A LIOUVILLE-TYPE THEOREM FOR CYLINDRICAL CONES

NICK EDELEN AND GÁBOR SZÉKELYHIDI

Abstract. Suppose that $\mathbf{C}_0^n \subset \mathbb{R}^{n+1}$ is a smooth strictly minimizing and strictly stable minimal hypercone (such as the Simons cone), $l \geq 0$, and M a complete embedded minimal hypersurface of $\mathbb{R}^{n+1+l}$ lying to one side of $\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^l$. If the density at infinity of M is less than twice the density of $\mathbf{C}$, then we show that $M = H(\lambda) \times \mathbb{R}^l$, where $\{H(\lambda)\}_\lambda$ is the Hardt-Simon foliation of $\mathbf{C}_0$. This extends a result of L. Simon, where an additional smallness assumption is required for the normal vector of M.

1. Introduction

Liouville type theorems, that is the rigidity properties of entire solutions of certain partial differential equations, are ubiquitous in geometric analysis. In this paper we prove a Liouville type theorem for minimal hypersurfaces lying on one side of a minimal cylindrical hypercone, extending a recent result of L. Simon [10].

To state the main result, let $\mathbf{C}_0^n \subset \mathbb{R}^{n+1}$ be a smooth strictly minimizing and strictly stable minimal hypercone (e.g. the Simons' cone), and let $\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^l$ for some $l \geq 0$. Write $\{H(\lambda)\}_\lambda$ for the Hardt-Simon foliation [5] associated to $\mathbf{C}_0$. See Section 2 for more details on the notation.

Our main result is the following.

Theorem 1.1. *If M is a smooth complete embedded minimal hypersurface of $\mathbb{R}^{n+l+1}$ lying to one side of $\mathbf{C}$ satisfying the density bound $\theta_M(\infty) < 2\theta_{\mathbf{C}}(0)$, then $M = H(\lambda) \times \mathbb{R}^l$ for some λ.*

Previously Simon [10] showed that the same conclusion holds under the additional assumption that the component ν_y of the normal vector to M in the $\mathbb{R}^l$ direction is sufficiently small. The $l = 0$ case of the Theorem is due to Hardt-Simon [5], who proved it for smooth $\mathbf{C}$ which are merely minimizing. The *existence* of a foliation associated to a minimizing hypercone $\mathbf{C}$ was first proven by [1] (for quadratic $\mathbf{C}$), [5] (for smooth $\mathbf{C}$), and just recently [17] (for any $\mathbf{C}$).

The Hardt-Simon foliation and Liouville theorems of [5, 10] have been of fundamental importance in the analysis of minimal hypersurfaces, including in results concerning generic regularity of stable or minimizing 7-dimensional hypersurfaces [5, 12, 2, 7], the construction of stable or minimizing singular minimal hypersurfaces [5, 11, 15], and local regularity/tangent cone uniqueness [9, 3, 14, 4].

Cylindrical cones $\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^l$ model generic singularities in the top stratum, and are also the simplest examples of tangent cones with non-isolated singular set. Let us also remark.

Remark 1.2. All known singular minimizing hypercones are either smooth (away from 0) and strictly stable and strictly minimizing, or cylindrical like we consider here (see e.g. [16] and the references therein). The most famous examples of singular minimizing hypercones are the Simons cones, and in the lowest singular dimension $n = 7$ these are in fact the only known examples.

Remark 1.3. With only cosmetic changes, Theorem 1.1 (and all the other lemmas/theorems in this paper) continue to hold for stationary integral varifolds in place of smooth, complete minimal surfaces. So, if V is a non-zero stationary integral $(n+l)$-varifold in $\mathbb{R}^{n+l+1}$ with $\theta_V(\infty) < 2\theta_{\mathbf{C}}(0)$ and $\mathrm{spt}V$ lying to one side of $\mathbf{C}$, then $V = [H(\lambda) \times \mathbb{R}^l]$ for some λ.

Using the standard decomposition of codimension-one currents into a sum of boundaries, and the strong maximum principle [6], Theorem 1.1 and Remark 1.3 imply directly the Corollary:

Corollary 1.4. *Let T be a mass-minimizing integral $(n+l)$-current in $\mathbb{R}^{n+l+1}$ with $\mathrm{spt}T$ lying to one side of $\mathbf{C}$ and satisfying $\theta_T(\infty) < \infty$. Then T is a finite union $\sum_i [H(\lambda_i)]$. In particular, if $T = \partial[E]$ is a boundary, then $T = [H(\lambda)]$ for some λ.*

Some of the basic ideas and strategies that we use originate from [14, 15], but the explicit nature of our comparison surfaces $T_\lambda := H(\lambda) \times \mathbb{R}^l$ allows for significant simplifications. A key technical tool is a geometric 3-annulus lemma (Lemma 5.1) for an excess $E(M, T_\lambda, R)$ defined for M with respect to T_λ at scale R. This in turn depends on a non-concentration estimate (Theorem 4.4) to reduce the estimate to the corresponding result for Jacobi fields.

Given the 3-annulus lemma, the argument can be summarized as follows: the fact that M lies on one side of $\mathbf{C}$ implies that the excess of M with respect to $T_0 = \mathbf{C}$ grows at most at rate $R^{\gamma-1+\epsilon}$ as $R \to \infty$ for any $\epsilon > 0$. Here r^γ is the growth rate of the only positive admissible Jacobi field on $\mathbf{C}_0$. At the same time we show that if λ is chosen appropriately then the excess of M with respect to T_λ at scale R grows at least at rate $R^{\gamma-1+\epsilon_0}$ for some $\epsilon_0 > 0$. This uses that fact that r^γ on $\mathbf{C}$ is generated by pushing into the T_λ, and is the smallest possible growth rate of admissible Jacobi fields on $\mathbf{C}$. Combining these two results we get a contradiction, unless the excess of M with respect to T_λ is zero, i.e. $M = T_\lambda$.

Acknowledgements N.E. was supported in part by NSF grant DMS-2204301. G. Sz. was supported in part by NSF grant DMS-2203218. We thank Otis Chodosh for helpful conversations.

2. Preliminaries

Throughout this paper $\mathbf{C}_0$ will be a smooth minimal hypercone in $\mathbb{R}^{n+1}$, l a non-negative integer, and $\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^l \subset \mathbb{R}^{n+l+1} \equiv \mathbb{R}^{n+1} \times \mathbb{R}^l = \{(x, y) : x \in \mathbb{R}^{n+1}, y \in \mathbb{R}^l\}$. When we write $u : \mathbf{C} \to \mathbb{R}$ we mean $u : \mathrm{reg}\mathbf{C} \to \mathbb{R}$. Define $B_\rho(\xi)$ to be the open Euclidean ball in $\mathbb{R}^{n+l+1}$ of radius ρ centered at ξ, $B_\rho \equiv B_\rho(0)$, and $A_{r,\rho} = B_r \setminus \overline{B_\rho}$ to be the open annulus centered at 0. Write ω_n for the volume of the Euclidean n-ball. Let $\eta_{X,\rho}(Y) = (Y - X)/\rho$ be the translation/rescaling map.

2.1. **Cylindrical cones.** The Jacobi operator on $\mathbf{C}_0$ is $L_{\mathbf{C}_0} f = \Delta_{\mathbf{C}_0} f + |A_{\mathbf{C}_0}|^2 f$. In polar coordinates $x = r\theta$ this becomes

$$L_{\mathbf{C}_0} = \partial_r^2 + r^{-1}(n-1)\partial_r + r^{-2}L, \quad L = \Delta_\Sigma + |A_\Sigma|^2,$$

so that $L_\Sigma = L + (n-1) = \Delta_\Sigma + |A_\Sigma|^2 + (n-1)$ is the Jacobi operator of $\Sigma \subset S^n$. Write $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ for the eigenvalues of L, and $\{\psi_j\}_{j\geq 1}$ for the corresponding $L^2(\Sigma)$-ON basis eigenfunctions, so that $L\psi_j + \lambda_j \psi_j = 0$. Define

$$\gamma_j^{\pm} = -(n-2)/2 \pm \sqrt{((n-2)/2)^2 + \lambda_j},$$

so that every linear combination $u(x = r\theta) = c_j^+ r^{\gamma_j^+} \psi_j(\theta) + c_j^- r^{\gamma_j^-} \psi_j(\theta)$ is a Jacobi field on $\mathbf{C}_0$. We assume $\mathbf{C}_0$ is *strictly stable*, which means that $\gamma_j^- < -(n-2)/2 < \gamma_j^+$. For shorthand we shall write $\gamma_j = \gamma_j^+$ and $\gamma = \gamma_1 = \gamma_1^+$.

Let $H_\pm$ be leaves of the Hardt-Simon foliation [5] of $\mathbf{C}_0$, lying on different sides of $\mathbf{C}_0$, so that each $H_\pm$ is oriented compatibly with $\mathbf{C}_0$ (i.e. so that $\nu_{H_\pm} \to \nu_{\mathbf{C}_0}$ as $r \to \infty$). We assume $\mathbf{C}_0$ is *strictly minimizing*, which means there is a radius $R_0(\mathbf{C}_0)$ so that (possibly after appropriately rescaling $H_\pm$)

$$H_\pm \setminus B_{R_0} = \mathrm{graph}_{\mathbf{C}_0}(\Psi_\pm), \tag{1}$$

where

$$\Psi_\pm(x = r\theta) = \pm r^\gamma \psi_1(\theta) + v_\pm, \quad |v_\pm| \le r^{\gamma - \alpha_0} \tag{2}$$

for some $\alpha_0(\mathbf{C}_0) > 0$ (see e.g. [5, Equation (10), p. 114]). It follows by standard elliptic estimates (see e.g. [15, Proposition 2.2]) that

$$|\nabla^i v_\pm| \le c(\mathbf{C}, i) r^{\gamma - i - \alpha_0}, \quad i = 0, 1, 2, \ldots. \tag{3}$$

We remark that (3) implies that if $h_{H_\pm}$ is the second fundamental form of $H_\pm$, then $|x||h_{H_\pm}| \le c(\mathbf{C})$.

