J. Math. Anal. Appl. 538 (2024) 128238

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Sobolev and Hölder estimates for homotopy operators of the $\overline{\partial}$-equation on convex domains of finite multitype ☆

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A R T I C L E I N F O

Article history:
Received 21 December 2022
Available online 20 February 2024
Submitted by X. Huang

Keywords:
Cauchy–Riemann equation
Integral representation
Convex domain of finite type
Negative Sobolev spaces

A B S T R A C T

We construct homotopy formulas for the $\overline{\partial}$-equation on convex domains of finite type that have optimal Sobolev and Hölder estimates. For a bounded smooth finite type convex domain $\Omega \subset \mathbb{C}^n$ that has q-type m_q for $1 \le q \le n$, our $\overline{\partial}$ solution operator $\mathcal{H}_q$ on $(0,q)$-forms has (fractional) Sobolev boundedness $\mathcal{H}_q : H^{s,p} \to H^{s+1/m_q,p}$ and Hölder–Zygmund boundedness $\mathcal{H}_q : \mathscr{C}^s \to \mathscr{C}^{s+1/m_q}$ for all $s \in \mathbb{R}$ and $1 < p < \infty$. We also demonstrate the L^p-boundedness $\mathcal{H}_q : H^{s,p} \to H^{s,pr_q/(r_q-p)}$ for all $s \in \mathbb{R}$ and $1 < p < r_q$, where $r_q := (n-q+1)m_q + 2q$.

1. Introduction

In this paper, we aim to prove the following.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{C}^n$ is a bounded smooth convex domain of finite type. Then, the operators $\mathcal{H}_q : \mathscr{S}'(\Omega; \wedge^{0,q}) \to \mathscr{S}'(\Omega; \wedge^{0,q-1})$ exist that map $(0,q)$-forms to $(0,q-1)$-forms with distributional coefficients, for $1 \le q \le n$ (we set $\mathcal{H}_{n+1} := 0$), such that:*

(i) (Homotopy formula) $f = \overline{\partial}\mathcal{H}_q f + \mathcal{H}_{q+1}\overline{\partial} f$ *for all $1 \le q \le n$ and $(0,q)$-forms $f \in \mathscr{S}'(\Omega; \wedge^{0,q})$.*

Moreover, suppose that Ω has q-type m_q (see Definition 3.1)*. Then, $\mathcal{H}_q$ has the following boundedness properties:*

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https://doi.org/10.1016/j.jmaa.2024.128238

(ii) (Sobolev and Hölder) *For every* $s \in \mathbb{R}$ *and* $1 < p < \infty$, $\mathcal{H}_q : H^{s,p}(\Omega; \wedge^{0,q}) \to H^{s+1/m_q,p}(\Omega; \wedge^{0,q-1})$ *and* $\mathcal{H}_q : \mathscr{C}^s(\Omega; \wedge^{0,q}) \to \mathscr{C}^{s+1/m_q}(\Omega; \wedge^{0,q-1})$.

(iii) (L^p-L^q estimates) *For every* $s \in \mathbb{R}$ *and* $1 < p < r_q$, $\mathcal{H}_q : H^{s,p}(\Omega; \wedge^{0,q}) \to H^{s,pr_q/(r_q-p)}(\Omega; \wedge^{0,q-1})$. Here $r_q := (n - q + 1) \cdot m_q + 2q$.

For an open subset $\Omega \subseteq \mathbb{C}^n$, we use $\mathscr{S}'(\Omega)$ for the space of extendable complex-valued distributions on Ω (see Notation 4.1 and Lemma 4.15) and $\mathscr{C}^\infty(\Omega)$ for the space of all bounded smooth complex functions on Ω (see Definition 4.4). For $s \in \mathbb{R}$ and $1 < p < \infty$, we use $H^{s,p}(\Omega)$ for the Sobolev–Bessel space and $\mathscr{C}^s(\Omega)$ for the Hölder–Zygmund space (see Definitions 4.3 and 4.4). When $1 < p < \infty$ and $k \ge 0$, $H^{k,p} = W^{k,p}$ is the usual Sobolev space; and $\mathscr{C}^s = C^s$ is the usual Hölder space when $s > 0$ is not an integer (see Remark 4.6).

In fact, we obtain a stronger estimate via Triebel–Lizorkin spaces (see Theorem 1.2). We also prove the corresponding L^p-L^q estimate for a strongly pseudoconvex domain in Section 7, which is new for negative Sobolev spaces (see Theorem 7.1).

For a bounded smooth convex domain $\Omega \subset \mathbb{C}^n$ of finite type m, Diederich–Fischer–Fornæss [20] constructed a solution operator H_q for the $\overline{\partial}$-equation from $(0, q)$ closed forms to $(0, q - 1)$-forms, which has boundedness $H_q : L^\infty \to C^{1/m}$. In particular, $\overline{\partial} H_q f = f$ for all L^∞ $\overline{\partial}$-closed $(0, q)$-forms f on Ω.

Based on their approach, subsequent authors obtained the following L^p and C^k-estimates.

- Fischer [23] proved that $H_q : L^p \cap \ker \overline{\partial} \to L^{\frac{p(mn+2)}{mn+2-p}}$ for $1 < p < mn + 2$.
- Hefer [34] improved the previous two results [20,23] using multitypes: if Ω has q-type m_q, then $H_q : L^\infty \cap \ker \overline{\partial} \to C^{1/m_q}$ and $H_q : L^p \cap \ker \overline{\partial} \to L^{\frac{p \cdot r_q}{r_q - p}}$ for $1 < p < (n-q+1) \cdot m_q + 2q$, where $r_q = (n-q+1) \cdot m_q + 2q$. Note that $m = m_1 \ge m_q$, and m_q is generally smaller.
- Alexandre [3] modified H_q to a new solution operator $\tilde{H}_q$ such that $\tilde{H}_q : C^k \cap \ker \overline{\partial} \to C^{k+1/m}$ ($\tilde{H}_q$ depends on k).

Our Theorem 1.1 implies all of the results above. In addition, we provide the following remarks.

- Our $\mathcal{H}_q$ is a solution operator to the Cauchy–Riemann equation on $(0, q)$-forms. When f is a $\overline{\partial}$ closed $(0, q)$-form, then the $(0, q - 1)$ form $u = \overline{\partial} \mathcal{H}_q f$ solves $\overline{\partial} u = f$. In addition, for estimates of $\mathcal{H}_q$, we do not require the domains to be the subspace of closed forms, whereas those in previous studies [20,23,34,3] were stated only on closed forms.
- Our estimates on $\mathscr{C}^s$-spaces imply that given by [20,3] because $C^k \subsetneq \mathscr{C}^k$ for $k \ge 1$ and $L^\infty \subsetneq \mathscr{C}^0$ (e.g., see [67, (2.5.7/11)]). For $q \ge 2$, our result shows the gain of the $\frac{1}{m_q}$ derivative, whereas in the studies by [20,3], the gain was only $\frac{1}{m} = \frac{1}{m_1}$.
- When $1 < p < r_q$, by taking $s = 0$, we see that Theorem 1.1 (iii) contains the L^p-L^q estimate in [34, Theorem 1.3]. We also have the boundedness $\mathcal{H}_q : L^{r_q} \to \mathrm{BMO}$ that recovers [23, Theorem 1.1 (ii)], see Remark 1.4.
- Even for a negative integer k, our operator $\mathcal{H}_q$ is defined on the distribution space $H^{k,p}$ and has $\frac{1}{m}$ gain $\mathcal{H}_q : H^{k,p} \to H^{k+\frac{1}{m},p}$ (in fact, to $H^{k+\frac{1}{m_q},p}$).
- The operator $\mathcal{H}_q$ is a "universal solution operator" in the sense that we have one operator that has $H^{s,p}$ and $\mathscr{C}^s$ boundedness for all s, rather than only a bounded range of s.

The estimates on Sobolev space of negative index were first achieved in [63] for the case of a smooth strongly pseudoconvex domain, where for each $1 \le q \le n$, we obtained a solution operator with $\frac{1}{2}$-estimate $H^{s,p} \to H^{s+\frac{1}{2},p}$ for all $s \in \mathbb{R}$ and $1 < p < \infty$.

In both [63] and the current paper, our solution operators are non-canonical because they do not come from the solutions of the $\overline{\partial}$-Neumann problem. However, for canonical solutions, it is comparably more difficult to discuss the boundedness (or even well-posedness) on negative function spaces because we need to use a version of a generalized trace to discuss the boundary value condition (see [56]), as mentioned by [28].

Note that the L^p-L^q estimates cannot be directly obtained from the $\frac{1}{m}$-estimates since the classical Sobolev estimate only yields $H^{\frac{1}{m},p} \hookrightarrow L^{\frac{2nmp}{2nm-p}}$, which is a larger space than $L^{\frac{p(mn+2)}{mn+2-p}}$.

Our solution operators follow from the construction of [20]. We recall that their solution operator H_q from [20] has the form

$$H_q f(z) := \int_{\Omega} B_{q-1}(z,\cdot) \wedge f - \int_{b\Omega} K_{q-1}(z,\cdot) \wedge f. \tag{1.1}$$

The first integral is the Bochner–Martinelli integral operator (see (2.7) for the definition of B_{q-1}), which is known to gain one derivative. The second integral is the main term. The construction of K_{q-1} is based on the Diederich–Fornæss support function $S(z,\zeta)$ (see (2.4) and (2.8)). We remark that a slight modification of $K(z,\zeta)$ is required in order to make it a bounded function for each ζ, as mentioned by [3]. See Lemma 2.2 and Remark 2.3.

Our solution operator replaces the boundary integral with integration of the commutator $[\overline{\partial}, \mathcal{E}]$ on the exterior neighborhood. The commutator was introduced by [50] and used later by [48] and recently by [30]:

$$\mathcal{H}_q f(z) := \int_{\mathcal{U}} B_{q-1}(z,\cdot) \wedge f + \int_{\mathcal{U}\setminus\overline{\Omega}} K_{q-1}(z,\cdot) \wedge [\overline{\partial}, \mathcal{E}]f, \tag{1.2}$$

where $\mathcal{U}$ is a sufficiently small neighborhood of $\overline{\Omega}$ and $\mathcal{E}$ is a suitably selected extension operator of Ω such that the extended functions are supported in $\mathcal{U}$.

In [45,50], the authors used $\mathcal{E}$ for the Seeley's half-space extension [59], which only works on smooth domains, and they extended $H^{s,p}$ and $\mathscr{C}^s$ functions for positive s. For the case of non-smooth domains, e.g., [30], the authors used $\mathcal{E}$ for the Stein's extension [60, Chapter VI], which is defined on Lipschitz domains, and also extended $H^{s,p}$ and $\mathscr{C}^s$ for positive s.

In our case, we choose $\mathcal{E}$ as the Rychkov extension operator, which works on Lipschitz domains and extends $H^{s,p}$ and $\mathscr{C}^s$ for **all** s (including $s < 0$) (see (4.6) and (4.14). The Rychkov's extension operator was first introduced to solve the $\overline{\partial}$-equation by [62].

To prove the $\frac{1}{m}$-estimates, in [20] and [23,34], the second integral in (1.1) was defined on the boundary, and thus we only need to consider the estimate of the tangential part of K_{q-1} with respect to ζ-variable, which is as follows in our notation (see Definition 2.6):

$$\int_{b\Omega} K_{q-1}(z,\cdot) \wedge f = \int_{b\Omega} K^{\top}_{q-1}(z,\cdot) \wedge f.$$

Moreover, to estimate (1.2), we need to deal with the normal part $K^{\perp}_{q-1} = K_{q-1} - K^{\top}_{q-1}$, which contributes to the major loss of the kernel. In general, based on the estimates in [20, Section 5], $K^{\perp}_{q-1}$ loses 1 more derivative than $K^{\top}_{q-1}$. Alexandre [3] gave a better control and showed that $\frac{1}{2} - \frac{1}{m}$ derivative is lost at most.

In this paper, we introduce the following decomposition (see Notation 2.7 and (2.12)), which simplifies Alexandre's approach:

$$\int_{\mathcal{U}\setminus\overline{\Omega}} K_{q-1}(z,\cdot) \wedge [\overline{\partial}, \mathcal{E}]f = \int_{\mathcal{U}\setminus\overline{\Omega}} K^{\top}_{q-1}(z,\cdot) \wedge [\overline{\partial}, \mathcal{E}]f + \int_{\mathcal{U}\setminus\overline{\Omega}} K^{\perp}_{q-1}(z,\cdot) \wedge ([\overline{\partial}, \mathcal{E}]f)^{\top}. \tag{1.3}$$

$[\overline{\partial}, \mathcal{E}]f$ has one derivative less than f, and the $\frac{1}{m}$-estimate of $K_{q-1}^{\top}(z, \cdot) \wedge [\overline{\partial}, \mathcal{E}]f$ essentially follows from [20]. Although $K_{q-1}^{\perp}$ loses one more derivative than $K_{q-1}^{\top}$, the tangential part of the commutator $[\overline{\partial}, \mathcal{E}]^{\top} f := ([\overline{\partial}, \mathcal{E}]f)^{\top}$ has the same regularity to f, which compensates for the estimate that we need (see Proposition 5.1 and Remark 5.2 (ii)).

Note that (1.3) is not needed in the case of strongly pseudoconvex domains because for the Leray map $\widehat{Q}(z, \zeta)$ (see Proposition 7.3), we only need the trivial estimates (7.7) and (7.8). See Remark 3.8.

For the case where $s < 1$ in Theorem 1.1, the commutator $[\overline{\partial}, \mathcal{E}]f$ may give a distribution rather than a classical function. In order to ensure that the integral operators make sense, we express the given forms as the derivatives of functions with positive index. For $k \geq 1$, we constructed the anti-derivative operators $\{\mathcal{S}^{k,\alpha}\}_{|\alpha| \leq k}$ in [61] such that if a function g is supported outside Ω, then $g = \sum_{|\alpha| \leq k} D^{\alpha} \mathcal{S}^{k,\alpha} g$ with all summands also supported outside Ω. See Proposition 4.13. Therefore, by integrating by parts,

$$\begin{aligned}\int\limits_{\mathcal{U} \setminus \overline{\Omega}} K_{q-1}^{(\top,\perp)}(z, \zeta) \wedge [\overline{\partial}, \mathcal{E}]^{(\top)} f(\zeta) d \operatorname{Vol}_{\zeta} &= \sum_{|\alpha| \leq k} \int\limits_{\mathcal{U} \setminus \overline{\Omega}} K_{q-1}^{(\top,\perp)}(z, \zeta) \wedge \big(D^{\alpha} \mathcal{S}^{k,\alpha} \circ [\overline{\partial}, \mathcal{E}]^{(\top)} f\big)(\zeta) d \operatorname{Vol}_{\zeta} \\ &= \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \int\limits_{\mathcal{U} \setminus \overline{\Omega}} D_{\zeta}^{\alpha} K_{q-1}^{(\top,\perp)}(z, \zeta) \wedge \big(\mathcal{S}^{k,\alpha} \circ [\overline{\partial}, \mathcal{E}]^{(\top)} f\big)(\zeta) d \operatorname{Vol}_{\zeta} .\end{aligned}$$

The method of trading derivatives between K_{q-1} and $[\overline{\partial}, \mathcal{E}]f$ was introduced by [63] for the estimates of strongly pseudoconvex domains.

The key step to prove Theorem 1.1 is to obtain the weighted estimates for $D_{z,\zeta}^{k}(K_{q-1}^{\top})(z, \zeta)$ and $D_{z,\zeta}^{k}(K_{q-1}^{\perp})(z, \zeta)$. See Theorem 2.9. Note that we take derivatives after we take ($\perp$ and $\top$) projections. The reduction from Theorem 1.1 to Theorem 2.9 is achieved by using the Hardy–Littlewood lemma (see Proposition 5.3 and Corollary 5.5 (iii)).

In fact, by combining Theorem 2.9 and Corollary 5.5 (iii), we have a stronger estimate of $\mathcal{H}_q$ in terms of Triebel–Lizorkin spaces (see Definition 4.5):

Theorem 1.2. *With $\mathcal{H}_q$ as in Theorem 1.1, the following boundedness properties hold for $1 \leq q \leq n-1$:*

$$\mathcal{H}_q : \mathscr{F}_{p,\infty}^{s}(\Omega; \wedge^{0,q}) \to \mathscr{F}_{p,\varepsilon}^{s+\frac{1}{m_q}}(\Omega; \wedge^{0,q-1}), \qquad \forall\, \varepsilon > 0, \quad 1 \leq p \leq \infty; \tag{1.4}$$

$$\mathcal{H}_q : \mathscr{F}_{p,\infty}^{s}(\Omega; \wedge^{0,q}) \to \mathscr{F}_{\frac{p r_q}{r_q - p},\varepsilon}^{s}(\Omega; \wedge^{0,q-1}), \qquad \forall\, \varepsilon > 0, \quad 1 \leq p \leq r_q. \tag{1.5}$$

Theorem 1.2 implies Theorem 1.1 (ii) and (iii) automatically for the case where $1 \leq q \leq n-1$ (see Remark 4.6).

Remark 1.3 *(Boundedness on Besov spaces).* Theorem 1.2 implies the $\frac{1}{m_q}$-estimate and higher order L^p-L^q estimates on Besov spaces via real interpolations.

By the elementary embedding (see Remark 4.6 (iii)), for every $s \in \mathbb{R}$ and $t \in (0, \infty]$, we have $\mathcal{H}_q : \mathscr{F}_{p,t}^{s} \to \mathscr{F}_{p,t}^{s+1/m_q}$ for $p \in [1, \infty]$ and $\mathcal{H}_q : \mathscr{F}_{p,t}^{s} \to \mathscr{F}_{\frac{p r_q}{r_q - p},t}^{s}$ for $p \in [1, r_q]$. In addition, we have real interpolations (e.g., see [66, Corollary 1.111]):

$$(\mathscr{F}_{p,t_0}^{s_0}(\Omega), \mathscr{F}_{p,t_1}^{s_1}(\Omega))_{\theta,t} = \mathscr{B}_{p,t}^{\theta s_1 + (1-\theta) s_0}(\Omega), \quad \forall p \in [1, \infty),\ t_0, t_1, t \in (0, \infty],\ \theta \in (0,1) \text{ and } s_0 \neq s_1;$$

$$(\mathscr{F}_{\infty,\infty}^{s_0}(\Omega), \mathscr{F}_{\infty,\infty}^{s_1}(\Omega))_{\theta,t} = \mathscr{B}_{\infty,t}^{\theta s_1 + (1-\theta) s_0}(\Omega), \quad \forall t \in (0, \infty],\ \theta \in (0,1) \text{ and } s_0 \neq s_1.$$

See [66, (1.368) and (1.369)]. Therefore (see [64, Definition 1.2.2/2 and Theorem 1.3.3] for example), for every $s \in \mathbb{R}$ and $t \in (0, \infty]$, we have $\mathcal{H}_q : \mathscr{B}_{p,t}^{s}(\Omega; \wedge^{0,q}) \to \mathscr{B}_{p,t}^{s+1/m_q}(\Omega; \wedge^{0,q-1})$ for $p \in [1, \infty]$, and $\mathcal{H}_q : \mathscr{B}_{p,t}^{s}(\Omega; \wedge^{0,q}) \to \mathscr{B}_{\frac{p r_q}{r_q - p},t}^{s}(\Omega; \wedge^{0,q-1})$ for $p \in [1, r_q]$.

Remark 1.4 *(Boundedness on BMO).* As a special case of (1.5), we recover the endpoint L^p-L^q boundedness $\mathcal{H}_q : L^{r_1}(\Omega; \wedge^{0,q}) \to \mathrm{BMO}(\Omega; \wedge^{0,q-1})$ from [23, Theorem 1.1 (ii)] (cf. [34, Theorem 1.3]). The definition of $\mathrm{BMO}(\Omega)$ used by Fischer [23] comes from [49, Section 4, Definition 3]. We recall that for an arbitrary open subset $U \subset \mathbb{R}^N$, $\mathrm{BMO}(U)$ and $\mathrm{bmo}(U)$ (see [12, Definition 1.2]) are spaces consisting of $f \in L^1_{\mathrm{loc}}(U)$ such that:

$$\|f\|_{\mathrm{BMO}(U)} := \sup_{B \subseteq U} \frac{1}{|B|} \int\limits_B \left|f - \tfrac{1}{|B|}\textstyle\int_B f\right| < \infty, \quad \|f\|_{\mathrm{bmo}(U)} := \|f\|_{\mathrm{BMO}(U)} + \sup_{B \subseteq U} \frac{1}{|B|} \int\limits_B |f| < \infty,$$

where B denotes the balls in $\mathbb{R}^N$.

Clearly, $\mathrm{bmo}(U) \subset L^1_{\mathrm{loc}}(U)$, whereas $\mathrm{BMO}(U) = \mathrm{bmo}(U)/\{c \cdot \mathbf{1}_U : c \in \mathbb{C}\}$ ignores the constant functions.

By [12, Theorem 1.4] (since Ω is bounded smooth), we have $\mathrm{bmo}(\Omega) = \{\tilde{f}|_\Omega : \tilde{f} \in \mathrm{bmo}(\mathbb{C}^n)\}$, and by [67, Theorem 2.5.8/2], we have $\mathrm{bmo}(\mathbb{C}^n) = \mathscr{F}^0_{\infty 2}(\mathbb{C}^n)$. Therefore, by Definition 4.5, for spaces on domains, we obtain $\mathrm{bmo}(\Omega) = \mathscr{F}^0_{\infty 2}(\Omega)$.

In addition, by Remark 4.6 (iii) and (vi), we have $\mathscr{F}^0_{\infty,\varepsilon} \subset \mathscr{F}^0_{\infty,2}$ and $\mathscr{F}^0_{r_q,2} = L^{r_q} \subset \mathscr{F}^0_{r_q,\infty}$. Therefore, we obtain the boundedness $\mathcal{H}_q : L^{r_q}(\Omega; \wedge^{0,q}) \to \mathrm{bmo}(\Omega; \wedge^{0,q-1})$. By taking the quotient of constant functions, we obtain a stronger one $\mathcal{H}_q : L^{r_q}(\Omega; \wedge^{0,q}) \to \mathrm{BMO}(\Omega; \wedge^{0,q-1})$. (Recall that $r_1 \geq r_q$ from Theorem 1.1 (iii) since $m_1 \geq m_q$.)

Obtaining the estimates for the $\overline{\partial}$-equation is a fundamental question in several complex variables. There are two major approaches can be applied. The first approach is the *$\overline{\partial}$-Neumann problem*, which defines the *canonical solutions*, and it was proposed by [31]. The estimate originated as the Hörmander [40] L^2-estimate and it was later developed by [42]. We refer the reader to [15] for a detailed discussion.

We use the second approach called *integral representations*, which yield non-canonical solutions but the expressions can be more explicit. This method was introduced for the $\overline{\partial}$-equation by Henkin [37] and Grauert & Lieb [29] in the study of strongly pseudoconvex domains. We refer the reader to [53] and [44] for a general discussion.

We briefly review the estimates for convex domains of finite type in the following. Studies in complex or real pseudo-ellipsoids were conducted by Range [52], Diederich–Fornæss–Wiegerinck [21], Chen–Krantz–Ma [13], and Fleron [26], and in the domain of real-analytic boundaries by Range [51] and Bruna-Castillo [5]. These are all special cases for general convex domains of finite type. The $\frac{1}{m}$-regularity was shown to be optimal by [13].

For the type conditions in convex domains, McNeal [47] introduced the *ε-extremal basis* and showed the equivalence between the line type and D'Angelo 1-type on convex domains, and it was later used to show the boundedness of $\overline{\partial}$-Neumann solutions by [49] (also see [7] for a short proof). McNeal's approach was used by Cumenge [16,17] and Wang [69] to obtain estimates for the $\overline{\partial}$-equation. For the type where $q \geq 2$, Yu [71] introduced a different basis from that of McNeal called the *ε-minimal basis*, and showed the equivalence of the line q-type, D'Angelo q-type, and Catlin's q-type (also see [35] for the connections between McNeal's ε-extremal basis and Yu's ε-minimal basis).

As mentioned in the beginning, the solution operators on convex domains of finite type are mainly derived from Diederich–Fischer–Fornæss [20] with the holomorphic supported function constructed by [18]. The Hölder estimate $L^\infty \to C^{\frac{1}{m}}$ was obtained by [20], and the anisotropic version was later obtained by Fischer [24] and Diederich–Fischer [19] on lineally convex domains of finite type. The L^p-estimate was first obtained by Fischer [23] and later partial progress was made by [1,2] and [11]. The $C^k \to C^{k+\frac{1}{m}}$ estimate was obtained by Alexandre [3]. The multitype notion was used by Hefer [34] who showed that on $(0,q)$-forms, the $\frac{1}{m}$-estimate could be automatically improved to the $\frac{1}{m_q}$-estimate if one considers the multitype of the domain (also see [4]).

The convex domains of infinite type were considered by studies by Range [55], Fornæss–Lee–Zhang [27], and Ha–Khanh–Raich [38], and recently by Ha [32,33]. Some of their constructions also used integral representations. It should be possible to improve their results using the Rychkov's extension operator, as applied in this paper.

For general finite type domains that are not necessarily convex, it is known that in $\mathbb{C}^2$, one can generally have the optimal $\frac{1}{m}$-estimate (see [25] and [14]; and also see [54] for an approach using integral representations). In higher dimensions, Catlin [8–10] showed that the canonical solution has boundedness $L^2 \to H^{\varepsilon,2}$ for some $\varepsilon > 0$ if and only if the domain has finite D'Angelo type. The general lower bounds for ε with respect to type m generally remain unknown. For further discussions of finite types and subelliptic estimates, we refer the reader to the survey by [22].

The paper is organized as follows. In Section 2, we recall the construction of the Diederich–Fornæness support function and the corresponding integral kernel, and we introduce the tangential and vertical projections for $d\bar{\zeta}$-forms. In Section 3, we review the ε-minimal basis and prove Theorem 2.9. In Section 4, we summarize the properties of function spaces and Rychkov's construction of the extension operator. In Section 5, we prove the boundedness of the tangential commutator, Proposition 5.1, and strong Hardy–Littlewood lemma, Proposition 5.3. In Section 6, we complete the proof of Theorems 1.1 and 1.2 using Theorem 2.9 and Corollary 5.5. In Section 7, we apply the proof techniques for Theorems 1.1 and 1.2 to the case of strongly pseudoconvex domains and prove Theorem 7.1.

In the following, we use $\mathbb{N} = \{0, 1, 2, \dots\}$ as the set of non-negative integers.

On a complex coordinate system $(z_1, \dots, z_n)$, ∂_z^α denotes the derivative on the holomorphic part $\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$, where $\alpha \in \mathbb{N}^n$, and D_z^β denotes the total derivative $\frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \dots \partial z_n^{\beta_n} \partial \bar{z}_1^{\beta_{n+1}} \dots \partial \bar{z}_n^{\beta_{2n}}}$, where $\beta \in \mathbb{N}^{2n}$.

We use the notation $x \lesssim y$ to denote that $x \le Cy$, where C is a constant that is independent of x, y, and $x \approx y$ for "$x \lesssim y$ and $y \lesssim x$." We use $x \lesssim_\varepsilon y$ to emphasize the dependence of C on the parameter ε.

For a function class $\mathscr{X}$ and a domain U, we use $\mathscr{X}(U) = \mathscr{X}(U; \mathbb{C})$ as the space of complex-valued functions in U that have regularity $\mathscr{X}$. We use $\mathscr{X}(U; \mathbb{R})$ if the functions are restricted to being real-valued. We use $\mathscr{X}(U; \wedge^{p,q})$ for the space of (complex-valued) (p, q)-forms on U that have regularity $\mathscr{X}$.

In the following, $U_1 = \{-T_1 < \varrho < T_1\}$ denotes a fixed neighborhood of $b\Omega$ (see Lemma 2.2).

Acknowledgment

The author would like to thank Xianghong Gong and Kenneth Koenig for their valuable support and comments.

2. Construction of homotopy formulas

Let $\Omega \subset \mathbb{C}^n$ be a smooth convex domain that has finite type m. We fix a defining function $\varrho \in C^\infty(\mathbb{C}^n; \mathbb{R})$ of Ω (i.e., $\Omega = \{\varrho < 0\}$ and $\nabla \varrho(\zeta) \neq 0$ for all $\zeta \in b\Omega$) such that the following holds.

(2.1) A $T_0 > 0$ exists and for every $-T_0 < t < T_0$, the domain $\Omega_t := \{\zeta : \varrho(\zeta) < t\}$ is convex and has the same complex affine q-type (see Definition 3.1) to $\Omega = \Omega_0$ for all $1 \le q \le n$.

This can be achieved by assuming that $0 \in \Omega$ (which can be achieved by passing to a translation) and requiring ϱ to have the homogeneity condition (also see [34, (2.1)]):

$$\varrho(\lambda\zeta) + 1 = \lambda(\varrho(\zeta) + 1) \text{ for all } \zeta \in b\Omega \text{ and all } \lambda \in \mathbb{R}_+ \text{ closed to } 1. \tag{2.2}$$

In this setting, Ω_t is simply the dilation of Ω, which shares the same (line, D'Angelo, or Catlin) type conditions.

We let $U_0 := \{\zeta : |\varrho(\zeta)| < T_0\}$ be a corresponding open neighborhood of $b\Omega$.

We recall the Diederich-F-ornæss holomorphic support function $S \in C^\infty(\mathbb{C}^n \times U_0; \mathbb{C})$ from [18], as follows.

