Michigan Math. J. (Advance Publication)
https://doi.org/10.1307/mmj/20216084

Global Newlander–Nirenberg Theorem for Domains with C^2 Boundary

Chun Gan & Xianghong Gong

Abstract. The Newlander–Nirenberg theorem says that a formally integrable complex structure is locally equivalent to the standard complex structure in the complex Euclidean space. In this paper, we consider two natural generalizations of the Newlander–Nirenberg theorem under the presence of a C^2 strictly pseudoconvex boundary. When a given formally integrable complex structure X is defined on the closure of a bounded strictly pseudoconvex domain with C^2 boundary $D \subset \mathbb{C}^n$, we show the existence of global holomorphic coordinate systems defined on $\overline{D}$ that transform X into the standard complex structure provided that X is sufficiently close to the standard complex structure. Moreover, we show that such closeness is stable under a small C^2 perturbation of ∂D. As a consequence, when a given formally integrable complex structure is defined on a one-sided neighborhood of some point in a C^2 real hypersurface $M \subset \mathbb{C}^n$, we prove the existence of local one-sided holomorphic coordinate systems provided that M is strictly pseudoconvex with respect to the given complex structure. We also obtain results when the structures are finite smooth.

1. Introduction

Given a formally integrable smooth (i.e. C^∞) almost complex structure defined on $\overline{D}$, where D is a bounded domain in $\mathbb{R}^{2n}$, we consider the problem of finding smooth global holomorphic coordinate systems on $\overline{D}$ compatible with the structure. By a smooth global holomorphic coordinate system on $\overline{D}$, we mean a smooth diffeomorphism sending $\overline{D}$ onto $\overline{D'}$ where D' is a domain in $\mathbb{C}^n$, while the diffeomorphism transforms the given complex structure into the standard complex structure on $\mathbb{C}^n$. The classical Newlander–Nirenberg theorem asserts the existence of local holomorphic coordinate systems for a formally integrable almost complex structure defined near an interior point of D. The main result of this paper is to show the existence of such global holomorphic coordinate systems when the structure is a small perturbation of the standard complex structure on $\overline{D}$, where D is a bounded, strictly pseudoconvex domain with C^2 boundary. When both boundary and the complex structure are C^∞, this result is due to R. Hamilton through a general program [7; 8; 9].

Received May 5, 2021. Revision received October 20, 2021.
The second author was partially supported by a grant from the Simons Foundation (award number: 505027) and NSF grant DMS-2054989.

We will use our global result to show the existence of local holomorphic coordinate systems on a small, one-sided neighborhood of a given point in a real hypersurface M: If the formally integrable almost complex structure is defined on $U \cup M$, where U is a domain in $\mathbb{C}^n$ and M is a piece of C^2 boundary of U that is strictly pseudoconvex with respect to the given complex structure on $U \cup M$, then for any boundary point $p \in M$ there is a smooth diffeomorphism, defined on a neighborhood of p in $U \cup M$, that transforms the complex structure into the standard one. When both the real hypersurface and the almost complex structure are C^∞, this result is due to Catlin [1] and Hanges and Jacobowitz [10].

Therefore, by restricting to $\partial D \in C^2$, we establish results for the complex structures on strictly pseudoconvex domains with the *minimum* smoothness required to define the strict Levi pseudoconvexity. For simplicity, we shall refer to the existence of global (resp. local) holomorphic coordinate systems as a global (resp. local) Newlander–Nirenberg theorem with (C^2) boundary.

To state our results more precisely, we first recall some definitions. Let $p \in \mathbb{R}^{2n}$ with $n \geq 2$. Let $X_{\overline{1}}, \dots, X_{\overline{n}}$ be vector fields defined near p and having C^1 complex coefficients. We say that $\{X_{\overline{\alpha}}\}_{\alpha=1}^n$ defines an almost complex structure near p if

$$X_{\overline{1}}, \dots, X_{\overline{n}}, \overline{X_{\overline{1}}}, \dots, \overline{X_{\overline{n}}} \text{ are } \mathbb{C}\text{-linearly independent at } p.$$

Here $\overline{X_{\overline{\alpha}}}$ denotes the complex conjugate of $X_{\overline{\alpha}}$ and

$$X_{\overline{\alpha}} = \sum_{\beta=1}^{n} \left\{ a_{\overline{\alpha}}^{\beta} \frac{\partial}{\partial z_\beta} + b_{\overline{\alpha}}^{\beta} \frac{\partial}{\partial \overline{z}_\beta} \right\}, \quad \alpha = 1, \dots, n.$$

Let $[X_{\overline{\alpha}}, X_{\overline{\beta}}] = X_{\overline{\alpha}} X_{\overline{\beta}} - X_{\overline{\beta}} X_{\overline{\alpha}}$ be the Lie bracket of $X_{\overline{\alpha}}, X_{\overline{\beta}}$. The almost complex structure is said to be *formally integrable* if in addition there exist functions $c_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}}$ such that

$$[X_{\overline{\alpha}}, X_{\overline{\beta}}] = c_{\overline{\alpha}\overline{\beta}}^{\overline{\gamma}} X_{\overline{\gamma}} \tag{1.1}$$

near p for $\alpha, \beta = 1, \dots, n$. Here we have used Einstein convention to sum over the repeated index γ, and we shall adapt this convention throughout the paper.

Recall that a domain $D \subset \mathbb{C}^n$ with C^2 boundary is said to be strictly pseudoconvex with respect to the standard complex structure at $p \in \partial D$ if there exists some open neighborhood U of p and a C^2 real-valued function $\rho : U \to \mathbb{R}$ such that the following hold: $D \cap U = \{z \in U : \rho < 0\}$, $\rho(p) = 0$, $d\rho(p) \neq 0$, and $\sum \frac{\partial^2 \rho}{\partial z_\alpha \partial \overline{z}_\beta}(p) t_\alpha t_{\overline{\beta}} > 0$ for all vectors $t \in \mathbb{C}^n \setminus \{0\}$ satisfying $\sum t_\alpha \frac{\partial \rho}{\partial z_\alpha}(p) = 0$. Finally, for $0 < a < \infty$, let $\| \cdot \|_{D,a}$ be the standard Hölder norm and let $|\cdot|_{D,a}$ be the Hölder–Zygmund norm of $\Lambda^a(\overline{D})$ for a domain $D \subset \mathbb{R}^n$. Note that Λ^a is the Hölder space C^a in equivalent norms when a is noninteger; see Section 3 for the definition of Zygmund spaces.

Our main result is the following.

THEOREM 1.1. *Let $5 < r \leq \infty$. Let D be a domain in $\mathbb{C}^n$ with C^2 boundary that is strictly pseudoconvex with respect to the standard complex structure on $\mathbb{C}^n$. Let $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta} \partial_\beta$, $\alpha = 1, \dots, n$ be $\Lambda^r(\overline{D})$ vector fields defining a formally* *integrable almost complex structure on* $\overline{D}$. *There exist positive constants* $\delta_r(D)$ *and* $\delta_5(D)$ *such that if*

$$\begin{aligned} |A|_{D,r} &< \delta_r(D), \quad r < \infty, \\ |A|_{D,5} &< \delta_5(D), \quad r = \infty, \end{aligned} \tag{1.2}$$

then there exists an embedding F *of* $\overline{D}$ *into* $\mathbb{C}^n$ *such that* $dF(X_{\bar 1}), \dots, dF(X_{\bar n})$ *are in the span of* $\frac{\partial}{\partial \overline{z}_1}, \dots, \frac{\partial}{\partial \overline{z}_n}$, *whereas* $F \in \Lambda^{r-1}(\overline{D})$ *if* $r < \infty$, $F \in C^\infty(\overline{D})$ *if* $r = \infty$, *and* $F(\overline{D})$ *is strictly pseudoconvex in* $\mathbb{C}^n$. *Moreover, the constants* $\delta_r(D)$, $\delta_5(D)$ *depend only on the* C^2 *norm of a given defining function of domain* D *and are lower stable* (*see Definition* 3.6) *under a small* C^2 *perturbation of* ∂D.

In fact, condition (1.2) can be relaxed. See Section 8 for more details. Notice that when the structure is smooth, we only need to control $|A|_{D,5}$ in order to achieve a smooth embedding.

The *lower stability* of $\delta_r(D)$ in the theorem means that for any domain $\widetilde{D}$ of which a defining function is sufficiently close to a given defining function of D in C^2 norm, we have

$$\delta_r(D) \leq C_r(D)\delta_r(\widetilde{D})$$

for some constant $C_r(D) > 0$ possibly dependent on D but independent of $\widetilde{D}$, and the theorem remains true for $\widetilde{D}$. This is an important ingredient in our proof of the local Newlander–Nirenberg theorem with C^2 boundary which we now describe.

Let U be an open subset of $\mathbb{C}^n$ and let $M \subset \partial U$ be a real C^2 hypersurface in $\mathbb{C}^n$. Let $X_{\bar 1}, \dots, X_{\bar n}$ be C^1 vector fields on $U \cup M$ that define a formally integrable almost complex structure on $U \cup M$. Let $T^{0,1}(U \cup M, X)$, denoted by $T^{0,1}_X$ also, be the span of $\{X_{\bar 1}, \dots, X_{\bar n}\}$ and $\Lambda^{0,1}(U \cup M, X)$ be its dual bundle. An integrable almost complex structure $\{X_{\overline{\alpha}}\}_{\alpha=1}^n$ induces a natural decomposition of the exterior derivative $d = \partial_X + \overline{\partial}_X$. Here $\overline{\partial}_X : \Lambda^{p,q}_X \to \Lambda^{p,q+1}_X$, $\overline{\partial}^2_X = 0$, ∂_X is its conjugate, and $\Lambda^{p,q}_X$ is the exterior algebra of smooth differential forms on $U \cup M$ of type (p, q) w.r.t $\{X_{\overline{\alpha}}\}_{\alpha=1}^n$.

We say that M is strictly pseudoconvex w.r.t. $(U \cup M, \{X_{\overline{\alpha}}\}_{\alpha=1}^n)$ if for each $p \in M$, there exists a C^2 function ρ, defined in a neighborhood ω of p such that $\omega \cap U = \{z \in \omega : \rho(z) < 0\}$, $\rho = 0$ on $M \cap \omega$, $d\rho(p) \neq 0$ and

$$\partial_X \overline{\partial}_X \rho(p)(v, \overline{v}) > 0, \quad \forall v \in T^{1,0}_p(U \cup M, X) \cap (T_p M \otimes \mathbb{C}), v \neq 0. \tag{1.3}$$

We now can state the following local Newlander–Nirenberg theorem with boundary.

THEOREM 1.2. *Let* $5 < r \leq \infty$. *Let* U *be a domain in* $\mathbb{C}^n$ *whose boundary contains a piece of* C^2 *real hypersurface* M, *and let* $X_{\bar 1}, \dots, X_{\bar n}$ *be* $\Lambda^r(U \cup M)$ *vector fields defining a formally integrable almost complex structure on* $U \cup M$. *Assume that* M *is strictly pseudoconvex with respect to* $(U \cup M, \{X_{\overline{\alpha}}\}_{\alpha=1}^n)$. *Then, for each* $p \in M$, *there exists a diffeomorphism* F *defined on a neighborhood* ω *of* p *in* $U \cup M$ *such that* $dF(X_{\bar 1}), \dots, dF(X_{\bar n})$ *are in the span of* $\frac{\partial}{\partial \overline{z}_1}, \dots, \frac{\partial}{\partial \overline{z}_n}$, *and* $F(\omega \cap M)$

is strictly pseudoconvex in $\mathbb{C}^n$, *whereas* $F \in \Lambda^{r-1}(\overline{\omega}) \cap \Lambda^{r+1}(\omega \cap U)$ *and* $F \in C^\infty(\overline{\omega})$ *if* $r = \infty$.

For $p \in U$, this is the classical Newlander–Nirenberg theorem [20; 21; 15; 13; 16; 27]. Very recently, Street [25] obtained a sharp result for the elliptic structures. By a result of Denson Hill [11], the local Newlander–Nirenberg theorem with boundary can fail for a suitable formal integrable smooth complex structure on a domain of which the boundary is smooth and has one negative Levi eigenvalue.

As mentioned above, under the assumptions that both boundary and almost complex structure are C^∞, the global Newlander–Nirenberg theorem with boundary was first proved by Hamilton [7] and the local version was shown by Catlin [1], Hanges and Jacobowitz [10] independently. In fact, Hamilton proved a more general version of Theorem 1.1 assuming that D is a relatively compact subset with smooth boundary in a complex manifold Y with $H^1(D, \mathcal{T}D) = 0$, where $\mathcal{T}D$ stands for the holomorphic tangent bundle of D. Catlin proved a local Newlander–Nirenberg theorem with smooth pseudoconvex boundary. We note that these results are all carried out in C^∞ category with $\partial D \in C^\infty$ using $\overline{\partial}$-Neumann-type methods.

To prove Theorem 1.1 under the minimum requirement of $\partial D \in C^2$, we will employ the homotopy formula methods together with a Nash–Moser type iteration. These techniques were originally employed by Webster [27; 28; 26] to prove the classical Newlander–Nirenberg theorem, the CR vector bundle problem, and the more difficult local CR embedding problem. These techniques together with a more precise interior regularity estimate for Henkin's integral solution operators of $\overline{\partial}_b$ on strictly pseudoconvex real hypersurfaces have been successfully used by the second-named author and Webster [4; 5; 6] to obtain a sharp version of the CR vector bundle problem and the local CR embedding problem. The second-named author [3] recently obtained a parameter version of Frobenius–Nirenberg theorem by using similar techniques. We also mention the work of Polyakov [23] who used similar techniques and obtained CR embeddings for a small perturbation of CR structures on compact regular 3-pseudoconcave CR submanifold M of some complex manifold X with $H^1(M, \mathcal{T}X|_M) = 0$.

The scheme of the proof of Theorem 1.1 is similar to the previous related work. However, we mention new features in the present work. First is the use of the estimate of gaining $\frac{1}{2}$ derivative for homotopy operators on the closure of a C^2 strictly pseudoconvex domain proved recently by the second-named author [2]. Note that in previous mentioned work, interior regularity estimates of $\overline{\partial}$, $\overline{\partial}_b$ for homotopy formulas are used. Another important difference is that the Nash–Moser smoothing operator [19] was applied to the interior of the domains before. In our case, we must find a way to use the Nash–Moser smoothing operator for the closure of the domain D since we are seeking global coordinate systems defined on $\overline{D}$. To use the smoothing, we simply extend the original complex structure to a neighborhood of $\overline{D}$; this simple extension, however, does not preserve the formal integrability of the extended complex structure outside D. The failure of the integrability is measured by the commutator $[\overline{\partial}, E]$, where E is an extension operator for functions on $\overline{D}$ constructed by Stein [24]. We shall make essential use of the vanishing order of $[\overline{\partial}, E]$ in our estimates. (See Sections 3 and 6 for details.) We remark here that this commutator term is the main source for losing one derivative in our results. The important commutator $[\overline{\partial}, E]$ was introduced by Peters [22] and has been used by Michel [17], Michel and Shaw [18], and others. It is also one of the main ingredients in the $\frac{1}{2}$-gain estimate [2] for a homotopy operator on a strictly pseudoconvex domain with C^2 boundary.

The plan of the paper is as follows. In Section 2, we first derive Theorem 1.2 from Theorem 1.1. In particular, we show that Catlin–Hanges–Jacobowitz's theorem is a consequence of Hamilton's theorem, the stability of $\delta_r(D)$ from Theorem 1.1, together with an initial normalization process constructed in Section 2. In Section 3, we recall basic facts about the standard Hölder–Zygmund norms, the Stein extension operator, Nash–Moser smoothing operators, and homotopy operators in [2]. In Section 4, we derive an approximate solution of the embedding via the homotopy formula. We then obtain necessary estimates for the approximate solution and the new almost complex structure in Sections 5 and 6. In Section 7, we describe the iteration scheme and verify the induction hypotheses. Finally, the convergence proof is carried out in Section 8.

2. A Reduction for Local Newlander–Nirenberg Theorem with Boundary

In this section, we derive Theorem 1.2 by using Theorem 1.1. To achieve this, we need some preparations. First, we show that one can define the strict pseudoconvexity with respect to the standard complex structure instead of the given almost complex structure near a reference point. Then we apply nonisotropic dilations to achieve the initial normalization condition: $|X_{\overline{\alpha}}z|_{\cdot} \leq \delta_r$ (the norm will be specified later), while the dilated hypersurface is close to the Heisenberg group near a reference point in C^2 norm. Here δ_r is the lower stability constant in Theorem 1.1 for some limiting domain under a nonisotropic dilation process. Finally, we construct a relatively compact C^2 strictly pseudoconvex domain U, which shares part of the boundary with M, and apply Theorem 1.1 to $(U, \{\widetilde{X}_{\overline{\alpha}}\}_{\alpha=1}^n)$, where $\{\widetilde{X}_{\overline{\alpha}}\}_{\alpha=1}^n$ is some suitable basis for the almost complex structure after dilation. We point out that the lower stability of δ_r under C^2 perturbation is crucial for this argument to work.

Throughout the paper, the Greek letters α, β, γ, and so on have range $1, 2, \ldots, n$ and Roman indices j, k, and so forth have range $1, 2, \ldots, n-1$. We denote by C_1, C_2, and so on constants bigger than 1 and by c_1, c_2, and so forth positive constants less than 1. We denote by $z = (z_1, \ldots, z_n)$ the standard coordinates of $\mathbb{C}^n$, whereas the standard complex structure on $\mathbb{C}^n$ is defined by $\partial_{\overline{\alpha}} := \frac{\partial}{\partial \overline{z}_\alpha}$, $1 \leq \alpha \leq n$. Set $\partial_\alpha := \frac{\partial}{\partial z_\alpha}$.

We will use various constants $C(D)$, $\delta(D)$, and so on, which depend only on a domain D. Thus it will be convenient to apply some standard procedures to find defining functions of a domain. A bounded domain D in $\mathbb{C}^n$ with C^k boundary with $k \geq 1$ is defined by a C^k function ρ on $\mathbb{C}^n$. Thus D is defined by $\rho < 0$ and $\nabla\rho \neq 0$ on ∂D. Locally D is defined by a C^k graph function R via

$$\rho = -y_n + R(z', x_n) < 0$$

after permuting $z_1, \dots, z_n$, and replacing z_n by iz_n or $-iz_n$ if necessary. The collection of such functions will be denoted by $\{R_i\}$. Using a partition of unity, we can construct a defining function ρ from the collection $\{R_i\}$. We may assume that $\rho = 1$ away from a neighborhood of $\overline{D}$. In such a way one can construct a defining function ρ that depends only on D, and we shall call such a ρ a *standard* defining function of D.

We start with the following elementary lemma showing how an almost complex structure changes with respect to a transformation of the form $F = I + f$, where I is the identity mapping. This lemma is essentially in Webster [27]; however, we present it here for convenience of the reader and for our later proofs.

LEMMA 2.1. *Let $\{X_{\overline{\alpha}}\}_{\alpha=1}^n$ be a C^1 almost complex structure defined near the origin of $\mathbb{R}^{2n}$.*

(i) *By a $\mathbb{R}$-linear change of coordinates of $\mathbb{C}^n$, the almost complex structure $\{X_{\overline{\alpha}}\}_{\alpha=1}^n$ can be transformed into $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta}\partial_\beta$ with $A(0) = 0$.*
(ii) *Let $F = I + f$ be a C^1 map with $f(0) = 0$ and Df small. The associated complex structure $\{dF(X_{\overline{\alpha}})\}$ has a basis $\{X'_{\overline{\alpha}}\}$ such that $X'_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A'^{\beta}_{\overline{\alpha}}\partial_\beta$. Moreover, $X_{\overline{\alpha}}F^\beta = (X_{\overline{\alpha}}F^{\overline{\gamma}})(A'^{\beta}_{\overline{\gamma}} \circ F)$. Equivalently, in the matrix form,*

$$A(z) + \partial_{\overline{z}} f + A(z)\partial_z f = (I + \partial_{\overline{z}}\overline{f(z)} + A(z)\partial_z\overline{f(z)})A' \circ F(z). \tag{2.1}$$

We remark that the formula in (ii) is valid when $F = I + f$ is a diffeomorphism of $\overline{D}$ onto $\overline{D'}$ when $\|f\|_{D,1}$ is sufficiently small.

Proof. (i) Let $U_\alpha = \frac{1}{2}(X_\alpha + X_{\overline{\alpha}})$ and $V_\alpha = \frac{\sqrt{-1}}{2}(X_{\overline{\alpha}} - X_\alpha)$. We would like to find another coordinate system $w_\alpha = u_\alpha + \sqrt{-1}v_\alpha$ such that $\frac{\partial}{\partial u_\alpha}|_{w=0} = U_\alpha(0)$ and $\frac{\partial}{\partial v_\alpha}|_{w=0} = V_\alpha(0)$. Since $\{U_\alpha, V_\alpha\}$ and $\{\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}\}$ both span $T_0\mathbb{C}^n$, then at 0 we have

$$U_\alpha = a_\alpha^\beta \frac{\partial}{\partial x_\beta} + b_\alpha^\beta \frac{\partial}{\partial y_\beta}, \qquad V_\alpha = c_\alpha^\beta \frac{\partial}{\partial x_\beta} + d_\alpha^\beta \frac{\partial}{\partial y_\beta},$$

where the coefficient matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is invertible. Set $x = au + bv$ and $y = cu + dv$. In new variables w, we have $X_{\overline{\alpha}} = A_{\overline{\alpha}}^{\overline{\gamma}}\partial_{\overline{\gamma}} + A_{\overline{\alpha}}^{\beta}\partial_\beta$ where $(A_{\overline{\alpha}}^{\overline{\gamma}}(0))$ is the identity matrix (δ_α^γ) and $A_{\overline{\alpha}}^\beta(0) = 0$. In particular, $(A_{\overline{\alpha}}^{\overline{\gamma}})$ is invertible near 0. We can then use a linear combination to achieve $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^\beta\partial_\beta$ with $A(0) = 0$.

