathcal{A}) &= c_1(S^m\mathcal{E}) = \binom{m+1}{2}c_1(\mathcal{E}), \\
c_2(\mathcal{A}) &= c_2(S^m\mathcal{E}) = \tfrac{1}{24}(3m+2)(m+1)m(m-1)c_1^2(\mathcal{E}) + \tfrac{1}{6}m(m+1)(m+2)c_2(\mathcal{E}).
\end{aligned}
\tag{A-1}
$$

For any vector sheaf $\mathcal{A}$ of rank $(m + 1)$ and a line bundle $\mathcal{L}$ on Y we have

$$
\begin{aligned}
c_1(\mathcal{A} \otimes \mathcal{L}) &= c_1(\mathcal{A}) + (m+1)c_1(\mathcal{L}), \\
c_2(\mathcal{A} \otimes \mathcal{L}) &= c_2(\mathcal{A}) + mc_1(\mathcal{A})c_1(\mathcal{L}) + \binom{m+1}{2}c_1^2(\mathcal{L});
\end{aligned}
\tag{A-2}
$$

see [Fulton 1998, p. 55].

Let $\mathcal{F}$ be a vector sheaf of rank r on Y. Writing $K = -c_1(T_Y)$ and $\chi = c_2(T_Y)$ for the Chern classes of the tangent bundle T_Y of Y, the Hirzebruch–Riemann–Roch theorem gives the Euler characteristic of $\mathcal{F}$ in terms of Chern classes of $\mathcal{F}$ and T_Y (see [Fulton 1998, Example 15.2.2]):

$$
\chi(Y, \mathcal{F}) = \left(-\tfrac{1}{2}c_1(\mathcal{F})K + \tfrac{1}{2}c_1^2(\mathcal{F}) - c_2(\mathcal{F}) + \tfrac{1}{12}r(K^2 + \chi)\right)[Y].
\tag{A-3}
$$

Here $[Y]$ is the fundamental class of Y. Together with (A-1) and (A-2), Hirzebruch–Riemann–Roch affords the Euler characteristic of $\mathcal{F} := S^m\mathcal{E} \otimes \mathcal{L}$ in terms of $c_1(\mathcal{E})$, $c_2(\mathcal{E})$, $c_1(\mathcal{L})$, K and χ:

$$
\begin{aligned}
\chi(Y, S^m\mathcal{E} \otimes \mathcal{L}) = {} &\tfrac{1}{6}\big(c_1(\mathcal{E})^2 - c_2(\mathcal{E})\big)m^3 - \tfrac{1}{4}\big(c_1(\mathcal{E})K - c_1(\mathcal{E})^2 - 2c_1(\mathcal{E})c_1(\mathcal{L}) + 2c_2(\mathcal{E})\big)m^2 \\
&+ \tfrac{1}{12}\big(K^2 - 3c_1(\mathcal{E})K + c_1(\mathcal{E})^2 - 6c_1(\mathcal{L})K \\
&\qquad\qquad + 6c_1(\mathcal{E})c_1(\mathcal{L}) + 6c_1(\mathcal{L})^2 - 4c_2(\mathcal{E}) + \chi\big)m \\
&\qquad\qquad\qquad + \tfrac{1}{12}\big(K^2 - 6c_1(\mathcal{L})K + 6c_1(\mathcal{L})^2 + \chi\big).
\end{aligned}
\tag{A-4}
$$

We specialize this result in two different ways. First, setting $\mathcal{E} = \Omega_Y^1$ and $\mathcal{L} = \mathcal{O}_Y$ and using $c_1(\Omega_Y^1) = K$ and $c_2(\Omega_Y^1) = \chi$, we get

$$
\chi(Y, S^m\Omega_Y^1) = \tfrac{1}{12}\big(2(K^2 - \chi)m^3 - 6\chi m^2 - (K^2 + 3\chi)m + K^2 + \chi\big).
\tag{A-5}
$$

Second, if E is an irreducible (-2)-curve on Y, then for $\mathcal{E} = \Omega_Y^1(\log(E))$ and $\mathcal{L}_h = -hE$ we have $c_1(\mathcal{E}) = K + E$, $c_2(\mathcal{E}) = \chi - 2$, and $c_1(\mathcal{L}_h) = -hE$, as well as $EK = 0$, leading to

$$\chi(Y, S^m \mathcal{E} \otimes \mathcal{L}_h) = \chi(Y, S^m \Omega_Y^1) - (m+1)\left(h^2 + hm - \tfrac{1}{2}m\right). \tag{A-6}$$

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