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OPTIMAL QUANTIZER SCHEDULING AND CONTROLLER SYNTHESIS FOR PARTIALLY OBSERVABLE LINEAR SYSTEMS*

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Abstract. In networked control systems, the sensory signals are often quantized before being transmitted to the controller. Consequently, performance is affected by the coarseness of this quantization process. Modern communication technologies allow users to obtain resolution-varying quantized measurements based on the prices paid. In this paper, we consider the problem of joint optimal controller synthesis and quantizer scheduling for a partially observed quantized-feedback linear-quadratic-Gaussian system, where the measurements are quantized before being sent to the controller. The system is presented with several choices of quantizers, along with the cost of using each quantizer. The objective is to jointly select the quantizers and synthesize the controller to strike an optimal balance between control performance and quantization cost. When the innovation signal is quantized instead of the measurement, the problem is decoupled into two optimization problems: one for optimal controller synthesis, and the other for optimal quantizer selection. The optimal controller is found by solving a Riccati equation and the optimal quantizer-selection policy is found by solving a linear program—both of which can be solved offline.

Key words. quantized optimal control, communication constrained control

MSC codes. 49N10, 93E03, 49N35, 93C41, 49K45, 94A05

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1. Introduction.

Networked control systems operating under finite data-rate constraints employ signal quantization to reduce the amount of data for communication. System-specific quantizers (encoders) and decoders are designed to compress signals with a finite number of bits and to incur minimal signal reconstruction errors, respectively. The available bit-rate to quantize the signals, as well as the choice of the quantizers and the decoders, determine the error in the reconstructed signal and, consequently, they affect the performance of the control system [20, 27]. Often, these quantizers are required to be time varying and their dynamics are tied to the dynamics of the control system for optimal performance [27].

Time-varying quantizers provide the flexibility to send high resolution quantized signals when needed, and use a coarser resolution otherwise. Typically, design of dynamic quantizers requires solving a joint optimization problem for the quantizer and the controller [40] to obtain optimal performance. Such codesign problems quickly become intractable due to the nonlinear/saturation behavior of the quantization process. Even the linear-quadratic optimal control problem—which is one of the simplest problems in optimal control for which an analytical closed-form solution exists – becomes intractable when quantized measurements are fed back to the controller. In [14] the authors show the lack of a *separation principle* for a linear-quadratic-Gaussian (LQG) system with quantized feedback. In [40] the authors demonstrate that a separation principle exists when *predictive quantizers* are used. Furthermore, [40] also

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demonstrated that the use of predictive quantizers can be made without loss of generality. Similar results on the separation principle can also be found in [3, 5, 29, 35]. The interested reader is referred to the article in [41] that specifically discusses the separation principle under quantization. While these works provide some characterization of the optimal quantizer, the exact solution of the optimal quantizer is not available. Owing to the intractability of the problem, prior works do not readily provide the optimal quantizers. An exception is [3], where an iterative method is proposed to find a quantizer and a controller for LQG systems. In principle, this iterative method converges in the special case of open-loop encoder systems. However, such a convergence is likely to happen at a local optimum. Besides, as mentioned in that work, this iterative method does not necessarily converge for the general case with partial side information. In summary, all these results point to the fact that finding the optimal quantizers is an intractable, still unsolved, problem.

One of the key motivations for this work is to revisit this decades-old problem from a new perspective and propose a new formulation where the problem becomes tractable and scalable. To circumvent the intractability associated with this problem, we formulate a modified problem by which one selects the optimal quantizer from a given finite collection of quantizers. This way, our formulation is one of quantizer scheduling/selection, where the best quantizer at each time instance is selected from a given finite set. Beyond computational tractability, another motivation for considering this framework stems from the fact that the optimal quantizer is time varying (with complex dynamics) and, therefore, physically changing/reconfiguring the quantizer on-the-fly is impractical for several applications (e.g., high-speed robotics). On the other hand, scheduling a quantizer from a given collection of quantizers is easy, practically feasible, and highly desirable. Furthermore, this set of quantizers can be optimized beforehand and could include any number, from a few quantizers to a large number of quantizers, depending on the scale of the application. Although the idea of considering a pre-designed set of quantizers is new to the controls community, such systems have already been used in signal processing applications [26, 30]. Last, we note that the vast majority of the existing work on quantized LQG systems is for a single system. The extension of the existing results to multiple systems is not straightforward. More importantly, the resource allocation problem (e.g., which system gets how much data rate) is extremely challenging even for a simple communication system. Our proposed framework, on the other hand, can provide a *scalable* formulation to this problem, where the controller synthesis for each system can be decoupled. The coupling occurs through the quantizer-selection part, where no more than a single system can select the same quantizer.

In this paper, we restrict ourselves to a single system, and to this end, we consider a partially observed linear system that can choose from a given set of quantizers to quantize its measurements (or a function thereof) and transmit the resulting quantized signal to the controller. The system can schedule different quantizers at different time instances to meet the need for time-varying quantizer resolution. We further assume that these quantizers are costly to use, and different quantizers have possibly different costs of operation. The performance of the system is thus measured by an expected quadratic cost plus the total cost of using the quantizers. Quantizers with higher resolution are generally more costly than ones with lower resolution. Therefore, better control performance can be achieved at the expense of a higher quantization cost. This way, our framework provides a control-quantization tradeoff, where the selection of the quantizers is not only dependent on the system's control objective but also depends on the incurred quantization cost.

In existing works (e.g., LQG optimal control [5, 34, 20], stability of systems [38, 28, 11, 7, 13, 16, 21], state estimation [37, 9, 12, 19], and Markov sources [42, 6, 15, 39]), the importance of quantization has been discussed and analyzed. However, for a given control objective, how to select and schedule from a set of available quantizers, which have a cost associated with them, has not been addressed. To the best of our knowledge, [25] is the first work where a joint optimization framework is considered to synthesize an optimal controller and schedule the optimal quantizers from a given set of costly quantizers. That work considered a fully observed linear system and considered two information structures for scheduling the quantizers, namely, the *perfect-measurement based* and the *quantized measurement based* quantizer-selection policies. In this work, we adopt the quantized measurement based quantizer-selection policy for partially observed linear systems with noisy sensors. Moreover, we consider quantizer-specific time-varying delays and analyze the effects of these delays in control performance. It is to be noted that while some existing works may have also considered delays, such delays, however, are either time invariant and independent of the quantizers or they do not introduce out-of-order packet arrivals. While keeping the relationship between the delay and the quantizer to be generic in our analysis, we also discuss the special case where the delay is proportional to the number of bits produced by the quantizers. That is, a coarsely quantized message is less delayed than a finely quantized one. Therefore, the question of quality-versus-freshness of data is naturally integrated within the proposed framework. Age-of-Information is an emerging topic in communication and information theory, where the freshness of the data is of paramount interest. We are able to couple measurement quality and freshness in the context of an LQG cost function and, more importantly, the formulated optimization problem helps in trading-off freshness-versus-quality through the choice of the quantizers. This inherently makes the problem difficult since the experienced delay is a function of the quantizer-selection process and this coupling makes the problem more complicated than the standard scenario where the delay is constant or not affected by the choice of the quantizer. The optimal quantizer scheduling in our case may result in an out-of-order delivery of the quantized signals to the controller, an effect that is quantified and rigorously incorporated into our analysis. Therefore, the control performance is affected by the selection of quantizers not only through the coarseness of the received messages, but also through their freshness. This delay and out-of-order delivery, along with the lack of perfect state measurements, require a different mathematical approach than that of [25].

Contributions. In contrast to the majority of existing works that consider infinite horizon problems and study the asymptotic behavior of the system, we focus on studying the joint quantizer-selection and controller design problem over a finite horizon. We show that quantizing the *innovation signal* separates the controller synthesis problem from the quantizer-selection problem, similarly to the case of predictive quantization. While the idea of innovation-quantization was originally proposed in [5] for a fully observed system with a deterministic initial state and later on used for the quantizer-selection problem in [25], in this work, we extend the innovation-quantization idea for partially observed systems with noisy measurements and uncertain initial states. Furthermore, we explicitly consider the quantizer induced time-varying delays in the arrival of the measurements at the controller. We study the optimal controller and show that the controller is of a certainty-equivalence type where the control gains can be computed offline and they do not depend on the parameters of the quantizers. The analysis of the quantizer-selection problem reveals that the optimal strategy for the selection of the quantizers can also be computed offline by solving a linear program. The objective function of the linear program encapsulates the tradeoff between coarser-but-faster measurement availability versus finer-but-delayed measurement availability. Furthermore, this tradeoff is coupled with the control cost function.

The rest of the paper is organized as follows: in section 2 we discuss some background on random variables; in section 3 we formally define the problem addressed in this paper; section 4 provides the structure of the optimal controller and the quantizer-selection scheme. Finally, we conclude the paper in section 7.

2. Preliminaries.

In this section we provide some background on random variables. In particular, Lemmas 2.1 and 2.2 will be used in our later derivations.

Define the probability space $(\Omega, \mathsf{F}, \mathsf{P})$, where Ω is the sample space, F is the set of events, and the measure $\mathsf{P}: \mathsf{F} \to [0,1]$ defines the probability of an event occurring. In this probability space, $X: \Omega \to \mathcal{X}$ is a random variable defined as a measurable function from the sample space Ω to a measurable space $\mathcal{X}$, such that for any measurable set $S \subseteq \mathcal{X}$, $X^{-1}(S) = \{\omega \in \Omega : X(\omega) \in S\} \in \mathsf{F}$. $\mathsf{E}[X]$ denotes the expected value of X with respect to P, defined as $\mathsf{E}[X] = \int_\Omega X(\omega)\mathrm{d}\mathsf{P}(\omega)$.

Let us define the space $\mathcal{H}$ of real-valued ($\mathcal{X} = \mathbb{R}$) random variables $X: \Omega \to \mathbb{R}$ such that $\mathcal{H} = \{X \mid \mathsf{E}[X^2] < \infty\}$. For $X, Y \in \mathcal{H}$, $\alpha X + \beta Y \in \mathcal{H}$ for all $\alpha, \beta \in \mathbb{R}$. The inner product in $\mathcal{H}$ is defined by $\langle X, Y\rangle = \mathsf{E}[XY]$.

Fact 1 [24, section 4.2]: $\mathcal{H}$ is a Hilbert space.

Let $X_1, \dots, X_\ell$ be a collection of ℓ random variables belonging to $\mathcal{H}$. The σ-field generated by these random variables is denoted by $\sigma(X_1, \dots, X_\ell)$, and the linear span of these random variables is denoted by $\mathcal{L}(X_1, \dots, X_\ell) \triangleq \{Y | Y = \sum_{i=1}^{\ell} c_i X_i, c_i \in \mathbb{R}\}$. The function $g(X_1, \dots, X_\ell): \mathbb{R}^\ell \to \mathbb{R}$ is a measurable function of the random variables $X_1, \dots, X_\ell$ if $g^{-1}(S) \in \sigma(X_1, \dots, X_\ell)$ for all measurable $S \subseteq \mathbb{R}$. Let $\mathcal{G}$ denote the set of all such measurable functions $g(X_1, \dots, X_\ell)$.

The following lemma is adapted from [31, Theorem 3.6].

LEMMA 2.1. *For any random variable Y, the solution to the optimization problem*

$$\inf_{g \in \mathcal{G}} \mathsf{E}[(Y-g)^2]$$

is $g^*(X_1, \dots, X_\ell) = \mathsf{E}[Y | X_1, \dots, X_\ell]$.

The following lemma, presented without proof, states that in the case of Gaussian random variables the conditional expectation can be represented as an affine combination of $X_1, \dots, X_\ell$.

LEMMA 2.2 (see [10, Chapter 11]). *Let $Y, X_1, \dots, X_\ell$ be jointly Gaussian random variables. Then, there exists $c_0, \dots, c_\ell \in \mathbb{R}$ such that*

$$\mathsf{E}[Y | X_1, \dots, X_\ell] = c_0 + \sum_{i=1}^{\ell} c_i X_i \in \mathcal{L}(1, X_1, \dots, X_\ell).$$

The study in [1] provides necessary and sufficient conditions for the conditional expectation $\mathsf{E}[Y | X_1, \dots, X_\ell]$ to be a linear function of $X_1, \dots, X_\ell$ when the variables are not jointly Gaussian. The previous definitions and lemmas can be extended to multidimensional random variables [24, 4, 31, 10].

3. Problem formulation.

Let us consider a discrete-time partially observed linear stochastic system

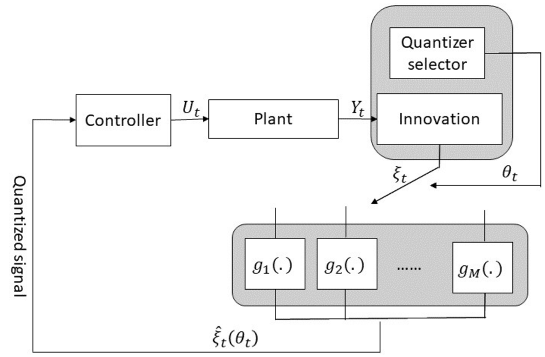

FIG. 1. *Schematic diagram of the system. The top-right gray block contains the quantizer selector that selects the optimal quantizer at each time, and the innovation block that produces the innovation signals from the measurements. The down-right gray block contains the set of M quantizers whose outputs are sent through the communication channel to the controller.*

$$X_{t+1} = A_t X_t + B_t U_t + W_t, \tag{3.1}$$

$$Y_t = C_t X_t + \nu_t, \tag{3.2}$$

where, for all $t \in \mathbb{N}_0$ $(= \mathbb{N} \cup \{0\})$, $X_t \in \mathbb{R}^n$, $U_t \in \mathbb{R}^m$, and $Y_t \in \mathbb{R}^p$, A_t, B_t, and C_t are matrices of compatible dimensions, $\{W_t\}_{t\in\mathbb{N}_0}$ and $\{\nu_t\}_{t\in\mathbb{N}_0}$ are two independent and identically distributed (i.i.d.) noise sequences in $\mathbb{R}^n$ and $\mathbb{R}^p$ with statistics $W_0 \sim \mathcal{N}(0, \mathcal{W})$ and $\nu_0 \sim \mathcal{N}(0, \mathcal{V})$, respectively, and W_k, ν_j are independent for all $j, k \in \mathbb{N}_0$. The initial state, X_0, is also a Gaussian random variable distributed according to $\mathcal{N}(\mu_0, \Sigma_x)$, and independent of the noises W_t and ν_t for all $t \in \mathbb{N}_0$. For notational convenience, we will write $X_0 = \mu_0 + W_{-1}$, where $W_{-1} \sim \mathcal{N}(0, \Sigma_x)$. Thus, X_0, W_k, W_ℓ, ν_i, and ν_j are independent random variables for all $k, \ell, i, j = 0, 1, \ldots$, such that $k \neq \ell$ and $i \neq j$. In what follows, we will consider A_t, B_t, and C_t to be time invariant in order to maintain notational brevity.

