KATOK'S SPECIAL REPRESENTATION THEOREM FOR MULTIDIMENSIONAL BOREL FLOWS

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ABSTRACT. Katok's special representation theorem states that any free ergodic measure-preserving $\mathbb{R}^d$-flow can be realized as a special flow over a $\mathbb{Z}^d$-action. It provides a multidimensional generalization of the "flow under a function" construction. We prove the analog of Katok's theorem in the framework of Borel dynamics and show that, likewise, all free Borel $\mathbb{R}^d$-flows emerge from $\mathbb{Z}^d$-actions through the special flow construction using bi-Lipschitz cocycles.

1. INTRODUCTION

1.1. **Overview.** Theorems of Ambrose and Kakutani [1,2] established a connection between measure-preserving $\mathbb{Z}$-actions and $\mathbb{R}$-flows by showing that any flow admits a cross-section and can be represented as a "flow under a function". Their construction provides a foundation for the theory of Kakutani equivalence (also called monotone equivalence) [7,11] on the one hand and study of the possible ceiling functions in the "flow under a function" representation [16,21] on the other.

The intuitive geometric picture of a "flow under a function" does not generalize to $\mathbb{R}^d$-flows for $d \geq 2$. However, Katok [12] re-interpreted it in a way that can readily be adapted to the multidimensional setup, calling flows appearing in this construction *special flows*. Despite their name, they aren't so special, since, as showed in the same paper, every free ergodic measure-preserving $\mathbb{R}^d$-flow is metrically isomorphic to a special flow. Just like the works of Ambrose and Kakutani, it opened gates for the study of multidimensional concepts of Kakutani equivalence [5] and stimulated research on tilings of flows [15,22].

Borel dynamics as a separate field goes back to the work of Weiss [31] and has blossomed into a versatile branch of dynamical systems. The phase space here is a standard Borel space $(X, \mathscr{B})$, i.e., a set X with a σ-algebra $\mathscr{B}$ of Borel sets for some Polish topology on X. Some of the key ergodic theoretical results have their counterparts in Borel dynamics, while others do not generalize. For example, Borel version of the Ambrose–Kakutani Theorem on the existence of cross-sections in $\mathbb{R}$-flows was proved by Wagh in [30] showing that, just like in ergodic theory, all free Borel $\mathbb{R}$-flows emerge as "flows under a function" over Borel $\mathbb{Z}$-actions. Likewise, Rudolph's two-valued theorem [21] generalizes to the Borel framework [25]. The theory of Kakutani equivalence, on the other hand, exhibits a different phenomenon. While being a highly non-trivial equivalence relation among measure-preserving flows [9,20], descriptive set theoretical version of Kakutani equivalence collapses entirely [19].

Considerable work has been done to understand the Borel dynamics of $\mathbb{R}$-flows, but relatively few things are known about multidimensional actions. This paper makes a contribution in this direction by showing that the analog of Katok's special representation theorem does hold for free Borel $\mathbb{R}^d$-flows.

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1.2. **Structure of the paper.** Constructions of orbit equivalent $\mathbb{R}^d$-actions often rely on (essential) hyperfiniteness and use covers of orbits of the flow by coherent and exhaustive regions. This is the case for the aforementioned paper of Katok [12], and related approaches have been used in the descriptive set theoretical setup as well (e.g., [26]). Particular assumptions on such coherent regions, however, depend on the specific application. Section 2 distills a general language of partial actions, in which many of the aforementioned constructions can be formulated. As an application we show that the orbit equivalence relation generated by a free $\mathbb{R}$-flow can also be generated by a free action of *any* non-discrete and non-compact Polish group (see Theorem 2.6). This is in a striking contrast with the actions of discrete groups, where a probability measure-preserving free $\mathbb{Z}$-action can be generated only by a free action of an amenable group.

Section 3 does the technical work of constructing Lipschitz maps that are needed for Theorem 3.12, which shows, roughly speaking, that up to an arbitrarily small bi-Lipschitz perturbation, any free $\mathbb{R}^d$-flow admits an integer grid—a Borel cross-section invariant under the $\mathbb{Z}^d$-action.

Finally, Section 4 discusses the descriptive set theoretical version of Katok's special flow construction and shows in Theorem 4.3 that, indeed, any free $\mathbb{R}^d$-flow can be represented as a special flow generated by a bi-Lipschitz cocycle with Lipschitz constants arbitrarily close to 1. This provides a Borel version of Katok's special representation theorem.

2. SEQUENCES OF PARTIAL ACTIONS

We begin by discussing the framework of partial actions suitable for constructing orbit equivalent actions. Throughout this section, X denotes a standard Borel space.

2.1. **Partial actions.** Let G be a standard Borel group, that is a group with a structure of a standard Borel space that makes group operations Borel. A **partial G-action**[1] is a pair (E,ϕ), where E is a Borel equivalence relation on X and $\phi : X \to G$ is a Borel map that is injective on each E-class: $\phi(x) \neq \phi(y)$ whenever xEy. The map ϕ itself may occasionally be refer to as a partial action when the equivalence relation is clear from the context.

The motivation for the name comes from the following observation. Consider the set

$$A_\phi = \big\{(g,x,y) \in G \times X \times X : xEy \text{ and } g\phi(x) = \phi(y)\big\}.$$

Injectivity of ϕ on E-classes ensures that for each $x \in X$ and $g \in G$ there is at most one $y \in X$ such that $(g,x,y) \in A_\phi$. When such a y exists, we say that the action of g on x is defined and set $gx = y$. Clearly, $(e,x,x) \in A_\phi$ for all $x \in X$, thus $ex = x$; also $g_2(g_1x) = (g_2g_1)x$ whenever all the terms are defined. The set A_ϕ is a graph of a total action $G \curvearrowright X$ if and only if for each $x \in X$ and $g \in G$ there does exist some $y \in X$ such that $(g,x,y) \in A_\phi$; in this case the orbit equivalence relation generated by the action coincides with E.

Example 2.1. An easy way of getting a partial action is by restricting a total one. Suppose we have a free Borel action $G \curvearrowright X$ with the corresponding orbit equivalence relation E_G and suppose that a Borel equivalence sub-relation $E \subseteq E_G$ admits a Borel selector—a Borel E-invariant map $\pi : X \to X$ such that $xE\pi(x)$ for all $x \in X$. If $\phi : X \to G$ is the map specified uniquely by the condition $\phi(x)\pi(x) = x$, then (E,ϕ) is a partial G-action.

Sub-relations E as in Example 2.1 are often associated with cross-sections of actions of locally compact second countable (lcsc) groups.

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[1] More precisely, we should call such (E,ϕ) a partial *free* action. Since we are mainly concerned with free actions in what follows, we choose to omit the adjective "free" in the definition.

2.2. **Tessellations of lcsc group actions.** Consider a free Borel action $G \curvearrowright X$ of a locally compact second countable group. A **cross-section** of the action is a Borel set $\mathscr{C} \subseteq X$ that intersects every orbit in a countable non-empty set. A cross-section $\mathscr{C} \subseteq X$ is

- **discrete** if $(Kx) \cap \mathscr{C}$ is finite for every $x \in X$ and compact $K \subseteq G$;
- U**-lacunary**, where $U \subseteq G$ is a neighborhood of the identity, if $Uc \cap \mathscr{C} = \{c\}$ for all $c \in \mathscr{C}$;
- **lacunary** if it is U-lacunary for some neighborhood of the identity U;
- **cocompact** if $K\mathscr{C} = X$ for some compact $K \subseteq G$.

Let $\mathscr{C}$ be a lacunary cross-section for $G \curvearrowright X$, which exists by [13, Corollary 1.2]. Any lcsc group G admits a compatible left-invariant proper metric [28], and any left-invariant metric d can be transferred to orbits due to freeness of the action via $\operatorname{dist}(x, y) = d(g, e)$ for the unique $g \in G$ such that $gx = y$. One can now define the so-called Voronoi tessellation of orbits by associating with each $x \in X$ the closest point $\pi_{\mathscr{C}}(x) \in \mathscr{C}$ of the cross-section $\mathscr{C}$ as determined by dist. Properness of the metric ensures that, for a ball $B_R \subseteq G$ of radius R, $B_R = \{g \in G : d(g, e) \leq R\}$, and any $x \in X$, the set $\mathscr{C} \cap B_R x$ is finite. Indeed, there can be at most $\lambda(B_{R+r})/\lambda(B_r)$ points in the intersection, where λ is a Haar measure on the group and $r > 0$ is so small that $B_r c \cap B_r c' = \varnothing$ whenever $c, c' \in \mathscr{C}$ are distinct.

Small care needs to be taken to address the possibility of having several closest points. For example, one may pick a Borel linear order on $\mathscr{C}$ and associated each x with the *smallest* closest point in the cross-section (see [23, Section 4] or [17, Section B.2] for the specifics). This way we get a Borel equivalence relation $E_{\mathscr{C}} \subseteq E_G$ whose equivalence classes are the cells of the Voronoi tessellation: $xE_{\mathscr{C}}y$ if and only if $\pi_{\mathscr{C}}(x) = \pi_{\mathscr{C}}(y)$.

Assumed freeness of the action $G \curvearrowright X$ allows for a natural identification of each Voronoi cell with a subset of the acting group via the map $\pi_{\mathscr{C}}^{-1}(c) \ni x \mapsto \phi_{\mathscr{C}}(x) \in G$ such that $\phi_{\mathscr{C}}(x)c = x$, which is exactly what the corresponding partial action from Example 2.1 does.

Our intention is to use partial actions to define total actions, and the example above may seem like going "in the wrong direction". The point, however, is that once we have a partial action $\phi : X \to G$, we can compose it with an arbitrary Borel injection $f : G \to G$ to get a different partial action $f \circ \phi$. This pattern is typical in the sense that new partial actions are often constructed by modifying those obtained as restrictions of total actions.

