

Reducing the LQG Cost With Minimal Communication

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Abstract—We study the linear quadratic Gaussian (LQG) control problem, in which the controller's observation of the system state is insufficient to attain a desired quadratic control cost. To achieve the desired LQG cost, we introduce a communication link from the observer (encoder) to the controller (decoder). We investigate the fundamental tradeoff between a desired LQG cost and the required communication (information) resources, measured with the conditional directed information. The optimization domain is all encoding–decoding policies, and our first result is the optimality of memoryless encoders that only transmit Gaussian measurements of the current system state. Additionally, it is shown that even if the controller's measurements are made available to the encoder it does not reduce the minimal directed information. The main result is a semidefinite programming (SDP) formulation for that optimization problem, which applies to general time-varying linear dynamical systems in the finite-horizon scenario, and to time-invariant systems at infinite horizon. For the latter scenario of time-invariant systems, we show that it is unnecessary to consider time-varying policies as a simple and optimal time-invariant policy can be directly constructed from the SDP solution. Our results extend a seminal work by Tanaka et al. to the scenario where the controller has access to a noisy measurement of the system state. We demonstrate the viability of this extra resource by illustrating that even low-quality measurements may have a significant impact on the required communication resources.

Index Terms—Communication channels, convex optimization, LQG, networked control systems, optimal control.

I. INTRODUCTION

NETWORKED control systems possess an inherent tension between the control performance and the resources that are allocated to communicate by different nodes of the system.

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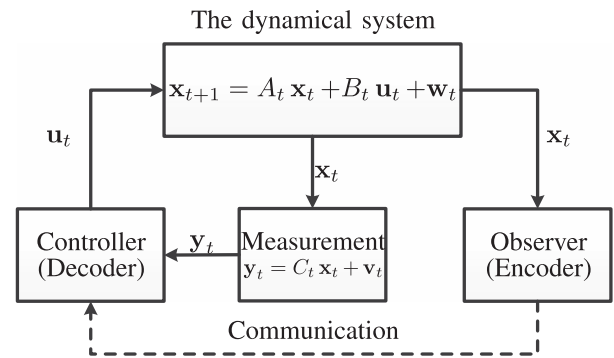


Fig. 1. LQG setting with a noisy observation y_t . The control performance (the quadratic cost) is improved using a communication link (the dashed line) from the observer to the controller.

Despite the great advances on important questions in this theme such as data rate theorems for stabilizability of dynamical systems [1], [2], [3], [4], [5], [6], [7], [8], there are still fundamental questions that remain open, such as the tradeoff between communication resources and the control cost [9], [10], [11], [12], [13], [14], [15]. In this article, we investigate this tradeoff on a simple, yet fundamental, topology consisting of the classical linear quadratic gaussian (LQG) setting with a communication link from an observer who has access to the system states to the controller.

The networked control setting investigated in this article is described in Fig. 1. It consists of an LQG system whose state is x_t with a noisy measurement y_t , and a communication link between a full observer and the controller. The operational objective is to reduce the achievable control cost at the expense of communication resources. The communication link introduced between an encoder and a decoder (colocated with the controller) serves as an information pipeline to the controller that also has an access to the LQG measurements y_t . Based on its (full) observation of the state, the encoder transmits extra information to the controller resulting in a reduction of the control cost. This problem can also be viewed as the rate-constrained LQG setting [16] with side information available to the controller (the measurement y_t) [14], [17], [18], [19]. The objective here is to characterize the minimal communication resources subject to a strict constraint on the control performance measured by a quadratic cost.

The communication (information) resources are measured by the conditional directed information. The directed information is

suitable for scenarios where the operations of the involved units are sequential, e.g., scalar and multiple inputs multiple outputs (MIMO) channels with feedback [20], [21], [22] and the causal rate distortion function in the context of control problems [10], [13]. The directed information is also closely related to the minimal expected length of variable-length codewords, as well as to the minimal blocklength of fixed-length quantizers in networked control systems [11, Sec. II-C]. From operational perspective, the conditional directed information considered here serves as a lower bound to the minimal expected length of variable-length prefix-free codewords [9], [18] (See also Section VI). The control performance in our problem formulation is measured by the quadratic cost of the states and control signals. The optimization problem is formulated for two scenarios corresponding to the finite-horizon and infinite-horizon regimes.

For the finite-horizon problem, we consider time-varying linear dynamical systems, and we formulate the minimal conditional directed information as a convex optimization problem. The optimization problem has a semidefinite programming (SDP) form (more precisely, $\max \log\text{-det}$ form) and can be implemented using standard solvers even for large horizons. We then show that the solution to the optimization problem can be realized by three design steps: 1) computation of LQG controller gains; 2) solution of the convex optimization problem; 3) a Kalman filter implemented at the controller. For the infinite-horizon problem where the dynamical system matrices are time-invariant, we show that the optimization problem can be formulated a finite-dimensional SDP. Most importantly, we show that the optimal encoding policy is a simple, time-invariant Gaussian measurement of the state that can be computed from the convex optimization.

Our results generalize the work by Tanaka et al. [16]. Specifically, we investigate the measurement-feedback LQG setting, while [16] assumed that the LQG measurement is absent ($y_t = 0$ in Fig. 1). Thus, the controller in our setting is based on the fusion of the communication link information and the LQG Gaussian measurement. Two key changes in our SDP formulation are the objective function that includes an additional term due to the study of conditional directed information rather than the directed information in [16], and a new linear matrix inequality (LMI) constraint which represents the error covariance reduction due to the LQG measurement y_t . To find the optimal policy structure, we study a relaxed optimization problem where the LQG measurements are available to the encoder as well. We then show that even in this relaxed scenario, the optimal encoder is a memoryless Gaussian measurement of the system state. Thus, the knowledge of the LQG measurements at the encoder does not reduce the minimal communication resources. This extends an observation made in [17] for the scalar setting.

The problem of control under communication constraints with side information has recently attracted much interest [14], [17], [18], [19], [23]. In [17], a scalar version of the problem in Fig. 1 was solved. In [18], a slightly less general problem than Fig. 1 was considered. They conjectured that a linear, memoryless policy is optimal and provided an SDP formulation for their optimization. Their conjecture and the SDP formulation are subsumed in the conference version of the current manuscript [24],

published prior to [18]. Additionally, [18] showed that the conditional directed information is within a constant gap from the operational problem of variable-length coding with side information available to the controller and the encoder. This is obtained by constructing a practical coding scheme and analyzing its performance. When the side information is available to the controller only (as we assume in our setting), the minimal directed information is a tighter lower bound [23], but it does not appear to have a computable form. In [14], the rate-distortion counterpart of the control problem studied here is considered. It is shown that if the policy is restricted to be linear and the LQG cost admits an upper bound at all times, a simple optimization problem can be realized for the corresponding rate-distortion problem. The result presented below in Theorem 1 confirms the optimality of the policy conjectured in [18] and of the linear policy assumed in [14].

The rest of this article is organized as follows. Section II introduces the notation, setting and the problem definition. Section III presents our main results. Section IV presents numerical examples. Section V provides their proofs. Finally, Section VI concludes this article

II. SETTING AND PROBLEM DEFINITION

A linear dynamical system is described by

$$\mathbf{x}_{t+1} = A_t \mathbf{x}_t + B_t \mathbf{u}_t + \mathbf{w}_t \quad t \geq 1 \quad (1)$$

where $\mathbf{x}_t \in \mathbb{R}^n$ is the system state, $\mathbf{u}_t \in \mathbb{R}^m$ is the control signal, and $\mathbf{w}_t \sim \mathcal{N}(0, W_t)$ is an i.i.d. disturbance with $W_t \succ 0$. The initial state $\mathbf{x}_1$ is distributed according to $P_{1|0} \succ 0$ and is independent of $\mathbf{w}_t$ for all t . A noisy measurement of the state is available to the controller

$$\mathbf{y}_t = C_t \mathbf{x}_t + \mathbf{v}_t \quad (2)$$

with $\mathbf{v}_t \sim \mathcal{N}(0, V_t)$. For a fixed time-horizon T , the LQG quadratic cost is defined as

$$J(\mathbf{x}^{T+1}, \mathbf{u}^T) = \mathbb{E} \left[\sum_{t=1}^T \mathbf{x}_{t+1}^* Q_t \mathbf{x}_{t+1} + \mathbf{u}_t^* R_t \mathbf{u}_t \right] \quad (3)$$

with $Q_t \succeq 0$ and $R_t \succ 0$, and superscripts denote vectors starting at time $t = 1$, e.g., $\mathbf{x}^{T+1} \triangleq (\mathbf{x}_1, \dots, \mathbf{x}_{T+1})$.

The objective is to design a system such that the LQG cost does not exceed a cost target denoted by Γ . Naturally, if the measurements $\mathbf{y}_t$ are sufficient to attain Γ , the classical solution to the LQG problem is satisfactory, and there is no need to expand. In the other extreme, the LQG cost is lower bounded by the LQG cost attained by a fully observer, i.e., $\mathbf{y}_t = \mathbf{x}_t$. Our interest lies in the scenario where Γ is below the optimal LQG cost attainable with the partial observer (2) but above the optimal LQG cost attainable with the full observer. In this case, the introduction of a communication/information link (see the dashed line in Fig. 1) between a full observer (encoder) and a controller (a decoder) will help to attain the desired LQG cost Γ .