In this work, we address the quantized output feedback LQG optimal control problem defined as follows. Referring to Figure 1, we assume that M quantizers are provided to quantize the measurement Y_t and transmit the quantized output to the controller. The range of the ith quantizer is denoted by $\mathcal{Q}^i = \{q_1^i, q_2^i, \cdots, q_{\ell_i}^i\}$, where each q_j^i is a symbol. Thus, the ith quantizer has ℓ_i quantization levels. Without any loss of generality, we assume that $\ell_1 \leq \cdots \leq \ell_M$. Associated with the ith quantizer, let $\mathcal{P}^i = \{\mathcal{P}_1^i, \mathcal{P}_2^i, \cdots, \mathcal{P}_{\ell_i}^i\}$ denote a partition of $\mathbb{R}^p$ such that $\mathcal{P}_j^i$ gets mapped to symbol q_j^i for each $j \in \{1, 2, \cdots, \ell_i\}$. Specifically, one may think of the ith quantizer as a mapping $g_i : \mathbb{R}^p \to \mathcal{Q}^i$ such that $g_i(y) = q_j^i$ if and only if $y \in \mathcal{P}_j^i$.

The quantized measurements are transmitted through a communication channel that has a finite data-rate. Consequently, some quantized measurements may need more than one time step to complete the sensor-to-controller transmission and the decoding at the controller's site [2] and, hence, the availability of that measurement to the controller will be delayed. Furthermore, quantized signals of different lengths may experience different amounts of delay and, hence, out-of-order measurement availability is inevitable [18]. In this work, we do not adhere to any particular model for characterizing this delay, rather, we simply consider the case where a quantized signal with a larger number of bits may experience a longer delay before it is available to the controller. That is, the delay d_i associated with the ith quantizer is nondecreasing with i, i.e., $d_1 \le d_2 \le \cdots \le d_M$. The number of quantization levels ℓ_i generally captures the resolution of the quantization, i.e., a higher ℓ_i typically means a better resolution and lesser quantization error, but, at the same time, it induces longer delay d_i. Therefore, this work will also reveal the tradeoff between choosing a coarser but faster quantization service versus a finer but delayed service. In fact, we will see later on that, for a finite-horizon optimal control problem, different resolution-delay (finer-delayed versus coarser-faster) characteristics are preferred at different times.

Associated with each quantizer there is an operating cost that must be paid in order to use this quantizer. Let $\lambda(\mathcal{Q}^i) = \lambda_i \in \mathbb{R}_+$ denote the cost associated with using the ith quantizer. This cost may also include communication cost or computation/data-processing cost or both. For example, $\lambda_i \propto \log_2 \ell_i$ represents the case where the cost is proportional to the code-length of the encoded quantizer output. This cost may also be related to the delay d_i associated with the quantizer. Furthermore, this cost may also be time varying to regulate the system's quantizer preference with the time-varying availability of the communication resources. Similarly, $\lambda_i \propto \phi(\mathcal{P}^i)$ represents a cost that is proportional to the average complexity of encoding an input to its right symbol q_j^i and decoding it at the controller ($\phi(\cdot)$ denotes the encoding and decoding computation complexity). In this work, we do not adhere to any specific structure for λ. We just assume that the values of λ_i's are given to us a priori. This cost can be appropriately designed depending on the applications.

Note that, in contrast to previous works [37, 13], we do not aim at designing a quantization scheme; rather, a set of quantizers is already given by some service provider. For a given horizon $[0, T]$, our objective is to find the optimal schedule for the quantizers. Also, we will assume that the costs λ_i are determined by the service provider and presented to us a priori. Designing such costs in order to regulate the use of the quantizers is an equally interesting problem for the service provider that will be addressed elsewhere. We will further assume that the communication channel between each quantizer and the controller always transmits the quantized information without any distortion.

The objective is to minimize a performance index that takes into account the quantization cost. Contrary to some of the existing literature on the quantization-based LQG problem [5, 33, 34, 35, 36, 22], in our case there are two decision makers instead of a single one: One decision maker (the controller) decides the input ($\{U_t\}_{t\in\mathbb{N}_0}$) to apply to the system, and the other decision maker (the quantizer selector) decides the quality and delay of the measurements (quantized state values) which are transmitted to the controller. To that end, we introduce a new decision variable θ_t^i for the quantizer selector in the following way:

$$\theta_t^i = \begin{cases} 1, & i\text{th quantizer is used at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Let us denote the vector $\theta_t \triangleq [\theta_t^1, \theta_t^2, \ldots, \theta_t^M]^\intercal \in \{0,1\}^M$, that characterizes the decision of the quantizer selector at time t. We enforce the quantizer selector to select exactly one quantizer at any time instance and, hence for all $t \in \mathbb{N}_0$, we have

$$\sum_{i=1}^{M} \theta_t^i = 1. \tag{3.3}$$

The decoded measurement(s) available to the controller at time t is denoted as $\hat{O}_t$. Note that $\hat{O}_t$ may contain delayed quantized measurements; also, several measurements may be made available simultaneously at the controller. The delay and

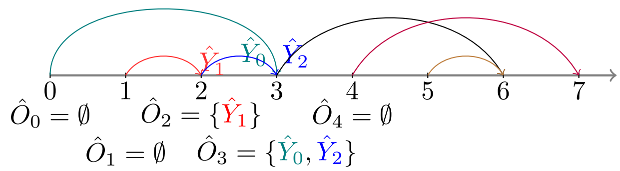

FIG. 2. *Out-of-order measurement availability at the controller when the second quantizer (with delay 3) is selected at times $t = 0, 3, 4$ and the first quantizer (with delay 1) is selected at other time instances. The new decoded measurements available at time t at the controller is $\hat{O}_t$, i.e., $\hat{O}_0 = \hat{O}_1 = \emptyset, \hat{O}_2 = \{\hat{Y}_1\}, \hat{O}_2 = \{\hat{Y}_0, \hat{Y}_2\}$, and so on. In this example, $\hat{Y}_1$ is available before $\hat{Y}_0$ and $\hat{Y}_5$ is available before $\hat{Y}_4$.*

out-of-order arrival are time varying and dependent on the choice of the θ_i's. Therefore, the θ_i's not only affect the coarseness of the quantization process, but also affect the delays in the measurement arrivals. For example, as shown in Figure 2, if there are two quantizers with $d_1 = 1$ and $d_2 = 3$, and if the second quantizer is selected at time 0 followed by the selection of the first quantizer at times $t = 1, 2$, then no decoded measurements are available at times $t = 0, 1$, i.e., $\hat{O}_0 = \hat{O}_1 = \emptyset$. The decoded information about Y_1, denoted as $\hat{Y}_1$, is available at time $t = 2$, i.e., $\hat{O}_2 = \{\hat{Y}_1\}$, and the decoded information about Y_0 and Y_2 are available simultaneously at time $t = 3$, i.e., $\hat{O}_3 = \{\hat{Y}_0, \hat{Y}_2\}$. Thus, $\hat{O}_t$ is a function of $\{\theta_0, \ldots, \theta_t\}$ (to be precise, $\hat{O}_t$ is only a function of $\{\theta_{t-d_i} : i = 1, \ldots, M, \; t - d_i \geq 0\}$). A detailed description of $\hat{O}_t$ will be provided later in section 4.2.

To streamline the discussion, we introduce the following sets at time t: $\mathcal{Y}_t \triangleq \{Y_0, Y_1, \ldots, Y_t\}$ is the measurement history set, $\hat{\mathcal{O}}_t \triangleq \{\hat{O}_0, \hat{O}_1, \cdots, \hat{O}_t\}$ is the set of the quantized measurement history at the controller, $\mathcal{U}_t \triangleq \{U_0, U_1, \cdots, U_t\}$ is the control history set, and $\Theta_t \triangleq \{\theta_0, \theta_1, \cdots, \theta_t\}$ is the quantization-selection history set.

The information available to the controller at time t is $\mathfrak{I}^c_t = \{\hat{\mathcal{O}}_t, \mathcal{U}_{t-1}\} = \mathfrak{I}^c_{t-1} \cup \{\hat{O}_t, U_{t-1}\}$, where $\mathfrak{I}^c_0 = \{\hat{O}_0\}$. It should be noted that $\mathfrak{I}^c_t$ depends on Θ_t through $\hat{\mathcal{O}}_t$. In classical optimal LQG control, the information available to the controller is not decided by any active decision maker, unlike the situation here. An admissible control strategy at time t is a measurable function from the Borel σ-field generated by $\mathfrak{I}^c_t$ to $\mathbb{R}^m$. Let us denote such strategies by $\gamma^u_t(\cdot)$ and the space they belong to by Γ^u_t.

On the other hand, the information available to the quantizer selector at time t is $\mathfrak{I}^q_t = \{\mathcal{Y}_t, \hat{\mathcal{O}}_{t-1}, \mathcal{U}_{t-1}, \Theta_{t-1}\} = \mathfrak{I}^q_{t-1} \cup \{Y_t, \hat{O}_{t-1}, U_{t-1}, \theta_{t-1}\}$, where $\mathfrak{I}^q_0 = \{Y_0\}$. We will use the information $\{\hat{\mathcal{O}}_{t-1}, \Theta_{t-1}\} \triangleq \bar{\mathfrak{I}}^q_t \subset \mathfrak{I}^q_t$ to schedule a quantizer at time t. Furthermore, we will quantize the *innovation* signal $\xi_t = Y_t - \mathsf{E}[Y_t | Y_0, \ldots, Y_{t-1}]$ at time t and send the quantized version to the controller. It should be noted that the proposed structure is suboptimal.[1] However, we impose this structure to make the problem tractable and obtain a solution that is computationally inexpensive. Otherwise, the general problem is intractable even for simple cases; for example, see [12, 8, 19, 32] and the references therein. Quantizing the innovation signal not only makes the problem tractable, but also allows us to show that a separation principle between control and quantizer-selection is retained. The existence of such a separation principle has been noted in earlier works as well; for example, see [5, 3, 35, 40, 41]. It is well known [17] that the information contained in the innovation signals $\{\xi_0, \ldots, \xi_t\}$ is the

[1]It can be shown (see [40], for example) that a predictive coding structure, i.e., quantizing $\sum_{k=0}^{t} \Psi(t,k)\xi_k$ maintains optimality. The fixed matrices $\Psi(t,k)$ are derived later in (4.8). However, the problem becomes highly complex and intractable due to the presence of past signals $\xi_0, \ldots, \xi_{t-1}$ when considering a predictive quantization scheme.

same as the information contained in the observations $\{Y_0, \dots, Y_t\}$. Therefore, designing an output-feedback controller is equivalent to designing an innovation-feedback controller. However, after quantization, the information contained in the quantized innovations is not necessarily the same as the information contained in the quantized outputs, and that is precisely why, in general, it cannot be claimed that the performance of the optimal output-quantized feedback controller will be the same as that of the optimal innovation-quantized feedback.