Define

$$H(t) = \begin{cases} |t|^{1/(1-\gamma)} H_{\mathrm{sign}(t)} & t \neq 0 \\ \mathbf{C}_0 & t = 0 \end{cases},$$

so that

$$H(t) \setminus B_{|t|^{1/(1-\gamma)} R_0} = \mathrm{graph}_{\mathbf{C}_0}(\Psi_t), \quad \Psi_t(x) = |t|^{1/(1-\gamma)} \Psi_{\mathrm{sign}(t)}(|t|^{-1/(1-\gamma)} x), \tag{4}$$

and hence

$$|\Psi_t(x) - t r^\gamma \psi_1(\theta)| \le |t|^{1 + \alpha_0/(1-\gamma)} r^{\gamma - \alpha_0}.$$

Lemma 2.2. *For sufficiently small ϵ (depending only on $\mathbf{C}_0$), we can write $(1+\epsilon)H_+$ as a graph over H_+ of the function $\Phi_{+,\epsilon}$, which we can expand as*

$$\Phi_{\epsilon,+} = \epsilon \Phi_+ + \epsilon^2 V_{+,\epsilon}, \tag{5}$$

where: Φ_+ is a positive Jacobi field on H_+ satisfying

$$\Phi_+(x + \Psi_+(x)\nu_{\mathbf{C}_0}(x)) = (1-\gamma) r^\gamma \psi_1(\theta) + O(r^{\gamma - \alpha_0}) \text{ for } x = r\theta \in \mathbf{C}_0 \setminus B_{R_0}, \tag{6}$$

$$\text{and } |\nabla^i \Phi_+(x)| \le c(\mathbf{C}, i)|x|^{\gamma - i}, \quad i = 0, 1, 2, \ldots; \tag{7}$$

and $V_{+,\epsilon}$ satisfies the estimates

$$|\partial_\epsilon^j \nabla^i V_{+,\epsilon}(x)| \le c(\mathbf{C}, i, j)|x|^{\gamma - i}. \tag{8}$$

The same statements hold with $(1-\epsilon)H_-$, $\Phi_{-,\epsilon}$, Φ_-, $V_{-,\epsilon}$ in place of $(1+\epsilon)H_+$, $\Phi_{+,\epsilon}$, Φ_+, $V_{+,\epsilon}$.

Proof. The decomposition (5) simply follows from the definition of Jacobi field. Positivity of Ψ_+ comes from the star-shapedness of H_+. For $x = r\theta$ with $r \gg 1$, we can write

$$\Phi_{\epsilon,+}(x + \Psi_+(x)\nu_{\mathbf{C}_0}(x))(1 + E_1(x)) = ((1+\epsilon)\Psi_+(E_2(x)/(1+\epsilon)) - \Psi_+(E_2(x))).$$

where $E_1, E_2 - id$ are smooth functions which are (at minimum) linearly controlled by $r^{-1}\Psi_+(x), \nabla\Psi_+(x)$. We compute:

$$\begin{aligned}&(1+\epsilon)\Psi_+(x/(1+\epsilon)) - \Psi_+(x)\\&= ((1+\epsilon)^{1-\gamma} - 1)r^\gamma\psi_1(\theta) + \int_1^{1+\epsilon}(v_+ - r\partial_r v_+)|_{x/\lambda}d\lambda\\&= ((1-\gamma)\epsilon + O(\epsilon^2))r^\gamma\psi_1(\theta) + O(\epsilon r^{\gamma-\alpha_0}).\end{aligned}$$

The bounds for Ψ_+ and $\nabla^i V_{+,\epsilon}$ follow by the above computations and standard elliptic estimates. □

2.3. **Minimal surfaces and varifolds.** It will be convenient to use the language of varifolds, see [8] for a standard reference. We shall write $||V||$ for the mass measure of a varifold, and given a countably-$(n+l)$-rectifiable set $M \subset \mathbb{R}^{n+l+1}$ we write $[M]$ for the integral $(n+l)$-varifold with mass measure $\mathcal{H}^{n+l}\llcorner M$.

Recall that the monotonicity formula for stationary (integral) $(n+l)$-varifolds in $\mathbb{R}^{n+l+1}$ says the density ratio

$$\theta_V(\xi,\rho) := \frac{||V||(B_\rho(\xi))}{\omega_{n+l}\rho^{n+l}}$$

is increasing in ρ, for any $\xi \in \mathbb{R}^{n+l+1}$, and is constant if and only if V is a cone over ξ. We define the density of V at a point ξ, resp. and at ∞, by

$$\theta_V(\xi) = \lim_{\rho\to 0}\theta_V(\xi,\rho), \quad \text{resp. } \theta_V(\infty) = \lim_{\rho\to\infty}\theta_V(0,\rho).$$

If $V = [M]$, we understand $||M||(U) \equiv ||V||(U)$, $\theta_M(\xi,\rho) \equiv \theta_{[M]}(\xi,\rho)$, etc.

Lemma 2.4. *Let V be a non-zero stationary integral varifold cone in $\mathbb{R}^{n+l+1}$, such that* spt$V$ *lies to one side of* $\mathbf{C}$. *Then $V = k[\mathbf{C}]$ for some integer $k \geq 1$.*

Proof. Follows by the maximum principles of [13], [6], since sing$\mathbf{C}$ has dimension at most $n+l-7$. □

2.5. **β-harmonic functions and Jacobi fields.** For $\beta > 0$, [10] introduced the notion of β-harmonic functions, which are functions $h(r,y)$ on $B_1^+ \subset \mathbb{R}^{1+l}_+ = \{(r,y) \in \mathbb{R}\times\mathbb{R}^l : r > 0\}$ solving

$$r^{-1-\beta}\partial_r(r^{1+\beta}\partial_r h) + \Delta_y h = 0, \tag{9}$$

and satisfying the integrability hypothesis

$$\int_{B_1^+} r^{-2}h^2 r^{1+\beta} < \infty. \tag{10}$$

[10] showed any such h extends analytically in r^2 and y to $\{(r,y) : r^2 + |y|^2 < 1, r \geq 0\}$, and in particular can be written as a sum of homogenous β-harmonic polynomials in r^2, y.

In spherical coordinates $(r,y) = \rho\omega$, where $\rho = \sqrt{r^2+|y|^2}$ and $\omega = (r,y)/\rho$, (9) becomes

$$\rho^{-l-1-\beta}\partial_\rho(\rho^{l+1+\beta}\partial_\rho h) + \rho^{-2}\omega_1^{-1-\beta}\mathrm{div}_{S^l}(\omega_1^{1+\beta}\nabla_{S^l}h) = 0. \tag{11}$$

Here $\omega_1 = \omega\cdot\partial_r \equiv r/\sqrt{r^2+|y|^2}$.

[10] showed that the β-harmonic homogenous polynomials $\{h_q\}$, when restricted to S^l_+, are $L^2(\omega_1^{1+\beta}d\omega)$-complete. So there is an $L^2(\omega_1^{1+\beta}d\omega)$-ON basis of functions $\{\phi_i\}_{i\geq 1}$, each being the restriction of a β-harmonic homogenous polynomial $h_i(\rho\omega) = \rho^{q_i}\phi_i(\omega)$. From (11), we get the eigenvalue-type equation

$$\omega_1^{-1-\beta}\mathrm{div}_{S^l_+}(\omega_1^{1+\beta}\nabla_{S^l}\phi_i) + q_i(q_i + l + \beta)\phi_i = 0 \tag{12}$$

on S^l_+.

For each $j \geq 1$, define $\beta_j = n - 2 + 2\gamma_j = 2\sqrt{((n-2)/2)^2 + \lambda_j}$. Let v be a Jacobi field on $\mathbf{C} \cap B_1$ satisfying

$$\int_{\mathbf{C}\cap B_1} |x|^{-2}v^2 < \infty.$$

For every j, let $h_j(r, y) = r^{-\gamma_j}\int_\Sigma v(r\theta, y)\phi_j(\theta)d\theta$. Then a straightforward computation shows each h_j is β_j-harmonic in $\mathbf{C} \cap B_1$, and hence admits an analytic expansion of the form

$$h_j = \sum_{i\geq 1} h_{ij}(r, y),$$

where each h_{ij} is a q_{ij}-homogenous β_j-harmonic polynomial, for some integer $q_{ij} \geq 0$. Moreover, all the $\{h_{ij}|_{S^l_+}\}_i$ are $L^2(\omega_1^{1+\beta_j}S^l_+)$-orthogonal.

Therefore v admits an expansion

$$v(r\theta, y) = \sum_{i,j\geq 1} r^{\gamma_j}\psi_j(\theta)h_{ij}(r, y), \tag{13}$$

which holds in the following senses: in $L^2(\Sigma)$ for every fixed (r, y); in $L^2(\mathbf{C} \cap B_\rho)$ for every $\rho < 1$; in $C^\infty_{loc}(\mathbf{C} \cap B_1 \setminus \{r = 0\})$. For every $0 < \rho < 1$ we have

$$\int_{\mathbf{C}\cap B_\rho} v^2 = \sum_{i,j\geq 0} c^2_{ij}\rho^{n+l+2\gamma_j+q_{ij}} =: \sum_{i\geq 1} a^2_i\rho^{n+l+2p_i}, \tag{14}$$

where $\gamma_1 = p_1 < p_2 < \ldots$. Note (14) implies that the function

$$\rho \mapsto \rho^{-n-l-2\gamma}\int_{\mathbf{C}\cap B_\rho} v^2 \tag{15}$$

is increasing in ρ.

We require a few helper lemmas about "tame" Jacobi fields.