The encoder is characterized by the space of sequences of Borel measurable stochastic kernels that can be compactly represented by the causally conditioned kernels

$$P(\mathbf{f}^T || \mathbf{x}^T) \triangleq \prod_{t=1}^T P(\mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{x}^t) \quad (4)$$

with $\mathbf{f}_t \in \mathbb{R}^n$. Similarly, the decoder (controller) is a causally conditioned probability distribution

$$P(\mathbf{u}^T || \mathbf{f}^T, \mathbf{y}^T) \triangleq \prod_{t=1}^T P(\mathbf{u}_t | \mathbf{u}^{t-1}, \mathbf{f}^t, \mathbf{y}^t). \quad (5)$$

By the construction, the encoder–decoder pair satisfies at all times

$$\begin{aligned} P(\mathbf{u}_t, \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{u}^{t-1}, \mathbf{x}^t, \mathbf{y}^t) \\ = P(\mathbf{u}_t | \mathbf{u}^{t-1}, \mathbf{f}^t, \mathbf{y}^t) P(\mathbf{f}_t | \mathbf{x}^t, \mathbf{f}^{t-1}). \end{aligned} \quad (6)$$

The overall joint distribution can be summarized using the one-step update

$$\begin{aligned} P(\mathbf{x}_t, \mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t | \mathbf{x}^{t-1}, \mathbf{y}^{t-1}, \mathbf{f}^{t-1}, \mathbf{u}^{t-1}) \\ = P(\mathbf{y}_t, \mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) P(\mathbf{u}_t, \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{u}^{t-1}, \mathbf{x}^t, \mathbf{y}^t). \end{aligned} \quad (7)$$

The communication resources are measured by the directed information from the encoder to the controller causally conditioned on the partial observations at the controller [25], [26]

$$I(\mathbf{x}^T \rightarrow \mathbf{f}^T || \mathbf{y}^T) = \sum_{t=1}^T I(\mathbf{x}^t; \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{y}^t) \quad (8)$$

where $I(X; Y | Z)$ is the mutual information between X and Y conditioned on Z .

The objective of this article is to solve the optimization problem

$$\begin{aligned} \min I(\mathbf{x}^T \rightarrow \mathbf{f}^T || \mathbf{y}^T) \\ \text{s.t. } J(\mathbf{x}^{T+1}, \mathbf{u}^T) \leq \Gamma \end{aligned} \quad (9)$$

where the minimum is over policies of the form (6).

When the measurement $\mathbf{y}_t$ is absent, the optimization problem in (9) simplifies to the directed information $I(\mathbf{x}^T \rightarrow \mathbf{f}^T)$ that was investigated in [9], [16]. To see that the conditional directed information measures the information encapsulated at the encoding policy, assume that the t th element in the conditional directed information satisfies

$$I(\mathbf{x}^t; \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{y}^t) = I(\mathbf{x}_t; \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{y}^t). \quad (10)$$

Then, the right hand side extracts the state uncertainty at the controller with and without the encoding variable $\mathbf{f}_t$, i.e., $I(\mathbf{x}_t; \mathbf{f}_t | \mathbf{f}^{t-1}, \mathbf{y}^t) = h(\mathbf{x}_t | \mathbf{f}^{t-1}, \mathbf{y}^t) - h(\mathbf{x}_t | \mathbf{f}^t, \mathbf{y}^t)$ where $h(\mathbf{x} | \mathbf{y})$ denotes the conditional differential entropy of $\mathbf{x}$ given $\mathbf{y}$. Specifically, the difference reflects the fact that $\mathbf{f}_t$ is costly while $\mathbf{y}_t$ is a natural occurrence of the dynamical system without any cost. These arguments are formalized in Theorem 1 and Lemma 1. We will also show a relation between the optimal conditional directed information and the Kalman filtering theory with two independent measurements.

III. RESULTS

This section presents our results. First, we provide a simple structure for the optimal policy in Theorem 1. Then, we present preliminaries on Kalman filtering theory to express the directed information in its terms. We then provide a semidefinite programming formulation of the optimization problem and present the optimal system design. Finally, Section III-E includes the formulation and the solution for the infinite-horizon problem.

A. Optimal Policy Structure

The first result is the optimal structure of the observer (encoder) and controller (decoder) policies:

Theorem 1 (Optimal policy structure): If the optimization problem in (9) is feasible, an optimal policy is given by

$$\begin{aligned} \mathbf{f}_t &= D_t \mathbf{x}_t + \mathbf{m}_t \\ \mathbf{u}_t &= -K_t \mathbb{E}[\mathbf{x}_t | \mathbf{f}^t, \mathbf{y}^t] \end{aligned} \quad (11)$$

where $\mathbf{m}_t \sim \mathcal{N}(0, M_t)$ is independent from $(\mathbf{x}^t, \mathbf{y}^t, \mathbf{f}^{t-1})$ and K_t is a constant given by the LQR controller (see (19), below).

Moreover, the knowledge of the measurements $\mathbf{y}^t$ at the encoder does not reduce the optimal directed information control problem in (9).

The theorem simplifies significantly the maximization domain from the general policy in (6) to the set $\{(D_t, M_t)\}_{t=1}^T$. After Theorem 2 below, we show an explicit construction of these optimization variables. It is also shown that D_t are wide matrices. The encoding rule reveals that $\mathbf{f}_t$ is an affine function of the state with an additive Gaussian noise that reduces the communication resources (i.e., the directed information). We emphasize that our problem formulation does not impose any structural constraints onto the encoding policy such as linear, memoryless, or having a Gaussian distribution. The control signal $\mathbf{u}_t$ is the standard LQG certainty equivalence controller. Thus, similar to the scalar case in [17], the separation principle between the controller gain and the estimation is preserved here too. The proof of Theorem 1 appears in Section V.

Theorem 1 extends [16, Th. 1] and recovers it when $\mathbf{y}_t$, the observation, is absent. The extension of [16] to our setting is not trivial (see e.g., [14], [18] for progress on that problem), and involves the study of a relaxed optimization problem where, at time t , the vector $\mathbf{y}^t$ is also available to the encoder. For this relaxed optimization problem, we show that the optimal policy is of the form (11). In other words, even if the side information is available at the encoder, it cannot reduce the conditional directed information. This is consistent with the observation made in [17] in the context of the scalar system.

B. Kalman Filter With Two (Independent) Measurements

As is evident from the optimal structure in Theorem 1, the encoding function $\mathbf{f}_t$ is a noisy measurement of the system state, and its additive noise is independent of the other measurement $\mathbf{y}_t$. Thus, the optimal system has a structure of an LQG setting with two independent observations. However, for the purpose

of optimizing the communication resources, f_t has a cost, while y_t is a natural occurrence of the system. In this section, we provide short preliminaries on Kalman filtering and present the conditional directed information in Kalman filtering terms.

Following a standard convention, we denote the error covariance matrices with respect to both measurements y_t and f_t as

$$\begin{aligned} P_{t|t-1} &\triangleq \text{Cov}(x_t - \mathbb{E}[x_t | f^{t-1}, y^{t-1}, u^{t-1}]) \\ P_{t|t} &\triangleq \text{Cov}(x_t - \mathbb{E}[x_t | f^t, y^t, u^{t-1}]). \end{aligned} \quad (12)$$

Since the communication resources should be measured with respect to the observation f_t only, we define the intermediate error covariance matrix corresponding to the prediction error after observing y_t only [Eqn. (14) shown at the bottom of this page]

$$P_{t|t-1}^+ \triangleq \text{Cov}(x_t - \mathbb{E}[x_t | f^{t-1}, y^t, u^{t-1}]) \quad (13)$$

The following lemma formalizes several relations between the error covariances.

Lemma 1 (Error covariance matrices): Let $P_{1|0} \succ 0$ be the covariance matrix of x_1 . Then, for a fixed policy $\{(D_t, M_t)\}_t$, the error covariance matrices can be updated as

$$(P_{t|t-1}^+)^{-1} = P_{t|t-1}^{-1} + \text{SNR}_t^Y \quad (15a)$$

$$\begin{aligned} P_{t|t} &= ((P_{t|t-1}^+)^{-1} + \text{SNR}_t^F)^{-1} \\ &= (I - L_t^F D_t) P_{t|t-1}^+ \end{aligned} \quad (15b)$$

$$P_{t+1|t} = A_t P_{t|t} A_t^T + W_t \quad (15c)$$

where $L_t^F = P_{t|t-1}^+ D_t^T (D_t P_{t|t-1}^+ D_t^T + M_t)^{-1}$, $\text{SNR}_t^F = D_t^T M_t^{-1} D_t$, and $\text{SNR}_t^Y = C_t^T V_t^{-1} C_t$.

The identities are standard in Kalman filtering theory e.g., [27, Ch. 9.3.2], and their proofs are omitted for brevity. Also note that the inverses exist since $W_t \succ 0$ and $P_{1|0} \succ 0$. It now follows that the directed information can be expressed as

$$\begin{aligned} I(x^T \rightarrow f^T || y^T) &= \sum_{t=1}^T h(x_t | y^t, f^{t-1}) - h(x_t | y^t, f^t) \\ &= \frac{1}{2} \sum_{t=1}^T \log \det(I - L_t^F D_t). \end{aligned} \quad (16)$$

Note that the matrix $(I - L_t^F D_t)$ is the multiplicative term of the error reduction when computing $P_{t|t}$ from $P_{t|t-1}^+$. Therefore, the conditional directed information measures the reduction in error covariance with respect to f_t only, as desired.