Fix suitably large constants $M_1, M_2, M_3 > 1$. For each $\zeta \in U_0$, we take a unitary matrix $\Phi(\zeta) \in \mathbb{C}^{n\times n}$ that is locally defined and smoothly dependent on ζ such that $\Phi(\zeta)\frac{\overline{\partial}\varrho(\zeta)}{|\overline{\partial}\varrho(\zeta)|} = [1, 0, \dots, 0]^\intercal$. We define[1] for $\zeta \in U_0$ and $w = [w_1, \dots, w_n]^\intercal \in \mathbb{C}^n (\simeq \mathbb{C}^{n\times 1})$:

$$S_\zeta^{\Phi(\zeta)}(\omega) := 3\omega_1 + M_1\omega_1^2 - \tfrac{1}{M_2}\sum_{j=1}^{m/2} M_3^{4^j}(-1)^j \sum_{|\alpha|=2j;\alpha_1=0} \frac{\partial^\alpha \varrho(\zeta + \Phi(\zeta)^\dagger \cdot w)}{\partial w^\alpha}\bigg|_{w=0} \cdot \frac{\omega^\alpha}{\alpha!}; \tag{2.3}$$

$$S(z,\zeta) := S_\zeta^{\Phi(\zeta)}\big(\Phi(\zeta)(z-\zeta)\big) \quad z \in \Omega. \tag{2.4}$$

Lemma 2.1. *In (2.4), $S(z,\zeta)$ with suitable constants $M_1, M_2, M_3 > 0$ satisfies the following.*

(i) ([20, Lemma 2.1]) *$S(z,\zeta)$ is a smooth function, holomorphic in z, and does not depend on the choice of the family $\{\Phi(\zeta) : \zeta \in U_0\}$.*

(ii) ([18, Corollary 2.4] and [23, Theorem 2.1]) *An $M_4 > 1$ exists such that*

$$\operatorname{Re} S(z,\zeta) \le M_4 \cdot \max(0, \varrho(z) - \varrho(\zeta)) - \tfrac{1}{M_4}|z-\zeta|^m, \quad \forall \zeta \in U_0, \quad z \in \Omega \cup U_0 \text{ such that } |z-\zeta| < \tfrac{1}{M_4}. \tag{2.5}$$

As mentioned by [3], $S(z,\zeta)$ may have zeroes in $\big(\Omega \times (U_0\backslash\overline{\Omega})\big) \cap \{|z-\zeta| \ge \frac{1}{M_4}\}$. We can make the following standard modification.

Lemma 2.2. *Let $S \in C^\infty_{\mathrm{loc}}(\mathbb{C}^n \times U_0; \mathbb{C})$ be as in (2.4). $T_1 \in (0, T_0]$ exist that are associated with the neighborhood $U_1 := \{\zeta : |\varrho(\zeta)| < T_1\}$ of $b\Omega$, a constant $M_5 > 1$, and a $\widehat{S} \in \mathscr{C}^\infty(\Omega \times U_1; \mathbb{C})$ such that:*

(i) *$\widehat{S}(\cdot,\zeta)$ is holomorphic in $z \in \Omega$ for all $\zeta \in U_1$,*

(ii) *$|\widehat{S}(z,\zeta)| \ge \frac{1}{M_5}$ for all $(z,\zeta) \in \Omega \times (U_1\backslash\overline{\Omega})$ such that $|z-\zeta| \ge \frac{1}{2M_4}$;*

(iii) *An $A \in \mathscr{C}^\infty(\Omega \times U_1; \mathbb{C})$ exists such that $\widehat{S}(z,\zeta) = A(z,\zeta) \cdot S(z,\zeta)$ and $\frac{1}{M_5} \le |A(z,\zeta)| \le M_5$ for all $(z,\zeta) \in \Omega \times (U_1\backslash\overline{\Omega})$ such that $|z-\zeta| \le \frac{1}{2M_4}$.*

Remark 2.3. Lemma 2.2 was not mentioned by [20], which might leave a gap when estimating the last integral in [20, Section 6]. There is a different modification $\widehat{S}(z,\zeta)$ in [34, Section 6] but it may not work in our situation.

Proof. We use the same construction from [39, Theorem 2.4.3].

Let $\delta_1 := \min\big(T_0, (2M_4)^{-m-2}(1 + \|\nabla\varrho\|_{L^\infty(U_0)})^{-1}\big) \in (0,1)$. By (2.5), we see that

$$-\operatorname{Re} S(z,\zeta) > \delta_1, \quad \text{whenever } \varrho(z), \varrho(\zeta) \in (-\delta_1, \delta_1) \text{ and } \tfrac{1}{2M_4} \le |z-\zeta| \le \tfrac{1}{M_4}.$$

Let $\chi_1 \in C_c^\infty((-\delta_1, \delta_1); [0,1])$ be such that $\chi_1\big|_{[-\frac{1}{2}\delta_1, \frac{1}{2}\delta_1]} \equiv 1$. Let $U_1' := \{\zeta : |\varrho(\zeta)| < \delta_1\}$, and we define a $(0,1)$-form $f(z,\zeta) = \sum_{j=1}^n f_j(z,\zeta) d\bar{z}_j$ for $z \in \Omega_{\delta_1} = \{\varrho < \delta_1\}$ and $\zeta \in U_1'$ by

$$f(z,\zeta) := \begin{cases} \overline{\partial}_z\big(\chi_1(|z-\zeta|) \cdot \log(-S(z,\zeta))\big), & \text{if } \frac{1}{2M_4} \le |z-\zeta| \le \frac{1}{M_4} \\ 0, & \text{otherwise} \end{cases}.$$

Since $S(z,\zeta)$ is smooth and holomorphic in z, we see that f is bounded smooth in the domain $\Omega_{\delta_1} \times U_1'$, and $f(\cdot,\zeta)$ is $\overline{\partial}$-closed for each $\zeta \in U_1'$.

[1] For a complex matrix A, we use $A^\dagger = \overline{A}^\intercal$ for the conjugate transpose. Thus, $A^\dagger = A^{-1}$ when A is unitary.

Therefore, either by applying [15, Theorem 11.2.7 and Lemma 11.2.6] since Ω_{δ_1} is convex, or by applying [39, Theorem 2.3.5] since we can find a strongly convex domain $\widetilde{\Omega}$ such that $\Omega_{\frac{1}{2}\delta_1} \subset \widetilde{\Omega} \subset \Omega_{\delta_1}$, a continuous solution operator $T : \mathscr{C}^\infty(\Omega_{\delta_1}; \wedge^{0,1}) \cap \ker \overline{\partial} \to \mathscr{C}^\infty(\Omega_{\frac{1}{2}\delta_1})$ exists such that $\overline{\partial} Tg = g$ in $\Omega_{\frac{1}{2}\delta_1}$ for every bounded smooth $\overline{\partial}$-closed form g in Ω_{δ_1}.

Now, for $z \in \Omega_{\frac{1}{2}\delta_1}$ and $\zeta \in U_1'$, we define

$$u(z,\zeta) := (Tf(\cdot,\zeta))(z); \qquad A(z,\zeta) := \exp(-u(z,\zeta));$$
$$\widehat{S}(z,\zeta) := \begin{cases} A(z,\zeta)S(z,\zeta), & \text{if } |z-\zeta| \le \frac{1}{2M_4} \\ -\exp\big(\chi_1(|z-\zeta|)\log(-S(z,\zeta)) - u(z,\zeta)\big), & \text{if } |z-\zeta| \ge \frac{1}{2M_4} \end{cases}.$$

We see that $\widehat{S} : \Omega_{\frac{1}{2}\delta_1} \times U_1' \to \mathbb{C}$ is holomorphic in z and bounded from below in $\{|z-\zeta| \ge \frac{1}{2M_4}\}$.

By taking $T_1 := \frac{1}{2}\delta_1$, $U_1 := \{|\varrho| < T_1\}$ and $M_5 := \max\Big(\frac{1}{\delta_1} \cdot \exp\big(\sup\limits_{\Omega_{T_1}\times U_1} u\big), \|S\|_{L^\infty(\Omega_{T_1}\times U_1)} \cdot \exp\big(\sup\limits_{\Omega_{T_1}\times U_1} (-u)\big)\Big)$, we obtain the estimates in (ii) and (iii), which completes the proof. $\square$

Now, we use $\widehat{S}(z,\zeta)$ to define the corresponding Leray map $\widehat{Q} = (\widehat{Q}_1, \dots, \widehat{Q}_n) \in C^\infty(\Omega \times U_1; \mathbb{C}^n)$ by the following: for $\zeta \in U_1$,

$$\begin{aligned} &\widehat{S}_\zeta^{\Phi(\zeta)}(w) := \widehat{S}(\zeta + \Phi(\zeta)^\dagger \cdot w, \zeta); && \widehat{Q}_{\zeta,j}^{\Phi(\zeta)}(w) := \int_0^1 \frac{\partial \widehat{S}_\zeta^{\Phi(\zeta)}}{\partial w_j}(tw)dt, \quad 1 \le j \le n; \\ &\widehat{Q}_\zeta^{\Phi(\zeta)}(w) := [\widehat{Q}_{\zeta,1}^{\Phi(\zeta)}(w), \dots, \widehat{Q}_{\zeta,n}^{\Phi(\zeta)}(w)]^\intercal; && \widehat{Q}(z,\zeta) := \Phi(\zeta)^\intercal \cdot \widehat{Q}_\zeta^{\Phi(\zeta)}\big(\Phi(\zeta)\cdot(z-\zeta)\big). \end{aligned} \tag{2.6}$$

By the same argument in [18, Lemma 2.1], $\widehat{Q}$ does not depend on the choice of unitary maps $\{\Phi(\zeta)\}$. In fact, we have $\widehat{Q}_j(z,\zeta) = \int_0^1 \frac{\partial \widehat{S}}{\partial z_j}(\zeta + t(z-\zeta), \zeta)dt$, and thus $\widehat{S}(z,\zeta) = \sum_{j=1}^n \widehat{Q}_j(z,\zeta)\cdot(z_j - \zeta_j)$.

Now, we identify the vector-valued function $\widehat{Q}(z,\zeta)$ with the 1-form $\sum_{j=1}^n \widehat{Q}_j(z,\zeta)d\zeta_j$ and we denote $b(z,\zeta) := \sum_{j=1}^n (\bar{\zeta}_j - \bar{z}_j)d\zeta_j$. The following notations for differential forms on $(z,\zeta) \in \Omega \times U_1$ are adapted from those used by [15]. In the following, $\overline{\partial} = \overline{\partial}_{z,\zeta}$,

$$B(z,\zeta) := \frac{b \wedge (\overline{\partial} b)^{n-1}}{(2\pi i)^n |z-\zeta|^{2n}} =: \sum_{q=0}^{n-1} B_q(z,\zeta); \tag{2.7}$$

$$K(z,\zeta) := \frac{b \wedge \widehat{Q}}{(2\pi i)^n} \wedge \sum_{k=1}^{n-1} (-1)^k \frac{(\overline{\partial} b)^{n-1-k} \wedge (\overline{\partial}\widehat{Q})^{k-1}}{|z-\zeta|^{2(n-k)}\widehat{S}^k} =: \sum_{q=0}^{n-2} K_q(z,\zeta). \tag{2.8}$$

B is an $(n, n-1)$ form where B_q is the component that has degree $(0,q)$ in z and $(n, n-1-q)$ in ζ; K is a $(n, n-2)$ form where K_q is the component that has degree $(0,q)$ in z and $(n, n-2-q)$ in ζ.

Lemma 2.4. *Let $\mathcal{E} : \mathscr{C}^\infty(\Omega) \to C_c^1(\Omega \cup U_1)$ be an extension operator such that $\operatorname{supp} \mathcal{E} f \Subset \Omega \cup U_1$ for all functions $f \in \mathscr{C}^\infty(\Omega)$. Then, the following integral is pointwisely defined:*

$$H_q f(z) := \int_{\Omega \cup U_1} B_{q-1}(z,\cdot) \wedge \mathcal{E} f + \int_{U_1 \setminus \overline{\Omega}} K_{q-1}(z,\cdot) \wedge [\overline{\partial}, \mathcal{E}] f, \quad 1 \le q \le n, \quad f \in \mathscr{C}^\infty(\Omega; \wedge^{0,q}), \quad z \in \Omega. \tag{2.9}$$

Moreover, $f = \overline{\partial} H_q f + H_{q+1} \overline{\partial} f$ for all $f \in \mathscr{C}^\infty(\Omega; \wedge^{0,q})$.

See [30, Proposition 2.1] or [15, Theorem 11.2.2] for a proof. Both references use the corresponding notation $K = \Omega^{01}(b, \widehat{Q})$. To integrate the bi-degree forms, we use the convention $\int_x u(x,y)dx^I \wedge dy^J := (\int_x u(x,y)dx^I)dy^J$, which we note is different from [53, Section III.1.9].

Lemma (2.4) does not guarantee that $f = \overline{\partial} H_q f + H_{q+1}\overline{\partial} f$ holds for distributions since $\mathcal{E}$ may not be defined on the space of distributions.

Definition 2.5. We construct the operator $\mathcal{H}_q$ from (2.9) by taking $\mathcal{E}$ as Rychkov's extension operator given in Definition 4.11.

Note that the Rychkov's extension operator is defined on the space $\mathscr{S}'(\Omega)$ of all extensible distributions.

The boundedness of $\mathcal{H}_q$ follows from the weighted estimates of the derivatives of the tangential part and the vertical part of $K_{q-1}(z,\zeta)$ with respect to ζ-variable.

Definition 2.6. Let $\varrho : U_1 \to (-T_1, T_1)$ be a defining function of Ω with non-vanishing gradient and let $b\Omega_t = \{\varrho = t\}$ (for $|t| < T_1$) be as given above. Let $1 \le p, q \le n$ and $\zeta_0 \in U$, and we define the $\overline{\partial}$-vertical projection $(-)^{\perp}_{\zeta_0}$ and $\overline{\partial}$-tangential projection $(-)^{\top}_{\zeta_0}$ at ζ_0 as the following surjective orthonormal projections:

$$(-)^{\perp}_{\zeta_0} : \bigwedge^{p,q}\mathbb{C}^n \twoheadrightarrow \bigwedge^p \mathbb{C}^n \otimes_{\mathbb{C}} \left(\mathrm{Span}\langle \overline{\partial}\varrho(\zeta_0)\rangle \wedge \bigwedge^{q-1}\mathbb{C}^n\right), \quad (-)^{\top}_{\zeta_0} : \bigwedge^{p,q}\mathbb{C}^n \twoheadrightarrow \bigwedge^p \mathbb{C}^n \otimes_{\mathbb{C}} \bigwedge^q T^{*0,1}_{\zeta_0}(b\Omega_{\varrho(\zeta_0)}).$$

For a (p,q)-form $f : U_1 \to \bigwedge^{p,q}\mathbb{C}^n$, we define $f^{\perp}(\zeta) := f(\zeta)^{\perp}_{\zeta}$ and $f^{\top}(\zeta) := f(\zeta)^{\top}_{\zeta}$ for $\zeta \in U_1$ naturally.

For a real hypersurface $M \subset \mathbb{C}^n$ and a $\zeta \in M$, $T^{*0,1}_{\zeta}M := T^{*0,1}_{\zeta}\mathbb{C}^n \cap \mathbb{C}T^*_{\zeta}M$ is the anti-holomorphic cotangent space of M at ζ.

Notation 2.7. For the bidegree form $K(z,\zeta)$, we use $K^{\top}(z,\zeta)$ and $K^{\perp}(z,\zeta)$ for the projections with respect to ζ-variable but not to z-variable, i.e., $K^{\top}_q(z,\zeta) := K_q(z,\cdot)^{\top}(\zeta)$ and $K^{\perp}_q(z,\zeta) := K_q(z,\cdot)^{\perp}(\zeta)$ for each q.

Remark 2.8. Let $\overline{\theta}_1, \dots, \overline{\theta}_n$ be $(0,1)$-forms defined on an open subset $U \subset U_1$, which form an orthonormal frame such that $\overline{\theta}_1 = \overline{\partial}\varrho/|\overline{\partial}\varrho|$. Let $(\overline{Z}_1, \dots, \overline{Z}_n)$ be the dual basis, which are $(0,1)$ vector fields on U. Therefore,

$$\mathrm{Span}(\overline{\theta}_2, \dots, \overline{\theta}_n) = \textstyle\coprod_{\zeta \in U} T^{*0,1}_{\zeta}(b\Omega_{\varrho(\zeta)}) \ (\subset T^{*0,1}U), \qquad \mathrm{Span}(\overline{Z}_2, \dots, \overline{Z}_n) = \textstyle\coprod_{\zeta \in U} T^{0,1}_{\zeta}(b\Omega_{\varrho(\zeta)}) \ (\subset T^{0,1}U).$$

We see that $\overline{Z}_1$ is uniquely determined by ϱ (which does not depend on $(\overline{\theta}_2, \dots, \overline{\theta}_n)$) and is globally defined on U_1:

$$\overline{Z}_1 = \frac{1}{|\overline{\partial}\varrho|}\sum_{j=1}^n \frac{\partial \varrho}{\partial \zeta_j}\frac{\partial}{\partial \overline{\zeta}_j}. \tag{2.10}$$

Let $f = \sum_{|J|=p,|K|=q} f_{J,K}\theta^J \wedge \overline{\theta}^K$ be a (p,q)-form on U, where $f_{J,K} = \langle Z_J \wedge \overline{Z}_K, f\rangle$, and we see that

$$f^{\perp} = \textstyle\sum_{|J|=p,|K'|=q-1} f_{J,1K'}\theta^J \wedge \overline{\theta}_1 \wedge \overline{\theta}^{K'}, \quad f^{\top} = \sum_{|J|=p,|K|=q;\, \min K \ge 2} f_{J,K}\theta^J \wedge \overline{\theta}^K.$$

Therefore, $f^{\perp}$ and $f^{\top}$ are still defined when f has distributional coefficients, and we have the following:

$$f^{\perp} = \overline{\theta}_1 \wedge \iota_{\overline{Z}_1} f = (f^{\perp})^{\perp}, \quad f^{\top} = f - f^{\perp} = (f^{\top})^{\top}. \tag{2.11}$$

Moreover, for a (p',q')-form g on U, one can see that

$$(f \wedge g)^{\top} = f^{\top} \wedge g^{\top}, \qquad f^{\perp} \wedge g^{\perp} = 0, \qquad \text{and thus} \quad f^{\perp} \wedge g = f^{\perp} \wedge g^{\top}. \tag{2.12}$$

We leave the proof to the reader.

The weighted estimates that we need are as follows.

Theorem 2.9 *(Weighted estimates for $K(z,\zeta)$). Let* $\operatorname{dist}(w) := \operatorname{dist}(w, b\Omega)$. *Let* $1 \le q \le n$. *Assume that* Ω *has q-type* $m_q < \infty$. *Let* $r_q := (n-q+1)\cdot m_q + 2q$ *and* $\gamma_q = \frac{r_q}{r_q-1}$.

Then, for any $k \ge 2$ *and* $0 < s < k-1-1/m_q$, *a* $C = C(\Omega, U_1, \widehat{S}, q, m_q, k, s) > 0$ *exists such that:*

$$\int_{U_1\setminus\overline{\Omega}} \operatorname{dist}(\zeta)^s |D^k_{z,\zeta}(K^\top_{q-1})(z,\zeta)| d\operatorname{Vol}(\zeta) \le C \operatorname{dist}(z)^{s+1+\frac{1}{m_q}-k}, \qquad \forall z \in \Omega; \tag{2.13}$$

$$\int_{\Omega} \operatorname{dist}(z)^s |D^k_{z,\zeta}(K^\top_{q-1})(z,\zeta)| d\operatorname{Vol}(z) \le C \operatorname{dist}(\zeta)^{s+1+\frac{1}{m_q}-k}, \qquad \forall \zeta \in U_1\setminus\overline{\Omega}; \tag{2.14}$$

$$\int_{U_1\setminus\overline{\Omega}} \operatorname{dist}(\zeta)^s |D^k_{z,\zeta}(K^\perp_{q-1})(z,\zeta)| d\operatorname{Vol}(\zeta) \le C \operatorname{dist}(z)^{s+\frac{2}{m_q}-k}, \qquad \forall z \in \Omega; \tag{2.15}$$

$$\int_{\Omega} \operatorname{dist}(z)^s |D^k_{z,\zeta}(K^\perp_{q-1})(z,\zeta)| d\operatorname{Vol}(z) \le C \operatorname{dist}(\zeta)^{s+\frac{2}{m_q}-k}, \qquad \forall \zeta \in U_1\setminus\overline{\Omega}; \tag{2.16}$$

$$\int_{U_1\setminus\overline{\Omega}} |\operatorname{dist}(\zeta)^s D^k_{z,\zeta}(K^\top_{q-1})(z,\zeta)|^{\gamma_q} d\operatorname{Vol}(\zeta) \le C \operatorname{dist}(z)^{(s+1-k)\gamma_q}, \qquad \forall z \in \Omega; \tag{2.17}$$

$$\int_{\Omega} |\operatorname{dist}(z)^s D^k_{z,\zeta}(K^\top_{q-1})(z,\zeta)|^{\gamma_q} d\operatorname{Vol}(z) \le C \operatorname{dist}(\zeta)^{(s+1-k)\gamma_q}, \qquad \forall \zeta \in U_1\setminus\overline{\Omega}; \tag{2.18}$$

$$\int_{U_1\setminus\overline{\Omega}} |\operatorname{dist}(\zeta)^s D^k_{z,\zeta}(K^\perp_{q-1})(z,\zeta)|^{\gamma_q} d\operatorname{Vol}(\zeta) \le C \operatorname{dist}(z)^{(s-k+\frac{1}{m_q})\gamma_q}, \qquad \forall z \in \Omega; \tag{2.19}$$

$$\int_{\Omega} |\operatorname{dist}(z)^s D^k_{z,\zeta}(K^\perp_{q-1})(z,\zeta)|^{\gamma_q} d\operatorname{Vol}(z) \le C \operatorname{dist}(\zeta)^{(s-k+\frac{1}{m_q})\gamma_q}, \qquad \forall \zeta \in U_1\setminus\overline{\Omega}. \tag{2.20}$$

We use $D^k_{z,\zeta} = \{\frac{\partial^{|\alpha+\beta+\gamma+\delta|}}{\partial z^\alpha \partial \bar z^\beta \partial \zeta^\gamma \partial \bar\gamma^\delta} : |\alpha+\beta+\gamma+\delta| \le k\}$ for the total derivatives among all variables acting on their coordinate components. Note that we take derivatives after we take ($\perp$ and $\top$) projections. We prove Theorem 2.9 in Section 3.

The estimates (2.15), (2.16), (2.19), and (2.20) are all not optimal. In practice, to prove Theorems 1.1 and 1.2, it is sufficient to replace the $\frac{2}{m_q}$-factors in (2.15) and (2.16) by any $\varepsilon + \frac{1}{m_q}$, and the $\frac{1}{m_q}$-factors in (2.19) and (2.20) by any ε, for all $\varepsilon > 0$. See Remark 3.10 (iii) for their improvements.

In Corollary 5.5 (iii), we show that $[\overline{\partial}, \mathcal{E}]^\top$ does not lose derivative (also see Remark 5.2). This technique is not necessary for the estimates for strongly pseudoconvex domains (see Remark 3.8).

By expanding $K_{q-1}(z,\zeta)$ from (2.8), we see that its coefficients are the (constant) linear combinations of

$$\frac{b(z,\zeta)\wedge \widehat{Q}(z,\zeta)\wedge \left(\overline{\partial}\widehat{Q}(z,\zeta)\right)^{k-1}}{\widehat{S}(z,\zeta)^k |z-\zeta|^{2(n-k)}}, \quad 1 \le k \le n-q. \tag{2.21}$$

We start with the estimates of the components in (2.21) in Section 3.

3. Estimates via ε-minimal bases

We recall some notations and definitions from [47,71].

Definition 3.1. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $\zeta \in b\Omega$. For $1 \leq q \leq n$, the (complex affine) q-type of Ω at ζ is

$$L_q(b\Omega, \zeta) := \sup\Big\{m \in \mathbb{R}_+ : \varliminf_{w\to 0; w\in H} \frac{\operatorname{dist}(\zeta + w, b\Omega)}{|w|^m} = 0 \text{ for all } q\text{-dim } \mathbb{C}\text{-linear subspace } H \leq \mathbb{C}^n\Big\}.$$

The (affine) q-type of Ω is the minimum of $L_q(b\Omega, \zeta)$ among all $\zeta \in b\Omega$, which we denote by m_q.

As mentioned by [34, Theorem 2.1], on convex domains, the affine types, D'Angelo types, and regular D'Angelo types all coincide. Moreover, if Ω has affine q-type $m_q < \infty$ for $1 \leq q \leq n$, then $(m_n, m_{n-1}, \dots, m_1)$ is the Catlin's multitype of Ω. See [47,7,71]. In particular, $m_1 \geq \dots \geq m_{n-1} \geq 2$ are all even integers and $m_n = 1$.

To study the $\frac{1}{m_q}$ gain on $(0, q)$-forms, especially for $q \geq 2$, we use the ε-minimal basis approach, which was introduced by [71] and used by [34].

Definition 3.2. Let $\Omega \subset \mathbb{C}^n$ be a finite type convex domain where the defining function ϱ is as given above. For $\zeta \in U_1$, $v \in \mathbb{C}^n$, and $\varepsilon > 0$, let

$$\tau(\zeta, v, \varepsilon) := \sup\{c > 0 : |\varrho(\zeta + \lambda v) - \varrho(\zeta)| \leq \varepsilon, \quad \forall \lambda \in \mathbb{C},\ |\lambda| \leq c\}.$$

An ε-*minimal basis* (or a *Yu-basis at the scale* ε) $(v_1, \dots, v_n)$ at $\zeta \in U_1$ is given recursively as follows: for $1 \leq k \leq n$, $v_k \in \mathbb{C}^n$ is a unit vector that minimizes to the following quantity of v:

$$\operatorname{dist}\big(\zeta, \{z \in (\zeta + \mathbb{C}\cdot v) : \varrho(z) = \varrho(\zeta) + \varepsilon\}\big), \quad \text{where } |v| = 1 \text{ and } v \perp \operatorname{Span}_{\mathbb{C}}\langle v_1, \dots, v_{k-1}\rangle.$$

For $k = 1$, we use $\operatorname{Span}_{\mathbb{C}} \varnothing = \{0\}$.

For an ε-minimal basis $(v_1, \dots, v_n)$ at ζ, we define $\tau_j(\zeta, \varepsilon) := \tau(\zeta, v_j, \varepsilon)$ for $1 \leq j \leq n$, and the ellipsoid $P_\varepsilon(\zeta) := \big\{\zeta + \sum_{j=1}^n a_j v_j : \sum_{j=1}^n \frac{|a_j|^2}{\tau_j(\zeta,\varepsilon)^2} < 1\big\} \subset \mathbb{C}^n$. For $c > 0$, we set $cP_\varepsilon(\zeta) := \{z \in \mathbb{C}^n : \zeta + \frac{z-\zeta}{c} \in P_\varepsilon(\zeta)\}$ for a dilation of $P_\varepsilon(\zeta)$ with the same center.

We use an ellipsoid rather than a rectangle to define $P_\varepsilon(\zeta)$ (cf. [20, Section 3] for example). One can see that $\{\tau_j(\zeta, \varepsilon)\}_{j=1}^n$ and $P_\varepsilon(\zeta)$ do not depend on the choice of the ε-minimal basis.

We recall the following from [47,71,20,34]. Recall Lemma 2.2 for $M_5 > 0$ and $U_1 \supset b\Omega$.

Lemma 3.3. *Assume that the finite type convex domain $\Omega \subset \mathbb{C}^n$ has q-type $m_q < \infty$. Then, a $C_0 > 1$ and an $\varepsilon_0 \in (0, \frac{1}{M_5})$ exist, and for every multi-index $\beta = (\beta', \beta'') \in \mathbb{N}^{2n}$, a $C_\beta > 0$ exists such that:*

(i) For every $\zeta \in \{|\varrho| < \varepsilon_0\}$, we have $P_{\varepsilon_0}(\zeta) \subseteq U_1$. Moreover, for every $0 < \varepsilon \leq \varepsilon_0$ and every ε-minimal basis $(v_1, \dots, v_n)$ at ζ (recall that $\tau_j(\zeta, \varepsilon) := \tau(\zeta, v_j, \varepsilon)$)*:*

$$P_\varepsilon(\zeta') \subseteq C_0 P_{\varepsilon/2}(\zeta) \text{ and } 2P_\varepsilon(\zeta') \subseteq P_{C_0\varepsilon}(\zeta), \quad \forall \zeta' \in P_\varepsilon(\zeta); \tag{3.1}$$

$$\tau_1(\zeta, \varepsilon) \leq \tau_2(\zeta, \varepsilon) \leq \tau_3(\zeta, \varepsilon) \leq \dots \leq \tau_n(\zeta, \varepsilon); \tag{3.2}$$

$$\tau_1(\zeta, \varepsilon) \geq \tfrac{1}{C_0}\varepsilon, \quad \tau_2(\zeta, \varepsilon) \geq \tfrac{1}{C_0}\varepsilon^{\frac{1}{2}}; \tag{3.3}$$

$$\tau_q(\zeta, \varepsilon) \leq C_0 \varepsilon^{1/m_{n+1-q}}, \quad \forall 1 \leq q \leq n; \tag{3.4}$$

(ii) For every $\zeta \in U_1$, $0 < \varepsilon \leq \varepsilon_0$ and ε-minimal basis $(v_1, \dots, v_n)$ at ζ,

(3.5)
$$\Big|\frac{\partial^{|\beta|}}{\partial w^{\beta'} \partial \bar{w}^{\beta''}} \varrho(\zeta + w_1 v_1 + \dots w_n v_n)\Big| \leq C_\beta \frac{\varepsilon}{\prod_{j=1}^n \tau_j(\zeta, \varepsilon)^{\beta'_j + \beta''_j}}, \quad \forall w \in \mathbb{C}^n \text{ such that } \sum_{j=1}^n \frac{|w_j|^2}{\tau_j(\zeta, \varepsilon)^2} < 1.$$

Remark 3.4.

(i) (3.4) is particularly useful for $q \geq 2$. If we only consider $(0,1)$-forms, we could use the ε-extremal basis (a.k.a. the McNeal-basis at the scale ε) to define $\tau_1, \dots, \tau_n$, where (3.1) and (3.3) remain true, but (3.4) is replaced by $\varepsilon^{1/m_{q-1}} \lesssim \tau_q \lesssim \varepsilon^{1/m_1}$ (e.g., see [34, Theorem 2.3]).