The admissible strategies for the selection of the quantizers are measurable functions from the Borel σ-field generated by $\bar{\mathfrak{I}}_t^q$ to $\{0,1\}^M$ and satisfying (3.3). Let us denote such strategies by $\gamma_t^\theta(\cdot)$, and the space they belong to by Γ_t^θ. Thus, the entire quantization process is characterized by the following two equations:

$$\xi_t = Y_t - \mathsf{E}[Y_t | Y_0, \dots, Y_{t-1}], \tag{3.4a}$$

$$\theta_t = \gamma_t^\theta(\bar{\mathfrak{I}}_t^q). \tag{3.4b}$$

For brevity, we will often use γ_t^u instead of $\gamma_t^u(\cdot)$ or $\gamma_t^u(\mathfrak{I}_t^c)$, and γ_t^θ in place of $\gamma_t^\theta(\cdot)$ or $\gamma_t^\theta(\bar{\mathfrak{I}}_t^q)$. Let γ^Θ denote the entire sequence $\{\gamma_0^\theta, \gamma_1^\theta, \dots, \gamma_{T-1}^\theta\}$ and let Γ^Θ denote the space γ^Θ belongs to. Likewise, $\gamma^{\mathcal{U}}$ and $\Gamma^{\mathcal{U}}$ are defined similarly. Let us also define $\mathfrak{I}^c = \{\mathfrak{I}_t^c\}_{t=0}^{T-1}$ and $\mathfrak{I}^q = \{\mathfrak{I}_t^q\}_{t=0}^{T-1}$. The cost function to be minimized cooperatively by the quantizer selector and the controller is a finite horizon expected quadratic criterion, given as

$$J(\mathcal{U}_{T-1}, \Theta_{T-1}) = \mathsf{E}\left[\sum_{t=0}^{T-1} (X_t^\intercal Q_1 X_t + U_t^\intercal R U_t + \theta_t^\intercal \Lambda) + X_T^\intercal Q_2 X_T\right], \tag{3.5}$$

where $\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_M]^\intercal$ is the cost for quantization, $Q_1, Q_2 \succeq 0$, $R \succ 0$, $\mathcal{U} = \gamma^{\mathcal{U}}(\mathfrak{I}^c) = \{\gamma_0^u(\mathfrak{I}_0^c), \gamma_1^u(\mathfrak{I}_1^c), \dots, \gamma_{T-1}^u(\mathfrak{I}_{T-1}^c)\}$ and $\Theta = \gamma^\Theta(\bar{\mathfrak{I}}^q) = \{\gamma_0^\theta(\bar{\mathfrak{I}}_0^q), \gamma_1^\theta(\bar{\mathfrak{I}}_1^q), \dots, \gamma_{T-1}^\theta(\bar{\mathfrak{I}}_{T-1}^q)\}$. For convenience, we will use the notation $\mathcal{U}$ for $\mathcal{U}_{T-1}$ and, likewise, we will use Θ for Θ_{T-1}. We seek to find the optimal strategies $\gamma^{\mathcal{U}*} = \{\gamma_0^{u*}, \gamma_1^{u*}, \dots, \gamma_{T-1}^{u*}\}$ and $\gamma^{\Theta*} = \{\gamma_0^{\theta*}, \gamma_1^{\theta*}, \dots, \gamma_{T-1}^{\theta*}\}$ that minimize (3.5). We will also rewrite (3.5) in terms of $\gamma^{\mathcal{U}}$ and γ^Θ as

$$\begin{aligned} J(\gamma^{\mathcal{U}}, \gamma^\Theta) = \mathsf{E}\Bigg[&\sum_{t=0}^{T-1} (X_t^\intercal Q_1 X_t + U_t^\intercal R U_t + \theta_t^\intercal \Lambda) + X_T^\intercal Q_2 X_T \\ &\,|\, U_t = \gamma_t^u(\mathfrak{I}_t^c), \theta_t = \gamma_t^\theta(\bar{\mathfrak{I}}_t^q)\Big]. \end{aligned} \tag{3.6}$$

4. Optimal control and quantization selection. In this section we find the optimal $\gamma^{\mathcal{U}*}$ and $\gamma^{\Theta*}$ that minimize the cost function (3.6) amongst all admissible strategies, that is,

$$(\gamma^{\mathcal{U}*}, \gamma^{\Theta*}) = \underset{\gamma^{\mathcal{U}} \in \Gamma^{\mathcal{U}}, \gamma^\Theta \in \Gamma^\Theta}{\arg\min} J(\gamma^{\mathcal{U}}, \gamma^\Theta). \tag{4.1}$$

Before proceeding further to solve (4.1), let us discuss, in some detail, the input for the quantization process (i.e., the innovation signal) since it will play a crucial role in the following analysis.

4.1. The innovation process. The control U_t is a function of the quantized innovations which are not Gaussian random variables. Therefore, the state X_t and the measurement Y_t are no longer Gaussian random variables under quantized innovation feedback. Although the innovation signal is a Gaussian random variable for partially observed classical LQG systems without quantization, in our case, this may no longer be true since the control is a function of quantized signals (which are not Gaussian random variables). We therefore need to independently verify whether the distribution of the innovation signal is Gaussian or not.

It can be verified that the innovation ξ_t is not affected by the control strategy, although, Y_t is affected. Furthermore, the innovation ξ_t retains its Gaussian distribution and the parameters of this distribution can be computed offline. This observation is presented in the following proposition.

PROPOSITION 4.1. *For all t, ξ_t is a Gaussian random variable with zero mean and covariance M_t such that*

$$\begin{aligned} M_{t+1} &= C\Sigma_{t+1|t}C^\intercal + \mathcal{V}, \\ \Sigma_{t+1|t} &= A\Sigma_t A^\intercal + \mathcal{W}, \quad \Sigma_{0|-1} = \Sigma_x, \\ \Sigma_{t+1} &= \Sigma_{t+1|t} - \Sigma_{t+1|t}C^\intercal M_{t+1}^{-1} C\Sigma_{t+1|t}. \end{aligned}$$

Moreover, the sequence of random variables $\{\xi_0, \dots, \xi_t\}$ is uncorrelated for all t.

Proof. The proof is presented in Appendix A. ◻

Proposition 4.1 is equivalent of the following facts:

1. The innovation sequence $\{\xi_t\}_{t\in\mathbb{N}_0}$ does not depend on the control history $\mathcal{U}_{t-1}$.
2. The innovation sequence is a Gaussian uncorrelated noise sequence with zero mean and covariance M_t.
3. Since the sequence of random variables $\{\xi_t\}_{t\in\mathbb{N}_0}$ is uncorrelated and Gaussian, each ξ_t and ξ_k is independent for all $k \neq t$.

4.2. Implications of delay. Let $g_i(\xi_t) \in \mathcal{Q}^i$ denote the quantized version of ξ_t if the ith quantizer is selected. Notice that $g_i(\xi_t) \in \mathcal{Q}^i$ is a random variable. The quantized information sent to the controller is

$$\hat{\xi}_t = \sum_{i=1}^{M} g_i(\xi_t)\theta_t^i, \tag{4.2}$$

and this information will be decoded and available at the controller at time $t + \sum_{i=1}^M \theta_t^i d_i$. It is noteworthy that, unlike the infinite horizon problem, the measurements arriving after time T are of no use to the controller for an optimal control problem defined over the horizon $[0, T]$. Therefore, even though a quantization cost is paid, such delayed information does not help in computing the control input and, thus, it does not help in reducing the objective cost (3.5). Therefore, the delay must be appropriately incorporated in the analysis so that the above scenario is avoided.

In addition, since the delays may result in out-of-order availability of the decoded signal to the controller, it is important that every quantized signal is time stamped, i.e., when the controller receives a decoded measurement $\hat{q}$ at time t, it should be able to uniquely determine which of the signals $\{\xi_0, \dots, \xi_t\}$ was quantized to produce this measurement along with the quantizer that was used. In order to uniquely decode which of the signals $\{\xi_0, \dots, \xi_t\}$ produced the data $\hat{q}$, the pair $(\hat{\xi}_t, i)$ will be sent at each time t, where i is the index of the quantizer that was used to quantize ξ_t. Consequently, if the pair $(\hat{q}, i)$ is received by the controller at time t, then the controller can immediately infer that the ith quantizer was used and that this signal is delayed by d_i units and, hence, $\hat{q}$ corresponds to ξ_{t-d_i}. Thus, $(\hat{q}, i)$ reveals that $\theta_{t-d_i}^i = 1$ and $\hat{q} = g_i(\xi_{t-d_i})$. At any time t, there can be at most M (delayed) new simultaneously available decoded measurements. We define the set of indexes present in $\hat{O}_t$ by

$$\mathrm{idx}_t = \{i : \exists q \in \mathbb{R}^p \text{ s.t. } (q,i) \in \hat{O}_t\} \subseteq \{1,\ldots,M\}.$$

Therefore, $\theta^i_{t-d_i} = 1$ if $i \in \mathrm{idx}_t$, otherwise $\theta^i_{t-d_i} = 0$. It follows that the new decoded measurements available to the controller at time t can be expressed as

$$\{\theta^1_{t-d_1},\ldots,\theta^M_{t-d_M}\} \cup \{\hat{\xi}_{t-d_i} : i \in \mathrm{idx}_t\}.$$

With a slight abuse of notation, the above set is equivalent to:

$$\left\{\theta^1_{t-d_1},\ldots,\theta^M_{t-d_M},\theta^1_{t-d_1}\hat{\xi}_{t-d_1},\ldots,\theta^M_{t-d_M}\hat{\xi}_{t-d_M}\right\}.$$

Notice that there is a bijective relationship between $\hat{O}_t$ and the set $\{\theta^1_{t-d_1},\ldots,\theta^M_{t-d_M}, \theta^1_{t-d_1}\hat{\xi}_{t-d_1},\ldots,\theta^M_{t-d_M}\hat{\xi}_{t-d_M}\}$. Therefore, for notational brevity, we will simply write

$$\hat{O}_t = \left\{\theta^1_{t-d_1},\ldots,\theta^M_{t-d_M},\theta^1_{t-d_1}\hat{\xi}_{t-d_1},\ldots,\theta^M_{t-d_M}\hat{\xi}_{t-d_M}\right\}. \tag{4.3}$$

Having characterized the effects of delays in the information available to the controller, we next discuss the optimal controller that minimizes cost (3.6).

4.3. Optimal control policy. Let us define the innovation history by $\Xi_t \triangleq \{\xi_0,\ldots,\xi_t\}$. With a slight abuse of notation, we also denote $\Xi_t = \sigma(\xi_0,\ldots,\xi_t)$ to be the σ-field generated by these innovation signals. We then define the state estimate by

$$\bar{X}_t \triangleq \mathsf{E}[X_t|\mathfrak{I}^c_t]. \tag{4.4}$$

Recall from section 3 that $\hat{\mathcal{O}}_t = \{\hat{O}_0,\hat{O}_1,\ldots,\hat{O}_t\}$. Based on (4.3), the set $\hat{\mathcal{O}}_t$ can now be expressed as $\hat{\mathcal{O}}_t = \{\vartheta_{0,t}\hat{\xi}_0,\vartheta_{1,t}\hat{\xi}_1,\ldots,\vartheta_{t,t}\hat{\xi}_t\} \cup_{k=0}^t \{\theta^i_{k-d_i} : i=1,\ldots,M, k \geq d_i\}$, where $\vartheta_{k,t}$, as expressed below, is an indicator of whether $\hat{\xi}_k$ is available to the controller by time instant t or not:

$$\vartheta_{k,t} = \sum_{i=0}^M \theta^i_k \mathbb{1}_{d_i \leq t-k}. \tag{4.5}$$

Clearly, if $t-k \geq d_M$ for some k, then the above expression for $\vartheta_{k,t}$ becomes $\vartheta_{k,t} = \sum_{i=0}^M \theta^i_k = 1$ ensuring that the quantized version of ξ_k is present at the controller.

Similarly to $\hat{\mathcal{O}}_t$, let us define the set $\mathcal{O}_t = \{\vartheta_{0,t}\xi_0,\vartheta_{1,t}\xi_1,\ldots,\vartheta_{t,t}\xi_t\} \cup_{k=0}^t \{\theta^i_{k-d_i} : i=1,\ldots,M, k \geq d_i\}$, which contains the innovation signals whose quantized versions are included in $\hat{\mathcal{O}}_t$. Similarly to $\hat{\mathcal{O}}_t$, the set $\mathcal{O}_t$ also contains the corresponding indexes of the quantizers that were used. Due to the construction of $\mathcal{O}_t$, $\hat{\mathcal{O}}_t$ does not contain any new information when $\mathcal{O}_t$ is given. Therefore, we have

$$\begin{aligned}\bar{X}_t &= \mathsf{E}[X_t|\mathfrak{I}^c_t] = \mathsf{E}[X_t|\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] = \mathsf{E}[\mathsf{E}[X_t|\mathcal{O}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}]|\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] \\ &= \mathsf{E}[\mathsf{E}[X_t|\mathcal{O}_t,\mathcal{U}_{t-1}]|\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}].\end{aligned} \tag{4.6}$$

In order to compute $\bar{X}_t$, we compute $\mathsf{E}[X_t|\mathcal{O}_t,\mathcal{U}_{t-1}]$ which is inside the outer expectation of the last equation.

LEMMA 4.1. *For any t,*

$$\mathsf{E}[X_t|\mathcal{O}_t,\mathcal{U}_{t-1}] = A^t\mu_0 + \sum_{k=0}^t \Psi(t,k)\vartheta_{k,t}\xi_k + \sum_{k=0}^{t-1} A^{t-1-k}BU_k, \tag{4.7}$$

and, for all $t \geq k$, the matrices $\Psi(t,k)$ are given by

$$\Psi(t,k) = A^{t-k}\Sigma_{k|k-1}C^{\mathsf{T}}M_k^{-1}. \tag{4.8}$$

Proof. The proof is given in Appendix B. ∎

Therefore, using Lemma 4.1 we obtain from (4.6) that

$$\begin{aligned}\bar{X}_t =&\mathsf{E}[\mathsf{E}[X_t|\mathcal{O}_t,\mathcal{U}_{t-1}]|\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}]\\ =&A^t\mu_0+\sum_{k=0}^{t}\Psi(t,k)\vartheta_{k,t}\mathsf{E}[\xi_k\mid\hat{\mathcal{O}}_t]+\sum_{k=0}^{t-1}A^{t-1-k}BU_k,\end{aligned} \tag{4.9}$$

where we have used the fact that U_t is a measurable function of $\mathfrak{I}_t^c=\{\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}\}$ and, hence, given $\hat{\mathcal{O}}_t$, the control history $\mathcal{U}_{t-1}$ does not provide any new information about ξ_k, i.e., $\mathsf{E}[\xi_k\mid\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}]=\mathsf{E}[\xi_k\mid\hat{\mathcal{O}}_t]$. Next, we focus on computing $\mathsf{E}[\xi_k\mid\hat{\mathcal{O}}_t]$. To that end, let us define $\bar{\xi}_t^i\triangleq\mathsf{E}[\xi_t|\hat{\xi}_t,\theta_t^i=1]$. Based on (3.3) and (4.2), we may write

$$\begin{aligned}\bar{\xi}_t^i&=\mathsf{E}[\xi_t|g_i(\xi_t),\theta_t^i=1]=\sum_{j=1}^{\ell_i}1_{g_i(\xi_t)=q_j^i}\mathsf{E}[\xi_t|g_i(\xi_t)=q_j^i,\theta_t^i=1]\\ &=\sum_{j=1}^{\ell_i}1_{g_i(\xi_t)=q_j^i}\mathsf{E}[\xi_t|\xi_t\in\mathcal{P}_j^i]=\sum_{j=1}^{\ell_i}1_{g_i(\xi_t)=q_j^i}\int_{\mathcal{P}_j^i}\xi\mathsf{P}_t(\mathrm{d}\xi|\mathcal{P}_j^i),\end{aligned}$$

where $1_{a=b}$ is an indicator function that is equal to 1 if and only if $a=b$, otherwise it equals 0. Therefore, $\bar{\xi}_t^i$ is a random variable taking values in the set $\{\int_{\mathcal{P}_j^i}\xi\mathsf{P}_t(\mathrm{d}\xi|\mathcal{P}_j^i)\,:\,j=1,\dots,\ell_i\}$ and it depends on the realization of ξ_t through $1_{g_i(\xi_t)=q_j^i}$. Using Proposition 4.1, one may compute $\mathsf{P}_t(\mathrm{d}\xi|\mathcal{P}_j^i)$ as follows:

$$\mathsf{P}_t(\mathrm{d}\xi|\mathcal{P}_j^i)=\begin{cases}\alpha_t e^{-\xi^\mathsf{T}M_t^{-1}\xi/2}\mathrm{d}\xi, & \xi\in\mathcal{P}_j^i,\\ 0, & \text{otherwise},\end{cases}$$
$$(\alpha_t)^{-1}=\sqrt{(2\pi)^p\det(M_t)}\mathsf{P}(\xi_t\in\mathcal{P}_j^i)=\int_{\mathcal{P}_j^i}e^{-\xi^\mathsf{T}M_t^{-1}\xi/2}\mathrm{d}\xi.$$