Lemma 2.6 ([14]). *Let v be a Jacobi field on $\mathbf{C} \cap B_1$ with $\sup_{\mathbf{C}\cap B_1} ||x|^{-\gamma}v| < \infty$. Then for every $\theta < 1$ we have the estimate*

$$\sup_{\mathbf{C}\cap B_\theta} ||x|^{-\gamma}v|^2 \leq c(\mathbf{C}, \theta)\int_{\mathbf{C}\cap B_1} v^2, \tag{16}$$

and

$$\int_{\mathbf{C}\cap B_1} v^2 \leq \int_{\mathbf{C}\cap B_1} |x|^{-2}v^2 \leq c(\mathbf{C}) \sup_{\mathbf{C}\cap B_1} ||x|^{-\gamma}v|^2. \tag{17}$$

Proof. We prove (16) for $\theta = 1/8$, and the statement for general θ will follow by standard elliptic estimates and an obvious covering argument. Pick $(x, y) =$

$(r\theta, y) \in \mathbf{C} \cap B_{1/8}$ with $r > 0$. By scale-invariant elliptic estimates and (15), we compute

$$\begin{aligned}|v(r\theta, y)|^2 &\leq c(\mathbf{C}) r^{-n-l} \int_{\mathbf{C}\cap B_{r/2}(x,y)} v^2 \\ &\leq c(\mathbf{C}) r^{-n-l} \int_{\mathbf{C}\cap B_r(0,y)} v^2 \\ &\leq c(\mathbf{C}) r^{2\gamma} \int_{\mathbf{C}\cap B_{1/4}(0,y)} v^2 \\ &\leq c(\mathbf{C}) r^{2\gamma} \int_{\mathbf{C}\cap B_{1/2}} v^2.\end{aligned}$$

To prove (17) simply use $-(n-2)/2 < \gamma < 0$. □

Lemma 2.7. *Let v be a non-negative Jacobi field on $\mathbf{C}$, with $\sup_{\mathbf{C}\cap B_R} ||x|^{-\gamma} v| < \infty$ for all R. Then $v = a|x|^\gamma \psi_1(\theta)$, for some constant a.*

Proof. We first note that if $h(r, y)$ is β-harmonic in $B_R \subset \mathbb{R}^{l+1}_+$, then by (e.g.) integrating (11), we have the mean-value equality

$$h(0,0) = \left(\int_{B_\rho^+} r^{1+\beta} dr dy \right)^{-1} \int_{B_\rho^+} h(r,y) r^{1+\beta} dr dy \tag{18}$$

for every $0 < \rho < R$. Since $h(\cdot, \cdot - y)$ is also β-harmonic, (18) holds with $h(0, y)$ in place of $h(0, 0)$ and $B_\rho^+(0, y) \subset B_R^+$ in place of B_ρ^+. Therefore if h is β-harmonic and non-negative in $\mathbb{R}^{l+1}_+$, we have

$$h(0, y) \leq \frac{\int_{B^+_{\rho+|y-y'|}(y')} r^{1+\beta} dr dy}{\int_{B_\rho(y)} r^{1+\beta} dr dy} h(0, y') = \left(\frac{\rho + |y - y'|}{\rho} \right)^{2+\beta+l} h(0, y')$$

for every $y, y' \in \mathbb{R}^l$ and $\rho >> 1$. Taking $\rho \to \infty$ and y, y' arbitrary implies $h(0, y) \equiv h(0, 0)$, and in particular for $h_j(r, y)$ as in (14) we deduce

$$h_1(0, y) = h_1(0, 0) \quad \forall y \in \mathbb{R}^l. \tag{19}$$

Now for every (r, y), we have $v(r\theta, y) = \sum_j r^{\gamma_j} h_j(r, y)\psi_j(\theta)$ in $L^2(\Sigma)$, and hence in $L^1(\Sigma)$. Using (18) we compute for every $\rho > 0$:

$$\int_{\mathbf{C}\cap B_\rho} |x|^\gamma v \psi_1 = \int_{B_\rho^+} h_1(r, y) r^{n-1+2\gamma} dr dy = c(\mathbf{C}) \rho^{n+l+2\gamma} h_1(0, 0). \tag{20}$$

Since $1/c(\mathbf{C}) \leq \psi_1 \leq c(\mathbf{C})$ and $v \geq 0$, we can use (19), (20), and standard elliptic estimates at scale $r/2$ to deduce

$$r^{-\gamma} v(r\theta, y) \leq c r^{-n-l-\gamma} \int_{B_{r/2}(r\theta,y)} v \tag{21}$$

$$\leq c r^{-n-l-2\gamma} \int_{B_{r/2}(r\theta,y)} |x|^\gamma v \psi_1 \tag{22}$$

$$\leq c r^{-n-l-2\gamma} \int_{B_r(0,y)} |x|^\gamma v \psi_1 \leq c h_1(0, y) = c(\mathbf{C}) h_1(0, 0). \tag{23}$$

If $v(r\theta, y) \neq ar^\gamma \psi_1(\theta)$ for some constant a, then from the expansion (13), (14) and equation (17) we must have

$$\rho^{2\alpha}/C \leq \rho^{-n-l-2\gamma} \int_{\mathbf{C}\cap B_\rho} v^2 \leq c(\mathbf{C}) \sup_{\mathbf{C}\cap B_\rho} |r^{-\gamma} v|^2$$

for some $\alpha > 0$ and some constant C independent of ρ. For $\rho \gg 1$ this contradicts (23). □

Lastly, we will require the following "baby" 3-annulus-type lemma.

Lemma 2.8. *Let $\{q_i \in \mathbb{R}\}_{i\in\mathbb{Z}}$ be an increasing sequence, and $\{b_i \in \mathbb{R}\}_{i\in\mathbb{Z}}$ be arbitrary. Fix $k \in \mathbb{Z}$, and $3\epsilon \in (0, q_{k+1} - q_k)$, and $T \geq 1/\epsilon$. Define $\psi(t) = \sum_i b_i^2 e^{2q_i t}$. Then*

$$\psi(t+T) \geq e^{2(q_k+\epsilon)T}\psi(t) \implies \psi(t+2T) \geq e^{2(q_{k+1}-\epsilon)T}\psi(t+T).$$

Proof. By replacing b_i^2 with $b_i^2 e^{2q_i t}$, it suffices to take $t = 0$. Observe that the first inequality implies

$$\sum_{i\geq k+1} b_i^2 e^{2q_i T} \geq (e^{2\epsilon T} - 1) \sum_{i\leq k} b_i^2 e^{2q_i T}.$$

Then we get

$$\begin{aligned}
\psi(2T) &= \sum_i b_i^2 e^{4q_i T} \\
&\geq e^{2(q_{k+1}-\epsilon)T} \sum_{i\geq k+1} b_i^2 e^{2q_i T} + e^{2(q_{k+1}-\epsilon)T}(e^{2\epsilon T}-1) \sum_{i\geq k+1} b_i^2 e^{2q_i T} \\
&\geq e^{2(q_{k+1}-\epsilon)T} \left(\sum_{i\geq k+1} b_i^2 e^{2q_i T} + (e^{2\epsilon T}-1)^2 \sum_{i\leq k} b_i^2 e^{2q_i T} \right) \\
&\geq e^{2(q_{k+1}-\epsilon)T}\psi(T).
\end{aligned}$$

□

3. Barriers

In this section we collate our functions and hypersurfaces we will use as barriers. We write $L_{H_\pm}$, $L_{H(\lambda)}$ for the Jacobi operator on $H_\pm$, $H(\lambda)$.

Lemma 3.1 ([15, Proposition 2.8])**.** *For any $a > \gamma$, there are functions $F_{\pm,a}$ on $H_\pm$ satisfying*

$$F_{\pm,a}(x + \Psi_\pm(x)\nu_{\mathbf{C}_0}(x)) = r^a \phi_1(\theta) \text{ for } x = r\theta \in \mathbf{C} \text{ and } r \gg 1, \tag{24}$$

$$|\nabla^i F_{\pm,a}| \leq c(\mathbf{C}_0, a, i) r^{a-i}, \quad \text{and } L_{H_\pm} F_{a,\pm} \geq c(\mathbf{C}_0, a)^{-1} r^{a-2}. \tag{25}$$

We extend $F_{\pm a}$ to an smooth a-homogenous function $F_a : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ by setting

$$F_a(x) = \begin{cases} \lambda^a F_{+,a}(\lambda^{-1}x) & x \in \lambda H_+ \\ r^a \phi_1 & x \in \mathbf{C}_0 \setminus \{0\} \\ \lambda^a F_{-,a}(\lambda^{-1}x) & x \in \lambda H_- \end{cases}, \quad \lambda > 0.$$

Each F_a satisfies

$$|D^i F_a| \leq c(\mathbf{C}_0, a, i)|x|^{a-i}, \quad L_{H(\lambda)}(F_a|_{H(\lambda)}) \geq |x|^{a-2}/c(\mathbf{C}_0, a). \tag{26}$$

Proof. See [15, Proposition 2.8] and [14, Lemma 5.7]. □

Write $\mathcal{M}_{T_\lambda}(G)$ for the mean curvature of the graph of G over T_λ, which is well-defined provided $|x|^{-1}G$ is sufficiently small. For $G, H \in C^2(T_\lambda)$, write $D\mathcal{M}_{T_\lambda}(G)[H]$ for the linearization of $\mathcal{M}_{T_\lambda}$ at G in the direction H, that is

$$D\mathcal{M}_{T_\lambda}(G)[H] = \frac{d}{dt}\Big|_{t=0} \mathcal{M}_{T_\lambda}(G + tH).$$

Provided $|x|^{-1}G$, ∇G, $|x|\nabla^2 G$ are sufficiently small then $D\mathcal{M}_{T_\lambda}(G)$ is a linear elliptic operator on $C^2(T_\lambda) \to C^0(T_\lambda)$. Of course if $G = 0$ then $D\mathcal{M}_{T_\lambda}(0) \equiv L_{T_\lambda}$ is the Jacobi operator on T_λ.

Lemma 3.2. *For $\gamma' > \gamma$, there are constants $\epsilon(\mathbf{C}, \gamma')$, $c(\mathbf{C}, \gamma')$ so that if $G : T_1 \to \mathbb{R}$ is a C^2 function satisfying $|\nabla^i G| \le \epsilon |x|^{\gamma - i}$ for $i = 0, 1, 2$, then*

$$D\mathcal{M}_{T_1}(G)[F_{\gamma'}|_{T_1}]|_{(x,y)} \ge \frac{1}{c}|x|^{\gamma'-2} > 0,$$

where $F_{\gamma'}$ is from Lemma 3.1. The same result also holds for T_{-1} in place of T_1.

Proof. Let us recall that the C^3-regularity scale $r_{C_3}(M, x)$ of a hypersurface $M \subset \mathbf{R}^N$ at a point $x \in M$ is defined to be the supremum of those $r > 0$ for which the translated and rescaled surface $r^{-1}(M - x)$ is the graph of a C^3 function u inside the unit ball, with $|u|_{C^3} \le 1$.