2.3. **Convergent sequences of partial actions.** A total action can be defined whenever we have a sequence of partial actions that cohere in the appropriate sense. Let G be a standard Borel group. A sequence (E_n, ϕ_n), $n \in \mathbb{N}$, of partial G-actions on X is said to be **convergent** if it satisfies the following properties:

- **monotonicity:** equivalence relations E_n form an increasing sequence, that is $E_n \subseteq E_{n+1}$ for all n;
- **coherence:** for each n the map $x \mapsto (\phi_n(x))^{-1}\phi_{n+1}(x)$ is E_n-invariant;
- **exhaustiveness:** for all $x \in X$ and all $g \in G$ there exist n and $y \in X$ such that $xE_n y$ and $g\phi_n(x) = \phi_n(y)$.

With such a sequence one can associate a free Borel (left) action $G \curvearrowright X$, called the **limit** of $(E_n, \phi_n)_n$, whose graph is $\bigcup_n A_{\phi_n}$. Coherence ensures that the partial action defined by ϕ_{n+1} is an extension of the one given by ϕ_n. Indeed, if $xE_n y$ are such that $g\phi_n(x) =$

$\phi_n(y)$, then also $xE_{n+1}y$ by monotonicity and, using coherence,

$$g\phi_{n+1}(x) = g\phi_n(x)(\phi_n(x))^{-1}\phi_{n+1}(x) = \phi_n(y)(\phi_n(y))^{-1}\phi_{n+1}(y) = \phi_{n+1}(y),$$

whence $A_{\phi_n} \subseteq A_{\phi_{n+1}}$. If C is an E_n-class, and $s = (\phi_n(x))^{-1}\phi_{n+1}(x)$ for some $x \in C$, then $\phi_{n+1}(C) = \phi_n(C)s$, so the image $\phi_n(C)$ gets shifted on the right inside $\phi_{n+1}(C)$. If we want to build a right action of the group, then $\phi_n(C)$ should be shifted on the left instead.

Finally, exhaustiveness guarantees that gx gets defined eventually: for all $g \in G$ and $x \in X$ there are n and $y \in X$ such that $(g, x, y) \in A_{\phi_n}$. It is straightforward to check that $\bigcup_n A_{\phi_n}$ is a graph of a total Borel action $G \curvearrowright X$. Equally easy is to check that the action is free, and its orbits are precisely the equivalence classes of $\bigcup_n E_n$.

This framework, general as it is, delegates most of the complexity to the construction of maps ϕ_n. Let us illustrate these concepts on essentially hyperfinite actions of lcsc groups.

2.4. **Hyperfinite tessellations of lcsc group actions.** In the context of Section 2.2, suppose that, furthermore, the restriction of the orbit equivalence relation E_G onto the cross-section $\mathscr{C}$ is hyperfinite, i.e., there is an increasing sequence of finite Borel equivalence relations F_n on $\mathscr{C}$ such that $\bigcup_n F_n = E_G|_{\mathscr{C}}$. We can use this sequence to define xE_ny whenever $\pi_{\mathscr{C}}(x)F_n\pi_{\mathscr{C}}(y)$, which yields an increasing sequence of Borel equivalence relations E_n such that $E_G = \bigcup_n E_n$.

The equivalence relations F_n admit Borel transversals, i.e., there are Borel sets $\mathscr{C}_n$ that pick exactly one point from each F_n-class. Just as in Section 2.2, we may define $\phi_n(x)$ to be such an element $g \in G$ that $gc = x$ for the unique $c \in \mathscr{C}_n$ satisfying xE_nc. This gives a convergent sequence of partial G-actions $(E_n, \phi_n)_n$ whose limit is the original action $G \curvearrowright X$.

2.5. **Partial actions revisited.** In practice, it is often more convenient to allow equivalence relations E_n to be defined on proper subsets of X. Let $X_n \subseteq X$, $n \in \mathbb{N}$, be Borel subsets, and suppose for each n, E_n is a Borel equivalence relation on X_n. We say that the sequence $(E_n)_n$ is **monotone** if the following conditions are satisfied for all $m \le n$:

- $E_m|_{X_m \cap X_n} \subseteq E_n|_{X_m \cap X_n}$;
- if $x \in X_m \cap X_n$ then the whole E_m-class of x is in X_n.

Partial action maps $\phi_n : X_n \to G$, where, as earlier, G is a standard Borel group, need to satisfy the appropriate versions of coherence and exhaustiveness:

- **coherence**: $X_m \cap X_n \ni x \mapsto (\phi_m(x))^{-1}\phi_n(x)$ is E_m-invariant for each $m < n$;
- **exhaustiveness**: for each $x \in X$ and $g \in G$ there exist n and $y \in X_n$ such that $x \in X_n$, xE_ny, and $g\phi_n(x) = \phi_n(y)$.

A sequence of partial G-actions $(X_n, E_n, \phi_n)_n$ will be called **convergent** if it satisfies the above properties of monotonicity, coherence, and exhaustiveness. Note that the condition $\bigcup_n X_n = X$ follows from exhaustiveness, so sets X_n must cover all of X.

Convergent sequences $(X_n, E_n, \phi_n)_n$ define total actions, which can be most easily seen by reducing this setup to the notationally simpler one given in Section 2.3. To this end, extend E_n to the equivalence relation $\hat{E}_n$ on all of X by

$$x\hat{E}_ny \iff \exists m \le n\ xE_my \text{ or } x = y;$$

and also extend ϕ_n to $\hat{\phi}_n : X \to G$ by setting $\hat{\phi}_n(x) = \phi_m(x)$ for the maximal $m \le n$ such that $x \in X_m$ or $\hat{\phi}_n(x) = e$ if no such m exists. It is straightforward to check that $(\hat{E}_n, \hat{\phi}_n)_n$ is a convergent sequence of partial G-actions in the sense of Section 2.3. By the **limit** of the sequence of partial actions $(X_n, E_n, \phi_n)_n$ we mean the limit of $(\hat{E}_n, \hat{\phi}_n)_n$ as defined earlier.

Remark 2.2. A variant of this generalized formulation, which we encounter in Proposition 2.4 below, occurs when sets X_n are nested: $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$. Monotonicity of equivalence relations then simplifies to $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ and coherence becomes equivalent to the E_n-invariant of maps $X_n \ni x \mapsto (\phi_n(x))^{-1}\phi_{n+1}(x) \in G$.

As was mentioned above, it is easy to create new partial actions simply by composing a partial action $\phi : X \to G$ with some Borel bijection $f : G \to G$ (or $f : G \to H$ if we choose to have values in a different group). However, an arbitrary bijection has no reasons to preserve coherence and extra care is necessary to maintain it.

Furthermore, in general we need to apply different modifications f to different E_n-classes, which naturally raises concern of how to ensure that construction is performed in a Borel way. In applications, the modification f applied to an E_n-class C, usually depends on the "shape" of C and the E_m-classes it contains, but does not depend on other E_n-classes. If there are only countably many such "configurations" of E_n-classes, resulting partial actions $f \circ \phi$ will be Borel as long as we consistently apply the same modification whenever "configurations" are the same. This idea can be formalized as follows.

2.6. **Rational sequences of partial actions.** Let $(E_n, \phi_n)_n$ be a convergent sequence of partial actions on X. For an E_n-class C, let $\mathscr{E}_m(C)$ denote the collection of E_m-classes contained in C. Given two E_n-classes C and C', we denote by $\phi_n(C) \equiv \phi_n(C')$ the existence for each $m \le n$ of a bijection $\mathscr{E}_m(C) \ni D \mapsto D' \in \mathscr{E}_m(C')$ such that $\phi_n(D) = \phi_n(D')$ for all $D \in \mathscr{E}_m(C)$. Collection of images $\{\phi_n(D) : D \in \bigcup_{m \le n} \mathscr{E}_m(C)\}$ constitutes the "configuration" of C referred to earlier.

We say that the sequence $(E_n, \phi_n)_n$ of partial actions is **rational** if for each n there exists a Borel E_n-invariant partition $X = \bigsqcup_k Y_k$ such that for each k one has $\phi_n(C) \equiv \phi_n(C')$ for all E_n-classes $C, C' \subseteq Y_k$.

Remark 2.3. This concept of rationality applies verbatim to convergent sequences of partial actions $(X_n, E_n, \phi_n)_n$ as described in Section 2.5. One can check that such a sequence is rational if and only if the sequence $(\hat{E}_n, \hat{\phi}_n)$ is rational.

2.7. **Generating the flow equivalence relation.** As an application of the partial actions formalism, we show that any orbit equivalence relation given by a free Borel $\mathbb{R}$-flow can also be generated by a free action of any non-discrete and non-compact Polish group. For this we need the following representation of an $\mathbb{R}$-flow as a limit of partial $\mathbb{R}$-actions.

Proposition 2.4. *Any free Borel $\mathbb{R}$-flow on X can be represented as a limit of a convergent rational sequence of partial $\mathbb{R}$-actions $(X_n, E_n, \phi_n)_n$ such that*

1. *both X_n and E_n are increasing: $X_0 \subseteq X_1 \subseteq \cdots$ and $E_0 \subseteq E_1 \subseteq \cdots$; (see Remark 2.2)*
2. *each E_{n+1}-class contains finitely many E_n-classes;*
3. *each E_0-class has cardinality of continuum;*
4. *for each E_{n+1}-class C the set $C \setminus X_n$ has cardinality of continuum.*

Proof. Any $\mathbb{R}$-flow admits a rational[2] $(-4,4)$-lacunary cross-section (see [24, Section 2]), which we denote by $\mathscr{C}$. Let $(E_{\mathscr{C}}, \phi_{\mathscr{C}})$ be the partial $\mathbb{R}$-action as defined in Section 2.2. If

[2] Rationality of the cross-section here means that the distance between any two points of $\mathscr{C}$ is a rational number. More generally, rationality of a cross-section $\mathscr{C}$ for an $\mathbb{R}^d$-action means $r \in \mathbb{Q}^d$ whenever $c + r = c'$ for some $c, c' \in \mathscr{C}$.

D is an $E_{\mathscr{C}}$-class, then $\phi_{\mathscr{C}}(D)$ is an interval. For $\epsilon > 0$, let D^{ϵ} consist of those $x \in D$ such that $\phi_{\mathscr{C}}(x)$ is at least ϵ away from the boundary points of $\phi_{\mathscr{C}}(D)$. In other words, D^{ϵ} is obtained by shrinking the class D by ϵ from each side.