Furthermore, from Proposition 4.1, we have that $\xi_t\sim\mathcal{N}(0,M_t)$. Since M_t can be computed offline, the prior distribution of ξ_t is known to the controller. After receiving the quantized value $\hat{\xi}_t$, the controller updates the distribution of ξ_t. If the quantized value of ξ_t, after being quantized by the ith quantizer, is $\hat{\xi}_t=q_j^i$, then the controller can infer that $\xi_t\in\mathcal{P}_j^i$. This is illustrated in Figure 3.

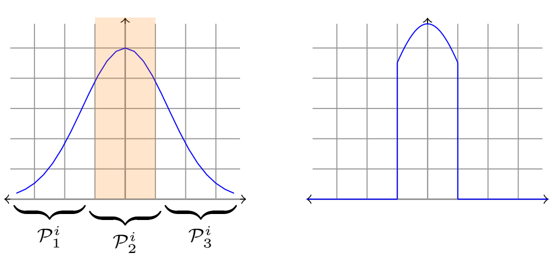

FIG. 3. *Left: The blue curve denotes the prior distribution $\mathsf{P}_t(d\xi)$. The partitions $\mathcal{P}_j^i$ for the ith quantizer are also shown, where $\mathcal{P}_2^i$ is highlighted with the orange block. Right: The posterior distribution $\mathsf{P}_t(d\xi|\mathcal{P}_2^i)$ of ξ_t is shown for the case when the received quantized measurement $\hat{\xi}_t$ is q_2^i, equivalently, $\xi_t\in\mathcal{P}_2^i$.*

The entity $\bar{\xi}_t^i$ is the expected value of ξ_t given that the ith quantizer was used in the process of quantization and the quantized value is $\hat{\xi}_t \in \mathcal{Q}^i$. We further define

$$\bar{\xi}_t \triangleq \mathsf{E}[\xi_t|\hat{\xi}_t, \theta_t] = \sum_{i=1}^{M} \theta_t^i \bar{\xi}_t^i \tag{4.10}$$

and

$$\tilde{\xi}_t \triangleq \xi_t - \bar{\xi}_t. \tag{4.11}$$

From this definition of $\bar{\xi}_t$, along with the constraint $\sum_{i=1}^M \theta_t^i = 1$, we have that $\bar{\xi}_t = \bar{\xi}_t^i$ if and only if the ith quantizer was selected at time t. The conditional covariance $\mathcal{M}_t(\theta_t) \triangleq \mathsf{E}[\tilde{\xi}_t \tilde{\xi}_t^{\mathsf{T}} \mid \theta_t]$ turns out to be

$$\begin{aligned} \mathcal{M}_t(\theta_t) =& \mathsf{E}\left[\xi_t \xi_t^{\mathsf{T}} - \xi_t \bar{\xi}_t^{\mathsf{T}} - \bar{\xi}_t \xi_t^{\mathsf{T}} + \bar{\xi}_t \bar{\xi}_t^{\mathsf{T}} \mid \theta_t\right] \\ =& \mathsf{E}[\xi_t \xi_t^{\mathsf{T}} \mid \theta_t] - \mathsf{E}[\bar{\xi}_t \bar{\xi}_t^{\mathsf{T}} \mid \theta_t] = \mathsf{E}[\xi_t \xi_t^{\mathsf{T}} \mid \theta_t] - \mathsf{E}[\bar{\xi}_t \bar{\xi}_t^{\mathsf{T}} \mid \theta_t], \end{aligned} \tag{4.12}$$

where we have used the fact that $\mathsf{E}[\xi_t \bar{\xi}_t^{\mathsf{T}} \mid \theta_t] = \mathsf{E}[\mathsf{E}[\xi_t \bar{\xi}_t^{\mathsf{T}} \mid \hat{\xi}_t, \theta_t] \mid \theta_t] = \mathsf{E}[\mathsf{E}[\xi_t|\hat{\xi}_t, \theta_t]\bar{\xi}_t^{\mathsf{T}} \mid \theta_t] = \mathsf{E}[\bar{\xi}_t \bar{\xi}_t^{\mathsf{T}} \mid \theta_t]$. By defining $F_t(\theta_t) \triangleq \mathsf{E}[\bar{\xi}_t \bar{\xi}_t^{\mathsf{T}} \mid \theta_t]$ and using the expression of $\bar{\xi}_t$ from (4.10), we obtain

$$F_t(\theta_t) = \mathsf{E}[\bar{\xi}_t \bar{\xi}_t^{\mathsf{T}} \mid \theta_t] = \sum_{i=1}^{M} \theta_t^i \mathsf{E}[\bar{\xi}_t^i \bar{\xi}_t^{i^{\mathsf{T}}}] = \sum_{i=1}^{M} \theta_t^i F_t^i, \tag{4.13}$$

where

$$F_t^i = \mathsf{E}[\bar{\xi}_t^i \bar{\xi}_t^{i^{\mathsf{T}}}] = \sum_{j=1}^{\ell_i} \mathsf{P}(\xi_t \in \mathcal{P}_j^i) \mathsf{E}[\xi_t|\xi_t \in \mathcal{P}_j^i] \mathsf{E}[\xi_t|\xi_t \in \mathcal{P}_j^i]^{\mathsf{T}}. \tag{4.14}$$

Therefore, using the definition of $F_t(\theta_t)$, we may rewrite (4.12) as $\mathcal{M}_t(\theta_t) = \mathsf{E}[\xi_t \xi_t^{\mathsf{T}} \mid \theta_t] - F_t(\theta_t)$ and, furthermore, we also obtain $\mathsf{E}[\mathcal{M}_t(\theta_t)] = M_t - \mathsf{E}[F_t(\theta_t)]$. The linear dependence of $F_t(\theta_t)$ on θ_t will be useful in designing a linear program for selecting the optimal quantizers, as shown later in the paper.

At this point, recall from Proposition 4.1 and the discussion thereafter that $\{\xi_t\}_{t\in\mathbb{N}_0}$ is a sequence of uncorrelated zero-mean Gaussian noises (hence, ξ_k, ξ_ℓ are independent for $k \neq \ell$) and $\{\hat{\xi}_t\}_{t\in\mathbb{N}_0}$ is the corresponding sequence of the quantized version of $\{\xi_t\}_{t\in\mathbb{N}_0}$. Therefore, ξ_k and $\hat{\xi}_\ell$ are independent for all $k \neq \ell$. Hence,

$$\mathsf{E}[\xi_k|\hat{\mathcal{O}}_t] = \begin{cases} \mathsf{E}[\xi_k|\hat{\xi}_k, \theta_k] = \bar{\xi}_k & \text{if } \hat{\xi}_k \in \hat{\mathcal{O}}_t, \\ \mathsf{E}[\xi_k] = 0 & \text{otherwise,} \end{cases} \tag{4.15}$$

where we have used the definition of $\bar{\xi}_t$ from (4.10). From this observation, and using Lemma 4.1, the expression of $\bar{X}_t$ is computed in the following lemma.

Lemma 4.2. *For any t, $\bar{X}_t = \mathsf{E}[X_t|\mathfrak{I}_t^c]$ is given by*

$$\bar{X}_t = A^t \mu_0 + \sum_{k=0}^{t} \Psi(t,k) \vartheta_{k,t} \bar{\xi}_k + \sum_{k=0}^{t-1} A^{t-1-k} B U_k. \tag{4.16}$$

Proof. Notice that, from (4.9) we have

$$\mathsf{E}[X_t|\mathfrak{I}_t^c] = A^t \mu_0 + \sum_{k=0}^{t} \Psi(t,k) \vartheta_{k,t} \mathsf{E}[\xi_k|\hat{\mathcal{O}}_t] + \sum_{k=0}^{t-1} A^{t-1-k} B U_k.$$

The lemma follows immediately after we substitute the expression for $\mathsf{E}[\xi_k|\hat{\mathcal{O}}_t]$ from (4.15) into the last equation and noting that $\vartheta_{k,t} = 1$ if $\hat{\xi}_k \in \hat{\mathcal{O}}_t$ and zero otherwise. ∎

Define the error $e_t \triangleq X_t - \bar{X}_t$. It follows from (4.16) that

$$e_t = A^t X_0 + \sum_{k=0}^{t-1} A^{t-k-1} W_k - A^t \mu_0 - \sum_{k=0}^{t} \Psi(t,k)\vartheta_{k,t}\bar{\xi}_k.$$

Notice that e_t does not depend on the control strategy $\gamma^{\mathcal{U}}$. However, it does depend on the quantizer-selection strategy γ^{Θ} through the last term in the above equation. Furthermore, for all t, $\mathsf{E}[e_t] = 0$ since $\mathsf{E}[\bar{X}_t] = \mathsf{E}[X_t]$ due to the law of total expectation.

At this point, we are ready to return to the cost function (3.6) and find the optimal controller and the optimal quantizer-selection policies.

Associated with the cost function (3.6), let us define the value function as follows:

$$V_k(\mathfrak{I}_k) = \min_{\{\gamma_t^u\}_{t=k}^{T-1}, \{\gamma_t^\theta\}_{t=k}^{T-1}} \mathsf{E}_\gamma \left[\sum_{t=k}^{T-1} (X_t^\intercal Q_1 X_t + U_t^\intercal R U_t + \theta_t^\intercal \Lambda) + X_T^\intercal Q_2 X_T \,|\mathfrak{I}_k \right], \tag{4.17a}$$

$$V_T(\mathfrak{I}_T) = \mathsf{E}_\gamma[X_T^\intercal Q_2 X_T \,|\mathfrak{I}_T], \tag{4.17b}$$

where the information set $\mathfrak{I}_k = \{\mathfrak{I}_k^c, \bar{\mathfrak{I}}_k^q\}$ and $\mathsf{E}_\gamma[\cdot]$ denotes the expectation under the strategy pair $\gamma = (\gamma^{\mathcal{U}}, \gamma^{\Theta})$. In the subsequent analysis, we will suppress the argument of V_k and the condition on $\mathfrak{I}_k$ in the expectation of (4.17) to maintain brevity. Using the dynamic programming principle,

$$V_k = \min_{\gamma_k^u \in \Gamma_k^u, \gamma_k^\theta \in \Gamma_k^\theta} \mathsf{E}_\gamma \Big[(X_k^\intercal Q_1 X_k + U_k^\intercal R U_k + \theta_k^\intercal \Lambda) + V_{k+1} \Big]. \tag{4.18}$$

If γ_k^{u*} and $\gamma_k^{\theta*}$ minimize the right-hand side of (4.18), then the optimal strategies are $U_k^* = \gamma_k^{u*}(\mathfrak{I}_k^c)$ and $\theta_k^* = \gamma_k^{\theta*}(\bar{\mathfrak{I}}_k^q)$. From (4.17), we also have that

$$\min_{\gamma^{\mathcal{U}} \in \Gamma^{\mathcal{U}}, \gamma^{\Theta} \in \Gamma^{\Theta}} J(\gamma^{\mathcal{U}}, \gamma^{\Theta}) = \mathsf{E}[V_0]. \tag{4.19}$$

The following theorem characterizes the optimal policy $\gamma_k^{u*}(\cdot)$ for all $k = 0, 1, \ldots, T-1$.

THEOREM 4.2 (optimal control policy). *Given the information $\mathfrak{I}_k^c$ to the controller at time k, the optimal control policy $\gamma_k^{u*} : \mathfrak{I}_k^c \to \mathbb{R}^m$ that minimizes the right-hand side of* (4.18) *has the following structure,*

$$U_k^* = \gamma_k^{u*}(\mathfrak{I}_k^c) = -L_k \bar{X}_k, \tag{4.20}$$

where $\bar{X}_k$ is computed in Lemma 4.2 *for all $k = 0, 1, \ldots, T-1$, and the matrices L_k and P_k are obtained by*

$$L_k = (R + B^\intercal P_{k+1} B)^{-1} B^\intercal P_{k+1} A, \tag{4.21a}$$

$$P_k = Q_1 + A^\intercal P_{k+1} A - L_k^\intercal (R + B^\intercal P_{k+1} B) L_k, \tag{4.21b}$$

$$P_T = Q_2. \tag{4.21c}$$

Proof. The proof of this theorem is based on the dynamic programming principle. Specifically, if there exist value functions V_k for all $k = 0, 1, \ldots, T$ that satisfy (4.18), then the optimal control U_k^* and the optimal quantizer selection θ_k^* are obtained by the policies γ_k^{u*} and $\gamma_k^{\theta*}$ that minimize (4.18).