The C^3 regularity scale of T_1 satisfies $|x|/c \le r_{C_3}(T_1, x, y) \le c|x|$, and $|x| \ge 1/c$ on T_1, for $c = c(\mathbf{C})$. It follows that provided $\epsilon(\mathbf{C})$ is sufficiently small, at any point $(x, y) = (r\theta, y) \in T_1$ we can write

$$\mathcal{M}_{T_1}(G) = L_{T_1}G + r^{-1}R(x, r^{-1}G, \nabla G, r\nabla^2 G)$$

where $R(x, z, p, q)$ is a smooth, uniformly bounded function which is quadratically controlled by z, p, q. More precisely, we have the bounds

$$|R(x, z, p, q)| \le c(|z|^2 + |p|^2 + |q|^2), \quad |\partial_z R| + |\partial_p R| + |\partial_q R| \le c(|z| + |p| + |q|).$$

For any $H \in C^2(T_1)$, we have

$$D\mathcal{M}_{T_1}(G)[H] = \frac{d}{ds}\Big|_{s=0} \mathcal{M}_{T_1}(G + sH) = L_{T_1}H + a_{ij}\nabla^2_{ij}H + r^{-1}b_i\nabla_i H + r^{-2}cH$$

with a_{ij}, b_i, c continuous functions satisfying

$$|a_{ij}| + |b_i| + |c| \le c\epsilon r^{\gamma-1}.$$

Taking $H = F_{\gamma'}|_{T_1}$ and using Lemma 3.1, we get

$$D\mathcal{M}_{T_1}(G)[F_{\gamma'}] \ge \left(\frac{1}{c(\mathbf{C}_0, \gamma')} - c\epsilon r^{\gamma-1}\right) r^{\gamma'-2} \ge \frac{1}{2c} r^{\gamma'-2}$$

provided $\epsilon(\mathbf{C}, \gamma')$ is chosen sufficiently small. The argument for T_{-1} is verbatim. $\square$

We say a set $A \subset \mathbb{R}^{n+l+1}$ lies above (resp. below) $H(\lambda) \times \mathbb{R}^l$ if $A \subset \cup_{\mu \ge \lambda} H(\mu)$ (resp. $A \subset \cup_{\mu \le \lambda} H(\mu)$). More generally, if $U \subset \mathbb{R}^{n+1+l}$ and $S \subset U$ divides U into two disjoint connected components $U_\pm$, and there are $\lambda_- < \lambda_+$ such that

$$U_+ \cap (H(\lambda_+) \times \mathbb{R}^l) \ne \emptyset = U_+ \cap (H(\lambda_-) \times \mathbb{R}^l) \tag{27}$$

$$U_- \cap (H(\lambda_-) \times \mathbb{R}^l) \ne \emptyset = U_- \cap (H(\lambda_+) \times \mathbb{R}^l), \tag{28}$$

then we say $A \subset U$ lies above S in U (resp. below S in U) if $A \subset \overline{U_+}$ (resp. $A \subset \overline{U_-}$).

If S is a smooth hypersurface of U, and $\overline{S}$ divides U into components $U_\pm$ as in the previous paragraph, we say S has positive (resp. negative) mean curvature if the mean curvature vector of $S \cap U$ never vanishes and always points into U_+ (resp. into U_-).

Theorem 3.3 ([15, Proposition 2.9]). *There is a large odd integer p and a constant $Q > 0$ depending only on $\mathbf{C}$ so that the following holds. Let I be an open set in $\mathbb{R}^l$, and let $f : I \to \mathbb{R}$ be a C^3 function satisfying $|f|_{C^3(I)} \leq K$ for some $K > Q$. Then for any $\epsilon < 1/Q$ there is a complete oriented hypersurface-without-boundary X_ϵ in $\{0 < |x| < K^{-Q^2}, y \in I\}$, satisfying:*

1. *X_ϵ is C^2 with negative mean curvature;*
2. *at any point $(0, y) \in \overline{X_\epsilon} \cap \{|x| = 0, y \in I\}$, the tangent cone of X_ϵ at $(0, y)$ is the graph of $-\epsilon|x|$ over $\mathbf{C}$;*
3. *X_ϵ varies continuously (in the Hausdorff distance) with ϵ, and for every $y' \in I$ the y-slice $X_\epsilon \cap \{y = y'\}$ is trapped between*

$$H(\epsilon f(y')^p - \epsilon) \text{ and } H(\epsilon f(y')^p + \epsilon). \tag{29}$$

In particular, if V is a stationary varifold in $U \subset \{y \in I, |x| < K^{-Q^2}\}$ which lies below X_ϵ in U, then $\mathrm{spt}V \cap X_\epsilon \cap U = \emptyset$.

Proof. This is proved with $l = 1$ in [15, Proposition 2.9]. When $l > 1$, the functions $f(y)$, $G(y)$, $E(y)$ (as defined in the proof [15]) becomes functions on $\mathbb{R}^l$, and so bounds on $G^{(i)}$ become bounds on $D^i G$, and the Jacobi operator $L_{H(\lambda)\times\mathbb{R}^l}$ on $H(\lambda) \times \mathbb{R}^l$ becomes $\Delta_y + L_{H(\lambda)}$. Otherwise the same proof carries over with only cosmetic changes. □

We shall also need the following computation.

Lemma 3.4. *Let S be a C^2 hypersurface in $\mathbb{R}^{n+1}$, and $f : \mathbb{R}^l \to \mathbb{R}_+$ a C^2 function. Define the new hypersurface $\tilde{S} \subset \mathbb{R}^{n+1+l}$ by*

$$\tilde{S} = \bigcup_{y \in \mathbb{R}^l} (f(y)S) \times \{y\}.$$

At any point $z = (f(y)x, y) \in \tilde{S}$, let ν be a choice of normal for S at x, and let $\tilde{\nu}$ be the normal of $\tilde{S}$ at z pointing in the same direction as ν. Then the mean curvature $\mathcal{M}_{\tilde{S}}$ of $\tilde{S}$ with respect to $\tilde{\nu}$ at z can be expressed as

$$\mathcal{M}_{\tilde{S}} = \frac{1}{\sqrt{E}} \left[\frac{\mathcal{M}_S}{f} + \frac{|Df|^2 h_S(x^T, x^T)}{fE} + (x \cdot \nu) \left(-\delta_{\alpha\beta} + \frac{(x \cdot \nu)^2 D_\alpha f D_\beta f}{E} \right) D^2_{\alpha\beta} f \right].$$

where $E = 1 + |Df|^2 (x \cdot \nu)^2$, and $\mathcal{M}_S$ is the mean curvature of S, and h_S is the second fundamental form of S.

Proof. Let $F(x^1, \ldots, x^n) : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a coordinate chart for S. Let $g_{ij} = \partial_i F \cdot \partial_j F$ be the induced metric in these coordinates, and $h_{ij} = -\partial^2_{ij} F \cdot \nu$ the second fundamental form. WLOG let us assume $g_{ij}(0) = \delta_{ij}$.

Define coordinate chart $\tilde{F}(x^1, \ldots, x^n, y^1, \ldots, y^l) : U \times \mathbb{R}^l \to \mathbb{R}^{n+l+1}$ for $\tilde{S}$ by $\tilde{F}(x, y) = (f(y)F(x), y)$. We have

$$\partial_i \tilde{F} = (f \partial_i F, 0), \quad \partial_\alpha \tilde{F} = ((\partial_\alpha f) F, e_\alpha)$$

where we abbreviate $\partial_i \equiv \frac{\partial}{\partial x^i}$, $\partial_\alpha \equiv \frac{\partial}{\partial y^\alpha}$, and write $\{e_\alpha\}$ for the standard basis vectors of $\mathbb{R}^l$. The metric $\tilde{g}$ in these coordinates at $(x, y) = (0, 0)$ is therefore

$$\tilde{g}_{\alpha\beta} = \delta_{\alpha\beta} + (\partial_\alpha f)(\partial_\beta f)|F|^2, \quad \tilde{g}_{\alpha i} = f(\partial_\alpha f)(F \cdot \partial_i F), \quad \tilde{g}_{ij} = f^2 \delta_{ij}.$$

One can verify directly that the metric inverse at $(x, y) = (0, 0)$ is then

$$\tilde{g}^{\alpha\beta} = \delta_{\alpha\beta} - \frac{(F \cdot \nu)^2}{E}(\partial_\alpha f)(\partial_\beta f), \quad \tilde{g}^{\alpha i} = -\frac{(\partial_\alpha f)(F \cdot \partial_i F)}{fE}$$
$$\tilde{g}^{ij} = f^{-2}\delta_{ij} + \frac{|Df|^2(F \cdot \partial_i F)(F \cdot \partial_j F)}{f^2 E}$$

where $E = 1 + |Df|^2(F \cdot \nu)^2$.

We have

$$\partial_i \partial_j \tilde{F} = (f \partial_i \partial_j F, 0), \quad \partial_i \partial_\alpha \tilde{F} = (\partial_\alpha f \partial_i F, 0), \quad \partial_\alpha \partial_\beta \tilde{F} = (\partial_\alpha \partial_\beta f F, 0).$$

Therefore, since by inspection $\tilde{\nu} = E^{-1/2}(\nu, -(\partial_\alpha f)(F \cdot \nu) e_\alpha)$ we can compute the second fundamental form of $\tilde{S}$ to be

$$\tilde{h}_{\alpha\beta} = \frac{-(\partial^2_{\alpha\beta} f)(F \cdot \nu)}{\sqrt{E}}, \quad \tilde{h}_{\alpha i} = 0, \quad \tilde{h}_{ij} = \frac{f h_{ij}}{\sqrt{E}}.$$

We deduce that, at $(x, y) = (0, 0)$, we have

$$\begin{aligned} \mathcal{M}_{\tilde{S}} &= \tilde{g}^{\alpha\beta}\tilde{h}_{\alpha\beta} + \tilde{g}^{ij}\tilde{h}_{ij} \\ &= \left(\delta_{\alpha\beta} - \frac{(F \cdot \nu)^2 D_\alpha f D_\beta f}{E}\right)\left(\frac{-D^2_{\alpha\beta} f(F \cdot \nu)}{\sqrt{E}}\right) \\ &\quad + f^{-2}\left(\delta_{ij} + \frac{|Df|^2(F \cdot \partial_i F)(F \cdot \partial_j F)}{E}\right)\frac{f h_{ij}}{\sqrt{E}} \end{aligned}$$

which, recalling that $g_{ij} = \delta_{ij}$ at $(0, 0)$ is the form required. □

4. Non-concentration

For shorthand let us write $T_\lambda = H(\lambda) \times \mathbb{R}^l$, so we also have $T_0 = \mathbf{C}$. We define the following notion of "distance from T_λ". This is effectively a non-linear version of the norm $\sup_U ||x|^{-\gamma} u|$, see Corollary 4.5.

Definition 4.1. Given subsets $M, U \subset B_1$, $\lambda \in \mathbb{R}$, define $D_{T_\lambda}(M; U)$ as the least $d \geq 0$ such that $M \cap U$ is trapped between $H(\lambda \pm d) \times \mathbb{R}^l$.