(ii) Let us show the existence of such a basis by determining the coefficient matrix A'. Since $T^{1,0}_{dF(X_{\overline{\alpha}})} = dFT^{1,0}_X$, we know $F_* X_{\overline{\alpha}} = C_{\overline{\alpha}}^{\overline{\beta}} X'_{\overline{\beta}}$ for some invertible matrix $(C_{\overline{\alpha}}^{\overline{\beta}})$. Apply both sides to $F^{\overline{\beta}}$ and use $X_{\overline{\beta}} = \partial_{\overline{\beta}} + A'^{\gamma}_{\overline{\beta}}\partial'_\gamma$. Then we have

$(C_{\overline{\alpha}}^{\overline{\beta}}) = (X_{\overline{\alpha}} F^{\overline{\beta}})$, which is invertible when Df is small. Consequently,

$$X_{\overline{\alpha}} F^{\beta} \partial_{\beta} + X_{\overline{\alpha}} F^{\overline{\beta}} \partial_{\overline{\beta}} = F_* X_{\overline{\alpha}} = C_{\overline{\alpha}}^{\overline{\beta}} X'_{\overline{\beta}}.$$

Comparing the coefficients of $\partial_{\overline{\beta}}$ and ∂_{β}, we see that $C_{\overline{\alpha}}^{\overline{\beta}} = X_{\overline{\alpha}} F^{\overline{\beta}}$ and

$$X_{\overline{\alpha}} F^{\beta} = (X_{\overline{\alpha}} F^{\overline{\gamma}})(A'^{\beta}_{\overline{\gamma}} \circ F).$$

Since $(C_{\overline{\alpha}}^{\overline{\gamma}})$ is invertible, we have $A'^{\beta}_{\overline{\gamma}} = (C^{-1})^{\overline{\alpha}}_{\overline{\gamma}} (X_{\overline{\alpha}} F^{\beta}) \circ F^{-1}$.

Now identity (2.1) follows from

$$\begin{aligned} X_{\overline{\alpha}} F^{\beta} &= (\partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\gamma} \partial_{\gamma})(z^{\beta} + f^{\beta}) \\ &= A_{\overline{\alpha}}^{\beta} + \partial_{\overline{\alpha}} f^{\beta} + A_{\overline{\alpha}}^{\gamma} \partial_{\gamma} f^{\beta} = (A + \overline{\partial} f + A \partial f)_{\overline{\alpha}}^{\beta}, \\ (X_{\overline{\alpha}} F^{\overline{\gamma}})(A'^{\beta}_{\overline{\gamma}} \circ F) &= (\partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\gamma} \partial_{\gamma})(z^{\overline{\beta}} + f^{\overline{\beta}})(A' \circ F) \\ &= ((I + \overline{\partial f} + A \partial \overline{f})(A' \circ F))_{\overline{\alpha}}^{\beta}. \end{aligned}$$ □

Using the integrability condition, we now remove the first order term in the Taylor expansion of A at the origin.

LEMMA 2.2. *Suppose that a C^1 almost complex structure defined by $\{X_{\overline{\alpha}}\}$ with $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta} \partial_{\beta}$ and $A(0) = 0$ satisfies the integrability condition (1.1) at 0. Then we can make a polynomial change of coordinates such that in the new coordinate system the almost complex structure is given by $X'_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A'^{\beta}_{\overline{\alpha}} \partial_{\beta}$ with $A'(0) = 0$ and $DA'(0) = 0$.*

Proof. Throughout the paper, we write $f(\zeta) = o(|\zeta|^k)$ if $\lim_{|\zeta| \to 0} f(x)/|\zeta|^k = 0$, where ζ are real or complex variables. We make a polynomial change of coordinates $F = I + f$, where

$$f^{\beta} = -\partial_{\gamma} A_{\overline{\alpha}}^{\beta}(0) z^{\gamma} \overline{z}^{\alpha} - \frac{1}{2} \partial_{\overline{\gamma}} A_{\overline{\alpha}}^{\beta}(0) \overline{z}^{\gamma} \overline{z}^{\alpha}.$$

According to Lemma 2.1, we have

$$A + \overline{\partial} f + A \partial f = (I + \overline{\partial f} + A \partial \overline{f}) A' \circ F.$$

Shrinking the domain if necessary, we can assume that $(I + \overline{\partial f} + A \partial \overline{f})$ and F are invertible. Therefore, in order to show $A'(z) = o(|z|)$, it suffices to show that $A(z) + \overline{\partial} f(z) + A(z) \partial \overline{f}(z) = o(|z|)$.

Since our structure satisfies the integrability condition at 0, then $[X_{\overline{\alpha}}, X_{\overline{\beta}}]$ is in the span of $X_{\overline{1}}, \dots, X_{\overline{n}}$ at 0. This implies that via $A(0) = 0$,

$$\partial_{\overline{\alpha}} A_{\overline{\beta}}^{\gamma}(0) = \partial_{\overline{\beta}} A_{\overline{\alpha}}^{\gamma}(0). \tag{2.2}$$

Plugging (2.2) into $A + \overline{\partial} f + A \partial f$, we get

$$\begin{aligned} &A_{\overline{\alpha}}^{\beta} + \partial_{\overline{\alpha}} f^{\beta} + A_{\overline{\alpha}}^{\gamma} \partial_{\gamma} f^{\beta} \\ &\quad = A_{\overline{\alpha}}^{\beta} - \partial_{\gamma} A_{\overline{\alpha}}^{\beta}(0) z^{\gamma} - \frac{1}{2} \partial_{\overline{\gamma}} A_{\overline{\alpha}}^{\beta}(0) \overline{z}^{\gamma} - \frac{1}{2} \partial_{\overline{\alpha}} A_{\overline{\gamma}}^{\beta}(0) \overline{z}^{\gamma} - A_{\overline{\alpha}}^{\rho} \partial_{\rho} A_{\overline{\gamma}}^{\beta}(0) \overline{z}^{\gamma} \end{aligned}$$

$$= A_{\overline{\alpha}}^{\beta} - \partial_\gamma A_{\overline{\alpha}}^{\beta}(0) z^\gamma - \partial_{\overline{\gamma}} A_{\overline{\alpha}}^{\beta}(0)\overline{z}^\gamma - A_{\overline{\alpha}}^{\rho}\partial_\rho A_{\overline{\gamma}}^{\beta}(0)\overline{z}^\gamma,$$

where we have used the integrability condition at 0 in the second equality. Now it is clear that the right-hand side vanishes up to second order at the origin, and thus the lemma follows. □

Note that in the formulation of Theorem 1.2, we require the boundary to be strictly pseudoconvex with respect to the given almost complex structure. However, in order to apply Theorem 1.1, it is important that the boundary is strictly pseudoconvex with respect to the standard complex structure. The next lemma shows that these two conditions are locally equivalent provided the given structure and the standard one agree up to second order at a reference point after some initial normalization.

LEMMA 2.3. *Let $M \subset \partial U$ be a C^2 real hypersurface. Let $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta}\partial_\beta$, $\alpha = 1, \dots, n$, define a formally integrable C^1 complex structure on the one-sided domain $U \cup M$. Suppose that $0 \in M$ and $A(z) = o(|z|)$. Assume that M is strictly pseudoconvex with respect to $(U \cup M, \{X_{\overline{\alpha}}\})$ (see (1.3) for definition). The following hold*:

- (i) *M is strictly pseudoconvex with respect to the standard complex structure near the origin.*
- (ii) *After a local polynomial change of coordinates that preserves the condition $A(z) = o(|z|)$, there exists a defining function r for M, defined near the origin, such that $\rho < 0$ on U, $\rho = 0$ on M, and*

$$\rho(z) = -y_n + |z'|^2 + h(z', x_n),$$

where $h = o(2)$ is a C^2 function.

Proof. (i) Since $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta}\partial_\beta$, $\alpha = 1, 2, \dots, n$, form a basis of $T_X^{0,1}\mathbb{C}^n$ near 0, we can find the dual frames ω^α, $\omega^{\overline{\alpha}}$ of X_α, $X_{\overline{\alpha}}$ near 0. Let $\Lambda_{0,X}^{p,q}$ denote the germ of smooth (p,q) forms with respect to the almost complex structure X at 0. More precisely, $u = u_{I\overline{J}}\omega^I \wedge \omega^{\overline{J}} \in \Lambda_{0,X}^{p,q}$, where I, J are multi-indices, $|I| = p$, $|J| = q$, and $u_{I\overline{J}}$ are elements in the germ of smooth functions at 0. Denote the decomposition of the exterior derivative with respect to $\{X_{\overline{\alpha}}\}$ by $d = \partial_X + \overline{\partial}_X$. See [14, P. 126]. Thus, for a function r,

$$dr = X_\alpha r\omega^\alpha + X_{\overline{\alpha}} r\omega^{\overline{\alpha}},$$

$$\partial_X\overline{\partial}_X r = (X_\alpha X_{\overline{\beta}} r + X_{\overline{\gamma}} r C_{\alpha\overline{\beta}}^{\overline{\gamma}})\omega^\alpha \wedge \omega^{\overline{\beta}},$$

where $C_{\alpha\overline{\beta}}^{\overline{\gamma}} = -\omega^{\overline{\gamma}}([X_\alpha, X_{\overline{\beta}}])$. Notice in particular that $C_{\alpha\overline{\beta}}^{\overline{\gamma}}(0) = 0$ since $A = o(|z|)$.

According to (1.3), we need to show that $\sqrt{-1}\partial_X\overline{\partial}_X r(0)$ is positive-definite on $\mathbb{L}_X$ if and only if $\sqrt{-1}\partial\overline{\partial}\rho(0) > 0$ on $\mathbb{L}$ where $\mathbb{L}_X = T_X^{1,0}\mathbb{C}^n \cap \mathbb{C}T_0M$ and $\mathbb{L} = T^{1,0}\mathbb{C}^n \cap \mathbb{C}T_0M$.

Write $r(z) = \mathrm{Im}(\sqrt{-1}c_\alpha z^\alpha) + O(2)$. Since $d\rho \neq 0$ at 0, we may assume $\frac{\partial\rho}{\partial y_n} \neq 0$ by permuting coordinates, which preserves conditions on $A_{\overline{\alpha}}^{\beta}$. Consequently, the linear transformation $(z', z^n) \to (z', \sqrt{-1}c_\alpha z^\alpha)$ preserves $A(z) = o(|z|)$, and we have

$$\rho = -y_n + O(2).$$

Applying the implicit function theorem to $\rho(z', x_n, y_n) = 0$, we obtain $y_n = F(z', x_n)$, where $F = O(2)$. More precisely, we can write

$$\begin{aligned} y_n = F(z', x_n) &= a_{j\overline{k}} z^j \overline{z}^j + 2\,\mathrm{Re}(b_{jk} z^j z^k + c_j z^j x^n) + o(|z'|^2 + x_n^2) \\ &= a_{j\overline{k}} z^j \overline{z}^k + 2\,\mathrm{Re}(b_{jk} z^j z^k + c_j z^j z^n) + o(|z'|^2 + x_n^2), \end{aligned}$$

where in the second line, we use the fact that $F = O(2)$.

Then it is easy to see that $L_i = X_i - \frac{X_i\rho}{X_n\rho} X_n$, $i = 1, \ldots, n-1$, form a basis for $\mathbb{L}_X$ near 0. Therefore, near the origin, we have

$$\begin{aligned} &\partial_X \overline{\partial}_X \rho(L_i, L_{\overline{j}}) \\ &\quad = (X_\alpha X_{\overline{\beta}}\rho + X_{\overline{\gamma}}\rho C_{\alpha\overline{\beta}}^{\overline{\gamma}})(\omega^\alpha \wedge \omega^{\overline{\beta}}) \left(X_i - \frac{X_i\rho}{X_n\rho} X_n, X_{\overline{j}} - \frac{X_{\overline{j}}\rho}{X_{\overline{n}}\rho} X_{\overline{n}} \right). \end{aligned}$$

Using the fact that $A = o(|z|)$, $C_{\alpha\overline{\beta}}^{\overline{\gamma}}(0) = 0$ and $\frac{\partial\rho}{\partial z_i}(0) = 0$ for $i = 1, 2, \ldots, n-1$, we have

$$\partial_X \overline{\partial}_X \rho(L_i, L_{\overline{j}})(0) = (\partial_\alpha \partial_{\overline{\beta}}\rho)(0)(dz^\alpha \wedge dz^{\overline{\beta}})(\partial_i, \partial_{\overline{j}}).$$

The proof is then complete by noticing that $\{\partial_i\}_{i=1}^{n-1}$ form a basis of $\mathbb{L}$ at 0 with respect to the standard complex structure.

(ii) According to the first part, it suffices to show the conclusion for a strictly pseudoconvex C^2 real hypersurface in $\mathbb{C}^n$. The proof is standard, but we include it here to ensure that the condition $A = o(|z|)$ is preserved, which is required in the next lemma.

Let us make a second order change of coordinates

$$z' \to z', \qquad z^n \to z^n - \sqrt{-1}(b_{jk} z^j z^k + c_{\alpha n} z^\alpha z^n),$$

which clearly preserves the condition $A(z) = o(|z|)$ according to part (ii) of Lemma 2.1. We have

$$\rho = -y_n + a_{jk} z^j \overline{z}^k + h(z', x_n), \quad h(z', x_n) = o(|z'|^2 + |x_n|^2).$$

Since M is strictly pseudoconvex, we see that the Hermitian matrix $(a_{j\overline{k}})$ is positive definite in z'. The final expression then follows from a complex linear change of coordinates. Note that the latter also preserves $A(z) = o(|z|)$. It is clear from our construction that we still have $h(0) = Dh(0) = D^2h(0) = 0$. □

We can now reformulate Theorem 1.2 in an equivalent form that requires the boundary to be strictly pseudoconvex with respect to the standard complex structure. Indeed, since the integrability condition of our almost complex structure holds at the origin by continuity, we can assume that $A = o(|z|)$ by Lemma 2.2.

Then, according to Lemma 2.3, the two assumptions are equivalent. Next, we achieve initial normalization by a nonisotropic dilation.

Recall that for $a \in (0, \infty)$, $\|\cdot\|_{D,r}$ stands for the standard Hölder norm on D and $|\cdot|_{D,r}$ stands for the Hölder–Zygmund norm (see definition in Section 3).

PROPOSITION 2.4. *Let $M \subset \partial U$ be a C^2 real hypersurface containing the origin. Let $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta}\partial_\beta$, $\alpha = 1, \dots, n$, in $\Lambda^r(U \cup M)$, $1 < r < \infty$, define an integrable almost complex structure on the one-sided domain $U \cup M$ with $A(z) = o(|z|)$. Assume that M is strictly pseudoconvex with respect to $(U \cup M, \{X_{\overline{\alpha}}\})$. Then, after a nonisotropic dilation $\phi_\varepsilon(z', z_n) = (\varepsilon^{-1}z', \varepsilon^{-2}z_n)$, where $\varepsilon > 0$ is sufficiently small, we have the following*:

(i) *There exist some open set $B \subset \mathbb{C}^n$ and a C^2 function $\rho_\varepsilon : B \to \mathbb{R}$ such that $D_\varepsilon = \{z \in B : \rho_\varepsilon(z) < 0\} \subset \phi_\varepsilon(U \cup M)$ is a connected C^2 strictly pseudoconvex domain that shares part of the boundary with $\phi_\varepsilon(M)$ near the origin. Moreover, there exists a C^2 function $\rho_0 : B \to \mathbb{R}$ such that* $\lim_{\varepsilon \to 0}\|\rho_\varepsilon - \rho_0\|_{B,2} = 0$ *and $D_0 := \{z \in B : \rho_0(z) < 0\}$ is also a connected C^2 strictly pseudoconvex domain.*

(ii) *On $\overline{D_\varepsilon}$ each $d\phi_\varepsilon X_{\overline{\beta}}$ is spanned by $X_{\overline{\alpha}}^{\varepsilon} = \partial_{\overline{\alpha}} + (A^{(\varepsilon)})_{\overline{\alpha}}^{\beta}\partial_\beta$, $\alpha = 1, \dots, n$, where $|A^{(\varepsilon)}|_{D_\varepsilon, r'}$ tends to 0 with ε for any finite $r' \leq r$.*

Proof. (i) By the second part of Lemma 2.3, we can assume that the defining function of $M \cup U$ is locally given by

$$\rho(z) = -y_n + |z'|^2 + h(z', x_n), \quad h = o(|z'|^2 + x_n^2),$$

and the condition $A(z) = o(|z|)$ is preserved. Let us denote $B_a = \{(z', z_n) \in \mathbb{C}^n : |z| < a\}$ for any $a > 0$.

We apply the dilation ϕ_ε to $U \cup M$. Then the new defining function for $M_\varepsilon := \phi_\varepsilon(M)$ can locally be written as

$$\hat{\rho}_\varepsilon(z) = -y_n + |z'|^2 + \varepsilon^{-2}h(\varepsilon z', \varepsilon^2 x_n), \quad h = o(|z'|^2 + x_n^2).$$

Without loss of generality, we may assume that $\hat{\rho}_\varepsilon$ are defined on B_3. Moreover, we have $\phi_\varepsilon(U \cup M) \cap B_2 = \{z \in B_2 : \hat{\rho}_\varepsilon(z) < 0\}$. Then we shall construct a C^2 strictly pseudoconvex domain D_ε such that

$$B_1 \cap \phi_\varepsilon(U \cup M) \subset \overline{D_\varepsilon} \subset B_2 \cap \overline{\phi_\varepsilon(U \cup M)}$$

as follows.

Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a smooth nondecreasing convex function such that $\chi = 0$ on $(-\infty, 1]$ and $\chi(4) = 1$. Moreover, we assume $0 < \chi'(x) \leq 1$ for $x \in (1, 4)$. Define $D_\varepsilon = \{z \in B_2 : \rho_\varepsilon(z) < 0\}$, where

$$\rho_\varepsilon(z) = -y_n + |z'|^2 + \varepsilon^{-2}h(\varepsilon z', \varepsilon^2 x_n) + 5\chi(|z|^2).$$

Note that $D_\varepsilon \subset \phi_\varepsilon(U \cup M)$ since we have added a nonnegative term to $\hat{\rho}_\varepsilon$.

We check that D_ε satisfies all the requirements. Since $\chi = 0$ on $(-\infty, 1]$, one has $B_1 \cap \phi_\varepsilon(U \cup M) \subset D_\varepsilon$. In order to show $\overline{D_\varepsilon} \subset B_2 \cap \overline{\phi_\varepsilon(U \cup M)}$, it suffices to show that for all $z \in \partial B_2$, we have $r_\varepsilon(z) > 0$, that is,

$$5\chi(4) = 5 > y_n - |z'|^2 - \varepsilon^{-2} h(\varepsilon z', \varepsilon^2 x_n).$$

This holds for ε small since $h(z', x_n) = o(|z'|^2 + x_n^2)$.

We want to prove that ρ_ε defines a strictly pseudoconvex domain with C^2 boundary.

Clearly, ρ_ε is a C^2 function. It then suffices to show that (a). $d\rho_\varepsilon(z) \neq 0$ for all $z \in \partial D_\varepsilon$; (b). $\lambda := \inf \frac{\partial^2 \rho_\varepsilon}{\partial z_i \partial \overline{z}_j}(z) t_i t_{\overline{j}} > 0$, where the infimum is taken for $z \in \partial D_\varepsilon$, $\sum_{j=1}^n t_j \partial_j \rho_\varepsilon = 0$, and $|t| = 1$.

Note that $-y_n + |z'|^2$ and $\chi(|z|^2)$ are plurisubharmonic. On $\partial D_\varepsilon \cap M_\varepsilon$, we have $\rho_\varepsilon = \hat{\rho}_\varepsilon$. It is clear that there is a positive constant c_0 such that when ε is small,

$$\lambda_0 := \inf_{t \in T_x M_\varepsilon, |t|=1, x \in \partial D_\varepsilon \cap M_\varepsilon} \frac{\partial^2 \hat{\rho}_\varepsilon}{\partial z_i \partial \overline{z}_j}(z) t_i t_{\overline{j}} > c_0.$$

We can find a neighborhood N of $M_\varepsilon \cap \partial D_\varepsilon$, independent of ε, such that

$$\lambda_1 := \inf_{t \in T_x M, |t|=1, x \in \partial D^\varepsilon \cap N} \frac{\partial^2 \rho_\varepsilon}{\partial z_i \partial \overline{z}_j}(z) t_i t_{\overline{j}} > c_0/2.$$

On $D_\varepsilon \setminus N$, we have $-y_n + |z'|^2 \leq -c_0'$, where c_0' is a positive constant, and hence

$$\chi(|z|^2) \geq c_0'/5, \quad z \in \partial D_\varepsilon \setminus N.$$

Note that $\chi(|z|^2)$ is strictly plurisubharmonic at z when $\chi(|z|^2) > 0$. Therefore,

$$\lambda_2 := \inf_{t \in T_x M_\varepsilon, |t|=1, x \in \partial D_\varepsilon \setminus N} \frac{\partial^2 \rho_\varepsilon}{\partial z_i \partial \overline{z}_j}(z) t_i t_{\overline{j}} > c_1 > 0.$$

This shows that $\lambda > \min(\lambda_0, \lambda_1, \lambda_2)/2$ when ε is small. Therefore, D_ε is a C^2 strictly pseudoconvex domain.

It is obvious that $\lim_{\varepsilon \to 0} \|\rho_\varepsilon - \rho_0\|_{B,2} = 0$. Note that r_0 is a convex function. Thus D_0 is connected, and the connectedness of D_ε follows easily from the C^2 convergence.

(ii) To find the new vector fields, we let $X_{\overline{\alpha}}^{(\varepsilon)} = \varepsilon(\phi_\varepsilon)_*(X_{\overline{\alpha}})$, $X_{\overline{n}}^{(\varepsilon)} = \varepsilon^2(\phi_\varepsilon)_*(X_{\overline{n}})$. Then, for $1 \leq j, k \leq n-1$, we have

$$X_{\overline{j}}^{(\varepsilon)} = \partial_{\overline{j}}' + (A^\varepsilon)_{\overline{j}}^k \partial_k' + \varepsilon^{-1} (A^\varepsilon)_{\overline{j}}^n \partial_n',$$
$$X_{\overline{n}}^{(\varepsilon)} = \partial_{\overline{n}}' + \varepsilon (A^\varepsilon)_{\overline{n}}^k \partial_k' + (A^\varepsilon)_{\overline{n}}^n \partial_n',$$

where $\partial.$ are vector fields associated to new coordinate and $(A^\varepsilon)(z) := A(\varepsilon z', \varepsilon^2 z_n)$.