Let us assume that the value function at time $k = 0, 1, \ldots, T-1$ is of the form

$$V_k = \mathsf{E}_\gamma[X_k^\intercal P_k X_k] + C_k + r_k, \tag{4.22}$$

where P_k is as in (4.21b) and, for all $k = 0, 1, \ldots, T-1$,

$$C_k = \min_{\{\gamma_t^\theta\}_{t=k}^{T-1}} \mathsf{E}_{\gamma^\theta}\left[\sum_{t=k}^{T-1} e_t^\intercal N_t e_t + \theta_t^\intercal \Lambda\right], \tag{4.23}$$

where $N_k \in \mathbb{R}^{n\times n}$ and $r_k \in \mathbb{R}$ are given by

$$N_k = L_k^\intercal (R + B^\intercal P_{k+1} B) L_k, \tag{4.24a}$$

$$r_k = r_{k+1} + \mathrm{tr}(P_{k+1}\mathcal{W}), \tag{4.24b}$$

$$r_T = 0. \tag{4.24c}$$

Equation (4.23) can be rewritten as

$$C_k = \min_{\gamma_k^\theta} \mathsf{E}_{\gamma^\theta}\left[e_k^\intercal N_k e_k + \theta_k^\intercal \Lambda + C_{k+1}\right], \qquad C_T = 0.$$

We first verify that V_{T-1} is of the form (4.22)

$$V_{T-1} = \min_{\gamma_{T-1}^u, \gamma_{T-1}^\theta} \mathsf{E}_\gamma\Big[X_{T-1}^\intercal Q_1 X_{T-1} + U_{T-1}^\intercal R U_{T-1} + \theta_{T-1}^\intercal \Lambda + X_T^\intercal P_T X_T\Big]. \tag{4.25}$$

Substituting into (4.25) the equation $X_T = AX_{T-1} + BU_{T-1} + W_{T-1}$, after some simplifications, yields

$$V_{T-1} = \min_{\gamma_{T-1}^u, \gamma_{T-1}^\theta} \mathsf{E}_\gamma\Big[\|U_{T-1} + L_{T-1}X_{T-1}\|^2_{(R+B^\intercal P_T B)} + \|X_{T-1}\|^2_{P_{T-1}} + \theta_{T-1}^\intercal \Lambda + \mathrm{tr}(P_T\mathcal{W})\Big],$$

where $\|L\|_K^2 \triangleq L^\intercal K L$ for any two matrices L and K of compatible dimensions. In the previous expression, $\|U_{T-1} + L_{T-1}X_{T-1}\|^2_{(R+B^\intercal P_T B)}$ is the only term that depends on U_{T-1}. Therefore, we seek $\gamma_{T-1}^u : \mathfrak{I}_{T-1}^c \to \mathbb{R}^m$ that minimizes the mean square error $\mathsf{E}\big[\|U_{T-1} + L_{T-1}X_{T-1}\|^2_{(R+B^\intercal P_T B)}\big]$. Thus, the optimal U_{T-1} is a minimum mean squared estimate of $-L_{T-1}X_{T-1}$ based on the σ-field generated by $\mathfrak{I}_{T-1}^c$. Hence, from Lemma 2.1,

$$U_{T-1}^* = \gamma_{T-1}^{u*}(\mathfrak{I}_{T-1}^c) = -L_{T-1}\mathsf{E}[X_{T-1}|\mathfrak{I}_{T-1}^c] = -L_{T-1}\bar{X}_{T-1}. \tag{4.26}$$

After substituting the optimal U_{T-1}^* into (4.25), we obtain

$$V_{T-1} = \min_{\gamma_{T-1}^\theta} \mathsf{E}_\gamma\Big[\|X_{T-1} - \bar{X}_{T-1}\|^2_{N_{T-1}} + \theta_{T-1}^\intercal \Lambda + \mathrm{tr}(P_T\mathcal{W}) + X_{T-1}^\intercal P_{T-1} X_{T-1}\Big].$$

The above expression of V_{T-1} can be rewritten as follows:

$$V_{T-1} = \min_{\gamma_{T-1}^\theta} \mathsf{E}_{\gamma^\theta}\left[e_{T-1}^\intercal N_{T-1} e_{T-1} + \theta_{T-1}^\intercal \Lambda\right] + \mathsf{E}\left[X_{T-1}^\intercal P_{T-1} X_{T-1}\right] + \mathrm{tr}(P_T\mathcal{W}).$$

Therefore, using the definitions of C_{T-1} and r_{T-1} from (4.23) and (4.24b), we obtain $V_{T-1} = \mathsf{E}[X_{T-1}^\intercal P_{T-1} X_{T-1}] + C_{T-1} + r_{T-1}$. Thus, V_{T-1} is of the form (4.22). Next, we prove the hypothesis (4.22) using mathematical induction. To that end, we assume that (4.22) is true for some $k+1$. Then,

$$\begin{aligned} V_k &= \min_{\gamma_k^u, \gamma_k^\theta} \mathsf{E}_\gamma\Big[(X_k^\intercal Q_1 X_k + U_k^\intercal R U_k + \theta_k^\intercal \Lambda) + V_{k+1}\Big] \\ &= \min_{\gamma_k^u, \gamma_k^\theta} \mathsf{E}_\gamma\Big[(X_k^\intercal Q_1 X_k + U_k^\intercal R U_k + \theta_k^\intercal \Lambda) + X_{k+1}^\intercal P_{k+1} X_{k+1} + r_{k+1} + C_{k+1}\Big]. \end{aligned}$$

Using (3.1), and after some simplifications, it follows that

$$V_k = \min_{\gamma_k^u,\gamma_k^\theta} \mathsf{E}_\gamma \Big[\|U_k + L_k X_k\|^2_{(R+B^\intercal P_{k+1}B)} + X_k^\intercal P_k X_k + \theta_k^\intercal \Lambda \\ + \mathrm{tr}(P_{k+1}\mathcal{W}) + r_{k+1} + C_{k+1}\Big]. \tag{4.27}$$

One may notice from the definition of e_k that it does not depend on the past control history $\mathcal{U}_k$ but, rather, it depends on the quantizer-selection history Θ_k. Thus, C_k does not depend on the control history $\mathcal{U}_k$. Furthermore, from (4.24a), (4.24b), and (4.21b), one notices that N_k, r_k, and P_k do not depend on the past (or future) decisions on the control or quantizer selection. Therefore, $\|U_k + L_k X_k\|^2_{(R+B^\intercal P_{k+1}B)}$ is the only term in the above expression of V_k that depends on U_k. Using Lemma 2.1, the optimal $\mathfrak{I}_k^c$-measurable control U_k^* that minimizes $\mathsf{E}\Big[\|U_k + L_k X_k\|^2_{(R+B^\intercal P_{k+1}B)}\Big]$ is given by

$$U_k^* = \gamma_k^{u*}(\mathfrak{I}_k^c) = -L_k \mathsf{E}\left[X_k|\mathfrak{I}_k^c\right] = -L_k \bar{X}_k. \tag{4.28}$$

After substituting the optimal control into (4.27) and using (4.24b), we obtain

$$\begin{aligned} V_k =& \mathsf{E}[X_k^\intercal P_k X_k] + \min_{\gamma_k^\theta} \mathsf{E}_\gamma\left[e_k^\intercal (L_k^\intercal (R + B^\intercal P_{k+1}B)L_k)e_k + \theta_k^\intercal \Lambda + C_{k+1}\right] + r_k \\ =& \mathsf{E}[X_k^\intercal P_k X_k] + \min_{\gamma_k^\theta} \mathsf{E}_{\gamma^\theta}\left[e_k^\intercal N_k e_k + \theta_k^\intercal \Lambda + C_{k+1}\right] + r_k = \mathsf{E}[X_k^\intercal P_k X_k] + C_k + r_k. \end{aligned}$$

Thus, the value function is indeed of the form (4.22) and, hence, the optimal control at time $k = 0, 1, \ldots, T-1$ is given by (4.28). This completes the proof. ∎

Remark 4.3. From Theorem 4.2, the optimal control is linear in $\bar{X}_k$. The optimal gain $-L_k$ can be computed offline without knowledge of $\gamma^{\Theta*}$. The effect of $\gamma^{\Theta*}$ on $\gamma^{\mathcal{U}*}$ is through the term $\bar{X}_k$, which can be computed online using (4.16).

Having computed the optimal controller, we now focus on solving for the optimal selection of the quantizers. To that end, from (4.19) and (4.22), we obtain

$$\min_{\gamma^{\mathcal{U}} \in \Gamma^{\mathcal{U}}, \gamma^\Theta \in \Gamma^\Theta} J(\gamma^{\mathcal{U}}, \gamma^\Theta) = \mathsf{E}[V_0] = \mu_0^\intercal P_0 \mu_0 + \mathrm{tr}(P_0 \Sigma_x) + r_0 + \mathsf{E}[C_0],$$

where, from (4.23), C_0 can be written as

$$C_0 = \min_{\{\gamma_t^\theta\}_{t=0}^{T-1}} \mathsf{E}_{\gamma^\theta}\left[\sum_{t=0}^{T-1} e_t^\intercal N_t e_t + \theta_t^\intercal \Lambda\right]. \tag{4.29}$$

Notice that the effect of the quantizer-selection policy γ^Θ on the cost $J(\gamma^{\mathcal{U}}, \gamma^\Theta)$ is reflected only through the term C_0. The optimal quantizer-selection policy can thus be found by performing the minimization associated with C_0 as represented in (4.29).

4.4. Optimal quantizer-selection policy. In this section, we study the optimal quantizer-selection policy $\gamma^{\Theta*}$, which can be found by solving (4.29). We may write $\mathsf{E}[e_t^\intercal N_t e_t] = \mathrm{tr}(N_t \mathsf{E}[e_t e_t^\intercal])$, and the following lemma computes $\mathsf{E}[e_t e_t^\intercal]$.

LEMMA 4.3. *For all* $t \in \mathbb{N}_0$,

$$\mathsf{E}[e_t e_t^\intercal] = \Sigma_t + \sum_{k=0}^{t} \Psi(t,k)(M_k - \mathsf{E}[\vartheta_{k,t} F_k(\theta_k)])\Psi(t,k)^\intercal.$$

Proof. The proof is given in Appendix C. ∎

Using Lemma 4.3, the cost C_0 can be simplified as

$$C_0 = \text{constant} + \min_{\{\gamma_t^\theta\}_{t=0}^{T-1}} \mathsf{E}_{\gamma^\theta}\left[\sum_{t=0}^{T-1} \operatorname{tr}\left(\Pi_t(\Theta)F_t(\theta_t)\right) + \theta_t^\intercal \lambda\right], \tag{4.30}$$

where the constant term is $\sum_{t=0}^{T-1}\left(\operatorname{tr}(\Sigma_t N_t) + \sum_{k=0}^{t} \operatorname{tr}(\tilde{N}_{k,t} M_k)\right)$ and

$$\tilde{N}_{k,t} = \Psi(t,k)^\intercal N_t \Psi(t,k), \tag{4.31a}$$

$$\Pi_t(\Theta) = -\sum_{\ell=t}^{T-1} \vartheta_{t,\ell} \tilde{N}_{t,\ell}. \tag{4.31b}$$

The optimal quantizer-selection policy is found by solving the mixed-integer-nonlinear program in (4.30).

At this point it may appear that the expression $\sum_{t=0}^{T-1} \operatorname{tr}(\Pi_t(\Theta)F_t(\theta_t))$ in (4.30) is a nonlinear function of Θ. However, we now show that after some simplifications, it can be written as a linear function of Θ. By expressing (4.30) as a linear function of Θ, we can recast (4.30) as a mixed-integer-linear-program (MILP), which further can be solved efficiently using existing efficient solvers [23].

To express (4.30) as an MILP, we construct a matrix $\Phi \in \mathbb{R}^{T\times M}$ as follows: for all $i = 0, \ldots, T-1$ and $j = 1, \ldots, M$, let

$$[\Phi]_{ij} = \begin{cases} 1 & \text{if } i \geq d_j, \\ 0 & \text{otherwise}, \end{cases} \tag{4.32}$$

where $[\Phi]_{ij}$ is the ijth component of the Φ matrix. It directly follows from the definition of Φ that $1_{d_j \leq t-k} = [\Phi]_{t-k,j}$. Consequently, we can express (4.5) as

$$\vartheta_{k,t} = \sum_{i=1}^{M} \theta_k^i [\Phi]_{t-k,i}.$$

Thus, $\Pi_t(\Theta)$ in (4.31b) can be rewritten as $\Pi_t(\Theta) = -\sum_{\ell=t}^{T-1}\sum_{i=1}^{M} \theta_t^i [\Phi]_{\ell-t,i}\tilde{N}_{t,\ell}$. Also, from (4.13), we have that $F_t(\theta_t) = \sum_{i=1}^{M} \theta_t^i F_t^i$. Thus,

$$\begin{aligned}
\operatorname{tr}(\Pi_t(\Theta)F_t(\theta_t)) &= -\operatorname{tr}\left(\sum_{i=1}^{M}\left(\theta_t^i \sum_{\ell=t}^{T-1}[\Phi]_{\ell-t,i}\tilde{N}_{t,\ell}\right) F_t(\theta_t)\right) \\
&= -\operatorname{tr}\left(\left(\sum_{i=1}^{M}\left(\theta_t^i \sum_{\ell=t}^{T-1}[\Phi]_{\ell-t,i}\tilde{N}_{t,\ell}\right)\right)\left(\sum_{j=1}^{M}\theta_t^j F_t^j\right)\right) \\
&\overset{(a)}{=} -\operatorname{tr}\left(\sum_{i=1}^{M}\theta_t^i\left(\sum_{\ell=t}^{T-1}[\Phi]_{\ell-t,i}\tilde{N}_{t,\ell}\right)F_t^i\right) = -\sum_{i=1}^{M}\theta_t^i\beta_t^i,
\end{aligned}$$

where $\beta_t^i = \operatorname{tr}((\sum_{\ell=t}^{T-1}[\Phi]_{\ell-t,i}\tilde{N}_{t,\ell})F_t^i)$ and the last equality follows from the fact that $\theta_t^i\theta_t^j = 0$ if $i \neq j$. Note that the coefficients β_t^i can be computed offline.