C. SDP Formulation

Despite the elegant representation of the objective function in (16), it is not clear whether (9) can be formulated as a convex optimization since its inverse includes a product of two optimization variables $(I - L_t^F D_t)^{-1} = I + P_{t|t-1}^+ \text{SNR}_t^F$. Our next result shows a convex optimization formulation for (9).

Theorem 2 (SDP formulation): For a fixed $P_{1|0} \succ 0$, the optimization problem (9) can be formulated as the convex optimization

$$\begin{aligned} \inf_{\{P_{t|t}, \Pi_t\}_{t=1}^T} \quad & \Lambda - \frac{1}{2} \sum_{t=1}^{T-1} \log \det(I + (A_t P_{t|t} A_t^T + W_t) \text{SNR}_t^Y) \\ & - \frac{1}{2} \sum_{t=1}^T \log \det \Pi_t \\ \text{s.t.} \quad & \text{Tr}(\Phi_1 P_{1|0}) + \sum_{t=1}^T \text{Tr}(\Theta_t P_{t|t}) + \text{Tr}(S_t W_t) \leq \Gamma \\ & \begin{bmatrix} P_{t|t} - \Pi_t & P_{t|t} A_t^T \\ A_t P_{t|t} & A_t P_{t|t} A_t^T + W_t \end{bmatrix} \succeq 0, \Pi_t \succ 0, \quad t < T \\ & P_{T|T} = \Pi_T \succeq 0 \\ & \Omega_t \succeq 0, \quad t = 1, \dots, T \text{ ((14) below)}, \end{aligned} \quad (17)$$

where the constant matrices $\Phi_t \triangleq A_t^T S_t A_t - K_t^T (B_t^T S_t B_t + R_t) K_t$ and $\{\Theta_t\}_{t=1}^T$ can be computed from (19) below, and the constant Λ is given by

$$\Lambda = -\frac{1}{2} \log \det(P_{1|0}^{-1} + \text{SNR}_1^Y) + \frac{1}{2} \sum_{t=1}^{T-1} \log \det W_t. \quad (18)$$

The optimization problem in Theorem 2 is convex optimization with respect to the decision variables $(P_{t|t}, \Pi_t)$, and can be solved using standard solvers, e.g., [28], [29], [30]¹. It will be shown in the proof of Theorem 2 in Section V below that the auxiliary decision variable Π_t evaluated at the optimal point is equal to $(P_{t|t}^{-1} + A_t^T W_t^{-1} A_t)^{-1}$. However, it is necessary to introduce this variable in order to convert the objective to have a standard convex form. Then, the equality constraint resulting from the change of variable $\Pi_t = (P_{t|t}^{-1} + A_t^T W_t^{-1} A_t)^{-1}$ can be (optimally) relaxed to an inequality that is equivalent to the LMI above. The optimization problem extends [16, Th. 1] to the case where the LQG measurement y_t is available to the controller, and recovers it by choosing $C_t = \text{SNR}_t^Y = 0$. In this case, the

¹Some solvers require to write the determinant of $I + (A_t P_{t|t} A_t^T + W_t) \text{SNR}_t^Y$ in a symmetric form which can be done using Sylvester's determinant theorem.

$$\begin{aligned} \Omega_1 &\triangleq \begin{bmatrix} P_{1|0} - P_{1|1} & P_{1|0} C_1^T \\ C_1 P_{1|0} & C_1 P_{1|0} C_1^T + V_1 \end{bmatrix} \\ \Omega_t &\triangleq \begin{bmatrix} (A_{t-1} P_{t-1|t-1} A_{t-1}^T + W_{t-1}) - P_{t|t} & (A_{t-1} P_{t-1|t-1} A_{t-1}^T + W_{t-1}) C_t^T \\ C_t (A_{t-1} P_{t-1|t-1} A_{t-1}^T + W_{t-1}) & C_t (A_{t-1} P_{t-1|t-1} A_{t-1}^T + W_{t-1}) C_t^T + V_t \end{bmatrix} \quad \text{for } t = 2, \dots, T. \end{aligned} \quad (14)$$

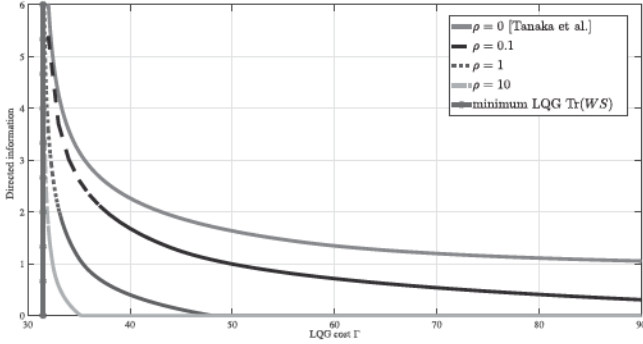


Fig. 2. Tradeoffs between the conditional directed information and the LQG cost when the SNR of the side information varies.

constraints on Ω_t simplify to $(A_{t-1}P_{t-1|t-1}A_{t-1}^T + W_{t-1}) - P_{t|t} \succeq 0$ and $P_{1|0} \succeq P_{1|1}$.

D. System Design

In this section, we construct a three-steps realizable policy using the results from the previous section.

1) Controller Gain: The controller gains are independent of the measurements and the variables from the optimization problem. The gains can be computed from a backward Riccati recursion, with the initial condition $S_T = Q_T$, as

$$\begin{aligned} S_{t-1} &= A_t^T S_t A_t - K_t^T (B_t^T S_t B_t + R_t) K_t + Q_{t-1} \\ K_t &= (B_t^T S_t B_t + R_t)^{-1} B_t^T S_t A_t \\ \Theta_t &= K_t^T (B_t^T S_t B_t + R_t) K_t. \end{aligned} \quad (19)$$

2) Covariance Matrices: Given the sequence $\{\Theta_t\}_{t=1}^T$, the optimal $\{P_{t|t}\}_{t=1}^T$ can be determined from the convex optimization problem in Theorem 2, and one can compute

$$\text{SNR}_t^F = P_{t|t}^{-1} - (A_{t-1}P_{t-1|t-1}A_{t-1}^T + W_{t-1})^{-1} - \text{SNR}_t^Y.$$

An application of the singular value decomposition (SVD) decomposition $\text{SNR}_t^F = D_t^T M_t^{-1} D_t$ determines the parameters $\{(D_t, M_t)\}_{t=1}^T$ of the optimal policy in Theorem 1.

3) Kalman Filter: The Kalman gain is defined as

$$L_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + N_t)^{-1} \quad (20)$$

$$\text{where } H_t \triangleq \begin{bmatrix} C_t \\ D_t \end{bmatrix}, N_t \triangleq \begin{bmatrix} V_t & 0 \\ 0 & M_t \end{bmatrix}.$$

The Kalman update is done in two steps

$$\begin{aligned} \hat{x}_{t+1|t} &= A_t \hat{x}_t + B_t u_t \\ \hat{x}_t &= \hat{x}_{t|t-1} + L_t \begin{bmatrix} y_t - C_t \hat{x}_{t|t-1} \\ f_t - D_t \hat{x}_{t|t-1} \end{bmatrix} \end{aligned} \quad (21)$$

where the control signal is $u_t = -K_t \hat{x}_t$.

E. Infinite-Horizon Setting

In this section, we formulate and solve the optimization problem (9) in the infinite-horizon regime. In this scenario, we consider time-invariant systems, i.e., $A_t = A$, $B_t = B$, $W_t =$

W , $C_t = C$, $V_t = V$ and time-invariant cost matrices $Q_t = Q$, $R_t = R$. The optimization problem is defined as

$$\begin{aligned} \inf \limsup_{T \rightarrow \infty} \frac{1}{T} J(x^T \rightarrow f^T || y^T) \\ \text{s.t. } \limsup_{T \rightarrow \infty} \frac{1}{T} J(x^{T+1}, u^T) \leq \Gamma \end{aligned} \quad (22)$$

where the infimum is taken with respect to the sequence of stochastic policies given in (6).

The solution structure is similar to the finite-horizon solution in Theorem 2. In particular, we construct a controller based on a solution to a convex optimization problem. We begin with the controller description.

1) Controller Gain: Assume that (A, B) is stabilizable and $(A, Q^{1/2})$ is observable on the unit circle. Then, we define $\bar{S}$ to be the unique stabilizing solution for the Riccati equation

$$A^T S A - S - A^T S B (B^T S B + R)^{-1} B^T S A + Q = 0. \quad (23)$$

By having the stabilizing solution, we can present the SDP-based system design in the infinite-horizon regime.

Theorem 3: If the pair (A, B) is stabilizable and the pair $(A, Q^{1/2})$ is observable on the unit circle, the infinite-horizon optimization problem (22) can be formulated as the convex optimization

$$\begin{aligned} \min_{P, \Pi} \quad & \frac{1}{2} \log \det W - \frac{1}{2} \log \det (I + \text{SNR}^Y (A P A^T + W)) \\ & - \frac{1}{2} \log \det \Pi \\ \text{s.t.} \quad & \text{Tr}(\Theta P) + \text{Tr}(W \bar{S}) \leq D, \\ & \begin{bmatrix} P - \Pi & P A^T \\ A P & A P A^T + W \end{bmatrix} \succeq 0, \quad \Pi \succ 0, \\ & \begin{bmatrix} A P A^T + W - P & (A P A^T + W) C^T \\ C (A P A^T + W) & C (A P A^T + W) C^T + V \end{bmatrix} \succeq 0 \end{aligned} \quad (24)$$

where $\bar{S}$ is given in (23), and $\Theta = K^T (B^T \bar{S} B + R) K$.

