

Exploring AC Power Flow Resultants through the Lens of Multi-homogeneous Algebraic Geometry

Masoud Barati

dept. Electrical and Computer Engineering

University of Pittsburgh

Pittsburgh, United States

masoud.barati@pitt.edu

Abstract—The computational aspects of power systems have been exploring the steady states of AC power flow for several decades. In this paper, we propose a novel approach to AC power flow calculation using resultants and discriminants for polynomials, which are primarily compiled for quadratic power flow equations. In the case of AC power flow nonlinear systems, it is not possible to determine the number of isolated solutions. However, for polynomial systems, the theorem of Bézout is the primary theorem of algebraic geometry. This study considers a certain multi-homogeneous structure in an algebraic geometry system to demonstrate that the theorem of Bézout is indeed a generalization of the fundamental theorem, among other results.

Index Terms—AC power flow, Algebraic Geometry, Bézout theorem, Determinant, Sylvester Resultant, Multi-homogeneous.

I. INTRODUCTION

THE AC power flow equations, also known as load flow equations, are a set of nonlinear algebraic equations used to determine the steady-state operating conditions of an AC power system. The equations relate the complex voltages and currents at all buses in the system, the complex power injections at the generator buses, and the complex power consumptions at the load buses. In general, the AC power flow equations can be written as [1]:

$$s_n = p_n + jq_n = v_n Y_{nn}^* v_n^* + v_n \sum_{m=1, m \neq n}^{NB} Y_{nm}^* v_m^* \quad (1)$$

where s_n , p_n , and q_n are the complex, active, and reactive power injections at bus n , respectively; v_n is the complex voltage at bus n ; Y_{nn} is the self-admittance of bus n ; Y_{nm} is the admittance between buses n and m ; and the sum is taken over all buses m connected to bus n .

These equations are typically solved iteratively using numerical methods, such as the Newton-Raphson and Guasse-Sidel methods until convergence is achieved. The solution provides the complex voltages and currents at all buses in the system under steady-state operating conditions.

The provided passage discusses the complex and non-linear nature of problems in the AC power flow model, as denoted by the notation $v_{(\cdot)}^*$ that represents complex conjugation. To obtain an algebraic system from equation (1), it is necessary to

reformulate the complex conjugate. This can be achieved by replacing all instances of v_k^* with independent variables u_k and filtering for “real” solutions where $u_k = v_k^* = v_{dk} - \sqrt{-1}v_{qk}$, after obtaining complex solutions. Consequently, this approach yields a specific structure that allows for various results to be proven.

The main contribution of the paper is a reformulation of the steady-state equations into a multi-homogeneous algebraic system. This methodology is vital to understanding the complex and non-linear nature of problems in the AC power flow model. The approach described herein provides an effective method for handling complex systems and deriving valuable insights into the steady-state operation of an AC power network.

This study presents both analytical and empirical findings. The analytical outcomes concentrate on the volume and composition of viable solutions, drawing upon the multi-homogeneous structures [15] as demonstrated in previous research [3], [4]. The article recognizes that earlier works have accomplished the algebraization of system 1 [6], [7], [12]–[15] and that homotopy-continuation methods have been extensively utilized [7]–[9], [16]–[23]. The methodology is comprehensively examined, providing innovative perspectives on the steady-state functioning of an AC power network grounded in established techniques. In summary, this study substantially contributes to the existing literature on AC power systems by offering analytical and empirical findings that augment the comprehension of viable solutions, along with their volume and organization.

The structure of this paper is as follows: Section II discusses the concept of Sylvester resultant, providing an overview of the Bezout number and resultant of the polynomial. In Section III, the comprehensive and tailored power flow equations for multi-homogeneous systems are expounded upon. Section IV presents the empirical results pertaining to the two-bus test system, covering aspects such as the number of roots and conditions for the non-uniqueness of optimal solution outcomes. These results rely on the Bertini technique [5] and resultant.

II. THE SYLVESTER RESULTANT

The concept of polynomials is commonly introduced in elementary algebra, and therefore, their fundamental characteristics and principles are widely recognized. However, as is

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typical of subjects taught at an elementary level, the scope of polynomials extends far beyond the content typically covered in secondary education. The investigation of polynomials frequently revolves around the identification of their roots or zeros. The examination of polynomials constitutes an integral component of abstract algebra and engenders compelling attributes, such as field extensions. Throughout this document, the notation F designates a field, and we commence with a straightforward definition.