Since $A(z) = o(|z|)$, then $|A^{(\varepsilon)}|_{D_\varepsilon, r'} \to 0$ as $\varepsilon \to 0$ for any finite $r' \leq r$. □

Assuming that Theorem 1.1 holds, we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. According to Lemma 2.3, we may assume without loss of generality that $M \cup U = \{z \in U_0 : \rho(z) \leq 0\}$, where $\rho(z) = -y_n + |z'|^2 + h(z', x_n)$, $h = o(2)$, and $h \in C^2(V)$ on some neighborhood V of the origin. Let ρ_ε and ϕ_ε be as in Proposition 2.4.

First, we let $5 < r < \infty$ and apply Lemma 2.4 to $\{U \cup M, \{X_{\overline{\alpha}}\}_{\alpha=1}^n\}$ with ε to be determined. Then we obtain a C^2 strictly pseudoconvex domain $D_\varepsilon \subset \phi_\varepsilon(U \cup M)$, which shares part of the boundary with M_ε. Moreover, there exists a new basis $\{X_{\overline{\alpha}}^{(\varepsilon)}\}_{\alpha=1}^n \in C^r(\overline{D_\varepsilon})$ for $\{(\phi_\varepsilon)_* X_{\overline{1}}^{(\varepsilon)}, \dots, (\phi_\varepsilon)_* X_{\overline{n}}^{(\varepsilon)}\}$ such that $|A^{(\varepsilon)}|_{D_\varepsilon, r}$ tends to 0 as $\varepsilon \to 0$ and a limiting C^2 strictly pseudoconvex domain with defining function ρ_0 such that $\|\rho_\varepsilon - \rho_0\|_{B_2,2}$ tends to 0 as $\varepsilon \to 0$.

According to Theorem 1.1, there exists $\delta_r(D_0) > 0$ that is lower stable under a small C^2 perturbation of D_0. Therefore, we can find ε sufficiently small such that

$$\begin{aligned} |A^{(\varepsilon)}|_{D_\varepsilon, r} &\le \delta_r(D_0)/C(D_0), \\ \delta_r(D_0) &\le C(D_0)\delta_r(r_\varepsilon). \end{aligned} \tag{2.3}$$

Here ε is chosen for the $C(D_0)$ in (2.3). Therefore, we have

$$|A^{(\varepsilon)}|_{D_\varepsilon, r} \le \delta_r(r_\varepsilon).$$

Consequently, we are able to apply Theorem 1.1 to $(D_\varepsilon, X_{\overline{\alpha}}^{(\varepsilon)})$ to obtain a $\Lambda^{r-1}(\overline{D_\varepsilon})$ diffeomorphism $F_\varepsilon : D_\varepsilon \to \mathbb{C}^n$ onto its image that sends the almost complex structure to the standard one. Since D_ε shares part of the boundary with M_ε, F_ε induces a diffeomorphism near $0 \in M_\varepsilon$ that sends the integrable almost complex structure to the standard one on one side of the domain. The embedding F is then given by $F_\varepsilon \circ \phi_\varepsilon^{-1}$.

Finally, we consider the case $r = \infty$. Notice that merely $|A|_{D,5} \le \delta_5(D_0)$ is required for the statement of Theorem 1.1 to be valid. Therefore, we do not need to control higher order derivatives of the error and the previous argument still applies. The proof of Theorem 1.2 is complete. □

3. Preliminaries

In this section, we present some preliminaries for the proof of Theorem 1.1. First, we recall some basic results for standard Hölder norms $\|\cdot\|_{D,a}$, $0 \le a < \infty$, and Hölder–Zygmund norms $|\cdot|_{D,a}$, $0 < a < \infty$, on domains $D \subset \mathbb{R}^n$ with cone property. We then introduce three main tools used in the proof: the Stein extension operator, the Nash–Moser smoothing operator, and the homotopy formula on strictly pseudoconvex domain with C^2 boundary in [2]. We also include necessary estimates for these operators for later use.

3.1. *Convexity of Hölder–Zygmund Norms*

We say that a domain D in $\mathbb{R}^n$ has the cone property if there exists some $C_* = C_*(D) > 0$ such that the following hold.

(1) Given two points p_0, p_1 in D, there exists a piecewise C^1 curve $\gamma(t)$ in D such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$, $|\gamma'(t)| \le C_*|p_1 - p_0|$ for all t except finitely many values.
(2) For each point $x \in \overline{D}$, D contains a cone V with vertex x, opening $\theta > C_*^{-1}$, and height $h > C_*^{-1}$.
(3) The diameter of D is less than C_*.

For a domain with cone property, the following Hölder estimates for interpolation, product rule, and chain rule are well known. For instance, see the appendices of [5; 2] or [12] for proofs and more details:

$$\|u\|_{D,(1-\theta)a+\theta b} \leq C_{a,b}\|u\|_{D,a}^{1-\theta}\|u\|_{D,b}^{\theta}, \qquad 0 \leq \theta \leq 1, \tag{3.1}$$

$$\|uv\|_{D,a} \leq C_a(\|u\|_{D,a}\|v\|_{D,0} + \|u\|_{D,0}\|v\|_{D,a}), \tag{3.2}$$

$$\|u \circ g\|_{D,a} \leq C_a(\|u\|_{\widetilde{D},a}\|g\|_{D,1}^{a} + \|u\|_{\widetilde{D},1}\|g\|_{D,a} + \|u\|_{\widetilde{D},0}). \tag{3.3}$$

If $(a,b) = \theta(a_1,b_1) + (1-\theta)(a_2,b_2)$, $0 \leq \theta \leq 1$, we have

$$\|u\|_{D,a}\|v\|_{D',b} \leq C_{a,b}(\|u\|_{D,a_1}\|v\|_{D',b_1} + \|u\|_{D,a_2}\|v\|_{D',b_2}). \tag{3.4}$$

We now recall the definition of Hölder–Zygmund spaces and basic properties. For $0 < r \leq 1$, the Hölder–Zygmund space $\Lambda^r(\mathbb{R}^n)$ is the set of functions $f \in L^\infty(\mathbb{R}^n)$ such that

$$|f|_{\mathbb{R}^n,r} := |f|_{L^\infty(\mathbb{R}^n)} + \sup_{0 \neq y \in \mathbb{R}^n} \frac{|\Delta_y^2 f|_{L^\infty(\mathbb{R}^n)}}{|y|^r}. \tag{3.5}$$

Here $\Delta_y f(x) := f(x+y) - f(x)$ and thus $\Delta_y^2 f(x) = f(x+2y) + f(x) - 2f(x+y)$. When $r > 1$, we define $\Lambda^r(\mathbb{R}^n)$ to be the set of functions $f \in C^{[r]-1}(\mathbb{R}^n)$ satisfying

$$|f|_{\mathbb{R}^n,r} := |f|_{L^\infty(\mathbb{R}^n)} + |\partial f|_{\mathbb{R}^n,r-1} < \infty.$$

For a noninteger r, $|\cdot|_{\mathbb{R}^n,r}$ is equivalent to the Hölder norm $\|\cdot\|_{\mathbb{R}^n;r}$; when $1 < r < 2$, $|\cdot|_{\mathbb{R}^n,r}$ is also equivalent to the norm defined by (3.5). See [24, Prop. 8, p. 146], and by the equivalence of the two norms one means

$$c_r\|f\|_{\mathbb{R}^n,r} \leq |f|_{\mathbb{R}^n,r} \leq C_r\|f\|_{\mathbb{R}^n,r}$$

for two positive numbers c_r, C_r depending only on r. Clearly, we have c_r tends to 0 when r tends to a positive integer and $C_r \leq 2$. Let F be a closed subset in $\mathbb{R}^n$. Let $r \in (0,\infty)$. We write $f \in \Lambda^r(F)$ if there exists $\widetilde{f} \in \Lambda^r(\mathbb{R}^n)$ such that $\widetilde{f}|_F = f$. Define $|f|_{F,r}$ to be the infimum of $|\widetilde{f}|_{\mathbb{R}^n,r}$ for all such extensions $\widetilde{f}$.

Next, we recall the extension operator constructed by Stein [24]. Given a bounded domain $D \subset \mathbb{R}^n$ with Lipschitz boundary (i.e. the boundary is locally the graph of some Lipschitz function), there exists an extension operator

$$E : \Lambda^r(\overline{D}) \to \Lambda^r(\mathbb{R}^n) \quad \text{with } |Ef|_{\mathbb{R}^n,r} \leq C_r(D)|f|_{\overline{D},r}, \forall r \in (0,\infty), \tag{3.6}$$

where the operator norm $C_r(D)$ depends only on the Lipschitz constants of finitely many graph functions of the boundary. In fact, Stein [24] proved the above estimates for Sobolev spaces. The estimates (3.6) for Zygmund spaces can be found in [2]. We refer the reader to [2; 24] for more details on the construction and the estimates. When there is no confusion, we write $|\cdot|_{\overline{D},r}$ as $|\cdot|_r$.

We now derive convexity in Hölder–Zygmund norms. It is clear that the second term in (3.5) satisfies

$$\sup_{y \in \mathbb{R}^n} \frac{|\Delta_y^2 f|_{L^\infty(\mathbb{R}^n)}}{|y|^{(1-\theta)r_0+\theta r_1}} \leq \left(\sup_{y \in \mathbb{R}^n} \frac{|\Delta_y^2 f|_{L^\infty(\mathbb{R}^n)}}{|y|^{r_0}}\right)^{1-\theta} \left(\sup_{y \in \mathbb{R}^n} \frac{|\Delta_y^2 f|_{L^\infty(\mathbb{R}^n)}}{|y|^{r_1}}\right)^{\theta}$$

for $r_0, r_1 \in (0,2)$ and $\theta \in (0,1)$. Then we get

$$|u|_{(1-\theta)a+\theta b} \le C_{b-a}|u|_a^{1-\theta}|u|_b^{\theta}, \quad 0\le\theta\le 1, \tag{3.7}$$

for $0<b-a<2$ and $\theta\in(0,1)$. Hence, it also holds for all positive a,b; indeed, suppose it holds for $0<b-a<d$. Suppose $a<c<b$ with $d/2<b-a<3d/2$ and $c=(1-\theta)a+\theta b$. We take e so that $0<\max(b-e,e-a)<d$. We may assume that $c<e$. Thus

$$|u|_c \le C|u|_a^{\frac{e-c}{e-a}}|u|_e^{\frac{c-a}{e-a}}, \qquad |u|_e \le C|u|_c^{\frac{b-e}{b-c}}|u|_b^{\frac{e-c}{b-c}}.$$

Eliminating $|u|_e$ and solving for $|u|_c$, we get (3.7). By (3.6) and (3.7), for D we get

$$|u|_{D,(1-\theta)a+\theta b} \le C_{a,b}(D)|u|_{D,a}^{1-\theta}|u|_{D,b}^{\theta}, \quad 0\le\theta\le1, a,b>0. \tag{3.8}$$

LEMMA 3.1. *Let D, $\widetilde{D}$ be connected bounded domains with Lipschitz boundary, and let g map D into $\widetilde{D}$. Suppose that $\|g\|_{D;1}<C$. Then we have*

$$\left|\prod_{j=1}^k u_j\right|_{D,a} \le C_a(D)C_{1/\varepsilon}\sum_{i\neq j}|u_i|_{D,a}\|u_j\|_{D,\varepsilon}\prod_{\ell\neq i,j}\|u_\ell\|_{D,0}, \quad a>0; \tag{3.9}$$

$$\left|\frac{1}{u}\right|_{D,a} \le C_a(D)(1+\|u^{-1}\|_0)^{[a]+2}(1+C_{1/\varepsilon}\|u\|_{D,\varepsilon}^{a+1})|u|_{D,a}; \tag{3.10}$$

$$|u\circ g|_{D,1} \le C(D)C(\widetilde{D})|u|_{\widetilde{D},1}(1+C_{1/\varepsilon}\|g\|_{D,1+\varepsilon}^{\frac{1}{1+\varepsilon}}); \tag{3.11}$$

$$\begin{aligned}|u\circ g|_{D,a} \le C_a(D)C_a(\widetilde{D})(&|u|_{\widetilde{D},a}(1+C_{1/\varepsilon}\|g\|_{D,1+\varepsilon}^{\frac{1+2\varepsilon}{1+\varepsilon}})\\ &+\|u\|_{\widetilde{D},1+\varepsilon}|g|_{D,a}+\|u\|_{\widetilde{D},0}), \quad a>1.\end{aligned} \tag{3.12}$$

Here, ε is any positive number and $C_{1/\varepsilon}$ is a positive constant depending on ε that tends to ∞ as $\varepsilon\to0$.

Proof. Note that stronger inequalities for Hölder norms are given by (3.1)–(3.3). We only need to verify the lemma when a is an integer. Here we need a bit more for low order derivatives for g. We will also employ the Stein extension operator.

For the product rule, by Stein extension it suffices to consider the case $D=\mathbb{R}^n$, and u, v have compact support in a ball of fixed radius.

Let $k=a-1\ge0$ be an integer. We have

$$(uv)^{(k)} = u^{(k)}v+uv^{(k)}+\sum_{0<j<k}B_{k,j}u^{(k-j)}v^{(j)}. \tag{3.13}$$

We remark that we do not have (3.4) for Zygmund norms. We have

$$\begin{aligned}&|\Delta_y^2(u^{(k)}v)(x)|\\ &\quad=|(\Delta_y^2u^{(k)}(x))v(x)+u^{(k)}(x)\Delta_y^2v(x)\\ &\qquad+(u^{(k)}(x+2y)-u^{(k)}(x))(v(x+2y)-v(x))\\ &\qquad+(u^{(k)}(x+y)-u^{(k)}(x))(v(x+y)-v(x))|\\ &\quad\le C|y|(|u|_a\|v\|_0+\|u\|_0|v|_a+\|u\|_{k+(1-\varepsilon)}\|v\|_\varepsilon+\|u\|_\varepsilon\|v\|_{k+1-\varepsilon}).\end{aligned}$$

We then use $\|u\|_{k+1-\varepsilon} \leq C_{1/\varepsilon}|u|_a$. For terms in the sum of (3.13), we use $1 \leq j < k$ to get

$$\begin{aligned}|u^{(k-j)}v^{(j)}|_1 &\leq C\|u^{(k-j)}v^{(j)}\|_1 \leq C'(\|u\|_{k-j+1}\|v\|_j + \|u\|_{k-j}\|v\|_{j+1})\\ &\leq C''(\|u\|_{k+1-\varepsilon}\|v\|_\varepsilon + \|u\|_\varepsilon\|v\|_{k+1-\varepsilon}).\end{aligned}$$

Here, we have used (3.2) and (3.4). We have verified (3.9) for $k = 2$. For $k > 2$, we can verify by a similar argument combining with (3.4).

To verify (3.10), it suffices to consider the case that a is a positive integer. Computing derivatives of u^{-1} of order $a-1$ and using the product rule (3.9), it suffices to verify it for $a = 1$. It is convenient to write $u^{-1} = g(u)$ where g is a smooth function. By the Taylor formula for g, we have

$$\begin{aligned}|\Delta_y^2(g(u))(x)| &\leq |g'(u(x))\Delta_y^2 u(x)|\\ &\quad + 2\|u^{-3}\|_{D,0}(|u(x+y)-u(x)|^2 + |u(x-y)-u(x)|^2).\end{aligned}$$

Note that $|u(x+y)-u(x)|^2 \leq \|u\|_\varepsilon\|u\|_{1-\varepsilon}|y|$. We can get (3.10).

We now verify (3.11). Let $\widetilde{u} = E_{\widetilde{D}}u$ and let $\widetilde{g} = E_D g$. Then

$$|u|_{D,1} \leq |\widetilde{u}|_1 \leq C(\widetilde{D})|u|_{\widetilde{D},1}, \qquad |g|_{D;1+\varepsilon} \leq 2\|\widetilde{g}\|_{1+\varepsilon} \leq C_\varepsilon(\widetilde{D})|g|_{D,1+\varepsilon}.$$

Thus $\widetilde{u} \circ \widetilde{g}$ is an extension of $u \circ g$. Let us drop all tildes in $\widetilde{u}$, $\widetilde{g}$. We have

$$\begin{aligned}&|u\circ g(x+h) + u\circ g(x-h) - 2u\circ g(x)|\\ &\quad\leq |u|_1|g(x+h)-g(x)| + |u(g(x-h)) - u(2g(x)-g(x+h))|\\ &\quad\leq C_1|u|_1\|g\|_1|h| + \|u\|_{\frac{1}{1+\varepsilon}}(C_n\|g\|_{1+\varepsilon}|h|^{1+\varepsilon})^{\frac{1}{1+\varepsilon}}\\ &\quad\leq C_1|u|_1\|g\|_1|h| + C_nC_{1/\varepsilon}|u|_1\|g\|_{1+\varepsilon}^{\frac{1}{1+\varepsilon}}|h|.\end{aligned}$$

Here we have used

$$g(x+h)+g(x-h)-2g(x) = h\cdot\int_0^1(\nabla g(x+th) - \nabla g(x-th)\,dt,$$
$$\|g\|_\alpha \leq C_{1/\alpha}|g|_1, \quad 0<\alpha<1.$$

Note that $C_{1/\alpha}$ is not bounded as α tends to 1^-. We have verified (3.11). To verify (3.12), it remains to verify it for integer $a \geq 2$. We have

$$\partial^{a-1}(u\circ g) = (\partial^{a-1}u)\circ g\partial g + \sum_{i=1}^{a-2}(\partial^i u)\circ g\partial^{a_1}g\cdots\partial^{a_i}g, \tag{3.14}$$

where $a_\ell \geq 1$ and $\sum a_l = a-2$. By (3.9) with $a = 1$ and (3.11), we get

$$|(\partial^{a-1}u)\circ g\partial g|_1 \leq C(|u|_a(1 + C_{1/\varepsilon}\|g\|_{D,1+\varepsilon}^{\frac{2+\varepsilon}{1+\varepsilon}}) + \|u\|_{a-1+\varepsilon}|g|_2).$$

When $a = 2$, we get the required estimate. When $a \geq 3$, we further estimate the last term by

$$\begin{aligned}\|u\|_{a-1+\varepsilon}\|g\|_2 &\leq C_{a,1/\varepsilon}(\|u\|_{a-\varepsilon}\|g\|_{1+2\varepsilon} + \|u\|_{1+2\varepsilon}\|g\|_{a-\varepsilon})\\ &\leq C'_{a,1/\varepsilon}(|u|_a\|g\|_{1+2\varepsilon} + \|u\|_{1+2\varepsilon}|g|_a).\end{aligned}$$

We now estimate the other terms in (3.14). We use $1 \leq i \leq a-2$. By (3.3), (3.2), and (3.4) for Hölder norms, we get

$$\begin{aligned}
&\|(\partial^i u)\circ g\partial^{a_1}g\cdots\partial^{a_i}g\|_1\\
&\quad\leq \|(\partial^i u)\circ g\|_1\|g\|_{a_1}\cdots\|g\|_{a_i}\\
&\qquad+\|u\|_i\sum_{j=1}^{i}\|g\|_{a_1}\cdots\|g\|_{a_j+1}\cdots\|g\|_{a_i}\\
&\quad\leq C\{\|u\|_{i+1+(a_1+\cdots+a_i-i-\varepsilon))}\|g\|_{1+\varepsilon}+C\|u\|_1\|g\|_{a-1+\varepsilon}\},
\end{aligned}$$

which gives us (3.12). □

We need the following more general chain rule estimate. The proof can be found in the appendix of [2] for Hölder norms. The similar estimate can be obtained analogously by using the above chair rule, product rule for the Zygmund spaces. We left the details to the reader.

LEMMA 3.2. *Let D_m be a sequence of Lipschitz domains in $\mathbb{R}^d$ of which $C_*(D_m)$ are bounded. Let $F_i = I + f_i$ map D_i into D_{i+1} with $\|f_i\|_1 \leq C_0$. Then*

$$\begin{aligned}
&\|u\circ F_m\circ\cdots\circ F_1\|_{D_0,r}\\
&\quad\leq C_r^m\left\{\|u\|_r+\sum_i\|u\|_1\|f_i\|_r+\|u\|_r\|f_i\|_1\right\},\quad r\geq 0;\\
&|u\circ F_m\circ\cdots\circ F_1|_{D_0,r}\\
&\quad\leq C_r^m\left\{|u|_r+\sum_i\|u\|_{1+\varepsilon}|f_i|_r+C_{1/\varepsilon}|u|_r\|f_i\|_{1+\varepsilon}^{\frac{1+2\varepsilon}{1+\varepsilon}}\right\},\quad r>1.
\end{aligned}$$

We also need to extend an inverse mapping estimate in Webster [27] to the Zygmund spaces. Note that

$$(\partial_x^a g)\circ F=\sum Q_\alpha(\partial f)\partial^{\alpha_1}f\cdots\partial^{\alpha_i}f,$$

where $i \geq 1$, $\alpha_j \geq 1$, $\alpha_1+\cdots+\alpha_i \leq a$, and $Q_\alpha(\partial f)$ are rational functions in ∂f with $\|Q_\alpha(\partial f)\|_{B_r,0} < C$.

LEMMA 3.3. *Let $F = I + f$ be a C^1 map from $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$ into $\mathbb{R}^n$ with*

$$f(0)=0,\qquad \|Df\|_{B_r,0}\leq\theta<\frac{1}{2}.$$

Let $r' = (1-\theta)r$. Then the range of F contains $B_{r'}$ and there exists a C^1 inverse $G = I + g$, which maps $B_{r'}$ injectively into B_r with

$$g(0)=0,\qquad \|Dg\|_{B_{r'},0}\leq 2\|Df\|_{B_r,0}.$$

Assume further that $f \in \Lambda^{a+1}(B_r)$. Then $g \in \Lambda^{a+1}(B_{r'})$ and

$$\begin{aligned}
&\|Dg\|_{B_{r'},a}\leq C_a\|Df\|_{B_r,a},\quad a\geq 0;\\
&|Dg|_{B_{r'},a}\leq C_a|Df|_{B_r,a}(1+C_{1/\varepsilon}\|f\|_{1+\varepsilon}^{\frac{1+2\varepsilon}{1+\varepsilon}}),\quad a>1.
\end{aligned}$$

In applications, the r in the lemma will be bounded between two absolute constants. Thus the constant C_a does not depend on r, r'. In fact, for convenience we will drop the requirement that $f(0)=0$ replacing with the condition that f has compact support in B_r. This allows us to take $r'=r$, too.