Moreover, let P be the optimal solution in (24) and compute

$$\text{SNR}^F = P^{-1} - (A P A^T + W)^{-1} - \text{SNR}^Y \quad (25)$$

and its SVD decomposition as $\text{SNR}^F = D^T M^{-1} D$. Then, optimal time-invariant encoder and decoder are given by

$$\begin{aligned} f_t &= D x_t + m_t \\ u_t &= -K \hat{x}_t \end{aligned} \quad (26)$$

where $m_t \sim N(0, M)$, $K = (B^T \bar{S} B + R)^{-1} B^T \bar{S} A$, and $\hat{x}_t$ is computed recursively using the Kalman filter in (21).

Theorem 3 shows that the optimization problem in the infinite-horizon regime is computationally simpler than the finite-horizon regime solved in Theorem 2. In the proof of Theorem 3, Theorem 1 is used for the structure of the optimal policy, however, it is interesting to note that we also show that a time-invariant law is optimal while in Theorem 2 the optimal policy is time-varying. The main idea to show this property is

the convexity of the objective. In particular, one can use Jensen's inequality to show that the evaluation of the objective at the convex combination of the decision variables is smaller than the averaged sum of objectives at all times. This fact can be exploited in the infinite-horizon regime to show that the convex combination of the decision variables satisfies the *stationary constraints* presented in Theorem 3. The proof of Theorem 3 is given in Section V-C.

IV. EXAMPLES

A. Side Information Reduces the Minimal Directed Information

In this section, we study a numerical example to show the benefits of side information and discuss the tradeoffs between communication resources and control performance. We set the matrices A, B, W, Q, R to be the same as those in [16, Sec. V]

$$\begin{aligned} A &= \begin{bmatrix} 0.12 & 0.63 & -0.52 & 0.33 \\ 0.26 & -1.28 & 1.57 & 1.13 \\ -1.77 & -0.30 & 0.77 & 0.25 \\ -0.16 & 0.20 & -0.58 & 0.56 \end{bmatrix} \\ B &= \begin{bmatrix} 0.66 & -0.58 & 0.03 & -0.20 \\ 2.61 & -0.91 & 0.87 & -0.07 \\ -0.64 & -1.12 & -0.19 & 0.61 \\ 0.93 & 0.58 & -1.18 & -1.21 \end{bmatrix} \\ W &= \begin{bmatrix} 4.94 & -0.10 & 1.29 & 0.35 \\ -0.10 & 5.55 & 2.07 & 0.31 \\ 1.29 & 2.07 & 2.02 & 1.43 \\ 0.35 & 0.31 & 1.43 & 3.10 \end{bmatrix} \end{aligned} \quad (27)$$

and the cost matrices Q, R are set to be identity matrices.

We start by studying an LQG system in which the side information to the decoder is given by $C = I$ and $V = \frac{1}{\rho}I$ with $\rho > 0$, so that $\text{SNR}^Y = \rho I$. For each $\rho = 0.1, \rho = 1$, and $\rho = 10$, we solve (24) for each LQG cost constraint Γ in the range $\Gamma \in [30, 90]$ and plot the optimal value of (24) as a function of Γ in Fig. 2. The case without side information studied in [16] can be equivalently viewed as the case with $\rho = 0$.

In Fig. 2, we can see that for any fixed Γ , the minimal conditional directed information decreases as ρ (the signal-to-noise ratio of the side information) increases. The red vertical line corresponds to the minimal cost that can be attained with clean observation available at the controller. The intersection with the LQG constraint axis corresponds to the LQG cost that is achieved without communication, i.e., using the side information only. It is also interesting to note that a fixed information level, the gain due to the presence of y_t increases for an increasing control cost.

In all curves with side information, the minimal directed information converges to zero as the LQG cost increases to infinity. However, in the case without side information, the curve converges to some constant known as the minimal rate needed to stabilize the system A [31]. This rate can be computed as $R = \sum_i \log_2 \max\{1, |\lambda_i(A)|\}$, where $\lambda_i(\cdot)$ denotes the i th

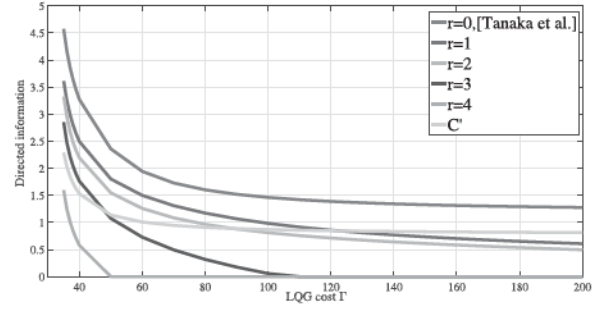


Fig. 3. Trade-off between the conditional directed information and the LQG cost with different side information SNR matrices.

eigenvalue of its argument. The fact that the curves converge to zero follow from the detectability of the pair $(A, CV^{1/2})$ (indeed, $CV^{1/2} = \rho^{-1/2}I$ is a full-rank so that the pair is observable). We proceed to study a scenario in which the side information implies that the pair is not detectable. Here, we fix the side information variance to be the identity matrix $V = I$ (i.e., $\rho = 1$), but change the observability matrix C according to two scenarios. In the first, the matrix C has dimensions $r \times 4$ for $0 \leq r \leq 4$, and is given by $C(r) = [0_{r \times (4-r)}, I_r]$. Clearly, if $r = 0$, there is no side information, and if $r = 4$ it is the full-observable matrix studied in Fig. 2 with $\rho = 1$. In the other case, we carefully choose C to be orthogonal to one of the *unstable eigenvectors* of A , i.e., the eigenvector whose corresponding eigenvalue is $\lambda_1 = -1.7124$. One choice of such

$$\text{a matrix is } C' = \begin{bmatrix} 1 & 1 & 1 & 3.75 \\ 2.11 & 1 & 1 & 1 \\ 1 & 1 & 0 & 4.56 \end{bmatrix}.$$

In Fig. 3, the minimal directed information is plotted as a function of the LQG cost Γ . As expected, it can be observed that the communication resources are decreasing as the side information dimension is increasing. For all observability matrices $C(r)$ with $1 \leq r \leq 4$, the curves tend to zero as the cost Γ grows to ∞ . On the other hand, the curves that correspond to $r = 0$ from [16], and the observability matrix C' tend to a constant when the cost is large. This constant can be calculated as the minimal rate needed to stabilize the system. In the blue curve, it is $R = \sum_i \log_2 \max\{1, |\lambda_i(A)|\} = 1.1685$ and for C' it is $R' = \log_2 |\lambda_1(A)| = 0.776$ where λ_1 is the only unstable eigenvalue that cannot be observed via C' .

B. Scalar Systems

For scalar systems, without the LQG measurement ($C = V = 0$), the solution to (24) [6], [9], [16] is

$$\frac{1}{2} \log \left(A^2 + \frac{W\Theta}{\Gamma - W\bar{S}} \right), \quad \forall \Gamma > W\bar{S} \quad (28)$$

where $\bar{S}$ is the unique solution to the Riccati equation and can be solved in closed-form as

$$\bar{S} = \frac{(A^2 + B^2 - 1) + \sqrt{(A^2 + B^2 - 1)^2 + 4B^2}}{2B^2}. \quad (29)$$

In the following result, we provide a closed-form for the scalar problem. The proof is in Section V-D below.

Corollary 1: When A, B, W, C, V are scalars, $Q = R = 1$ and $|A| > 1$, the optimal value of the optimization (24) is

$$\frac{1}{2} \log \left(A^2 + \frac{W\Theta}{\Gamma - W\bar{S}} \right) - \frac{1}{2} \log \left(1 + \text{SNR}^Y \left(W + \frac{A^2(\Gamma - W\bar{S})}{\Theta} \right) \right) \quad (30)$$

when $W\bar{S} < \Gamma \leq W\bar{S} + \Theta P^*$; and is 0 when $\Gamma > W\bar{S} + \Theta P^*$, where P^* is the unique positive solution to the quadratic equation

$$A^2 \text{SNR}^Y P^2 + (1 - A^2 + \text{SNR}^Y W)P - W = 0 \quad (31)$$

and $\bar{S}$ is given in (29) and $\Theta = \frac{(AB\bar{S})^2}{1+B^2\bar{S}}$.

By comparing (28) and (30), the information gain due to the presence of the LQG measurement is the nonnegative expression

$$\frac{1}{2} \log \left(1 + \text{SNR}^Y \left(W + \frac{A^2(\Gamma - W\bar{S})}{\Theta} \right) \right). \quad (32)$$

Note that the gain is an increasing function of SNR^Y . Also, the gain is upper bounded by (28) which is achieved with equality when $\Gamma = W\bar{S} + \Theta P^*$ since P^* satisfies $1 + \text{SNR}^Y(W + A^2 P^*) = A^2 + \frac{W}{P^*}$ [see (31)].