(ii) In (3.5), $\sum_{j=1}^n \frac{|w_j|^2}{\tau_j(\zeta,\varepsilon)^2} < 1$ is the same as stating that $\zeta + w_1 v_1 + \cdots + w_n v_n \in P_\varepsilon(\zeta)$. This estimate is only useful when $\sum_{j=1}^n \frac{1}{m_{n+1-j}}(\beta'_j + \beta''_j) < 1$; otherwise, the right-hand side of (3.5) is bounded from below (or even tends to ∞ as $\varepsilon \to 0$) whereas the left-hand side is always uniformly bounded.

To prove Theorem 2.9, we need to estimate $|K^\top_{q-1}(z,\zeta)|$ and $|K^\perp_{q-1}(z,\zeta)|$ inside an ellipsoid $P_\varepsilon(\zeta)$. Recall ε_0 in Lemma 3.3 and $\widehat{Q}$ in (2.6).

Lemma 3.5. *By retaining the notations in Lemma 3.3, a $C_1 > 0$ exists that satisfies the following:*

(i) $|\widehat{S}(z,\zeta)| \geq \frac{1}{C_1}\varepsilon$, for every $\zeta \in U_1$, $0 < \varepsilon \leq \varepsilon_0$ and $z \in \Omega_{\varrho(\zeta)} \backslash P_\varepsilon(\zeta)$.

(ii) Let $\zeta_0 \in U_1$, $0 < \varepsilon \leq \varepsilon_0$, and let $\Psi_0 \in \mathbb{C}^{n\times n}$ be a unitary matrix such that its n column vectors (with order) form an ε-minimal basis at ζ_0. Let $\widehat{Q}_{\Psi_0}(z,\zeta) := \overline{\Psi}_0 \cdot \widehat{Q}(\Psi_0 \cdot z, \Psi_0 \cdot \zeta)$. Then,

(3.6)
$$|\widehat{Q}_{\Psi_0,j}(z,\zeta_0)| \leq \frac{C_1\varepsilon}{\tau_j(\zeta_0,\varepsilon)}, \quad \left|\frac{\partial}{\partial\bar{\zeta}_k}\widehat{Q}_{\Psi_0,j}(z,\zeta_0)\right| \leq \frac{C_1\varepsilon}{\tau_j(\zeta_0,\varepsilon)\tau_k(\zeta_0,\varepsilon)}, \quad \textit{for } z \in P_\varepsilon(\zeta_0), \quad 1 \leq j,k \leq n.$$

Proof. See [20, Lemma 4.2] or [34, Proposition 4.1] for (i). Note that the modification from S to $\widehat{S}$ in Lemma 2.2 ensures that $|\widehat{S}(z,\zeta)|$ is bounded from below when $|z-\zeta|$ is large.

For (3.6), since we fix the point ζ_0 and the number ε, by passing to a unitary coordinate change, we can assume that $\Psi_0 = I_n$. In particular, $\widehat{Q}_{\Psi_0} = \widehat{Q}$.

(3.6) is a weaker version of [3, Lemma 8] where (under the assumption that $\Psi_0 = I_n$) it was proved that $|\partial_{\bar{\zeta}_k}\widehat{Q}_j(z,\zeta_0)| \lesssim \frac{\varepsilon}{\tau_j\tau'_k}$. In the statement, $\tau'_1 = \varepsilon^{\frac{1}{2}} \gg \varepsilon \approx \tau_1$ and $\tau'_k = \tau_k$ for $k \geq 2$. [3, Lemma 8] is stated with ε-extremal bases, but the result is still true if we replace it by ε-minimal bases. There is no additional change to the proof.

Alternatively, we define $Q = [Q_1, \dots, Q_n]^\intercal$ as the $\widehat{Q}$ from (2.6) with $\widehat{S}(z,\zeta)$ replaced by $S(z,\zeta)$, i.e. $Q_j(z,\zeta) = \int_0^1 \frac{\partial S}{\partial z_j}(\zeta + t(z-\zeta),\zeta)dt$. Then, we have

$$\widehat{Q}_j = A\cdot Q_j + \partial_{z_j}A\cdot S, \quad \text{and} \quad \partial_{\bar{\zeta}_k}\widehat{Q}_j = A\cdot\widehat{Q}_j + \partial_{\bar{\zeta}_k}A\cdot\widehat{Q}_j + \partial_{z_j}A\cdot\partial_{\bar{\zeta}_k}S + \partial^2_{z_j\bar{\zeta}_k}A\cdot S,$$

where $A \in C^\infty$ is as given in Lemma 2.2 (iii). By [20, Lemma 5.1], we have $|Q_j| \lesssim \varepsilon/\tau_j$ and $|\partial_{\bar{\zeta}_k}Q_j| \lesssim \varepsilon/(\tau_j\tau_k)$, and thus (3.6) follows from the fact that $A \in C^2$ and $S(z,\zeta) = \sum_{j=1}^n Q_j(z,\zeta)(z_j - \zeta_j)$. Again, [20, Lemma 5.1] is stated with ε-extremal bases but the result is still true if we replace it by ε-minimal bases. There is no additional change to the proof. $\square$

Corollary 3.6. *We retain the notations from Lemma 3.5 (ii), and we identify the column vector function $\widehat{Q}_{\Psi_0} = [\widehat{Q}_{\Psi_0,1}, \dots, \widehat{Q}_{\Psi_0,n}]$ with the $(1,0)$-form $\widehat{Q}_{\Psi_0} = \sum_{j=1}^n \widehat{Q}_{\Psi_0,j}(z,\zeta)d\zeta_j$. Then, the estimates using (3.7) and (3.8) imply the following.*

A $C'_1 > 0$ exists that does not depend on $\zeta_0 \in U_1$ and $0 < \varepsilon \leq \varepsilon_0$, such that for every $a \in \{0,1\}$, $1 \leq b \leq n$ and every $z \in P_\varepsilon(\zeta_0)$,

$$\left|(\widehat{Q}_{\Psi_0})^a \wedge (\overline{\partial}\widehat{Q}_{\Psi_0})^b(z,\zeta_0) \mod d\bar{\zeta}_1\right| \leq C'_1 \frac{\varepsilon^{a+b-1}}{\prod_{l=2}^{b+1}\tau_l(\zeta_0,\varepsilon)^2}; \tag{3.7}$$

$$\left|(\widehat{Q}_{\Psi_0})^a \wedge (\overline{\partial}\widehat{Q}_{\Psi_0})^b(z,\zeta_0)\right| \le C_1' \frac{\varepsilon^{a+b-2}\tau_{b+1}(\zeta_0,\varepsilon)}{\prod_{l=2}^{b+1}\tau_l(\zeta_0,\varepsilon)^2}. \tag{3.8}$$

By (3.7), we mean that if $(\widehat{Q}_{\Psi_0})^a \wedge (\overline{\partial}\widehat{Q}_{\Psi_0})^b(z,\zeta) = \sum_{|J|=a+b;|K|=v} f_{JK}(z,\zeta)d\zeta^J \wedge d\overline{\zeta}^K$, then $|f_{JK}| \lesssim \varepsilon^{a+b-1}/\prod_{l=2}^{b+1}\tau_l^2$ for all index sets J and $K=(k_1,\dots,k_b)$ such that $k_1,\dots,k_b \ge 2$.

Remark 3.7. This is essentially [20, Lemma 5.5] or [34, Lemma 4.2], which used the $(1,0)$-form $Q(z,\zeta)$ (see the proof of Lemma 3.5) instead of the $\widehat{Q}(z,\zeta)$ in the statement.

Remark 3.8. In the case of strongly pseudoconvex domains, we informally have $\tau_2 \approx \cdots \approx \tau_n \approx \varepsilon^{\frac{1}{2}}$, which means that $|(\widehat{Q}_{\Psi_0})^a \wedge (\overline{\partial}\widehat{Q}_{\Psi_0})^b \mod d\bar{\zeta}_1| \lesssim \varepsilon^{a-1}$ and $|(\widehat{Q}_{\Psi_0})^a \wedge (\overline{\partial}\widehat{Q}_{\Psi_0})^b| \lesssim \varepsilon^{a-\frac{3}{2}}$. Since $\varepsilon^{a-1}, \varepsilon^{a-\frac{3}{2}} \gtrsim 1$, we recall from Remark 3.4 (ii) that these estimates become unnecessary.

Proof of Corollary 3.6. Again, we assume that $\Psi_0 = I_n$. By writing $\widehat{Q}^a \wedge (\overline{\partial}\widehat{Q})^b = \sum_{|J|=a+b;|K|=v} f_{JK}d\zeta^J \wedge d\overline{\zeta}^K$, we have $f_{j_1\dots j_{a+b},k_1\dots k_b} = \pm\prod_{p=1}^b \frac{\partial\widehat{Q}_{j_p}}{\partial\bar{\zeta}_{k_p}}\prod_{q=b+1}^{a+b}\widehat{Q}_{j_q}$. By (3.6) we have

$$\left|f_{j_1\dots j_{a+b},k_1\dots k_b}\right| \lesssim \frac{\varepsilon^{a+b}}{\prod_{p=1}^b \tau_{j_p}\tau_{k_p}\prod_{q=b+1}^{a+b}\tau_{j_q}}. \tag{3.9}$$

In the differential form, $(j_1,\dots,j_{a+b})$ and $(k_1,\dots,k_b)$ are two collections of distinct indices. Therefore, we can assume that $(1\le)j_{b+1} < \cdots < j_{a+b} < j_1 < \cdots < j_b$ and $k_1 < \cdots < k_b$. In (3.10), we modular $d\bar{\zeta}_1$, where we only need the case of $k_1 \ge 2$.

By (3.2) and (3.3), $\varepsilon \approx \tau_1 \le \cdots \le \tau_n$, and thus for the case where $k_1 \ge 2$,

$$|f_{j_1\dots j_{a+b},k_1\dots k_{a+b}}| \lesssim \begin{cases} \left.\dfrac{\varepsilon^{b+1}}{\tau_{j_{b+1}}\prod_{p=1}^b\tau_{j_p}\tau_{k_p}}\right|_{\substack{j_{b+1}=1\\ j_p=k_p=p+1}} \approx \dfrac{\varepsilon^b}{\prod_{l=2}^{b+1}\tau_l^2} & \text{when } a=1\\ \left.\dfrac{\varepsilon^b}{\prod_{p=1}^b\tau_{j_p}\tau_{k_p}}\right|_{\substack{j_p=p\\ k_p=p+1}} \approx \dfrac{\varepsilon^{b-1}}{\prod_{l=2}^b\tau_l^2\cdot\tau_{b+1}} & \text{when } a=0\end{cases} \le \frac{\varepsilon^{a+b-1}}{\prod_{l=2}^{b+1}\tau_l^2}.$$

This proves (3.7).

Similarly, for the case where $k_1 \ge 1$,

$$|f_{j_1\dots j_{a+b},k_1\dots k_{a+b}}| \lesssim \begin{cases} \left.\dfrac{\varepsilon^{b+1}}{\tau_{j_{b+1}}\prod_{p=1}^b\tau_{j_p}\tau_{k_p}}\right|_{\substack{j_{b+1}=1\\ j_p=p+1\\ k_p=p}} \approx \dfrac{\varepsilon^{b-1}}{\prod_{l=2}^b\tau_l^2\cdot\tau_{b+1}} & \text{when } a=1\\ \left.\dfrac{\varepsilon^b}{\prod_{p=1}^b\tau_{j_p}\tau_{k_p}}\right|_{j_p=k_p=p} \approx \dfrac{\varepsilon^{b-2}}{\prod_{l=2}^b\tau_l^2} & \text{when } a=0\end{cases} \le \frac{\varepsilon^{a+b-2}\tau_{b+1}}{\prod_{l=2}^{b+1}\tau_l^2}.$$

This proves (3.8). □

By taking pullback from $\zeta \mapsto \Psi_0\cdot\zeta$, we have the following.

Lemma 3.9. *For every $j \ge 0$, a $C_j > 0$ exists such that for every $1 \le k \le n$, $0 < \varepsilon \le \varepsilon_0$, $\zeta \in U_1\backslash\overline{\Omega}$, and $z \in \Omega_{\varrho(\zeta)} \cap P_\varepsilon(\zeta)\backslash P_{\varepsilon/2}(\zeta)$,*

$$\left|D^j_{z,\zeta}\Big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^k}{\widehat{S}^{k+1}}\Big)^\top(z,\zeta)\right| \le C_j\frac{\varepsilon^{-1-j}}{\prod_{l=2}^{k+1}\tau_l(\zeta,\varepsilon)^2}; \tag{3.10}$$

$$\left|D^j_{z,\zeta}\Big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^k}{\widehat{S}^{k+1}}\Big)(z,\zeta)\right| \le C_j\frac{\varepsilon^{-2-j}\tau_{k+1}(\zeta,\varepsilon)}{\prod_{l=2}^{k+1}\tau_l(\zeta,\varepsilon)^2}, \tag{3.11}$$

where $D^j = \{\partial_z^\alpha \partial_\zeta^\beta \partial_{\bar\zeta}^\gamma\}_{|\alpha+\beta+\gamma|\le j}$ *is the collection of differential operators acting on the components.*

Remark 3.10.

(i) There is no $\bar z$-derivative since the fractions are holomorphic in $z \in \Omega_{\varrho(\zeta)}(\supset \Omega)$.

(ii) In fact, (3.10) and (3.11) correspond to the terms $\frac{\varepsilon^{-j}}{\prod_{i=0}^k \tau_{\nu_i}\prod_{i=1}^k \tau_{\mu_i}}$ and $\frac{\varepsilon^{-j-\frac12}}{\prod_{i=0}^k \tau_{\nu_i}\prod_{i=1}^{k-1} \tau_{\mu_i}}$ in [3, Lemma 9], respectively. By obtaining a refined estimate on the normal direction, where [3] introduced a notion of $\tau_1' := \varepsilon^{\frac12}$, we can improve (3.11) by a factor of $\varepsilon^{\frac12}$. However, (3.11) is sufficient for our proof.

(iii) If we consider the anisotropic estimates, we can show that for $\alpha, \beta, \gamma \in \mathbb{N}^n$, $\zeta \in U_1\backslash\overline{\Omega}$ and $z \in \Omega\cap P_\varepsilon(\zeta)$, if the standard coordinate basis is ε-minimal at ζ, then

$$\left|\partial_z^\alpha \partial_\zeta^\beta \partial_{\bar\zeta}^\gamma \Big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^k}{\widehat{S}^{k+1}}\Big)^\top(z,\zeta)\right| \lesssim_{\alpha,\beta,\gamma} \frac{\varepsilon^{-1-\alpha_1-\beta_1-\frac12\gamma_1}}{\prod_{l=2}^{k+1}\tau_l(\zeta,\varepsilon)^2}\prod_{j=2}^n \tau_j(\zeta,\varepsilon)^{-\alpha_j-\beta_j-\gamma_j};$$

$$\left|\partial_z^\alpha \partial_\zeta^\beta \partial_{\bar\zeta}^\gamma \Big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^k}{\widehat{S}^{k+1}}\Big)(z,\zeta)\right| \lesssim_{\alpha,\beta,\gamma} \frac{\varepsilon^{-\frac32-\alpha_1-\beta_1-\frac12\gamma_1}}{\tau_{k+1}(\zeta,\varepsilon)\prod_{l=2}^{k}\tau_l(\zeta,\varepsilon)^2}\prod_{j=2}^n \tau_j(\zeta,\varepsilon)^{-\alpha_j-\beta_j-\gamma_j}.$$

The proof requires Alexandre's estimate for normal derivatives in [3, Section 2].

Proof of Lemma 3.9. For convenience, we write[2] $\overline{\partial}^\top\widehat{Q} := (\overline{\partial}\widehat{Q})^\top$ throughout the proof.

By Lemma 3.5 (i) and Lemma 2.2 (iii), we have $|\widehat{S}| \gtrsim \varepsilon$ for $z \in \Omega_{\varrho(\zeta)}\backslash P_{\varepsilon/2}(\zeta)$. By applying the trivial estimate $|D^j\widehat{S}| \lesssim_j 1$ for $j \ge 1$, we see that

$$|D^j(\widehat{S}^{-k})(z,\zeta)| \lesssim_j |\widehat{S}(z,\zeta)|^{-k-j} \lesssim_j \varepsilon^{-k-j} \text{ when } z \in (\Omega\cap P_\varepsilon(\zeta))\backslash P_{\varepsilon/2}(\zeta). \tag{3.12}$$

Therefore, by applying product rules, for $z \in (\Omega_{\varrho(\zeta)}\cap P_\varepsilon(\zeta))\backslash P_{\varepsilon/2}(\zeta)$, we uniformly have

$$\begin{aligned}
&\left|D^j_{z,\zeta}\frac{\widehat{Q}\wedge(\overline{\partial}^\top\widehat{Q})^k}{\widehat{S}^{k+1}}\right| \lesssim_j \sum_{q=0}^k \sum_{\substack{j_0+\cdots+j_{q+1}=j\\ j_2,\dots,j_q\ge1}} \left|D^{j_0}\frac{1}{\widehat{S}^{k+1}}\right|\left|(D^{j_1}\widehat{Q})\wedge(\overline{\partial}^\top\widehat{Q})^{k-q}\right|\prod_{p=2}^{q+1}|D^{j_p}(\overline{\partial}\widehat{Q})|\\
&\lesssim_j \sum_{q=0}^k\Big(\sum_{j_0=0}^{j-q}\left|D^{j_0}\frac{1}{\widehat{S}^{k+1}}\right|\cdot\left|\widehat{Q}\wedge(\overline{\partial}^\top\widehat{Q})^{k-q}\right| + \sum_{\substack{1\le j_1\le j-q\\ j_0\le j-q-j_1}}\left|D^{j_0}\frac{1}{\widehat{S}^{k+1}}\right|\cdot|D^{j_1}\widehat{Q}|\cdot\left|(\overline{\partial}^\top\widehat{Q})^{k-q}\right|\Big),\\
&\qquad (|D^{j_p}(\overline{\partial}\widehat{Q})| \lesssim_j 1)\\
&\lesssim_{k,\widehat{S}} \sum_{q=0}^k\Big(\sum_{j_0=1}^{j-q}\varepsilon^{-k-j_0-1}\left|\widehat{Q}\wedge(\overline{\partial}^\top\widehat{Q})^{k-q}\right| + \sum_{j_0=1}^{j-q-1}\varepsilon^{-k-j_0-1}\left|(\overline{\partial}^\top\widehat{Q})^{k-q}\right|\Big), \quad (\text{by (3.12) and } |D^{j_1}\widehat{Q}| \lesssim_j 1).
\end{aligned}$$

Now, we fix $\zeta = \zeta_0$ and ε. The left-hand side of (3.10) is invariant under a change of unitary coordinate system, so we can assume that the standard basis is ε-minimal at ζ_0, and thus $\overline{\partial}^\top\widehat{Q}_j(z,\zeta_0) = \sum_{k=2}^n \frac{\partial\widehat{Q}_j}{\partial\overline{\zeta}_k}(z,\zeta_0)d\overline{\zeta}_k$. By applying (3.7), we obtain

$$\left|D^j_{z,\zeta}\Big(\frac{\widehat{Q}\wedge(\overline{\partial}^\top\widehat{Q})^k}{\widehat{S}^{k+1}}\Big)\right| \lesssim \sum_{q=0}^k\Big(\sum_{j_0=1}^{j-q}\varepsilon^{-k-j_0-1}\frac{\varepsilon^{k-q}}{\prod_{l=2}^{k-q+1}\tau_l^2} + \sum_{j_0=1}^{j-q-1}\varepsilon^{-k-j_0-1}\frac{\varepsilon^{k-q-1}}{\prod_{l=2}^{k-q+1}\tau_l^2}\Big) \lesssim \frac{\varepsilon^{-1-j}}{\prod_{l=2}^{k+1}\tau_l^2}.$$

[2] On a $(1,0)$-form f, we have $(\overline{\partial}f)^\top = \overline{\partial}_b f$ on the boundary $b\Omega$. For orthonormal $(0,1)$-forms $(\overline{\theta}_1,\dots,\overline{\theta}_n)$, and its dual basis $(\overline{Z}_1,\dots,\overline{Z}_n)$ in Remark 2.8, we have $\overline{\partial}_b(f_k d\zeta_k) = \sum_{j=2}^n(\overline{Z}_j f)\overline{\theta}_j\wedge d\zeta_k$. One can interpret $\overline{\partial}^\top|_{b\Omega_t}$ as $\overline{\partial}_b$ on each leaf $b\Omega_t \subset U_1$.

This completes the proof of (3.10).

After replacing $\overline{\partial}^{\top}\widehat{Q}$ by $\overline{\partial}\widehat{Q}$ and (3.7) by (3.8), the above argument yields (3.11). $\square$

The same estimates hold if we swap z and ζ.

Corollary 3.11. *By enlarging the constant $C_j > 0$ in Lemma 3.9 if necessary, the estimates of (3.10) and (3.11) with $\tau_j(\zeta,\varepsilon)$ replaced by $\tau_j(z,\varepsilon)$ hold for all $z \in \Omega$ and $\zeta \in P_\varepsilon(z)\backslash(P_{\varepsilon/2}(z) \cup \Omega)$.*

Proof. Indeed, we still have $z \in \Omega$ and $\zeta \in U_1\backslash\overline{\Omega}$. By (3.1), $\zeta \in P_\varepsilon(z)\backslash P_{\varepsilon/2}(z)$ implies $z \in P_{C_0\varepsilon}(\zeta)\backslash P_{\varepsilon/(2C_0)}(\zeta)$ and $\tau_j(\zeta,\varepsilon) \geq \frac{1}{C_0}\tau_j(z,\varepsilon)$, where C_0 is as given in Lemma 3.3. The results then follow from Lemma 3.9. $\square$

Recall that $K_{q-1}(z,\zeta)$ in (2.8). Let $r_q := (n-q+1)m_q + 2q$ and $\gamma_q := \frac{r_q}{r_q-1}$. Since $m_1 \geq \cdots \geq m_{n-1} \geq 2 > m_n = 1$, we see that

$$r_1 \geq r_2 \geq \cdots \geq r_n, \qquad \text{thus } 1 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n. \tag{3.13}$$

We can now integrate K_{q-1} on some ε-minimal ellipsoids.

Lemma 3.12. *For every $j \geq 0$, a $C_j \geq 0$ exists such that for every $1 \leq q \leq n-1$, $z \in \Omega$, $\zeta \in U_1\backslash\overline{\Omega}$ and $0 < \varepsilon \leq \varepsilon_0$,*

$$\int\limits_{\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)} |D^j(K^{\top}_{q-1})(w,\zeta)| d\operatorname{Vol}_w + \int\limits_{P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)} |D^j(K^{\top}_{q-1})(z,w)| d\operatorname{Vol}_w \leq C_j\varepsilon^{\frac{1}{m_q}+1-j}; \tag{3.14}$$

$$\int\limits_{\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)} |D^j K^{\perp}_{q-1}(w,\zeta)| d\operatorname{Vol}_w + \int\limits_{P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)} |D^j K^{\perp}_{q-1}(z,w)| d\operatorname{Vol}_w \leq C_j\varepsilon^{\frac{2}{m_q}-j}; \tag{3.15}$$

$$\int\limits_{\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)} |D^j K^{\top}_{q-1}(w,\zeta)|^{\gamma_q} d\operatorname{Vol}_w + \int\limits_{P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)} |D^j K^{\top}_{q-1}(z,w)|^{\gamma_q} d\operatorname{Vol}_w \leq (C_j\varepsilon^{1-j})^{\gamma_q}; \tag{3.16}$$

$$\int\limits_{\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)} |D^j K^{\perp}_{q-1}(w,\zeta)|^{\gamma_q} d\operatorname{Vol}_w + \int\limits_{P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)} |D^j K^{\perp}_{q-1}(z,w)|^{\gamma_q} d\operatorname{Vol}_w \leq (C_j\varepsilon^{\frac{1}{m_q}-j})^{\gamma_q}, \tag{3.17}$$

where $D^j = \{\partial_z^{\alpha'}\partial_{\bar{z}}^{\alpha''}\partial_\zeta^{\beta'}\partial_{\bar{\zeta}}^{\beta''} : |\alpha'|+|\alpha''|+|\beta'|+|\beta''| \leq j\}$ is the collection of derivatives of all variables.

Proof. Since $K_{q-1} = K^{\top}_{q-1} + K^{\perp}_{q-1}$ (see Remark 2.8) and the right-hand side of (3.14) is smaller than the right-hand side of (3.15), it is sufficient to estimate (3.15) with $K^{\perp}_{q-1}$ replaced by K_{q-1}. The same replacement also works for (3.17).

From (2.8), we recall that K_{q-1} are linear combinations of (2.21). From Definition 2.6 and Remark 2.8, we recall that $K^{\top}_{q-1}$ are also the linear combinations of (2.21) with $\overline{\partial}\widehat{Q}$ replaced by $\overline{\partial}^{\top}\widehat{Q} = (\overline{\partial}\widehat{Q})^{\top}$. Therefore, by Lemma 3.9 and Corollary 3.11, for z and ζ in the assumption, and w in the integrands (for z and ζ, respectively),

$$|D^j K^{\top}_{q-1}(w,\zeta)| \lesssim \sum_{k=1}^{n-q} \frac{\varepsilon^{-1-j}}{\prod_{l=2}^k \tau_l(\zeta,\varepsilon)^2}\frac{1}{|w-\zeta|^{2n-2k-1}}, \quad |D^j K^{\top}_{q-1}(z,w)| \lesssim \sum_{k=1}^{n-q} \frac{\varepsilon^{-1-j}}{\prod_{l=2}^k \tau_l(z,\varepsilon)^2}\frac{1}{|w-z|^{2n-2k-1}};$$

$$|D^j K_{q-1}(w,\zeta)| \lesssim \sum_{k=1}^{n-q} \frac{\varepsilon^{-2-j}}{\prod_{l=2}^k \tau_l(\zeta,\varepsilon)^2}\frac{\tau_{k+1}(\zeta,\varepsilon)}{|w-\zeta|^{2n-2k-1}}, \quad |D^j K_{q-1}(z,w)| \lesssim \sum_{k=1}^{n-q} \frac{\varepsilon^{-2-j}}{\prod_{l=2}^k \tau_l(z,\varepsilon)^2}\frac{\tau_{k+1}(z,\varepsilon)}{|w-z|^{2n-2k-1}}.$$

By taking an ε-minimal basis at ζ and at z, respectively, $P_\varepsilon(\zeta)$ and $P_\varepsilon(z)$ are mapped into the subset $\{u : |u_1| < \tau_1, \dots, |u_n| < \tau_n\}$, where $\tau_l = \tau_l(\zeta,\varepsilon)$ or $\tau_l(z,\varepsilon)$. By (3.4), we have $\tau_l \lesssim \varepsilon^{1/m_{n-l+1}}$. Recall that $1 = m_n < m_{n-1} \le \dots \le m_1$ from Definition 3.1.

Therefore, (3.14) and (3.15) are given by the following (also see [34, Section 3]):

$$
\begin{aligned}
&\int\limits_{P_\varepsilon(\zeta)} \sum_{k=1}^{n-q} \frac{d\,\mathrm{Vol}(w)}{|w-\zeta|^{2n-2k-1}\prod_{l=2}^{k}\tau_l(\zeta,\varepsilon)^2} + \int\limits_{P_\varepsilon(z)} \sum_{k=1}^{n-q} \frac{d\,\mathrm{Vol}(w)}{|w-z|^{2n-2k-1}\prod_{l=2}^{k}\tau_l(\zeta,\varepsilon)^2} \\
&\approx \sum_{k=1}^{n-q} \int\limits_{|w_1|<\tau_1,\dots,|w_n|<\tau_n} \frac{d\,\mathrm{Vol}(w_1,\dots,w_n)}{\left(\prod_{l=2}^{k}\tau_l^2\right)\cdot\left(\sum_{l=1}^{n}|w_l|\right)^{2n-2k-1}} \\
&\lesssim \sum_{k=1}^{n-q} \tau_1^2 \int\limits_{|w_{k+1}|<\tau_{k+1},\dots,|w_n|<\tau_n} \frac{d\,\mathrm{Vol}(w_{k+1},\dots,w_n)}{\left(\sum_{l=k+1}^{n}|w_l|\right)^{2n-2k-1}} \qquad (3.18) \\
&\lesssim \varepsilon^2 \sum_{k=1}^{n-q} \int_0^{\tau_{k+1}} t\,dt \int_0^{\infty} \frac{s^{2n-2k-3}ds}{(t+s)^{2n-2k-1}} \qquad (\tau_1 \approx \varepsilon,\ t = |w_{k+1}|,\ s = |(w_{k+2},\dots,w_n)|) \\
&\lesssim \varepsilon^2 \sum_{k=1}^{n-q} \int_0^{\tau_{k+1}} dt = \varepsilon^2 \sum_{k=1}^{n-q} \tau_{k+1} \lesssim \varepsilon^2 \sum_{k=1}^{n-q} \varepsilon^{\frac{1}{m_{n-k}}} \approx \varepsilon^{2+\frac{1}{m_q}} \qquad (\text{by } (3.4)).
\end{aligned}
$$

By multiplying ε^{-1-j}, we obtain (3.14). Note that by (3.4), $\varepsilon^{-2-j} \max\limits_{1\le k\le n-q} \tau_{k+1}^2 \lesssim \varepsilon^{\frac{2}{m_q}-2-j}$, and thus (3.15) follows.