The restriction of the orbit equivalence relation onto $\mathscr{C}$ is hyperfinite. This fact is true in the much wider generality of actions of locally compact Abelian groups [4]. Specifically for $\mathbb{R}$-flows, $E|_{\mathscr{C}}$ is generated by the first return map—a Borel automorphism of $\mathscr{C}$ that sends a point in $\mathscr{C}$ to the next one according to the order of the $\mathbb{R}$-flow. The first return map is well defined and is invertible, except for the orbits, where $\mathscr{C}$ happens to have the maximal or the minimal point. The latter part of the space evidently admits a Borel selector and is therefore smooth, hence won't affect hyperfiniteness of the equivalence relation. It remains to recall the standard fact that orbit equivalence relations of $\mathbb{Z}$-actions are hyperfinite (see, for instance, [6, Theorem 5.1]), and thus so is the restriction $E|_{\mathscr{C}}$.

In particular, we can represent the $\mathbb{R}$-flow as the limit of a convergent sequence of partial actions $(E'_n, \phi'_n)_n$ as described in Section 2.4. Note that $(E'_n, \phi'_n)_n$ is necessarily rational by rationality of $\mathscr{C}$. Such a sequence satisfies items (2) and (3), but fails (4). We fix this by shrinking equivalence classes to achieve proper containment. Let $(\epsilon_n)_n$ be a strictly decreasing sequence of positive reals such that $1 > \epsilon_0$ and $\lim_n \epsilon_n = 0$. Put $X'_n = \bigcup D^{\epsilon_n}$, where the union is taken over all $E_{\mathscr{C}}$-classes D. Note that sets X'_n fail to cover X, because the boundary points of any $E_{\mathscr{C}}$-class do not belong to any of X'_n. Put $Y = X \setminus \bigcup_n X'_n$ and let $X_n = X'_n \cup Y$. Clearly, $(X_n)_n$ is an increasing sequence of Borel sets and $\bigcup_n X_n = X$.

Finally, set $E_n = E'_n|_{X_n}$ and $\phi_n : X_n \to \mathbb{R}$ to be $\phi'_n|_{X_n}$. The sequence $(X_n, E_n, \phi_n)_n$ of partial $\mathbb{R}$-actions satisfies the conditions of the proposition. $\square$

All non-smooth orbit equivalence relations produced by free Borel $\mathbb{R}$-flows are Borel isomorphic to each other [14, Theorem 3]. Theorem 2.6 will show that this orbit equivalence relation can also be generated by a free action of any non-compact and non-discrete Polish group.

Let G be a group. We say that a set $A \subseteq G$ **admits infinitely many disjoint right translates** if there is a sequence $(g_n)_n$ of elements of G such that $Ag_m \cap Ag_n = \emptyset$ for all $m \neq n$.

Lemma 2.5. *Let G be a non-compact Polish group. There exists a neighborhood of the identity $V \subseteq G$ such that for any finite $F \subseteq G$ the set VF admits infinitely many disjoint right translates.*

Proof. We begin with the following characterization of compactness established independently by Solecki [27, Lemma 1.2] and Uspenskij [29]: a Polish group G is non-compact if and only if there exists a neighborhood of the identity $U \subseteq G$ such that $F_1 U F_2 \neq G$ for all finite $F_1, F_2 \subseteq G$. Let $V \subseteq G$ be a symmetric neighborhood of the identity such that $V^2 \subseteq U$. We claim that such a set V has the desired property. Pick a finite $F \subseteq G$, set $g_0 = e$ and choose g_n inductively as follows. Let $F_1 = F^{-1}$ and $F_{2,n} = F \cdot \{g_k : k < n\}$. The defining property of U assures existence of $g_n \not\in F_1 U F_{2,n}$. Translates $(VFg_n)_n$ are then pairwise disjoint, for if $VFg_m \cap VFg_n \neq \emptyset$ for $m < n$, then $g_n \in F^{-1}V^{-1}VFg_m \subseteq F_1 U F_{2,n}$, contradicting the construction. $\square$

Theorem 2.6. *Let E be an orbit equivalence relation given by a free Borel $\mathbb{R}$-flow on X. Any non-discrete non-compact Polish group G admits a free Borel action $G \curvearrowright X$ such that $E_G = E$.*

Proof. Let $(X_n, E_n, \phi_n)_n$ be a convergent sequence of partial $\mathbb{R}$-actions as in Proposition 2.4 and let $V \subseteq G$ be given by Lemma 2.5. Choose a countable dense $(h_n)_n$ in G so that $\bigcup_n V h_n = G$. Since the sequence of partial $\mathbb{R}$-actions is rational, one may pick for each n a Borel E_n-invariant partition $X_n = \bigsqcup_k Y_{n,k}$ such that $\phi_n(C) \equiv \phi_n(C')$ for all E_n-classes $C, C' \subseteq Y_{n,k}$. We construct a convergent sequence of partial G-actions $(X_n, E_n, \psi_n)_n$ such that for each n and k there exists a finite set $F \subseteq G$ such that $\{h_i : i < n\} \subseteq F$ and $\psi_n(C) = VF$ for all E_n-classes $C \subseteq Y_{n,k}$.

For any E_0-class C, both $\phi_0(C) \subseteq \mathbb{R}$ and $V \subseteq G$ are Borel sets of the same cardinality. We may therefore pick a Borel bijection $f_k : \phi_0(C) \to V$ where $C \subseteq Y_{0,k}$. For the base of the inductive construction we set $\psi_0|_{Y_k} = f_k \circ \phi_0$. Suppose that $\psi_m : X_m \to G$, $m \le n$, have been constructed.

We now construct ψ_{n+1}. Let C be an E_{n+1}-class and let $D_1, \ldots, D_l$ be a complete list of E_n-classes contained in C. By the inductive assumption, there are finite sets $F_1, \ldots, F_l \subseteq G$ such that $\psi_n(D_i) = VF_i$. Let $\tilde{F} = \bigcup_{i \le l} F_i$. By the choice of V, there are elements $g_1, \ldots, g_l \in G$ such that $V\tilde{F}g_i$, are pairwise disjoint for $1 \le i \le l$. Pick a finite $F \subseteq G$ large enough that $\tilde{F}g_i \subseteq F$, $\{h_i : i < n+1\} \subseteq F$, and $VF \setminus \bigcup_{i \le l} V\tilde{F}g_i$ has cardinality of continuum (the latter can be achieved, for instance, by assuring that one more disjoint translate of $V\tilde{F}$ is inside VF). Note that $\phi_{n+1}(C \setminus X_n) = \phi_{n+1}(C) \setminus \bigcup_{i \le l} \phi_{n+1}(D_i)$ has cardinality of continuum by the properties guaranteed by Proposition 2.4. Pick any Borel bijection

$$f : \phi_{n+1}(C) \setminus \bigcup_{i \le l} \phi_{n+1}(D_i) \to VF \setminus \bigcup_i \psi_n(D_i) g_i$$

and define ψ_{n+1} by the conditions $\psi_{n+1}|_{D_i} = \psi_n|_{D_i} \cdot g_i$ and $\psi_{n+1}|_{C \setminus \bigcup_{i \le l} D_i} = f \circ \phi_{n+1}$. Just as in the base case, the same modification f works for all classes E_{n+1}-classes C, C' such that $\phi_{n+1}(C) \equiv \phi_{n+1}(C')$, which ensures Borelness of the construction.

It is now easy to check that $(X_n, E_n, \psi_n)_n$ is a convergent sequence of partial G-actions, hence its limit is a free Borel action $G \curvearrowright X$ such that $E_G = E$. □

Remark 2.7. Theorem 2.6 highlights difference with actions of discrete groups, since a free Borel $\mathbb{Z}$-action that preserves a finite measure cannot be generated by a free Borel action of a non-amenable group (see, for instance, [32, Proposition 4.3.3] or [10, Proposition 2.5(ii)]).

However, if we consider hyperfinite equivalence relations without any finite invariant measures, then we do have the analog for $\mathbb{Z}$-actions. There exists a unique up to isomorphism non-smooth hyperfinite Borel equivalence relation without any finite invariant measures and it can be realized as an orbit equivalence relation of a free action of any infinite countable group [6, Proposition 11.2].

3. LIPSCHITZ MAPS

Our goal in this section is to prove Theorem 3.12, which shows that any free Borel $\mathbb{R}^d$-flow is bi-Lipschitz orbit equivalent to a flow with an integer grid. Sections 3.1–3.3 build the necessary tools to construct such an orbit equivalence.

Recall that a map $f : X \to Y$ between metric spaces (X, d_Y) and (Y, d_Y) is K**-Lipschitz** if $d_Y(f(x_1), f(x_2)) \le K d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, and it is (K_1, K_2)**-bi-Lipschitz** if f is injective, K_2-Lipschitz, and f^{-1} is K_1^{-1}-Lipschitz, which can equivalently be stated as

$$K_1 d_X(x_1, x_2) \le d_Y(f(x_1), f(x_2)) \le K_2 d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

The **Lipschitz constant** of a Lipschitz map f is the smallest K with respect to which f is K-Lipschitz.

3.1. **Linked sets.** Given two Lipschitz maps $f : A \to A'$ and $g : B \to B'$ that agree on the intersection $A \cap B$, the map $f \cup g : A \cup B \to A' \cup B'$, in general, may not be Lipschitz. The following condition is sufficient to ensure that $f \cup g$ is Lipschitz with the Lipschitz constant bounded by the maximum of the constants of f and g.

Definition 3.1. Let (X, d) be a metric space and $A, B \subseteq X$ be its subsets. We say that A and B are **linked** if for all $x \in A$ and $y \in B$ there exists $z \in A \cap B$ such that $d(x, y) = d(x, z) + d(z, y)$.