Remark 1: In [17], the rate distortion problem which corresponds to the control problem studied in this article has been solved for the scalar case. To reveal [17, Th. 7] from Corollary 1, let $d \triangleq \frac{\Gamma - W\bar{S}}{\Theta}$ in order to write (30) as

$$-\frac{1}{2} \log d \left(\text{SNR}^Y + \frac{1}{A^2 d + W} \right). \quad (33)$$

V. PROOFS

In this section, we prove our results. We start with Theorem 1 on the optimal policy structure.

A. Proof of Theorem 1 (Optimal Policy Structure)

The proof follows from the following claims that will be shown consecutively thereafter.

- 1) Instead of minimizing over stochastic kernels $P(\mathbf{u}_t | \mathbf{u}^{t-1}, \mathbf{f}^t, \mathbf{y}^t)$ in (5), it is sufficient to minimize over $\mathbf{u}_t$ that is a deterministic function of $\mathbf{f}^t, \mathbf{y}^t$.
- 2) The minimization domain is relaxed by allowing encoders of the form $P(\mathbf{f}_t | \mathbf{y}^t, \mathbf{x}^t, \mathbf{f}^{t-1})$ instead of $P(\mathbf{f}_t | \mathbf{x}^t, \mathbf{f}^{t-1})$ [in (4)]. In words, the new encoder has additional access to the observation $\mathbf{y}_t$.
- 3) It is sufficient to minimize the relaxed optimization problem over $P(\mathbf{f}_t | \mathbf{y}^t, \mathbf{x}_t, \mathbf{f}^{t-1})$, i.e., to let the encoder depend on $\mathbf{x}_t$ rather the tuple $\mathbf{x}^t$.
- 4) It is sufficient to minimize the relaxed optimization problem over Gaussian encoder outputs, i.e.,

$$\mathbf{f}_t = D_t \mathbf{x}_t + \lambda_t \mathbf{y}^t + \gamma_t \mathbf{f}^{t-1} + \mathbf{m}_t \quad (34)$$

where $\mathbf{m}_t \sim \mathcal{N}(0, M_t)$.

- 5) It is sufficient to minimize the relaxed optimization problem over

$$\mathbf{f}_t = D_t \mathbf{x}_t + \mathbf{m}_t. \quad (35)$$

- 6) The optimal control is $\mathbf{u}_t = -K_t \mathbb{E}[\mathbf{x}_t | \mathbf{f}^t, \mathbf{y}^t]$, where K_t is the control gain.

By claim 5, the minimizer of the relaxed optimization problem is in the original minimization domain (6). Thus, both optimization problems have a common minimizer, and $\mathbf{u}_t$ is a composition of a Kalman filter and certainty equivalence controller.

Claim 1: From the functional representation lemma [32], one can write $\mathbf{u}_1 = \Lambda(\mathbf{f}_1, \mathbf{y}_1, W_1)$ for some deterministic function $\Lambda(\cdot)$ and random variable W_1 that is independent of $(\mathbf{f}_1, \mathbf{x}_1, \mathbf{y}_1)$. Let $\tilde{\mathbf{f}}_1 \triangleq (\mathbf{f}_1, W_1)$, and note that $I(\mathbf{x}_1; \mathbf{f}_1 | \mathbf{y}_1) = I(\mathbf{x}_1; \tilde{\mathbf{f}}_1 | \mathbf{y}_1)$. Moreover, the joint distribution of $\mathbf{u}_1$ and $\mathbf{x}_1$ is unaffected by absorbing the controller's randomness to the encoder (stochastic) mapping so the LQG cost remains the same. This procedure can be inductively repeated to derandomize $\mathbf{u}_t$ at all times.

Claim 2: Trivial, since the minimization domain is increased.

Claim 3: Here we show that a lower bound on the objective function can be achieved using encoders of the form $P(\mathbf{f}_t | \mathbf{y}^t, \mathbf{x}_t, \mathbf{f}^{t-1})$. Consider the lower bound

$$\begin{aligned} I(\mathbf{x}^T \rightarrow \mathbf{f}^T | \mathbf{y}^T) &= \sum_{t=1}^T I(\mathbf{x}^t; \mathbf{f}_t | \mathbf{y}^t, \mathbf{f}^{t-1}) \\ &\geq \sum_{t=1}^T I(\mathbf{x}_t; \mathbf{f}_t | \mathbf{y}^t, \mathbf{f}^{t-1}). \end{aligned} \quad (36)$$

For a fixed sequence of deterministic mappings that characterize $\mathbf{u}_t$, the lower bound (36) and the LQG cost are fully determined by $\{P(\mathbf{x}_t, \mathbf{y}^t, \mathbf{f}^t)\}_{t \geq 1}$.

We will now show by induction that $P(\mathbf{x}_t, \mathbf{y}^t, \mathbf{f}^t)$ is determined by $\{P(\mathbf{f}_i | \mathbf{y}^i, \mathbf{x}_i, \mathbf{f}^{i-1})\}_{i \leq t}$. For $t = 1$, this claim is trivial. For the inductive step, assume that $P(\mathbf{x}_{t-1}, \mathbf{y}^{t-1}, \mathbf{f}^{t-1})$ is determined by $\{P(\mathbf{f}_i | \mathbf{y}^i, \mathbf{x}_i, \mathbf{f}^{i-1})\}_{i < t}$. Now, consider

$$P(\mathbf{x}_t, \mathbf{y}^t, \mathbf{f}^t) = P(\mathbf{f}_t | \mathbf{x}_t, \mathbf{y}^t, \mathbf{f}^{t-1}) P(\mathbf{x}_t, \mathbf{y}^t, \mathbf{f}^{t-1})$$

and note that $P(\mathbf{x}_t, \mathbf{y}^t, \mathbf{f}^{t-1})$ can be written as

$$\int_{\mathbf{x}_{t-1}} P(\mathbf{y}_t | \mathbf{x}_t) P(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) P(\mathbf{x}_{t-1}, \mathbf{y}^{t-1}, \mathbf{f}^{t-1})$$

which is fixed by the sequence $\{P(\mathbf{f}_i | \mathbf{y}^i, \mathbf{x}_i, \mathbf{f}^{i-1})\}_{i < t}$ due to the measurement characteristics (2), the fact that $\mathbf{u}_{t-1}$ is a deterministic function of $(\mathbf{y}^{t-1}, \mathbf{f}^{t-1})$ and the induction hypothesis.

Claim 4: First, the differential entropy from (36) is rewritten for $t > 1$ as

$$\begin{aligned} h(\mathbf{x}_t | \mathbf{y}^t, \mathbf{f}^{t-1}) &= h(\mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1} | \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) \\ &\quad - h(\mathbf{y}_t | \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) - h(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) \\ &= h(\mathbf{y}_t, \mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) + h(\mathbf{x}_{t-1} | \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) \\ &\quad - h(\mathbf{y}_t | \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) - h(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{y}^{t-1}, \mathbf{f}^{t-1}). \end{aligned} \quad (37)$$

We proceed to lower bound the mutual information using (37). To that end, we define a Gaussian probability measure $\mathbb{G}(\cdot)$ with first and second order moments that are identical to the ones under the measure $\mathbb{P}$. All differential entropy are computed with respect to $\mathbb{P}$ unless a subscript $\mathbb{G}$ appears. Consider the lower

bound

$$\begin{aligned}
& \sum_{t=1}^T I(\mathbf{x}_t; \mathbf{f}_t | \mathbf{y}^t, \mathbf{f}^{t-1}) \\
& \stackrel{(a)}{=} h(\mathbf{x}_1) - h(\mathbf{x}_T | \mathbf{y}^T, \mathbf{f}^T) + \sum_{t=2}^T h(\mathbf{y}_t, \mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) \\
& \quad - h(\mathbf{y}_t | \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) - h(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) \\
& \geq h(\mathbf{x}_1) - h_{\mathbb{G}}(\mathbf{x}_T | \mathbf{y}^T, \mathbf{f}^T) + \sum_{t=2}^T h(\mathbf{y}_t, \mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_{t-1}) \\
& \quad - h_{\mathbb{G}}(\mathbf{y}_t | \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) - h_{\mathbb{G}}(\mathbf{x}_{t-1} | \mathbf{x}_t, \mathbf{y}^{t-1}, \mathbf{f}^{t-1}) \quad (38)
\end{aligned}$$

where (a) follows from (37), and the inequality follows from the general observation that $h_P(\mathbf{x}|\mathbf{y}) \leq h_{\mathbb{G}}(\mathbf{x}|\mathbf{y})$ for any probability measure $\mathbb{G}$ that has the same mean and covariance as P . In particular, consider

$$\begin{aligned}
& -h_P(\mathbf{x}|\mathbf{y}) + h_{\mathbb{G}}(\mathbf{x}|\mathbf{y}) \\
& = \int \log(\mathbb{P}(\mathbf{x}|\mathbf{y})) d\mathbb{P}(\mathbf{x}, \mathbf{y}) - \int \log(\mathbb{G}(\mathbf{x}|\mathbf{y})) d\mathbb{G}(\mathbf{x}, \mathbf{y}) \\
& = \int \log(\mathbb{P}(\mathbf{x}|\mathbf{y})) d\mathbb{P}(\mathbf{x}, \mathbf{y}) - \int \log(\mathbb{G}(\mathbf{x}|\mathbf{y})) d\mathbb{P}(\mathbf{x}, \mathbf{y}) \\
& \geq 0 \quad (39)
\end{aligned}$$

where in the second equality the measure can be changed since $\log \mathbb{G}(\mathbf{x}|\mathbf{y})$ is a quadratic function of $\mathbf{x}, \mathbf{y}$ and $\mathbb{P}$ and $\mathbb{G}$ have identical first and second moments. This is a direct extension of a classical result for the unconditioned, scalar case [33, Th. 8.6.5].