The following follows directly from the definition:

$$D_{T_\lambda}(M; U) \leq D_{T_{\lambda'}}(M; U) + |\lambda - \lambda'|. \tag{30}$$

Note that D scales like $\rho^{1-\gamma}$ in the sense that for $c > 0$ we have

$$D_{cT_\lambda}(cM; cU) = c^{1-\gamma} D_{T_\lambda}(M; U).$$

We define a scale-invariant "excess" quantity which will be our main mechanism for measuring decay/growth.

Definition 4.2. Given $R > 0$, $\lambda \in \mathbb{R}$, and subset $M \subset B_R$, define the excess of M in B_R w.r.t. T_λ to be

$$E(M, T_\lambda, R) = D_{R^{-1}T_\lambda}(R^{-1}M; B_1) \equiv R^{\gamma-1} D_{T_\lambda}(M; B_R).$$

We remark that $E(T_\lambda, \mathbf{C}, R) = R^{\gamma-1}|\lambda|$ and that for M that is the graph of u over T_λ we can think of $E(M, T_\lambda, R)$ as equivalent to $\sup_{B_R} R^{\gamma-1}\big||x|^{-\gamma}u\big|$.

Lemma 4.3. *If $d = D_{T_\lambda}(M; U) \leq \beta|\lambda|$ for some $\beta(\mathbf{C})$ sufficiently small, then $M \cap U$ is trapped between the graphs* $\mathrm{graph}_{T_\lambda}(\pm c(\mathbf{C})d|x|^\gamma)$. *Conversely, if $M \cap U$ is trapped between* $\mathrm{graph}_{T_\lambda}(\pm d|x|^\gamma)$ *and $d \leq \beta|\lambda|$, then $D_{T_\lambda}(M; U) \leq c(\mathbf{C})d$.*

Proof. By scale-invariance it suffices to consider the case when $\lambda = \pm 1$, in which case the Lemma follows straightforwardly from Lemma 2.2. □

The main Theorem of this Section is the following non-concentration result. We emphasize that c_0 in (31), (32) is independent of s.

Theorem 4.4 (Non-concentration). *Given any $s \in (0, 1/4]$ and $\theta \in (0, 1)$, there are constants $c_0(\mathbf{C}, \theta)$, $r_0(\mathbf{C}, \theta, s)$, $\delta_0(\mathbf{C}, \theta, s)$ so that the following holds. Let M be a complete minimal hypersurface in B_1, such that $D_{T_\lambda}(M; B_1) < \delta_0$ for $|\lambda| < \delta_0$, and $M \cap B_1 \cap \{r \geq r_0\}$ is trapped between* $\mathrm{graph}_{T_\lambda}(\pm b|x|^\gamma)$ *for $b < \delta_0$. Then*

$$D_{T_\lambda}(M; B_\theta) \leq c_0(b + sD_{T_\lambda}(M; B_1)) \tag{31}$$

If B_1 is replaced by $A_{1,\rho}$ in our assumptions, for some $\rho \in (0, 1/2]$, then instead we get

$$D_{T_\lambda}(M; A_{\theta,\theta^{-1}\rho}) \leq c_0(b + sD_{T_\lambda}(M; A_{1,\rho})). \tag{32}$$

(with c_0, r_0, δ_0 depending on ρ also).

Proof of Theorem 4.4. For ease of notation write $d = D_{T_\lambda}(M; B_1)$. We need to break the proof into two cases, depending on whether $d \gtrsim |\lambda|$ (when M is about as close to $\mathbf{C}$ as it is to T_λ), or whether $d << |\lambda|$ (when M is much closer to T_λ than to $\mathbf{C}$). In the first case we will use the barrier surfaces constructed in Theorem 3.3. In the second case we will construct barrier surfaces as graphs over T_λ. At the end of the proof we will explain the (very minor) changes required to get (32).

Fix $\gamma < \gamma' < \min\{\gamma + 1/2, 0\}$. Throughout the proof

$$1/2 \geq \beta(\mathbf{C}, \theta, \gamma') \gg r_0(\mathbf{C}, \theta, \beta, \gamma', s) \gg \delta_0(\mathbf{C}, \theta, \beta, \gamma', s, r_0)$$

are small constants which we shall choose as we proceed, but can a posteriori be fixed.

We first claim that $D_{T_\lambda}(M; B_1 \cap \{|x| \geq r_0\}) \leq c(\mathbf{C}, \beta)b$. If $b \leq \beta|\lambda|$ this follows from Lemma 4.3, provided $\beta(\mathbf{C})$ is sufficiently small. Suppose now $b > \beta|\lambda|$. Then provided $\delta_0(\mathbf{C}, r_0)$ is sufficiently small, $M \cap B_1 \cap \{|x| \geq r_0\}$ is trapped between the graphs of $\pm c(\mathbf{C})(b+|\lambda|)|x|^\gamma$ over $\mathbf{C} \cap \{|x| \geq r_0/2\}$ in $\{|x| \geq r_0/2\}$, and hence trapped between the graphs of $\pm c(\mathbf{C})(b/\beta)|x|^\gamma$ over $\mathbf{C} \cap \{|x| \geq r_0/2\}$. But then provided $\delta_0(\mathbf{C}, r_0, \beta)$ is sufficiently small, $\mathrm{graph}_{\mathbf{C}\cap\{|x|\geq r_0/2\}}(c(\mathbf{C})(b/\beta)|x|^\gamma)$ is trapped between $H(\pm c(\mathbf{C})b/\beta) \times \mathbb{R}^l$ in $\{|x| \geq r_0/2\}$. Combined with the inequality $|\lambda| \leq b/\beta$, our initial claim follows.

We shall henceforth work towards proving the estimate

$$D_{T_\lambda}(M; \{|x| \leq r_0\} \cap \{|y| \leq \theta^2\}) \leq c(\mathbf{C}, \beta, \theta)(b + sd). \tag{33}$$

Provided $r_0(\mathbf{C})$ is sufficiently small, (33) combined with our initial claim will imply (31) (with θ^2 in place of θ). For ease of notation let us define the domains

$$\Omega_1 = \{|x| \leq r_0\} \cap \{|y| \leq \theta - s\}, \quad \Omega_2 = \{|x| \leq r_0\} \cap \{|y| \leq \theta^2\}.$$

We now break into two cases as outlined at the start of the proof.

Case 1: $\bar{d} := d + s^{-1}b > \beta|\lambda|$. Here we use the barriers constructed in Theorem 3.3. Note first that $\bar{d} \leq 2\delta_0/s$, so by ensuring $\delta_0(\bar{d}, s)$ is small, we can assume $\bar{d}$ is small also.

Fix p as in Theorem 3.3, and fix $\sigma : \mathbb{R} \to \mathbb{R}$ a smooth function satisfying $|\sigma(z)^p - z| \leq 1/10$. Define

$$f(y) = \sigma(t^{-1}\lambda + h(y)), \quad h(y) = (\theta - |y|)^{-1}. \tag{34}$$

Note that on Ω_1 we have $h \geq 4$ and $|D^k h| \leq c(k, s, \theta)$.

Provided $t \geq s|\lambda|$, we have $|f|_{C^3(\Omega_1)} \leq c(\mathbf{C}, s, \theta)$. Therefore there are $t_0(\mathbf{C})$, $r_0(\mathbf{C}, s, \theta)$ so that for every $s|\lambda| \leq t \leq t_0$, there are surfaces X_t defined in Ω_1 with negative mean curvature, as constructed in Theorem 3.3. Each y-slice $X_t \cap \{y = y'\}$ is trapped between $H(tf(y')^p \pm t) \times \{y'\}$ in $\mathbb{R}^{n+1} \times \{y'\}$, and hence (recalling our definition of σ, f) is trapped between $H(\lambda + 2th(y')) \times \{y'\}$ and $H(\lambda + th(y')/2) \times \{y'\}$.

Provided $\delta_0(\mathbf{C})$ is sufficiently small, we have $\lambda + t_0 h(y')/2 \geq t_0/4$. Therefore since $D_{\mathbf{C}}(M; B_1) \leq |\lambda| + D_{T_\lambda}(M; B_1) \leq 2\delta_0 < t_0/8$ (taking $\delta_0(\mathbf{C})$ smaller as necessary), we deduce M lies below X_{t_0} in Ω_1. Set $t_1 = \beta^{-2}(b + sd) = \beta^{-2}s\bar{d}$, and note that $t_1 \leq 2\beta^{-2}\delta_0 < t_0$ for $\delta_0(\beta, \mathbf{C})$ small. We claim that M lies below X_{t_1} in $\partial\Omega_1$, provided $\beta(\mathbf{C})$ is chosen sufficiently small.

Let $S_1 = \partial\Omega_1 \cap \{|y| = \theta - s\}$. In S_1 we have $h \geq 1/s$, and so X_{t_1} lies above $H(\lambda + \beta^{-2}\bar{d}/2) \times \mathbb{R}^l$ in S_1. But of course $\beta^{-2}\bar{d}/2 \geq d$, and so $H(\lambda + \beta^{-2}\bar{d}/2) \times \mathbb{R}^l$ lies above M in S_1.

Let $S_2 = \partial\Omega_1 \cap \{|x| = r_0\}$. In S_2, X_{t_1} lies above $H(\lambda + \beta^{-2}\bar{d}) \times \mathbb{R}^l$, and hence above $H(\beta^{-2}\bar{d}/2) \times \mathbb{R}^l$. On the other hand, provided $\bar{d}(\beta)$, $\delta_0(\beta, \bar{d}, r_0)$ are sufficiently small, in S_2 M lies below $\mathrm{graph}_{T_\lambda}(b|x|^\gamma)$, which in S_2 lies below $\mathrm{graph}_{\mathbf{C}}(c\beta^{-1}\bar{d}|x|^\gamma)$, which in S_2 lies below $H(c\beta^{-1}\bar{d}) \times \mathbb{R}^l$. Our claim follows by ensuring $\beta(\mathbf{C})$ is small.

Since $t \mapsto X_t$ is continuous in the Hausdorff distance, by Theorem 3.3 and the previous claim we can bring t from t_0 down to t_1 to deduce M lies below X_{t_1} in Ω_1. In particular, since on Ω_2 we have $h \leq c(\theta)$, we deduce that each y-slice of $M \cap \Omega_2$ lies below $H(\lambda + c(\mathbf{C}, \beta, \theta)(b + sd))$. Repeating the above argument with the orientations reversed implies that $M \cap \Omega_2$ is trapped between $H(\lambda \pm c(\mathbf{C}, \beta, \theta)(b + sd))$. This proves Case 1.