Similarly, (3.16) and (3.17) are given by the following (cf. the control of L_k in [34, Section 4]):

$$
\begin{aligned}
&\int\limits_{P_\varepsilon(\zeta)} \sum_{k=1}^{n-q} \frac{d\,\mathrm{Vol}(w)}{|w-\zeta|^{(2n-2k-1)\gamma_q}\prod_{l=2}^{k}\tau_l(\zeta,\varepsilon)^{2\gamma_q}} + \int\limits_{P_\varepsilon(z)} \sum_{k=1}^{n-q} \frac{d\,\mathrm{Vol}(w)}{|w-z|^{(2n-2k-1)\gamma_q}\prod_{l=2}^{k}\tau_l(\zeta,\varepsilon)^{2\gamma_q}} \\
&\lesssim \sum_{k=1}^{n-q} \int\limits_{|w_1|<\tau_1,\dots,|w_n|<\tau_n} \frac{d\,\mathrm{Vol}(w_1,\dots,w_n)}{\left(\prod_{l=2}^{k}\tau_l^{2\gamma_q}\right)\cdot\left(\sum_{l=1}^{n}|w_l|\right)^{(2n-2k-1)\gamma_q}} \\
&\lesssim \varepsilon^2 \sum_{k=1}^{n-q} \prod_{l=2}^{k} \frac{1}{\tau_l^{2(\gamma_q-1)}} \int\limits_{|w_{k+1}|<\tau_{k+1},\dots,|w_n|<\tau_n} \frac{d\,\mathrm{Vol}(w_{l+1},\dots,w_n)}{\left(\sum_{l=1}^{n}|w_l|\right)^{(2n-2k-1)\gamma_q}} \qquad (3.19) \\
&\lesssim \varepsilon^2 \sum_{k=1}^{n-q} \varepsilon^{-\frac{1}{2}\cdot 2(k-1)(\gamma_q-1)} \int_0^{\tau_{k+1}} t\,dt \int_0^{\infty} \frac{s^{2n-2k-3}ds}{(t+s)^{(2n-2k-1)\gamma_q}} \qquad (\varepsilon^{\frac{1}{2}} \lesssim \tau_2 \le \tau_3 \le \dots) \\
&\lesssim \varepsilon^{2\gamma_q} \sum_{k=1}^{n-q} \varepsilon^{2-2\gamma_q-(k-1)(\gamma_q-1)} \int_0^{\tau_{k+1}} t^{(2n-2k-1)(1-\gamma_q)}dt \\
&\lesssim \varepsilon^{2\gamma_q} \sum_{k=1}^{n-q} \varepsilon^{(k+1)(1-\gamma_q)} \tau_{k+1}^{(2n-2k-1)(1-\gamma_q)+1} \\
&\lesssim \varepsilon^{2\gamma_q} \sum_{p=q}^{n-1} \varepsilon^{(n-p+1)(1-\gamma_q)+\frac{1}{m_p}(2p-1)(1-\gamma_q)+\frac{1}{m_p}} \qquad (\tau_{k+1} \lesssim \varepsilon^{\frac{1}{m_{n-k}}},\ p = n-k)
\end{aligned}
$$

$$\lesssim \varepsilon^{2\gamma_q} \sum_{p=q}^{n-1} \varepsilon^{(n-p+1)(1-\gamma_p)+\frac{1}{m_p}(2p-1)(1-\gamma_p)+\frac{1}{m_p}} \qquad \text{(by (3.13)).}$$

The last inequality above equals $\varepsilon^{2\gamma_q}$ itself because for every $1 \le p \le n-1$,

$$(n-p+1)(1-\gamma_p)+\tfrac{1}{m_p}(2p-1)(1-\gamma_p)+\tfrac{1}{m_p} = \tfrac{(n-p+1)m_p+2p-1}{m_p}(1-\gamma_p)+\tfrac{1}{m_p} = -\tfrac{r_p-1}{m_p(r_p-1)}+\tfrac{1}{m_p} = 0. \tag{3.20}$$

Thus, by multiplying $\varepsilon^{-(1+j)\gamma_q}$ to (3.19), we obtain (3.16). Again, by (3.4), $\varepsilon^{-1-j} \max\limits_{1\le k\le n-q} \tau_{k+1} \lesssim \varepsilon^{-1-j+\frac{1}{m_q}}$. By multiplying $\varepsilon^{(\frac{1}{m_q}-1-j)\gamma_q}$ to (3.19), we obtain (3.17). □

We can now prove Theorem 2.9 by taking the sums over ε-minimal ellipsoids.

Proof of Theorem 2.9. Note that the constants $\varepsilon_0, C_0 > 0$ in Lemma 3.3 depend only on Ω, ϱ, and $\widehat{S}$, but not on z and ζ. We can replace the domains of the integrals (2.13) - (2.20) by $z \in \Omega \cap P_{\varepsilon_0}(\zeta)$ and $\zeta \in P_{\varepsilon_0}(z)\backslash\overline{\Omega}$. Indeed, by construction,

$$\sup_{z\in\Omega,\zeta\in U_1\backslash\overline{\Omega};\, |z-\zeta|\ge\varepsilon_0/C_0} |D^k_{z,\zeta}K^\top_{q-1}(z,\zeta)| + |D^k_{z,\zeta}K^\perp_{q-1}(z,\zeta)| < \infty, \quad k \ge 0.$$

The proofs of (2.13) and (2.14) both follow from the same argument, and similarly for the other equations, and thus we only need to prove (2.13), (2.15), (2.17), and (2.19).

Let $z \in \Omega$ with $\operatorname{dist}(z) < \varepsilon_0$. Let $J \in \mathbb{Z}$ be a unique number such that $2^{-J}\varepsilon_0 \le \varrho(z) < 2^{1-J}\varepsilon_0$. Therefore, $P_{2^{-J}\varepsilon_0}(z) \subseteq \Omega$ and $\zeta \in P_\varepsilon(z) \Rightarrow \operatorname{dist}(\zeta) \lesssim \varepsilon$ for all $0 < \varepsilon \le \varepsilon_0$.

By applying (3.14), we obtain (2.13):

$$\begin{aligned} \int\limits_{P_{\varepsilon_0}(z)\backslash\overline{\Omega}} \operatorname{dist}(\zeta)^s |D^k K^\top_{q-1}(z,\zeta)| d\operatorname{Vol}_\zeta &\lesssim_k \sum_{j=1}^{J} \int\limits_{P_{2^{1-j}\varepsilon_0}(z)\backslash(P_{2^{-j}\varepsilon_0}(z)\cup\Omega)} (2^{-j}\varepsilon_0)^s |D^k K^\top_{q-1}(z,\zeta)| d\operatorname{Vol}_\zeta \\ &\lesssim_k \sum_{j=1}^{J} (2^{-j}\varepsilon_0)^s (2^{-j}\varepsilon_0)^{\frac{1}{m_q}+1-k} \lesssim_{\varepsilon_0} 2^{-J(s+1+\frac{1}{m_q}-k)} \approx \operatorname{dist}(z)^{s+1+\frac{1}{m_q}-k}. \end{aligned} \tag{3.21}$$

By applying (3.16), we obtain (2.17):

$$\begin{aligned} \int\limits_{P_{\varepsilon_0}(z)\backslash\overline{\Omega}} |\operatorname{dist}(\zeta)^s D^k K^\top_{q-1}(z,\zeta)|^{\gamma_q} d\operatorname{Vol}_\zeta &\lesssim_k \sum_{j=1}^{J} \int\limits_{P_{2^{1-j}\varepsilon_0}(z)\backslash(P_{2^{-j}\varepsilon_0}(z)\cup\Omega)} (2^{-j}\varepsilon_0)^{s\gamma_q} |D^k K^\top_{q-1}(z,\zeta)|^{\gamma_q} d\operatorname{Vol}_\zeta \\ &\lesssim_k \sum_{j=1}^{J} (2^{-j}\varepsilon_0)^{s\gamma_q} (2^{-j}\varepsilon_0)^{(1-k)\gamma_q} \lesssim_{\varepsilon_0} 2^{-J(s+1-k)\gamma_q} \approx \operatorname{dist}(z)^{(s+1-k)\gamma_q}. \end{aligned} \tag{3.22}$$

Using (3.15) and (3.17), the same arguments show that

$$\int\limits_{P_{\varepsilon_0}(z)\backslash\overline{\Omega}} \operatorname{dist}(\zeta)^s |D^k K^\perp_{q-1}(z,\zeta)| d\operatorname{Vol}(\zeta) \lesssim 2^{-J(s+\frac{2}{m_q}-k)} \approx \operatorname{dist}(z)^{s+\frac{2}{m_q}-k};$$

$$\int\limits_{P_{\varepsilon_0}(z)\backslash\overline{\Omega}} |\operatorname{dist}(\zeta)^s D^k K^\perp_{q-1}(z,\zeta)|^{\gamma_q} d\operatorname{Vol}(\zeta) \lesssim 2^{-J(s-k+\frac{1}{m_q})\gamma_q} \approx \operatorname{dist}(z)^{(s-k+\frac{1}{m_q})\gamma_q}.$$

These two equations give (2.15) and (2.19). The proof is now complete. □

Theorem 2.9 implies the following weighted boundedness between Sobolev spaces.

Corollary 3.13. *Let $1 \le q \le n-1$ and $\alpha \in \mathbb{N}^{2n}(= \mathbb{N}^n_\zeta \times \mathbb{N}^n_{\bar\zeta})$. We define integral operators $\mathcal{K}^\top_{q,\alpha}, \mathcal{K}^\perp_{q,\alpha} : L^1(U_1 \backslash \overline{\Omega}; \wedge^{0,q+1}) \to C^0_{\mathrm{loc}}(\Omega; \wedge^{0,q-1})$ by*

$$\mathcal{K}^\top_{q,\alpha} g(z) := \int_{U_1\backslash\overline{\Omega}} (D^\alpha_\zeta (K^\top_{q-1}))(z,\cdot) \wedge g; \quad \mathcal{K}^\perp_{q,\alpha} g(z) := \int_{U_1\backslash\overline{\Omega}} (D^\alpha_\zeta (K^\perp_{q-1}))(z,\cdot) \wedge g. \tag{3.23}$$

Let $\mathrm{dist}(w) := \mathrm{dist}(w, b\Omega)$. *Then, for every $k \ge 0$ and $1 < s < k + |\alpha| - \frac{1}{m_q}$* (in particular, $k + |\alpha| \ge 2$),

$$\mathcal{K}^\top_{q,\alpha} : L^p(U_1\backslash\overline{\Omega}, \mathrm{dist}^{1-s}; \wedge^{0,q+1}) \to W^{k,p}(\Omega, \mathrm{dist}^{k+|\alpha|-\frac{1}{m_q}-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le \infty; \tag{3.24}$$

$$\mathcal{K}^\perp_{q,\alpha} : L^p(U_1\backslash\overline{\Omega}, \mathrm{dist}^{\frac{1}{m_q}-s}; \wedge^{0,q+1}) \to W^{k,p}(\Omega, \mathrm{dist}^{k+|\alpha|-\frac{1}{m_q}-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le \infty; \tag{3.25}$$

$$\mathcal{K}^\top_{q,\alpha} : L^p(U_1\backslash\overline{\Omega}, \mathrm{dist}^{1-s}; \wedge^{0,q+1}) \to W^{k,\frac{pr_q}{r_q-p}}(\Omega, \mathrm{dist}^{k+|\alpha|-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le r_q; \tag{3.26}$$

$$\mathcal{K}^\perp_{q,\alpha} : L^p(U_1\backslash\overline{\Omega}, \mathrm{dist}^{\frac{1}{m_q}-s}; \wedge^{0,q+1}) \to W^{k,\frac{pr_q}{r_q-p}}(\Omega, \mathrm{dist}^{k+|\alpha|-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le r_q. \tag{3.27}$$

Remark 3.14. Using integrating by parts, we obtain the relation

$$\mathcal{K}^{(\top,\perp)}_{q,\alpha} g = (-1)^{|\alpha|} \mathcal{K}^{(\top,\perp)}_{q,0} \circ D^\alpha g, \quad \text{for all } g \in C^\infty_c(U_1\backslash\overline{\Omega}; \wedge^{0,q+1}).$$

Therefore, (3.24) - (3.27) can be restated as: for every $k, l \ge 0$ and $1 < s < k + l - \frac{1}{m_q}$ (in particular, $k + l \ge 2$),

$$\mathcal{K}^\top_{q,0} : \widetilde{W}^{l,p}(\overline{U}_1\backslash\Omega, \mathrm{dist}^{1-s}; \wedge^{0,q+1}) \to W^{k,p}(\Omega, \mathrm{dist}^{k+l-\frac{1}{m_q}-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le \infty;$$

$$\mathcal{K}^\perp_{q,0} : \widetilde{W}^{l,p}(\overline{U}_1\backslash\Omega, \mathrm{dist}^{\frac{1}{m_q}-s}; \wedge^{0,q+1}) \to W^{k,p}(\Omega, \mathrm{dist}^{k+l-\frac{1}{m_q}-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le \infty;$$

$$\mathcal{K}^\top_{q,0} : \widetilde{W}^{l,p}(\overline{U}_1\backslash\Omega, \mathrm{dist}^{1-s}; \wedge^{0,q+1}) \to W^{k,\frac{pr_q}{r_q-p}}(\Omega, \mathrm{dist}^{k+l-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le r_q;$$

$$\mathcal{K}^\perp_{q,0} : \widetilde{W}^{l,p}(\overline{U}_1\backslash\Omega, \mathrm{dist}^{\frac{1}{m_q}-s}; \wedge^{0,q+1}) \to W^{k,\frac{pr_q}{r_q-p}}(\Omega, \mathrm{dist}^{k+l-s}; \wedge^{0,q-1}), \qquad \forall 1 \le p \le r_q,$$

where $\widetilde{W}^{l,p}(\overline{U}, \varphi) := \{g \in W^{l,p}(\mathbb{R}^N, \varphi) : g|_{\overline{U}^c} = 0\}$ follow the notations in Definition 4.5.

Corollary 3.13 follows almost immediately by Schur's test.

Lemma 3.15 *(Schur's test). Let (X, μ) and (Y, ν) be two measure spaces. Let $G \in L^1_{\mathrm{loc}}(X \times Y, \mu \otimes \nu)$, $1 \le \gamma < \infty$, and $A > 0$, which satisfy*

$$\operatorname*{essup}_{y\in Y} \int_X |G(x,y)|^\gamma d\mu(x) \le A^\gamma; \quad \operatorname*{essup}_{x\in X} \int_X |G(x,y)|^\gamma d\mu(y) \le A^\gamma.$$

Then, the integral operator $Tf(y) := \int_X G(x,y) f(x) d\mu(x)$ has boundedness $T : L^p(X, d\mu) \to L^q(Y, d\nu)$, with operator norm $\|T\|_{L^p \to L^q} \le A$ for all $1 \le p, q \le \infty$ such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{\gamma} - 1$.

For example, see [53, Appendix B]. Note that the norm of $L^p(X, \mu)$ is $(\int_X |f|^p d\mu)^{\frac{1}{p}}$, and the norm of $L^p(\Omega, \varphi)$ in Definition 4.2 is $(\int_\Omega |\varphi f|^p d\,\mathrm{Vol})^{1/p}$. When $p < \infty$, we have the correspondence $d\mu = |\varphi|^p \cdot d\,\mathrm{Vol}$.

Proof of Corollary 3.13. For $\beta \in \mathbb{N}^{2n}_{z,\bar{z}}$, we have $D^\beta_z \mathcal{K}^\top_{q,\alpha} g(z) = \int (D^\beta_z D^\alpha_\zeta K^\top_{q-1})(z,\cdot) \wedge g$.

By applying Lemma 3.15 to (2.13) and (2.14) with $(X,\mu) = (U_1 \backslash \overline{\Omega}, \mathrm{Vol}_\zeta)$, $(Y,\nu) = (\Omega, \mathrm{Vol}_z)$, $\gamma = 1$ and $G(\zeta,z) = \mathrm{dist}_{\Omega^c}(z)^{|\alpha|+k-s-\frac{1}{m_q}} \cdot (D^k_z D^\alpha_\zeta K^\top_{q-1})(z,\zeta) \cdot \mathrm{dist}_\Omega(\zeta)^{s-1}$, we see that for every $k \geq 0$ and $1 < s < k + |\alpha| - 1/m_q$,

$$\left[g \mapsto \mathrm{dist}^{|\alpha|+|\beta|-s-\frac{1}{m_q}} \cdot \left(D^k_z \mathcal{K}^\top_{q,\alpha}(\mathrm{dist}^{s-1} \cdot g)\right)\right] : L^p(U_1 \backslash \overline{\Omega}) \to L^p(\Omega), \quad \forall 1 \leq p \leq \infty.$$

This is the same as stating that $D^k_z \mathcal{K}^\top_{q,\alpha} : L^p(U_1 \backslash \overline{\Omega}, \mathrm{dist}^{1-s}) \to L^p(\Omega, \mathrm{dist}^{|\alpha|+k-s-1/m_q})$. Thus, (3.24) follows.

Similarly, by applying Lemma 3.15 to (2.17) and (2.18) with $(X,\mu) = (U_1 \backslash \overline{\Omega}, \mathrm{Vol}_\zeta)$, $(Y,\nu) = (\Omega, \mathrm{Vol}_z)$, $\gamma = \frac{r_q}{r_q - 1}$ and $G(\zeta,z) = \mathrm{dist}_{\Omega^c}(z)^{|\alpha|+k-s} \cdot (D^k_z D^\alpha_\zeta K^\top_{q-1})(z,\zeta) \cdot \mathrm{dist}_\Omega(\zeta)^{s-1}$, we obtain (3.26).

After repeating the same arguments and replacing $\mathrm{dist}(\zeta)^{s-1}$ by $\mathrm{dist}(\zeta)^{s-\frac{1}{m_q}}$ and $K^\top$ by $K^\perp$, we obtain (3.25) and (3.27). $\square$

Remark 3.16. By keeping track of the proof, the implied constants in Theorem 2.9 and the operator norms in Corollary 3.13 depend only on C_0 in Lemma 3.3, C_1 in Lemma 3.5, and the upper bound of $\|\varrho\|_{C^{m+k+2}}$.

More generally, whenever the smooth holomorphic support function $\widehat{S}(z,\zeta)$ as well as the corresponding Leray map $\widehat{Q}_j(z,\zeta) := \int_0^1 \frac{\partial S}{\partial z_j}(\zeta + t(z-\zeta), \zeta) dt$ $(j = 1, \dots, n)$ satisfy the estimates in Lemma 3.5, then the kernel $K(z,\zeta)$ given by (2.8) would fulfill the same weighted estimates as in Theorem 2.9.

4. Function spaces and extension operators

In this section, we focus on the real domain $\mathbb{R}^N \simeq \mathbb{C}^n$ where $N = 2n$.

Notation 4.1. We denote $\mathscr{S}'(\mathbb{R}^N)$ as the space of tempered distributions, and for an arbitrary open subset $U \subseteq \mathbb{R}^n$, we denote $\mathscr{S}'(U) := \{\tilde{f}|_U : \tilde{f} \in \mathscr{S}'(\mathbb{R}^N)\} \subsetneq \mathscr{D}'(U)$ as the space of distributions in U that can be extended to tempered distributions in $\mathbb{R}^N$ (also see [58, (3.1) and Proposition 3.1]).

First, we recall the classical Sobolev and Hölder spaces. The characterizations in Definitions 4.3 and 4.4 are not used directly in this paper.

Definition 4.2 *(Weighted Sobolev)*. Let $U \subseteq \mathbb{R}^N$ be an arbitrary open set. Let $\varphi : U \to [0,\infty)$ be a non-negative continuous function, and for $k \geq 0$ and $1 \leq p \leq \infty$, we define,

$$W^{k,p}(U,\varphi) := \{f \in W^{k,p}_{\mathrm{loc}}(U) : \|f\|_{W^{k,p}(U,\varphi)} < \infty\},$$

$$\|f\|_{W^{k,p}(U,\varphi)} := \left(\sum_{|\alpha|\leq k} \int_U |\varphi \partial^\alpha f|^p\right)^{\frac{1}{p}} \quad 1 \leq p < \infty; \quad \|f\|_{W^{k,\infty}(U,\varphi)} := \sup_{|\alpha|\leq k} \|\varphi \partial^\alpha f\|_{L^\infty(U)}. \tag{4.1}$$

We define $W^{k,p}(U) := W^{k,p}(U, \mathbf{1})$ where $\mathbf{1} = \mathbf{1}_{\mathbb{R}^N}$ is the constant function.

Definition 4.3 *(Sobolev-Bessel)*. Let $s \in \mathbb{R}$. For $1 < p < \infty$, we define the Bessel potential space $H^{s,p}(\mathbb{R}^N)$ as the set of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^N)$ such that

$$\|f\|_{H^{s,p}(\mathbb{R}^N)} := \|(I - \Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^N)} < \infty.$$

We use the standard (negative) Laplacian $\Delta = \sum_{j=1}^N \partial^2_{x_j}$.

On an open subset $U \subseteq \mathbb{R}^N$, we define $H^{s,p}(U) := \{\tilde{f}|_U : \tilde{f} \in H^{s,p}(\mathbb{R}^N)\}$ for $s \in \mathbb{R}$, $1 < p < \infty$ with norm $\|f\|_{H^{s,p}(U)} := \inf_{\tilde{f}|_U = f} \|\tilde{f}\|_{H^{s,p}(\mathbb{R}^N)}$.

Definition 4.4 *(Hölder–Zygmund).* Let $U \subseteq \mathbb{R}^N$ be an open subset. We define the Hölder–Zygmund space $\mathscr{C}^s(U)$ for $s \in \mathbb{R}$ as follows.

- For $0 < s < 1$, $\mathscr{C}^s(U)$ consists of all $f \in C^0(U)$ such that $\|f\|_{\mathscr{C}^s(U)} := \sup_U |f| + \sup_{x,y\in U} \frac{|f(x)-f(y)|}{|x-y|^s} < \infty$.
- $\mathscr{C}^1(U)$ consists of all $f \in C^0(U)$ such that $\|f\|_{\mathscr{C}^1(U)} := \sup_U |f| + \sup_{x,y\in U;\frac{x+y}{2}\in U} \frac{|f(x)+f(y)-2f(\frac{x+y}{2})|}{|x-y|} < \infty$.
- For $s > 1$ recursively, $\mathscr{C}^s(U)$ consists of all $f \in \mathscr{C}^{s-1}(U)$ such that $\nabla f \in \mathscr{C}^{s-1}(U;\mathbb{C}^N)$. We define $\|f\|_{\mathscr{C}^s(U)} := \|f\|_{\mathscr{C}^{s-1}(U)} + \sum_{j=1}^N \|D_j f\|_{\mathscr{C}^{s-1}(U)}$.
- For $s \le 0$ recursively, $\mathscr{C}^s(U)$ consists of all distributions that have the form $g_0 + \sum_{j=1}^N \partial_j g_j$ where $g_0,\dots,g_N \in \mathscr{C}^{s+1}(U)$. We define $\|f\|_{\mathscr{C}^s(U)} := \inf\{\sum_{j=0}^N \|g_j\|_{\mathscr{C}^{s+1}(U)} : f = g_0 + \sum_{j=1}^N \partial_j g_j \in \mathscr{D}'(U)\}$.
- We define $\mathscr{C}^\infty(U) := \bigcap_{s>0} \mathscr{C}^s(U)$ as the space of bounded smooth functions.

In this paper we consider a more general version of the function spaces: the Triebel–Lizorkin spaces.

Definition 4.5 *(Triebel–Lizorkin).* Let $\lambda = (\lambda_j)_{j=0}^\infty$ be a sequence of Schwartz functions that satisfy the following.

(4.2) The Fourier transform $\hat\lambda_0(\xi) = \int_{\mathbb{R}^n} \lambda_0(x) 2^{-2\pi i x\xi} dx$ satisfies $\operatorname{supp}\hat\lambda_0 \subset \{|\xi| < 2\}$ and $\hat\lambda_0|_{\{|\xi|<1\}} \equiv 1$.
(4.3) $\lambda_j(x) = 2^{jn}\lambda_0(2^j x) - 2^{(j-1)n}\lambda_0(2^{j-1}x)$ for $j \ge 1$.

Let $0 < p,q \le \infty$ and $s \in \mathbb{R}$, and we define the Triebel–Lizorkin norm $\|\cdot\|_{\mathscr{F}^s_{pq}(\lambda)}$ as

$$\|f\|_{\mathscr{F}^s_{pq}(\lambda)} := \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{L^p(\mathbb{R}^N;\ell^q(\mathbb{N}))} = \Bigg(\int_{\mathbb{R}^N} \Big(\sum_{j=0}^\infty |2^{js}\lambda_j * f(x)|^q\Big)^{\frac{p}{q}} dx\Bigg)^{\frac{1}{p}}, \qquad p < \infty; \tag{4.4}$$

$$\|f\|_{\mathscr{F}^s_{\infty q}(\lambda)} := \sup_{x\in\mathbb{R}^N, J\in\mathbb{Z}} 2^{NJ\frac{1}{q}} \|(2^{js}\lambda_j * f)_{j=\max(0,J)}^\infty\|_{L^q(B(x,2^{-J});\ell^q)}, \qquad p = \infty. \tag{4.5}$$

For $q = \infty$ we take the usual modifications, where we replace the ℓ^q sum by the supremum over j.

We define $\mathscr{F}^s_{pq}(\mathbb{R}^N)$, with its norm given by a fixed choice of λ.

For an arbitrary open subset $U \subseteq \mathbb{R}^N$, we define

$$\mathscr{F}^s_{pq}(U) := \{\tilde f|_U : \tilde f \in \mathscr{F}^s_{pq}(\mathbb{R}^N)\} \qquad \text{with} \quad \|f\|_{\mathscr{F}^s_{pq}(U)} := \inf_{\tilde f|_U = f} \|\tilde f\|_{\mathscr{F}^s_{pq}(\mathbb{R}^N)};$$

$$\widetilde{\mathscr{F}}^s_{pq}(\overline{U}) := \{f \in \mathscr{F}^s_{pq}(\mathbb{R}^N) : f|_{\overline{U}^c} = 0\} \qquad \text{as a closed subspace of } \mathscr{F}^s_{pq}(\mathbb{R}^N).$$

Remark 4.6.

(i) When p or $q < 1$, (4.4) and (4.5) are only quasi-norms. For convenience, we still use the terminology "norms" to refer them.
(ii) Different choices of λ result in equivalent norms (see [67, Proposition 2.3.2] and [68, Propositions 1.3 and 1.8]).
(iii) We have the embedding $\mathscr{F}^s_{pq_1}(\mathbb{R}^N) \hookrightarrow \mathscr{F}^s_{pq_2}(\mathbb{R}^N) \hookrightarrow \mathscr{F}^{s-\delta}_{pq_1}(\mathbb{R}^N)$ for all $0 < p \le \infty$, $s \in \mathbb{R}$, $q_1 \le q_2$, $\delta > 0$.
We have the embedding $\mathscr{F}^s_{p_1q_1}(\mathbb{R}^N) \hookrightarrow \mathscr{F}^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_1q_2}(\mathbb{R}^N)$ for all $p_1 < p_2 \le \infty$, $s \in \mathbb{R}$, $0 < q_1, q_2 \le \infty$.
See [68, Corollary 2.7] for an illustration. By taking restrictions to an arbitrary open subset U, we have $\mathscr{F}^s_{pq_1}(U) \hookrightarrow \mathscr{F}^s_{pq_2}(U) \hookrightarrow \mathscr{F}^{s-\delta}_{pq_1}(U)$ for $q_1 \le q_2$ and $\delta > 0$, and $\mathscr{F}^s_{p\infty}(U) \hookrightarrow \mathscr{F}^{s-1}_{r\varepsilon}(U)$ for $\varepsilon > 0$ and $0 < p \le r \le Np/\max(N-p,0)$.

(iv) When $p = q = \infty$, (4.5) can be written as $\|f\|_{\mathscr{F}^s_{\infty\infty}(\lambda)} = \sup_{j\geq 0} \|2^{js}\lambda_j * f\|_{L^\infty(\mathbb{R}^N)}$, which is more commonly referred to as the *Besov norm* $\mathscr{B}^s_{\infty\infty}$ (also see [68, (1.15)] and [67, Remark 2.3.4/3]).

(v) For an arbitrary open subset $U \subseteq \mathbb{R}^N$, we have $\mathscr{F}^s_{pq}(\overline{U}^c) = \mathscr{F}^s_{pq}(\mathbb{R}^N)/\widetilde{\mathscr{F}}^s_{pq}(\overline{U})$, or equivalently $\mathscr{F}^s_{pq}(U) = \mathscr{F}^s_{pq}(\mathbb{R}^N)/\widetilde{\mathscr{F}}^s_{pq}(U^c)$ (also see [64, Remark 4.3.2/1]).

(vi) When $\Omega \subseteq \mathbb{R}^N$ is either a bounded Lipschitz domain or the total space, we have following (see [67, Sections 2.5.6 and 2.5.7] and [66, Theorem 1.122]),

- $H^{s,p}(\Omega) = \mathscr{F}^s_{p2}(\Omega)$ for $s \in \mathbb{R}$ and $1 < p < \infty$;
- $W^{k,p}(\Omega) = \mathscr{F}^k_{p2}(\Omega)$ for $k \in \mathbb{N}$ and $1 < p < \infty$;
- $\mathscr{C}^s(\Omega) = \mathscr{F}^s_{\infty\infty}(\Omega)$ for $s \in \mathbb{R}$;
- $\mathscr{C}^{k+s}(\Omega) = C^{k,s}(\Omega)$ for $k \in \mathbb{N}$ and $0 < s < 1$.

We sketch the proof of $\mathscr{C}^s = \mathscr{F}^s_{\infty\infty}$ for $s < 0$ on $\mathbb{R}^N$ and on bounded Lipschitz domains in the following.

Proof of $\mathscr{C}^s = \mathscr{F}^s_{\infty\infty}$ when $s \leq 0$. We fix an $s \leq 0$ as follows.