Lemma 3.2. *Let (X, d) be a metric space, $f : A \to A'$, $g : B \to B'$ be K-Lipschitz maps between subsets of X and suppose that $f|_{A\cap B} = g|_{A\cap B}$. If A and B are linked, then $f \cup g : A \cup B \to A' \cup B'$ is K-Lipschitz.*

Proof. Set $h = f \cup g : A \cup B \to A' \cup B'$. It suffices to check the K-Lipschitz condition for h at $x \in A$ and $y \in B$. Since A and B are linked, we may pick $z \in A \cap B$ such that $d(x, z) + d(z, y) = d(x, y)$. Then

$$\begin{aligned} d(h(x), h(y)) &\le d(h(x), h(z)) + d(h(z), h(y)) = d(f(x), f(z)) + d(g(z), g(y)) \\ &\le Kd(x, z) + Kd(z, y) = Kd(x, y), \end{aligned}$$

and so h is K-Lipschitz. □

Recall that a metric space (X, d) is **geodesic** if for all points $x, y \in X$ there exists a geodesic between them—an isometry $\tau : \big[0, d(x, y)\big] \to X$ such that $\tau(0) = x$ and $\tau\big(d(x, y)\big) = y$. For geodesic metric spaces, closed sets $A, B \subseteq X$ are always linked whenever the boundary of one of them is contained in the other. The boundary of a set A will be denoted by ∂A, and $\operatorname{int} A$ will stand for the interior of A.

Lemma 3.3. *Suppose (X, d) is a geodesic metric space. If $A, B \subseteq X$ are closed and satisfy $\partial A \subseteq B$, then A and B are linked.*

Proof. Pick $x \in A$, $y \in B$. If either $x \in A \cap B$ or $y \in A \cap B$, then the linking condition is fulfilled by $z = x$ or $z = y$, so we assume that $x \in A \setminus B$ and $y \in B \setminus A$. Let $\tau : [0, d(x, y)] \to X$ be a geodesic from x to y. Since $y \not\in A$, there must be some t_0 such that $\tau(t_0) \in \partial A \subseteq B$. Then $z = \tau(t_0) \in A \cap B$ satisfies $d(x, z) + d(z, y) = d(x, y)$ since τ is geodesic, showing that A and B are linked. □

3.2. **Inductive step.** The following lemma encompasses the inductive step in the construction of the forthcoming Theorem 3.12.

Lemma 3.4. *Let (X, d) be a geodesic metric space and $A \subseteq X$ be a closed set. Suppose $(A_i)_{i=1}^n$ are pairwise disjoint closed subsets of A and $h_i : A_i \to A_i$ are (K_1, K_2)-bi-Lipschitz maps such that $h_i|_{\partial A_i}$ is the identity map for each $1 \le i \le n$. The map $g : A \to A$ given by*

$$g(x) = \begin{cases} h_i(x) & \text{if } x \in A_i, \\ x & \text{otherwise} \end{cases}$$

is (K_1, K_2)-bi-Lipschitz.

Proof. Set $A_0 = A \setminus \bigcup_{i=1}^n \operatorname{int} A_i$ and $h_0 : A_0 \to A_0$ be the identity map. Note that $A_0 \cap A_i = \partial A_i$ and $h_0|_{A_0\cap A_i} = h_i|_{A_0\cap A_i}$ are both identity maps for all $1 \le i \le n$. Let $g_1 : A_0 \cup A_1 \to A_0 \cup A_1$ be given by

$$g_1(x) = \begin{cases} h_1(x) & \text{if } x \in A_1, \\ h_0(x) & \text{if } x \in A_0. \end{cases}$$

Sets A_0 and A_1 are linked by Lemma 3.3, and therefore Lemma 3.2 applies to both h_0, h_1 and h_0^{-1}, h_1^{-1}, thus showing that g_1 is (K_1, K_2)-bi-Lipschitz.

It remains to apply the same argument inductively, constructing $g_k : \bigcup_{i\le k} A_i \to \bigcup_{i\le k} A_i$ for $k \le n$ as

$$g_k(x) = \begin{cases} g_{k-1}(x) & \text{if } x \in A_i \text{ for some } i \le k-1, \\ h_k(x) & \text{if } x \in A_k, \end{cases}$$

and using Lemma 3.3 and Lemma 3.2 to verify that each g_k is (K_1, K_2)-bi-Lipschitz. The map g is equal to g_n, and the lemma follows. □

3.3. **Lipschitz shifts.** Let $(X, \|\cdot\|)$ be a normed space and let $A \subseteq X$ be a closed bounded subset. We begin with the following elementary and well-known observation regarding Lipschitz perturbations of the identity map.

Lemma 3.5. *If $\xi : A \to X$ is a K-Lipschitz map, $K < 1$, then $A \ni x \mapsto x + \xi(x) \in X$ is $(1-K, 1+K)$-bi-Lipschitz.*

Proof. The statement is justified by the following chain of inequalities

$$\begin{aligned}(1-K)||x-y|| &= ||x-y|| - K||x-y|| \le ||x-y|| - ||\xi(x) - \xi(y)|| \\ &\le ||(x-y) + (\xi(x) - \xi(y))|| = ||(x+\xi(x)) - (y+\xi(y))|| \\ &\le ||x-y|| + ||\xi(x) - \xi(y)|| \le ||x-y|| + K||x-y|| = (1+K)||x-y||\end{aligned}$$

and so $x \mapsto x + \xi(x)$ is $(1-K, 1+K)$-bi-Lipschitz. □

For the rest of Section 3.3, we fix a vector $v \in X$ and a real $K > ||v||$. Let the function $f_{A,K,v} : A \to X$ be given by

$$f_{A,K,v}(x) = x + \frac{d(x, \partial A)}{K} v,$$

where $d(x, \partial A)$ denotes the distance from x to the boundary of A. This function (as well as its variant to be introduced shortly) is $(1 - K^{-1}||v||, 1 + K^{-1}||v||)$-bi-Lipschitz. To simplify the notation, we set $\alpha^+ = 1 + K^{-1}||v||$ and $\alpha^- = 1 - K^{-1}||v||$.

Lemma 3.6. *The function $f_{A,K,v}$ is an (α^-, α^+)-bi-Lipschitz homeomorphism onto A.*

Proof. Let $f_{A,K,v}$ be denoted by f for brevity. Since $X \ni x \mapsto d(x, \partial A) \in \mathbb{R}$ is 1-Lipschitz, the map $A \ni x \mapsto \frac{d(x,\partial A)}{K} v \in X$ is $K^{-1}||v||$-Lipschitz, and so f is (α^-, α^+)-bi-Lipschitz by Lemma 3.5.

It remains to show that $f(A) = A$. We may assume that $v \neq 0$, for otherwise the statement is obvious. Clearly, $f|_{\partial A}$ is the identity map. For $x \in A$, let $\Lambda_x = \{\lambda \in \mathbb{R} : x + \lambda v \in A\}$, which is a closed and bounded subset of $\mathbb{R}$, and let $S_x = A \cap (x + \mathbb{R}v) = \{x + \lambda v : \lambda \in \Lambda_x\}$. Consider the map $\zeta : \Lambda_x \to \mathbb{R}$ given by $\zeta(\lambda) = \lambda + \dfrac{d(x + \lambda v, \partial A)}{K}$. Note that $x + \lambda v \in \partial A$ whenever $\lambda \in \partial \Lambda_x$, so $\zeta|_{\partial \Lambda_x}$ is the identity map. Moreover, if $\lambda_0 \in \partial \Lambda_x$ and $\lambda < \lambda_0$ then

$$\begin{aligned}\zeta(\lambda) = \lambda + \frac{d(x + \lambda v, \partial A)}{K} &\le \lambda + K^{-1}||(x + \lambda v) - (x + \lambda_0 v)|| \\ &= \lambda + K^{-1}||v||(\lambda_0 - \lambda) \le \lambda + \lambda_0 - \lambda = \lambda_0.\end{aligned} \tag{1}$$

In particular, if $I = [a, b] \subseteq \Lambda_x$ is a closed interval such that $a, b \in \partial \Lambda_x$, then $\zeta(I) = I$. Indeed, $\zeta(\lambda) \ge a$ and $\zeta(\lambda) \le b$ for all $\lambda \in I$ by Eq. (1), thus $\zeta : I \to I$ is a continuous function that fixes the endpoints of the interval, and so must be surjective by the Intermediate Value Theorem.

The interior $\operatorname{int}\Lambda_x$ can be written as a countable disjoint union of open intervals (a_n, b_n), $a_n, b_n \in \partial\Lambda_x$. We have just shown that $\zeta([a_n, b_n]) = [a_n, b_n]$, which, when coupled with $\zeta|_{\partial\Lambda_x}$ being the identity function, yields $\zeta(\Lambda_x) = \Lambda_x$. The latter translates into $f(S_x) = S_x$, for if $y = x + \lambda v \in S_x$, then $f(y) = x + \zeta(\lambda)v$. Finally, $A = \bigcup_{x\in A} S_x$, and therefore $f(A) = f(\bigcup_{x\in A} S_x) = \bigcup_{x\in A} f(S_x) = \bigcup_{x\in A} S_x = A$. □

Fix a real $L > 0$ and let $A^L = \{x \in A : d(x, \partial A) \ge L\}$ be the set of those elements that are at least L units of distance away from the boundary of A.