Definition 1 characterizes a polynomial $f(x) \in F[x]$ as the product of factors $(x - \alpha_i)$ for i ranging from 0 to n , with a_n representing the coefficient of the highest degree term, x^n , and a_0 the constant term. Notably, the degree of $f(x)$, $\deg(f(x))$, equals n , with $a_n \neq 0$.

$$f(x) = \prod_{i=0}^n (x - \alpha_i) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad (2)$$

The roots of $f(x)$ are denoted by α_i and each coefficient a_i belongs to the field F . Interested parties can consult reference [24] for additional information on polynomials, their properties, and polynomial rings.

Our aim is to determine the intersections of algebraic curves of AC power flow equations for real- and reactive powers at each node of the power grid such as p_i, q_i when $i \in \{1, \dots, NB\}$ belongs to $\mathbb{P}$ and $\mathbb{Q}$, where $\mathbb{P}$ and $\mathbb{Q}$ correspond to the vanishing sets of the multivariate polynomials p_i and q_i when $i \in \{1, \dots, NB\}$.

$$\mathbb{P} = \{p_i \mid i \in 1, \dots, NB\} \quad (3)$$

$$\mathbb{Q} = \{q_i \mid i \in 1, \dots, NB\}$$

For the purpose of maintaining mathematical rigor in this section, we restrict our attention to bivariate functions in both sets $\mathbb{P}$ and $\mathbb{Q}$, denoted as $f_p = p(x, y)$ and $f_q = q(x, y)$, respectively. Our primary objective is to identify the common zeros of p and q . We begin by examining the simpler question of whether the line $x = \alpha$ intersects $\mathbb{P}$ and $\mathbb{Q}$. To address this issue algebraically, we investigate whether the univariate polynomials $p(\alpha, y)$ and $q(\alpha, y)$ possess a common zero or in the other words these two equations have the same root which is the AC power flow solution of p and q . To solve this problem, we employ the technique of resultant calculus, which provides a solution that can be generalized to the bivariate scenario.

A. Common Zeros of Bivariate Polynomials

In algebraic geometry, the study of common zeros of bivariate polynomials plays a fundamental role. Given two bivariate polynomials $p(x, y)$ and $q(x, y)$ defined over a field $\mathbb{P} = \mathbb{R}[x, y]$, the common zeros of p and q are the solutions $(x, y) \in \mathbb{R}[x, y]$ that satisfy both $p(x, y) = 0$ and $q(x, y) = 0$ simultaneously. Geometrically, these common zeros correspond to the points of intersection between the algebraic curves defined by p and q .

Our focus now turns to the problem of determining the common zeros of two bivariate polynomials $p \in \mathbb{R}[x, y]$ and $q \in \mathbb{R}[x, y]$. When the degree of p and q is no more than

two in one of the variables, say y , a simple method can be employed. We first solve $q(x, y) = 0$ for y , and then substitute the resulting expression for y into $p(x, y) = 0$, effectively eliminating one of the variables.

We consider p and q as polynomials in y with coefficients in $\mathbb{R}[x]$. Specifically,

$$p(x, y) = \sum_{0 \leq k \leq n} p_i(x) y^k, \quad (4)$$

$$q(x, y) = \sum_{0 \leq k \leq m} q_i(x) y^k \quad (5)$$

Where $p_i(x)$ and $q_j(x)$ are polynomials in x and n, m denote the degree of p and q in y , respectively. We wish to compute the resultant of p and q with respect to y , which is a polynomial in x that characterizes the common zeros of p and q . Through the use of the substitution $x \mapsto \varphi$, we obtain univariate polynomials that facilitate the computation of the common zeros of the bivariate polynomials in question.

$$p(\varphi, y) = \sum_{0 \leq k \leq n} p_i(\varphi) y^k, \quad (6)$$

$$q(\varphi, y) = \sum_{0 \leq k \leq m} q_i(\varphi) y^k \quad (7)$$

If the resultant of the univariate polynomials $p(\varphi, y)$ and $q(\varphi, y)$ is equal to zero, then these polynomials share a common root. The resultant is a mathematical construct that encapsulates the information regarding the common zeros of the bivariate polynomials p and q .