From the previous derivation, C_0 in (4.30) becomes

$$C_0 = \alpha + \min_{\{\gamma_t^\theta\}_{t=0}^{T-1}} \mathsf{E}_{\gamma^\theta}\left[\sum_{t=0}^{T-1} c_t^\intercal \theta_t\right], \tag{4.33}$$

where the constants are $\alpha = \sum_{t=0}^{T-1}\big(\mathrm{tr}(\Sigma_t N_t) + \sum_{k=0}^{t}\mathrm{tr}(\tilde{N}_{k,t}M_k)\big)$ and $c_t = [c_t^1, \ldots, c_t^M]^\intercal$ with $c_t^i = \lambda_i - \beta_t^i$. Notice that, in (4.33), the cost function is linear in θ_t, and the coefficients c_t^i are deterministic (and can be computed offline). Therefore, it is sufficient to look for a deterministic strategy to minimize the linear cost $\sum_{t=0}^{T-1} c_t^\intercal \theta_t$, as the class of deterministic strategies contains an optimal solution for $\min_{\{\gamma_t^\theta\}_{t=0}^{T-1}} \mathsf{E}_{\gamma^\theta}\left[\sum_{t=0}^{T-1} c_t^\intercal \theta_t\right]$. The following lemma presents an MILP formulation to obtain the optimal quantizer-selection policy.

Lemma 4.4. *The optimal quantizer-selection strategy is found by solving the following MILP*

$$\min_{\Theta} \sum_{t=0}^{T-1} c_t^\intercal \theta_t, \tag{4.34a}$$

$$\text{s.t.} \quad \sum_{i=1}^{M} \theta_t^i = 1, \quad \theta_t^i \in \{0,1\}, \quad t = 0, \ldots, T-1, \quad i = 1, \ldots, M. \tag{4.34b}$$

Proof. The proof directly follows from the derivation of (4.33) and the subsequent discussion. ∎

Notice that in the optimization problem above there is no constraint coupling θ_k and θ_ℓ, and the cost function in (4.34a) is also decoupled in θ_k and θ_ℓ for all $k \neq \ell \in \{0, \ldots, T-1\}$. Therefore, the optimal θ_t at time t can be found by minimizing $c_t^\intercal \theta_t$ subject to the constraints $\sum_{i=1}^{M} \theta_t^i = 1$, $\theta_t^i \in \{0,1\}$. Thus, the optimal quantizer-selection strategy for this problem turns out to be remarkably simple: if $i^* = \arg\min_{i=1,\ldots,M}\{c_t^1, \ldots, c_t^M\}$, then the optimal strategy is to use the i^*th quantizer[2] such that

$$\gamma_t^{\theta *} = \theta_t^* = [1_{i^*=1}, \ldots, 1_{i^*=M}]^\intercal.$$

This result is summarized in the following theorem.

Theorem 4.4 (optimal quantizer-selection). *At time t, the jth quantizer is optimal if and only if $c_t^j = \min\{c_t^1, \ldots, c_t^M\}$, where, for all $i = 1, \ldots, M$,*

$$c_t^i = \lambda_i - \mathrm{tr}\left(\left(\sum_{\ell=t}^{T-1} [\Phi]_{\ell-t,i}\tilde{N}_{t,\ell}\right) F_t^i\right),$$

and $\tilde{N}_{t,\ell}, [\Phi]_{\ell-t,i}$, and F_t^i are defined in (4.31a), (4.32), *and* (4.14), *respectively.*

The following remark is immediate from Theorem 4.4.

Remark 4.5. The optimal strategy for selecting the quantizers can be computed offline. This requires an offline computation of $\tilde{N}_{t,\ell}$ and F_t^i, but it does not require knowledge of the optimal control strategy.

4.5. Discussion and remarks. Let us delve into the cost $c_t^\intercal \theta_t$ in (4.34) to discuss how the three factors, namely, the cost of quantization, the quantization resolution, and the delay, affect the cost function. The coefficients c_t^i which determine the optimal quantizer-selection strategy at time t have two components, namely, λ_i, and β_t^i, where λ_i is the cost for using the ith quantizer, and β_t^i captures the trade-off between quantization quality and the associated delays. Let us discuss each of

[2]In case there exists multiple minimizers for $\mathrm{argmin}_{i=1,\ldots,M}\{c_t^1, \ldots, c_t^M\}$, one of these minimizers can be chosen randomly without affecting optimality.

these two terms in greater detail. First, c_t^i being proportional to the cost λ_i, reflects the fact that lower quantization cost is desirable. The quantity β_t^i is arguably more interesting. Note that β_t^i is of the form $\mathrm{tr}(G_t^i F_t^i)$, where for all i, G_t^i is a positive (semi)definite matrix whose expression can be easily identified from the expression of β_t^i. Moreover, since $1 \geq [\Phi]_{i,1} \geq [\Phi]_{i,2} \geq \cdots \geq [\Phi]_{i,M} \geq 0$ for all $i = 0, \ldots, T-1$, we have $G_t^1 \succeq G_t^2 \succeq \cdots \succeq G_t^M$. On the other hand, by using the ith quantizer, the reduction in *uncertainty covariance* is F_t^i. By uncertainty covariance we mean the following: before the arrival of any measurement $(\hat{\xi}_t)$, ξ_t is a Gaussian distributed random variable with covariance M_t. Once a quantized version $(\hat{\xi}_t)$ of ξ_t arrives at the controller, the controller receives information on the realization of the random variable ξ_t. Specifically, at this point, the controller knows the region $\mathcal{P}_j^i \subset \mathbb{R}^p$ wherein the random variable ξ_t belongs. Therefore, the posterior distribution of ξ_t changes after receiving $\hat{\xi}_t$, and the difference between the covariance of this posterior distribution and the prior distribution is F_t^i if the ith quantizer is used. Needless to say, had there been a quantizer that could ensure $\hat{\xi}_t = \xi_t$, i.e., no loss during quantization for every realization of ξ_t, then the reduction in covariance is exactly M_t and the posterior distribution of ξ_t at the controller is a Dirac measure around $\hat{\xi}_t$. The use of quantized measurements is similar to operating somewhere in-between open-loop and closed-loop control. In open loop, no measurement is sent, and in closed loop, the exact measurement is sent without any distortion. By means of quantization, the controller receives *something* but not *everything.* Furthermore, since $\beta_t^i \geq 0$ and since it appears with a negative sign in the cost function, it is clearly desirable to choose a quantizer that would maximize β_t^i. The matrix F_t^i directly reflects how much reduction in covariance will occur if the ith quantizer is used. The matrix G_t^i, on the other hand, incorporates the delay associated with the ith quantizer. As i is increased from 1 to M, G_t^i decreases $\mathrm{tr}(G_t^i F_t^i)$, reflecting the fact that a smaller delay is preferable. However, as i is varied, F_t^i shows the variation in covariance reduction. For example, if the reduction in covariance increases with the increase in ℓ_i, then F_t^i is attempting to increase $\mathrm{tr}(G_t^i F_t^i)$ as i is varied from 1 to M. Thus, there is a dual behavior between F_t^i and G_t^i as i changes, and this duality is captured by the parameters of the channel and the quantizers, namely, $\mathcal{P}^i$, ℓ^i, and the delay d_i.

We conclude this section with a few more remarks.

Remark 4.6. The cost function in (4.34) resembles the component $\sum_{t=0}^{T-1} \Lambda^\intercal \theta_t$ in (3.6), except that all the state and control costs are absorbed in the coefficients c_t^i. Here c_t^i can be viewed as the *adjusted cost* for operating the ith quantizer at time t, and the adjustment factor is β_t^i, which can be computed offline.

Remark 4.7. The approach allows for the case when the set of available quantizers contains a quantizer $\mathcal{Q}^0$ with only one quantization level, i.e., $\ell_0 = 1$, $\mathcal{P}^0 = \{\mathcal{P}_1^0 = \mathbb{R}^p\}$, and quantization cost $\lambda_0 = 0$. This quantizer produces the same quantized output for every input signal, hence, providing the option to remain open loop. For such a quantizer, it can be verified from (4.14) that $F_t^0 = 0$ for all t. Therefore, $c_t^0 = \lambda_0 - \beta_t^0 = 0$ for all t, and the selection of this quantizer at any time t reflects the fact that the optimal strategy is not to send any information to the controller at that time. If the quantization costs are very high,[3] $\lambda_i \gg 1$, the optimal choice of the quantizers

[3]Alternatively, the quantization cost is higher than the reward from using quantization, i.e., $\lambda_i > \beta_t^i$ for all t.

would be $\mathcal{Q}^0$ and, hence, the controller will not be receiving any information, which in principle, is equivalent to open-loop control.

4.6. Choice of the quantizers. In this work, we assume that the set of quantizers are given a priori. However, from our analysis it follows that the final LQG cost depends on the quantizer parameters through the variables β_t^i for $i = 1, \ldots, M$ and $t = 0, \ldots, T-1$. Specifically, notice that $\beta_t^i = \mathrm{tr}((\sum_{\ell=t}^{T-1}[\Phi]_{\ell-t,i}\tilde{N}_{t,\ell})F_t^i)$ and, thus, the quantizer resolution and the delay affect β_t^i through F_t^i and $[\Phi]_{\ell-t,i}$, respectively. Equation (4.14) directly relates the quantization cells $\mathcal{P}_j^i$ and F_t^i. Therefore, in principle, one can choose to optimize over $\mathcal{P}_j^i$'s to find β_t^i's, even though such an optimization can be computationally very expensive.

5. Special cases. In this section, we simplify some of the expressions obtained in section 4.4 by considering some special cases. In particular, we show that the expression of c_t^i in Theorem 4.4 can be substantially simplified under these special cases. These simplifications will be helpful for fast and efficient computation of the optimal solution for the optimal quantizer scheduling problem described in Theorem 4.4. To this end, we consider (i) the constant-delay case, and (ii) the full observation case.

5.1. Constant-delay. In this section, we consider the case where $d_1 = d_2 = \cdots = d_M = d$, i.e., the delay induced by each quantizer is the same. Intuitively, since the delay is not affected by the choice of the quantizer, then the quantizer selection problem should reduce to a tradeoff between the quantization cost and the quality of quantization. To see this, let us first note that $[\Phi]_{i,1} = \cdots = [\Phi]_{i,M} = 1_{i\geq d}$ for all $i = 0, \ldots, T-1$. Therefore,

$$\begin{aligned}\beta_t^i =& \mathrm{tr}\left(\left(\sum_{\ell=t}^{T-1}[\Phi]_{\ell-t,i}\tilde{N}_{t,\ell}\right)F_t^i\right) = \mathrm{tr}\left(\left(\sum_{\ell=t}^{T-1}1_{\ell-t\geq d}\tilde{N}_{t,\ell}\right)F_t^i\right)\\ &= \mathrm{tr}\left(\left(\sum_{\ell=t+d}^{T-1}\tilde{N}_{t,\ell}\right)F_t^i\right) = \mathrm{tr}\big(H(t,d)F_t^i\big),\end{aligned}$$

where $H(t,d) = \sum_{\ell=t+d}^{T-1}\tilde{N}_{t,\ell} \succeq 0$. Thus, for fixed t and d, whether the ith quantizer is optimal at time t is entirely determined by F_t^i, where we recall that F_t^i represents the uncertainty covariance reductions. Also notice that $H(t,d) = 0$ for all $t \geq T-d$, and hence $\beta_t^i = 0$. Therefore, the optimal selection for the quantizers for $t \geq T-d$ would be the one with the lowest λ_i. This is due to the fact that the quantized information $\hat{\xi}_{T-d}, \hat{\xi}_{T-d+1}, \ldots$ will not be available at the controller before time $T-1$, and hence these quantized measurements would be of no use to the controller. Therefore, the quality of the quantization for time $T-d$ onward is immaterial to the controller and, hence, the lowest cost quantizer would be optimal.

5.2. Full observation. For the full observation case, we substitute $\mathcal{V} = 0$ and $C = I$ in the analysis presented above. As a direct consequence, one can verify that $\xi_t = W_{t-1}$ for all t. Therefore, $\{\xi_t \sim \mathcal{N}(0, \mathcal{W})\}_{t\in\mathbb{N}_0}$ are i.i.d. signals and, consequently, the matrices F_t^i given in (4.13) will be time invariant, i.e., $F_1^i = \cdots = F_T^i \triangleq F^i$.