Conversely, the lower bound can be achieved by choosing $\mathbf{f}_t$ with a Gaussian distribution. Specifically, for some fixed inputs $\{P(\mathbf{f}_t | \mathbf{y}^t, \mathbf{x}_t, \mathbf{f}^{t-1})\}_{t \geq 1}$, a jointly Gaussian distribution is formed by borrowing the first and second order statistics of the joint. Let $D_t \mathbf{x}_t + \lambda_t \mathbf{y}^t + \gamma_t \mathbf{f}^{t-1}$ be the linear minimum mean square estimator of $\mathbf{f}_t$ and $\mathbf{m}_t$ be its error covariance. Then, $\mathbf{f}_t \sim \mathcal{N}(D_t \mathbf{x}_t + \lambda_t \mathbf{y}^t + \gamma_t \mathbf{f}^{t-1}, \mathbf{m}_t)$, and it can be shown that the second-order statistics of $\{P(\mathbf{x}_t, \mathbf{y}^t, \mathbf{f}^t)\}_{t \geq 1}$ are unaffected since the relation between the random variables are all linear.

Finally, note that the LQG cost depends on $\{P(\mathbf{u}_t, \mathbf{x}_t)\}_{t=1}^T$ via its second moments. Since the second moments of $\mathbb{G}$ and P are the same, the LQG cost is unaffected. To summarize, we showed that we may restrict the optimization domain to Gaussian inputs of the form $\mathbf{f}_t = D_t \mathbf{x}_t + \lambda_t \mathbf{y}^t + \gamma_t \mathbf{f}^{t-1} + \mathbf{m}_t$ without loss of optimality.

Claim 5: By Claim 4, the objective of the relaxed optimization problem can be written as

$$\begin{aligned}
& \sum_t I(\mathbf{x}_t; D_t \mathbf{x}_t + \lambda_t \mathbf{y}^t + \gamma_t \mathbf{f}^{t-1} + \mathbf{m}_t | \mathbf{y}^t, \mathbf{f}^{t-1}) \\
& = \sum_t I(\mathbf{x}_t; D_t \mathbf{x}_t + \mathbf{m}_t | \mathbf{y}^t, \mathbf{f}^{t-1}) \quad (40)
\end{aligned}$$

where the equality follows since $\lambda_t \mathbf{y}^t + \gamma_t \mathbf{f}^{t-1}$ is constant when conditioned on $(\mathbf{y}^t, \mathbf{f}^{t-1})$. For the LQG cost, since $\mathbf{u}_t$ is a

deterministic function of $(\mathbf{y}^t, \mathbf{f}^{t-1})$, the effect of $\lambda_t \mathbf{y}^t + \gamma_t \mathbf{f}^{t-1}$ can be embedded into the controller's function.

Claim 6: In the previous steps, we showed that $\mathbf{f}_t = D_t \mathbf{x}_t + \mathbf{m}_t$ is optimal. We now show that $\mathbf{u}_t$ has no affect on the objective function. Therefore, for a fixed $\mathbf{f}_t$, we have a classical LQG problem whose solution is just a Kalman filter with the control gain defined in (19). Consider the objective

$$\begin{aligned}
& \sum_{t=1}^T I(\mathbf{x}_t; \mathbf{f}_t | \mathbf{y}^t, \mathbf{f}^{t-1}) \\
& = \sum_{t=1}^T h(\mathbf{x}_t | \mathbf{y}^t, \mathbf{f}^{t-1}) - h(\mathbf{x}_t | \mathbf{y}^t, \mathbf{f}^t) \\
& = \frac{1}{2} \sum_{t=1}^T \log \det(P_{t|t-1}^+) - \log \det(P_{t|t}) \\
& \stackrel{(a)}{=} \frac{1}{2} \sum_{t=1}^T \log \det(P_{t|t-1}^+) + \log \det((P_{t|t-1}^+)^{-1} + \text{SNR}_t^F) \\
& = \frac{1}{2} \sum_{t=1}^T \log \det(I + P_{t|t-1}^+ \text{SNR}_t^F) \quad (41)
\end{aligned}$$

where (a) follows from Lemma 1. Also, by Lemma 1, $P_{t|t-1}^+$ depends on the choice of SNR_t^F only. Therefore, the objective is unaffected by $\mathbf{u}_t$.

B. Proof of Theorem 2

Using Lemma 1, the optimization problem can be written as

$$\begin{aligned}
& \min \frac{1}{2} \sum_{t=1}^T \log \det(P_{t|t-1}^+) - \log \det(P_{t|t}) \\
& \text{s.t. } \text{Tr}(\Phi_1 P_{1|0}) + \sum_{t=1}^T \text{Tr}(\Theta_t P_t) + \text{Tr}(S_t W_t) \leq \Gamma \\
& (P_{t|t-1}^+)^{-1} = (P_{t|t-1})^{-1} + \text{SNR}_t^Y, t = 1, \dots, T \\
& P_{t|t}^{-1} = (P_{t|t-1}^+)^{-1} + \text{SNR}_t^F, t = 1, \dots, T \\
& P_{t+1|t} = A_t P_{t|t} A_t^T + W_t \quad (42)
\end{aligned}$$

where the minimization is over the covariance matrices $\{P_{t|t-1}^+\}_{t=1}^T$. We first rewrite the objective in a convex form, consider

$$\begin{aligned}
& \frac{1}{2} \sum_{t=1}^T \log \det(P_{t|t-1}^+) - \log \det(P_{t|t}) \\
& = \frac{1}{2} \log \det(P_{1|0}^+) - \frac{1}{2} \log \det(P_{T|T}) \\
& \quad + \frac{1}{2} \sum_{t=1}^{T-1} \log \det(P_{t+1|t}^+) - \log \det(P_{t|t}). \quad (43)
\end{aligned}$$

Each term in the sum can be written as

$$\log \det(P_{t+1|t}^+) - \log \det(P_{t|t})$$

$$\begin{aligned}
&\stackrel{(a)}{=} -\log \det(I + P_{t+1|t} \text{SNR}_t^Y) + \log \det P_{t+1|t} P_{t|t}^{-1} \\
&\stackrel{(b)}{=} -\log \det(I + P_{t+1|t} \text{SNR}_t^Y) + \log \det W_t \\
&\quad + \log \det(P_{t|t}^{-1} + A_t^T W_t^{-1} A_t) \\
&\stackrel{(c)}{=} -\log \det(I + P_{t+1|t} \text{SNR}_t^Y) + \log \det W_t \\
&\quad + \inf_{\Pi_t} \log \det \Pi_t^{-1} \\
&\quad \text{s.t. } 0 \prec \Pi_t \preceq (P_{t|t}^{-1} + A_t^T W_t^{-1} A_t)^{-1} \\
&\stackrel{(d)}{=} \inf_{\Pi_t} -\log \det(I + P_{t+1|t} \text{SNR}_t^Y) \\
&\quad - \log \det \Pi_t + \log \det W_t \\
&\quad \text{s.t. } \begin{bmatrix} P_{t|t} - \Pi_t & P_{t|t} A_t^T \\ A_t P_{t|t} & A_t P_{t|t} A_t^T + W_t \end{bmatrix} \succeq 0, \Pi_t \succ 0 \quad (44)
\end{aligned}$$

where (a) follows from Lemma 1, (b) follows from Lemma 1 and Sylvester's determinant theorem, (c) follows from introducing an auxiliary positive definite matrix Π_t and from the monotonicity of $\log \det(\cdot)$ and, finally, (d) follows from the matrix inversion lemma and Schur complement.

We will now convert the constraints to have a standard LMI form. First, note that the objective has no dependence on SNR_t^F . Thus, we can reduce this variable in the constraints of the optimization in (42) as

$$\begin{aligned}
(P_{t|t-1}^+)^{-1} &= (P_{t|t-1})^{-1} + \text{SNR}_t^Y \\
P_{t|t}^{-1} &\succeq (P_{t|t-1}^+)^{-1} \\
P_{t+1|t} &= A_t P_{t|t} A_t^T + W_t. \quad (45)
\end{aligned}$$

The first two constraints can be combined as

$$\begin{aligned}
P_{t|t}^{-1} &\succeq (P_{t|t-1})^{-1} + \text{SNR}_t^Y \\
&= (P_{t|t-1})^{-1} + C_t^T V_t^{-1} C_t. \quad (46)
\end{aligned}$$

By taking the inverse of both sides and applying the matrix inversion lemma, we can equivalently write the resulted inequality, using the Schur complement of a matrix, as

$$\Omega_t = \begin{bmatrix} P_{t|t-1} - P_{t|t} & P_{t|t-1} C_t^T \\ C_t P_{t|t-1} & C_t P_{t|t-1} C_t^T + V_t \end{bmatrix} \succeq 0. \quad (47)$$

The derivation is completed by substituting in (47) $P_{t+1|t} = A_t P_{t|t} A_t^T + W_t$ for $t = 1, \dots, T-1$.