Let $k > -s/2$ be an integer. For an $f \in \mathscr{F}^s_{\infty\infty}(\mathbb{R}^N)$, we have $f = (I - \Delta)^k (I - \Delta)^{-k} f$, where by [67, Theorem 2.3.8], $(I - \Delta)^{-k} f \in \mathscr{F}^{s+2k}_{\infty\infty}(\mathbb{R}^N) = \mathscr{C}^{s+2k}(\mathbb{R}^N)$, and thus $f \in \mathscr{C}^s(\mathbb{R}^N)$. Conversely, for an $f = \sum_{|\alpha|\leq\lceil -s\rceil+1} D^\alpha g_\alpha \in \mathscr{C}^s(\mathbb{R}^N)$, where $g \in \mathscr{C}^{s+\lceil -s\rceil+1}(\mathbb{R}^N) = \mathscr{F}^{s+\lceil -s\rceil+1}_{\infty\infty}(\mathbb{R}^N)$, by [67, Theorem 2.3.8], $D^\alpha g_\alpha \in \mathscr{F}^s_{\infty\infty}(\mathbb{R}^N)$ thus $f \in \mathscr{F}^s_{\infty\infty}(\mathbb{R}^N)$.

For a bounded Lipschitz Ω, since we have $\mathscr{C}^{s+2k}(\Omega) = \mathscr{F}^{s+2k}_{\infty\infty}(\Omega)$ and $\mathscr{C}^{s+\lceil -s\rceil+1}(\Omega) = \mathscr{F}^{s+\lceil -s\rceil+1}_{\infty\infty}(\Omega)$, then by taking restrictions on both sides, we obtain $\mathscr{C}^s(\Omega) = \mathscr{F}^s_{\infty\infty}(\Omega)$. □

Remark 4.6 (vi) can be illustrated via the extension operator. Our convex domain $\Omega \subset \mathbb{C}^n$ is smooth and we can use the so-called *half-space extension* (see e.g. Remark 5.2 and [67, Sections 2.9 and 3.3.4]). In our case, it is preferable to use the *Rychkov's extension*, which can also work on Lipschitz domains.

Definition 4.7. Let $\omega \subset \mathbb{R}^N$ be a *special Lipschitz domain*,[3] i.e., $\omega = \{(x_1, x') : x_1 > \sigma(x')\}$, for some $\sigma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that $\|\nabla\sigma\|_{L^\infty} < 1$.

The *Rychkov's universal extension operator* $E = E_\omega$ for ω is given by the following:

$$E_\omega f := \sum_{j=0}^{\infty} \psi_j * (\mathbf{1}_\omega \cdot (\phi_j * f)), \qquad f \in \mathscr{S}'(\omega), \tag{4.6}$$

where $(\psi_j)_{j=0}^\infty$ and $(\phi_j)_{j=0}^\infty$ are families of Schwartz functions that satisfy the following properties:

(4.7) *Scaling condition*: $\phi_j(x) = 2^{(j-1)N}\phi_1(2^{j-1}x)$ and $\psi_j(x) = 2^{(j-1)N}\psi_1(2^{j-1}x)$ for $j \geq 2$;

(4.8) *Moment condition*: $\int \phi_0 = \int \psi_0 = 1$, $\int x^\alpha \phi_0(x)dx = \int x^\alpha \psi_0(x)dx = 0$ for all multi-indices $|\alpha| > 0$, and $\int x^\alpha \phi_1(x)dx = \int x^\alpha \psi_1(x)dx = 0$ for all $|\alpha| \geq 0$;

(4.9) *Approximate identity*: $\sum_{j=0}^\infty \phi_j = \sum_{j=0}^\infty \psi_j * \phi_j = \delta_0$ is the Dirac delta measure;

(4.10) *Support condition*: ϕ_j, ψ_j are all supported in the negative cone $-\mathbb{K} := \{(x_1, x') : x_1 < -|x'|\}$.

The family $(\phi_j, \psi_j)_{j=0}^\infty$ exists (see [58, Theorem 4.1(b) and Proposition 2.1]).

Proposition 4.8. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Let $(\phi_j)_{j=0}^\infty$ be as given in Definition 4.7.*

[3] In previous studies such as [66, Definition 1.103], the condition $\|\nabla\sigma\|_{L^\infty} < 1$ was not required, which could be achieved through an invertible linear transformation.

(i) ([58, Theorem 4.1]) $E_\omega : \mathscr{F}^s_{pq}(\omega) \to \mathscr{F}^s_{pq}(\mathbb{R}^N)$ *is bounded provided that* $(p,q) \notin \{\infty\} \times (0,\infty)$.

(ii) ([58, Theorem 3.2] and [70, Theorem 1]) *There are intrinsic norms*

$$\|f\|_{\mathscr{F}^s_{pq}(\omega)} \approx_{p,q,s} \|(2^{js}\phi_j * f)_{j=0}^\infty\|_{L^p(\omega;\ell^q(\mathbb{N}))}, \qquad \textit{provided } p<\infty;$$

$$\|f\|_{\mathscr{F}^s_{\infty q}(\omega)} \approx_{q,s} \sup_{x\in\mathbb{R}^N; J\in\mathbb{Z}} 2^{NJ\frac{1}{q}} \|(2^{js}\phi_j * f)_{j=\max(0,J)}^\infty\|_{L^q(\omega\cap B(x,2^{-J});\ell^q)}, \qquad \textit{for } p=\infty.$$

(iii) ([61, Proposition 6.6] and [70, Theorem 2]) *For every* $m \geq 0$*, an equivalent norm exists via derivatives* $\|f\|_{\mathscr{F}^s_{pq}(\omega)} \approx_{p,q,s,m} \sum_{|\alpha|\leq m} \|D^\alpha f\|_{\mathscr{F}^{s-m}_{pq}(\omega)}$.

In fact, $E_\omega : \mathscr{F}^s_{\infty q}(\omega) \to \mathscr{F}^s_{\infty q}(\mathbb{R}^N)$ is also bounded (see [72]). We do not need this result in this paper.

In our case, we work on the bounded domains instead of special type domains. For a bounded domain Ω, we define its extension operator $\mathcal{E}_\Omega$ via partition of unity. For completeness, we give the construction below.

Notation 4.9 *(Objects for partition of unity).* Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $\mathcal{U} \Supset \Omega$ be a fixed open neighborhood. We use the following objects, which can all be obtained by the standard partition of unity argument:

- $(U_\nu)_{\nu=0}^M$ are finitely many bounded smooth open sets in $\mathcal{U} \subseteq \mathbb{R}^n$;
- $(\Phi_\nu : \mathbb{R}^N \to \mathbb{R}^N)_{\nu=1}^M$ as invertible affine linear transformations;
- $(\chi_\nu)_{\nu=0}^M$ are C_c^∞-functions on $\mathbb{R}^N$ that take values in $[0,1]$;
- $(\omega_\nu)_{\nu=1}^M$ are special Lipschitz domains on $\mathbb{R}^N$.

They have the following properties:

(4.11) $b\Omega \subset \bigcup_{\nu=1}^M U_\nu$ and $U_0 \Subset \Omega \Subset \bigcup_{\nu=0}^M U_\nu \Subset \mathcal{U}$;
(4.12) $\chi_\nu \in C_c^\infty(U_\nu)$ for $0 \leq \nu \leq M$, and $\sum_{\nu=0}^M \chi_\nu^2 \equiv 1$ in a neighborhood of $\overline{\Omega}$;
(4.13) For each $1 \leq \nu \leq M$, $U_\nu = \Phi_\nu(B^N(0,1))$ and $U_\nu \cap \Omega = U_\nu \cap \Phi_\nu(\omega_\nu)$.

The partition of unity argument requires nothing other than the following fact.

Lemma 4.10. *Let* $U \subseteq \mathbb{R}^N$ *be an arbitrary open subset, let* Φ *be an invertible linear transform, and let* $\chi \in \mathscr{C}^\infty(\mathbb{R}^N)$ *be a bounded smooth function.*

Then, $Tf(x) := \chi(x)f(\Phi(x))$ *defines a bounded linear map* $T : \mathscr{F}^s_{pq}(U) \to \mathscr{F}^s_{pq}(\Phi^{-1}(U))$ *for all* $s \in \mathbb{R}$ *and* $0 < p,q \leq \infty$.

Proof. We have the boundedness $[g \mapsto g\circ\Phi] : \mathscr{F}^s_{pq}(\mathbb{R}^N) \to \mathscr{F}^s_{pq}(\mathbb{R}^N)$ from [68, Theorem 2.25] and $[g \mapsto \chi g] : \mathscr{F}^s_{pq}(\mathbb{R}^N) \to \mathscr{F}^s_{pq}(\mathbb{R}^N)$ from [68, Theorem 2.28]. For $f \in \mathscr{F}^s_{pq}(U)$, let $\tilde{f} \in \mathscr{F}^s_{pq}(\mathbb{R}^N)$ be an extension of f, and we see that $\chi \cdot (\tilde{f} \circ \Phi)$ is an extension to Tf with respect to the domain $\Phi^{-1}(U)$. Therefore, by taking restrictions to U and $\Phi^{-1}(U)$, respectively, we obtain the boundedness of T. $\square$

Definition 4.11. Let Ω be a bounded Lipschitz domain and let $\mathcal{U} \Supset \Omega$ be an open subset. Let $(\chi_\nu)_{\nu=0}^M$, $(\omega_\nu)_{\nu=1}^M$ and $(\Phi_\nu)_{\nu=1}^M$ be as given above. We define the extension operator $\mathcal{E} = \mathcal{E}_\Omega$ for Ω as

$$\mathcal{E}_\Omega f := \chi_0^2 \cdot f + \sum_{\nu=1}^M \chi_\nu \cdot (E_{\omega_\nu}[(\chi_\nu f)\circ\Phi_\nu]\circ\Phi_\nu^{-1}), \tag{4.14}$$

where E_{ω_ν} is given by (4.6).

Remark 4.12. By combining Proposition 4.8 and Lemma 4.10, $\mathcal{E}_\Omega : \mathscr{F}^s_{pq}(\Omega) \to \mathscr{F}^s_{pq}(\mathbb{R}^N)$ is also bounded for all $0 < p, q \le \infty$ and $s \in \mathbb{R}$ such that $(p, q) \notin \{\infty\} \times (0, \infty)$ (for the proof, see [61, Lemma 6.3] for example). The $\mathscr{F}^s_{\infty q}$-boundedness is also true (see [72] and [70, Remark 20]), but we do not need it because we only require the extensions on $\mathscr{F}^s_{p\infty}$ in applications.

Before doing the weighted estimates, if f has low regularity, then we need to express $[\overline{\partial}, \mathcal{E}]f$ and $([\overline{\partial}, \mathcal{E}]f)^\top$ as the sum of derivatives of good functions, and move those derivatives to $K_{q-1}(z, \zeta)$ via integration by parts. To ensure that there are no boundary terms when integrating by parts, we need the following.

Proposition 4.13 *(Anti-derivatives with support). Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, and for any $k \ge 1$, the operators $\mathcal{S}^{k,\alpha}_\Omega : \mathscr{S}'(\mathbb{R}^N) \to \mathscr{S}'(\mathbb{R}^N)$, $|\alpha| \le k$ exist such that*

(i) $\mathcal{S}^{k,\alpha}_\Omega : \mathscr{F}^s_{pq}(\mathbb{R}^N) \to \mathscr{F}^{s+k}_{pq}(\mathbb{R}^N)$ for all $0 < p, q \le \infty$ and $s \in \mathbb{R}$ such that $(p, q) \notin \{\infty\} \times (0, \infty)$.
(ii) $g = \sum_{|\alpha| \le k} D^\alpha \mathcal{S}^{k,\alpha} g$ for all $g \in \mathscr{S}'(\mathbb{R}^N)$.
(iii) If $g \in \mathscr{S}'(\mathbb{R}^N)$ satisfies $g|_\Omega = 0$, then $\mathcal{S}^{k,\alpha} g|_\Omega = 0$ for all $|\alpha| \le k$.

In particular, $\mathcal{S}^{k,\alpha}_\Omega : \widetilde{\mathscr{F}}^s_{pq}(\Omega^c) \to \widetilde{\mathscr{F}}^{s+k}_{pq}(\Omega^c)$.

See [61, Proposition 1.7] and its proof is in [61, Section 6.3].

In fact, the result is also true for $\mathscr{F}^s_{\infty q}$-cases, which can be obtained by replacing [61, Theorem 1.5 (i)] with [70, Proposition 17] in its proof. We do not need this result in this paper.

The condition (iii) is non-trivial here. If we only want conditions (i) and (ii) to be satisfied, we can consider the decomposition $f = (I - \Delta)^m((I - \Delta)^{-m} f) = \sum_{|\gamma| \le 2m} c_\gamma D^\gamma((I - \Delta)^{-m} f)$ on $\mathbb{R}^N$.

Remark 4.14. In practice, we consider the composition $\mathcal{S}^{k,\alpha} \circ [\overline{\partial}, \mathcal{E}]f$, where $\mathcal{E}$ is an extension operator of Ω such that $\operatorname{supp} \mathcal{E}f \Subset \mathcal{U}$ for some fixed open bounded (smooth) neighborhood $\mathcal{U} \Supset \Omega$. Clearly, $[\overline{\partial}, \mathcal{E}]f$ is supported in $\mathcal{U} \backslash \overline{\Omega}$, and thus $\operatorname{supp}(\mathcal{S}^{k,\alpha}_\Omega \circ [\overline{\partial}, \mathcal{E}]f) \subseteq \Omega^c$. In order to obtain a better support condition,

$$\operatorname{supp}(\mathcal{S}^{k,\alpha} \circ [\overline{\partial}, \mathcal{E}]f) \subseteq \overline{\mathcal{U}} \backslash \Omega,$$

we can apply the proposition to the domain $\Omega \cup (\mathcal{V} \backslash \overline{\mathcal{U}})$, where $\mathcal{V} \Supset \mathcal{U}$ is a larger bounded smooth domain that makes $\mathcal{V} \backslash \overline{\mathcal{U}}$ a bounded Lipschitz domain, and then take a smooth cutoff outside $\mathcal{V}$.

Thus, $\{\mathcal{S}^{k,\alpha}\}_{|\alpha| \le k} : \widetilde{\mathscr{F}}^s_{pq}(\overline{\mathcal{U}} \backslash \Omega) \to \widetilde{\mathscr{F}}^{s+k}_{pq}(\overline{\mathcal{U}} \backslash \Omega)$ exist such that $g = \sum_{|\alpha| \le k} D^\alpha \mathcal{S}^{k,\alpha} g$.

In Theorem 1.1, we claim that $\mathcal{H}_q$ is defined on the large space $\mathscr{S}'(\Omega; \wedge^{0,q})$, which can be implied by its definedness on all Hölder spaces with negative indices.

Lemma 4.15. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then, $\mathscr{S}'(\Omega) = \bigcup_{s<0} \mathscr{C}^s(\Omega)$.*

In fact, we can replace $\mathscr{C}^s(\Omega)$ by $H^{s,p}(\Omega)$ or even $\mathscr{F}^s_{pq}(\Omega)$ for $1 \le p, q \le \infty$. Indeed, by Remark 4.6, we have $\mathscr{C}^s(\Omega) = \mathscr{F}^s_{\infty\infty}(\Omega)$ and the embedding $\mathscr{C}^{s+1}(\Omega) \subset \mathscr{F}^s_{pq}(\Omega) \subset \mathscr{C}^{s-N}(\Omega)$.

Proof. By Remark 4.6, (vi) $\mathscr{C}^s(\Omega) = \mathscr{F}^s_{\infty\infty}(\Omega) = \mathscr{F}^s_{\infty\infty}(\mathbb{R}^N)|_\Omega \subset \mathscr{S}'(\mathbb{R}^N)|_\Omega = \mathscr{S}'(\Omega)$ for all s, and thus $\mathscr{S}'(\Omega) \supseteq \bigcup_{s<0} \mathscr{C}^s(\Omega)$.

Conversely, for an $f \in \mathscr{S}'(\Omega)$, we take an extension $\tilde{f} \in \mathscr{S}'(\mathbb{R}^N)$. We can assume that $\tilde{f}$ has compact support, which can be shown by replacing $\tilde{f}$ with $\chi\tilde{f}$, where $\chi \in C^\infty_c(\mathbb{R}^N)$ satisfies $\chi|_\Omega \equiv 1$. Now, by the structure theorem of distributions (e.g., see [57, Theorem 6.27]), $M \ge 0$ and $\{g_\alpha\}_{|\alpha| \le M} \subset C^0_c(\mathbb{R}^N)$ exist such that $\tilde{f} = \sum_{|\alpha| \le M} D^\alpha g_\alpha$. Clearly, $g_\alpha \in \mathscr{C}^0(\mathbb{R}^N)$, and thus $\tilde{f} \in \mathscr{C}^{-M}(\mathbb{R}^N)$ and we see that $f \in \mathscr{C}^{-M}(\Omega) \subset \bigcup_{s<0} \mathscr{C}^s(\Omega)$. This completes the proof. $\square$

5. Tangential commutator estimate and strong Hardy–Littlewood lemma

To reduce Theorem 1.1 to Theorem 2.9, we need the following two results: the Propositions 5.1 and 5.3. We need to show that the tangential part of the commutator $[\overline{\partial},\mathcal{E}]^\top f=([\overline{\partial},\mathcal{E}]f)^\top$ does not lose derivative.

Proposition 5.1 *(Tangential projection of commutator). Let $\omega=\{x_1>\sigma(x')\}\subset\mathbb{R}^N$ be a special Lipschitz domain. Let $E=E_\omega$ be the extension operator in Definition 4.7.*

Let $X=\sum_{\nu=1}^N X^\nu\frac{\partial}{\partial x_\nu}$ be a smooth vector field on $\mathbb{R}^N$ such that $X(x)\in T_x(b\omega)$ for almost every $x\in b\omega$ (in the sense of surface measure). Then, we have the following boundedness:

$$\big\langle X,[d,E]\big\rangle=\sum_{\nu=1}^N X^\nu\cdot[D_\nu,E]:\mathscr{F}_{pp}^s(\omega)\to\widetilde{\mathscr{F}}_{pp}^s(\omega^c),\qquad \textit{for every } 1\le p\le\infty,\quad s\in\mathbb{R}.$$

In particular, $\langle X,[d,E]\rangle:\mathscr{F}_{p\infty}^s(\omega)\to\widetilde{\mathscr{F}}_{p\varepsilon}^{s-\delta}(\omega^c)$ for every $\varepsilon,\delta>0$.

Remark 5.2.

(i) There is a different kind of commutator estimate in [62, Theorem 4.1], where we prove a smoothing estimate $[D,E]:\mathscr{F}_{p\infty}^s(\omega)\to L^1(\mathbb{R}^N;\mathbb{C}^N)$ for $1\le p\le\infty$ and $s>1-\frac{1}{p}$.

(ii) In fact, $\langle X,[d,E]\rangle:\mathscr{F}_{p\infty}^s(\omega)\to\widetilde{\mathscr{F}}_{p\varepsilon}^s(\omega^c)$ is bounded, i.e., we can take $\delta=0$. In applications, a δ-loss is sufficient and the proof is simpler for $\mathscr{F}_{pp}^s$ spaces.

(iii) Proposition 5.1 can be intuitively understood if we let E be the standard half-space extension for $\omega=\mathbb{R}_+^N:=\{x_1>0\}$. Recall that for an integer $M\ge1$, the half-space extension is given by

$$E^Mf(x_1,x'):=\begin{cases}\sum_{j=-M}^M a_jf(-b_jx_1,x') & x_1<0\\ f(x) & x_1>0\end{cases},$$

$$\text{where } b_j>0 \text{ and } \sum_{j=-M}^M a_j(-b_j)^k=1 \text{ for all } |k|\le M.$$

We have the boundedness $E^M:\mathscr{F}_{pq}^s(\mathbb{R}_+^N)\to\mathscr{F}_{pq}^s(\mathbb{R}^N)$ for all $1\le p,q\le\infty$ and $-M<s<M$ (see [67, Theorem 2.9.2]). We can obtain the construction with $M=\infty$ and also have the boundedness $E^\infty:\mathscr{F}_{pq}^s(\mathbb{R}_+^N)\to\mathscr{F}_{pq}^s(\mathbb{R}^N)$ (see the recent paper by Lu and the current author [46]). It is clear that $[D_\nu,E^\infty]\equiv0$ in the domain for all $2\le\nu\le N$, and thus $\langle X,[d,E^\infty]\rangle\equiv0$ if $X^1\equiv0$.

We can use E^∞ with partition of unity (see (4.14)) or the method in [46, Theorem 26] to define the extension operator $\mathcal{E}$ for the smooth domain $\Omega\subset\mathbb{C}^n$. It is possible that a modification of Corollary 5.5 (iii) still holds.

We also need the Hardy–Littlewood type lemma, which gives the embeddings between the fractional Sobolev spaces (or more generally the Triebel–Lizorkin spaces for Theorem 1.2) and the weighted Sobolev spaces.

Proposition 5.3 *(Strong Hardy-Littlewood lemma). Let $\omega\subset\mathbb{R}^N$ be a special Lipschitz domain. Let $\delta(x):=\min(1,\operatorname{dist}(x,b\omega))$ for $x\in\mathbb{R}^N$. We have the following embeddings:*

(i) $\widetilde{\mathscr{F}}_{p\infty}^s(\overline{\omega})\hookrightarrow L^p(\omega,\delta^{-s})$ for all $1\le p\le\infty$ and $s>0$;

(ii) $W^{m,p}(\omega,\delta^{m-s})\hookrightarrow\mathscr{F}_{p\varepsilon}^s(\omega)$ for all $\varepsilon>0$, $1\le p\le\infty$, $m\in\mathbb{N}$ and $s<m$.

First, we prove Proposition 5.3 and then Proposition 5.1.

Remark 5.4. The proof of Proposition 5.3 is standard if we replace $\mathscr{F}^s_{p\varepsilon}$ and $\mathscr{F}^s_{p\infty}$ by classical Sobolev or Hölder spaces (e.g., see [43]). Our result is stronger since from Remark 4.6, we have $\mathscr{F}^s_{p\varepsilon} \subsetneq H^{s,p} = \mathscr{F}^s_{p2} \subsetneq \mathscr{F}^s_{p\infty}$ for $1 < p < \infty$ and $\mathscr{F}^s_{\infty\varepsilon} \subsetneq \mathscr{C}^s = \mathscr{F}^s_{\infty\infty}$.

The result (i) is not new (see [41] for a more general version). We also refer the reader to [65, Chapter 5.8] for a proof on smooth domains, which contains the discussion of the case where $p < 1$.

First, we give the application of Propositions 5.1 and 5.3 in our setting.

Corollary 5.5. *Let $\Omega = \{\varrho < 0\} \subset \mathbb{C}^n$ be a bounded Lipschitz domain and let $\mathcal{U} \Supset \Omega$ be a bounded open neighborhood such that ϱ has a non-vanishing gradient in $\mathcal{U}\backslash\overline{\Omega}$. Let $\operatorname{dist}(z) := \operatorname{dist}(z, b\Omega)$ for $z \in \mathbb{C}^n$ and let $\mathcal{E} = \mathcal{E}_\Omega$ be as given in Definition 4.11 and its images are supported in $\mathcal{U}$. The following linear maps are bounded.*

(i) $\widetilde{\mathscr{F}}^s_{p\infty}(\overline{\mathcal{U}\backslash\Omega}) \hookrightarrow L^p(\mathcal{U}\backslash\overline{\Omega}, \operatorname{dist}^{-s})$ *for all* $1 \le p \le \infty$ *and* $s > 0$.
In particular, $\widetilde{H}^{s,p}(\overline{\mathcal{U}\backslash\Omega}) \hookrightarrow L^p(\mathcal{U}\backslash\overline{\Omega}, \operatorname{dist}^{-s})$, $\widetilde{\mathscr{C}}^s(\overline{\mathcal{U}\backslash\Omega}) \hookrightarrow L^\infty(\mathcal{U}\backslash\overline{\Omega}, \operatorname{dist}^{-s})$ *for* $1 < p < \infty$, $s > 0$.
(ii) $W^{k,p}(\Omega, \operatorname{dist}^{k-s}) \hookrightarrow \mathscr{F}^s_{p\varepsilon}(\Omega)$ *for all* $\varepsilon > 0$, $1 \le p \le \infty$, $k \in \mathbb{N}$, *and* $s < k$.
In particular, $W^{k,p}(\Omega, \operatorname{dist}^{k-s}) \hookrightarrow H^{s,p}(\Omega)$ *and* $W^{k,\infty}(\Omega, \operatorname{dist}^{k-s}) \hookrightarrow \mathscr{C}^s(\Omega)$ *for* $1 < p < \infty$, $s < k$.
(iii) If Ω *is a smooth domain, then* $[\overline{\partial}, \mathcal{E}]^\top : \mathscr{F}^s_{pp}(\Omega) \to \widetilde{\mathscr{F}}^s_{pp}(\overline{\mathcal{U}\backslash\Omega}; \mathbb{C}^n)$ *for all* $1 \le p \le \infty$, $s \in \mathbb{R}$.
In particular, $[\overline{\partial}, \mathcal{E}]^\top : H^{s,p}(\Omega) \to \widetilde{H}^{s-\delta,p}(\overline{\mathcal{U}\backslash\Omega}; \mathbb{C}^n)$ *and* $[\overline{\partial}, \mathcal{E}]^\top : \mathscr{C}^s(\Omega) \to \widetilde{\mathscr{C}}^s(\overline{\mathcal{U}\backslash\Omega}; \mathbb{C}^n)$ *for* $p \in (1, \infty)$ *and* $\delta > 0$.

$\widetilde{H}^{s,p}(\overline{U}) := \{f \in H^{s,p}(\mathbb{R}^N) : f|_{\overline{U}^c} = 0\} \subset H^{s,p}(\mathbb{R}^N)$ and $\widetilde{\mathscr{C}}^s(\overline{U}) := \{f \in \mathscr{C}^s(\mathbb{R}^N) : f|_{\overline{U}^c} = 0\} \subset \mathscr{C}^s(\mathbb{R}^N)$ follow the notations given in Definitions 4.3, 4.4 and 4.5.

Recall that $H^{s,p} = \mathscr{F}^s_{p2} \hookrightarrow \mathscr{F}^{s-\delta/2}_{pp} \hookrightarrow \mathscr{F}^{s-\delta}_{p2} = H^{s-\delta,p}$ and $\mathscr{C}^s = \mathscr{F}^s_{\infty\infty}$ from Remark 4.6. The boundedness holds immediately for Sobolev and Hölder spaces.

Proof. From Notation 4.9, recall that we have $U_\nu \cap \Omega = U_\nu \cap \Phi_\nu(\omega_\nu)$ for $1 \le \nu \le M$ and $f = \sum_{\nu=0}^M \chi_\nu^2 f$. By Lemma 4.10, $[f \mapsto \chi_\nu f] : \mathscr{F}^s_{pq}(\Omega) \to \mathscr{F}^s_{pq}(U_\nu \cap \Omega)$ are all bounded for $0 \le \nu \le M$.

For each $1 \le \nu \le M$, by Proposition 5.3, we have $\widetilde{\mathscr{F}}^s_{p\infty}(\overline{U_\nu \cap \Omega}) \hookrightarrow L^p(U_\nu \cap \Omega, \operatorname{dist}^{-s}_{b\Omega})$ for $s > 0$, and $W^{m,p}(\Phi_\nu(\omega_\nu), \operatorname{dist}^{m-s}_{b\Omega}) \xrightarrow{(-)|_{U_\nu}} \mathscr{F}^s_{p\varepsilon}(U_\nu \cap \Omega)$ for $s < m$.

For $\nu = 0$, we have the trivial estimates $[f \mapsto \chi_0 f] : \mathscr{F}^s_{p\infty}(\Omega) \to L^p_c(U_0) \hookrightarrow L^p(\Omega, \operatorname{dist}^{-s})$ for $s < m$, and $[f \mapsto \chi_0 f] : W^{m,p}(\Omega, \operatorname{dist}^{m-s}) \to W^{m,p}_c(U_0) \hookrightarrow \mathscr{F}^s_{p\varepsilon}(\Omega)$ for $s < m$.

Therefore, for every $1 \le p \le \infty$, $m \in \mathbb{N}$, and $\varepsilon > 0$,

$$\begin{aligned} &\widetilde{\mathscr{F}}^s_{p\infty}(\overline{\Omega}) \xrightarrow{f \mapsto (\chi_\nu f)_{\nu=0}^M} \bigoplus_{\nu=0}^M \widetilde{\mathscr{F}}^s_{p\infty}(\overline{U_\nu \cap \Omega}) \hookrightarrow \bigoplus_{\nu=0}^M L^p(U_\nu \cap \Omega, \operatorname{dist}^{-s}_{b\Omega}) \\ &\xrightarrow{(g_\nu)_{\nu=0}^M \mapsto \sum_{\nu=0}^M \chi_\nu g_\nu} L^p(\Omega, \operatorname{dist}^{-s}) \end{aligned}, \quad s > 0;$$

$$\begin{aligned} &W^{m,p}(\Omega, \operatorname{dist}^{m-s}) \xrightarrow{f \mapsto (\chi_\nu f)_\nu} \bigoplus_{\nu=0}^M W^{m,p}(U_\nu \cap \Omega, \operatorname{dist}^{m-s}_{b\Omega}) \\ &\hookrightarrow \bigoplus_{\nu=0}^M \mathscr{F}^s_{p\varepsilon}(U_\nu \cap \Omega) \xrightarrow{(g_\nu)_\nu \mapsto \sum_\nu \chi_\nu g_\nu} \mathscr{F}^s_{p\varepsilon}(\Omega) \end{aligned}, \quad s < m.$$

The second composition map gives $W^{m,p}(\Omega, \operatorname{dist}^{m-s}) \hookrightarrow \mathscr{F}^s_{p\varepsilon}(\Omega)$, which completes the proof of (ii).