3.2. *Estimates on the Commutator*

For our application, we need to consider the commutator $[\overline{\partial}, E] := \overline{\partial}E - E\overline{\partial}$.

PROPOSITION 3.4. *Let D be a bounded C^1 domain in $\mathbb{R}^n$ and E be the Stein extension operator for D satisfying* (3.6). *Moreover, let $U = D + \eta \cdot \vec{N}$, where $\vec{N}$ is the outer unit normal vector of D and* $0<\eta<1$. *Then we have the following estimates*:

$$\|[\overline{\partial}, E]u\|_{U,a} \le C_b \eta^{b-a-1}\|u\|_{D,b}, \quad b-1 \ge a \ge 1;$$
$$|[\overline{\partial}, E]u|_{U,a} \le C_b \eta^{b-a-1}|u|_{D,b}, \quad b-1 \ge a > 1.$$

Proof. First, let k, l be integers such that $0 \le k \le l$. Notice that for any function $f \in C^l(U)$ that vanishes on D, we have the point-wise estimate for the kth derivatives

$$|f^{(k)}(x)| \le C_l \operatorname{dist}(x, \partial D)^{l-k}\|f\|_{U,l}, \quad \forall x \in U \setminus \overline{D}.$$

Indeed, fix any $x \in U\setminus\overline{D}$. We may assume without loss of generality that $0 \in \partial D$ and $|x| = \operatorname{dist}(0, x) = \operatorname{dist}(x, \partial D)$. Let $\gamma(t) = tx$, $0 \le t \le 1$, be the line segment that connects $0, x$. Let $N = l-k$. Then, by the fundamental theorem of calculus,

$$f^{(k)}(x) = \int_0^1 \frac{d}{dt_1}\cdots\int_0^1 \frac{d}{dt_N} f^{(k)}(t_N\cdots t_1 x)\, dt_N\cdots dt_1.$$

Consequently,

$$|f^{(k)}(x)| \le C_N|x|^N\|f^{k+N}\|_{U,0} \le C_N \operatorname{dist}(x, \partial D)^{l-k}\|f\|_{U,l}.$$

In particular,

$$\|f\|_{U,k} \le C_{\ell-k}\eta^{\ell-k}\|f\|_{U,\ell}.$$

Now, let $0 \le \alpha, \beta < 1$ and $k+\alpha \le l+\beta$. We claim that

$$\|f\|_{U,k,\alpha} \le C_{l-k}\eta^{l-k+\beta-\alpha}\|f\|_{U,l,\beta}. \tag{3.15}$$

Indeed, assume first that $k=l$ and $\alpha \le \beta$. Then, for any $|x-y| \le \eta$, we have

$$\frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} \le \|f\|_{U,l,\beta}|x-y|^{\beta-\alpha} \le \|f\|_{U,l,\beta}\eta^{\beta-\alpha}. \tag{3.16}$$

If $|y-x| > c\eta$, we use $f^{(k)}|_{\overline{D}} = 0$. We find $y^* \in \partial D$ such that $|y^*-y| = \operatorname{dist}(y, D)$. Then we get (3.16) from

$$\begin{aligned}\frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} &\le C\eta^{-\alpha}(|f^{(k)}(x)| + |f^{(k)}(y)|)\\ &\le C\eta^{-\alpha}\|f\|_{\ell+\beta}(|x|^\beta + |y-y^*|^\beta).\end{aligned}$$

Assume now that $l > k$. Note that any two points x, y in U can be connected by a smooth curve of length at most $C(D)|x-y|$. Putting the above together, we obtain

$$\begin{aligned}\frac{|f^{(k)}(x)-f^{(k)}(y)|}{|x-y|^{\alpha}} &\leq C(D)\|f\|_{U,k+1}|x-y|^{1-\alpha} \leq C_{l-k}C(D)\|f\|_{U,l}\eta^{l-k-\alpha}\\ &\leq C_{l-k}C(D)\|f\|_{U,l+\beta}\eta^{l-k-\alpha+\beta}.\end{aligned}$$

Finally, we apply (3.15) to $f = [\overline{\partial}, E]u$ with $k+\alpha = a$, $l+\beta = b-1$. Notice also that $\|[\overline{\partial}, E]u\|_{U,b-1} \leq C_b\|u\|_{D,b}$, where C_b is the operator norm for $E : C^b(D) \to C^b(\mathbb{R}^n)$. Then

$$\|[\overline{\partial}, E]u\|_{U,a} \leq C_{b-a}C(D)\eta^{b-a-1}\|u\|_{D,b}.$$

We now consider the Zygmund space case. Here we use real interpolation on operator norms. Let E_U be a Stein extension for functions on U. We have

$$E_U[\overline{\partial}, E_D]u = E_U\overline{\partial}E_Du - E_UD\overline{\partial}u.$$

Thus, we can write the Hölder estimates as

$$\|E_U[\overline{\partial}, E_D]u\|_{\mathbb{C}^n,a} \leq C\eta^{b-a-1}\|u\|_b, \quad \forall u \in \Lambda^b(\mathbb{C}^n).$$

We remark that the inequality is trivial when $b = a+1$. Thus we assume that $b > a+1$. We also have $a > 0$. We take nonintegers a_i, b_i satisfying $b_0 < b < b_1$, $0 < a_0 < a < a_1$, and $b_i > a_i + 1$. Furthermore,

$$a = (a_0+a_1)/2, \qquad b = (b_0+b_1)/2.$$

We have $|[\overline{\partial}, E]u|_{a_i} \leq C_i\eta^{b_i-a_i-1}|u|_{b_i}$. Since $u \to E_U[\overline{\partial}, E]u$ is a linear operator, we get via interpolation of operator norms (see for instance [2])

$$|[\overline{\partial}, E]u|_a \leq |E_U[\overline{\partial}, E]\widetilde{u}|_a \leq C(C_0\eta^{b_0-a_0-1})^{1/2}(C_1\eta^{b_1-a_1-1})^{1/2}|\widetilde{u}|_b$$

for any $\widetilde{u} \in \Lambda^b$ with $\widetilde{u}|_D = u$. This gives us the last inequality. The proposition is proven. □

Let $U_0 = D_0 + t_0\cdot\vec{N}$, where $\vec{N}$ is the unit outer normal vector of the boundary. Fix $L \in \mathbb{N}$. Moser constructed in [19] a smoothing operator $S_t : C^0(U_0) \to C^\infty(D_0)$,

$$S_tu(x) = \int_{|y|<1} u(y)\chi_t(x-y)\,dy, \quad x \in D_0, 0 < t < t_0/C, \tag{3.17}$$

where $\int \chi(z)\,dz = 1$, $\chi_t(z) = \chi(z/t)$, $\operatorname{supp}\chi \subset \{z \in \mathbb{R}^{2n} : |z| < 1/2\}$, and

$$\int z^I\chi(z)\,dz = 0, \quad 0 < |I| \leq L. \tag{3.18}$$

Therefore, S_t is a convolutional operator, and for $0 < t < t_0/C$, we have

$$\|S_tu\|_{D_0,a} \leq C_at^{b-a}\|u\|_{U_0,b}, \quad 0 \leq b \leq a < \infty; \tag{3.19}$$

$$\|(I-S_t)u\|_{D_0,a} \leq C_bt^{b-a}\|u\|_{U_0,b}, \quad 0 \leq a, 0 \leq b-a < L. \tag{3.20}$$

Here the last inequality relies on (3.18). Via interpolation as explained in the proof of Proposition 3.4 and applied to linear operator S_t and $I - S_t$, we get from the above two inequalities (3.19)–(3.20) the following for Zygmund norms:

$$|S_t u|_{D_0,a} \le C_a t^{b-a} |u|_{U_0,b}, \quad 0 < b \le a < \infty; \tag{3.21}$$

$$|(I - S_t)u|_{D_0,a} \le C_b t^{b-a} |u|_{U_0,b}, \quad a > 0, 0 \le b - a < L. \tag{3.22}$$

3.3. Stability of Constants

We first recall the homotopy operators on a C^2 strictly pseudoconvex domain constructed in [2]. Let D_0 be a C^2 strictly pseudoconvex domain in $\mathbb{C}^n$ and $\mathcal{U} \subset \mathbb{C}^n$ be some open neighborhood of $\overline{D_0}$. Then, for any $\phi \in \Lambda^r_{(0,1)}(\overline{D_0})$ with $r > 1$, we have the homotopy formula

$$\phi = \overline{\partial} P_{D_0,\mathcal{U}}\phi + Q_{D_0,\mathcal{U}}\overline{\partial}\phi, \quad \text{on } D_0, \tag{3.23}$$

where

$$P_{D_0,\mathcal{U}}\phi(z) = \int_{\zeta\in\mathcal{U}} \Omega^0_{0,0}(\zeta, z) \wedge E_{D_0}\phi(\zeta) + \int_{\zeta\in\mathcal{U}\setminus D_0} \Omega^{01}_{0,0}(\zeta, z) \wedge [\overline{\partial}, E_{D_0}]\phi(\zeta).$$

Here, $\Omega^0_{0,0}(\zeta, z)$, $\Omega^{01}_{0,0}(\zeta, z)$ are forms of types $(0, 0)$ and $(0, 1)$ in z, respectively. Moreover, $E_{D_0}g$ has the form $\chi E_{D_0}g$ with the latter $E_{D_0}g$ being the actual Stein extension of g, and χ is a smooth function that has compact support in $\mathcal{U}$ and equals 1 near $\overline{D_0}$; see the proof in [2, Prop. 2.1]. Thus we have the following estimate by [2, Thm. 1.1]:

$$|P_{D_0,\mathcal{U}}\phi|_{D_0,r+1/2} \le C_r(D_0)\theta_0^{-r-\mu}|\phi|_{D_0,r}, \quad r > 1, \tag{3.24}$$

where $\theta_0 = \operatorname{dist}(D_0, \partial\mathcal{U})$, and μ is some constant depending only on the dimension and $C(D_0) > 0$ is another constant depending on a C^2 norm of the defining function. Similar formula and estimates hold with Q in place of P and ϕ, a $(0, 2)$ form. We refer to [2] for more details on these operators and estimates.

In our application, we will also apply estimates on P, Q to a sequence of domains D_j such that $\operatorname{dist}(D_j, \partial\mathcal{U})$ are bounded below by a fixed positive number depending on the initial domain D_0. Consequently, we can absorb $\theta_0^{-s-\mu}$ into the coefficient in the estimate. We shall also drop the subscript in D_0 for simplicity if no confusion is caused. We remark that the constant C_s in (3.24) depends on s and it may not be bounded as s tends to some special values such as a positive integer.

We make a remark about the stability of a constant under C^2 perturbation of the domains.

REMARK 3.5. Let $D_0 := \{x \in \mathcal{U} : \rho_0(x) < 0\} \subset \mathcal{U} \subset \mathbb{R}^n$ be a domain with C^2 boundary where $\mathcal{U}$ is some open neighborhood of D_0 and ρ_0 is a (standard) C^2 defining function. Let

$$\mathcal{D}_{\varepsilon_0} = \{\rho \in C^2(\mathcal{U}) : \|\rho - \rho_0\|_{\mathcal{U},2} \le \varepsilon_0\}.$$

Here $\varepsilon_0 > 0$ is sufficiently small such that, for all $\rho \in \mathcal{D}_{\varepsilon_0}$, we have $d\rho(x) \neq 0$ on $\{x \in \mathcal{U} : \rho(x) = 0\}$.

Suppose that there is a function

$$\mathcal{C} : \mathcal{D}_{\varepsilon_0} \to (0, \infty).$$

DEFINITION 3.6. We say $\mathcal{C}$ is *upper stable* (resp. *lower stable*) under small C^2 perturbation of ρ_0 if there exist $\varepsilon(D_0) > 0$ and a constant $C_0(D_0) > 1$ possibly dependent on ρ_0, such that

$$\mathcal{C}(\rho) \leq C_0(D_0)\mathcal{C}(\rho_0) \quad (\text{resp. } \mathcal{C}(\rho_0) \leq C_0(D_0)\mathcal{C}(\rho))$$

for all ρ satisfying $\|\rho - \rho_0\|_{\mathcal{U},2} \leq \varepsilon(D_0)$.

The following are examples of *upper* stable mappings that will be used for our purposes.

(1) Recall that we introduce *standard* defining functions for C^k domains with $k \geq 1$ in Section 2. By being standard, we mean that the defining functions depend only on the domains in construction. We will write $\mathcal{C}(\rho_0)$ when ρ_0 is a standard definition function of D_0. There are other ways to construct standard definition functions. For instance, we can replace ρ_0 by a Whitney extension of $\rho_0|_{\overline{D_0}}$ so that $\rho_0 \in C^\infty$ away from $\overline{D}_0$. There are other ways to construct definition functions. For instance the Stein extension can also be used.

(2) The operator norms of Stein extension operator between $\Lambda^r(D_0) \to \Lambda^r(D_0)$ for some $r > 1$ are upper stable under small C^2 perturbation of the domain D_0. Indeed, it is well known that the operator norm $C_r(D_0)$ only depends on the Lipschitz constant of D_0; see [24, Section 3.3, p. 189] for L^p Sobolev spaces L^p_k and [2] for Hölder–Zygmund spaces. Thus $C_r(\widetilde{D}) < C_0 C_r(D_0)$ when $\partial\widetilde{D}$ has a C^1 defining function $\widetilde{\rho}$ with $\|\widetilde{\rho} - \rho_0\|_1 < \varepsilon(D_0)$ sufficiently small for some constant $C_0(D_0, \varepsilon)$.

(3) The constants in estimates (3.1), (3.2), (3.3), (3.8), (3.9), (3.12) and Lemma 3.2 are also upper stable under small C^2 perturbation of D provided that D is a C^1 domain. This should follow in principle from the proofs of these inequalities. Alternatively, it also follows from the above remark. Indeed, let $\widetilde{D} = \{z \in \mathcal{U} : \widetilde{\rho}(z) < 0\}$, where $\|\widetilde{\rho} - \rho\|_{\mathcal{U},2} \leq \varepsilon$ for some $\varepsilon(\rho) > 0$. Let $E_{\widetilde{D}}$ be the Stein extension operator on $\widetilde{D}$. Then we have

$$\begin{aligned}\|u\|_{\widetilde{D},l} \leq \|E_{\widetilde{D}}u\|_{\mathcal{U},l} &\leq C_{a,b}(\mathcal{U})\|E_{\widetilde{D}}u\|^{\theta}_{\mathcal{U},a}\|E_{\widetilde{D}}u\|^{1-\theta}_{\mathcal{U},b}\\ &\leq C_{a,b}(\mathcal{U})C'_a(\widetilde{D})C''_b(\widetilde{D})\|u\|^{\theta}_{\widetilde{D},a}\|u\|^{1-\theta}_{\widetilde{D},b}\end{aligned}$$

for $l = \theta a + (1-\theta)b$. Consequently, the stability of

$$C_{a,b}(\widetilde{D}) = C_{a,b}(\mathcal{U})C'_a(\widetilde{D})C''_b(\widetilde{D})$$

follows from the stability of $C'_a(\widetilde{D})$, $C''_b(\widetilde{D})$ from the Stein extension operator. The proofs for (3.2), (3.3), (3.8), (3.9), and (3.12) are similar. For the coefficients in Lemma 3.2, we simply notice they are finite products of the constants from (3.1), (3.2), (3.3) and some dimensional constants. Consequently, they are also stable under small C^2 perturbation.

(4) It is easy to see that the operator norms for Nash–Moser smoothing operator (3.17) are upper stable under small C^2 perturbation of the domain D_0.

(5) The operator norms for (3.23) in (3.24) are upper stable under small C^2 perturbation of the domain D_0. This was proved in [2, Theorem 5.2].

The *upper* stability of these constants in (1)–(5) is important for the convergence of our iteration process in Section 7 and also for the proof of the *lower* stability of $\delta_r(D_0)$. Notice that the latter condition already played an important role in the proof of Theorem 1.2.

4. Approximate Solution via Homotopy Formula

Let D_0 be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Given the initial integrable almost complex structure $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta}\partial_\beta$ on $\overline{D}_0$, we wish to find a transformation defined on $\overline{D}_0$ to transform the complex structure into a new complex structure closer to the standard complex structure, whereas $\overline{D}_0$ is transformed to a new domain that is still C^2 strictly pseudoconvex.

According to Lemma 2.1, after a perturbation of the form $F = I + f$ with Df small, the new structure $\{\partial_{\overline{\alpha}} + \hat{A}_{\overline{\alpha}}^{\beta}\partial_\beta\}$ has the matrix form

$$\hat{A} \circ F = (I + \overline{\partial f} + A\overline{\partial f})^{-1}(A + \overline{\partial} f + A\partial f).$$

We first formally decide the correction f following Webster [27]. Then we indicate the obstructions and make necessary modifications.

From now on, we shall regard $A_{\overline{\alpha}}^{\beta}$ as the coefficients of (0, 1) forms by simply identifying $A_{\overline{\alpha}}^{\beta}\partial_\beta$ with $A^\beta := A_{\overline{\alpha}}^{\beta} d\overline{z}^{\overline{\alpha}}$, where $\beta = 1, \dots, n$. We can then apply the homotopy formula (3.23) componentwise to $A := (A^1, \dots, A^n)$ and write

$$A = \overline{\partial} PA + Q\overline{\partial} A.$$

For Newton's method, we would take $f = -PA$. Then

$$A + \overline{\partial} f + A\partial f = \overline{\partial} PA + Q\overline{\partial} A - \overline{\partial} PA + A\partial f = Q\overline{\partial} A + A\partial f.$$

Using the integrability condition $\overline{\partial} A = [A, \partial A]$ and product rule (3.9), formally we would have $|\hat{A}| \leq |A|^2$. This is used in Webster's proof of the classical Newlander–Nirenberg theorem [27]. However, similar to Webster [28], Gong and Webster [6], and Gong [3], the homotopy operator P does not gain the full derivative lost in applying $\overline{\partial}$ to A as one can see from (3.24). Therefore, we need to apply a smoothing operator to $-PA$ so that the iteration does not terminate within finitely many steps. Note also that the transformation F must be defined on $\overline{D_0}$. Consequently, we need to use the Nash–Moser smoothing method in a way different from the above mentioned work. Namely, we first extend PA to a larger domain via the Stein extension operator and then apply the smoothing operator S_t in (3.17). This ensures that the new complex structure is defined on the closure $\overline{D}_1 = F(\overline{D}_0)$ where the new structure is still formally integrable and has the same regularity as the original complex structure. We first remark that the new domain D_1 as well as future iterated domains in $\mathbb{C}^n$, which are small perturbations of D_0, need to be controlled to apply, for instance, the upper stability of various constants. The control of these domains will be achieved in Lemma 7.1. Therefore, we modify $f = -PA$ and take

$$f = -S_t E_{D_0} P_{D_0,U_1} A. \tag{4.1}$$

Here we assume that

$$\operatorname{dist}(D_0, \partial B_0) \geq c_0^*,$$

and via a cut-off function, we assume that the Stein extension $E_{D_0}u$ of functions u has compact support in

$$B_0 = \{z \in \mathbb{C}^n, |z| < \sigma_0\},$$

where B_0 is a fixed large ball containing D_0, D_1 and their neighborhoods $\mathcal{U}$. Note that f defined by (4.1) still has compact support, provided

$$t < c_0^{**}.$$

Consequently, we have the following identities on D_0:

$$\begin{aligned}
A + \overline{\partial} f + A\partial f &= A - \overline{\partial} S_t EPA + A\partial f \\
&= A - S_t \overline{\partial} EPA + [S_t, \overline{\partial}]EPA + A\partial f \\
&= A - S_t E\overline{\partial} PA + S_t[E, \overline{\partial}]PA + A\partial f \\
&= A - S_t EA + S_t EQ\overline{\partial} A + S_t[E, \overline{\partial}]PA + A\partial f \\
&= (I - S_t)EA + S_t EQ\overline{\partial} A + S_t[E, \overline{\partial}]PA + A\partial f,
\end{aligned}$$

where in the third equality, we use $[S_t, \overline{\partial}] = 0$ on D_0 when acting on $C^1(U_0)$.

According to the above computation and Lemma 2.1, our new error $\hat{A}$ satisfies

$$\begin{aligned}
&\hat{A} \circ F \\
&= (I + \overline{\partial f} + A\overline{\partial} f)^{-1}\{(I - S_t)EA + S_t EQ\overline{\partial} A + S_t[E, \overline{\partial}]PA + A\partial f\}.
\end{aligned} \tag{4.2}$$

We shall denote

$$I_1 = (I - S_t)EA, \qquad I_2 = S_t EQ\overline{\partial} A, \tag{4.3}$$

$$I_3 = -S_t[\overline{\partial}, E]PA, \qquad I_4 = A\partial f, \qquad I_5 = \overline{\partial f} + A\partial f, \tag{4.4}$$

$$\widetilde{A} = (I + I_5)^{-1}(I_1 + I_2 + I_3 + I_4). \tag{4.5}$$

Then we have

$$\hat{A} \circ F = \widetilde{A}, \quad \text{on } \overline{D}_0; \qquad \hat{A} = \widetilde{A} \circ G, \quad \text{on } \overline{D}_1.$$

Here, $G = F^{-1}$ maps $D_1 := F(D_0)$ onto D_0.

Before proceeding, let us briefly discuss the plan for proving Theorem 1.1. In Section 5, we estimate lower order norm $|f|_{D_0,s}$ for some $s > 2$ of the transformation $F = I + f$ defined by (4.1) and give a rough estimate of the new complex structure $\hat{A}$ on the closure of the new domain D_1. In Section 6, we refine our estimates on the lower and high order norms $|\widetilde{A}|_{D_0,s}$, $|\widetilde{A}|_{D_0,r}$ with $s < r$ by estimating $I_1, \ldots, I_5$ defined by (4.3), (4.4), (4.5). In Section 7, we describe the iteration scheme and verify the induction hypotheses. We shall obtain uniform control for the gradient, second order derivatives of f, and the Levi form of the defining function for iterated domains. In Section 8, we run the iteration and determine all the parameters in order to achieve optimal regularity result. Finally, we show the convergence of the composition of a sequence of transformations on $\overline{D_0}$ in Λ^k norm for suitable k.