To summarize, we showed that the optimization problem (up to the constant Λ) is

$$\begin{aligned}
&\min_{\{P_{t|t}\}_{t=1}^T, \{\Pi_t\}_{t=1}^{T-1}} \Lambda - \frac{1}{2} \log \det(P_{T|T}) \\
&+ \frac{1}{2} \sum_{t=1}^{T-1} -\log \det(I + (\text{SNR}_t^Y)^{\frac{1}{2}} \\
&\times P_{t+1|t} (\text{SNR}_t^Y)^{\frac{1}{2}}) - \log \det \Pi_t
\end{aligned}$$

$$\begin{aligned}
&\text{s.t. } \text{Tr}(\Phi_1 P_{1|0}) + \sum_{t=1}^T \text{Tr}(\Theta_t P_t) + \text{Tr}(S_t W_t) \leq d \\
&\begin{bmatrix} P_{t|t} - \Pi_t & P_{t|t} A_t^T \\ A_t P_{t|t} & A_t P_{t|t} A_t^T + W_t \end{bmatrix} \succeq 0, \Pi_t \succ 0 \\
&\Omega_t \succeq 0, \quad (48)
\end{aligned}$$

where $\Lambda = \frac{1}{2} \sum_{t=1}^{T-1} \log \det W_t + \frac{1}{2} \log \det(P_{1|0}^+)$. To obtain the closed form in Theorem 2, we substitute $(P_{1|0}^+)^{-1} = P_{1|0}^{-1} + \text{SNR}_1^Y$ and define $\Pi_T \triangleq P_{T|T}$.

C. Proof of Theorem 3

To simplify notation, we define the objective of the optimization problem as

$$\begin{aligned}
f_s(P) &\triangleq \frac{1}{2} \log \det W - \frac{1}{2} \log \det(I + (APA^T + W)\text{SNR}^Y) \\
&\quad + \frac{1}{2} \log \det(P^{-1} + A^T W^{-1} A) \quad (49)
\end{aligned}$$

and the constraints set as

$$\mathcal{D}_0 \triangleq \{P \succ 0 \mid \text{Tr}(\Theta P) + \text{Tr}(W \bar{S}) \leq D, \Omega(P) \succeq 0\}, \quad (50)$$

where

$$\Omega(P) \triangleq \begin{bmatrix} APA^T + W - P & (APA^T + W)C^T \\ C(APA^T + W) & C(APA^T + W)C^T + V \end{bmatrix}.$$

We also define

$$\begin{aligned}
R_T(P) &\triangleq \frac{1}{T} \left[\frac{1}{2} \log \det(I + (APA^T + W)\text{SNR}^Y) \right. \\
&\quad \left. - \frac{1}{2} \log \det(APA^T + W) \right] \quad (51)
\end{aligned}$$

and present two technical lemmas needed for the proof of Theorem 3.

Lemma 2: For a sequence of matrices $\{P_{t|t}\}_{t \geq 1}$, let $\bar{P}_T \triangleq \frac{1}{T} \sum_{t=1}^T P_{t|t}$ be their uniform convex combination. Then, for any sequence $\{P_{t|t}\}_{t \geq 1}$ that satisfies $\Omega_t \succeq 0$, there exists $\{T_i\}_{i \geq 1}$ such that $\lim_{i \rightarrow \infty} \bar{P}_{T_i} \in \mathcal{D}_0$.

Lemma 3: Let T_i be a sequence as in Lemma 2, and $\{P_{T_i|T_i}\}_{i \geq 1}$ is a sequence that satisfies the constraints in (17). Then, under the conditions of Theorem 3,

$$\limsup_{i \rightarrow \infty} R_{T_i}(P_{T_i|T_i}) \geq 0.$$

The proofs of Lemma 2 and Lemma 3 appear below. We are now ready to prove the main result in this section.

Proof of Theorem 3: The optimal structure for the policy derived in Theorem 1 is true for any time horizon. Therefore, we can utilize Theorem 1 to write the optimization problem over the time-varying decision variables policy as

$$\begin{aligned}
&\inf_{\{P_{t|t} \succ 0\}_{t \in \mathbb{N}}} \limsup_{T \rightarrow \infty} \frac{1}{T} (\Lambda_T - \frac{1}{2} \log \det P_{T|T}) \\
&+ \frac{1}{2T} \sum_{t=1}^{T-1} \log \det(P_{t|t}^{-1} + A^T W^{-1} A)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2T} \sum_{t=1}^{T-1} \log \det(I + (AP_{t|t}A^T + W)\text{SNR}^Y) \\
& \text{s.t. } \limsup_{T \rightarrow \infty} \frac{1}{T} \text{Tr}(\Phi_1 P_{1|0}) \\
& + \frac{1}{T} \sum_{t=1}^T \text{Tr}(\Theta_t P_{t|t}) + \text{Tr}(S_t W) \leq \Gamma \\
& \Omega_t = \begin{bmatrix} P_{t|t-1} - P_{t|t} & P_{t|t-1}C_t^T \\ C_t P_{t|t-1} & C_t P_{t|t-1}C_t^T + V_t \end{bmatrix} \succeq 0, \quad t \in \mathbb{N} \quad (52)
\end{aligned}$$

where the constant matrices Φ and Θ_t are given in Theorem 2, and the constant Λ_T is given by

$$\Lambda_T = -\frac{1}{2} \log \det(P_{1|0}^{-1} + \text{SNR}^Y) + \frac{1}{2} \sum_{t=1}^{T-1} \log \det W. \quad (53)$$

By taking the limit in (52) over the time-independent quantities, we have $\frac{1}{T} \Lambda_T \rightarrow \frac{1}{2} \log \det W$ and $\frac{1}{T} \text{Tr}(\Phi_1 P_{1|0}) \rightarrow 0$. The LQG cost constraint is simplified by noting that S_t converges to the stabilizing solution of the Riccati equation in (23), $\bar{S}$, by the assumption that the (A, B) is stabilizable and $(A, Q^{1/2})$ is controllable on the unit circle. This in turn implies that $\Theta_t \rightarrow \Theta$.

Next, we define

$$\begin{aligned}
f_{r,T}(\{P_{t|t}\}_{t=1}^T) & \triangleq \frac{1}{2T} \sum_{t=1}^T \log \det(P_{t|t}^{-1} + A^T W^{-1} A) \\
& + \frac{1}{2} \log \det W - \frac{1}{2T} \sum_{t=1}^T \log \det(I + (AP_{t|t}A^T + W)\text{SNR}^Y) \\
& \quad (54)
\end{aligned}$$

in order to compactly express the optimization problem as

$$\begin{aligned}
& \min_{\{P_{t|t}\}_{t \in \mathbb{N}}} \limsup_{T \rightarrow \infty} R_T(P_{T|T}) + f_{r,T}(\{P_{t|t}\}_{t=1}^T) \\
& \text{s.t. } \limsup_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \text{Tr}(\Theta P_{t|t}) + \text{Tr}(\bar{S}W) \leq \Gamma \\
& \Omega_t = \begin{bmatrix} P_{t|t-1} - P_{t|t} & P_{t|t-1}C_t^T \\ C_t P_{t|t-1} & C_t P_{t|t-1}C_t^T + V_t \end{bmatrix} \succeq 0 \\
& P_{t|t-1} = AP_{t-1|t-1}A^T + W \quad \forall t \in \mathbb{N} \quad (55)
\end{aligned}$$

where $R_T(\cdot)$ was defined in (51).

We can now present the main steps that constitute the proof of the lower bound in Theorem 3

$$\begin{aligned}
& \min \limsup_{T \rightarrow \infty} R_T + f_{r,T}(\{P_{t|t}\}_{t=1}^T) \\
& \stackrel{(a)}{\geq} \min \limsup_{i \rightarrow \infty} R_{T_i} + f_{r,T_i}(\{P_{t|t}\}_{t=1}^{T_i}) \\
& \stackrel{(b)}{\geq} \min \limsup_{i \rightarrow \infty} f_{r,T_i}(\{P_{t|t}\}_{t=1}^{T_i}) \\
& \stackrel{(c)}{\geq} \min \limsup_{i \rightarrow \infty} f_s(\bar{P}_{T_i})
\end{aligned}$$

$$\stackrel{(d)}{\geq} \min_{P \in \mathcal{D}_0} f_s(P) \quad (56)$$

where:

- a) follows relaxing to the limit supremum from T_i to T ;
- b) follows from the nonnegativity of $\limsup R_{T_i}$, shown below as Lemma 3;
- c) follows from the convexity of the function $-\log \det(I + (APA^T + W)\text{SNR}^Y) - \log \det(P^{-1} + A^T W^{-1} A)$;
- d) follows from Lemma 2.

Note that the left-hand side of (56) is the optimization problem derived in (55). Thus, we showed a single-letter lower bound to the optimization problem (22). Furthermore, for a matrix $P \in \mathcal{D}_0$, the time-invariant cartesian product $\bigotimes_{t \geq 1} P$ satisfies the time-dependent constraints in (55) and the resulting objective in the optimization problem (55) is $f_s(P)$. These steps conclude that (24) is a lower to (22).