By suitably shrinking $\mathcal{U}$, we can assume that $\mathcal{U}$ is bounded Lipschitz, and thus $\mathcal{U}\backslash\overline{\Omega}$ is also bounded Lipschitz. The first composition map then gives $\widetilde{\mathscr{F}}^s_{p\infty}(\overline{\mathcal{U}\backslash\Omega}) \hookrightarrow L^p(\mathcal{U}\backslash\overline{\Omega}, \mathrm{dist}^{-s}_{b\Omega\cup b\mathcal{U}}) \hookrightarrow L^p(\mathcal{U}\backslash\overline{\Omega}, \mathrm{dist}^{-s}_{b\Omega})$ for $s > 0$, which completes the proof of (i).

To prove (iii), for convenience, we write $E_\nu g := (E_{\omega_\nu}[g \circ \Phi_\nu]) \circ \Phi_\nu^{-1}$, where g is defined on $\Phi_\nu(\omega_\nu)$ (from (4.13), we recall that $U_\nu \cap \Omega = U_\nu \cap \Phi_\nu(\omega_\nu)$), and thus $\mathcal{E} = \chi_0^2 + \sum_{\nu=1}^M \chi_\nu \circ E_\nu \circ \chi_\nu$, and we have

$$[\overline{\partial}, \mathcal{E}] = 2(\chi_0\overline{\partial}\chi_0) + \sum_{\nu=1}^M \big((\overline{\partial}\chi_\nu) \circ E_\nu \circ \chi_\nu + \chi_\nu \circ E_\nu \circ (\overline{\partial}\chi_\nu) + \chi_\nu \circ [\overline{\partial}, E_\nu] \circ \chi_\nu\big), \tag{5.1}$$

where the function χ_ν is the linear map (pointwise multiplier) $[g \mapsto \chi_\nu g]$.

Since χ_ν are smooth, by Proposition 4.8 (i) and Lemma 4.10, all terms in (5.1), except for $\chi_\nu \circ [\overline{\partial}, E_\nu] \circ \chi_\nu$, have the boundedness $\mathscr{F}^s_{pp}(\Omega) \to \mathscr{F}^s_{pp}(\mathbb{C}^n)$ for all $1 \le p \le \infty$ and $s \in \mathbb{R}$. It is sufficient to show that for $1 \le \nu \le M$, we have $\chi_\nu \circ [\overline{\partial}, E_\nu] \circ \chi_\nu : \mathscr{F}^s_{pp}(\Omega) \to \mathscr{F}^s_{pp}(\mathbb{C}^n)$.

From Remark 2.8, recall that by partition of unity, we can find $M' \ge n - 1$, smooth $(0,1)$-vector fields $\overline{W}_1, \dots, \overline{W}_{M'}$ and $(0,1)$-forms $\overline{\eta}_1, \dots, \overline{\eta}_{M'}$ on $\mathcal{U}\backslash\overline{\Omega}$ such that $\overline{W}_\mu(\zeta) \in T^{0,1}_\zeta(b\Omega_{\varrho(\zeta)})$ for all $1 \le \mu \le M'$ and $\zeta \in \mathcal{U}\backslash\overline{\Omega}$, and $\alpha^\top = \sum_{\mu=1}^{M'} \langle, \overline{W}_\mu, \alpha\rangle \cdot \overline{\eta}_\mu$ holds for all $(0,1)$-form α on $\mathcal{U}\backslash\overline{\Omega}$.

Since $T^{0,1}(b\Omega) \subset \mathbb{C}T(b\Omega)$ and $T_\zeta(\Phi_\nu(b\omega_\nu)) = T_\zeta(b\Omega)$ for $\zeta \in U_\nu \cap b\Omega$, we have $(\chi_\nu W_\mu)(\zeta) \in \mathbb{C}T_\zeta(\Phi_\nu(b\omega_\nu))$ for $\zeta \in U_\nu \cap b\Omega$. By Proposition 5.1, $\langle \overline{W}_\mu, \chi_\nu \circ [d, E_\nu]\rangle = \langle \chi_\nu \cdot \overline{W}_\mu, [\overline{\partial}, E_\nu]\rangle : \mathscr{F}^s_{pp}(\Phi_\nu(\omega_\nu)) \to \widetilde{\mathscr{F}}^s_{pp}(\Phi_\nu(\omega_\nu)^c; \mathbb{C}^n)$ is bounded. Therefore,

$$\langle \overline{W}_\mu, \chi_\nu \circ [\overline{\partial}, E_\nu] \circ \chi_\nu\rangle : \mathscr{F}^s_{pp}(\Omega) \xrightarrow{\chi_\nu} \mathscr{F}^s_{pp}(\Phi_\nu(\omega_\nu)) \xrightarrow{\langle \chi_\nu \cdot \overline{W}_\mu, [\overline{\partial}, E_\nu]\rangle} \widetilde{\mathscr{F}}^s_{pp}(\Phi_\nu(\omega_\nu)^c; \mathbb{C}^n) \xrightarrow{(-)|_{U_\nu}} \mathscr{F}^s_{pp}(\Omega^c; \mathbb{C}^n).$$

We obtain $\langle \overline{W}_\mu, \chi_\nu \circ [\overline{\partial}, E_\nu] \circ \chi_\nu\rangle : \mathscr{F}^s_{pp}(\Omega) \to \widetilde{\mathscr{F}}^s_{pp}(\overline{\mathcal{U}\backslash\Omega}; \mathbb{C}^n)$ since $\operatorname{supp} \chi_\nu \Subset \mathcal{U}$, which gives (iii). $\square$

Remark 5.6. Using the notations from Remark 2.8, for a $(0, q)$-form $f = \sum_J f_J d\bar{z}^J$, we have

$$\begin{aligned}
[\overline{\partial}, \mathcal{E}]^\top f &= \sum_{|K'|=q+1, |J|=q} \langle \overline{Z}_{K'}, [\overline{\partial}, \mathcal{E}](f_J d\bar{z}^J)\rangle \\
&= \sum_{j=1}^n \sum_{k=2}^n \sum_{|J|=|K|=q; \min K \ge 2} ([\overline{\partial}_j, \mathcal{E}] f_J)\langle \overline{Z}_k, d\bar{z}^j\rangle \langle \overline{Z}_K, d\bar{z}^J\rangle \overline{\theta}^k \wedge \overline{\theta}^K \\
&= \sum_{k=2}^n \sum_{|K|=q; \min K \ge 2} \big(\langle \overline{Z}_k, [\overline{\partial}, \mathcal{E}] f_J\rangle\big) \cdot \langle \overline{Z}_K, d\bar{z}^J\rangle \overline{\theta}^k \wedge \overline{\theta}^K.
\end{aligned}$$

Therefore, to estimate $[\overline{\partial}, \mathcal{E}]^\top f$, it is sufficient to estimate its components $[\overline{\partial}, \mathcal{E}]^\top f_J$.

We prove Proposition 5.3 first, but we only prove the case where $\varepsilon = 1$ for (ii) and leave the proof of $\varepsilon < 1$ to the appendix.

Proof of Proposition 5.3. Write $\omega = \{x_1 > \sigma(x')\}$. We define the outer strips

$$S_k = S_k^\omega := \{(x_1, x') : -2^{\frac{1}{2}-k} < x_1 - \sigma(x') < -2^{-\frac{1}{2}-k}\} \subset \omega^c \text{ for } k \in \mathbb{Z}_+, \tag{5.2}$$

and $S_0 := \{(x_1, x') : x_1 - \sigma(x') < -2^{-\frac{1}{2}}\}$. Recall that $\delta(x) = \min(1, \mathrm{dist}(x, b\omega))$ in the assumption. Recall that a special Lipschitz domain satisfies $\|\nabla\sigma\|_{L^\infty} < 1$, and thus (also see [61, (5.3)])

$$2^{-1-k} \le \delta(x) \le 2^{\frac{1}{2}-k}, \quad \forall k \ge 0, \quad x \in S_k. \tag{5.3}$$

Since $\{S_k\}_{k=0}^\infty$ is a partition of ω^c up to zero measured sets, we have $\|g\|_{L^p(\omega)} = \|(\|g\|_{L^p(S_k)})_{k=0}^\infty\|_{\ell^p(\mathbb{N})}$.

By the assumption (4.10), we see that ϕ_0, ϕ_1 are supported in $-\mathbb{K} \cap \{x_1 < -c_1\}$ for some $c_1 > 0$. By the assumption (4.7), we obtain $\operatorname{supp} \phi_j \subset -\mathbb{K} \cap \{x_1 < -c_1 2^{-j}\}$ for all $j \geq 0$. A simple calculation shows that (see [61, Lemma 5.3])

$$\exists R \in \mathbb{Z}_+ \text{ such that } \operatorname{supp} \phi_j + \omega^c \subseteq \{x_1 - \sigma(x') < -2^{-j-R}\}. \tag{5.4}$$

Let $f \in \widetilde{\mathscr{F}}^s_{p\infty}(\omega^c)$. We have $\operatorname{supp} f \subseteq \omega^c$. By (4.9), we have $f = \sum_{j=0}^\infty \phi_j * f$. Using (5.4), we see that $f(x) = \sum_{j=k-R}^\infty \phi_j * f(x)$ for $k \geq 0$ and $x \in S_k$. Thus,

$$\begin{aligned}
&\|f\|_{L^p(\overline{\omega}^c,\delta^{-s})} = \big\|\big(\|f\|_{L^p(S_k,\delta^{-s})}\big)_{k=0}^\infty\big\|_{\ell^p(\mathbb{N})} \leq \big\|\big(2^{(k+1)s}\|f\|_{L^p(S_k)}\big)_{k=0}^\infty\big\|_{\ell^p(\mathbb{N})} \\
\leq &\left\|\left(\left\|2^{(k+1)s}\sum_{j=k-R}^\infty |\phi_j * f|\right\|_{L^p(S_k)}\right)_{k=0}^\infty\right\|_{\ell^p(\mathbb{N})} = \left\|\left(\left\|\sum_{j=k-R}^\infty 2^{(k-j+1)s}|2^{js}\phi_j * f|\right\|_{L^p(S_k)}\right)_{k=0}^\infty\right\|_{\ell^p(\mathbb{N})} \\
\leq &\sum_{l=-R}^\infty 2^{(1-l)s}\left\|\left(\left\|\sup_{j\geq k-R}|2^{js}\phi_j * f|\right\|_{L^p(S_k)}\right)_{k=0}^\infty\right\|_{\ell^p(\mathbb{N})} \lesssim_{R,s} \left\|\sup_{j\in\mathbb{N}}|2^{js}\phi_j * f|\right\|_{L^p(\omega^c)} = \|f\|_{\mathscr{F}^s_{p\infty}(\phi)}.
\end{aligned}$$

We use the convention of $\phi_j * f = 0$ for $j \leq -1$. By [58, Proposition 1.2 (i)], $\|f\|_{\mathscr{F}^s_{p\infty}(\phi)}$ is an equivalent norm for $\mathscr{F}^s_{p\infty}(\mathbb{R}^N)$. This completes the proof of (i).

For (ii), we prove the case where $\varepsilon = 1$ by duality argument. We leave the proof of $0 < \varepsilon < 1$ to the appendix, which is obtained as a direct proof without using duality.

Let $\mathring{\mathscr{F}}^s_{pq}(\mathbb{R}^N)$ be the norm closure of $C_c^\infty(\mathbb{R}^N)$ in $\mathscr{F}^s_{pq}(\mathbb{R}^N)$ and let $\mathring{\mathscr{F}}^s_{pq}(\overline{\omega}) := \{f \in \mathring{\mathscr{F}}^s_{pq}(\mathbb{R}^N) : f|_{\overline{\omega}^c} = 0\}$ be its subspace. Clearly, $\mathring{\mathscr{F}}^s_{pq}(\overline{\omega}) \subseteq \widetilde{\mathscr{F}}^s_{pq}(\overline{\omega})$. We see that $\mathring{\mathscr{F}}^s_{pq}(\overline{\omega}) = \overline{C_c^\infty(\omega)}^{\mathscr{F}^s_{pq}(\mathbb{R}^N)}$, and the proof follows according to the same argument for [64, Theorem 4.3.2/1 Proof Step 2] via partition of unity and translations.

In addition, by [68, Remark 1.5], we have $\mathscr{F}^{-s}_{p1}(\mathbb{R}^N) = \mathring{\mathscr{F}}^s_{p'\infty}(\mathbb{R}^N)'$ for all $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, where $p' = \frac{p}{p-1}$. Therefore, (also see [64, Theorem 2.10.5/1]) for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$,

$$\begin{aligned}
\mathring{\mathscr{F}}^s_{p'\infty}(\overline{\omega})' =&\mathscr{F}^{-s}_{p1}(\mathbb{R}^N)/\{f : \langle f, \phi\rangle = 0, \forall \phi \in \mathring{\mathscr{F}}^s_{p'\infty}(\mathbb{R}^N),\ \phi|_{\overline{\omega}^c} = 0\} \\
=&\mathscr{F}^{-s}_{p1}(\mathbb{R}^N)/\{f : \langle f, \phi\rangle = 0, \forall \phi \in C_c^\infty(\omega)\} = \mathscr{F}^{-s}_{p1}(\mathbb{R}^N)/\{f : f|_\omega = 0\} = \mathscr{F}^{-s}_{p1}(\omega).
\end{aligned}$$

By result (i), we have $\mathscr{F}^t_{p'\infty}(\overline{\omega}) \hookrightarrow L^{p'}(\omega, \delta^{-t})$ for $t > 0$. In particular, $\mathring{\mathscr{F}}^t_{p'\infty}(\overline{\omega}) \hookrightarrow \overline{C_c^\infty(\omega)}^{L^{p'}(\omega,\delta^{-t})}$. Clearly, $\overline{C_c^\infty(\omega)}^{L^{p'}(\omega,\delta^{-t})} = L^{p'}(\omega, \delta^{-t})$ if $1 < p \leq \infty$, and by taking the adjoint, we obtain $L^p(\omega, \delta^t) \hookrightarrow \mathscr{F}^{-t}_{p1}(\omega)$ for $1 < p \leq \infty$, which is (ii) at $m = 0$ and $s = -t < 0$.

For $p = 1$, we have $\overline{C_c^\infty(\omega)}^{L^{p'}(\omega,\delta^{-t})} = \{f \in C^0(\overline{\omega}) : \lim_{x\to b\omega} \operatorname{dist}_{b\omega}(x)^{-t}f(x) = 0 \text{ uniformly}\}$. Thus, the adjoint gives the embedding $\{f \in \mathscr{M}_{\mathrm{loc}}(\omega) : \|\delta^t f\|_{\mathscr{M}} < \infty\} \hookrightarrow \mathscr{F}^{-t}_{11}(\omega)$ from the space of locally finite Borel measures.[4] Since $L^1(\omega, \delta^t) \subset \{f \in \mathscr{M}_{\mathrm{loc}}(\omega) : \|\delta^t f\|_{\mathscr{M}} < \infty\}$ is a closed subspace, we obtain $L^1(\omega, \delta^t) \hookrightarrow \mathscr{F}^{-t}_{11}(\omega)$, which is (ii) for $m = 0$, $p = 1$, and $s = -t < 0$.

For $m \geq 1$, we recall that $\|f\|_{W^{m,p}(\omega,\delta^{s-m})} \approx \sum_{|\alpha|\leq m}\|D^\alpha f\|_{L^p(\omega,\delta^{s-m})}$. Therefore, for every $m \geq 0$ and $s < m$, we have $\sum_{|\alpha|\leq m}\|D^\alpha f\|_{\mathscr{F}^{s-m}_{p1}(\omega)} \lesssim \|f\|_{L^p(\omega,\delta^{s-m})}$. In addition, by Proposition 4.8 (iii), we have $\|f\|_{\mathscr{F}^s_{p1}(\omega)} \approx \sum_{|\alpha|\leq m}\|D^\alpha f\|_{\mathscr{F}^{s-m}_{p1}(\omega)}$. By combining them, we obtain (ii) for $\varepsilon = 1$ and the proof is complete. $\square$

To prove Proposition 5.1, we use a version of the Heideman-type estimate [36].

[4] We use $\mathscr{M}_{\mathrm{loc}}(\Omega)$ for the space of locally finite signed Borel measures, where the norm $\|\cdot\|_{\mathscr{M}}$ is the total variation in a measure.

Lemma 5.7. *Let $\phi = (\phi_j)_{j=0}^\infty$ and $\psi = (\psi_j)_{j=0}^\infty$ be as given in Definition 4.7. Then, for any $M > 0$, $\alpha, \beta \in \mathbb{N}$, and $g \in \mathscr{C}^\infty(\mathbb{R}^N)$, a $C = C_{\phi,\psi,M,\alpha,\beta,g} > 0$ exists such that for every $j, k \in \mathbb{N}$, $1 \le p \le \infty$, and $f \in L^p(\mathbb{R}^N)$,*

$$\|\phi_j * (g \cdot (D^\alpha \psi_k * f)) - \phi_j * D^\alpha \psi_k * (gf)\|_{L^p(\mathbb{R}^N)} \le C 2^{k|\alpha| - M|j-k| - k} \|f\|_{L^p(\mathbb{R}^N)}. \tag{5.5}$$

We can write the left-hand side of (5.5) as $\left\|\phi_j * \left([g \cdot (-), D^\alpha \psi_k * (-)]\{f\}\right)\right\|_{L^p}$ in terms of the commutator.

Proof. The direct computation yields

$$\begin{aligned}
&\phi_j * (g \cdot (D^\alpha \psi_k * f))(x) - \phi_j * D^\alpha \psi_k * (gf)(x) \\
=&\phi_j * \left[t \mapsto \int f(y)(g(t) - g(y)) D^\alpha \psi_k(t-y) dy\right](x) \\
=& \int\limits_{\mathbb{R}^N} f(y) dy \int\limits_{\mathbb{R}^N} (g(t) - g(y)) D^\alpha \psi_k(t-y) \phi_j(x-t) dt =: \int\limits_{\mathbb{R}^N} K_{jk}(x,y) f(y) dy,
\end{aligned}$$

where $K_{jk}(x,y) = \int_{\mathbb{R}^N} (g(t) - g(y)) D^\alpha \psi_k(t-y) \phi_j(x-t) dt$.

By Schur's test Lemma 3.15 with $\gamma = 1$ and $(X, \mu) = (Y, \nu) = (\mathbb{R}^N, dx)$, we need to prove that

$$\sup_x \int |K_{jk}(x,y)| dy + \sup_y \int |K_{jk}(x,y)| dx \lesssim_{\phi,\psi,\alpha,\beta,g,M} 2^{k|\alpha| - M|j-k| - k}.$$

Let $M' \ge 0$ be as selected later and by Taylor's expansion, we can write

$$K_{jk}(x,y) = \int \Big(\sum_{0 < |\gamma| \le M'} \frac{s^\gamma}{\gamma!} D^\gamma g(y) + R_{M'}(y,s) \Big) D^\alpha \psi_k(s) \phi_j(x-y-s) ds,$$

where $R_{M'}(y,s) := g(y+s) - \sum_{0 \le |\gamma| \le M'} \frac{s^\gamma}{\gamma!} D^\gamma g(y)$ is the Taylor's remainder in s-variable. Therefore,

$$|K_{jk}(x,y)| \lesssim_{M',g} \sum_{0 < |\gamma| \le M'} \left| \left(\left[s \mapsto s^\gamma D^\alpha \psi_k(s) \right] * \phi_j \right)(x-y) \right| + \left| \left(\left(R_{M'}(y,\cdot) D^\alpha \psi_k \right) * \phi_j \right)(x-y) \right|. \tag{5.6}$$

In addition, note that for every $\gamma \in \mathbb{N}^N$, we have the scaling $\phi_k(x) = 2^{(k-1)N} \phi_1(2^{k-1}x)$ for $k \ge 1$ and

$$x^\gamma D^\alpha \psi_k(x) = 2^{(k-1)(N + |\alpha| - |\gamma|)} (2^{k-1}x)^\gamma D^\alpha \psi_1(2^{k-1}x), \quad k \ge 1. \tag{5.7}$$

Both ϕ_k and $x^\gamma D^\alpha \psi_k$ have infinite moment vanishing for $k \ge 1$, so by [6, Lemma 2.1] again (also see [62, Lemma 4.4] with $l \to +\infty$), for every $M > 0$ and $|\gamma| > 0$, we have,

$$\|\phi_j * (s^\gamma D^\alpha \psi_k)\|_{L^1} \lesssim_{\phi,\psi,\alpha,\beta,\gamma,M} 2^{k(|\alpha| - |\gamma|) - M|j-k|} \le 2^{k(|\alpha| - 1) - M|j-k|}, \quad \text{for all } j, k \ge 0. \tag{5.8}$$

Moreover, by [6, Lemma 2.1] (also see [61, Proposition 3.5]), for every $M > 0$, we have,

$$\|\phi_j * h\|_{L^1} \lesssim_{\phi,\alpha,M} 2^{-jM} \sup_{|\gamma| \le 2M+N; x \in \mathbb{R}^N} (1 + |x|^{2M+N}) |D^\gamma h(x)|, \quad \text{for all } j \ge 0, \quad h \in \mathscr{S}(\mathbb{R}^N). \tag{5.9}$$

By taking $h = h_{M',y} = R_{M'}(y,\cdot) D^\alpha \psi_k$ in (5.9) and applying Taylor's theorem to $R_{M'}(y,\cdot)$, we obtain

$$\begin{aligned}
(1 + |s|^{2M+N}) \sum_{|\gamma| \le 2M+N} |D^\gamma h_{M',y}(s)| &\lesssim (1 + |s|^{2M+N}) \sum_{|\beta|, |\gamma| \le 2M+N} |D_s^\beta R_{M'}(y,s)| |D^{\alpha+\gamma} \psi_k(s)| \\
&\lesssim (1 + |s|^{2M+N}) |s|^{M' - 2M - N} \sum_{|\gamma| \le 2M+N} |D^{\alpha+\gamma} \psi_k(s)|,
\end{aligned} \tag{5.10}$$

uniformly in $s, y \in \mathbb{R}^N$ whenever $M' > 2M + N$.

For $k \geq 1$, by using the scaling property (5.7) and the fact that ψ_1 is Schwartz, for every s and y, we uniformly have

$$
\begin{aligned}
&(1+|s|^{2M+N})|s|^{M'-2M-N} \textstyle\sum_{|\gamma|\leq 2M+N} |D^{\alpha+\gamma}\psi_k(s)| \\
\lesssim &(1+|s|^{2M+N})|s|^{M'-2M-N} \textstyle\sum_{|\gamma|\leq 2M+N} 2^{kN+k|\alpha+\gamma|}(1+|2^k s|)^{-M'} \\
\lesssim &\tfrac{1+|s|^{2M+N}}{1+|2^k s|^{2M+N}} \cdot \tfrac{|s|^{M'-2M-N}}{1+|2^k s|^{M'-2M-N}} 2^{k(2M+2N)+k|\alpha|} \lesssim 2^{-k(M'-2M-N)} 2^{k(2M+2N+|\alpha|)}.
\end{aligned}
\tag{5.11}
$$

In summary, by (5.9), (5.10), and (5.11),

$$
\sup_{y\in\mathbb{R}^N} \left\| \left(R_{M'}(y,\cdot) D^\alpha \psi_k\right) * \phi_j \right\|_{L^1(\mathbb{R}^N)} \lesssim 2^{-jM} 2^{-kM'} 2^{k(4M+3N+|\alpha|)}. \tag{5.12}
$$

Therefore, by (5.8) and by taking $M' = 4M + 3N + 1$ in (5.12), we have

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \left(|K_{jk}(x,t)| + |K_{jk}(t,x)|\right) dt \\
&\lesssim \sum_{0<|\gamma|\leq M'} \|\phi_j * (s^\gamma D^\alpha \psi_k)\|_{L^1(\mathbb{R}^N)} + \sup_y \|\phi_j * (R_{M'}(y,\cdot) D^\alpha \psi_k)\|_{L^1(\mathbb{R}^N)} \\
&\lesssim_M 2^{k(|\alpha|-1)-M|j-k|} + 2^{-jM+k(4M+3N+|\alpha|-M')} \\
&\lesssim_M 2^{k(|\alpha|-1)-M|j-k|} + 2^{k(|\alpha|-1)-M(j+k)} \lesssim 2^{k(|\alpha|-1)-M|j-k|}.
\end{aligned}
$$

This completes the proof. $\square$

Proof of Proposition 5.1. First, we claim that $X\mathbf{1}_\omega = \sum_{\nu=1}^N X^\nu D_\nu \mathbf{1}_\omega = 0$ holds in the sense of distributions. We use approximation. The assumption that $X(x) \in \mathbb{C}T_x(b\omega)$ for almost every $x \in b\omega$ gives

$$
X^1(\sigma(x'), x') = \textstyle\sum_{\nu=2}^N X^\nu(\sigma(x'), x') \cdot \frac{\partial \sigma}{\partial x_\nu}(x'), \quad \text{for almost every } x' \in \mathbb{R}^{N-1}.
$$

For $\delta > 0$, we define

$$
\begin{aligned}
&h_\delta(x) := 0 \quad \text{when } x_1 \leq \sigma(x') - \tfrac{\delta}{2}; \qquad h_\delta(x) := \tfrac{x_1 - \sigma(x')}{\delta} \quad \text{when } |x_1 - \sigma(x')| \leq \tfrac{\delta}{2}; \\
&h_\delta(x) := 1 \quad \text{when } x_1 \geq \sigma(x') + \tfrac{\delta}{2}.
\end{aligned}
$$

Clearly, $h_\delta \to \mathbf{1}_\omega$ as $\delta \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 < p < \infty$.

Clearly, $Xh_\delta(x) = 0$ for $|x_1 - \sigma(x')| > \frac{\delta}{2}$. For $|x_1 - \sigma(x')| < \frac{\delta}{2}$,

$$
\begin{aligned}
Xh_\delta(x) &= \textstyle\sum_{\nu=1}^N X^\nu(x) D_\nu h_\delta(x) = \delta^{-1} X^1(x) - \delta^{-1} \sum_{\nu=2}^N X^\nu(x) D_\nu \sigma(x') \\
&= \tfrac{X^1(x) - X^1(\sigma(x'), x')}{\delta} - \textstyle\sum_{\nu=2}^N \tfrac{X^\nu(x) - X^\nu(\sigma(x'), x')}{\delta} D_\nu \sigma(x') \\
&= (D_1 X^1)(\sigma(x'), x') - \textstyle\sum_{\nu=2}^N (D_1 X^\nu)(\sigma(x'), x') D_\nu \sigma(x') + O(\delta).
\end{aligned}
$$

We conclude that $Xh_\delta \in L^\infty(\mathbb{R}^N)$ is uniformly bounded in δ and $Xh_\delta = 0$ outside a δ-neighborhood of $b\Omega$. Therefore, $Xh_\delta \xrightarrow{L^p_{\mathrm{loc}}} 0$ for all $p < \infty$, and hence $X\mathbf{1}_\omega = \lim_{\delta\to 0} Xh_\delta = 0$ as distributions.

Now, we can rewrite $\langle X, [d, E]\rangle$ as

$$\langle X, [d, E]\rangle f = \sum_{\nu=1}^{N}\sum_{k=0}^{\infty} X^\nu \cdot (\psi_k * ((D_\nu \mathbf{1}_\omega)\cdot(\phi_k * f)))$$

$$= \sum_{\nu=1}^{N}\sum_{k=0}^{\infty} X^\nu \cdot (\psi_k * ((D_\nu \mathbf{1}_\omega)\cdot(\phi_k * f))) - \psi_k * ((X^\nu D_\nu \mathbf{1}_\omega)\cdot(\phi_k * f))$$

$$= \sum_{\nu=1}^{N}\sum_{k=0}^{\infty} \Big(X^\nu \cdot (D_\nu\psi_k * (\mathbf{1}_\omega \cdot (\phi_k * f)) - D_\nu\psi_k * ((X^\nu \mathbf{1}_\omega)\cdot(\phi_k * f))$$

$$- \big(X^\nu \cdot (\psi_k * (\mathbf{1}_\omega \cdot (D_\nu\phi_k * f))) - \psi_k * ((X^\nu\mathbf{1}_\omega)\cdot(D_\nu\phi_k * f))\big) - \psi_k * ((D_\nu X^\nu)\cdot \mathbf{1}_\omega \cdot (\phi_k * f))\Big)$$

$$= \sum_{\nu=1}^{N}\sum_{k=0}^{\infty} \Big([X^\nu, D_\nu\psi_k * (-)]\big\{\mathbf{1}_\omega(\phi_k * f)\big\} - [X^\nu, \psi_k * (-)]\big\{\mathbf{1}_\omega(D_\nu\phi_k * f)\big\} + \psi_k * ((D_\nu X^\nu)\mathbf{1}_\omega(\phi_k * f))\Big).$$

Note that $(\phi_j)_{j=0}^\infty$ satisfies conditions (4.7), (4.8), and (4.9). By [58, Proposition 1.2], we have

$$\|\langle X, [d, E]\rangle f\|_{\mathscr{F}^s_{pp}(\mathbb{R}^N)} \approx \big\|\big(2^{js}\|\phi_j * (\langle X, [d, E]\rangle f)\|_{L^p(\mathbb{R}^N)}\big)_{j=0}^\infty\big\|_{\ell^p(\mathbb{N})}. \tag{5.13}$$

Moreover, by applying Lemma 5.7, we obtain

$$\big\|\phi_j * \big([X^\nu \cdot (-), D^\alpha\psi_k * (-)]\big\{\mathbf{1}_\omega(D^\beta\phi_k * f)\big\}\big)\big\|_{L^p(\mathbb{R}^N)} \lesssim_M 2^{j|\alpha| - M|j-k| - k}\|D^\beta\phi_k * f\|_{L^p(\omega)}. \tag{5.14}$$

By [6, Lemma 2.1] (also see [61, Corollary 3.6]), we have

$$\|\phi_j * \psi_k((D_\nu X^\nu)\mathbf{1}_\omega(\phi_k * f))\|_{L^p(\mathbb{R}^N)} \le \|\phi_j * \psi_k\|_{L^1}\|D_\nu X^\nu\|_{L^\infty}\|\phi_k * f\|_{L^p(\omega)} \lesssim_M 2^{-M|j-k|}\|\phi_k * f\|_{L^p(\omega)}. \tag{5.15}$$

By plugging (5.14) and (5.15) into (5.13), for every $M > 0$, we obtain

$$\begin{aligned}
&\|\langle X, [d, E]\rangle f\|_{\mathscr{F}^s_{pp}(\mathbb{R}^N)} \\
\lesssim_M &\Big\|\Big(2^{js}\sum_{k=0}^\infty \big(2^{-M|j-k|+k-k}\|\phi_k * f\|_{L^p(\omega)} + 2^{-M|j-k|-k}\|D\phi_k * f\|_{L^p(\omega)} \\
&\quad + 2^{-M|j-k|}\|\phi_k * f\|_{L^p(\omega)}\big)\Big)_{j=0}^\infty\Big\|_{\ell^p(\mathbb{N})} \\
\lesssim_M &\Big\|\Big(\sum_{k=0}^\infty 2^{-(M-|s|)|j-k|}2^{ks}\|\phi_k * f\|_{L^p(\omega)}\Big)_{j=0}^\infty\Big\|_{\ell^p(\mathbb{N})} \\
&\quad + \Big\|\Big(\sum_{k=0}^\infty 2^{-(M-|s|)|j-k|}2^{k(s-1)}\|D\phi_k * f\|_{L^p(\omega)}\Big)_{j=0}^\infty\Big\|_{\ell^p(\mathbb{N})} \\
\le &\|(2^{-(M-|s|)|j|})_{j=-\infty}^\infty\|_{\ell^1}\Big(\big\|\big(2^{ks}\|\phi_k * f\|_{L^p(\omega)}\big)_{k=0}^\infty\big\|_{\ell^p} + \big\|\big(2^{k(s-1)}\|\phi_k * Df\|_{L^p(\omega)}\big)_{k=0}^\infty\big\|_{\ell^p}\Big).
\end{aligned}$$

The last inequality above is obtained by Young's inequality on $\mathbb{Z}$. Since M is arbitrary, by taking $M > |s|$, we have $\|(2^{-(M-|s|)|j|})_{j=-\infty}^\infty\|_{\ell^1} < \infty$.