5. Change of Coordinates and New Complex Structure

Let $A \in \Lambda^r(\overline{D_0})$ be the error term in the original almost complex structure where $1 < r < \infty$. Let $\frac{3}{2} < m \le r + \frac{1}{2}$ and $1 < \ell \le r$. We allow r, ℓ, m, and s below be nonintegers.

Recall that f is defined by (4.1), and $I_1, \dots, I_5$ are defined by (4.3)–(4.5). We start by deriving the following two estimates for $f = F - I$ via (3.21)–(3.22):

$$\begin{aligned} |f|_{D_0,m} &= |S_t EPA|_{D_0,m} \le C'_m |EPA|_{U_0,m} \le C'_m C''_m |PA|_{D_0,m} \\ &\le C'_m C''_m C'''_m |A|_{D_0,m-1/2}, \end{aligned} \tag{5.1}$$

$$\begin{aligned} |f|_{D_0,\ell+1} &= |S_t EPA|_{D_0,\ell+1} \le C'_{\ell+1} t^{-\frac{1}{2}} |EPA|_{U_0,\ell+1/2} \\ &\le C'_{\ell+1} C''_{\ell+1} C'''_{\ell+\frac{1}{2}} t^{-\frac{1}{2}} |A|_{D_0,\ell}, \end{aligned} \tag{5.2}$$

where $C'_\bullet$ is the constant from the Nash–Moser smoothing operator (3.19), which is independent of the domain D_0, $C''_\bullet$ is the constant from Stein's extension operator (3.6), and $C'''_\bullet$ is the constant from estimate (3.24). Constants in the second estimate have similar meaning.

Let us first describe how we control the norms in iteration. Let $3/2 < s < 3$. We need to get rapid convergence in low order derivatives of f. This will be measured by the s-norn $|A_0|_{D_0,s}$. There are two estimates (5.1) and (5.2) which are available to control the lower order derivatives of f. We will use (5.1) to control the second-order derivatives of f and thus the Levi forms of the domains in iteration. We will use (5.2) to control the $(s+1)$-norm of f. Let $\sigma_0 > 0$ be any number large enough such that

$$\overline{D_0} \subset \mathcal{U} \subset B_0,$$

where $\mathcal{U}$ is an open neighborhood of $\overline{D_0}$ and $B_0 = \{z \in \mathbb{C}^n : |z| < \sigma_0\}$. We shall still denote by f the extension Ef to B_0 where E is the Stein extension operator. We may assume that Ef has compact support in B_0.

To simplify our notation, we denote by $C_m(D_0)$ finite products of *upper stable* constants. Notice that by definition of upper stability, finite products of upper stable constants is still upper stable. We will then use $C^*_m(D_0)$ to indicate in the context when these constants are fixed for the rest of the paper.

According to (5.1) and (5.2), we have

$$\|f\|_{B_0,2} \le C^*_2 |A|_{D_0,s}, \tag{5.3}$$

$$|f|_{B_0,s+1} \le C^*_s t^{-1/2} |A|_{D_0,s}. \tag{5.4}$$

Let us first assume that

$$|A|_{D_0,s} < \frac{1}{C_s^{**}}, \qquad C_s^{**} > \sqrt{8n}C_2^*, \qquad s > \frac{3}{2}. \tag{5.5}$$

Here, we have fixed C_2^*, C_s^* and we will adjust the constant C_s^{**} a few times, which will be indicated sometime for clarity. By (5.3) and (5.5), we have for the operator norm of Df,

$$\|Df\|_{B_0,0} \leq \sqrt{2n}\|f\|_{B_0,2} \leq \frac{\sqrt{2n}C_2^*}{C_s^{**}} < \frac{1}{2}. \tag{5.6}$$

We now apply Lemma 3.3 to $F = I + f$ to estimate its inverse G. Recall that f has compact support in B_0. Therefore F is a diffeomorphism from B_0 onto itself. Let $F^{-1} = G$ be its inverse mapping defined on B_0. Estimate (5.6) also ensures that the constants in Stein extension for $F(D_0)$ and the convexity of norms are equivalent to the constants for D_0. However, in the next section we will impose a stronger condition ensuring that $F(D_0)$ remains strictly pseudoconvex.

From (4.2)–(4.5), we have

$$\hat{A} \circ F = (I + I_5)^{-1}(I_1 + I_2 + I_3 + I_4) =: \widetilde{A}, \tag{5.7}$$

where $I_1 = (I - S_t)EA$, $I_2 = S_t EQ\overline{\partial}A$, $I_3 = -S_t[\overline{\partial}, E]PA$, $I_4 = A\partial f$, and $I_5 = \overline{\partial f} + A\partial f$.

We wish to estimate $|\hat{A}|_{D_1,\ell}$ in terms of Λ^s and Λ^r norms of f, A. We do this by first applying the chain rule to $\hat{A} = \widetilde{A} \circ G$ and reducing the problem to estimating $|\widetilde{A}|_{D_0,\ell}$. Then, we use the convexity of Hölder norms to further reduce the problem to estimating $|\widetilde{A}|_{D_0,r}$, $|\widetilde{A}|_{D_0,s}$.

According to (5.3), (5.4), (5.6) and Lemma 3.3, we have

$$\|g\|_{B_0,2} \leq C_2\|f\|_{B_0,2} \leq C_s|A|_{D_0,s}, \tag{5.8}$$

$$|g|_{B_0,s+1} \leq C_s|f|_{B_0,s+1} \leq C_s t^{-1/2}|A|_{D_0,s}. \tag{5.9}$$

By (5.1), (5.6), and Lemma 3.3, we obtain

$$|g|_{D_1,m} \leq C_m|f|_{D_0,m} \leq C_m|A|_{D_0,m-1/2}. \tag{5.10}$$

Let $D_1 = F(D_0)$. We apply chain rule estimate (3.12) to $\hat{A}$ on D_1 together with (5.8) and (5.5) to obtain

$$|\hat{A}|_{D_1,m} = |\widetilde{A} \circ G|_{D_1,m} \leq C_m(|\widetilde{A}|_{D_0,m} + |\widetilde{A}|_{D_0,1+\varepsilon} \cdot |g|_{D_1,m}). \tag{5.11}$$

We take $\varepsilon \in (0, 1/2)$ to be a small positive number. Let us assume that

$$|I_i|_s < 1/2, \quad i = 1, \dots, 5. \tag{5.12}$$

Then we have $|\tilde{A}|_s < C$. Using (5.10) and (5.11), we have

$$|\hat{A}|_{D_1,m} \leq C_m|\widetilde{A}|_{D_0,m}.$$

Therefore, it suffices to estimate $\widetilde{A}$, assuming (5.5) and (5.12).

According to the convexity of Hölder–Zygmund norms (3.8) with $a = s$, $b = r$ and $\ell = (1-\theta)s + \theta r$ for some $0 \leq \theta \leq 1$, we have the following estimates for intermediate derivatives:

$$|\widetilde{A}|_{D_0,\ell} \leq C_{r,s,\ell}|\widetilde{A}|^{1-\theta}_{D_0,s}|\widetilde{A}|^{\theta}_{D_0,r}.$$

Consequently, we can reduce the problem of estimating the intermediate derivatives to estimating $|\widetilde{A}|_{D_0,s}$ and $|\widetilde{A}|_{D_0,r}$, which we shall often refer to as low and high order derivative estimates.

To this end, we apply product rule (3.9) to the Λ^m norm of $\widetilde{A} = \hat{A} \circ F$. We get

$$\begin{aligned}|\widetilde{A}|_{D_0,m} &\leq C_m\left(|(I+I_5)^{-1}|_{D_0,m}\sum_{i=1}^{4}\|I_i\|_{D_0,\varepsilon} + \|(I+I_5)^{-1}\|_{D_0,\varepsilon}\sum_{i=1}^{4}|I_i|_{D_0,m}\right)\\ &\leq C_m\left(|(I+I_5)^{-1}|_{D_0,m}\sum_{i=1}^{4}|I_i|_{D_0,s} + |(I+I_5)^{-1}|_{D_0,s}\sum_{i=1}^{4}|I_i|_{D_0,m}\right).\end{aligned}$$

When $m = s$, we have the low order estimate

$$|\widetilde{A}|_{D_0,s} \leq 2C_s|(I+I_5)^{-1}|_{D_0,s}\cdot\sum_{i=1}^{4}|I_i|_{D_0,s}. \tag{5.13}$$

Similarly, when $m = r$, we have the high order estimate

$$\begin{aligned}&|\widetilde{A}|_{D_0,r}\\ &\quad\leq C_r\left(|(I+I_5)^{-1}|_{D_0,r}\sum_{i=1}^{4}|I_i|_{D_0,s} + |(I+I_5)^{-1}|_{D_0,s}\sum_{i=1}^{4}|I_i|_{D_0,r}\right).\end{aligned} \tag{5.14}$$

We shall begin to estimate the right-hand sides in the next section.

6. Estimate of $I_1, \ldots, I_5$ and $\hat{A}$

Let $A \in \Lambda^r$, $\frac{3}{2} < r \leq \infty$, be the error term in the original complex structure. Let $\frac{3}{2} < s < 3$. In this section, we assume that $\frac{3}{2} < m \leq r$ if $r < \infty$ and $\frac{3}{2} < m < \infty$ if $A \in C^\infty$. We also replace the initial conditions (5.5) and (5.12) by the stronger conditions

$$t^{-1/2}|A|_{D_0,s} \leq \frac{1}{C_s^{**}}, \qquad t^{r-s-1/2}|A|_{D_0,r} \leq \frac{1}{C_r^{**}}, \tag{6.1}$$

where $t \in (0,1)$ and C_s^{**}, C_r^{**} that are larger than 1 will be adjusted several times in this section. When $r = \infty$, we take $r = 5$ in the above condition.

When $r < \infty$, we choose our smoothing operator S_t depending on r. More precisely, in (3.17), we choose $L = \lceil r \rceil$ where $\lceil \cdot \rceil$ means rounding up to the closest integer. By (3.22) for the smoothing operator and (3.6) for the extension operator, we get

$$|I_1|_{D_0,m} = |(I - S_t)EA|_{D_0,m} \leq C_r t^{r-m}|EA|_{U_0,r} \leq C_r t^{r-m}|A|_{D_0,r}.$$

Substituting s into m, we have

$$|I_1|_{D_0,s} \le C_r t^{r-s}|A|_{D_0,r}, \tag{6.2}$$

$$|I_1|_{D_0,r} \le C_r|A|_{D_0,r}, \tag{6.3}$$

where the former inequality will be used to prove the rapid convergence of (5.13) and the latter will be used to control the growth rate of (5.14).

When $r=\infty$, we construct another smoothing operator by choosing $L=3$ in the construction of S_t. Then, for all $\frac{3}{2}<m<\infty$, we have

$$|I_1|_{D_0,m} = |(I-S_t)EA|_{D_0,m} \le C_m|A|_{D_0,m}.$$

To estimate I_2, we need to use the integrability condition $\overline{\partial}A=[A,\partial A]$ on D_0 and estimate (5.1). We will also apply (3.24) to $\overline{\partial}A\in\Lambda^{s-1}(D_0)$ in the following estimate, which forces us to impose a stronger condition:

$$s>2.$$

By $\overline{\partial}A=[A,\partial A]$, we have by (3.21), (3.24), and (3.9)

$$\begin{aligned}
|I_2|_{D_0,m} &= |S_tEQ\overline{\partial}A|_{D_0,m} \le C_s' t^{-1/2}|EQ\overline{\partial}A|_{U_0,m-1/2}\\
&\le C_m' C''_{m-\frac{1}{2}} t^{-1/2}|Q\overline{\partial}A|_{D_0,m-1/2}\\
&\le C_m' C''_{m-\frac{1}{2}} C_{m-\frac{1}{2}} t^{-1/2}|\overline{\partial}A|_{D_0,m-1}\\
&\le C_m t^{-1/2}(|A|_{D_0,m-1}\|A\|_{D_0,1+\varepsilon}+|A|_{D_0,m}\|A\|_{D_0,\varepsilon})\\
&\le C_m t^{-1/2}|A|_{D_0,s}|A|_{D_0,m}.
\end{aligned}$$

Using initial condition (6.1) and applying the above estimate with s, r in place of m, we get

$$|I_2|_{D_0,s} \le C_s t^{-1/2}|A|^2_{D_0,s}, \tag{6.4}$$

$$|I_2|_{D_0,r} \le C_r|A|_{D_0,r}. \tag{6.5}$$

When $r=\infty$, we have

$$|I_2|_{D_0,m} \le C_m|A|_{D_0,m}.$$

For the estimate of I_3, we have

$$\begin{aligned}
|I_3|_{D_0,m} &= |S_t[\overline{\partial},E]PA|_{D_0,m} \le C_m'|[\overline{\partial},E]PA|_{U_0,m}\\
&\le C''_{r,m} t^{r+1/2-m-1}|PA|_{D_0,r+1/2}\\
&\le C_r t^{r-m-1/2}|A|_{D_0,r},
\end{aligned}$$

where we used Lemma 3.4 in the second inequality recalling $U_0=D_0+t\vec{N}$ with $\vec{N}$ being the unit outer normal vector of ∂D_0. Here we emphasize that D_0+tN needs to be contained in the domain $\mathcal{U}$ that appears in the integral operator P in (3.24), which is satisfied since $\mathcal{U}$ can be chosen to contain D_0 and its small C^2 perturbation. Applying the above estimate with s, r in place of m, we have

$$|I_3|_{D_0,s} \le C_r t^{r-s-1/2}|A|_{D_0,r}, \quad r-\frac{1}{2}\ge s>2, \tag{6.6}$$

$$|I_3|_{D_0,r} \le C_r t^{-1/2}|A|_{D_0,r}. \tag{6.7}$$

We remark here that $t^{-1/2}$ in the coefficient of (6.7) is the main obstruction that prevents us from having a linear growth in [26; 2] for the high order norms.

When $r=\infty$, we have

$$|I_3|_{D_0,m} \leq C_m t^{-1/2}|A|_{D_0,m}.$$

The estimate for I_4 is more involved. Recall that $f=-S_tEPA$. By (5.4) we have

$$\begin{aligned}|I_4|_{D_0,m} = |A\partial f|_{D_0,m} &\leq C_{m+1}(|A|_{D_0,m}\|f\|_{D_0,1+\varepsilon}^{\frac{1+2\varepsilon}{1+\varepsilon}} + \|A\|_{D_0,\varepsilon}|f|_{D_0,m+1})\\ &\leq C_{m+1}(|A|_{D_0,m}|A|_{D_0,s} + \|A\|_{D_0,0}t^{-1/2}|A|_{D_0,m})\\ &\leq C_{m+1}t^{-1/2}|A|_{D_0,s}|A|_{D_0,m}.\end{aligned}$$

Using initial condition (6.1) and applying the above estimate twice with r, s in place of m, respectively, we get

$$|I_4|_{D_0,s} \leq C_{s+1}t^{-1/2}|A|^2_{D_0,s}, \tag{6.8}$$

$$|I_4|_{D_0,r} \leq C_{r+1}|A|_{D_0,r}. \tag{6.9}$$

When $r=\infty$, we have

$$|I_4|_{D_0,m} \leq C_{m+1}|A|_{D_0,m}.$$

Finally, we need to estimate the low and high order derivatives for I_5 and $(I+I_5)^{-1}$:

$$|I_5|_{D_0,m} = |\overline{\partial f} + A\overline{\partial} f|_{D_0,m} \leq C_{m+1}t^{-1/2}|A|_{D_0,m}.$$

Using the initial condition (6.1) for C_s^{**} sufficiently large and applying the above estimate with r in place of m, we get by (5.4)

$$|I_5|_{D_0,s} \leq 1/C_s, \tag{6.10}$$

$$|I_5|_{D_0,r} \leq C_{r+1}t^{-1/2}|A|_r. \tag{6.11}$$

Recall that in our notation all constants C_r, C_s, and so on are larger than 1 and constants c_r, c_s, and so forth are positive and less than 1. When $r=\infty$, we have

$$|I_5|_{D_0,m} \leq C_{m+1}t^{-1/2}|A|_m.$$

We now consider $(I+I_5)^{-1}-I$. By the matrix inversion formula $(I+I_5)^{-1} = \det(I+I_5)^{-1}(A_{ij})$, where (A_{ij}) is the transpose of the adjugate matrix of $I+I_5$. Notice that every entry in $(I+I_5)^{-1}-I$ is a polynomial in $(\det(I+I_5))^{-1}$ and entries of I_5 without constant term and with fixed degree. Therefore, we can now estimate $|(I+I_5)^{-1}-I|_{D_0,m}$ using product and quotient rules (3.9) and (3.10) to show that

$$|(I+I_5)^{-1}-I|_{D_0,m} \leq C_m(1+\|I_5\|_{D_0,\varepsilon})^{n+m+2}\frac{|I_5|_{D_0,m}}{(1-C_*\|I_5\|_{D_0,0})^{m+2}}, \quad C_*>0,$$

if $\|I_5\|_{D_0,0} < C_*/2$. Consequently, by (6.10),

$$|(I+I_5)^{-1}|_{D_0,m} \leq C_m(1+|I_5|_{D_0,m}), \quad m>0.$$

Recall that $C_s>1$. By (6.10), we have the estimate

$$|(I+I_5)^{-1}|_{D_0,s} \leq C_s(1+|I_5|_{D_0,s}) \leq 2C_s. \tag{6.12}$$

Letting $m = r$, we have

$$|(I + I_5)^{-1}|_{D_0,r} \leq C_r(1 + |I_5|_{D_0,r}). \tag{6.13}$$

Similarly, when $r = \infty$,

$$|(I + I_5)^{-1}|_{D_0,m} \leq C_m(1 + |I_5|_{D_0,m}).$$

Let C_s^{**}, C_r^{**} be sufficiently large in the initial condition (6.1), and we may assume that s-norms of I_1, I_2, I_3, I_4, $(I + I_5)^{-1}$ are uniformly bounded by some positive constant C_r.

Using s-norm derivative estimates (6.1), (6.2), (6.4), (6.6), (6.8), (6.11), and (6.12) in the product rule formula (5.13), we obtain

$$|\widetilde{A}|_{D_0,s} \leq C_r(t^{r-s-1/2}|A|_{D_0,r} + t^{-1/2}|A|^2_{D_0,s}).$$

Then by using r-norm estimates (6.1), (6.3), (6.5), (6.7), (6.9), and (6.13) in the product rule formula (5.14), we obtain

$$|\widetilde{A}|_{D_0,r} \leq C_r t^{-1/2}|A|_{D_0,r}, \quad r > 2.$$

Similarly, when $r = \infty$, we have

$$|\widetilde{A}|_{D_0,m} \leq C_m t^{-1/2}|A|_{D_0,m}, \quad m > 2.$$

Noticing that by (5.7), (5.9), and (6.1), $\hat{A} = \widetilde{A} \circ G$ and $\|g\|_{1+\varepsilon} < \frac{1}{2}$, we apply (5.9) and (5.11) to get

$$|\hat{A}|_{D_1,s} \leq C_r^*(t^{r-s-1/2}|A|_{D_0,r} + t^{-1/2}|A|^2_{D_0,s}), \quad r - \frac{1}{2} \geq s > 2, \tag{6.14}$$

$$|\hat{A}|_{D_1,r} \leq C_r^* t^{-1/2}|A|_{D_0,r}, \quad r > 2. \tag{6.15}$$

And when $r = \infty$,

$$|\hat{A}|_{D_1,m} \leq C_m^* t^{-1/2}|A|_{D_0,m}, \quad m > 2. \tag{6.16}$$

We have derived the estimates for new A and f, g under assumption (6.1), where the C_s^{**} is now fixed for the rest of the proof. Moreover, C_2^*, C_s^* have been fixed in (5.3), (5.4) and C_r^*, C_m^* have been fixed in (6.14), (6.15), and (6.16).

7. Levi Form of Iterated Domains

Let us summarize what we have achieved so far under assumption (6.1) on error A_0. Let D_0 be a strictly pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$, and let $X_{(0)} = \overline{\partial} + A_0\partial \in \Lambda^r(\overline{D_0})$ be the initial perturbed integrable almost complex structure where $\overline{\partial} = (\partial_{\overline{1}}, \dots, \partial_{\overline{n}})$ is the standard complex structure on $\mathbb{C}^n$ and ∂ is its conjugate. In Section 4, we defined $F_0 = I + f_0$ to be our first approximate solution where

$$f_0 = -S_{t_0}E_{D_0}P_{D_0}A_0$$

for some $t_0 > 0$ to be determined. Let $D_1 := F_0(D_0)$ be the new domain and A_1 be the error for the new almost complex structure on D_1 where

$$A_1 \circ F_0 = (I + \overline{\partial} f_0 + A_0\partial \overline{f}_0)^{-1}(A_0 + \overline{\partial} f_0 + A_0\partial f_0)$$

by Lemma 2.1. We then obtained estimates (6.14), (6.15) for the new error A_1 in terms of certain low and high order norms of the previous error A_0.

We would like to repeat the above procedure on D_1 to further reduce the new error. However, in order to define the approximate solution $F_1 = I + f_1$ on D_1 via the homotopy formula, we have to show that D_1 is still a strictly pseudoconvex domain with C^2 boundary. This is true provided that the initial error is small enough. In fact, we shall set up an iteration scheme and prove a general statement in the next proposition.

Without loss of generality, we assume that

$$D_0 = \{z \in \mathcal{U} : \rho_0(z) < 0\} \subset \overline{D_0} \subset \mathcal{U} \subset B_0,$$

where ρ_0 is some C^2 defining function of D_0, $\mathcal{U}$ is some open neighborhood of D_0 and $B_0 = \{z \in \mathbb{C}^n : |z| < 100\}$.

Next, we discuss how the Levi form of a C^2 domain is controlled by a sequence of C^2 diffeomorphism.