$$\det \begin{pmatrix} p_n(\varphi) & \dots & p_0(\varphi) & & \\ & \ddots & & \ddots & \\ & & p_n(\varphi) & \dots & p_0(\varphi) \\ q_m(\varphi) & \dots & q_0(\varphi) & & \\ & \ddots & & \ddots & \\ & & q_m(\varphi) & \dots & q_0(\varphi) \end{pmatrix} \quad (8)$$

An alternate method for determining the determinant involves preserving the entries as polynomials in variable x . The resulting polynomial in x can then be evaluated by substituting the value of φ for x . To compute this polynomial, we first construct the Sylvester matrix of the given polynomials p and q with respect to variable y . This approach provides a more efficient and computationally feasible method for computing the determinant. The equation (9) represents the result of applying the Sylvester method to compute the resultant of two polynomials p and q with respect to variable y . The resultant is denoted by $\text{res}_y(p, q)$, which is computed as the determinant of the Sylvester matrix $\text{Syl}_y(p, q)$. The Sylvester matrix is constructed by placing the coefficients of p and q in the appropriate positions of the matrix, resulting in a square matrix with $n + m$ rows and columns. The curly brace with the subscript m indicates that the matrix has m rows. The determinant of this matrix yields the desired resultant, which is a polynomial in variable x . It suffices to demonstrate that if the determinant of the Sylvester matrix of polynomials p and q with respect to variable x is zero, then there exists a

vector in the kernel of $\text{Syl}_x(p, q)$ that has the form specified in the aforementioned equation. This implies that there exists a common root φ for p and q , thereby establishing the converse of the aforementioned result.

The notion of root products and companion matrices is grounded on the assumption that $p(x)$ and $q(x)$ have roots π_j and ζ_k , respectively. Based on this assumption, the following expression can be constructed as $\prod_{j=1}^n \prod_{k=1}^m (\pi_j - \zeta_k)$. The product obtained as a result of this construction is zero if and only if there exists a common root of both p and q . It may seem like computing this product requires explicit knowledge of the roots; however, this can be circumvented by

$$\text{res}_y(p, q) = \det[\text{Syl}_y(p, q)] = \det \begin{pmatrix} p_n(x) & \dots & p_0(x) & & \\ & \ddots & & \ddots & \\ q_m(x) & p_n(x) & \dots & p_0(x) & \\ & \dots & q_0(x) & & \\ & \ddots & & \ddots & \\ & q_m(x) & \dots & & q_0(x) \end{pmatrix} \quad (9)$$

The resulting product can be expressed in terms of the coefficients of p and q rather than their roots. This approach makes it possible to determine whether p and q share a common root without having to compute the roots explicitly, which can be particularly useful in cases where the roots are difficult to compute or not known.

$$\begin{aligned} p_n^m q_m^n \prod_{j=1}^n \prod_{k=1}^m (\pi_j - \zeta_k) &= p_n^m \prod_{j=1}^n q(\pi_j) \\ &= p_n^m \det q(\mathcal{C}_p) \\ &= (-1)^{nm} q_m^n \prod_{k=1}^m p(\zeta_k) \\ &= (-1)^{nm} q_m^n \det p(\mathcal{C}_q) \\ &= p_n^m q_m^n \det(\mathcal{C}_p \otimes I_m - I_n \otimes \mathcal{C}_q) \end{aligned} \quad (10)$$

$p(\mathcal{C}_q)$ refers to the matrix obtained by substituting the elements of the companion matrix $\mathcal{C}_q$ of polynomial $q(x)$ into the polynomial $p(x)$. This is done by replacing every occurrence of x in the entries of $\mathcal{C}_q$ with the polynomial $p(x)$. The resulting matrix is a square matrix of the same size as $\mathcal{C}_q$, but with polynomial entries obtained by evaluating $p(x)$ at the corresponding entries of $\mathcal{C}_q$. The determinant of this matrix, $\det p(\mathcal{C}_q)$, is equal to the value of $p(x)$ evaluated at each root of $q(x)$. Through a recognized association with Kronecker products, it is possible to express (10) in an alternative form. This representation may offer a more efficient or manageable means of addressing the equation in certain scenarios.