In the last step, we show that there exists a time-invariant policy that achieves the lower bound in (55). Namely, we show that for each covariance matrix $P \in \mathcal{D}_0$ in (56), there exists a time-invariant policy that makes the objective in (55) equal to $f_s(P)$. To this end, we construct a time-invariant policy given by the pair (D, M) as follows: Given P , compute (D, M) matrices using the SVD decomposition $D^T M^{-1} D = P^{-1} - (APA^T + W)^{-1} - \text{SNR}^Y$ with $M \succ 0$. By construction, we have that P is a solution to the Riccati equation

$$APA^T - P + W - APH^T(HPH^T + \tilde{V})^{-1}HPA^T = 0 \quad (57)$$

where $H \triangleq \begin{bmatrix} C \\ D \end{bmatrix}$, $\tilde{V} \triangleq \begin{bmatrix} V & 0 \\ 0 & M \end{bmatrix}$. The closed-loop Riccati equation of (57) can be rewritten as

$$(A - \Xi H)P(A - \Xi H)^T - P + W + \Xi \tilde{V} \Xi^T = 0 \quad (58)$$

where $\Xi = APH^T(HPH^T + \tilde{V})^{-1}$. From the assumption $W \succ 0$, the closed-loop system $(A - \Xi H)$ is stable. Therefore, (A, H) is detectable and P is the unique maximal solution to the Riccati equation. The detectability of (A, H) and $P_{1|0} \succeq P$ ² guarantee the convergence of the forward Riccati recursion

$$\begin{aligned}
P_{t|t} &= AP_{t-1|t-1}A^T + W \\
& - AP_{t-1|t-1}H^T(HP_{t-1|t-1}H^T + \tilde{V})^{-1}HP_{t-1|t-1}A^T
\end{aligned} \quad (59)$$

to the maximal solution of the Riccati equation. The proof is completed by computing the limit in (55) using the convergence of (59). This concludes the proof of Theorem 3. ■

Proof of Lemma 2: For $\epsilon > 0$, define $\mathcal{D}_\epsilon \triangleq \{P \succ 0 | \text{Tr}(\Theta P) + \text{Tr}(W\bar{S}) \leq \Gamma \text{ and } (60)\}$.

$$\begin{bmatrix} APA^T + W - P & (APA^T + W)C^T \\ C(APA^T + W) & C(APA^T + W)C^T + V \end{bmatrix} \succeq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. \quad (60)$$

²If this is not the case, one can increase the covariance by ignoring the measurements for several time instances

By the assumption, $\Omega_t \succeq 0$, for all $t \geq 1$. Consider the time-invariant constraint

$$\begin{aligned} \Omega(\bar{P}_T) &= \frac{1}{T} \sum_{t=1}^{T+1} \Omega_t \\ &+ \begin{bmatrix} -\frac{1}{T}P_{1|0} + \frac{1}{T}P_{T+1|T+1} & -\frac{1}{T}P_{1|0}C^T \\ -\frac{1}{T}CP_{1|0} & -\frac{1}{T}CP_{1|0}C^T - \frac{1}{T}V \end{bmatrix} \\ &\succeq \begin{bmatrix} -\frac{1}{T}P_{1|0} & -\frac{1}{T}P_{1|0}C^T \\ -\frac{1}{T}CP_{1|0} & -\frac{1}{T}CP_{1|0}C^T - \frac{1}{T}V \end{bmatrix}. \end{aligned} \quad (61)$$

By the boundedness of $P_{1|0}$ and V , for each $\epsilon > 0$, there exists T_ϵ such that $\bar{P}_T \in \mathcal{D}_\epsilon$ for all $T \geq T_\epsilon$. The LQG constraint is trivially satisfied at $\bar{P}_T$ by the linearity of the trace operator. The boundedness of $\mathcal{D}_\epsilon$ and $\mathcal{D}_0$ follows from the boundedness of a larger set in [34]. Therefore, these sets are compact, and there exists a limit point in $\cap_{\epsilon>0} \mathcal{D}_\epsilon$ equal to $\mathcal{D}_0$. ■

Proof of Lemma 3: We can write

$$\begin{aligned} R_{T_i}(P_{T_i|T_i}) &= \frac{1}{T_i} \left[\frac{1}{2} \log \det(I + (AP_{T_i|T_i}A^T + W)\text{SNR}^Y) \right. \\ &\quad \left. - \frac{1}{2} \log \det(AP_{T_i|T_i}A^T + W) \right] \\ &= \frac{1}{T_i} \frac{1}{2} \log \det(P_{T_i|T_i}^{-1} + \text{SNR}^Y)^{-1}. \end{aligned} \quad (62)$$

In order to show the nonnegativity of the limit, one can follow the steps in [34, Lemma 3] along with the fact that the limiting error covariance $\lim_{i \rightarrow \infty} \bar{P}_{T_i}$, must be bounded in all directions that are not orthogonal to A . ■

D. Proof of Corollary 1

Proof: When A, B, W, C, V are scalars and $Q = R = 1$, $\bar{S}, K, \Theta$ also become scalars and the optimization (24) reduces to

$$\begin{aligned} \inf_{P, \Pi \in \mathbb{R}} & \frac{1}{2} \log W - \frac{1}{2} \log(1 + \text{SNR}^Y(A^2P + W)) \\ & - \frac{1}{2} \log \Pi \end{aligned}$$

$$\text{s.t. } \Theta P + W\bar{S} \leq \Gamma$$

$$\begin{aligned} \begin{bmatrix} P - \Pi & AP \\ AP & A^2P + W \end{bmatrix} &\succeq 0, \quad \Pi > 0 \\ \begin{bmatrix} A^2P + W - P & C(A^2P + W) \\ C(A^2P + W) & C^2(A^2P + W) + V \end{bmatrix} &\succeq 0. \end{aligned} \quad (63)$$

Since $|A| > 1$, we can easily verify using (29) that $\bar{S} > 0$ (in fact, $\bar{S} > 1$) and thus $K > 0$ and $\Theta > 0$. To simplify the PSD constraints, notice that a 2×2 matrix X is positive semidefinite if and only if $\text{Tr}(X) \geq 0$ and $\det(X) \geq 0$. With this observation, we can simplify the constraints further as

$$0 < P \leq \min \left\{ \frac{\Gamma - W\bar{S}}{\Theta}, P^* \right\}$$

$$0 < \Pi \leq \min \left\{ P + A^2P + W, \frac{WP}{A^2P + W} \right\} \quad (64)$$

where P^* is defined as the unique positive solution to (31). Since

$$\frac{WP}{A^2P + W} \leq \frac{W}{A^2} \leq W < P + A^2P + W \quad (65)$$

the constraint on Π is further simplified to $0 < \Pi \leq \frac{WP}{A^2P + W}$. Also notice that for any fixed feasible P , $\log \Pi$ is maximized at $\frac{WP}{A^2P + W}$. Then, the optimization problem is further simplified as

$$\inf_{P \in \mathbb{R}: (64)} \frac{1}{2} \log \left(A^2 + \frac{W}{P} \right) - \frac{1}{2} \log \left(1 + \frac{C^2}{V} (A^2P + W) \right). \quad (66)$$

Let $g(P)$ be the function such that the objective function is written as $-\frac{1}{2} \log g(P)$, i.e.,

$$g(P) \triangleq \frac{P(1 + \frac{C^2}{V}(A^2P + W))}{A^2P + W}. \quad (67)$$

It then suffices to show that $g(P)$ is an increasing function in $P > 0$. This can be accomplished by rewriting $g(P)$ as

$$g(P) = \frac{1}{A^2} \left(\frac{C^2}{V} (A^2P + W) - \frac{W}{A^2P + W} - \frac{C^2W}{V} + 1 \right) \quad (68)$$

which is increasing in $P > 0$ since $W > 0$. Therefore, when $W\bar{S} < \Gamma \leq W\bar{S} + \Theta P^*$, the optimal value is given by

$$-\frac{1}{2} \log g \left(\frac{\Gamma - W\bar{S}}{\Theta} \right) \quad (69)$$

which equals (30). When $\Gamma > W\bar{S} + \Theta P^*$, the optimal value is given by

$$-\frac{1}{2} \log g(P^*). \quad (70)$$

Finally, one can verify that $g(P^*) = 1$, which implies that the optimal value equals 0 for $\Gamma > W\bar{S} + \Theta P^*$ using the fact that P^* is the solution to (31). ■

VI. CONCLUSION

In this article, we formulated and solved an optimization problem for reducing the LQG cost using an additional communication link. It was shown that the optimal encoding law is a memoryless Gaussian measurement of the state, while the optimal controller inherits the standard LQG control law. The simple policy structure is utilized to show that the minimization of the conditional directed information subject to a control constraint can be formulated as a standard convex optimization problem. For the finite-horizon regime, that convex optimization problem consists of a sequence of decision variables, while in the infinite-horizon regime, it simplifies to a single-letter optimization problem. The examples illustrate the benefits of the LQG setting with side information compared to the setting without side information even if the measurement has a low SNR.

The optimization problem studied in this article serves as a lower bound to the operational problem of minimizing the expected length of prefix-free codewords subject to a constraint on the control cost (see, e.g., [18], [23]). In [18], it is shown that if the LQG measurement is available to the encoder and the controller, then our optimization problem also serves as an upper bound to the operational problem (up to an additive constant term). A natural research direction is the construction of upper bounds to the operational problem in the practical scenario where the LQG measurements are available to the controller only.

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