It will be convenient to extend the defining function of a domain to a larger and fixed domain. Let ρ_0 be a C^m defining function of D_0 on $\mathcal{U}$. Suppose that $\partial\mathcal{U} \in C^1$ and D_0 is relatively compact in $\mathcal{U}$. Define

$$\widetilde{E}u = \chi E_{\mathcal{U}} u + (1 - \chi), \tag{7.1}$$

where $\chi \geq 0$ is a smooth function that equals 1 on $\mathcal{U}_1$ for some $\mathcal{U}_1$ and has compact support in B_0 and $E_{\mathcal{U}}\rho_0 > 0$ on $\overline{\mathcal{U}_1}$, and furthermore $\overline{\mathcal{U}} \subset \mathcal{U}_1 \subset B_0$.

LEMMA 7.1. *Fix a positive integer m. Let $D_0 \subset \mathcal{U} \subset B_0 \subset \mathbb{R}^N$ with $\overline{D_0} \subset \mathcal{U}$. Suppose that D_0 admits a C^m defining function ρ_0 satisfying*

$$D_0 = \{x \in \mathcal{U} \colon \rho_0(x) < 0\},$$

where $\rho_0 > 0$ on $\overline{\mathcal{U}} \setminus D_0$ and $\nabla\rho_0 \neq 0$ on ∂D_0. Let $F_j = I + f_j$ be a C^m diffeomorphism that maps B_0 onto B_0 and maps D_j onto D_{j+1}. Let $\rho_1 = (\widetilde{E}\rho_0) \circ F_0^{-1}$ and $\rho_{j+1} = \rho_j \circ F_j^{-1}$ for $j > 0$, which are defined on B_0. For any $\varepsilon > 0$, there exists

$$\delta = \delta(\rho_0, \varepsilon, m) > 0$$

such that if

$$\|f_j\|_{B_0,m} \leq \frac{\delta}{(j+1)^2}, \quad 0 \leq j < L,$$

then we have the following:

(i) *$\tilde{F}_j = F_j \circ \cdots \circ F_0$ and ρ_{j+1} satisfy*

$$\begin{aligned} \|\tilde{F}_{j+1} - \tilde{F}_j\|_{B_0,m} &\leq C_m \frac{\delta}{(j+1)^2}, \quad 0 \leq j < L, \\ \|\tilde{F}_{j+1}^{-1} - \tilde{F}_j^{-1}\|_{B_0,m} &\leq C'_m \frac{\delta}{(j+1)^2}, \quad 0 \leq j < L, \\ \|\rho_{j+1} - \rho_0\|_{\mathcal{U},m} &\leq \varepsilon, \quad 0 \leq j < L. \end{aligned} \tag{7.2}$$

(ii) *All D_j are contained in $\mathcal{U}$ and*

$$\operatorname{dist}(\partial D_j, \partial D) \leq C\varepsilon, \qquad \operatorname{dist}(D_j, \partial\mathcal{U}) \geq \operatorname{dist}(D_0, \partial\mathcal{U}) - C\varepsilon. \tag{7.3}$$

In particular, when $L = \infty$, $\widetilde{F}_j$ converges in C^m to a C^m diffeomorphism from B_0 onto itself, whereas ρ_j converges in C^m of B_0 as $\widetilde{F}_j^{-1}$ converges in C^m norm on the set.

Proof. Let us denote $\tilde{E}\rho_0$ from (7.1) by ρ_0.

(i) Let $\tilde{F}_i = I + \tilde{f}_i$. We have $\tilde{f}_{i+1} = f_{i+1} \circ \tilde{F}_i + \tilde{f}_i$. By the chain rule, we get

$$\|\tilde{f}_{i+1} - \tilde{f}_i\|_{B_0,m} \le C_m \|f_{i+1}\|_m \prod_i (1 + \|f_i\|_m)^{2m} \le C'_m \|f_{i+1}\|_m \le C_m \frac{\delta}{(i+1)^2}.$$

Let $F_i^{-1} = I + g_i$ and $\tilde{F}_i^{-1} = I + \tilde{g}_i$. On B_0, we can use the identity $\tilde{F}_{i+1}^{-1} - \tilde{F}_i^{-1} = \tilde{F}_{i+1}^{-1} - F_{i+1} \circ \tilde{F}_{i+1}^{-1}$. Thus

$$\tilde{g}_{i+1} - \tilde{g}_i = -f_{i+1} \circ \tilde{G}_{i+1}.$$

By the chain rule, we get $\|G_{i+1}\|_m \le C_m(1 + \|\tilde{f}_{i+1}\|_m)^{2m} \le C'_m$. Then we obtain

$$\|\tilde{g}_{i+1} - \tilde{g}_i\|_m \le C_m \|f_{i+1}\|_m, \qquad \|\tilde{g}_i\|_m \le C_m \delta.$$

So far we have not used any assumption on δ other than the condition that $\delta < C$. To verify (7.2), we must use the uniform continuity of the m-th derivatives of ρ_0. Let D_K be a derivative of order k. We have

$$\begin{aligned} \rho_{i+1} - \rho_0 &= \rho_0 \circ \tilde{G}_i - \rho_0, \\ D_K(\rho_{i+1} - \rho_0) &= (D_K \rho_0) \circ \tilde{G}_i - D_K \rho_0 + \sum P_{K,K'}(\partial_x \tilde{g}_i, \dots, \partial_x^k \tilde{g}_i) D_{K'} \rho_0, \end{aligned}$$

where $P_{K,K'}(\partial_x \tilde{g}_i, \dots, \partial_x^k \tilde{g}_i)$ is a polynomial without constant term. Thus its sup norm is bounded by

$$C_m \|\tilde{g}_i\|_m \le C'_m \delta.$$

Applying the chain rule, we bound the sup norm of $D_{K'}\rho_i$ by $C\|\rho_0\|_m$. By the uniform continuity of $D_K \rho_0$ and the estimate

$$\|\tilde{g}_i\|_0 \le C_m \delta,$$

we therefore obtain $\|(D_K \rho_0) \circ \tilde{G}_i - D_K \rho_0\|_0 < \varepsilon/2$ when δ is sufficiently small.

(ii) Applying (7.2) for $m = 1$ implies that when $\varepsilon < \varepsilon(\rho_0)$ and $\varepsilon(\rho_0) > 0$ is sufficiently small, we have (7.3), where C depends only on $\nabla \rho_0$. □

To use the Levi form of ρ at z, it will be convenient to define

$$T_z^{1,0}\rho = \left\{ \sum t_j \partial_{z_j} : \sum t_j \partial_{z_j} \rho(z) = 0 \right\}, \qquad \left| \sum t_j \partial_{z_j} \right| = \sqrt{\sum |t_j|^2}.$$

LEMMA 7.2. *Let D be a relatively compact C^2 domain in $\mathcal{U}$ defined by a C^2 function ρ. There are $\varepsilon = \varepsilon(\rho) > 0$ and a neighborhood $\mathcal{N} = \mathcal{N}(\rho)$ of ∂D such* *that if* $\|\tilde\rho-\rho\|_{\mathcal{U},2}<\varepsilon$, *then we have*

$$\begin{aligned}&\inf_{\tilde z,\tilde t,|\tilde t|=1}\{L\tilde\rho(\tilde z,\tilde t)\colon \tilde t\in T^{1,0}_{F(z)}\tilde\rho,\tilde z\in\mathcal{N}\}\\ &\quad\geq\inf_{z,t,|t|=1}\{L\rho(z,t)\colon t\in T^{1,0}_z\rho, z\in\mathcal{N}\}-C\varepsilon.\end{aligned}$$

Furthermore, $\tilde D=\{z\in\mathcal{U}\colon \tilde\rho<0\}$ *is a* C^2 *domain with* $\partial\widetilde{D}\subset\mathcal{N}(\rho)$.

Proof. Since ρ is a C^2 defining function of D, there exists a neighborhood $\mathcal{N}$ of ∂D such that $\nabla\rho(z)\neq 0$ for $z\in\mathcal{N}$. Let $\tilde z\in\mathcal{N}$. Without loss of generality, we assume that $\tilde z=0$ and $\frac{\partial\rho}{\partial z_n}\neq 0$ near the origin. When δ is small, we still have $\frac{\partial\tilde\rho}{\partial z_n}\neq 0$ near the origin. Consequently, we know that $T_0^{1,0}\tilde\rho$ is spanned by $\{\partial_i-\frac{\partial_i\tilde\rho(0)}{\partial_n\tilde\rho(0)}\partial_n\}_{i=1}^n$. Let

$$\tilde t_i=\frac{\partial_i-\frac{\partial_i\tilde\rho(0)}{\partial_n\tilde\rho(0)}\partial_n}{|\partial_i-\frac{\partial_i\tilde\rho(0)}{\partial_n\tilde\rho(0)}\partial_n|},\qquad t_i=\frac{\partial_i-\frac{\partial_i\rho(0)}{\partial_n\rho(0)}\partial_n}{|\partial_i-\frac{\partial_i\rho(0)}{\partial_n\rho(0)}\partial_n|}.$$

Then we have

$$\left|\sum_{i,j}\frac{\partial^2\tilde\rho(0)}{\partial z_i\partial\overline{z_j}}\tilde t_i\overline{\tilde t_j}-\sum_{i,j}\frac{\partial^2\rho(0)}{\partial z_i\partial\overline{z_j}}t_i\overline{t_j}\right|\leq C\varepsilon.$$

Shrinking $\mathcal{N}$ if necessary, we have $\rho>\varepsilon'$ on $\partial\mathcal{N}\setminus D$. Taking $\varepsilon<\varepsilon'/2$, we conclude that $\partial\widetilde{D}$ is contained in $\mathcal{N}$. □

Before stating our main result in this section, let us fix some notations that will appear in the next proposition. Let $\mathcal{U}$ be the same open neighborhood of D_0 that appears in the homotopy formula (3.23) and

$$B_0=\{z\in\mathbb{C}^n : |z|<100\}.$$

Recall that the constants C_2^*, C_s^*, C_s^{**}, C_r^{**}, C_r^*, C_m^* that appear in (5.3), (5.4), (6.1), (6.14), (6.15), and (6.16) have been fixed. Next, recall that for a bounded strictly pseudoconvex domain D_0 with a C^2 *standard* defining function ρ_0, there is positive $\varepsilon(D_0)$ such that if $\|\rho-\rho_0\|_2<\varepsilon(D_0)$, then all the bounds in the estimates for Stein extension, Nash–Moser smoothing operator, and the homotopy operator in [2] are *upper stable* for domains with defining function ρ_0. See Remark 3.5. In particular, the domain defined by $\rho<0$ is strictly pseudoconvex when $\varepsilon(D_0)$ is sufficiently small. Finally, let

$$\delta(\rho_0,\varepsilon)=\delta(\rho_0,\varepsilon,2),\qquad \delta(\rho_0)=\delta(\rho_0,\varepsilon(D_0),2)\tag{7.4}$$

be the constants from Lemma 7.1.

PROPOSITION 7.3. *Let* $2<s<3$ *and* $s+\sqrt{2}+\frac{3}{2}<r<\infty$. *Let* C_2^*, C_s^{**}, C_r^*, $\varepsilon(D_0)$, $\delta(\rho_0)$ *be the constants stated above, and let positive numbers* α, β, d, γ,

κ satisfy

$$r-s-\frac{1}{2}-\gamma-\kappa>\alpha d+\beta, \qquad \alpha(d-1)>\frac{1}{2}+\kappa, \\ \beta(2-d)>\frac{1}{2}+\kappa. \tag{7.5}$$

Let $\overline{\partial}:=(\partial_{\overline{1}},\dots,\partial_{\overline{n}})^t$ be the standard complex structure on $\mathbb{C}^n$, and let ∂ be its conjugate. Let D_0 be a bounded strictly pseudoconvex domain with a C^2 defining function ρ_0 on $\mathcal{U}$ and $X_{(0)}=\overline{\partial}+A_0\partial\in\Lambda^r(\overline{D_0})$ be a formally integrable almost complex structure. There exists a constant

$$\hat{t}_0:=\hat{t}_0(r,s,\alpha,\beta,d,\kappa,C_2^*,C_s^{**},C_r^{**},C_r^*,\varepsilon(D_0),\delta(\rho_0))\in(0,1/2)$$

such that if $0<t_0\leq\hat{t}_0$ and

$$|A_0|_{\overline{D_0},s}\leq t_0^{\alpha}, \tag{7.6}$$

$$|A_0|_{\overline{D_0},r}\leq t_0^{-\gamma}, \tag{7.7}$$

then the following statements are true for $i=0,1,2\dots$:

(i) *There exists a C^∞ diffeomorphism $F_i=I+f_i$ from B_0 onto itself with $F_i^{-1}=I+g_i$ such that f_i, g_i satisfy*

$$|g_i|_{B_0,s+1}\leq C_s|f_i|_{B_0,s+1}, \qquad |f_i|_{B_0,s+1}\leq C_s^*t_i^{-1/2}a_i, \tag{7.8}$$

where

$$t_{i+1}=t_i^d, \quad i\geq 0.$$

(ii) *Set $\rho_{i+1}=\rho_i\circ F_i^{-1}$. Then $D_{i+1}:=F_i(D_i)=\{z\in\mathcal{U}:\rho_{i+1}<0\}$ and*

$$\|\rho_{i+1}-\rho_0\|_{\mathcal{U},2}\leq\varepsilon(D_0), \tag{7.9}$$

$$\operatorname{dist}(D_{i+1},\partial\mathcal{U})\geq\operatorname{dist}(D_0,\partial\mathcal{U})-C\varepsilon. \tag{7.10}$$

(iii) *$(F_i|_{\overline{D_i}})_*(X_{(i)})$ is in the span of $X_{(i+1)}:=\overline{\partial}+A_{i+1}\partial$ on $\overline{D_{i+1}}$. Moreover, $a_{i+1}=|A_{i+1}|_{D_{i+1},s}$ and $L_{i+1}=|A_{i+1}|_{D_{i+1},r}$ satisfy*

$$a_{i+1}\leq t_{i+1}^{\alpha}, \qquad L_{i+1}\leq L_0t_{i+1}^{-\beta}.$$

(iv) *If in addition $A_0\in C^\infty(\overline{D}_0)$, then for any $m>1$ and $M_i=|A_i|_{D_i,m}$, we have*

$$|f_i|_{D_i,m+\frac{1}{2}}\leq C_mM_i.$$

Moreover, there exist some $\eta(d)>0$ independent of m and $N=N(m,d)\in\mathbb{N}$ such that, for all $i>N$,

$$M_i\leq M_Nt_i^{-\eta}. \tag{7.11}$$

By Lemma 7.2, the assertions (i), (ii) clearly imply that D_i is a strictly pseudoconvex domain with C^2 boundary. According to the remark at the end of Section 3, the assertion (ii) implies that we can choose the constants C_2^*, C_s^*, C_s^{**}, C_r^*, C_m^* to be independent of D_i provided that $\varepsilon=\varepsilon(D)$ is sufficiently small. This is important for our iteration to converge. The last two assertions roughly say that we have rapid decay in the low order norm and rapid growth in the high order norm.

This is different from the previous work, for example [26; 3], where the high order norm grows only linearly. We refer the reader to [26; 3] for the precise definition of rapid and linear growths. Here we would like to point out that for this reason the parameters α, β, d, γ, κ, s will be carefully chosen in the end to both accommodate the constraints obtained in the iteration procedure and achieve optimal regularity results for the convergence.

Proof of Proposition 7.3. We are given $\alpha > 0$, $\beta > 0$, $d > 1$, $\gamma > 0$, $\kappa > 0$ satisfying (7.5). We will see at the end of the proof that such α, β, d, γ, κ exist when $r - s > \sqrt{2} + 3/2$. It is also clear that $\alpha > 1/2$, $\beta > 1/2$, and $1 < d < 2$.

For the moment, we require $\hat{t}_0 \in (0, \frac{1}{2})$. We will further adjust $\hat{t}_0$ a few times and indicate its explicit dependency on parameters mentioned in the statement of the proposition. This will be used in the next section to prove the lower stability of $\delta_r(D_0)$, which appears in Theorem 1.1.

We consider first the case when $i = 0$.

Let E_0 be the Stein extension operator on D_0 and let $S_{t_0} : C^0(\mathcal{U}) \to C^\infty(D_0)$ be the Nash–Moser smoothing operator. Let $P_{D_0,\mathcal{U}}$, $Q_{D_0,\mathcal{U}}$ be the homotopy operators defined in Section 3 (we shall abbreviate them as P_0, Q_0 for simplicity). We defined in Section 4 that

$$f_0 = -S_{t_0} E_0 P_0 A_0, \qquad F_0 = I + f_0.$$

By an abuse of notation, we still denote by F_0 its extension $E_0 F_0$ to $C^\infty(\mathbb{C}^n)$.

Assume that (6.1) holds, that is,

$$t_0^{-1/2} |A_0|_{D_0,s} \leq \frac{1}{C_s^{**}}, \qquad t_0^{r-s-1/2} |A_0|_{D_0,r} \leq \frac{1}{C_s^{**}}. \tag{7.12}$$

Then, according to (5.9) and (5.4), we have

$$\begin{aligned} |g_0|_{B_0,s+1} &\leq C_s |f_0|_{B_0,s+1}, \\ |f_0|_{B_0,s+1} &\leq C_s^* t_0^{-1/2} |A_0|_{D_0,s}. \end{aligned}$$

To achieve (7.12), we require that

$$\hat{t}_0 \leq \left(\frac{1}{C_s^{**}}\right)^{\frac{2}{2\alpha-1}}, \qquad \hat{t}_0 \leq \left(\frac{1}{C_r^{**}}\right)^{\frac{1}{\beta}}. \tag{7.13}$$

Then it is clear that (7.12) is ensured by (7.6)–(7.7) and (7.13) since $\alpha > 1/2$ and $\beta > 0$ are fixed, $r - s - \gamma - 1/2 > \beta > 0$ and

$$t_0^{-1/2} |A_0|_{D_0,s} \leq t_0^{\alpha - 1/2} \leq \frac{1}{C_s^{**}}, \qquad t_0^{r-s-1/2} |A_0|_{D_0,r} \leq t_0^{r-s-\gamma-1/2} \leq \frac{1}{C_r^{**}}$$

when $t_0 \leq \hat{t}_0$.

Now we verify (ii) when $i = 0$. Let $\delta = \delta(\rho_0)$ be the constant that appears in (7.4). Assume that

$$\hat{t}_0 \leq \left(\frac{\delta}{C_2^*}\right)^2. \tag{7.14}$$

Then, according to (5.3) and (7.6), for $0 < t_0 \le \hat{t}_0$, we have

$$\|f_0\|_{B_0,2} \le C_2^* a_0 \le C_2^* t_0^{\alpha}.$$

Consequently, we obtain from (7.2)

$$\|\rho_1 - \rho_0\|_{\mathcal{U},2} \le \varepsilon(D_0).$$

By Lemma 7.1 (ii), we have

$$\operatorname{dist}(D_1, \partial U) \ge \operatorname{dist}(D_0, \partial U) - C\varepsilon.$$

To verify (iii) for $i = 0$, let us recall what we proved in Section 6. Since (6.1) is satisfied for $i = 0$ by (7.13), then from (6.14) and (6.15) we know that $a_1 = |A_1|_{D_1,s}$, $L_1 = |A_1|_{D_{i+1},r}$ satisfy

$$a_1 \le C_r^* \cdot (t_0^{r-s-1/2} L_0 + t_0^{-1/2} a_0^2), \tag{7.15}$$

$$L_1 \le C_r^* \cdot (t_0^{-1/2} L_0). \tag{7.16}$$

We would like to show that

$$a_1 \le t_1^{\alpha}, \qquad L_1 \le L_0 t_1^{-\beta}, \tag{7.17}$$

where $\alpha > 1/2$, $\beta > 0$ and $t_1 = t_0^d$ for some $d > 1$. We remark that (7.17) also imply that (7.12) hold when t_0, A_0 are replaced by t_1, A_1, respectively.

Here we must use (7.7) in addition to (7.6). Recall that the positive parameters $\alpha, \beta, \kappa, \gamma$ have been given such that

$$\{(\alpha, \beta, d, s) : \alpha d + \beta < r - s - 1/2 - \kappa - \gamma\} \ne \emptyset.$$

Next, choose $\hat{t}_0 \in (0, 1/2)$ so that

$$\hat{t}_0 \le \left(\frac{1}{2C_r^*}\right)^{\frac{1}{\kappa}}. \tag{7.18}$$

Note that this implies $2C_r^* t_0^{\kappa} \le 1$ for $0 < t_0 < \hat{t}_0$. Then it is easy to see from (7.15) and (7.16) the following inequalities:

$$a_1 \le \frac{1}{2}(t_0^{r-s-1/2} t_0^{-\kappa-\gamma} + t_0^{2\alpha} t_0^{-1/2} t_0^{-\kappa}) \le t_0^{\alpha d} = t_1^{\alpha},$$
$$L_1 \le t_0^{-1/2} t_0^{-\kappa} L_0 \le t_0^{-\beta d} L_0 = t_1^{-\beta} L_0.$$

Here, we have used (7.6)–(7.7) and assumed the following constraints on α, β, d, κ, γ:

$$\alpha d < r - s - 1/2 - \kappa - \gamma,$$
$$\alpha(2 - d) > 1/2 + \kappa, \quad \alpha > 0,$$
$$\beta d > 1/2 + \kappa, \quad \beta > 0.$$

We have verified (iii) for $i = 0$ assuming the intersection of these constraints is nonempty. We will see in the induction step that this is true provided that $2 < s < 3$ and $s + \frac{3}{2} + \sqrt{2} < r < \infty$.

Part (iv) will be proved separately at the end of the proposition.

Now, assume that the induction hypotheses hold for some $i - 1 \in \mathbb{N}$, $i \ge 1$.

(i) By induction hypotheses (i), (ii) and Lemma 7.2, we know that D_i is a C^2 strictly pseudoconvex domain. Therefore, we can apply the construction of the approximate solution defined in Section 4 on D_i

$$f_i = S_{t_i} E_i P_i A_i, \qquad F_i = I + f_i,$$

where E_i is the Stein extension operator on D_i, $S_{t_i} : C^0(\mathcal{U}) \to C^\infty(D_i)$ is the Nash–Moser smoothing operator, and $P_i = P_{D_i,\mathcal{U}}$, $Q_i = Q_{D_i,\mathcal{U}}$ are the homotopy operators defined in Section 3. Moreover, by induction hypothesis (ii), we can assume that C_s^* is independent of $1, 2, \dots, i$. Therefore, estimates (5.3) and (5.4) hold for f_i,

$$\|f_i\|_{B_0,2} \le C_2^* a_i,$$
$$|f_i|_{B_0,s+1} \le C_s^* t_i^{-1/2} a_i.$$

Notice that $f_i = E_i f_i$ has compact support in B_0. Obviously, we have by $(ii)_{i-1}$

$$C_2^* a_i \le C_2^* t_i^{1/2} \le C_2^* t_0^{1/2} \le 1/2.$$

Then by Lemma 3.3, F_i is a diffeomorphism from B_0 to itself and $G_i := F_i^{-1}$ exists on B_0.