Lemma 3.7. *$f_{A,K,v}|_{A^L} = f_{A^L,K,v} + LK^{-1}v$ and $f_{A,K,v}(A^L) = A^L + LK^{-1}v$.*

Proof. Let $f_{A,K,v}$ and $f_{A^L,K,v}$ be denoted simply by f_A and f_{A^L} respectively. Since any normed space X is geodesic, for any $x \in A^L$ we have $d(x, \partial A) = d(x, \partial A^L) + L$, and therefore

$$\begin{aligned} f_A(x) &= x + \frac{d(x,\partial A)}{K} v = x + \frac{d(x,\partial A^L) + L}{K} v \\ &= x + K^{-1} d(x, \partial A^L) v + LK^{-1} v = f_{A^L}(x) + LK^{-1} v. \end{aligned}$$

Since $f_{A^L}(A^L) = A^L$ by Lemma 3.6, we get $f_A(A^L) = f_{A^L}(A^L) + LK^{-1}v = A^L + LK^{-1}v$. □

A truncated shift function $h_{A,K,v,L} : A \to X$ is defined by

$$h_{A,K,v,L}(x) = \begin{cases} f_{A,K,v}(x) & \text{for } x \in A \setminus A^L, \\ x + LK^{-1}v & \text{for } x \in A^L. \end{cases}$$

Lemma 3.8. *The function $h_{A,K,v,L}$ is an (α^-, α^+)-bi-Lipschitz homeomorphism onto A.*

Proof. First, $h_{A,K,v,L}(A) = A$ follows from Lemma 3.6 and Lemma 3.7. Let B be the closure of $A \setminus A^L$, and $g : A^L \to X$ be the map $x \mapsto x + LK^{-1}v$. Note that $\partial A^L = \{x \in A : d(x, \partial A) = L\}$, $f_{A,K,v}|_{\partial A^L} = g|_{\partial A^L}$, both of these maps are α^+-Lipschitz, and B and A^L are linked by Lemma 3.3. Therefore, the map $h_{A,K,v,L} = f_{A,K,v}|_B \cup g|_{A^L}$ is α^+-Lipschitz in view of Lemma 3.2. The lower Lipschitz constant in the bi-Lipschitz condition follows by applying the same argument to functions $f^{-1}_{A,K,v}$ and g^{-1} instead. □

3.4. **Lipschitz equivalence to grid flows.** The maps $h_{A,K,v,L}$ can be used to show that any free Borel $\mathbb{R}^d$-flow is bi-Lipschitz equivalent to a flow admitting an integer grid. This is the content of Theorem 3.12, but first we formulate the properties of partial actions needed for the construction. This is an adaption of the so-called unlayered toast construction announced in [8]. The proof given in [18, Appendix A] for $\mathbb{Z}^d$-actions, transfers to $\mathbb{R}^d$-flows.

For the rest of the paper, we fix a norm $\|\cdot\|$ on $\mathbb{R}^d$ and let $d(x, y) = \|x - y\|$ be the corresponding metric on $\mathbb{R}^d$. Recall that $B_R(r) \subseteq \mathbb{R}^d$ denotes a closed ball of radius R centered at $r \in \mathbb{R}^d$.

Lemma 3.9. *Let $K > 0$ be a positive real. Any free $\mathbb{R}^d$-flow on a standard Borel space X is a limit of a rational convergent sequence of partial actions $(X_n, E_n, \phi_n)_n$ (see Section 2.5) such that for each E_n-class C*

1. *$\phi_n(C)$ is a closed and bounded subset of $\mathbb{R}^d$ and $B_K(0) \subseteq \phi_n(C)$;*
2. *the set of E_m-classes, $m \le n$, contained in C is finite;*
3. *$d(\phi_n(D), \partial\phi_n(C)) \ge K$ for any E_m-class D such that $D \subseteq C$.*

Before outlining the proof, we need to introduce some notation. Let $E_1,\dots,E_n$ be equivalence relations on $X_1,\dots,X_n$ respectively. By $E_1\vee\cdots\vee E_n$ we mean the equivalence relation E on $\bigcup_{i\le n} X_i$ generated by E_i, i.e., xEy whenever there exist $x_1,\dots,x_m$ and for each $1\le i\le m$ there exists $1\le j(i)\le n$ such that $x_1=x$, $x_m=y$ and $x_i E_{j(i)} x_{i+1}$ for all $1\le i<m$.

If E is an equivalence relation on $Y\subseteq X$ and $K>0$, we define the relation E^{+K} on $Y^{+K}=\{x\in X:\mathrm{dist}(x,y)\le K \text{ for some } y\in Y\}$ by $x_1E^{+K}x_2$ if and only if there are $y_1,y_2\in Y$ such that $\mathrm{dist}(x_1,y_1)\le K$, $\mathrm{dist}(x_2,y_2)\le K$ and y_1Ey_2. Note that in general, E^{+K} may not be an equivalence relation if two E-classes get connected after the "fattening". However, E^{+K} is an equivalence relation if $\mathrm{dist}(C_1,C_2)>2K$ holds for all distinct E-classes C_1,C_2.

Proof of Lemma 3.9. One starts with a sufficiently fast-growing sequence of radii a_n (say, $a_n=K1000^{n+1}$ is fast enough) and chooses using [3] (see also [18, Lemma A.2]) a sequence of Borel $B_{a_n}(0)$-lacunary cross-sections $\mathscr{C}_n\subseteq X$ such that

$$\forall x\in X\ \forall\epsilon>0\ \exists^\infty n \text{ such that } \mathrm{dist}(x,\mathscr{C}_n)<\epsilon a_n, \tag{2}$$

where $\mathrm{dist}(x,\mathscr{C}_n)=\inf\{\mathrm{dist}(x,c):c\in\mathscr{C}_n\}$ and $\exists^\infty$ stands for "there exist infinitely many". We may assume without loss of generality that cross-sections $\mathscr{C}_n$ are rational in the sense that if $c_1+r=c_2$ for some $c_1,c_2\in\bigcup_n\mathscr{C}_n$ then $r\in\mathbb{Q}^d$. This can be achieved by moving elements of $\mathscr{C}_n$ by an arbitrarily small amount (see [24, Lemma 2.4]) which maintains the property given in Eq. (2). Rationality of cross-sections guarantees that the sequence of partial actions constructed below is rational.

One now defines X_n and E_n inductively with the base $X_0=\mathscr{C}_0+B_{a_0/10}(0)$, and xE_0y if and only if there is $c\in\mathscr{C}_0$ such that $x,y\in c+B_{a_0/10}(0)$. For the inductive step, begin with $\tilde X_n=\mathscr{C}_n+B_{a_n/10}(0)$ and $\tilde E_n$ being given analogously to the base case: $x\tilde E_n y$ if and only if there is some $c\in\mathscr{C}_n$ such that $\mathrm{dist}(x,c)\le a_n/10$ and $\mathrm{dist}(y,c)\le a_n/10$. Set $E'_n=\tilde E_n\vee E^{+K}_{n-1}\vee\cdots\vee E^{+K}_0$ and let $X'_n=\tilde X_n\cup\bigcup_{i=0}^{n-1}X_i^{+K}$ be the domain of E'_n. Finally, let X_n be the E'_n-saturation of $\tilde X_n$, i.e., $x\in X_n$ if and only if there exists $y\in\tilde X_n$ such that xE'_ny. Put $E_n=E'_n|_{X_n}$.

An alternative description of an E_n-class is as follows. One starts with an $\tilde E_n$-class C_n and joins it first with all E^{+K}_{n-1}-classes D that intersect C_n. Let the resulting $\tilde E_n\vee E^{+K}_{n-1}$-class be denoted by C_{n-1}. Next we add all E^{+K}_{n-2}-classes that intersect C_{n-2} producing an $\tilde E_n\vee E^{+K}_{n-1}\vee E^{+K}_{n-2}$-class C_{n-2}. The process terminates with an E_n-class C_0.

It is easy to check inductively that the diameter of any E_n-class C satisfies $\mathrm{diam}(C)\le a_n/3$ and therefore $\mathrm{dist}(C_1,C_2)\ge a_n/3\gg 2K$ for all distinct E_n-classes C_1,C_2 by the lacunarity of $\mathscr{C}_n$. The latter shows that E^{+K}_n is an equivalence relation on X^{+K}_n.

Monotonicity of the sequence $(X_n,E_n)_n$ is evident from the construction. Eq. (2) is crucial for establishing the fact that $\bigcup_n X_n=X$. Indeed, for each $x\in X$ there exists some n such that $\mathrm{dist}(x,\mathscr{C}_n)<a_n/10$ and thus also $x\in\tilde X_n\subseteq X_n$.

The maps $\phi_n:X_n\to\mathbb{R}^d$, needed to specify partial $\mathbb{R}^d$-actions, are defined by the condition $\phi_n(x)c=x$ for the unique $c\in\mathscr{C}_n$ such that cE_nx. Note that $d(\phi_n(D),\partial\phi_n(C))\ge K$ for any E_m-class D, $m<n$, that is contained in an E_n-class C is a consequence of the fact that $D^{+K}\subseteq C$ by the construction. The convergent sequence of partial actions $(X_n,E_n,\phi_n)_n$ therefore satisfies the desired properties. □

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be free $\mathbb{R}^d$-flows on X that generate the same orbit equivalence relation, $E_{\mathfrak{F}_1}=E_{\mathfrak{F}_2}$, and let $\rho=\rho_{\mathfrak{F}_1,\mathfrak{F}_2}:\mathbb{R}^d\times X\to\mathbb{R}^d$ be the associated cocycle map, defined for $x\in X$ and $r\in\mathbb{R}^d$ by the condition $x+_2r=x+_1\rho(r,x)$. We say that the cocycle ρ is

(K_1, K_2)-bi-Lipschitz if such is the map $\rho(\cdot, x) : \mathbb{R}^d \to \mathbb{R}^d$ for all $x \in X$:

$$K_1 ||r_2 - r_1|| \le ||\rho(r_2, x) - \rho(r_1, x)|| \le K_2 ||r_2 - r_1||. \tag{3}$$

Since $\rho(r_2, x) - \rho(r_1, x) = \rho(r_2 - r_1, x +_1 r_1)$, Lipschitz condition (3) for a cocycle can be equivalently and more concisely stated as

$$K_1 \le \frac{||\rho(r, x)||}{||r||} \le K_2 \quad \text{for all } x \in X \text{ and } r \in \mathbb{R}^d \setminus \{0\}. \tag{4}$$

Remark 3.10. Note that cocycles $\rho_{\mathfrak{F}_1, \mathfrak{F}_2}$ and $\rho_{\mathfrak{F}_2, \mathfrak{F}_1}$ are connected via the identities

$$\rho_{\mathfrak{F}_1, \mathfrak{F}_2}(\rho_{\mathfrak{F}_2, \mathfrak{F}_1}(r, x), x) = r \qquad \text{and} \qquad \rho_{\mathfrak{F}_2, \mathfrak{F}_1}(\rho_{\mathfrak{F}_1, \mathfrak{F}_2}(r, x), x) = r.$$

In particular, if $\rho_{\mathfrak{F}_1, \mathfrak{F}_2}$ is (K_1, K_2)-bi-Lipschitz, then $\rho_{\mathfrak{F}_2, \mathfrak{F}_1}$ is (K_2^{-1}, K_1^{-1})-bi-Lipschitz.