For all $t \in \mathbb{N}_0$, $\Sigma_t = 0$, $\Sigma_{t+1|t} = M_{t+1} = \mathcal{W}$. This also implies that, for all $t \geq k$,

$$\Psi(t,k) = A^{t-k} \text{ and } \tilde{N}_{k,t} = A^{t-k^{\mathsf{T}}}N_tA^{t-k}.$$

Therefore, the state estimate can be written as

$$\begin{aligned}\bar{X}_t &= A^t\mu_0+\sum_{k=0}^{t}\Psi(t,k)\vartheta_{k,t}\bar{\xi}_k+\sum_{k=0}^{t-1}A^{t-1-k}BU_k\\ &= A^t\mu_0+\sum_{k=0}^{t}A^{t-k}\vartheta_{k,t}\bar{\xi}_k+\sum_{k=0}^{t-1}A^{t-1-k}BU_k\\ &= A\bar{X}_{t-1}+BU_{t-1}+\vartheta_{t,t}\bar{\xi}_t+\sum_{k=0}^{t-1}A^{t-k}(\vartheta_{k,t}-\vartheta_{k,t-1})\bar{\xi}_k. \qquad (5.1)\end{aligned}$$

The expression for β_t^i is now given by

$$\beta_t^i = \mathrm{tr}\left(\left(\sum_{\ell=t}^{T-1}[\Phi]_{\ell-t,i}\tilde{N}_{t,\ell}\right)F_t^i\right) = \mathrm{tr}\left(\left(\sum\nolimits_{\ell=t+d_i}^{T-1} A^{\ell-t^{\intercal}}N_\ell A^{\ell-t}\right)F^i\right).$$

Let us define a symmetric matrix Υ_t as follows,

$$\Upsilon_t = A^{\intercal}\Upsilon_{t+1}A+N_t, \quad \Upsilon_T=0,$$

which allows us to rewrite $\beta_t^i = \mathrm{tr}(\Upsilon_{\min\{t+d_i,T\}}F^i)$. We conclude this section by discussing the constant delay case for fully observed systems.

Under the assumption of constant delay, i.e., $d_1=\cdots=d_M=d$, we obtain $\beta_t^i=\mathrm{tr}(\Upsilon_{\min\{t+d,T\}}F^i)$. Furthermore, $\vartheta_{k,t}=1$ if and only if $t-k\geq d$, otherwise, $\vartheta_{k,t}=0$. This implies, from (5.1) that, for all $t\in\mathbb{N}_0$,

$$\bar{X}_t = \begin{cases} A\bar{X}_{t-1}+BU_{t-1}+A^d\bar{\xi}_{t-d} & \text{if } t\geq d,\\ A\bar{X}_{t-1}+BU_{t-1} & \text{otherwise.}\end{cases} \qquad (5.2)$$

6. Numerical examples.

In this section, we illustrate our theory on a linearized inverted pendulum system whose discretized equations of motion are given by[4]

$$X_{t+1} = \begin{bmatrix}1 & 0.05\\ 0.5 & 0.95\end{bmatrix}X_t+\begin{bmatrix}0\\0.05\end{bmatrix}U_t++\begin{bmatrix}0\\1\end{bmatrix}W_t, \qquad (6.1a)$$

$$Y_t = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}X_t+\nu_t, \qquad (6.1b)$$

where $X_0\sim\mathcal{N}(0,I)$, $W_t\sim\mathcal{N}(0,0.05)$, and $\nu_t\sim\mathcal{N}(0,\frac{1}{4}I)$. The control cost has parameters $Q=Q_f=0.5\,I$, $R=0.5$, and the time horizon is set to $T=50$.

The simulation was performed with three quantizers $(\mathcal{Q}^1,\mathcal{Q}^2,\mathcal{Q}^3)$, where $\mathcal{Q}^i$ has 2^i numbers of quantization levels, i.e., $\mathcal{Q}^1=\{0,1\}$, $\mathcal{Q}^2=\{00,01,10,11\}$, and so on. The partitions associated with the quantizers are $\mathcal{P}^1=\{\mathbb{R}_+\times\mathbb{R},\ \mathbb{R}_{<0}\times\mathbb{R}\}$, $\mathcal{P}^2=\{\mathbb{R}_+\times\mathbb{R}_+,\ \mathbb{R}_+\times\mathbb{R}_{<0},\ \mathbb{R}_{<0}\times\mathbb{R}_+,\ \mathbb{R}_{<0}\times\mathbb{R}_{<0}\}$, and $\mathcal{P}^3=\{[0,1)\times\mathbb{R}_+,\ [1,\infty)\times\mathbb{R}_+, [0,1)\times\mathbb{R}_{<0},\ [1,\infty)\times\mathbb{R}_{<0}, [-1,0)\times\mathbb{R}_+,\ (-\infty,-1)\times\mathbb{R}_+, [-1,0)\times\mathbb{R}_{<0},\ (-\infty,-1)\times\mathbb{R}_{<0}\}$. The costs associated with the quantizers are $\Lambda=[10,11,12]^{\intercal}$.

We consider two scenarios, where in the first scenario the delays associated with the quantizers are $d_i=1$ for all i, and in the second scenario $d_i=i$ for all i. Under these

[4]Let the angular displacement and velocity of the pendulum be denoted as x_1 and x_2. The dynamics of the system are given by $\dot{x_1}=x_2$ and $dx_2=(-\mathrm{g}\sin(x_1)/l-kx_2+u)dt+dw$, where $\mathrm{g}=10\mathrm{ms}^{-2}$ is the gravitational acceleration, $l=1$m is the length of the pendulum, $k=1\mathrm{s}^{-1}$ is the damping coefficient, u is the input and dw is a standard Brownian motion. We consider a linearized model around $x_1=\pi$ for the system operating at 20 Hz.

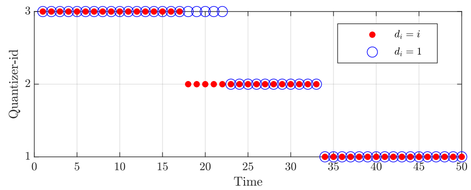

FIG. 4. *Optimal selection of quantizers over time.*

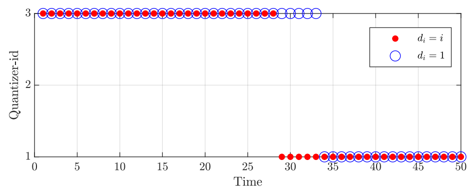

FIG. 5. *Optimal selection of quantizers over time when the second dimension of Y_t is pure noise.*

conditions, the optimal selection for the quantizers is plotted in Figure 4. Although both plots in Figure 4 portray similar behavior, there are a few differences (see the time interval $[18, 22]$) in the optimal selection of the quantizers due to the delays. We notice that during the interval from $t = 18$ to 22, $\mathcal{Q}^3$ is optimal when $d_3 = 1$, whereas $\mathcal{Q}^2$ is optimal when $d_3 = 3$. The reason behind this is the fact that the quantized output of both $\mathcal{Q}^3$ and $\mathcal{Q}^2$ will be available with the same delay when $d_i = 1$ for all i, whereas the quantized output of $\mathcal{Q}^3$ will reach later than that of $\mathcal{Q}^2$ when $d_i = i$, although the quantized output of $\mathcal{Q}^3$ will be less distorted than that of $\mathcal{Q}^2$. During the time period $[18, 22]$, it turned out to be beneficial to have a coarser measurement faster than a finer, more delayed, measurement. This simple example reflects the combined (dual) effect of the quantization resolution and the associated delays in the optimal choice of the quantizers.

We next considered the same example while observing the angular position only. We modify the Y_t equation as follows: $Y_t = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right] X_t + \nu_t$. In this case, although Y_t is two dimensional, the second dimension of the observation is pure noise and does not contain any useful state information. The optimal selection for the quantizers is shown in Figure 5. We notice that $\mathcal{Q}^2$ is a finer version of $\mathcal{Q}^1$ along the second dimension only. There is no difference in the quantization quality between $\mathcal{Q}^1$ and $\mathcal{Q}^2$ for the first dimension. Since the second dimension of Y_t is pure noise, $\mathcal{Q}^2$ is never selected for this problem (since $\mathcal{Q}^1$ performs equally well as $\mathcal{Q}^2$ with lesser cost).

7. Conclusions. In this work, we have considered a quantization-based partially observed LQG problem with a quantization cost. The problem is to choose an optimal quantizer among a set of available quantizers that minimizes the combined cost of quantization and control performance. The number of bits required to represent the quantized value increases as the quantization resolution gets better, and hence the delay in transmitting the measurement also increases. We illustrate how the quality of quantization and delay together emerge in the cost function and we demonstrate their dual role in the optimal solution.

We have shown that the optimal controller exhibits a separation principle and it has a linear relationship with the estimate of the state. The optimal gains for the controller are found by solving the classical Riccati equation associated with the LQG problem. We have also shown that the optimal selection of the quantizers can be found by solving a linear program that can be solved offline independently from the calculation of the optimal controller gain. Furthermore, the special cases of full observation and constant delay are also discussed. The possibility of the system to remain open loop at time t by not sending any quantized information is discussed as well in Remark 4.7.

Appendix A.

Proof of Proposition 4.1. Let us consider a state process $X_t^{\text{new}} \triangleq X_t - \sum_{k=0}^{t-1} A^{t-1-k}BU_k - A^t\mu_0$ and an observation process $Y_t^{\text{new}} \triangleq CX_t^{\text{new}} + V_t$. Therefore,

$$X_{t+1}^{\text{new}} = AX_t^{\text{new}} + W_t, \tag{A.1a}$$

$$Y_t^{\text{new}} = CX_t^{\text{new}} + V_t, \tag{A.1b}$$

$$X_0^{\text{new}} = X_0 - \mu_0 = W_{-1} \sim \mathcal{N}(0, \Sigma_x). \tag{A.1c}$$

Here X_t^{new} is the process associated with X_t, which is independent of the control strategy. Using this definition of X_t^{new} and Y_t^{new}, we have $X_t = X_t^{\text{new}} + \varphi(t,\mathcal{U}_{t-1})$ and $Y_t = Y_t^{\text{new}} + C\varphi(t,\mathcal{U}_{t-1})$, where $\varphi(t,\mathcal{U}_{t-1}) = \sum_{k=0}^{t-1} A^{t-1-k}BU_k + A^t\mu_0$. Therefore, the information sets $(\mathcal{Y}_{t-1},\mathcal{U}_{t-1})$ and $(Y_0^{\text{new}},\dots,Y_{t-1}^{\text{new}},\mathcal{U}_{t-1})$ are equivalent, i.e., one can be constructed from the other.

The innovation process associated with system (A.1) is given by

$$\xi_t^{\text{new}} = Y_t^{\text{new}} - \mathsf{E}[Y_t^{\text{new}}|Y_0^{\text{new}},\dots,Y_{t-1}^{\text{new}}].$$

Let ξ_t be the innovation process associated with the system (3.1). It can be shown that $\xi_t^{\text{new}} = \xi_t$ for all t. In order to prove this statement, notice that

$$\begin{aligned}
\xi_t &= Y_t - \mathsf{E}[Y_t|\mathcal{Y}_{t-1},\mathcal{U}_{t-1}] \\
&= Y_t^{\text{new}} + C\varphi(t,\mathcal{U}_{t-1}) - \mathsf{E}[Y_t^{\text{new}}|\mathcal{Y}_{t-1},\mathcal{U}_{t-1}] - \mathsf{E}[C\varphi(t,\mathcal{U}_{t-1})|\mathcal{Y}_{t-1},\mathcal{U}_{t-1}] \\
&= Y_t^{\text{new}} - \mathsf{E}[Y_t^{\text{new}}|Y_0^{\text{new}},\dots,Y_{t-1}^{\text{new}},\mathcal{U}_{t-1}] \\
&= Y_t^{\text{new}} - \mathsf{E}[Y_t^{\text{new}}|Y_0^{\text{new}},\dots,Y_{t-1}^{\text{new}}] = \xi_t^{\text{new}}.
\end{aligned}$$

Thus, ξ_t does not depend on the control history $\mathcal{U}_{t-1}$.

The standard results of Kalman filtering hold for the process X_t^{new} with observation Y_t^{new}. It follows that $\{\xi_t^{\text{new}}\}_{t\in\mathbb{N}_0}$ is a sequence of uncorrelated Gaussian noises. Thus, using standard Kalman filtering theory, we define

$$e_t^{\text{new}} = X_t^{\text{new}} - \mathsf{E}[X_t^{\text{new}}|Y_0^{\text{new}},\dots,Y_{t-1}^{\text{new}}], \tag{A.2a}$$

$$\Delta_t^{\text{new}} = X_t^{\text{new}} - \mathsf{E}[X_t^{\text{new}}|Y_0^{\text{new}},\dots,Y_t^{\text{new}}], \tag{A.2b}$$

$$\Sigma_{t|t-1} = \mathsf{E}[e_t^{\text{new}}e_t^{\text{new}\intercal}], \tag{A.2c}$$

$$\Sigma_t = \mathsf{E}[\Delta_t^{\text{new}}\Delta_t^{\text{new}\intercal}]. \tag{A.2d}$$

Moreover,

$$\mathsf{E}[X_t^{\text{new}}|Y_0^{\text{new}},\dots,Y_t^{\text{new}}] = \mathsf{E}[X_t^{\text{new}}|Y_0^{\text{new}},\dots,Y_{t-1}^{\text{new}}] + K_t\xi_t^{\text{new}},$$

where K_t is the Kalman gain at time t. Thus, $\Delta_t^{\text{new}} = e_t^{\text{new}} - K_t\xi_t^{\text{new}} = (I - K_tC)e_t^{\text{new}} - K_tV_t$. The initial conditions are $e_0^{\text{new}} = X_0^{\text{new}} \sim \mathcal{N}(0,\Sigma_x)$ and $\Sigma_{0|-1} = \Sigma_x$. Therefore, $\mathsf{E}[\xi_t^{\text{new}}] = 0$ and $M_t, \Sigma_{t|t-1}$ and Σ_t satisfy

$$\begin{aligned}
M_t &= \mathsf{E}[(Ce_t^{\text{new}} + V_t)(Ce_t^{\text{new}} + V_t)^\intercal] = C\Sigma_{t|t-1}C^\intercal + \mathcal{V}, \\
\Sigma_{t|t-1} &= \mathsf{E}[e_t^{\text{new}}e_t^{\text{new}\intercal}] = \mathsf{E}[(A\Delta_{t-1}^{\text{new}} + W_{t-1})(A\Delta_{t-1}^{\text{new}} + W_{t-1})^\intercal] = A\Sigma_{t-1}A^\intercal + \mathcal{W}, \\
\Sigma_t &= \mathsf{E}[(I - K_tC)e_t^{\text{new}}e_t^{\text{new}\intercal}(I - K_tC)^\intercal] + K_tC\mathcal{V}C^\intercal K_t^\intercal \\
&= (I - K_tC)\Sigma_{t|t-1}(I - K_tC)^\intercal + K_t\mathcal{V}K_t^\intercal = \Sigma_{t|t-1} - \Sigma_{t|t-1}C^\intercal M_t^{-1}C\Sigma_{t|t-1},
\end{aligned}$$

where $K_t = \Sigma_{t|t-1} C^\intercal M_t^{-1}$ is the Kalman gain. This concludes the proof. $\square$

Appendix B.