By Proposition 4.8 (ii), we have $\big\|\big(2^{ks}\|\phi_k * f\|_{L^p(\omega)}\big)_{k=0}^\infty\big\|_{\ell^p} \approx \|f\|_{\mathscr{F}^s_{pp}(\omega)}$ and $\big\|\big(2^{k(s-1)}\|\phi_k * Df\|_{L^p(\omega)}\big)_{k=0}^\infty\big\|_{\ell^p} \approx \|Df\|_{\mathscr{F}^{s-1}_{pp}(\omega)} \lesssim \|f\|_{\mathscr{F}^s_{pp}(\omega)}$. Therefore, $\|\langle X, [d, E]\rangle f\|_{\mathscr{F}^s_{pp}(\mathbb{R}^N)} \lesssim \|f\|_{\mathscr{F}^s_{pp}(\omega)}$ and the proof is complete. $\square$

6. Proof of the theorems

We now consider the complex domain $\mathbb{C}^n$ and assume that $\Omega \subset \mathbb{C}^n$ is a bounded smooth domain of finite type. Let U_1 be a fixed neighborhood of $b\Omega$ obtained from Lemma 2.2, m_q be the q-type of Ω, and $r_q := (n-q+1)\cdot m_q + 2q$ for $1 \le q \le n$.

Recall the space $\widetilde{\mathscr{F}}^s_{p\infty}(\overline{\mathcal{U}})$ in Definition 4.5. We recall that the Bochner–Martinelli kernels always gain one derivative.

Lemma 6.1. *Let $\mathcal{U} \subset \mathbb{C}^n$ be a bounded set. Then, the Bochner–Martinelli integral $\mathcal{B}_q g(z) = \int_{\mathcal{U}} B_{q-1}(z,\cdot)\wedge g$ has boundedness $\mathcal{B}_q : \widetilde{\mathscr{F}}^s_{pr}(\overline{\mathcal{U}};\wedge^{0,q}) \to \mathscr{F}^{s+1}_{pr}(\mathcal{U};\wedge^{0,q-1})$ for all $0 < p,r \le \infty$ and $s \in \mathbb{R}$ such that $(p,r) \notin \{\infty\}\times(0,\infty)$.*

The boundedness $\mathcal{B}_q : \widetilde{\mathscr{F}}^s_{\infty q} \to \mathscr{F}^{s+1}_{\infty q}$ is also true by using [68, Theorem 1.22].

Proof. The proof is standard. We can see that $B_{q-1}(z,\zeta)$ is simply the linear combination of the derivatives of the Newtonian potential $G(z-\zeta) := -\frac{(n-2)!}{4\pi^n}|z-\zeta|^{2-2n}$. We need to prove $[f \mapsto G*f] : \widetilde{\mathscr{F}}^s_{pr}(\overline{\mathcal{U}}) \to \mathscr{F}^{s+2}_{pr}(\mathcal{U})$ for all $0 < p,r \le \infty$ and $s \in \mathbb{R}$ such that $(p,r) \notin \{\infty\}\times(0,\infty)$.

Indeed, let $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ be such that its Fourier transform has compact support. We define $G_0 := \chi * G$ and $G_\infty = G - G_0$. We see that the Fourier transform $\hat{G}_\infty(\xi) = (1-\hat{\chi}(\xi))|\xi|^{-2}$ ($\xi \in \mathbb{R}^{2n}$) is a bounded smooth function. Therefore, by Hörmander–Mikhlin multiplier theorem (see [67, Theorem 2.3.7]), we have $[f \mapsto (I-\Delta)G_\infty * f] : \mathscr{F}^s_{pr}(\mathbb{R}^{2n}) \to \mathscr{F}^s_{pr}(\mathbb{R}^{2n})$. Note that $(I-\Delta)^{-2} : \mathscr{F}^s_{pr}(\mathbb{R}^{2n}) \to \mathscr{F}^{s+2}_{pr}(\mathbb{R}^{2n})$ is bounded (see [67, Theorem 2.3.8]). Therefore, $[f \mapsto G_\infty * f] : \widetilde{\mathscr{F}}^s_{pr}(\overline{\mathcal{U}}) \to \mathscr{F}^{s+2}_{pr}(\mathcal{U})$ is also bounded.

In addition, the Fourier support $\operatorname{supp}\hat{G}_0 \subseteq \operatorname{supp}\hat{\chi}$ is compact, so we see that $G_0 \in C^\infty_{\mathrm{loc}}(\mathbb{R}^{2n})$. $\mathcal{U}$ is a bounded set, so we have $[f \mapsto G_0 * f] : \{f \in \mathscr{S}'(\mathbb{R}^{2n}) : \operatorname{supp} f \subseteq \overline{\mathcal{U}}\} \to C^\infty_{\mathrm{loc}}(\mathbb{R}^{2n})$; in particular, $[f \mapsto G_0 * f] : \widetilde{\mathscr{F}}^s_{pr}(\overline{\mathcal{U}}) \to \mathscr{F}^{s+2}_{pr}(\mathcal{U})$.

Now, $[f \mapsto G * f] : \widetilde{\mathscr{F}}^s_{pr}(\overline{\mathcal{U}}) \to \mathscr{F}^{s+2}_{pr}(\mathcal{U})$ is bounded. The boundedness of $\mathcal{B}_q : \mathscr{F}^s_{pr} \to \mathscr{F}^{s+1}_{pr}$ follows from the fact that $\nabla : \mathscr{F}^{s+2}_{pr}(\mathbb{R}^{2n}) \to \mathscr{F}^{s+1}_{pr}(\mathbb{R}^{2n};\mathbb{C}^{2n})$ (see [67, Theorem 2.3.8]). $\square$

Proof of Theorems 1.1 and 1.2. We prove the definedness of $\mathcal{H}_q$ on $\mathscr{S}'(\Omega;\wedge^{0,q})$ and the homotopy formula $f = \overline{\partial}\mathcal{H}_q f + \mathcal{H}_{q+1}\overline{\partial} f$ for $f \in \mathscr{S}'$ after giving the proof of the boundedness of $\mathcal{H}_q$ on Triebel–Lizorkin spaces.

Recall the Rychkov's extension operator $\mathcal{E} = \mathcal{E}_\Omega$ in Definition 4.7 and the anti-derivative operators $\mathcal{S}^{k,\alpha} = \mathcal{S}^{k,\alpha}_\Omega$ in Proposition 4.13 (also see Remark 4.14). For $k \ge 0$, we define $\mathcal{H}^k_q := \mathcal{B}_q \circ \mathcal{E} + \mathcal{H}^{k,\top}_q + \mathcal{H}^{k,\perp}_q$ as follows, where $\mathcal{B}_q$ is given in Lemma 6.1 and $\mathcal{K}^\top_{q,\alpha}, \mathcal{K}^\perp_{q,\alpha}$ are in Corollary 3.13,

$$\mathcal{H}^{k,\top}_q f(z) := \sum_{|\alpha|\le k}(-1)^{|\alpha|}\mathcal{K}^\top_{q,\alpha}\circ \mathcal{S}^{k,\alpha}[\overline{\partial},\mathcal{E}]f = \sum_{|\alpha|\le k}(-1)^{|\alpha|}\int_{U_1\setminus\overline{\Omega}} (D^\alpha_\zeta(K^\top_{q-1}))(z,\cdot)\wedge \mathcal{S}^{k,\alpha}[\overline{\partial},\mathcal{E}]f;$$

$$\mathcal{H}^{k,\perp}_q f(z) := \sum_{|\alpha|\le k}(-1)^{|\alpha|}\mathcal{K}^\perp_{q,\alpha}\circ \mathcal{S}^{k,\alpha}[\overline{\partial},\mathcal{E}]^\top f = \sum_{|\alpha|\le k}(-1)^{|\alpha|}\int_{U_1\setminus\overline{\Omega}} (D^\alpha_\zeta(K^\perp_{q-1}))(z,\cdot)\wedge \mathcal{S}^{k,\alpha}[\overline{\partial},\mathcal{E}]^\top f.$$

Clearly, $\mathcal{H}^0_q = \mathcal{H}_q$. We now prove $\mathcal{H}^{k,(\top,\perp)}_q = \mathcal{H}^{0,(\top,\perp)}_q$ for all $k \ge 0$; in particular, that $\mathcal{H}^k_q = \mathcal{H}_q$ holds.

By applying Lemma 6.1, for every $1 \le p,r \le \infty$ and $s \in \mathbb{R}$ such that $(p,r) \notin \{\infty\}\times[1,\infty)$, we have

$$\mathcal{B}_q\circ\mathcal{E} :\ \mathscr{F}^s_{pr}(\Omega;\wedge^{0,q}) \xrightarrow{\mathcal{E}} \widetilde{\mathscr{F}}^s_{pr}(\overline{U_1\cup\Omega};\wedge^{0,q}) \xrightarrow{\mathcal{B}_q} \mathscr{F}^{s+1}_{pr}(U_1\cup\Omega;\wedge^{0,q-1}) \xrightarrow{(-)|_\Omega} \mathscr{F}^{s+1}_{pr}(\Omega;\wedge^{0,q-1}). \tag{6.1}$$

When $q = n$, we have $\mathcal{H}_n = \mathcal{B}_n\circ\mathcal{E}$ since $K_n(z,\zeta) \equiv 0$. Therefore, by Remark 4.6 (vi), we obtain the boundedness $\mathcal{H}_n : H^{s,p}(\Omega;\wedge^{0,n}) \to H^{s+1,p}(\Omega;\wedge^{0,n-1})$ and $\mathcal{H}_n : \mathscr{C}^s(\Omega;\wedge^{0,n}) \to \mathscr{C}^{s+1}(\Omega;\wedge^{0,n-1})$ for all $1 < p < \infty$ and $s \in \mathbb{R}$, which is Theorem 1.1 (ii) when $q = n$.

We have the Sobolev embedding $H^{s+1,p}(\mathbb{C}^n) \hookrightarrow H^{s,\frac{2np}{p-2n}}(\mathbb{C}^n)$ for $1 < p < 2n$ (see [68, Corollary 2.7]). By taking restrictions on Ω, we have $\mathcal{H}_n : H^{s,p}(\Omega; \wedge^{0,n}) \to H^{s,\frac{2np}{p-2n}}(\Omega; \wedge^{0,n-1})$. Since $r_n = 1 + 2n > 2n$, then the embedding $H^{s,\frac{2np}{p-2n}}(\Omega; \wedge^{0,n-1}) \hookrightarrow H^{s,\frac{pr_n}{p-r_n}}(\Omega; \wedge^{0,n-1})$ follows immediately, which proves Theorem 1.1 (iii) when $q = n$.

Now, we assume that $q \le n-1$ in the following.

By applying Lemma 6.1, and Remarks 4.12 and 4.6 (iii) to (6.1), we see that

$$\mathcal{B}_q \circ \mathcal{E} : \mathscr{F}^s_{p\infty}(\Omega; \wedge^{0,q}) \to \widetilde{\mathscr{F}}^s_{p\infty}(\overline{U_1 \cup \Omega}; \wedge^{0,q}) \to \mathscr{F}^{s+1}_{p\infty}(\Omega; \wedge^{0,q-1}) \hookrightarrow \begin{cases} \mathscr{F}^{s+\frac{1}{m_q}}_{p\varepsilon}(\Omega; \wedge^{0,q-1}) & \text{for all } 1 \le p \le \infty \\ \mathscr{F}^s_{\frac{pr_q}{r_q-p},\varepsilon}(\Omega; \wedge^{0,q-1}) & \text{when } 1 \le p \le r_q \end{cases}.$$

Note, that if we write $f = \sum_{|I|=q} f_I d\overline{\zeta}^I$, then $[\overline{\partial}, \mathcal{E}]^\top f$ is the linear combination of $[\overline{\partial}, \mathcal{E}]^\top f_I$. Therefore, by Corollary 5.5 (iii), we have $[\overline{\partial}, \mathcal{E}]^\top : \mathscr{F}^s_{p\infty}(\Omega; \wedge^{0,q}) \to \widetilde{\mathscr{F}}^{s-1/m_q}_{p\infty}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1})$.

For every $s > 1-k$ and integer $l > \max(0, s+1)$, by applying Remarks 4.12 and 4.14, and Corollaries 5.5 and 3.13, for every $1 \le p \le \infty$ and $\varepsilon > 0$, we have,

$$\begin{aligned}
\mathcal{H}^{k,\top}_q : \mathscr{F}^s_{p\infty}(\Omega; \wedge^{0,q}) &\xrightarrow{[\overline{\partial},\mathcal{E}]} \widetilde{\mathscr{F}}^{s-1}_{p\infty}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1}) \\
&\xrightarrow{\mathcal{S}^{k,\alpha}} \widetilde{\mathscr{F}}^{s-1+k}_{p\infty}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1}) \xhookrightarrow{s>1-k} L^p(U_1 \backslash \overline{\Omega}, \mathrm{dist}^{1-k-s}; \wedge^{0,q+1}) \\
&\xrightarrow{\mathcal{K}^\top_{q,\alpha}} \begin{cases} W^{l,p}(\Omega, \mathrm{dist}^{l-\frac{1}{m_q}-s}; \wedge^{0,q-1}) \xhookrightarrow{l>s+\frac{1}{m_q}} \mathscr{F}^{s+\frac{1}{m_q}}_{p\varepsilon}(\Omega, \wedge^{0,q-1}) & \text{for all } 1 \le p \le \infty \\ W^{l,\frac{pr_q}{r_q-p}}(\Omega, \mathrm{dist}^{l-s}; \wedge^{0,q-1}) \xhookrightarrow{l>s} \mathscr{F}^s_{\frac{pr_q}{r_q-p},\varepsilon}(\Omega, \wedge^{0,q-1}) & \text{when } 1 \le p \le r_q \end{cases};
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^{k,\perp}_q : \mathscr{F}^s_{p\infty}(\Omega; \wedge^{0,q}) &\hookrightarrow \mathscr{F}^{s-\frac{1}{m_q}}_{pp}(\Omega; \wedge^{0,q}) \xrightarrow{[\overline{\partial},\mathcal{E}]^\top} \widetilde{\mathscr{F}}^{s-\frac{1}{m_q}}_{pp}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1}) \hookrightarrow \widetilde{\mathscr{F}}^{s-\frac{1}{m_q}}_{p\infty}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1}) \\
&\xrightarrow{\mathcal{S}^{k,\alpha}} \widetilde{\mathscr{F}}^{s+k-\frac{1}{m_q}}_{p\infty}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1}) \xhookrightarrow{s>\frac{1}{m_q}-k} L^p(U_1 \backslash \overline{\Omega}, \mathrm{dist}^{\frac{1}{m_q}-k-s}; \wedge^{0,q+1}) \\
&\xrightarrow{\mathcal{K}^\perp_{q,\alpha}} \begin{cases} W^{l,p}(\Omega, \mathrm{dist}^{l-\frac{1}{m_q}-s}; \wedge^{0,q-1}) \xhookrightarrow{l>s+\frac{1}{m_q}} \mathscr{F}^{s+\frac{1}{m_q}}_{p\varepsilon}(\Omega, \wedge^{0,q-1}) & \text{for all } 1 \le p \le \infty \\ W^{l,\frac{pr_q}{r_q-p}}(\Omega, \mathrm{dist}^{l-s}; \wedge^{0,q-1}) \xhookrightarrow{l>s} \mathscr{F}^s_{\frac{pr_q}{r_q-p},\varepsilon}(\Omega, \wedge^{0,q-1}) & \text{when } 1 \le p \le r_q \end{cases}.
\end{aligned}$$

In particular, we see that $\mathcal{H}^{k,\top}_q, \mathcal{H}^{k,\perp}_q$ are both defined on $\bigcup_{s>1-k; 1\le p\le\infty} \mathscr{F}^s_{p\infty}(\Omega; \wedge^{0,q})$, which completes the proof of Theorem 1.2 after we show that $\mathcal{H}_q = \mathcal{H}^k_q$ for all k.

By integrating by parts, for $f \in \mathscr{C}^\infty(\Omega; \wedge^{0,q})$,

$$\mathcal{H}^{k,\top}_q f(z) = \sum_{|\alpha|\le k} \int_{U_1 \backslash \overline{\Omega}} K^\top_{q-1}(z, \cdot) \wedge D^\alpha \mathcal{S}^{k,\alpha}[\overline{\partial}, \mathcal{E}] f = \mathcal{H}^{0,\top}_q f(z). \tag{6.2}$$

There is no boundary term because of Proposition 4.13 (iii).

In the same manner, we obtain $\mathcal{H}^{k,\perp}_q f = \mathcal{H}^{0,\perp}_q f$. Therefore, $\mathcal{H}^k_q f = \mathcal{H}^0_q f (= \mathcal{H}_q f)$ for all $f \in \mathscr{C}^\infty(\Omega; \wedge^{0,q})$.

For $s > 1-k$, $1 \le p \le \infty$ and $f \in \mathscr{F}^s_{p\infty}(\Omega; \wedge^{0,q})$, we can find a smooth sequence $\{f_j\}_{j=0}^\infty \subset \mathscr{C}^\infty(\Omega; \wedge^{0,q})$ such that $f_j \xrightarrow{\mathscr{F}^{s'}_{p\infty}} f$ for some $s' \in (1-k, s)$. Therefore, $\lim_{j\to\infty} \mathcal{H}_q f_j = \lim_{j\to\infty} \mathcal{H}^k_q f_j = \mathcal{H}^k_q f$ gives the definedness of $\mathcal{H}_q f$, and we see that $\mathcal{H}_q = \mathcal{H}^k_q$ for all $k \ge 0$. This completes the proof of Theorem 1.2.

Moreover, for this $(f_j) \subset \mathscr{C}^\infty$, by Lemma 2.4, we see that $f_j = \overline{\partial}\mathcal{H}^{k+1}_q f_j + \mathcal{H}^{k+1}_{q+1}\overline{\partial} f_j = \overline{\partial}\mathcal{H}_q f_j + \mathcal{H}_{q+1}\overline{\partial} f_j$. Therefore, by taking the limit, we see that $f = \overline{\partial}\mathcal{H}_q f + \mathcal{H}_{q+1}\overline{\partial} f$ also holds.

By Lemma 4.15, $\mathcal{H}_q$ is defined on $\bigcup_s \mathscr{C}^s(\Omega;\wedge^{0,q}) = \mathscr{S}'(\Omega;\wedge^{0,q})$. Therefore, $f = \overline{\partial}\mathcal{H}_q f + \mathcal{H}_{q+1}\overline{\partial} f$ holds for all $f \in \mathscr{S}'(\Omega;\wedge^{0,q})$. Now, we prove Theorem 1.1 (i).

Theorems 1.1 (ii) and (iii) follow from the inclusions $\mathscr{F}^s_{p1}(\Omega) \hookrightarrow H^{s,p}(\Omega) = \mathscr{F}^s_{p2}(\Omega) \hookrightarrow \mathscr{F}^s_{p\infty}(\Omega)$ and $\mathscr{F}^s_{\infty 1}(\Omega) \hookrightarrow \mathscr{C}^s(\Omega) = \mathscr{F}^s_{\infty\infty}(\Omega)$, as discussed in Remarks 4.6 (iii) and (vi). $\square$

7. An additional result for smooth strongly pseudoconvex domains

We summarize the techniques from Sections 3 and 4. We see that the corresponding results for Theorem 1.1 for bounded smooth strongly pseudoconvex domains also hold.

Theorem 7.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth strongly pseudoconvex domain. The operators $\mathcal{H}_q : \mathscr{S}'(\Omega;\wedge^{0,q}) \to \mathscr{S}'(\Omega;\wedge^{0,q-1})$ for $1 \le q \le n$ exist such that $f = \overline{\partial}\mathcal{H}_q f + \mathcal{H}_{q+1}\overline{\partial} f$ for all $f \in \mathscr{S}'(\Omega;\wedge^{0,q})$ (we set $\mathcal{H}_{n+1} = 0$).*

Moreover, for every $1 \le q \le n$, $s \in \mathbb{R}$, and $\varepsilon > 0$, $\mathcal{H}_q$ has the following boundedness:

$$\mathcal{H}_q : \mathscr{F}^s_{p,\infty}(\Omega;\wedge^{0,q}) \to \mathscr{F}^{s+\frac12}_{p,\varepsilon}(\Omega;\wedge^{0,q-1}), \qquad \forall\, 1 \le p \le \infty; \tag{7.1}$$

$$\mathcal{H}_q : \mathscr{F}^s_{p,\infty}(\Omega;\wedge^{0,q}) \to \mathscr{F}^s_{\frac{(2n+2)p}{2n+2-p},\varepsilon}(\Omega;\wedge^{0,q-1}), \qquad \forall\, 1 \le p \le 2n+2. \tag{7.2}$$

In particular, for every $1 \le q \le n$ and $s \in \mathbb{R}$, we have the boundedness $\mathcal{H}_q : \mathscr{C}^s(\Omega;\wedge^{0,q}) \to \mathscr{C}^{s+\frac12}(\Omega;\wedge^{0,q-1})$, $\mathcal{H}_q : H^{s,p}(\Omega;\wedge^{0,q}) \to H^{s+\frac12,p}(\Omega;\wedge^{0,q-1})$ for all $1 < p < \infty$, and $\mathcal{H}_q : H^{s,p}(\Omega;\wedge^{0,q}) \to H^{s,\frac{(2n+2)p}{2n+2-p}}(\Omega;\wedge^{0,q-1})$ for all $1 < p < 2n+2$.

Remark 7.2. Theorem 7.1 improves the result in [63, Theorem 1.2], which proves $\mathcal{H}_q : H^{s,p} \to H^{s+\frac12,p}$ for all $s \in \mathbb{R}$ and $1 < p < \infty$. For negative s, the boundedness $\mathcal{H}_q : H^{s,p} \to H^{s,\frac{(2n+2)p}{2n+2-p}}$ is new. Recall that these two results are not comparable since the Sobolev embedding $H^{s+\frac12,p} \hookrightarrow H^{s,\frac{4np}{4n-p}}$ is not contained in $H^{s,\frac{(2n+2)p}{2n+2-p}}$.

By keeping check on the proof using regularized distance functions, we can show that the results for non-smooth domains are true, where if $k \ge 0$ is an integer and $b\Omega \in C^{k+2}$, then $\mathcal{H}_q : H^{s,p} \to H^{s,\frac{(2n+2)p}{2n+2-p}}$ is still true for all $1 < p < 2n+2$ and $s > \frac1p - k$. We refer the reader to [62,63].

To prove Theorem 7.1, we repeat the arguments used in Section 3. Note that for $1 \le q \le n-1$, the (D'Angelo or Catlin) q-type of Ω is always 2. Later in the proof, we show that the $\top$ and $\perp$ projections are not needed for the estimates.

Recall that we can choose a smooth defining function for Ω such that it is plurisubharmonic in a neighborhood of $b\Omega$ (e.g., see [15, Theorem 3.4.4]). In particular, a $T_0 > 0$ exists such that $\Omega_t := \{\varrho < t\}$ is smooth strongly pseudoconvex for all $-T_0 < t < T_0$.

We recall the standard Henkin–Ramírez function for strongly pseudoconvex domains.

Proposition 7.3. *Let $\Omega \subset \mathbb{C}^n$ be a smooth strongly pseudoconvex domain. We have a number $T_1 \in (0, T_0]$ associated with a neighborhood $U_1 := \{|\varrho| < T_1\}$ of $b\Omega$, a $c \in (0, \frac12 T_1)$, and a map $\widehat{Q} \in \mathscr{C}^\infty(\Omega \times U_1;\mathbb{C}^n)$ that is holomorphic in z such that the associated support function $\widehat{S}(z,\zeta) := \widehat{Q}(z,\zeta)\cdot(z-\zeta)$ satisfies:*

$$|\widehat{S}(z,\zeta)| \ge \varrho(\zeta) - \varrho(z) + \tfrac1c |z-\zeta|^2, \qquad \forall z \in \Omega,\ \zeta \in U_1\backslash\overline{\Omega} \text{ such that } |z-\zeta| \le c; \tag{7.3}$$

$$|\widehat{S}(z,\zeta)| \ge c, \qquad \forall z \in \Omega,\ \zeta \in U_1\backslash\overline{\Omega} \text{ such that } |z-\zeta| \ge c. \tag{7.4}$$

See [44, Theorem III.7.15], [39, Theorem 2.4.3], or [30, Proposition 5.1].

For these $\widehat{S}$ and $\widehat{Q}$, we still use $K(z,\zeta)$ from (2.8). We define our solution operators $\mathcal{H}_q$ in Definition 2.5, where the image function of the Rychkov's extension operator $\mathcal{E}$ is always supported in $U_1 \cup \Omega$. Thus,

$$\mathcal{H}_q f(z) = \int_{U_1\cup\Omega} B_{q-1}(z,\cdot)\wedge \mathcal{E}f + \int_{U_1\setminus\overline{\Omega}} K_{q-1}(z,\cdot)\wedge[\overline{\partial},\mathcal{E}]f. \tag{7.5}$$

We define a naive version of $P_\varepsilon(\zeta)$ adapted to (Ω,ϱ) (cf. Definition 3.2), as follows.

Definition 7.4. For $\zeta\in\mathbb{C}^n$ and $\varepsilon>0$, we define

$$P_\varepsilon(\zeta) := \{z\in B(\zeta,\varepsilon^{\frac12}) : |\varrho(z)-\varrho(\zeta)| < \tfrac{2}{c}\varepsilon\}.$$

Clearly, when $\zeta\in U_1$ and $\varepsilon<\frac12 T_1$, the set $P_\varepsilon(\zeta)$ is non-empty. Moreover, the definition has symmetry (cf. (3.1)):

$$z\in P_\varepsilon(\zeta) \text{ if and only if } \zeta\in P_\varepsilon(z). \tag{7.6}$$

Informally, this is saying $\tau_1(\zeta,\varepsilon) := \frac{2}{c}\varepsilon$ and $\tau_2(\zeta,\varepsilon)=\cdots=\tau_n(\zeta,\varepsilon) := \varepsilon^{\frac12}$. In the strongly pseudoconvex case, there is no difference between using the ε-extremal basis and ε-minimal basis.

Now, (7.3) implies that there an $\varepsilon_0>0$ exists such that the corresponding result for Lemma 3.5 (i) holds:

$$|\widehat{S}(z,\zeta)| \gtrsim \varepsilon, \quad \text{for all } 0<\varepsilon\le\varepsilon_0, \quad \zeta\in U_1\setminus\overline{\Omega} \text{ and } z\in\Omega\setminus P_\varepsilon(\zeta).$$

We do not need the corresponding estimates in Lemma 3.5 (ii), and we see that the trivial estimates for Corollary 3.6 hold (also see Remark 3.8):

$$|\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^k| + |(\overline{\partial}\widehat{Q})^k| \lesssim 1. \tag{7.7}$$

Thus, a simpler version of Lemma 3.9 and Corollary 3.11 holds, where for $\varepsilon\in(0,\varepsilon_0]$, $1\le k\le n$, $j\ge 0$, and for all $(z,\zeta)\in\Omega\times(U_1\setminus\overline{\Omega})$ such that $z\notin P_\varepsilon(\zeta)$, we have $\big|D^j_{z,\zeta}\big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^{k-1}}{\widehat{S}^k}\big)(z,\zeta)\big| \lesssim_j \varepsilon^{-j-k}$. Therefore,

$$|D^j_{z,\zeta}K_{q-1}(z,\zeta)| \lesssim \sum_{k=1}^{n-q}\frac{\varepsilon^{-j-k}}{|z-\zeta|^{2n-2k-1}}, \qquad \forall (z,\zeta)\in\Omega\times(U_1\setminus\overline{\Omega}) \text{ such that } z\notin P_\varepsilon(\zeta). \tag{7.8}$$

Recall (7.6) and the statement above is the same as $(z,\zeta)\in\Omega\times(U_1\setminus\overline{\Omega})$ such that $\zeta\notin P_\varepsilon(z)$.