(ii) Let $\delta = \delta(\rho_0, \varepsilon, s)$ be the constant that appeared in Lemma 7.1. Let

$$D_{i+1} = \{z \in \mathcal{U} : \rho_{i+1}(z) < 0\},$$

where $\rho_{i+1}(z) = \rho_i \circ G_i$. Since D_i is strictly pseudoconvex by induction, by (5.3) we get

$$\|f_i\|_{B_0,2} \le C_2^* a_i \le C_2^* t_i^{\alpha},$$

where we used induction hypothesis (iv) for $i-1$ in the last inequality. Notice that we have

$$C_2^* t_i^{\alpha} \le \frac{\delta}{(i+1)^2},$$

assuming that

$$C_2^* \hat{t}_0^{\frac{1}{2}d^i} \le \frac{\delta}{(i+1)^2}. \tag{7.19}$$

This has been achieved for $i = 0$ in (7.14). We show how to achieve this condition for all i assuming $\hat{t}_0$ is sufficiently small. Indeed, assume that we have achieved (7.19) for $i-1$. Then

$$C_2^* \hat{t}_0^{\frac{1}{2}d^i} = C_2^* \hat{t}_0^{\frac{1}{2}d^{i-1}} \hat{t}_0^{\frac{1}{2}d^{i-1}(d-1)} \le \frac{\delta}{i^2} \hat{t}_0^{\frac{1}{2}d^{i-1}(d-1)} \le \frac{\delta}{(i+1)^2},$$

where the last inequality holds for all $i \ge 1$ by requiring that

$$\hat{t}_0 \le 4^{-\frac{2}{d-1}}. \tag{7.20}$$

Consequently, we obtain from (7.2)

$$\|\rho_{i+1} - \rho_0\|_{\mathcal{U},2} \le \varepsilon(D_0).$$

Note that (7.10) follows from (7.9) and (7.3).

(iii) The verification for (iii) in the general case is similar to the case when $i = 0$. However, as we will see, extra constraints on the parameters α, β, d, κ, γ appear, when $i > 0$.

According to induction hypothesis (iv) and (7.13), we have

$$t_i^{-1/2}|A_i|_{D_i,s} \leq t_i^{\alpha-1/2} \leq \frac{1}{C_s^{**}}.$$

Moreover, since D_i is a C^2 strictly pseudoconvex domain, we know from Section 5 that (6.14), (6.15) are valid for $a_{i+1} = |A_{i+1}|_{D_{i+1},s}$, $L_{i+1} = |A_{i+1}|_{D_{i+1},r}$:

$$\begin{aligned} a_{i+1} &\leq C_r^* \cdot (t_i^{r-s-1/2} L_i + t_i^{-1/2} a_i^2), \\ L_{i+1} &\leq C_r^* \cdot (t_i^{-1/2} L_i). \end{aligned}$$

Here, the constant C_r^* does not depend on i by induction hypothesis (ii). Notice that

$$a_i \leq t_i^{\alpha}, \qquad L_i \leq L_0 t_i^{-\beta}, \qquad 2C_r^* t_i^{\kappa} \leq 1,$$

where the first two inequalities are nothing but induction hypothesis (iv), and the last condition follows easily from (7.18) since $t_i < t_0$. Then, after a computation similar to the case when $i = 0$, we obtain

$$\begin{aligned} a_{i+1} &\leq \frac{1}{2}(t_i^{r-s-1/2} t_i^{-\beta} t_i^{-\kappa-\gamma} + t_i^{2\alpha} t_i^{-1/2} t_i^{-\kappa}) \leq t_i^{\alpha d} = t_{i+1}^{\alpha}, \\ L_{i+1} &\leq t_i^{-1/2} t_i^{-\kappa} t_i^{-\beta} L_0 \leq t_i^{-\beta d} L_0 = t_{i+1}^{-\beta} L_0, \end{aligned}$$

which also gives us (7.12) when a_0, A_0 are replaced by a_{i+1}, A_{i+1}, where we have assumed

$$\alpha d + \beta < r - s - 1/2 - \kappa - \gamma, \tag{7.21}$$

$$\alpha(2-d) > 1/2 + \kappa, \quad \alpha > 0, \tag{7.22}$$

$$\beta(d-1) > 1/2 + \kappa, \quad \beta > 0. \tag{7.23}$$

Note that the first and third constraints are more restrictive than the ones for $i = 0$.

Before proceeding to the proof of (iv), we briefly discuss how the parameters α, β, γ, and so on can be chosen to satisfy conditions (7.5).

Since the first condition in (7.5) is about the difference $r - s$, we therefore introduce

$$\xi = r - s - 1/2. \tag{7.24}$$

Let $\mathcal{D}(\xi, d, \kappa, \gamma) \subset \mathbb{R}^2$ be the set of (α, β) such that (7.21), (7.22), and (7.23) are satisfied. We must determine the values of ξ, d, κ, γ so that $\mathcal{D}(\xi, d, \kappa, \gamma)$ is nonempty.

We readily notice that $1 < d < 2$, $\alpha > 1/2 + \kappa$, $\beta > 1/2 + \kappa$, and $r > \frac{7}{2}$ since $s > 2$. We consider the limiting domain for fixed ξ, d and $\kappa = \gamma = 0$

$$\begin{aligned} &\mathcal{D}(\xi, d, 0, 0) \\ &\quad = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha d + \beta < \xi, \alpha(2-d) > \frac{1}{2}, \beta(d-1) > \frac{1}{2}\right\}. \end{aligned} \tag{7.25}$$

We first determine the condition on ξ, d so that $\mathcal{D}(\xi, d, 0, 0)$ is nonempty. By the defining equations of $\mathcal{D}(\xi, d, 0, 0)$, the latter is nonempty if and only if ξ, defined by (7.24), satisfies

$$\xi > p(d), \qquad p(d) := \frac{d}{2(2-d)} + \frac{1}{2(d-1)}, \quad 1 < d < 2.$$

Here, the first inequality is just the first inequality in (7.25), after α, β are solved from the last two inequalities in (7.25). Note that on interval $(1, 2)$, p is a strictly convex function that attains minimum value $p(\sqrt{2}) = \sqrt{2} + 1$. This implies that

$$\xi = r - s - \frac{1}{2} > p\big(\sqrt{2}\big) = \sqrt{2} + 1.$$

Therefore, we obtain the minimum smoothness requirement for our complex structure

$$r > s + \frac{3}{2} + \sqrt{2} > \frac{7}{2} + \sqrt{2}.$$

Notice that $\mathcal{D}(\xi, d, \kappa, \gamma)$ is still nonempty for sufficiently small κ, γ. Consequently, we have found a set of values α, β, d, κ, γ so that the constraints are satisfied. However, we remark here that our goal is to obtain the convergence of $|A_j|_{D_j,\ell}$ where $s \le \ell \le r$ for ℓ to be as large as possible. To achieve this, we need to optimize our choice of the constants α, β, d, κ, γ together with s. This will be done in the next section.

(iv) The case when $A \in C^\infty$ needs an additional estimate. We still keep all previous assumptions. In particular, r, s are fixed finite numbers. Thus we have (i), (ii), (iii). Recall that in Section 6 we constructed the smoothing operator S_t by choosing $L = 3$ in (3.17) if $A \in C^\infty$. Let $M_i := |A_i|_m$.

Since D_i is strictly pseudoconvex, it follows from (5.1) that

$$|f_i|_{m+\frac{1}{2}} \le C_m M_i.$$

Since (6.1) holds for f_i, it follows from (6.16) that

$$M_{i+1} \le C_m^* \cdot (t_i^{-1/2} M_i).$$

We would like to show (7.26), that is, there exist some $\eta = \eta(d)$ and $N = N(m, d)$ such that for all $i > N$, we have

$$M_i \le M_N t_i^{-\eta}.$$

Note that this holds trivially for $i = N$.

Let $N = N(m, d) \in \mathbb{N}$ be sufficiently large so that, for all $i > N$,

$$C_m^* t_i^{\lambda} \le 1.$$

Then we have an estimate that is almost identical to the estimate of L_{i+1} for $i > N(m, d)$,

$$M_{i+1} \le t_i^{-\lambda} t_i^{-1/2} M_i \le t_i^{-\lambda - 1/2 - \eta} M_N \le t_i^{-\eta d} M_N = t_{i+1}^{-\eta} M_N, \tag{7.26}$$

where we have fixed an η satisfying

$$\eta(d-1) > 1/2 + \lambda. \qquad \square$$

8. Optimal Regularity and Convergence of Iteration

Let us first explain what we mean by optimal regularity. Here we will use the interpolation methods in Moser [19] and Webster [27].

Let $2 < s < 3$ and $s + \frac{3}{2} + \sqrt{2} < r < \infty$. Assume that the given initial integrable almost complex structure $X_0 = \overline{\partial} + A_0 \partial$ is in $\Lambda^r(\overline{D_0})$. Moreover, we assume that conditions (7.6) and (7.7) from Proposition 7.3 are satisfied. That is,

$$|A_0|_{\overline{D_0},s} \le t_0^{\alpha}, \qquad |A_0|_{\overline{D_0},r} \le t_0^{-\gamma} \tag{8.1}$$

for some α, γ, t_0 satisfying the requirements in Proposition 7.3.

By convexity of Hölder–Zygmund norms and (7.17), we can control the intermediate derivatives $\ell = (1-\theta)s + \theta r$ for $0 < \theta < 1$,

$$|A_{j+1}|_{D_{j+1},\ell} \le C_r |A_{j+1}|_{D_{j+1},s}^{1-\theta} |A_{j+1}|_{D_{j+1},r}^{\theta} \le C_r t_{j+1}^{(1-\theta)\alpha - \theta\beta}, \tag{8.2}$$

where $j \in \mathbb{Z}^+$. To achieve the convergence of $|A_{j+1}|_{D_{j+1},\ell}$, we need $(1-\theta)\alpha > \theta\beta$. Therefore, $0 \le \theta < \frac{\alpha}{\alpha+\beta} < 1$. Let $\theta_0 = \frac{\alpha}{\alpha+\beta}$, and we would like to maximize

$$\ell(\alpha, \beta, r, s, d) = s + \theta_0(r-s) = s + \frac{\alpha}{\alpha+\beta}(r-s)$$

under the constraints (7.21), (7.22), and (7.23). Notice that we cannot achieve the maximum value ℓ_0 for ℓ, because $\theta < \theta_0$.

Recall that $\xi = r - s - 1/2$. Recall from the proof of induction hypothesis (iv) in Proposition 7.3 the following facts. We have defined $\mathcal{D}(\xi, d, \kappa, \gamma) \subset \mathbb{R}^2$ to be the set of (α, β) such that (7.21), (7.22), and (7.23) are satisfied. That is,

$$\begin{aligned}&\mathcal{D}(\xi, d, \kappa, \gamma)\\ &\quad = \left\{(\alpha, \beta) : \alpha d + \beta + \kappa + \gamma < \xi, \alpha(2-d) > \frac{1}{2} + \kappa, \beta(d-1) > \frac{1}{2} + \kappa\right\}.\end{aligned}$$

We consider the limit domain when $\kappa = \gamma = 0$,

$$\mathcal{D}(\xi, d, 0, 0) = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha d + \beta < \xi, \alpha(2-d) > \frac{1}{2}, \beta(d-1) > \frac{1}{2}\right\}$$

and

$$p(d) := \frac{d}{2(2-d)} + \frac{1}{2(d-1)}, \quad 1 < d < 2,$$

which is a strictly convex function on $(1,2)$ and attains minimum value $p(\sqrt{2}) = \sqrt{2} + 1$. It is clear that $\mathcal{D}(\xi, d, 0, 0)$ is nonempty if and only if

$$\xi > p(d). \tag{8.3}$$

It is also easy to see that the closure of $\mathcal{D}(\xi, d, 0, 0)$ is

$$\overline{\mathcal{D}(\xi, d, 0, 0)} = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha d + \beta \le \xi, \alpha(2-d) \ge \frac{1}{2}, \beta(d-1) \ge \frac{1}{2}\right\}.$$

We write

$$\ell(\alpha, \beta, r, s, d) = r - \widetilde{\ell}(\alpha, \beta, \xi, d), \qquad \widetilde{\ell}(\alpha, \beta, \xi, d) := \frac{\xi + \frac{1}{2}}{\frac{\alpha}{\beta} + 1}. \tag{8.4}$$

On $\overline{\mathcal{D}(\xi, d, 0, 0)}$, let us minimize

$$\widetilde{\ell}(\alpha, \beta, \xi, d) = \frac{\xi + \frac{1}{2}}{\frac{\alpha}{\beta} + 1}.$$

Since $\overline{\mathcal{D}(\xi, d, 0, 0)}$ is a compact set, the continuous function $\widetilde{\ell}$ achieves minimum at some point $(\alpha_\infty, \beta_\infty) \in \overline{\mathcal{D}(\xi, d, 0, 0)}$. It is clear that β_∞ takes the smallest possible value and α_∞ takes the largest possible value. Thus,

$$\beta_\infty = \frac{1}{2(d-1)}, \qquad \alpha_\infty = \frac{\xi - \beta_\infty}{d}.$$

Thus, we have

$$\begin{aligned}\widetilde{\ell}(\alpha_\infty, \beta_\infty, \xi, d) &= \frac{\xi + \frac{1}{2}}{2\frac{d-1}{d}(\xi - \beta_\infty) + 1} \\ &= \frac{d(\xi + \frac{1}{2})}{2(d-1)\xi + (d-1)} \\ &= \frac{1}{2}\left(1 + \frac{1}{d-1}\right).\end{aligned}$$

Fix any $r > \frac{7}{2} + \sqrt{2}$ and $\xi = r - s - \frac{1}{2}$. In order to achieve the optimal regularity, that is, to find the largest ℓ (equivalently the smallest $\widetilde{\ell}$) for a given r, we would like to find the largest d that satisfies (8.3). Since $p(d)$ is a strictly convex function, the largest d for a given ξ is therefore achieved by the larger solution $d(\xi)$ where $p(d(\xi)) = \xi$. By a simple computation, we have

$$d(\xi) = \frac{1 + 3\xi + \sqrt{\xi^2 - 2\xi - 1}}{1 + 2\xi} = 1 + \frac{1}{2 + \frac{1}{\xi}} + \frac{\sqrt{1 - \frac{2}{\xi} - \frac{1}{\xi^2}}}{2 + \frac{1}{\xi}}.$$

It follows that $d(\xi)$ is an increasing function in ξ. In particular, it approaches 2^- and $\widetilde{\ell}(\alpha_\infty, \beta_\infty, \xi, d)$ tends to 1 as ξ and hence r tends to $+\infty$.

We observe that for any given $r > \frac{7}{2} + \sqrt{2}$, $d(\xi)$ is maximized when $\xi = r - s - 1/2$ takes the maximum value

$$\xi_\infty := r - \frac{5}{2}$$

for $s = 2$. Let $d_\infty = d(\xi_\infty)$. Then at $(\alpha_\infty, \beta_\infty, \xi_\infty, d_\infty) \in \mathcal{D}(0, 0)$, $\widetilde{\ell}$ achieves its minimum $\widetilde{\ell}_\infty$, where $\mathcal{D}(0, 0) = \cup_{(\xi, d)} D(\xi, d, 0, 0)$ for all (ξ, d) satisfies condition (8.3).

In particular, when

$$r = 5,$$

a simple computation shows that

$$\xi_\infty = 5 - 5/2 = 5/2, \qquad d_\infty = d(\xi_\infty) = 3/2, \qquad \widetilde{\ell}_\infty = 3/2.$$

By monotonicity of $d(\xi)$, this implies that we actually need $r > 5$ to ensure that $\widetilde{\ell}_\infty < 3/2$. Therefore, we now assume $r > 5$. We can take $\xi = r - s_*$ for suitable $2 < s_* < s$, the above values then satisfy

$$d_\infty \geq 3/2 + c_0, \qquad \widetilde{\ell}_\infty \leq 3/2 - \widetilde{c}_0$$

for sufficiently small $c_0, \widetilde{c}_0 > 0$.

In summary, we have proved the following: Let $\mathcal{D}(\kappa, \gamma) = \cup_{\xi,d}\mathcal{D}(\xi, d, \kappa, \gamma)$ be the set of (α, β, ξ, d) in $\mathbb{R}^4$ satisfying (7.21), (7.22), and (7.23). Let $r > 5$. We choose $2 < s_* < s$ such that

$$\xi_* = r - s_* - 1/2 > 5/2.$$

Consequently,

$$d(\xi_*) > 3/2.$$

Then we choose $3/2 < d_* < d(\xi_*)$ so that $\xi_* > p(d_*)$. This ensures that $\mathcal{D}(\xi_*, d_*, 0, 0)$ is nonempty. Therefore, there exist parameters α_*, β_*, ξ_*, d_* so that $\mathcal{D}(0, 0)$ is nonempty. Notice that κ, γ can be arbitrarily close to 0. Then, for sufficiently small κ, γ, we have $(\alpha_*, \beta_*, \xi_*, d_*) \in \mathcal{D}(\kappa, \gamma)$. Moreover, since $d_* > 3/2$,

$$\widetilde{\ell}(\alpha_*, \beta_*, \xi_*, d_*) = \frac{1}{2}\left(1 + \frac{1}{d_* - 1}\right) \leq 3/2 - 2c^* \tag{8.5}$$

for sufficiently small $c^* > 0$. It is clear that c^* depends only on the choice of ξ_*, d_* specified above. Note that (8.5) implies

$$\ell_* = \ell(\alpha_*, \beta_*, \xi_*, d_*) \geq r - \frac{3}{2} + 2c^*.$$

Let $\ell = s_* + \theta(r - s_*)$ and choose $0 \leq \theta < \frac{\alpha_*}{\alpha_* + \beta_*}$ such that

$$\ell > \ell_* - c^*. \tag{8.6}$$

Suppose that (8.1) is satisfied for the above choices of α_* and γ. Then

$$|A_0|_{D_0,s_*} \leq |A_0|_{D_0,s} \leq t_0^{\alpha_*}, \qquad |A_0|_{D_0,r} \leq t_0^{-\gamma}.$$

Consequently, we know from (8.2) that

$$|A_j|_{D_j,\ell} \leq C_r t_j^{a_\ell}, \quad a_\ell > 0, \tag{8.7}$$

where $a_\ell = (1-\theta)\alpha_* - \theta\beta_*$ for the above ℓ that, according to (8.4)–(8.6), satisfies

$$\ell > r - \frac{3}{2} + c^*. \tag{8.8}$$

We now discuss briefly the case when the given structure X is smooth. Here our goal is to minimize the value of r instead of minimizing ℓ_∞ to impose the Λ^r norm of initial A_0 to be small. More precisely we require the initial conditions (6.1) to be satisfied for the sequence A_j. Therefore, for this simple purpose, we only need $r - s - 1 > p(d)$ and $s > 2$. With $p(d)$ having minimum value $\sqrt{2} + 1$, we can fix $r = 5$ (in fact any $r > \frac{7}{2} + \sqrt{2}$) and then choose $2 < s < 3$ and α, β, γ, κ, d easily to fulfill the requirements (7.5). We leave the details to the reader.

Finally, we are ready to show the convergence of the sequence $\widetilde{F}_j = F_{j-1} \circ \cdots \circ F_0$ to some embedding F on $\overline{D_0}$ in $\Lambda^{r-1}(\overline{D_0})$ for any $r > 5$ (including

$r = \infty$). Moreover, F maps the perturbed almost complex structure to the standard one and $F(D_0) := D$ is still a C^2 strictly pseudoconvex domain in $\mathbb{C}^n$.

PROPOSITION 8.1. *Let $5 < r < \infty$ and $2 < s < r - 3$. Let D_0 be a C^2 strictly pseudoconvex domain in $\mathbb{C}^n$ and $X_{\overline{\alpha}} = \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta}\partial_\beta \in \Lambda^r(\overline{D_0})$, $\alpha = 1, \dots, n$, be a formally integrable almost complex structure on D_0. There exist constants $\alpha > 1/2$, $\gamma \in (0, 1)$, and $\hat{t}_0 \in (0, 1/2)$ such that if*

$$|A|_{D_0,s} \le t_0^{\alpha} \quad \text{and} \quad |A|_{D_0,r} \le t_0^{-\gamma}, \tag{8.9}$$

where $0 < t_0 \le \hat{t}_0$, then the following statements are true.

(i) *There is a sequence of mappings $\widetilde{F}_j$ converging to some embedding $F : \overline{D_0} \to \mathbb{C}^n$ in $\Lambda^{\ell+\frac{1}{2}}(\overline{D_0})$ for any $0 \le \ell \le r - 3/2 + c^*$. Here $c^* > 0$ is the same constant that appeared in (8.8). In particular, $F \in C^{r-1}(\overline{D_0})$.*
(ii) *If in addition $A \in C^\infty(\overline{D_0})$, then $F \in C^\infty(\overline{D_0})$ under (8.9) and the weaker condition $r - 3/2 - \sqrt{2} > s > 2$.*
(iii) *$F_*(X_{\overline{\alpha}})$ are in the span of $\partial_{\overline{1}}, \dots, \partial_{\overline{n}}$ and $F(\overline{D_0})$ is strictly pseudoconvex.*
(iv) *The $\delta_r(D_0) := \hat{t}_0^{\alpha}$ is lower stable under a small C^2 perturbation of ∂D_0.*

Proof. We may assume that $2 < s < 3$.