Proof of Lemma 4.1. The information contained in $(\mathcal{Y}_t, \mathcal{U}_{t-1})$ is the same as that contained in $(\Xi_t, \mathcal{U}_{t-1})$, where $\Xi_t = \{\xi_0, \ldots, \xi_t\}$. Therefore,

$$\begin{aligned}\mathsf{E}[X_t|\mathcal{Y}_t, \mathcal{U}_{t-1}] = \mathsf{E}[X_t|\Xi_t, \mathcal{U}_{t-1}] &= \mathsf{E}[X_t^{\text{new}}|\Xi_t, \mathcal{U}_{t-1}] + \sum_{k=0}^{t-1} A^{t-1-k} B U_k + A^t \mu_0 \\ &= \mathsf{E}[X_t^{\text{new}}|\Xi_t^{\text{new}}] + \sum_{k=0}^{t-1} A^{t-1-k} B U_k + A^t \mu_0,\end{aligned}$$

where $\Xi_t^{\text{new}} = \{\xi_t^{\text{new}}\}_{t\in\mathbb{N}_0} = \{\xi_t\}_{t\in\mathbb{N}_0} = \Xi_t$. It follows from Kalman filtering that

$$\mathsf{E}[X_t^{\text{new}}|\Xi_t^{\text{new}}] = \mathsf{E}[X_t^{\text{new}}|\Xi_{t-1}^{\text{new}}] + K_t \xi_t^{\text{new}} = A\mathsf{E}[X_{t-1}^{\text{new}}|\Xi_{t-1}^{\text{new}}] + K_t \xi_t^{\text{new}},$$

since W_{t-1} is independent of Ξ_{t-1}^{new}. We need to show that

$$\mathsf{E}[X_t^{\text{new}}|\Xi_t^{\text{new}}] = \sum_{k=0}^{t} \Psi(t,k) \xi_k^{\text{new}} \tag{B.1}$$

for some $\Psi(t,k)$. We show this by induction. To this end, notice that (B.1) is true for $t = 0$ with $\Psi(0,0) = \Sigma_x C^\intercal (C \Sigma_x C^\intercal + \mathcal{V})^{-1}$, where Σ_x is the covariance of the initial state X_0. Next, if (B.1) is true for $t = \tau$, then we have that, for $t = \tau + 1$,

$$\begin{aligned}\mathsf{E}[X_{\tau+1}^{\text{new}}|\Xi_{\tau+1}^{\text{new}}] &= A\mathsf{E}[X_\tau^{\text{new}}|\Xi_\tau^{\text{new}}] + K_{\tau+1}\xi_{\tau+1}^{\text{new}} \\ &= A\sum_{k=0}^{\tau} \Psi(\tau,k)\xi_k^{\text{new}} + K_{\tau+1}\xi_{\tau+1}^{\text{new}} = \sum_{k=0}^{\tau+1} \Psi(\tau+1,k)\xi_k^{\text{new}},\end{aligned}$$

where $K_{\tau+1}$ is the Kalman gain at time $\tau+1$, $\Psi(\tau+1,k) = A\Psi(\tau,k)$ for all $k = 0,\ldots,\tau$, and $\Psi(\tau+1,\tau+1) = K_{\tau+1}$. Therefore, for all $t \geq k$, $\Psi(t,k) = A^{t-k}K_k = A^{t-k}\Sigma_{k|k-1}C^\intercal M_k^{-1}$, and

$$\begin{aligned}\mathsf{E}[X_t|\mathcal{Y}_t, \mathcal{U}_{t-1}] &= \mathsf{E}[X_t^{\text{new}}|\Xi_t^{\text{new}}] + \sum_{k=0}^{t-1} A^{t-1-k} B U_k + A^t \mu_0 \\ &= \sum_{k=0}^{t}\Psi(t,k)\xi_k^{\text{new}} + \sum_{k=0}^{t-1} A^{t-1-k} B U_k + A^t \mu_0 \\ &= \sum_{k=0}^{t}\Psi(t,k)\xi_k + \sum_{k=0}^{t-1} A^{t-1-k} B U_k + A^t \mu_0.\end{aligned}$$

The set $\mathcal{O}_t$ may not contain all the elements of Ξ_t due to delays. In fact, for $k \leq t$, we have that $\xi_k \in \mathcal{O}_t$ if and only if $\vartheta_{k,t} = 1$. Since ξ_k and ξ_t are independent for $t \neq k$, we have

$$\mathsf{E}[\xi_k|\mathcal{O}_t] = \begin{cases} \xi_k & \text{if } \xi_k \in \mathcal{O}_t, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we can write $\mathsf{E}[\xi_k|\mathcal{O}_t] = \vartheta_{k,t}\xi_k$. Thus,

$$\begin{aligned}\mathsf{E}[X_t|\mathcal{O}_t,\mathcal{U}_{t-1}] &= \mathsf{E}[\mathsf{E}[X_t|\Xi_t,\mathcal{U}_{t-1}]|\mathcal{O}_t,\mathcal{U}_{t-1}]\\ &= \mathsf{E}\left[\sum_{k=0}^{t}\Psi(t,k)\xi_k|\mathcal{O}_t,\mathcal{U}_{t-1}\right] + \sum_{k=0}^{t-1}A^{t-1-k}BU_k + A^t\mu_0\\ &= \sum_{k=0}^{t}\Psi(t,k)\vartheta_{k,t}\xi_k + \sum_{k=0}^{t-1}A^{t-1-k}BU_k + A^t\mu_0.\end{aligned}$$

This completes the proof. □

Appendix C.

Proof of Lemma 4.3. Let us define $\Delta_t = \mathsf{E}[e_t \mid \mathcal{Y}_t, \hat{\mathcal{O}}_t, \mathcal{U}_{t-1}]$, and notice that,

$$\mathsf{E}[e_t e_t^\intercal \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] = \mathsf{E}[(e_t-\Delta_t)(e_t-\Delta_t)^\intercal \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] + \mathsf{E}[\Delta_t\Delta_t^\intercal \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}],$$

since $\mathsf{E}[\Delta_t(e_t-\Delta)^\intercal \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] = \Delta_t\mathsf{E}[(e_t-\Delta)^\intercal \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] = 0$. Taking expectations on both sides of the last equation, we obtain

$$\mathsf{E}[e_t e_t^\intercal] = \mathsf{E}[(e_t-\Delta_t)(e_t-\Delta_t)^\intercal] + \mathsf{E}[\Delta_t\Delta_t^\intercal]. \tag{C.1}$$

Substituting the expression for $\bar{X}_t$ from (4.9) into $e_t = X_t - \bar{X}_t$ yields

$$e_t = X_t - \sum_{k=0}^{t-1}A^{t-1-k}BU_k - A^t\mu_0 - \sum_{k=0}^{t}\Psi(t,k)\vartheta_{k,t}\bar{\xi}_k.$$

Therefore,

$$\begin{aligned}\Delta_t &= \mathsf{E}[e_t \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}]\\ &= \mathsf{E}[X_t \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] - \sum_{k=0}^{t-1}A^{t-1-k}BU_k - A^t\mu_0 - \sum_{k=0}^{t}\Psi(t,k)\vartheta_{k,t}\bar{\xi}_k\\ &= \sum_{k=0}^{t}\Psi(t,k)\xi_k - \sum_{k=0}^{t}\Psi(t,k)\vartheta_{k,t}\bar{\xi}_k = \sum_{k=0}^{t}\Psi(t,k)(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t]),\end{aligned} \tag{C.2}$$

where we have used $\mathsf{E}[X_t \mid \mathcal{Y}_t,\hat{\mathcal{O}}_t,\mathcal{U}_{t-1}] = \mathsf{E}[X_t \mid \mathcal{Y}_t,\mathcal{U}_{t-1}]$ since $\hat{\mathcal{O}}_t$ is a $\mathcal{Y}_t$-measurable function and we have also used (4.15) to write $\vartheta_{k,t}\bar{\xi}_k$ as $\mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t]$. Using the expression of Δ_t from (C.2), we obtain

$$\begin{aligned}e_t - \Delta_t &= X_t - \sum_{k=0}^{t-1}A^{t-1-k}BU_k - A^t\mu_0 - \sum_{k=0}^{t}\Psi(t,k)\xi_k\\ &= X_t^{\text{new}} - \mathsf{E}[X_t^{\text{new}}|\Xi_t^{\text{new}}] = \Delta_t^{\text{new}},\end{aligned}$$

where X_t^{new}, $\mathsf{E}[X_t^{\text{new}} \mid \Xi_t^{\text{new}}]$, and Δ_t^{new} are defined in (A.1) and (B.1) and (A.2), respectively. Thus, we may rewrite (C.1) as follows

$$\begin{aligned}\mathsf{E}[e_t e_t^\intercal] &= \mathsf{E}[(e_t-\Delta_t)(e_t-\Delta_t)^\intercal] + \mathsf{E}[\Delta_t\Delta_t^\intercal]\\ &= \mathsf{E}[\Delta_t^{\text{new}}\Delta_t^{\text{new}\intercal}] + \sum_{k=0}^{t}\sum_{\ell=0}^{t}\Psi(t,k)\mathsf{E}[(\xi_k - \mathsf{E}[\xi_k|\hat{\mathcal{O}}_t])(\xi_\ell - \mathsf{E}[\xi_\ell|\hat{\mathcal{O}}_t])^\intercal]\Psi(t,\ell)^\intercal\\ &= \Sigma_t + \sum_{k=0}^{t}\sum_{\ell=0}^{t}\Psi(t,k)\mathsf{E}[(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t])(\xi_\ell - \mathsf{E}[\xi_\ell \mid \hat{\mathcal{O}}_t])^\intercal]\Psi(t,\ell)^\intercal,\end{aligned} \tag{C.3}$$

where we used the definition $\Sigma_t = \mathsf{E}[\Delta_t^{\text{new}}\Delta_t^{\text{new}\mathsf{T}}]$ from (A.2).

To further simplify (C.3), we recall that ξ_k and ξ_ℓ are independent random variables when $k \neq \ell$ and, therefore, we obtain

$$\begin{aligned}\mathsf{E}[(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t])(\xi_\ell - \mathsf{E}[\xi_\ell \mid \hat{\mathcal{O}}_t])^\mathsf{T}] &= \mathsf{E}\left[\mathsf{E}[(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t])(\xi_\ell - \mathsf{E}[\xi_\ell \mid \hat{\mathcal{O}}_t])^\mathsf{T} \mid \xi_k, \hat{\mathcal{O}}_t]\right] \\ &= \mathsf{E}\left[(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t])\mathsf{E}[(\xi_\ell - \mathsf{E}[\xi_\ell \mid \hat{\mathcal{O}}_t])^\mathsf{T} \mid \xi_k, \hat{\mathcal{O}}_t]\right] \\ &= \mathsf{E}[(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t])(\mathsf{E}[\xi_\ell \mid \hat{\mathcal{O}}_t] - \mathsf{E}[\xi_\ell \mid \hat{\mathcal{O}}_t])^\mathsf{T}] = 0\end{aligned}$$

for all $k \neq \ell$. On the other hand, for $k = \ell$, we obtain

$$\begin{aligned}&\mathsf{E}[(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t])(\xi_k - \mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t])^\mathsf{T}] = \mathsf{E}[\xi_k\xi_k^\mathsf{T}] - \mathsf{E}[\mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t]\mathsf{E}[\xi_k \mid \hat{\mathcal{O}}_t]^\mathsf{T}] \\ &\quad \overset{(a)}{=} M_k - \mathsf{E}[\vartheta_{k,t}\bar{\xi}_k\bar{\xi}_k^\mathsf{T}] = M_k - \mathsf{E}[\mathsf{E}[\vartheta_{k,t}\bar{\xi}_k\bar{\xi}_k^\mathsf{T} \mid \theta_k]] \\ &\quad \overset{(b)}{=} M_k - \mathsf{E}[\vartheta_{k,t}\mathsf{E}[\bar{\xi}_k\bar{\xi}_k^\mathsf{T} \mid \theta_k]] \overset{(c)}{=} M_k - \mathsf{E}[\vartheta_{k,t}F_k(\theta_k)],\end{aligned}$$

where (a) follows from (4.15) and the fact that $\vartheta_{k,t}^2 = \vartheta_{k,t}$ since $\vartheta_{k,t} \in \{0,1\}$, and (b) follows from the fact that $\vartheta_{k,t}$ is a deterministic function of θ_k due to (4.5), and finally, (c) follows from (4.13). Consequently, (C.3) reduces to the following equation:

$$\mathsf{E}[e_t e_t^\mathsf{T}] = \Sigma_t + \sum_{k=0}^{t} \Psi(t,k)(M_k - \mathsf{E}[\vartheta_{k,t}F_k(\theta_k)])\Psi(t,k)^\mathsf{T}. \qquad \square$$

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