Definition 3.11. Let $\mathfrak{F}$ be a free $\mathbb{R}^d$-flow on X. An **integer grid** for the flow $\mathfrak{F}$ is a $\mathbb{Z}^d$-invariant Borel subset $Z \subseteq X$ whose intersection with each orbit of the flow is a $\mathbb{Z}^d$-orbit. In other words, $Z + \mathbb{R}^d = X$, $Z + \mathbb{Z}^d = Z$, and $z_1 + \mathbb{Z}^d = z_2 + \mathbb{Z}^d$ for all $z_1, z_2 \in Z$ such that $z_1 E_{\mathfrak{F}} z_2$.

Not every flow admits an integer grid, but, as the following theorem shows, each flow is bi-Lipschitz equivalent to the one that does.

Theorem 3.12. *Let $\mathfrak{F}_1$ be a free Borel $\mathbb{R}^d$-flow on X. For any $\alpha > 1$ there exists a free Borel $\mathbb{R}^d$-flow $\mathfrak{F}_2$ on X that admits an integer grid, induces the sames orbit equivalence as does $\mathfrak{F}_1$, i.e., $E_{\mathfrak{F}_1} = E_{\mathfrak{F}_2}$, and whose associated cocycle $\rho_{\mathfrak{F}_1, \mathfrak{F}_2}$ is (α^{-1}, α)-bi-Lipschitz.*

Proof. Let R be so big that the ball $B_R(0) \subseteq \mathbb{R}^d$ satisfies $\mathbb{Z}^d + B_R(0) = \mathbb{R}^d$. Choose $K > 0$ large enough to ensure that $\alpha^- = 1 - K^{-1}R > \alpha^{-1}$, and therefore also $\alpha^+ = 1 + K^{-1}R < \alpha$. Let $(X_n, E_n, \phi_n)_n$ be a rational convergent sequence of partial actions produced by Lemma 3.9 for the chosen value of K. For an E_n-class C, let C' denote the collection of all $x \in C$ that are at least K-distance away from the boundary of C:

$$C' = \{x \in C : d(\phi_n(x), \partial\phi_n(C)) \ge K\}.$$

If D is an E_m-class such that $D \subseteq C$, then item (3) of Lemma 3.9 guarantees the inclusion $D \subseteq C'$. Let $X'_n = \bigcup C'$, where the union is taken over all E_n-classes C, and set $E'_n = E_n|_{X'_n}$, $\phi'_n = \phi_n|_{X'_n}$. Note that $(X'_n, E'_n, \phi'_n)_n$ is a rational convergent sequence of partial actions whose limit is the flow $\mathfrak{F}_1$. The flow $\mathfrak{F}_2$ will be constructed as the limit of partial actions (X'_n, E'_n, ψ_n), where maps ψ_n will be defined inductively and will satisfy $\psi_n(C') = \phi_n(C')$ for all E_n-classes C. The sets $Z_n = \psi_n^{-1}(\mathbb{Z}^d)$ will satisfy $Z_m \cap X'_n \subseteq Z_n$ for $m \le n$, and $Z = \bigcup_n Z_n$ will be an integer grid for $\mathfrak{F}_2$.

For the base of the construction, set $\psi_0 = \phi'_0$ and $Z_0 = \psi_0^{-1}(\mathbb{Z}^d)$. Next, consider a typical E_1-class C with $D_1, \dots, D_l$ being a complete list of E_0-classes contained in it (see Figure 1). Consider the set $\tilde{Z}_{C'} = \phi_1^{-1}(\mathbb{Z}^d) \cap C'$, which is the integer grid inside C' (marked by dots in Figure 1). Each of the D_i-classes comes with the grid $\tilde{Z}_{D'_i} = \psi_0^{-1}(\mathbb{Z}^d) \cap D'_i$ constructed at the previous stage (depicted by crosses in Figure 1). The coherence condition for partial actions guarantees existence of some $s_i \in \mathbb{R}^d$, $i \le l$, such that

$$\phi_1(D'_i) = \phi_0(D'_i) + s_i = \psi_0(D'_i) + s_i.$$

In general, the grid $\tilde{Z}_{C'}$ does not contain $\tilde{Z}_{D'_i}$, but for each $i \le l$, we can find a vector $v_i \in \mathbb{R}^d$ of norm $||v_i|| \le R$ such that $\tilde{Z}_{D'_i} +_1 v_i \subseteq \tilde{Z}_{C'}$. More precisely, we take for v_i any vector in $B_R(0)$ such that $s_i + v_i \in \mathbb{Z}^d$, which exists by the choice of R. Let

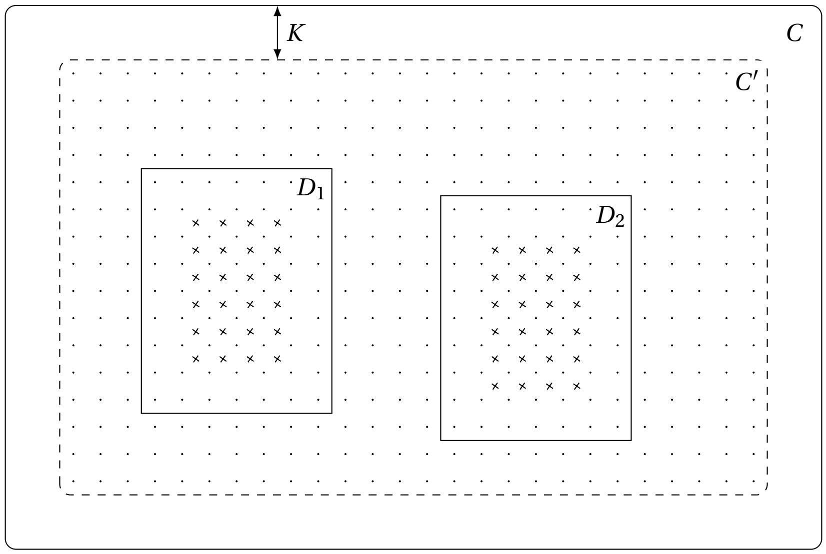

FIGURE 1. Construction of the integer grid

$h_i : \phi_1(D_i) \to \phi_1(D_i)$ be the function $h_{\phi_1(D_i),K,v_i,K}$, which is (α^-, α^+)-bi-Lipschitz by Lemma 3.8. Finally, define $g_1 : \phi_1(C') \to \phi_1(C')$ to be

$$g_1(r) = \begin{cases} h_i(r) & \text{if } r \in \phi_1(D_i), \\ r & \text{otherwise.} \end{cases}$$

Lemma 3.4 has been tailored specifically to show that g_1 is (α^-, α^+)-bi-Lipschitz. We set $\psi_1|_{C'} = g_1 \circ \phi_1|_{C'}$. Note that

$$\begin{aligned} \psi_1(D'_i) &= g_1 \circ \phi_1(D'_i) = h_i \circ \phi_1(D'_i) = \phi_1(D'_i) + KK^{-1}v_i \\ &= \phi_0(D'_i) + s_i + v_i = \psi_0(D'_i) + s_i + v_i, \end{aligned} \tag{5}$$

which validates coherence and, in view of $s_i + v_i \in \mathbb{Z}^d$, gives $\psi_1^{-1}(\mathbb{Z}^d) \cap D'_i = \psi_0^{-1}(\mathbb{Z}^d) \cap D'_i$ for all $i \le l$.

While we have provided the definition of ψ_1 on a single E_1-class C, the same construction can be done in a Borel way across all E_1-classes C using rationality of the sequence of partial actions just like we did in Theorem 2.6. If we let $Z_1 = \psi_1^{-1}(\mathbb{Z}^d)$, then $Z_0 \cap X_1 \subseteq Z_1$ by Eq. (5).

The general inductive step is analogous. Suppose that we have constructed maps ψ_k for $k \le n$. An E_{n+1}-class C contains finitely many subclasses $D_1, \ldots, D_l$, where D_i is an E_{m_i}-class, $m_i < n$, and no D_i is contained in a bigger E_m-class for some $m_i < m < n$. By coherence and inductive assumption, there exist $s_i \in \mathbb{R}^d$, $i \le l$, such that

$$\phi_{n+1}(D'_i) = \phi_{m_i}(D'_i) + s_i = \psi_{m_i}(D'_i) + s_i.$$

Choose vectors $v_i \in B_R(0)$ to satisfy $s_i + v_i \in \mathbb{Z}^d$, set $h_i : \phi_{n+1}(D_i) \to \phi_{n+1}(D_i)$ to be $h_{\phi_{n+1}(D_i),K,v_i,K}$, and define an (α^-, α^+)-bi-Lipschitz function g_{n+1} by

$$g_{n+1}(r) = \begin{cases} h_i(r) & \text{if } r \in \phi_{n+1}(D_i), \\ r & \text{otherwise.} \end{cases}$$

Finally, set $\psi_{n+1}|_{C'} = g_{n+1} \circ \phi_{n+1}|_{C'}$ and extend this definition to a Borel map $\psi_{n+1} : X'_{n+1} \to \mathbb{R}^d$ using the rationality of the sequence of partial actions. Coherence of the maps $(\psi_k)_{k \le n+1}$ and the inclusion $Z_m \cap X'_{n+1} \subseteq Z_{n+1}$ for $m \le n+1$ follow from the analog of Eq. (5).