Case 2: $d + s^{-1}b \leq \beta|\lambda|$. Here we construct graphical barriers for M over T_λ. There is no loss in generality in assuming $\lambda > 0$. For ease of notation write $\Phi(x, \epsilon) = \Phi_{\epsilon,+}(x)$ for the graphing function of $(1 + \epsilon)H_+$ over H_+ as in Lemma 2.2, and set $\mu = |\lambda|^{1/(1-\gamma)}$. Define for $i = 1, 2$ the domains $\tilde{\Omega}_i = \{(x, y) : (x/2, y) \in \Omega_i\}$.

First note that on T_λ we have the inequality $|x| \geq \mu/c(\mathbf{C})$. Second, recall that the C^3 regularity scale of T_λ at x is comparable to $|x|$. Third, note that by Lemma 4.3 (ensuring $\beta(\mathbf{C})$ is small) we know that

$$M \text{ is trapped between the graphs } \mathrm{graph}_{T_\lambda}(\pm c_1 d|x|^\gamma) \text{ in } B_1, \tag{35}$$

for some constant $c_1(\mathbf{C})$, and hence M is trapped between $\mathrm{graph}_{T_\lambda}(\pm c_1\beta|\lambda||x|^\gamma)$ in B_1.

For $A(\mathbf{C}, \gamma')$ a large constant to be determined later, let $\eta(t) : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function satisfying $\eta(t) = t$ for $|t| < A\beta/2$, $\eta(t) \equiv \mathrm{sign}(t)A\beta$ for

$|t| \geq A\beta$, and $|\eta'| \leq 10$. For $t \in [0,1]$ define $G_t(x,y) : T_\lambda \cap \{|y| < \theta\} \to \mathbb{R}$ by

$$G_t(x,y) = \mu\Phi(\mu^{-1}x, \eta(th(y))) - \eta(t)|\lambda| F_{\gamma'}(x), \tag{36}$$

where $F_{\gamma'}$ as in Lemma 3.1. The G_t will define our graphical barriers.

From Lemma 2.2, for $(x,y) = (r\theta, y) \in T_\lambda \cap \tilde{\Omega}_1$ we have

$$\mu\Phi(\mu^{-1}x, \eta(th(y))) = \eta(th(y))(\mu\Phi_+(\mu^{-1}x) \pm c(\mathbf{C})A\beta|\lambda|r^\gamma),$$

and so, ensuring $\beta(A, \mathbf{C})$ is small, we get

$$\eta(th(y))|\lambda|r^\gamma/c \leq \mu\Phi(\mu^{-1}x, \eta(th(y))) \leq c\eta(th(y))|\lambda|r^\gamma.$$

for $c = c(\mathbf{C})$. Since $|F_{\gamma'}(x)| \leq c|x|^{\gamma'}$ and $\eta(th(y)) \geq \eta(t)$, ensuring $r_0(\gamma', \mathbf{C})$ is small, we deduce that

$$\eta(th(y))|\lambda||x|^\gamma/c \leq G_t(x,y) \leq c\eta(th(y))|\lambda||x|^\gamma \leq cA\beta|\lambda||x|^\gamma \tag{37}$$

on $T_\lambda \cap \tilde{\Omega}_1$, for $c = c(\mathbf{C})$.

By a similar computation, recalling that $\gamma' > \gamma$ and $|x| = r$, we have

$$\begin{aligned} |\nabla G_t(x,y)| &\leq c\eta|\lambda|r^{\gamma-1} + ct|Dh||\lambda|r^\gamma + c\eta|\lambda|r^{\gamma'-1} \\ &\leq (cA\beta + ct|Dh|r)|\lambda|r^{\gamma-1}, \end{aligned}$$

and

$$|\nabla^2 G_t(x,y)| \leq (cA\beta + c|Dh|r + c|D^2h|r^2)|\lambda|r^{\gamma-2},$$

where $c = c(\mathbf{C}, \gamma')$. Ensuring $r_0(\mathbf{C}, \gamma', \beta, s, \theta)$ is sufficiently small, and recalling the bound $|x| \geq \mu/c(\mathbf{C})$ on T_λ, we get for $i = 0, 1, 2$ the bounds

$$|\nabla^i G_t(x,y)| \leq c(\mathbf{C}, \gamma')A\beta|\lambda||x|^{\gamma-i} \leq c(\mathbf{C}, \gamma')A\beta|x|^{1-i} \quad \text{on } T_\lambda \cap \tilde{\Omega}_1. \tag{38}$$

In particular, ensuring $\beta(\mathbf{C}, \gamma', A)$ is sufficiently small we get that $\mathrm{graph}_{T_\lambda}(G_t)$ is a smooth hypersurface without boundary in Ω_1.

We aim to show the graph of G_t has negative mean curvature in Ω_1. We first compute

$$\mathcal{M}_{T_\lambda}(G_t) = \mathcal{M}_{T_\lambda}(\mu\Phi(\mu^{-1}x, \eta(th(y)))) - \eta(t)|\lambda| \int_0^1 D\mathcal{M}_{T_\lambda}(G_{t,s})[F_{\gamma'}]ds =: I + II$$

where $G_{t,s}(x,y) = \mu\Phi(\mu^{-1}x\eta(th(y))) - s\eta(t)|\lambda|F_{\gamma'}(x)$. We claim that, at $(x,y) \in T_\lambda \cap \tilde{\Omega}_1$, with $|x| = r$, we have

$$|I| \leq c(t^2|Dh|^2 + t|D^2h|)|\lambda|r^\gamma, \quad II \leq -\eta(t)|\lambda|r^{\gamma'-2}/c \tag{39}$$

for $c = c(\mathbf{C}, \gamma')$. Bounds (39) will imply that on $T_\lambda \cap \tilde{\Omega}_1$ and for $0 < t \leq A\beta$ we have

$$\mathcal{M}_{T_\lambda}(G_t) \leq c(\mathbf{C}, \gamma')t|\lambda|(r|h|_{C^2(\tilde{\Omega}_1)} - 1)r^{\gamma'-2} < 0, \tag{40}$$

provided we ensure $r_0(\mathbf{C}, \gamma', \theta, s)$ is chosen sufficiently small.

Let us prove our claim for $|I|$, i.e. the first inequality in (39). By construction, $\mathrm{graph}_{T_\lambda}(\mu\Phi(\mu^{-1}x, \eta(th(y)))) \cap \{|y| < \theta\}$ coincides with $S \cap \{|y| < \theta\}$ where S is the hypersurface

$$S = \bigcup_{|y|<\theta} [(1 + \eta(th(y)))H(\lambda)] \times \{y\}.$$

Since $\max\{\eta(th(y))|x|, \mu\Phi(\mu^{-1}x, \eta(th(y)))\} \leq |x|/2$ provided $\beta(\mathbf{C}, A)$ is sufficiently small, it will suffice to prove the bound

$$|\mathcal{M}_S((1+\eta(th(y)))x, y)| \leq c(t^2|Dh|^2 + t|D^2h|)|\lambda||x|^\gamma \tag{41}$$

for any $((1+\eta(th(y)))x, y) \in S \cap \{|y| < \theta\}$, where $\mathcal{M}_S$ is the mean curvature of S. From Lemma 3.4, for the same x, y as above we have the bound

$$|\mathcal{M}_S| \leq c(l)|x \cdot \nu_{H(\lambda)}(x)||D^2\eta(th(y))| + c(l)|D\eta(th(y))|^2|h_{H(\lambda)}(x^T, x^T)|, \tag{42}$$

where $h_{H(\lambda)}$ is the second fundamental form of $H(\lambda)$, and $\nu_{H(\lambda)}$ the unit normal. Trivially we have

$$|D\eta(th(y))|^2 \leq ct^2|Dh|^2, \quad |D^2\eta(th(y))| \leq ct|D^2h| + ct^2|Dh|^2, \tag{43}$$

$$\text{and} \quad |h_{H(\lambda)}(x^T, x^T)| \leq c|x^T|^2/|x| \leq c|x|, \tag{44}$$

for $c = c(\mathbf{C})$.

If $|x| \leq R_0\mu$ (for R_0 as in (1)) then since $\mu/c(\mathbf{C}) \leq |x|$ also, the bound (41) follows from (42), (43), (44) and the inequality $|x| \leq R_0|\lambda||x|^\gamma$. If $|x| > R_0\mu$, then near x, $H(\lambda)$ is graphical over $\mathbf{C}_0$ by the function Ψ_λ as in (4). From (2), (3) we have

$$|x \cdot \nu_{H(\lambda)}(x)| \leq c|\nabla\Psi_\lambda||x| \leq c|\lambda||x|^\gamma, \quad |h_{H(\lambda)}(x^T, x^T)| \leq c|\nabla^2\Psi_\lambda||x|^2 \leq c|\lambda||x|^\gamma, \tag{45}$$

and the bound (41) follows from (42), (43), (45).

We consider now the bound for II. By similar computations as before, we have

$$|\nabla^i G_{t,s}(x, y)| \leq c(\mathbf{C}, \gamma')A\beta|\lambda||x|^{\gamma - i}, \quad (i = 0, 1, 2), \quad \text{on } T_\lambda \cap \tilde{\Omega}_1,$$

for any $s, t \in [0, 1]$. By scaling and the definition of $F_{\gamma'}$ we have

$$D\mathcal{M}_{T_\lambda}(G_{t,s})[F_{\gamma'}]|_{(x,y)} = \mu^{-2}D\mathcal{M}_{T_1}(G^\mu_{t,s})[F_{\gamma'}(\mu\cdot)]|_{(\mu^{-1}x,y)} \tag{46}$$

$$= \mu^{\gamma'-2}D\mathcal{M}_{T_1}(G^\mu_{t,s})[F_{\gamma'}]|_{(\mu^{-1}x,y)}, \tag{47}$$

where $G^\mu_{t,s}(\xi, \zeta) = \mu^{-1}G_{t,s}(\mu\xi, \zeta)$. Using (38) we have on $T_1 \cap \mu^{-1}\tilde{\Omega}_1$,

$$|\nabla^i G^\mu_{t,s}| \leq \mu^{-1+\gamma}cA\beta|\lambda||x|^{\gamma-i} \leq cA\beta|x|^{\gamma-i} \quad (i = 0, 1, 2),$$

and therefore provided $\beta(\mathbf{C}, \gamma', A)$ is sufficiently small, we can apply Lemma 3.2 to deduce

$$\mu^{\gamma'-2}D\mathcal{M}_{T_1}(G^\mu_{t,s})[F_{\gamma'}]|_{(\mu^{-1}x,y)} \geq |x|^{\gamma'-2}/c.$$

This proves the bound for II in (39), completing the proof of our claim and hence the inequality (40).