By integrating on $P_\varepsilon(\zeta)$ and $P_\varepsilon(z)$, we obtain the estimates corresponding to (3.14) and (3.15) (recall (3.18) with $\tau_1\approx\varepsilon$ and $\tau_2=\cdots=\tau_n=\varepsilon^{\frac12}$):

$$\int_{\Omega\cap P_\varepsilon(\zeta)\setminus P_{\frac{\varepsilon}{2}}(\zeta)} |D^jK_{q-1}(w,\zeta)|d\,\mathrm{Vol}_w + \int_{P_\varepsilon(z)\setminus(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)} |D^jK_{q-1}(z,w)|d\,\mathrm{Vol}_w$$
$$\lesssim_j \sum_{k=1}^{n-q}\int_{|w_1|<\varepsilon,|w_2|<\varepsilon^{\frac12},\dots,|w_n|<\varepsilon^{\frac12}} \frac{d\,\mathrm{Vol}(w_1,\dots,w_n)}{\varepsilon^{j+k}\big(\sum_{l=1}^n|w_l|\big)^{2n-2k-1}} \lesssim \varepsilon^{\frac32-j}.$$

In our case, the types for Ω are $m_1=\cdots=m_{n-1}=2$. Therefore, the numbers $r_q=(n-q+1)m_q+2q$, and $\gamma_q=\frac{r_q}{r_q-1}$ are indeed $r_q=2n+2$ and $\gamma_q=\frac{2n+2}{2n+1}$ for all $1\le q\le n-1$. Using (3.19) and (3.20), we obtain the estimates corresponding to (3.16) and (3.17):

$$\int_{\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)} |D^j K_{q-1}(w,\zeta)|^{\frac{2n+2}{2n+1}} d\operatorname{Vol}_w + \int_{P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)} |D^j K_{q-1}(z,w)|^{\frac{2n+2}{2n+1}} d\operatorname{Vol}_w$$

$$\lesssim_j \sum_{k=1}^{n-q} \int_{|w_1|<\varepsilon,|w_2|<\varepsilon^{\frac12},\dots,|w_n|<\varepsilon^{\frac12}} \frac{d\operatorname{Vol}(w_1,\dots,w_n)}{\left(\varepsilon^{j+k}\sum_{l=1}^n |w_l|\right)^{(2n-2k-1)\gamma_q}} \lesssim \varepsilon^{(1-j)\gamma_q} = \varepsilon^{(1-j)\frac{2n+2}{2n+1}}.$$

Therefore, by integrating on the dyadic shells $P_{2^{1-j}\varepsilon}(\zeta)\backslash P_{2^{-j}\varepsilon}(\zeta)$ or $P_{2^{1-j}\varepsilon}(z)\backslash P_{2^{-j}\varepsilon}(z)$, and using (3.21) and (3.22), we obtain the weighted estimates (cf. Theorem 2.9), that for every $k\ge 2$, $0<s<k-\frac32$, and $1\le q\le n-1$,

$$\int_{U_1\backslash\overline{\Omega}} \operatorname{dist}(\zeta)^s |D^k_{z,\zeta}K_{q-1}(z,\zeta)| d\operatorname{Vol}(\zeta) \lesssim_{k,s} \operatorname{dist}(z)^{s+\frac32-k}, \qquad \forall z\in\Omega;$$

$$\int_{\Omega} \operatorname{dist}(z)^s |D^k_{z,\zeta}K_{q-1}(z,\zeta)| d\operatorname{Vol}(z) \lesssim_{k,s} \operatorname{dist}(\zeta)^{s+\frac32-k}, \qquad \forall \zeta\in U_1\backslash\overline{\Omega};$$

$$\int_{U_1\backslash\overline{\Omega}} |\operatorname{dist}(\zeta)^s D^k_{z,\zeta}K_{q-1}(z,\zeta)|^{\frac{2n+2}{2n+1}} d\operatorname{Vol}(\zeta) \lesssim_{k,s} \operatorname{dist}(z)^{(s+1-k)\frac{2n+2}{2n+1}}, \qquad \forall z\in\Omega;$$

$$\int_{\Omega} |\operatorname{dist}(z)^s D^k_{z,\zeta}K_{q-1}(z,\zeta)|^{\frac{2n+2}{2n+1}} d\operatorname{Vol}(z) \lesssim_{k,s} \operatorname{dist}(\zeta)^{(s+1-k)\frac{2n+2}{2n+1}}, \qquad \forall \zeta\in U_1\backslash\overline{\Omega}.$$

By Schur's test (Lemma 3.15), for $\alpha\in\mathbb{N}^{2n}_\zeta$ and $1\le q\le n-1$, the integral operator

$$\mathcal{K}_{q,\alpha}g(z) := \int_{U_1\backslash\overline{\Omega}} (D^\alpha_{z,\zeta}K_{q-1})(z,\cdot)\wedge g \quad \left(=(-1)^{|\alpha|}\mathcal{K}_{q,0}\circ D^\alpha g(z)\right)$$

has the boundedness (cf. Corollary 3.13), that for every $k\ge 0$ and $1<s<k+|\alpha|-\frac12$ (in particular, $k+|\alpha|\ge 2$),

$$\mathcal{K}_{q,\alpha}: L^p(U_1\backslash\overline{\Omega},\operatorname{dist}^{1-s};\wedge^{0,q+1}) \to W^{k,p}(\Omega,\operatorname{dist}^{k+|\alpha|-\frac12-s};\wedge^{0,q-1}), \qquad \forall 1\le p\le\infty; \tag{7.9}$$

$$\mathcal{K}_{q,\alpha}: L^p(U_1\backslash\overline{\Omega},\operatorname{dist}^{1-s};\wedge^{0,q+1}) \to W^{k,\frac{(2n+2)p}{p-2n-2}}(\Omega,\operatorname{dist}^{k+|\alpha|-s};\wedge^{0,q-1}), \qquad \forall 1\le p\le 2n+2. \tag{7.10}$$

Recall the notations $\mathcal{B}_q$ in Lemma 6.1 and $\mathcal{S}^{k,\alpha}$ in Proposition 4.13. By rewriting (7.5) and using the same argument in (6.2) we have

$$\mathcal{H}_q f = \mathcal{B}_q\circ\mathcal{E}f + \sum_{|\alpha|\le k}(-1)^{|\alpha|}\mathcal{K}_{q,\alpha}\circ\mathcal{S}^{k,\alpha}\circ[\overline{\partial},\mathcal{E}]f, \qquad \forall k\ge 0. \tag{7.11}$$

By Remark 4.12, Lemma 6.1, and Remark 4.6 (iii), for every $1\le q\le n$, $s\in\mathbb{R}$, $1\le p\le\infty$, and $\varepsilon>0$,

$$\mathcal{B}_q\circ\mathcal{E}: \mathscr{F}^s_{p\infty}(\Omega;\wedge^{0,q}) \xrightarrow{\mathcal{E}} \widetilde{\mathscr{F}}^s_{p\infty}(\overline{U_1\cup\Omega};\wedge^{0,q}) \xrightarrow{\mathcal{B}_q} \mathscr{F}^{s+1}_{p\infty}(\Omega;\wedge^{0,q-1})$$
$$\hookrightarrow \begin{cases} \mathscr{F}^{s+\frac12}_{p\varepsilon}(\Omega;\wedge^{0,q-1}) & \forall p\in[1,\infty] \\ \mathscr{F}^{s}_{\frac{(2n+2)p}{2n+2-p},\varepsilon}(\Omega;\wedge^{0,q-1}) & \forall p\in[1,2n+2] \end{cases}.$$

In particular, we have (7.1) and (7.2) for $q=n$ since $\mathcal{K}_{n,\alpha}\equiv 0$.

Choose integers $k, l \geq 0$ such that $k > 1 - s$ and $l > s + \frac{1}{2}$. By applying (7.9), (7.10), and Corollary 5.5 to the summands in (7.11), for $1 \leq q \leq n-1$, $1 \leq p \leq \infty$, and $\varepsilon > 0$, we have,

$$\mathscr{F}^s_{p\infty}(\Omega; \wedge^{0,q}) \xrightarrow{[\overline{\partial}, \mathcal{E}]} \widetilde{\mathscr{F}}^{s-1}_{p\infty}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1}) \xrightarrow{\mathcal{S}^{k,\alpha}} \widetilde{\mathscr{F}}^{s-1+k}_{p\infty}(\overline{U_1 \backslash \Omega}; \wedge^{0,q+1}) \xhookrightarrow{s>1-k} L^p(U_1 \backslash \overline{\Omega}, \operatorname{dist}^{k-1-s}; \wedge^{0,q+1})$$
$$\xrightarrow{\mathcal{K}_{q,\alpha}} \begin{cases} W^{l,p}(\Omega, \operatorname{dist}^{l-\frac{1}{2}-s}; \wedge^{0,q-1}) \xhookrightarrow{l>s+\frac{1}{2}} \mathscr{F}^{s+\frac{1}{2}}_{p\varepsilon}(\Omega, \wedge^{0,q-1}) & \text{for all } 1 \leq p \leq \infty \\ W^{l,\frac{(2n+2)p}{2n+2-p}}(\Omega, \operatorname{dist}^{l-s}; \wedge^{0,q-1}) \xhookrightarrow{l>s} \mathscr{F}^s_{\frac{(2n+2)p}{2n+2-p},\varepsilon}(\Omega, \wedge^{0,q-1}) & \text{when } 1 \leq p \leq 2n+2 \end{cases}.$$

This completes the proofs of (7.1) and (7.2).

The Hölder bound $\mathcal{H}_q : \mathscr{C}^s \to \mathscr{C}^{s+\frac{1}{2}}$ and Sobolev bounds $\mathcal{H}_q : H^{s,p} \to H^{s+\frac{1}{2},p}$, $\mathcal{H}_q : H^{s,p} \to H^{s,\frac{(2n+2)p}{2n+2-p}}$ follow from the inclusions $\mathscr{F}^s_{p1}(\Omega) \hookrightarrow H^{s,p}(\Omega) = \mathscr{F}^s_{p2}(\Omega) \hookrightarrow \mathscr{F}^s_{p\infty}(\Omega)$ and $\mathscr{F}^s_{\infty 1}(\Omega) \hookrightarrow \mathscr{C}^s(\Omega) = \mathscr{F}^s_{\infty\infty}(\Omega)$, as discussed in Remarks 4.6 (iii) and (vi). $\square$

Appendix A. Proof of Proposition 5.3 (ii) for $\varepsilon < 1$

Let $\omega = \{x_1 > \sigma(x')\} \subset \mathbb{R}^N$ be a special Lipschitz domain. To provide a direct proof of Proposition 5.3 (ii), we define the inner strips (cf. (5.2)) of ω, where we define $P_0 = P_0^\omega := \{(x_1, x') : x_1 - \sigma(x') > 2^{-\frac{1}{2}}\}$ and

(A.1) $$P_k = P_k^\omega := \{(x_1, x') : 2^{-\frac{1}{2}-k} < x_1 - \sigma(x') < 2^{\frac{1}{2}-k}\} \subset \omega^c \text{ for } k \in \mathbb{Z}_+;$$

(A.2) $$P_{<k} = P_{<k}^\omega := \{(x_1, x') : x_1 - \sigma(x') > 2^{\frac{1}{2}-k}\} \subset \omega^c \text{ for } k \in \mathbb{Z}_+.$$

Now, $\{P_k\}_{k=0}^\infty$ is a partition of ω up to zero measured sets. Similar to (5.3) (recall that $\delta(x) = \min(1, \operatorname{dist}(x, b\omega))$),

(A.3) $$2^{-1-k} \leq \delta(x) \leq 2^{\frac{1}{2}-k} \text{ for all } k \geq 0, \quad x \in P_k.$$

Similar to (5.4) (see [61, Lemma 5.3]), we can also find an $R > 0$ such that $\operatorname{supp} \phi_j + P_k \subseteq \omega^c$ whenever $k \geq j + R$. Thus,

(A.4) $$\mathbf{1}_{P_k} \cdot (\phi_j * f) = \mathbf{1}_\omega \cdot (\phi_j * (f \cdot \mathbf{1}_{P_{<\min(j+R,k)}})), \quad \forall j \geq 0.$$

The duality argument does not work for the case where $\varepsilon < 1$ since $\mathscr{F}^s_{p\varepsilon}$ is no longer a locally convex space. In the proof of the case where $1 \leq p < \infty$, we need a version of the locally constant principle.

Lemma 7.5 *(Locally constant). Let $\phi = (\phi_j)_{j=0}^\infty \subset \mathscr{S}(\mathbb{R}^N)$ satisfy the scaling condition (4.7) (in Definition 4.7). Then, for any $M > 0$, a $C_{\phi,M} > 0$ exists such that for every $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^N)$,*

(A.5) $$|\phi_j * f(x)| \leq C_{\phi,M} 2^{\frac{Nj}{p}} \sum_{v \in \mathbb{Z}^N} \frac{1}{(1+|v|)^M} \|f\|_{L^p(x+2^{-j}v+[0,2^{-j}]^N)}, \quad \forall j \geq 0, \quad x \in \mathbb{R}^N.$$

In particular, if ϕ also satisfies the support condition (4.10), then a $C_\phi > 0$ exists, such that for every $1 \leq p \leq \infty$, $j \geq 0$ and $0 \leq k \leq j + R$,

(A.6) $$\|\phi_j * f\|_{L^p(P_k)} \leq C_\phi \min(1, 2^{\frac{j-k}{p}}) \|f\|_{L^p(P_{<j+R})}, \quad \forall f \in L^p(P_{<j+R}),$$

where $R > 0$ is given in (A.4) and $\operatorname{dist}(x) = \operatorname{dist}(x, b\omega)$.

In particular, $\|\phi_j * f\|_{L^\infty(P_k)} \lesssim \|f\|_{L^\infty(P_{<j+R})}$.

Proof. We denote $Q_{j,v} := 2^{-j}v + [0, 2^{-j}]^N$ for $j \geq 0$ and $v \in \mathbb{Z}^N$. Therefore, for every $x \in \mathbb{R}^N$,

$$\phi_j * f(x) = \sum_{v \in \mathbb{Z}^N} \phi_j * (f \cdot \mathbf{1}_{(x+Q_{j,v})})(x) = \sum_{v \in \mathbb{Z}^N} (\phi_j \cdot \mathbf{1}_{B(0,2^{-j}\max(0,|v|-\sqrt{N}))^c}) * (f \cdot \mathbf{1}_{(x+Q_{j,v})})(x).$$

Therefore, by Hölder's inequality, $|\phi_j * f(x)| \leq \sum_v \|\phi_j\|_{L^{p'}(B(0,2^{-j}\max(0,|v|-\sqrt{N}))^c)} \|f\|_{L^p(x+Q_{j,v})}$.

Note that ϕ_0, ϕ_1 are Schwartz, so for $j \in \{0,1\}$ and $l \in \mathbb{Z}$, we have $\int_{|x|>2^l} |\phi_j(x)| dx \lesssim_M 2^{-M\max(0,l)}$, and thus $\|\phi_j\|_{L^{p'}(B(0,2^l)^c)} \lesssim_M 2^{-M\max(0,l)}$ for $j \in \{0,1\}$. Therefore, by the scaling condition (4.7), for every $j \geq 0$, we have,

$$\begin{aligned}\|\phi_j\|_{L^{p'}(B(0,2^{-j}\max(0,|v|-\sqrt{N}))^c)} &\leq 2^{\frac{Nj}{p}} (\|\phi_0\|_{L^{p'}(B(0,\max(|v|-\sqrt{N}))^c)} + \|\phi_0\|_{L^{p'}(B(0,\frac{1}{2}\max(|v|-\sqrt{N}))^c)}) \\ &\lesssim_M \tfrac{2^{Nj/p}}{(1+|v|)^M}.\end{aligned}$$

Therefore, $|\phi_j * f(x)| \lesssim_M 2^{Nj/p} \sum_v (1+|v|)^{-M} \|f\|_{L^p(x+Q_{j,v})}$, which gives (A.5).

To prove (A.6), by Fubini's theorem,

$$\begin{aligned}\int_{P_k} |\phi_j * f|^p &= \int_{\mathbb{R}^{N-1}} \int_{2^{-\frac{1}{2}-k}}^{2^{\frac{1}{2}-k}} |\phi_j * f(t+\sigma(x'), x')|^p dt dx' \\ &\leq 2^{-k} \sup_{2^{-1/2-k}<t<2^{1/2-k}} \int_{\mathbb{R}^{N-1}} \left|(\phi_j * f)(t+\sigma(x'), x')\right|^p dx'.\end{aligned}$$

Therefore, by taking $M > N$,

$$\begin{aligned}&\|\phi_j * f\|_{L^p(P_k)} \leq 2^{-\frac{k}{p}} \sup_{2^{-1/2-k}<t<2^{1/2-k}} \left\|x' \mapsto \left|(\phi_j * f)(t+\sigma(x'), x')\right|\right\|_{L^p(\mathbb{R}^{N-1}_{x'})} \\ &\leq 2^{-\frac{k}{p}} 2^{\frac{j}{p}} \sup_{2^{-1/2-k}<t<2^{1/2-k}} \left\|u' \mapsto \left\|\phi_j * (f\mathbf{1}_{P_{<j+R}})\right\|_{L^p((t+\sigma(u'),u')+B(0,2^{-j}\sqrt{N}))}\right\|_{\ell^p(2^{-j}\mathbb{Z}^{N-1}_{u'})} \\ &\leq 2^{\frac{j-k}{p}} \left\|u' \mapsto \left\|\phi_j * (f\mathbf{1}_{P_{<j+R}})\right\|_{L^p((\sigma(u'),u')+B(0,2^{1-j}\sqrt{N}))}\right\|_{\ell^p(2^{-j}\mathbb{Z}^{N-1}_{u'})} \\ &\lesssim_M 2^{\frac{j-k}{p}} 2^{\frac{Nj}{p}} \sum_{v\in\mathbb{Z}^N} (1+|v|)^{-M} \left\|u' \mapsto \left\|f\mathbf{1}_{P_{<j+R}}\right\|_{L^p((\sigma(u'),u')+B(0,2^{1-j}\sqrt{N})+Q_{j,v})}\right\|_{\ell^p(2^{-j}\mathbb{Z}^{N-1}_{u'})} \\ &\leq 2^{\frac{j-k}{p}} 2^{\frac{Nj}{p}} \sum_{v\in\mathbb{Z}^N} (1+|v|)^{-M} \left\|u \mapsto \left\|f\mathbf{1}_{P_{<j+R}}\right\|_{L^p(u+B(0,2^{2-j}\sqrt{N})+Q_{j,v})}\right\|_{\ell^p(2^{-j}\mathbb{Z}^N_u)} \\ &\lesssim_M 2^{\frac{j-k}{p}} 2^{\frac{Nj}{p}} \left|B(0,2^{2-j}\sqrt{N})\right| \left\|\left((1+|v|)^{-M}\right)_{v\in\mathbb{Z}^N}\right\|_{\ell^1(\mathbb{Z}^N)} \|f\mathbf{1}_{P_{<j+R}}\|_{L^p(\mathbb{R}^N)} \lesssim_M 2^{\frac{j-k}{p}} \|f\|_{L^p(P_{<j+R})}.\end{aligned}$$

This completes the proof of (A.6). □

Proof of Proposition 5.3 (ii). We only need to prove $m = 0$. Indeed, if the case when $m = 0$ is true, then by Proposition 4.8 (iii), for every $s < m$, we have

$$\|f\|_{\mathscr{F}^s_{p\varepsilon}(\omega)} \approx_{s,m,\varepsilon} \sum_{|\alpha|\leq m} \|D^\alpha f\|_{\mathscr{F}^{s-m}_{p\varepsilon}(\omega)} \overset{\text{case } m=0}{\lesssim} \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^p(\omega,\delta^{m-s})} \approx \|f\|_{W^{m,p}(\delta^{m-s})}.$$

Now, assume that $m = 0$ and $0 < \varepsilon < 1$. By Proposition 4.8 (ii), we have $\|f\|_{\mathscr{F}^s_{p\varepsilon}(\omega)} \approx \|(2^{js}\phi_j * f)_{j=0}^\infty\|_{L^p(\omega;\ell^\varepsilon)}$ for $1 \leq p < \infty$ and $\|f\|_{\mathscr{F}^s_{\infty\varepsilon}(\omega)} \approx \sup_{x,J} 2^{NJ\frac{1}{\varepsilon}} \|(2^{js}\phi_j * f)_{j=\max(0,J)}^\infty\|_{L^\varepsilon(\omega\cap B(x,2^{-J});\ell^\varepsilon)}$. We need to show that for every $t > 0$ and $\varepsilon > 0$,

$$\big\|\big(2^{-jt}\phi_j * f\big)_{j=0}^{\infty}\big\|_{L^p(\omega;\ell^\varepsilon(\mathbb{N}))} \lesssim_{\phi,p,\varepsilon,t} \|\delta^t f\|_{L^p(\omega)}, \quad 1<p<\infty; \tag{A.7}$$

$$2^{NJ}\sup_{x\in\omega, J\in\mathbb{Z}} \big\|\big(2^{-jt}\phi_j * f\big)_{j=\max(0,J)}^{\infty}\big\|^{\varepsilon}_{L^\varepsilon(\omega\cap B(x,2^{-J});\ell^\varepsilon(\mathbb{N}))} \lesssim_{\phi,\varepsilon,t} \|\delta^t f\|^{\varepsilon}_{L^\infty(\omega)}. \tag{A.8}$$

First, we prove (A.7) using Lemma 7.5. For $k\ge 0$, recall that $P_k, P_{<k}$ from (A.1):

$$\begin{aligned}
&\big\|\big(2^{-jt}\phi_j * f\big)_{j=0}^{\infty}\big\|_{L^p(P_k;\ell^\varepsilon(\mathbb{N}))} = \big\|\big(2^{-jt}\phi_j * (f\mathbf{1}_{P_{<\min(j+R,k)}})\big)_{j=0}^{\infty}\big\|_{L^p(P_k;\ell^\varepsilon(\mathbb{N}))}\\
\le&\big\|\big(2^{-jt}\phi_j * (f\mathbf{1}_{P_{<\min(j+R,k)}})\big)_{j=0}^{\infty}\big\|_{\ell^\varepsilon(\mathbb{N};L^p(P_k))} && \text{(by (A.9) below)}\\
\le&2^{\frac{1}{\varepsilon}}\Big(\big\|\big(2^{-jt}\|\phi_j * (f\mathbf{1}_{P_{<j+R}})\|_{L^p(P_k)}\big)_{j=0}^{k-R}\big\|_{\ell^\varepsilon} + \big\|\big(2^{-jt}\|\phi_j * (f\mathbf{1}_{P_{<k}})\|_{L^p(P_k)}\big)_{j=\max(0,k-R)}^{\infty}\big\|_{\ell^\varepsilon}\Big)\\
\lesssim_{\varepsilon,\phi,M}&\big\|\big(2^{-jt}2^{\frac{j-k}{p}}\|f\|_{L^p(P_{<j+R})}\big)_{j=0}^{k-R}\big\|_{\ell^\varepsilon} + \big\|\big(2^{-jt}\|f\|_{L^p(P_{<k})}\big)_{j=\max(0,k-R)}^{\infty}\big\|_{\ell^\varepsilon} && \text{(by (A.6))}\\
\lesssim_R&\big\|\big(2^{\frac{j-k}{p}}2^{-jt}\|f\|_{L^p(P_{<j})}\big)_{j=1}^{k}\big\|_{\ell^\varepsilon} + \|(2^{ls})_{l=-R}^{\infty}\|_{\ell^\varepsilon}2^{-tk}\|f\|_{L^p(P_{<k})}\\
\lesssim_{\varepsilon,R,t}&\big\|\big(2^{-\frac{|j-k|}{p}}2^{-jt}\|f\|_{L^p(P_{<j})}\big)_{j=1}^{\infty}\big\|_{\ell^\varepsilon}.
\end{aligned}$$

The first inequality is a variant of Minkowski's inequality and since we assume that $0<\varepsilon\le 1$,

$$\|(g_j)_j\|_{L^p(\ell^\varepsilon)} = \|(|g_j|^\varepsilon)_j\|^{\frac{1}{\varepsilon}}_{L^{p/\varepsilon}(\ell^1)} = \|(|g_j|^\varepsilon)_j\|^{\frac{1}{\varepsilon}}_{\ell^1(L^{p/\varepsilon})} = \|(g_j)_j\|_{\ell^\varepsilon(L^p)}. \tag{A.9}$$

Therefore,

$$\begin{aligned}
&\|f\|_{\mathscr{F}^{-t}_{p\varepsilon}(\omega)} \approx \big\|\big(2^{-jt}\phi_j * f\big)_{j=0}^{\infty}\big\|_{L^p(\omega;\ell^\varepsilon(\mathbb{N}))} = \big\|\big(\|(2^{-jt}\phi_j * f)_{j=0}^{\infty}\|_{L^p(P_k;\ell^\varepsilon(\mathbb{N}))}\big)_{k=0}^{\infty}\big\|_{\ell^p(\mathbb{N})}\\
\lesssim_{\varepsilon,p,t}&\big\|\big(\textstyle\sum_{j=0}^{\infty}2^{-\frac{\varepsilon}{p}|j-k|}2^{-jt\varepsilon}\|f\|^{\varepsilon}_{L^p(P_{<j})}\big)_{k=0}^{\infty}\big\|^{1/\varepsilon}_{\ell^{p/\varepsilon}(\mathbb{N})}\\
\le&\big\|(2^{-\frac{\varepsilon}{p}|j|})_{j=-\infty}^{\infty}\big\|^{1/\varepsilon}_{\ell^1}\big\|\big(2^{-jt\varepsilon}\|f\|^{\varepsilon}_{L^p(P_{<j})}\big)_{j=0}^{\infty}\big\|^{1/\varepsilon}_{\ell^{p/\varepsilon}(\mathbb{N})} && \text{(by Young's inequality)}\\
\lesssim_{\varepsilon,p}&\big\|\big(2^{-jt}\|f\|_{L^p(P_{<j})}\big)_{j=0}^{\infty}\big\|_{\ell^p(\mathbb{N})} = \big\|\big(\textstyle\sum_{j=k}^{\infty}2^{-jt}\|f\|_{L^p(P_k)}\big)_{k=0}^{\infty}\big\|_{\ell^p(\mathbb{N})}\\
\approx&\big\|\big(2^{-kt}\|f\|_{L^p(P_k)}\big)_{k=0}^{\infty}\big\|_{\ell^p(\mathbb{N})} \approx \big\|\big(\|\delta^t f\|_{L^p(P_k)}\big)_{k=0}^{\infty}\big\|_{\ell^p(\mathbb{N})} = \|f\|_{L^p(\omega,\delta^t)}. && \text{(by } t>0 \text{ and (A.3))}
\end{aligned}$$

This completes the proof of (A.7).

To prove (A.8), let $x\in\omega$ and let $k_0\ge 0$ be such that $x\in\overline{P_{k_0}}$. We separate the discussion of the norm $\|(2^{-jt}\phi_j * f)_{j\ge J}\|_{L^\varepsilon(\omega\cap B(x,2^{-J});\ell^\varepsilon)}$ between $J\le k_0+1$ and $J\ge k_0+2$.

When $J\le k_0+1$, we have $|P_k\cap B(x,2^{-J})|\lesssim 2^{-k}2^{-(N-1)J}$ if $k\ge J-2$ and $P_k\cap B(x,2^{-J})=\varnothing$ if $k\le J-3$. By (A.4) and (A.6), $\|\phi_j * f\|_{L^\infty(P_k)}\lesssim_{\phi,R} 2^{t\min(j+R,k)}\|f\|_{L^\infty(\omega,\delta^t)}$, and thus

$$\begin{aligned}
&2^{NJ}\textstyle\sum_{j=\max(0,J)}^{\infty}\int_{\omega\cap B(x,2^{-J})}2^{-jt\varepsilon}|\phi_j * f|^\varepsilon = 2^{NJ}\sum_{j=\max(0,J)}^{\infty}\sum_{k=J-2}^{\infty}\int_{P_k\cap B(x,2^{-J})}2^{-jt\varepsilon}|\phi_j * f|^\varepsilon\\
\lesssim&2^{NJ}\|f\|^{\varepsilon}_{L^\infty(\omega,\delta^t)}\textstyle\sum_{k=J-2}^{\infty}\sum_{j=\max(0,J)}^{\infty}|P_k\cap B(x,2^{-J})|2^{-jt\varepsilon}2^{t\varepsilon\min(j+R,k)}\\
=&2^{NJ}\|f\|^{\varepsilon}_{L^\infty(\omega,\delta^t)}\textstyle\sum_{k=J-2}^{\infty}2^{-k}2^{(N-1)J}\big(k-(J-2)+\sum_{j=k}^{\infty}2^{t\varepsilon(k-j)}\big)\\
\approx&\|f\|^{\varepsilon}_{L^\infty(\omega,\delta^t)}2^{tJ\varepsilon}\lesssim\|f\|^{\varepsilon}_{L^\infty(\omega,\delta^t)}2^{tk_0\varepsilon}.
\end{aligned}$$

When $J\ge k_0+2$, we have $B(x,2^{-J})\subset P_{>k_0+2}$, and thus

$$\begin{aligned}
&2^{NJ}\textstyle\sum_{j=\max(0,J)}^{\infty}\int_{\omega\cap B(x,2^{-J})}2^{-jt\varepsilon}|\phi_j * f|^\varepsilon = 2^{NJ}\sum_{j=\max(0,J)}^{\infty}\int_{P_{>k_0+2}\cap B(x,2^{-J})}2^{-jt\varepsilon}|\phi_j * f|^\varepsilon\\
\lesssim&2^{NJ}|B(x,2^{-J})|\textstyle\sum_{j=k_0+2}^{\infty}2^{-jt\varepsilon}2^{k_0t\varepsilon}\|f\|^{\varepsilon}_{L^\infty(P_{>k_0+2})}\lesssim\|f\|^{\varepsilon}_{L^\infty(\omega,\delta^t)}2^{tk_0\varepsilon}.
\end{aligned}$$

This completes the proof of (A.8), and thus the whole proof. □

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