(i) Let us first determine the constants α, γ, and $\hat{t}_0$. Recall $\hat{t}_0$ from Proposition 7.3 where

$$\hat{t}_0 := \hat{t}_0(r, s, \alpha, \beta, d, \kappa, C_2^*, C_s^{**}, C_r^{**}, C_r^*, \varepsilon(D_0), \delta(\rho_0)) \in (0, 1/2).$$

Notice that r, C_2^*, C_s^{**}, C_r^*, $\varepsilon(D_0)$, $\delta(\rho_0)$ have been specified before the proof of Proposition 7.3. Choose κ, γ and $(\alpha, \beta, d, \xi) \in \mathcal{D}(\kappa, \gamma)$ such that (8.5) is satisfied. Then $\hat{t}_0$ that appeared in Proposition 7.3 is determined by the constraints (7.13), (7.14), (7.18), and (7.20). These constraints will be written down explicitly when proving the stability of $\delta_r(D)$ in (iv).

By assumption, for $0 < t_0 \le \hat{t}_0$, we have

$$|A_0|_{D_0,s} \le t_0^{\alpha}, \qquad |A_0|_{D_0,r} \le t_0^{-\gamma}.$$

Consequently, Proposition 7.3 is now valid for such choices of $\hat{t}_0$ and A_0. Moreover,

$$|f_j|_{D_j,\ell+\frac{1}{2}} \le C_\ell |A_j|_{D_j,\ell} \le C_\ell t_j^{a_\ell}, \quad a_\ell > 0$$

according to (5.1) and (8.7).

Consider the composition $\widetilde{F}_{j+1} = F_j \circ F_{j-1} \circ \cdots \circ F_0$, where $F_j = I + f_j$ for $j \ge 0$. Let $\ell = r - 3/2 + c^*$. We use Lemma 3.2 to estimate

$$\begin{aligned} |\widetilde{F}_{j+1} - \widetilde{F}_j|_{D_0,\ell+\frac{1}{2}} &= |f_j \circ F_{j-1} \circ \cdots \circ F_0|_{D_0,\ell+\frac{1}{2}} \\ &\le (C_\ell)^j \Big\{ |f_j|_{\ell+\frac{1}{2}} + \sum_i (\|f_j\|_2 |f_i|_{\ell+\frac{1}{2}} + |f_j|_{\ell+\frac{1}{2}} |f_i|_s) \Big\} \\ &\le (C_\ell)^j C |f_j|_{\ell+\frac{1}{2}} \le C_r^j t_j^{a_\ell} \quad \text{for some } a_\ell > 0. \end{aligned} \tag{8.10}$$

This shows that $|\widetilde{F}_{j+1}-\widetilde{F}_j|_{D_0,\ell+\frac{1}{2}}$ is a Cauchy sequence since $\sum_j C_\ell^j t_j^{a_\ell}$ clearly converges. We denote the limit mapping by F.

(ii) The case $r=\infty$ needs a separate argument because the construction of smoothing operator S_t depends on the r in the finite smooth case.

We are going to use (7.8), (7.11) from Proposition 7.3 and convexity (3.8) without the optimization process. Indeed, let

$$\eta(d), N(m,d) \in \mathbb{N}$$

be the same constants from (v) in Proposition 7.3. Then, for $\ell=(1-\theta)s+\theta m$, $j>N(m,d)$, we have

$$|f_{j+1}|_{D_{j+1},\ell+\frac{1}{2}} \leq C_m |f_{j+1}|^{1-\theta}_{D_{j+1},s+\frac{1}{2}} |f_{j+1}|^{\theta}_{D_{j+1},m+\frac{1}{2}} \leq C_m' t_{j+1}^{(1-\theta)\alpha-\theta\eta}.$$

We have the convergence provided that $(1-\theta)\alpha-\theta\eta>0$, which can be achieved by choosing any $0<\theta<\frac{\alpha}{\alpha+\eta}<1$. For instance, we can choose $\theta=\frac{\alpha}{2(\alpha+\eta)}$.

Then we can apply the same argument (8.10) to see that $F\in\Lambda^{\ell+1/2}(\overline{D_0})$ where

$$\ell+1/2 = s+\frac{\alpha}{2(\alpha+\eta)}(m-s)+1/2 > r_0-1-1/2$$

for m sufficiently large. Since m can be arbitrarily large and θ is independent of m, we conclude that $F\in\Lambda^\ell(\overline{D_0})$ for all ℓ. This implies that $F\in C^\infty(\overline{D_0})$.

(iii) By part (iv) in Proposition 7.3, we see that F transforms the formally integrable almost complex structure into the standard complex structure. By (ii) in Proposition 7.3, we know that $D:=F(D_0)$ is a strictly pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$.

Finally, we show that F is a diffeomorphism. Since F is Λ^{r-1}, it suffices to check the Jacobian of $F(x)$ for $x\in D_0$.

$$|DF-I|_{D_0,0} \leq \sum_{j=0}^{\infty} |D\widetilde{F}_{j+1}-D\widetilde{F}_j|_{D_0,0} \leq \sum_{j=0}^{\infty} |\widetilde{F}_{j+1}-\widetilde{F}_j|_{D_0,1} \leq \frac{1}{2},$$

where the last inequality follows from (8.10).

(iv) Let $\varepsilon(D_0)$ be the size of second order perturbation of ρ_0 such that we have *upper stability* of $C_2^*(\rho_0)$, $C_s^*(\rho_0)$, $C_s^{**}(\rho_0)$, $C_r^{**}(\rho_0)$, $C_r^*(\rho_0)$.

Recall that $\hat{t}_0$ is determined by the constraints (7.13), (7.14), (7.18), and (7.20). More specifically,

$$\hat{t}_0 \leq \min\left\{\left(\frac{1}{C_s^{**}}\right)^{\frac{2}{2\alpha-1}}, \left(\frac{1}{C_r^{**}}\right)^{\frac{1}{\beta}}, \left(\frac{\delta(\rho_0)}{C_2^*}\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2C_r^*}\right)^{\frac{1}{\kappa_0}}, \left(\frac{1}{4}\right)^{\frac{2}{d-1}}\right\}.$$

Here $C_2^*(\rho_0)$, $C_s^{**}(\rho_0)$, $C_r^{**}(\rho_0)$, $C_r^*(\rho_0)$ are upper stable constants and $\delta(\rho_0)$ is given by (7.4) and satisfies the properties in Lemma 7.1. See also Section 3 for details on upper stability.

Let us replace $\delta(\rho_0)$ by a smaller quantity $\delta^*(D_0)$ defined by

$$\delta^*(D_0) := \min\left\{\frac{\varepsilon(D_0)}{4C''}, \delta\left(\rho_0, \frac{\varepsilon(D_0)}{2}, 2\right)\right\},$$

where C'' is an absolute constant determined later. Then we have $0 < t_0 \le \hat{t}_0$ for

$$t_0(D_0) := \min\left\{\left(\frac{1}{C_s^{**}}\right)^{\frac{2}{2\alpha-1}}, \left(\frac{1}{C_r^{**}}\right)^{\frac{1}{\beta}}, \left(\frac{\delta^*(D_0)}{C_2^*}\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2C_r^*}\right)^{\frac{1}{\kappa}}, \left(\frac{1}{4}\right)^{\frac{2}{d-1}}\right\}.$$

Define $\delta_r(D_0) = t_0^\alpha(D_0)$, that is,

$$\begin{aligned}&\delta_r(D_0)\\ &:= \min\left\{\left(\frac{1}{C_s^{**}}\right)^{\frac{2}{2\alpha-1}}, \left(\frac{1}{C_r^{**}}\right)^{\frac{1}{\beta}}, \left(\frac{\delta^*(D_0)}{C_2^*}\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2C_r^*}\right)^{\frac{1}{\kappa}}, \left(\frac{1}{4}\right)^{\frac{2}{d-1}}\right\}^\alpha. \quad (8.11)\end{aligned}$$

Finally, we show that $\delta_r(D_0)$ is *lower stable* under C^2 perturbation. Let $(\widetilde{D}, \widetilde{X})$ be a pair of strictly pseudoconvex domains and formally integrable complex structures that satisfy the conditions of the proposition. Let $\widetilde{D} = \{z \in \mathcal{U} : \tilde{\rho} < 0\}$ and $\delta_r(\widetilde{D})$ be the corresponding stability constant to be determined.

Recall that we say $\delta_r(D_0)$ is lower stable under C^2 perturbation of ρ_0 if the following holds. There exist

$$\varepsilon^*(\rho_0) > 0, \qquad C(\rho_0) > 0$$

such that if $\|\tilde{\rho} - \rho_0\|_{\mathcal{U},2} \le \varepsilon^*(\rho_0)$, then we can choose $\delta_r(\widetilde{D})$ satisfying

$$\delta_r(D_0) \le C(\rho_0)\delta_r(\widetilde{D}). \tag{8.12}$$

We start by choosing

$$\varepsilon^*(\rho_0) := \frac{\varepsilon(D_0)}{4C''},$$

where as mentioned above $C'' > 1$ is an absolute constant to be determined.

Next, for the domain $\widetilde{D}$, we define

$$\begin{aligned}&\delta_r(\widetilde{D}) := \hat{t}_0^\alpha(\widetilde{D})\\ &:= \min\left\{\left(\frac{1}{C_s^{**}}\right)^{\frac{2}{2\alpha-1}}, \left(\frac{1}{C_r^{**}}\right)^{\frac{1}{\beta}}, \left(\frac{\hat{\delta}(\widetilde{D})}{C_2^*}\right)^{\frac{1}{\alpha}}, \left(\frac{1}{2C_r^*}\right)^{\frac{1}{\kappa}}, \left(\frac{1}{4}\right)^{\frac{2}{d-1}}\right\}^\alpha, \quad (8.13)\end{aligned}$$

where C_2^*, C_s^{**}, C_r^{**}, C_r^* depend on $\widetilde{D}$, and

$$\hat{\delta}(\widetilde{D}) := \delta\left(\tilde{\rho}, \frac{\varepsilon(D_0)}{2}, 2\right).$$

Note that the second argument of the last expression does not depend on $\widetilde{D}$.

Notice that, by definition of upper stability, the reciprocal of an upper stable constant is lower stable. It also follows from the definition that taking *minimum* of lower stable constants or raising to certain fixed positive power does not change lower stability.

Therefore, in order to prove that $\delta_r(D_0)$ is lower stable, it suffices to show that if the initial domain $\widetilde{D}$ has a defining function $\tilde{\rho}$ satisfying

$$\|\tilde{\rho} - \rho_0\|_{\mathcal{U},2} < \varepsilon^*(D_0),$$

then the following hold:

(1) Proposition 7.3 holds for the pair $(\widetilde{D}, \widetilde{X})$ with $\varepsilon(\widetilde{D})$, $\delta(\tilde{\rho}, \varepsilon(\widetilde{D}), 2)$ and $\hat{t}_0$ being replaced by $\varepsilon(D_0)/2$, $\hat{\delta}(\widetilde{D}) = \delta(\tilde{\rho}, \varepsilon(D_0)/2, 2)$ and $\hat{\delta}(\widetilde{D})^{1/\alpha}$, and the rest of the statements remain unchanged.

(2) Let $\tilde{\rho}_1, \tilde{\rho}_2, \dots,$ be the sequence of defining functions for domains obtained in the previous assertion for the initial domain $\widetilde{D}$ with a defining function $\tilde{\rho}$ satisfying $\|\tilde{\rho} - \rho_0\|_{\mathcal{U},2} < \varepsilon^*(D_0)$. Then

$$\|\tilde{\rho} - \rho_0\|_{\mathcal{U},2} < \varepsilon(D_0), \qquad \|\tilde{\rho}_j - \rho_0\|_{\mathcal{U},2} \leq \varepsilon(D_0).$$

Moreover, we get an embedding for $(\widetilde{D}, \widetilde{X})$ with the given $\delta_r(\widetilde{D})$ in (8.13).

(3) We can use the same set of parameters α, d, κ for initial defining functions ρ_0, $\tilde{\rho}$.

(4) Finally, we have $\delta^*(D_0) \leq \hat{\delta}(\widetilde{D})$, that is,

$$\delta\left(\rho_0, \frac{\varepsilon(D_0)}{4}, 2\right) \leq \delta\left(\tilde{\rho}, \frac{\varepsilon(D_0)}{2}, 2\right). \tag{8.14}$$

In other words, $\delta(\rho_0, \frac{\varepsilon(D_0)}{4}, 2)$ fulfills the requirements for $\delta(\tilde{\rho}, \frac{\varepsilon(D_0)}{2}, 2)$. Notice here the difference in the domains (with different defining functions) and the difference in the scales of perturbation. Clearly, (8.11), (8.13), and (8.14) imply immediately (8.12).

These assertions follow in principle from the proofs. However, let us point out how to achieve them.

To see the first assertion, we only need to argue that we can replace $\varepsilon(\widetilde{D})$ by $\varepsilon(D_0)/2$. The rest of the changes are obvious. One way to see this is to give a precise estimate of how $\varepsilon(\widetilde{D})$ depends on the defining function. However, we give an alternative argument based on the proof of Proposition 7.3 itself.

Notice that the function $\varepsilon(\widetilde{D})$, replacing $\varepsilon(D_0)$ in Proposition 7.3, is two fold. On the one hand, we need to control the Levi forms of a sequence of domains. On the other hand, in order to get convergence, we need to make sure that we can use the same coefficients C_2^*, C_s^*, C_s^{**}, C_r^* in the estimates during the iteration despite that the domains $\widetilde{D}_j$ are changing with $\widetilde{D}$.

Let f_j be the sequence of corrections in Proposition 7.3 for $\widetilde{D}$. Then they are guaranteed to satisfy the requirements in Lemma 7.1 when (7.14) and (7.20) are satisfied. These two conditions are achieved by our choice of $\hat{t}_0$. Here Lemma 7.1 is applied to $\tilde{\rho}$, $\hat{\delta}(\widetilde{D})$ and the sequence f_j.

Consequently, by Lemma 7.1 applied to $\widetilde{D}$ and $\varepsilon = \varepsilon(\widetilde{D})$, we have

$$\|\tilde{\rho}_j - \rho_0\|_{\mathcal{U},2} = \|\tilde{\rho}_j - \tilde{\rho} + \tilde{\rho} - \rho_0\|_{\mathcal{U},2} \leq \frac{\varepsilon(D_0)}{2} + \varepsilon^*(D_0) < \varepsilon(D_0), \tag{8.15}$$

provided we can verify

$$\|\tilde{\rho}_j - \tilde{\rho}\|_{\mathcal{U},2} < \frac{\varepsilon(D_0)}{2}. \tag{8.16}$$

Therefore, the sequence of domains defined by $\tilde{\rho}_j$ is strictly pseudoconvex provided that $\varepsilon(D_0)$ is sufficiently small. Note also that by (8.15) we have

$$C_2^*(\tilde{\rho}_j) \leq C(\rho_0) C_2^*(\rho_0).$$

Similar estimates hold for C_s^*, C_s^{**}, C_r^*.

Thus we have verified the second assertion. However, strictly speaking, one should check (8.15) by induction in $j \in \mathbb{N}$ as we did in the original proof of Proposition 7.3. However, since the ideas are the same, we leave the detail to the interested reader. We will verify (8.16) below and show how $\hat{\delta}(\widetilde{D})$, $\delta_r(\tilde{D})$ are chosen.

For the third assertion, we note that the choices of α, d, κ depend only on the constraints (7.21), (7.22), (7.23) and the optimization process. Therefore, they can be chosen uniformly.

Finally, we will show that $\delta^*(D_0) \leq \hat{\delta}(\widetilde{D})$, that is,

$$\delta\left(\rho_0, \frac{\varepsilon(D_0)}{4}, 2\right) \leq \delta\left(\tilde{\rho}, \frac{\varepsilon(D_0)}{2}, 2\right),$$

which amounts to verifying that $\delta(\rho_0, \frac{\varepsilon(D_0)}{4}, 2)$ fulfills the requirements for $\delta(\tilde{\rho}, \frac{\varepsilon(D_0)}{2}, 2)$.

Indeed, let $F_j = I + f_j$ be the sequence of diffeomorphisms that satisfy the condition of Lemma 7.1 in which D_0, δ are replaced by $\widetilde{D}$, $\delta^*(D_0)$. Thus, we can assume

$$\|f_j\|_{B_0,2} \leq \frac{\delta^*(D_0)}{(j+1)^2}.$$

Let $\tilde{F}_j = I + \tilde{f}_j = F_j \circ \cdots F_0$ and $\tilde{G}_j = I + \tilde{g}_j = F_j^{-1} \circ \cdots F_0^{-1}$. Let $|K| = |K'| = 2$. Set $\rho_j = \rho \circ \tilde{G}_j$, $\tilde{\rho}_j = \tilde{\rho} \circ G_j$, $\rho' = \tilde{\rho} - \rho$, and $\rho'_j = \rho' \circ \tilde{G}_j$. We have

$$\|\tilde{g}_j\|_{\mathcal{U},2} \leq C_2 \delta^*(D_0) \leq C_2$$

and

$$\begin{aligned}
\|\tilde{\rho}_j - \tilde{\rho}\|_{\mathcal{U},2} &\leq \|\rho_j - \rho\|_{\mathcal{U},2} + \|\rho' \circ \tilde{G}_j\|_{\mathcal{U},2} + \|\rho'\|_{\mathcal{U},2} \\
&\leq \frac{\varepsilon(D_0)}{4} + (1 + C_2'\|\tilde{g}_j\|_{\mathcal{U},2})^2 \|\rho'\|_{\mathcal{U},2} \\
&\leq \frac{\varepsilon(D_0)}{4} + C_2'' \varepsilon^*(D_0) \leq \frac{\varepsilon(D_0)}{2}.
\end{aligned}$$

Therefore, we have

$$\|\tilde{\rho}_j - \tilde{\rho}\|_{\mathcal{U},2} \leq \varepsilon(D_0)/2, \qquad \|\tilde{\rho}_j - \rho_0\|_{\mathcal{U},2} < \varepsilon(D_0).$$

This completes the proof of assertion (4) and also assertion (2).

Having verified all four assertions, we conclude that $\delta_r(D_0)$, $\delta_r(\widetilde{D})$ defined by (8.11) and (8.13) are lower stable at D_0 under small C^2 perturbation. This completes the proof. □

Acknowledgments. The authors were grateful to the referee for helpful suggestions which improved exposition.

References

[1] D. Catlin, *A Newlander–Nirenberg theorem for manifolds with boundary,* Michigan Math. J. 35 (1988), no. 2, 233–240.

[2] X. Gong, *Hölder estimates for homotopy operators on strictly pseudoconvex domains with C^2 boundary,* Math. Ann. 374 (2019), no. 1–2, 841–880.

[3] ______, *A Frobenius–Nirenberg theorem with parameter,* J. Reine Angew. Math. 759 (2020), 101–159.

[4] X. Gong and S. M. Webster, *Regularity for the CR vector bundle problem I,* Pure Appl. Math. Q. 6 (2010), no. 4, Special Issue: In honor of Joseph J. Kohn. Part 2, 983–998.

[5] ______, *Regularity for the CR vector bundle problem II,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 10 (2011), no. 1, 129–191.

[6] ______, *Regularity in the local CR embedding problem,* J. Geom. Anal. 22 (2012), no. 1, 261–293.

[7] R. S. Hamilton, *Deformation of complex structures on manifolds with boundary. I. The stable case,* J. Differential Geom. 12 (1977), no. 1, 1–45.

[8] ______, *Deformation of complex structures on manifolds with boundary. II. Families of noncoercive boundary value problems,* J. Differential Geom. 14 (1979), no. 3, 409–473 (1980).

[9] ______, *The inverse function theorem of Nash and Moser,* Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 1, 65–222.

[10] N. Hanges and H. Jacobowitz, *A remark on almost complex structures with boundary,* Amer. J. Math. 111 (1989), no. 1, 53–64.

[11] C. D. Hill, *Counterexamples to Newlander–Nirenberg up to the boundary,* Several complex variables and complex geometry, part 3 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., 52, pp. 191–197, Amer. Math. Soc., Providence, RI, 1991.

[12] L. Hörmander, *The boundary problems of physical geodesy,* Arch. Ration. Mech. Anal. 62 (1976), no. 1, 1–52.

[13] L. Hörmander, *An introduction to complex analysis in several variables,* Third edition, N.-Holl. Math. Libr., 7, North-Holland Publishing Co., Amsterdam, 1990.

[14] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, volume 2,* Wiley Publ. Appl. Stat., Wiley, 1996.

[15] ______, *Foundations of differential geometry, volume 2,* Wiley Publ. Appl. Stat., Wiley, 1996.

[16] ______, *Foundations of differential geometry, volume 2,* Wiley Publ. Appl. Stat., Wiley, 1996.

[17] J. Michel, *Integral representations on weakly pseudoconvex domains,* Math. Z. 208 (1991), no. 3, 437–462.

[18] J. Michel and M.-C. Shaw, *A decomposition problem on weakly pseudoconvex domains,* Math. Z. 230 (1999), no. 1, 1–19.

[19] J. Moser, *On invariant curves of area-preserving mappings of an annulus,* Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1962 (1962), 1–20.

[20] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds,* Ann. of Math. (2) 65 (1957), 391–404.

[21] A. Nijenhuis and W. B. Woolf, *Some integration problems in almost-complex and complex manifolds,* Ann. of Math. (2) 77 (1963), 424–489.

[22] ______, *Some integration problems in almost-complex and complex manifolds,* Ann. of Math. (2) 77 (1963), 424–489.

[23] P. L. Polyakov, *Versal embeddings of compact 3-pseudoconcave CR submanifolds,* Math. Z. 248 (2004), no. 2, 267–312.

[24] E. M. Stein, *Singular integrals and differentiability properties of functions,* Princeton Math. Ser., 30, Princeton University Press, Princeton, N.J., 1970.

[25] B. Street, *Sharp regularity for the integrability of elliptic structures,* J. Funct. Anal. 278 (2020), 108290, 55 pp.

[26] S. Webster, *On the proof of Kuranishi's embedding theorem,* Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), no. 3, 183–207.

[27] S. M. Webster, *A new proof of the Newlander–Nirenberg theorem,* Math. Z. 201 (1989), no. 3, 303–316.

[28] ——, *The integrability problem for CR vector bundles,* Several complex variables and complex geometry, part 3 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., 52, pp. 355–368, Amer. Math. Soc., Providence, RI, 1991.

C. Gan
Department of Mathematics
University of Wisconsin-Madison
Madison, WI 53706
USA

chun@math.wisc.edu

X. Gong
Department of Mathematics
University of Wisconsin-Madison
Madison, WI 53706
USA

gong@math.wisc.edu