We now use G_t to control $D_{T_\lambda}(M; \Omega_2)$. First note that if $0 < t \leq A\beta$, then (37) implies

$$G_t(x, y) \geq \min\{th(y), A\beta\}|\lambda||x|^\gamma/c \geq \min\{t, A\beta\}|\lambda||x|^\gamma/c \quad \text{on } T_\lambda \cap \tilde{\Omega}_1, \tag{48}$$

for $c = c(\mathbf{C}, \gamma')$. Therefore, by ensuring $A(\mathbf{C}, \gamma')$ is sufficiently large from (35) we know that M lies below $\mathrm{graph}_{T_\lambda}(G_{A\beta})$ in Ω_1. Set $t_1 = \min\{\beta^{-2}|\lambda|^{-1}(b + sd), A\beta\}$. We claim that, provided $\beta(\mathbf{C})$ is chosen sufficiently small, M lies below $\mathrm{graph}_{T_\lambda}(G_t)$ in $\partial\Omega_1$ for every $t_1 \leq t \leq A\beta$.

We prove this claim. Of course if $t_1 = A\beta$ there is nothing to show, so let us assume $t_1 < A\beta$. Suppose $(x, y) + G_t(x, y)\nu_{T_\lambda}(x, y) \in \partial\Omega_1 \cap S_1$. Then $|y| = \theta - s$

and (by (38)) $|x| < 2r_0$. Since $h(y) \geq 1/s$, we can use (48) and our assumption $d + s^{-1}b \leq \beta|\lambda|$ to estimate

$$G_t(x, y) \geq \min\{(\beta^{-2}|\lambda|^{-1}sd)s^{-1}|\lambda||x|^\gamma/c, A\beta|\lambda||x|^\gamma/c\} \geq c_1 d|x|^\gamma$$

provided $\beta(\mathbf{C})$, $A(\mathbf{C})^{-1}$ are chosen sufficiently small. Therefore $\mathrm{graph}_{T_\lambda}(G_t)$ lies above $\mathrm{graph}_{T_\lambda}(c_1 d|x|^\gamma)$ in S_1, and hence lies above M in S_1.

Suppose $(x, y) + G_t(x, y)\nu_{T_\lambda}(x, y) \in \partial\Omega_1 \cap S_2$. Then as before $(x, y) \in \tilde{\Omega}_1$, and we can estimate instead

$$G_t(x, y) \geq \min\{(\beta^{-2}|\lambda|^{-1}b)|\lambda||x|^\gamma/c, A\beta|\lambda||x|^\gamma/c\} \geq b|x|^\gamma,$$

again ensuring $\beta(\mathbf{C}), A(\mathbf{C})^{-1}$ are small. We deduce $\mathrm{graph}_{T_\lambda}(G_t)$ lies above $\mathrm{graph}_{T_\lambda}(b|x|^\gamma)$ in S_2, and hence by our assumptions lies above M in S_2. This finishes the proof of our claim.

By our last two claims and the negative mean curvature (40) we can bring t from $A\beta$ down to t_1 and deduce by the maximum principle [13] that M lies below $\mathrm{graph}_{T_\lambda}(G_t)$ in Ω_1. In particular, since $h \leq c(\mathbf{C}, \theta)$ on $\tilde{\Omega}_2$ from (37) we get that M lies below $\mathrm{graph}_{T_\lambda}(c(\mathbf{C}, \theta, \gamma')(b + sd)|x|^\gamma)$ in Ω_2. Repeating the argument with the orientation swapped, we deduce M is trapped between $\mathrm{graph}_{T_\lambda}(\pm c(\mathbf{C}, \theta, \gamma')(b+ sd)|x|^\gamma)$ in Ω_2. Since $b + sd \leq s\beta|\lambda|$, ensuring $\beta(\mathbf{C}, \theta, \gamma')$ is sufficiently small we can apply Lemma 4.3 to finish the proof of Case 2.

With $A_{1,\rho}$ in place of B_1: To get (32), we only need to modify our definition of h, Ω_1, Ω_2. In this case, we define

$$h(y) = (|y| - \theta^{-1}\rho)^{-1} + (\theta - |y|)^{-1},$$

and

$$\Omega_1 = \{r \leq r_0\} \cap \{|y| \in [(\theta^{-1} + s)\rho, \theta - s]\}, \quad \Omega_2 = \{r \leq r_0\} \cap \{|y| \in [\theta^{-2}\rho, \theta^2]\}.$$

The proof for (32) is then verbatim to the proof above for (31), of course replacing B_1 with $A_{1,\rho}$ wherever it occurs, and allowing all constants to depend on ρ also. $\square$

The main utility of Theorem 4.4 is in the below Corollary 4.5 concerning inhomogeneous blow-up limits, in particular in the lower bound of Item 3.

Corollary 4.5. *Let M_i be a sequence of complete minimal hypersurfaces in B_1, and $\lambda_i \to 0$. Suppose that*

$$D_{T_{\lambda_i}}(M_i; B_1) \to 0, \quad (1/2)||\mathbf{C}||(B_1) \leq ||M_i||(B_1) \leq (3/2)||\mathbf{C}||(B_1),$$

and let μ_i be a sequence such that $\sup_i \mu_i^{-1} D_{T_{\lambda_i}}(M_i; B_1) < \infty$.

Then, first, there are $\tau_i \to 0$ so that

$$M_i \cap B_{1-\tau_i} \cap \{|x| \geq \tau_i\} = \mathrm{graph}_{T_{\lambda_i}}(u_i),$$

for $u_i : B_{1-\tau_i/2} \cap \{|x| \geq \tau_i/2\} \to \mathbb{R}$ smooth functions satisfying

$$|x|^{-1}|u_i| + |\nabla u_i| + |x||\nabla^2 u_i| \leq \tau_i.$$

Second, passing to a subsequence, we can find a Jacobi field v on $\mathbf{C} \cap B_1$ so that for any given $\theta < 1$ we have:

(1) *$\mu_i^{-1} u_i \to v$ smoothly on compact subsets of $\mathbf{C} \cap B_1 \cap \{|x| > 0\}$;*
(2) *$\sup_{\mathbf{C} \cap B_1} ||x|^{-\gamma} v| \leq c(\mathbf{C}) \liminf_i \mu_i^{-1} D_{T_{\lambda_i}}(M_i; B_1)$;*
(3) *$\limsup_i \mu_i^{-1} D_{T_{\lambda_i}}(M_i; B_{\theta^2}) \leq c(\mathbf{C}, \theta) \sup_{\mathbf{C} \cap B_\theta} ||x|^{-\gamma} v|$.*

Third, given any $\rho \in (0, 1/2]$, the above Corollary also holds with $A_{1,\rho}$, $A_{1-\tau_i,\rho+\tau_i}$, $A_{1-\tau_i/2,\rho+\tau_i/2}$, $A_{\theta,\theta^{-1}\rho}$, $A_{\theta^2,\theta^{-2}\rho}$ in place of B_1, $B_{1-\tau_i}$, $B_{1-\tau_i/2}$, B_θ, B_{θ^2} (resp.), in which case all constants depend on ρ also.

Remark 4.6. Since $2\gamma > -n+2$, and by Lemma 2.6, we have for every $\theta < 1$:

$$\frac{1}{c(\mathbf{C},\theta)} \sup_{\mathbf{C}\cap B_\theta} ||x|^{-\gamma} v|^2 \leq \int_{\mathbf{C}\cap B_1} |v|^2 \leq \int_{\mathbf{C}\cap B_1} |x|^{-2}|v|^2 \leq c(\mathbf{C}) \sup_{\mathbf{C}\cap B_1} ||x|^{-\gamma} v|^2.$$

Proof. The existence of τ_i, u_i follows from the definition of D, the constancy theorem, and Allard's theorem by a standard argument. For convenience write $U_i = B_{1-\tau_i} \cap \{|x| > \tau_i\}$ and $d_i = D_{T_{\lambda_i}}(M_i; B_1)$. After passing to a subsequence we can assume that $\Gamma = \lim_i \mu_i^{-1} d_i$ exists, and for all i, either $d_i > \beta|\lambda_i|$ or $d_i \leq \beta|\lambda_i|$, for β a small number to be determined momentarily.

By definition of D, for all i, $M_i \cap B_1$ is trapped between $H(\lambda_i \pm d_i) \times \mathbb{R}^l$. If $d_i > \beta|\lambda_i|$, then by ensuring $\tau_i \to 0$ sufficiently slowly, from equations (1), (2) we get that $M_i \cap B_1 \cap \{|x| > \tau_i\}$ is trapped in $\{|x| > \tau_i\}$ between $\mathrm{graph}_{\mathbf{C}\cap\{|x|>\tau_i/2\}}(\pm c(\mathbf{C})(d_i + |\lambda_i|)|x|^\gamma)$, and hence $|u_i| \leq c(d_i + |\lambda_i|)|x|^\gamma \leq c(\mathbf{C},\beta) d_i |x|^\gamma$. If $d_i \leq \beta|\lambda_i|$, then provided $\beta(\mathbf{C})$ is sufficiently small Lemma 4.3 implies $M_i \cap B_1$ is trapped between $\mathrm{graph}_{T_\lambda}(\pm c(\mathbf{C}) d_i |x|^\gamma)$, and hence $|u_i| \leq c(\mathbf{C}) d_i |x|^\gamma$.

Either way, we have that

$$\sup_{T_{\lambda_i}\cap U_i} ||x|^{-\gamma} u_i| \leq c(\mathbf{C},\beta) d_i, \tag{49}$$

and hence by standard elliptic theory we can pass to a subsequence, find a Jacobi field v on $\mathbf{C} \cap B_1$, and get smooth convergence $\mu^{-1} u_i \to v$ on compact subsets of $\mathbf{C} \cap B_1 \cap \{|x| > 0\}$. The estimate (49) implies

$$\sup_{\mathbf{C}\cap U} ||x|^{-\gamma} v| \leq c(\mathbf{C},\beta)\Gamma \quad \forall U \subset\subset \mathbf{C} \cap B_1 \cap \{|x| > 0\},$$

which proves Items 1, 2.