It remains to check the bi-Lipschitz condition for the resulting cocycle $\rho_{\mathfrak{F}_1,\mathfrak{F}_2}$. It is easier to work with the cocycle $\rho_{\mathfrak{F}_2,\mathfrak{F}_1}$, which for $x, x+r \in X'_n$ satisfies

$$\rho_{\mathfrak{F}_2,\mathfrak{F}_1}(r,x) = g_n(\phi_n(x)+r) - g_n(\phi_n(x)),$$

and is therefore (α^-,α^+)-bi-Lipschitz, because so is g_n. Hence, $\rho_{\mathfrak{F}_2,\mathfrak{F}_1}$ is also (α^{-1},α)-bi-Lipschitz, because $\alpha^{-1} < \alpha^- < \alpha^+ < \alpha$ by the choice of K. Finally, we apply Remark 3.10 to conclude that $\rho_{\mathfrak{F}_1,\mathfrak{F}_2}$ is also (α^{-1},α)-bi-Lipschitz. □

Restricting the action of $\mathfrak{F}_2$ onto the integer grid Z, we get the following corollary.

Corollary 3.13. *Let $\mathfrak{F}$ be a free Borel $\mathbb{R}^d$-flow on X. For any $\alpha > 1$ there exist a cross-section $Z \subseteq X$ and a free $\mathbb{Z}^d$-action T on Z such that the cocycle $\rho = \rho_{\mathfrak{F},T} : \mathbb{Z}^d \times X \to \mathbb{R}^d$ given by $T_n x = x + \rho(n,x)$ is (α^{-1},α)-bi-Lipschitz.*

4. Special representation theorem

The main goal of this section is to formulate and prove a Borel version of Katok's special representation theorem [12] that connects free $\mathbb{R}^d$-flows with free $\mathbb{Z}^d$-actions. We have already done most of the work in proving Theorem 3.12, and it is now a matter of defining special representations in the Borel context and connecting them to our earlier setup.

4.1. Cocycles.

Given a Borel action $G \curvearrowright X$, a (Borel) **cocycle** with values in a group H is a (Borel) map $\rho : G \times X \to H$ that satisfies the **cocycle identity**:

$$\rho(g_2 g_1, x) = \rho(g_2, g_1 x)\rho(g_1, x) \quad \text{for all } g_1, g_2 \in G \text{ and } x \in X.$$

We are primarily concerned with the Abelian groups $\mathbb{Z}^d$ and $\mathbb{R}^d$ in this section, so the cocycle identity will be written additively. A cocycle $\rho : G \times X \to H$ is said to be **injective** if $\rho(g,x) \ne e_H$ for all $g \ne e_G$ and all $x \in X$, where e_G and e_H are the identity elements of the corresponding groups. Suppose that furthermore the groups G and H are locally compact. We say that ρ **escapes to infinity** if for all $x \in X$, $\lim_{g\to\infty} \rho(g,x) = +\infty$ in the sense that for any compact $K_H \subseteq H$ there exists a compact $K_G \subseteq G$ such that $\rho(g,x) \not\in K_H$ whenever $g \not\in K_G$.

Example 4.1. Suppose $a_H : H \curvearrowright X$ and $a_G : G \curvearrowright Y$, $Y \subseteq X$, are free actions of groups G and H on standard Borel spaces, and suppose that we have containment of orbit equivalence relations $E_G \subseteq E_H$. For each $y \in Y$ and $g \in G$, there exists a unique $\rho_{a_H,a_G}(g,y) \in H$ such that $a_H(\rho_{a_H,a_G}(g,y),y) = a_G(g,y)$. The map $(g,y) \mapsto \rho_{a_H,a_G}(g,y)$ is an injective Borel cocycle. We have already encountered two instances of this idea in Section 3.4.

4.2. Flow under a function.

Borel $\mathbb{R}$-flows and $\mathbb{Z}$-actions are tightly connected through the "flow under a function" construction. Let $T : Z \to Z$ be a free Borel automorphism of a standard Borel space and $f : Z \to \mathbb{R}^{>0}$ be a positive Borel function. There is a natural definition of a flow $\mathfrak{F} : \mathbb{R} \curvearrowright X$ on the space $X = \{(z,t) : z \in Z, 0 \le t < f(z)\}$ under the graph of f. The action $(z,t) + r$ for a positive r is defined by shifting the point (z,t) by

r units upward until the graph of f is reached, then jumping to the point $(Tz, 0)$, and continuing to flow upward until the graph of f at Tz is reached, etc. More formally,

$$(z,t)+r=\big(T^k z, t+r-\sum_{i=0}^{k-1} f(T^i z)\big)$$

for the unique $k \geq 0$ such that $\sum_{i=0}^{k-1} f(T^i z) \leq t+r < \sum_{i=0}^{k} f(T^i z)$; for $r \leq 0$ the action is defined by "flowing backward", i.e.,

$$(z,t)+r=\big(T^{-k} z, t+r+\sum_{i=1}^{k} f(T^{-i} z)\big)$$

for $k \geq 0$ such that $0 \leq t+r+\sum_{i=1}^{k} f(T^{-i}z) < f(T^{-k}z)$. The action is well-defined provided that the fibers within the orbits of T have infinite cumulative lengths:

$$\sum_{i=0}^{\infty} f(T^i z) = +\infty \quad \text{and} \quad \sum_{i=0}^{\infty} f(T^{-i} z) = +\infty \quad \text{for all } z \in Z. \tag{6}$$

The appealing geometric picture of the "flow under a function" does not generalize to higher dimensions, but admits an interpretation as the so-called special flow construction suggested in [12].

4.3. **Special flows.** Let T be a free $\mathbb{Z}^d$-action on a standard Borel space Z and let $\rho : \mathbb{Z}^d \times Z \to \mathbb{R}^d$ be a Borel cocycle. One can construct a $\mathbb{Z}^d$-action $\hat{T}$, the so-called principal $\mathbb{R}^d$-extension, defined on $Z \times \mathbb{R}^d$ via $\hat{T}_n(z,r) = (T_n z, r+\rho(n,z))$. An easy application of the cocycle identity verifies axioms of the action. While the action T will typically have complicated dynamics, the action $\hat{T}$ admits a Borel transversal as long as the cocycle ρ escapes to infinity.

Lemma 4.2. *If the cocycle ρ satisfies* $\lim_{n\to\infty} ||\rho(n,z)|| = +\infty$ *for all $z \in Z$, then the action $\hat{T}$ has a Borel transversal.*

Proof. Let $Y_k = \{(z,r) \in Z \times \mathbb{R}^d : ||r|| \leq k\}$. We claim that each orbit of $\hat{T}$ intersects Y_k in a finite (possibly empty) set. Indeed, cocycle values escaping to infinity yield for any $(z,r) \in Z \times \mathbb{R}^d$ a number N so large that $||r+\rho(n,z)|| > k$ whenever $||n|| \geq N$. In particular, $||n|| \geq N$ implies $\hat{T}_n(z,r) = (T_n z, r+\rho(n,z)) \notin Y_k$. Hence, the intersection of the orbit of (z,r) with Y_k is finite.

Set $Y = \bigsqcup_{k\in\mathbb{N}}\big(Y_k \setminus \bigcup_{n\in\mathbb{Z}^d} \hat{T}_n Y_{k-1}\big)$. Each orbit of $\hat{T}$ intersects Y in a finite and necessarily non-empty set, so $E_{\hat{T}}|_Y$ is a finite Borel equivalence relation. A Borel transversal for $E_{\hat{T}}|_Y$ is also a transversal for the action of $\hat{T}$. □

We assume now that the cocycle ρ satisfies the assumptions of Lemma 4.2, and $X = (Z \times \mathbb{R}^d)/E_{\hat{T}}$ therefore carries the structure of a standard Borel space as a push-forward of the factor map $\pi : Z \times \mathbb{R}^d \to X$, which sends a point to its $E_{\hat{T}}$-equivalence class.

There is a natural $\mathbb{R}^d$-flow $\hat{\mathfrak{F}}$ on $Z \times \mathbb{R}^d$ which acts by shifting the second coordinate: $(z,r) +_{\hat{\mathfrak{F}}} s = (z, r+s)$. This flow commutes with the $\mathbb{Z}^d$-action $\hat{T}$ and therefore projects onto the flow $\mathfrak{F}$ on X given by the condition $\pi((z,r) +_{\hat{\mathfrak{F}}} s) = \pi(z,r) +_{\mathfrak{F}} s$. We say that $\mathfrak{F}$ is the **special flow** over T generated by the cocycle ρ. Freeness of T implies freeness of $\mathfrak{F}$.

The construction outlined above, works just as well in the context of ergodic theory, where the space Z would be endowed with a finite measure ν preserved by the action T. The product of ν with the Lebesgue measure on $\mathbb{R}^d$ induces a measure μ on X, which is preserved by the flow $\mathfrak{F}$. Furthermore, μ is finite provided the cocycle ρ is integrable in the sense of [12, Condition (J), p. 122]. Katok's special representation theorem asserts that, up to a null set, any free ergodic measure-preserving flow can be obtained via this process. Furthermore, the cocycle can be picked to be bi-Lipschitz with Lipschitz constants arbitrarily close to 1.

As will be shown shortly, such a representation result continues to hold in the framework of descriptive set theory, and every free Borel $\mathbb{R}^d$-flow is Borel isomorphic to a special flow over some free Borel $\mathbb{Z}^d$-action. Moreover, just as in Katok's original work, Theorem 4.3 provides some significant control on the cocycle that generates the flow, tightly coupling the dynamics of the $\mathbb{Z}^d$-action with the dynamics of the flow it produces. But first, we re-interpret the construction in different terms.