B. Bézout matrix and resultant of polynomial real and reactive power flow functions

The Bézout matrix is a mathematical object associated with two given polynomials $p(x)$ and $q(x)$, as mentioned

previously. In this context, we examine a function of two variables. This function of two variables is a polynomial in v_d and v_q , and its symmetry under the interchange of v_d and v_q is readily apparent. $\text{Bez}_x(p, q)$ indicates the symmetric $d \times d$ matrix that expresses a polynomial in the standard basis of monomials. Specifically, this means that:

$$B(v_d, v_q) := \frac{p(v_d)q(v_q) - p(v_q)q(v_d)}{v_d - v_q} \quad (11)$$

The equation (12) expresses how the Bézout matrix $\text{Bez}_x(p, q)$ can be constructed using the function $B(v_d, v_q)$, which is defined as the dot product of a row vector and a column vector of monomials. Specifically, the Bézout matrix is a $d \times d$ symmetric matrix, where d is the maximum degree of $p(x)$ and $q(x)$, and each entry is obtained by evaluating the function $B(v_d, v_q)$ at the corresponding values of v_d and v_q . By constructing the Bézout matrix in this way, we can easily obtain information about the common roots of $p(x)$ and $q(x)$ through the determinant of the matrix or its rank.

$$B(v_d, v_q) = \begin{bmatrix} 1 \\ v_d \\ \vdots \\ v_d^{d-1} \end{bmatrix}^T \text{Bez}_x(p, q) \begin{bmatrix} 1 \\ v_q \\ \vdots \\ v_q^{d-1} \end{bmatrix} \quad (12)$$

The different constructions mentioned in this context include the Sylvester matrix (9), the Bezout matrix, the product of the leading coefficients, and the Kronecker product (10). Each of these constructions leads to a polynomial that is a function of the coefficients of $p(x)$ and $q(x)$.

$$\text{Res}_y(p, q) = \det[\text{Syl}_y(p, q)] = eq. \quad (10)$$

$$\begin{aligned} &= \frac{(-1)^{\binom{n}{2}}}{p_n^{n-m}} \det[\text{Bez}_y(p, q)] \end{aligned} \quad (13)$$

It can be shown that all these different constructions yield the same polynomial, which is the resultant of p and q . This polynomial has the property that it vanishes if and only if $p(x)$ and $q(x)$ have a common root. Therefore, the resultant provides a way to determine whether two polynomials have a common root, and it plays an important role in algebraic geometry and polynomial computations. The resultant of two polynomials $p(x)$ and $q(x)$ is a homogeneous multivariate polynomial that involves $n+m+2$ variables p_j and q_k , where n and m are the degrees of p and q , respectively. The resultant has integer coefficients and a degree of $n+m$ and is denoted as $\text{Res}_y(p, q)$. It is a well-known fact that the resultant is zero if and only if the polynomials p and q share a common root. Although the definition is not symmetric in its two arguments, we have the relation $\text{Res}_y(p, q) = (-1)^{nm} \text{Res}(q, p)$, but this does not affect the check for whether it is zero.

III. MULTI-HOMOGENEOUS SYSTEM

It should be noted that to derive an algebraic system from the steady-state equations (1), it is necessary to re-express the complex conjugate. This can be achieved by substituting all instances of v_n^* with independent variables u_n , and subsequently filtering for solutions where $u_n = v_n^*$ after obtaining complex solutions. Such solutions are referred to as “real”. Let G denote the set of slack generators for which $|v_n|$ is specified, and let $1 \in G$ correspond to a reference node with phase 0. Notably, the use of variables v_n and u_n leads to a multi-homogeneous structure with variable groups v_n and u_n .

$$\begin{aligned} \sum_{k=1}^{NB} (Y_{n,k} v_n u_k + Y_{n,k}^* u_n v_k) &= 2p_n & n \in N \setminus G \\ \sum_{k=1}^{NB} (Y_{n,k} v_n u_k - Y_{n,k}^* u_n v_k) &= 2q_n & n \in N \setminus G \\ v_1 = u_1 = |v_1| & & n \in \{1\} \end{aligned} \quad (14)$$

IV. SIMULATION RESULTS

We consider a two-bus test system in Fig. 1. We assume node 1 is slack bus with $v_{d1} = 1$, $v_{q1} = 0$ and node 2 is PQ bus with $p = q = 1$ and line parameter $g = -b = 0.05$. All values are represented in a per-unit system. The simulation results of the program are provided in GitHub¹. The polynomial equations of this system can be represented in (15).