4.4. **Flows generated by admissible cocycles.** Let the map $Z \ni z \mapsto (z,0) \in Z \times \{0\}$ be denoted by ι. If the cocycle ρ is injective, then $\pi \circ \iota : Z \to \pi(Z \times \{0\}) = Y$ is a bijection and Y intersects every orbit of $\mathfrak{F}$ in a non-empty countable set. The $\mathbb{Z}^d$-action T on Z can be transferred via $\pi \circ \iota$ to give a free $\mathbb{Z}^d$-action $T' = \pi \circ \iota \circ T \circ \iota^{-1} \circ \pi^{-1}$ on Y. Let $\rho' = \rho_{T',\mathfrak{F}} : \mathbb{Z}^d \times Y \to \mathbb{R}^d$ be the cocycle of the action $\pi \circ \iota \circ T \circ \iota^{-1} \circ \pi^{-1}$; in other words

$$T'_n(y) = (\pi \circ \iota \circ T_n \circ \iota^{-1} \circ \pi^{-1})(y) = y +_{\mathfrak{F}} \rho'(n, y) \quad \text{for all } n \in \mathbb{Z}^d \text{ and } y \in Y. \tag{7}$$

If $y = (\pi \circ \iota)(z)$ for $z \in Z$, then Eq. (7) translates into

$$\pi(T_n z, 0) = \pi(z, \rho'(n, y)).$$

Since $\pi(T_n z, 0) = \pi(z, \rho(-n, T_n z)) = \pi(z, -\rho(n, z))$, we conclude that $\rho'(n, y) = -\rho(n, z)$, where $y = (\pi \circ \iota)(z)$. In particular, Y is a discrete cross-section for the flow $\mathfrak{F}$ precisely because ρ escapes to infinity.

Conversely, if $\mathfrak{F}$ is any free $\mathbb{R}^d$-flow on a standard Borel space X, and $Z \subseteq X$ is a discrete cross-section with a $\mathbb{Z}^d$-action T on it, then $\mathfrak{F}$ is isomorphic to the special flow over T generated by the (necessarily injective) cocycle $-\rho_{T,\mathfrak{F}}$.

Let us say that a cocycle ρ is **admissible** if it is both injective and escapes to infinity. The discussion of the above two paragraphs can be summarized by saying that, up to a change of sign in the cocycles, representing a flow as a special flow generated by an admissible cocycle is the same thing as finding a free $\mathbb{Z}^d$-action on a discrete cross-section of the flow.

For instance, given any free $\mathbb{Z}^d$-action T on Z, we may consider the admissible cocycle $\rho(n, z) = -n$ for all $z \in Z$ and $n \in \mathbb{Z}^d$. The set $Y = \pi(Z \times \{0\})$ is then an integer grid for the flow $\mathfrak{F}$ (in the sense of Definition 3.11). Conversely, any flow that admits an integer grid is isomorphic to a special flow generated by such a cocycle.

4.5. **Special representation theorem.** Restriction of the orbit equivalence relation of any $\mathbb{R}^d$-flow onto a cross-section gives a hyperfinite equivalence relation [10, Theorem 1.16], and therefore can be realized as an orbit equivalence relation by a free Borel $\mathbb{Z}^d$-action (as long as the restricted equivalence relation is aperiodic). Since any free flow admits a discrete (in fact, lacunary) aperiodic cross-section, it is isomorphic to a special flow over *some* action generated by *some* cocycle. In general, however, the structure of the $\mathbb{Z}^d$-orbit and the corresponding orbit of the flow have little to do with each other. Theorem 3.12 and Corollary 3.13 allow us to improve on this and find a special representation generated by a bi-Lipschitz cocycle.

For comparison, Katok's theorem [12] can be formulated in the parlance of this section as follows.

Theorem (Katok)**.** *Pick some $\alpha > 1$. Any free ergodic measure-preserving $\mathbb{R}^d$-flow on a standard Lebesgue space is isomorphic to a special flow over a free ergodic measure-preserving $\mathbb{Z}^d$-action generated by an (α^{-1}, α)-bi-Lipschitz cocycle.*

As is the case with all ergodic theoretical results, isomorphism is understood to hold up to a set of measure zero. We may now conclude with a Borel version of Katok's special representation theorem, which holds for all free Borel $\mathbb{R}^d$-flows and establishes isomorphism on all orbits.

Theorem 4.3. *Pick some $\alpha > 1$. Any free Borel $\mathbb{R}^d$-flow is isomorphic to a special flow over a free Borel $\mathbb{Z}^d$-action generated by an (α^{-1}, α)-bi-Lipschitz cocycle.*

Proof. Let $\mathfrak{F}$ be a free Borel $\mathbb{R}^d$-flow on X. Corollary 3.13 gives a cross-section $Z \subseteq X$ and a $\mathbb{Z}^d$-action T on it such that the cocycle $\rho_{\mathfrak{F},T} : \mathbb{Z}^d \times X \to \mathbb{R}^d$ is (α^{-1}, α)-bi-Lipschitz. By the discussion in Section 4.4, this gives a representation of the flow as a special flow over T generated by the cocycle $-\rho_{\mathfrak{F},T}$, which is also (α^{-1}, α)-bi-Lipschitz. □

REFERENCES

[1] Warren Ambrose, *Representation of ergodic flows*, Annals of Mathematics. Second Series **42** (1941), 723–739.

[2] Warren Ambrose and Shizuo Kakutani, *Structure and continuity of measurable flows*, Duke Mathematical Journal **9** (1942), 25–42.

[3] Charles M. Boykin and Steve Jackson, *Borel boundedness and the lattice rounding property*, Contemporary Mathematics (Su Gao, Steve Jackson, and Yi Zhang, eds.), vol. 425, American Mathematical Society, Providence, Rhode Island, 2007, pp. 113–126.

[4] Michael R. Cotton, *Abelian group actions and hypersmooth equivalence relations*, Annals of Pure and Applied Logic **173** (2022), no. 8, 103122.

[5] Andrés del Junco and Daniel J. Rudolph, *Kakutani equivalence of ergodic $\mathbb{Z}^n$ actions*, Ergodic Theory and Dynamical Systems **4** (1984), no. 1, 89–104.

[6] Randall Dougherty, Steve Jackson, and Alexander S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Transactions of the American Mathematical Society **341** (1994), no. 1, 193–225.

[7] Jacob Feldman, *New K-automorphisms and a problem of Kakutani*, Israel Journal of Mathematics **24** (1976), no. 1, 16–38. MR 0409763

[8] Su Gao, Steve Jackson, Edward Krohne, and Brandon Seward, *Forcing constructions and countable Borel equivalence relations*, The Journal of Symbolic Logic **87** (2022), no. 3, 873–893. MR 4472517

[9] Marlies Gerber and Philipp Kunde, *Anti-classification results for the Kakutani equivalence relation*, September 2021, Comment: 72 pages, 3 figures.

[10] Steve Jackson, Alexander S. Kechris, and Alain Louveau, *Countable Borel equivalence relations*, Journal of Mathematical Logic **2** (2002), no. 1, 1–80.

[11] Anatole B. Katok, *Monotone equivalence in ergodic theory*, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **41** (1977), no. 1, 104–157, 231. MR 0442195

[12] ———, *The special representation theorem for multi-dimensional group actions*, Dynamical Systems, Vol. I—Warsaw, 1977, pp. 117–140. Astérisque, No. 49. MR 0492181

[13] Alexander S. Kechris, *Countable sections for locally compact group actions*, Ergodic Theory and Dynamical Systems **12** (1992), no. 2, 283–295.

[14] ———, *Countable sections for locally compact group actions. II*, Proceedings of the American Mathematical Society **120** (1994), no. 1, 241–247.

[15] Bryna Kra, Anthony Quas, and Ayşe Şahin, *Rudolph's two step coding theorem and Alpern's lemma for $\mathbb{R}^d$ actions*, Transactions of the American Mathematical Society **367** (2015), no. 6, 4253–4285. MR 3324927

[16] Ulrich Krengel, *On Rudolph's representation of aperiodic flows*, Ann. Inst. H. Poincaré Sect. B (N.S.) **12** (1976), no. 4, 319–338.

[17] François Le Maître and Konstantin Slutsky, *L^1 full groups of flows*, August 2021.

[18] Andrew S. Marks and Spencer T. Unger, *Borel circle squaring*, Annals of Mathematics **186** (2017), no. 2, 581–605.

[19] Benjamin D. Miller and Christian Rosendal, *Descriptive Kakutani equivalence*, Journal of the European Mathematical Society (JEMS) **12** (2010), no. 1, 179–219.

[20] Donald S. Ornstein, Daniel J. Rudolph, and Benjamin Weiss, *Equivalence of measure preserving transformations*, Memoirs of the American Mathematical Society **37** (1982), no. 262, xii+116. MR 653094
[21] Daniel J. Rudolph, *A two-valued step coding for ergodic flows*, Mathematische Zeitschrift **150** (1976), no. 3, 201–220.
[22] ______, *Rectangular tilings of $\mathbb{R}^n$ and free $\mathbb{R}^n$-actions*, Dynamical Systems (College Park, MD, 1986–87), Lecture Notes in Math., vol. 1342, Springer, Berlin, 1988, pp. 653–688.
[23] Konstantin Slutsky, *Lebesgue orbit equivalence of multidimensional Borel flows: A picturebook of tilings*, Ergodic Theory and Dynamical Systems **37** (2017), no. 6, 1966–1996. MR 3681992
[24] ______, *On time change equivalence of Borel flows*, Fundamenta Mathematicae **247** (2019), no. 1, 1–24. MR 3984276
[25] ______, *Regular cross sections of Borel flows*, Journal of the European Mathematical Society (JEMS) **21** (2019), no. 7, 1985–2050. MR 3959857
[26] ______, *Smooth orbit equivalence of multidimensional Borel flows*, Advances in Mathematics **381** (2021), Paper No. 107626, 22. MR 4215747
[27] Sławomir Solecki, *Actions of non-compact and non-locally compact Polish groups*, The Journal of Symbolic Logic **65** (2000), no. 4, 1881–1894. MR 1812189
[28] Raimond A. Struble, *Metrics in locally compact groups*, Compositio Mathematica **28** (1974), 217–222.
[29] Vladimir V. Uspenskij, *On subgroups of minimal topological groups*, Topology and its Applications **155** (2008), no. 14, 1580–1606. MR 2435151
[30] Vivek M. Wagh, *A descriptive version of Ambrose's representation theorem for flows*, Indian Academy of Sciences. Proceedings. Mathematical Sciences **98** (1988), no. 2-3, 101–108.
[31] Benjamin Weiss, *Measurable dynamics*, Conference in Modern Analysis and Probability (New Haven, Conn., 1982), Contemp. Math., vol. 26, Amer. Math. Soc., Providence, RI, 1984, pp. 395–421.
[32] Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984. MR 776417

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