$$\begin{aligned} p &= v_d^2 - v_q^2 - v_d - 5v_q + 10 = 0 \\ q &= -5v_d^2 + 5v_q^2 + 5v_d - v_q - 10 = 0 \end{aligned} \quad (15)$$

The determinant of resultant is given by $-4(169v_d^2 - 169v_d - 10)$. By computing the determinant and set it equal to zero, $v_d^{(1)} = \frac{1}{2} - \frac{\sqrt{209}}{26}$ with $v_q^{(1)} = -0.8776$ and $v_d^{(2)} = \frac{1}{2} + \frac{\sqrt{209}}{26}$ with $v_q^{(2)} = 0.6633$. the distance between the two power flow equation solutions, $v^{(1)} \approx 0.87919076$ and $v^{(2)} \approx 1.24652515$, reveals that one value is smaller than the other, with $v^{(1)} < v^{(2)}$. If we assume a lower bound of 0.9, then $v^{(1)}$ would represent an unstable voltage result. The distance between these two solutions can be used to gauge how close

the stable operating point is to the nose point of voltage collapse. In essence, this signifies that,

$$\begin{aligned} d(v^{(1)}, v^{(2)}) &= |v^{(2)} - v^{(1)}| \\ &= 1.24652515 - 0.87919076 \approx 0.36733439. \end{aligned} \quad (16)$$

Thus, the approximate distance between $v^{(1)}$ and $v^{(2)}$ is 0.36733439, a relatively small value. This suggests that the system's operating point is nearing voltage collapse, as indicated by the closeness of the values.

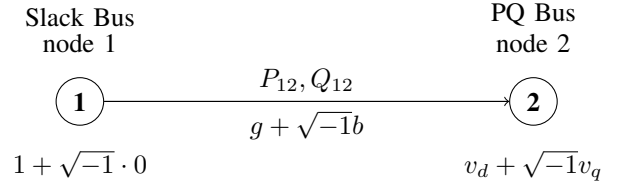


Fig. 1. Two-bus power system diagram.

V. CONCLUSION

The presented structural results are expected to facilitate the development of more efficient solvers for the relevant nonlinear AC power flow problems. One possible approach is to utilize the Algebraic Geometry theorem in constructing systems for polynomial equations, thereby enabling the study of larger zero-dimensional systems. Furthermore, there is a need to develop methods for solving the optimal power flow problem, with a complexity that is only super-polynomial by using the resultant and multi-homogeneous system concept. Addressing these challenges can potentially lead to significant advancements in the analysis of power systems.

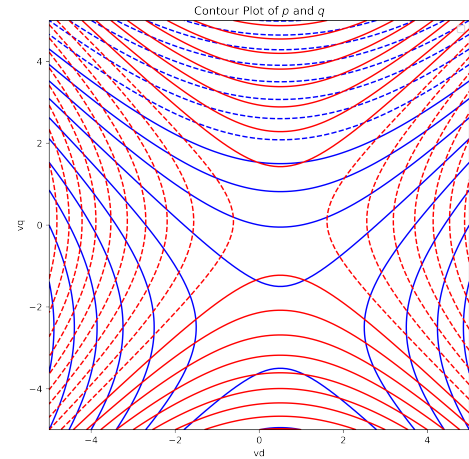


Fig. 2. The algebraic functions of p and q in two-bus test system

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¹<https://github.com/msdbarati/Resultant-of-AC-Power-Flow-Under-Multi-homogeneous-of-Algebraic-Geometry-Lense>

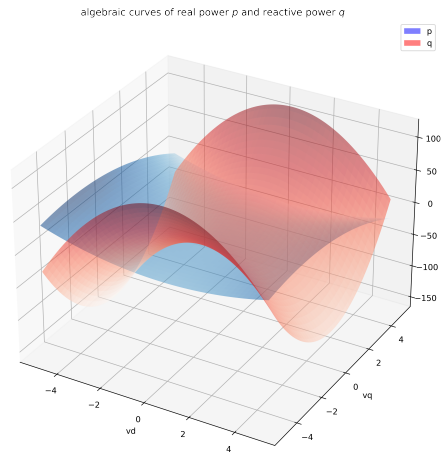


Fig. 3. The algebraic geometry representation of surfaces p and q

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