To prove Item 3, we use Theorem 4.4 and our hypotheses, to deduce that for every $s > 0$ there is an $r_0 > 0$ so that for $i \gg 1$ we have

$$D_{T_{\lambda_i}}(M_i; B_{\theta^2}) \leq c_0 \sup_{T_{\lambda_i}\cap B_\theta \cap\{|x|>r_0\}} ||x|^{-\gamma} u_i| + c_0 s D_{T_{\lambda_i}}(M_i; B_1),$$

where $c_0 = c_0(\mathbf{C},\theta)$ is independent of s. We can therefore take a limit as $i \to \infty$, and then as $s \to 0$, we deduce Item 3. ∎

5. Geometric 3-annulus lemma

Lemma 5.1. *Given $\epsilon < \epsilon_0(\mathbf{C})/16$, we can find an $R_0(\mathbf{C},\epsilon) > 1$ so that for every $R \geq R_0$, there is a $\delta_0(\mathbf{C},\epsilon,R) > 0$ so that the following holds.*

Let $|\lambda| < \delta_0$, and let M be a complete minimal hypersurface in B_R, such that

$$E(M,\mathbf{C},R) < \delta_0, \quad \theta_M(0,R) \leq (3/2)\theta_{\mathbf{C}}(0). \tag{50}$$

Then:

$$E(M,T_\lambda,1) \geq E(M,T_\lambda,1/R) R^{\gamma-1+\epsilon} \tag{51}$$

$$\implies E(M,T_\lambda,R) \geq E(M,T_\lambda,1) R^{\gamma-1+\epsilon_0-\epsilon}. \tag{52}$$

Proof. Set $\epsilon_0 = \min\{p_2 - p_1, 1\}$ for p_i as in (14). Assume $R_0 \geq e^{2/\epsilon}$, so that we can write $R = R_*^k$ for some integer $k \geq 1$ and some $R_* \in [e^{2/\epsilon}, e^{4/\epsilon})$. We will show the Lemma holds provided $k(\mathbf{C}, \epsilon)$ (and hence R_0) is sufficiently large, to be determined below.

Suppose the Lemma failed. Then we have sequences $\delta_i \to 0$, $\lambda_i \to 0$, and complete minimal hypersurfaces M_i in B_R so that (50) holds but

$$E(M_i, T_{\lambda_i}, 1/R) \leq E(M_i, T_{\lambda_i}, 1) R^{-\gamma+1-\epsilon}$$
$$\text{and } E(M_i, T_{\lambda_i}, R) \leq E(M_i, T_{\lambda_i}, 1) R^{\gamma-1+\epsilon_0-\epsilon}.$$

Since (52) vacuously holds if $E(M, T_\lambda, 1) = 0$, there is no loss in assuming $M_i \cap B_1 \neq \emptyset$ for all i. Then by our hypotheses (50), standard compactness of stationary integral varifolds, and the constancy theorem, we deduce $M_i \to [\mathbf{C}]$ as varifolds in B_R. By Allard's theorem we can find an exhaustion U_i of $B_R \setminus \{|x| = 0\}$ so that

$$M_i \cap U_i = \mathrm{graph}_{T_{\lambda_i}}(u_i)$$

for smooth functions u_i. By Corollary 4.5, after passing to a subsequence, the rescaled functions $E(M_i, T_{\lambda_i}, 1)^{-1} u_i$ converge on compact subsets of $\mathbf{C} \cap B_R \setminus \{r = 0\}$ to a Jacobi field v on $\mathbf{C} \cap B_R$ satisfying

$$\sup_{\mathbf{C} \cap B_{1/R}} (1/R)^{\gamma-1} ||x|^{-\gamma} v| \leq c(\mathbf{C}) R^{-\gamma+1-\epsilon},$$
$$\sup_{\mathbf{C} \cap B_2} ||x|^{-\gamma} v| \geq 1/c(\mathbf{C}),$$
$$\sup_{\mathbf{C} \cap B_R} R^{\gamma-1} ||x|^{-\gamma} v| \leq c(\mathbf{C}) R^{\gamma-1+\epsilon_0-\epsilon}.$$

Define

$$S(i)^2 = R_*^{i(-n-l)} \int_{B_{R_*^i}} v^2.$$

Then from Lemma 2.6 we have

$$S(-k) \leq c(1/R)^\gamma \sup_{B_{1/R}} ||x|^{-\gamma} v| \leq c R_*^{-k(\gamma+\epsilon)} \tag{53}$$

$$S(1) \geq c^{-1} R_*^{-n-l} \sup_{B_2} ||x|^{-\gamma} v| \geq 1/c(\mathbf{C}, \epsilon) \tag{54}$$

$$S(k) \leq c R_*^{k(\gamma+\epsilon_0-\epsilon)}. \tag{55}$$

We claim that, for any $\eta > 0$, provided $k(\mathbf{C}, \epsilon, \eta) \in \mathbb{N}$ is chosen sufficiently large, then we have $S(1) \leq \eta$, which will contradict (54) for $\eta(\mathbf{C}, \epsilon)$ sufficiently small. We prove this claim. First assume

$$S(1) \geq R_*^{(k+1)(\gamma+\epsilon/2)} S(-k). \tag{56}$$

Then by Lemma 2.8 and our choice of R_*, we have

$$S(k) \geq R_*^{(k-1)(\gamma+\epsilon_0-\epsilon/2)} S(1),$$

which implies

$$S(1) \leq c(\mathbf{C}, \epsilon) R_*^{-k\epsilon/2} \leq \eta, \tag{57}$$

provided we ensure $k(\mathbf{C}, \epsilon, \eta)$ is large. On the other hand, if (56) fails, then we again have (57) (for perhaps a larger constant $c(\mathbf{C}, \epsilon)$, and hence a larger $k(\mathbf{C}, \epsilon, \eta)$). This proves our claim, and finishes the proof of Lemma 5.1. □

6. Growth of entire hypersurfaces

Proposition 6.1. *There are constants $\epsilon_1(\mathbf{C})$, $\rho_1(\mathbf{C})$ so that for every $\rho \leq \rho_1$, and $\eta > 0$, we can find a $\delta_1(\mathbf{C}, \eta, \rho)$ satisfying the following. Let M be a complete minimal hypersurface in B_1 satisfying*

$$E(M, \mathbf{C}, 1) < \delta_1, \quad ||M||(B_1) \leq (3/2)||\mathbf{C}||(B_1). \tag{58}$$

Then we can find a $\lambda \in (-\eta, \eta)$ so that

$$E(M, T_\lambda, \rho) \leq \rho^{\gamma-1+\epsilon_1} E(M, T_\lambda, 1), \tag{59}$$

and $E(M, T_\lambda, 1) \leq c(\mathbf{C}) E(M, \mathbf{C}, 1)$.

Proof. Suppose the Proposition failed. Then for ϵ_1, ρ_1 to be determined later, we can find sequences $\delta_i \to 0$, and complete minimal hypersurfaces M_i in B_1 satisfying (58) but failing (59) for all $\lambda \in (-\eta, \eta)$. Let λ'_i minimize $\lambda \mapsto E(M_i, T_\lambda, 1)$. Trivially $\lambda'_i \to 0$ and $E(M_i, T_{\lambda'_i}, 1) \leq E(M_i, \mathbf{C}, 1)$.

By standard compactness and the constancy theorem, $M_i \to \kappa[\mathbf{C}]$ as varifolds for $\kappa \in \{0, 1\}$. Since (59) is trivially satisfied if $M \cap B_\rho = \emptyset$, by our contradiction hypothesis we have $M_i \cap B_\rho \neq \emptyset$ for all i, and hence $\kappa = 1$. Allard's theorem implies therefore $M_i \to \mathbf{C}$ smoothly on compact subsets of $B_1 \setminus \{|x| = 0\}$.

For U_i an exhaustion of $B_1 \setminus \{|x| = 0\}$, we can write $M_i \cap U_i = \mathrm{graph}_{T_{\lambda'_i}}(u'_i)$. Passing to a subsequence, by Corollary 4.5 we can get convergence $E(M, T_{\lambda'_i}, 1)^{-1} u'_i \to v$, for some Jacobi field on $\mathbf{C} \cap B_1$ with $\sup_{\mathbf{C} \cap B_1} ||x|^{-\gamma} v| \leq c(\mathbf{C})$, and hence (by Lemma 2.6) $\int_{\mathbf{C} \cap B_1} |x|^{-2} v^2 \leq c(\mathbf{C})$.

By (13), (14), we can write $v(r\theta, y) = a r^\gamma \psi_1(\theta) + z(r\theta, y)$, for $|a| \leq c(\mathbf{C})$ and z satisfying the decay

$$\rho^{-n-l} \int_{\mathbf{C} \cap B_\rho} z^2 \leq c(\mathbf{C}) \rho^{2\gamma + 4\epsilon_1}$$

for some $\epsilon_1(\mathbf{C}) > 0$ determined by the spectral decomposition of $\mathbf{C}_0$. Using Lemma 2.6 we deduce

$$\rho^\gamma \sup_{\mathbf{C} \cap B_\rho} ||x|^{-\gamma} z| \leq c(\mathbf{C}) \left(\rho^{-n-l} \int_{\mathbf{C} \cap B_{2\rho}} z^2 \right)^{1/2} \leq c(\mathbf{C}) \rho^{\gamma + 2\epsilon_1} \tag{60}$$

for all $\rho \leq 1/2$.

Let $\lambda_i = \lambda'_i + a E(M, T_{\lambda'_i}, 1)$. By altering U_i as necessary, we can write $M_i \cap U_i = \mathrm{graph}_{T_{\lambda_i}}(u_i)$, and it's straightforward to check that $E(M, T_{\lambda'_i}, 1)^{-1} u_i \to v - a r^\gamma \psi_1 = z$ smoothly on compact subsets of $\mathbf{C} \cap B_1 \setminus \{|x| = 0\}$.

Using property (30) we have $E(M, T_{\lambda_i}, 1) \leq c(\mathbf{C}) E(M, T_{\lambda'_i}, 1)$, and by definition of λ'_i we have $E(M, T_{\lambda'_i}, 1) \leq E(M, T_{\lambda_i}, 1)$. Therefore, after passing to a subsequence, we can assume

$$\frac{E(M, T_{\lambda'_i}, 1)}{E(M, T_{\lambda_i}, 1)} \to b, \quad 1/c(\mathbf{C}) \leq b \leq 1.$$

In particular, we have smooth convergence $E(M, T_{\lambda_i}, 1)^{-1} u_i \to bz$.

By (60) and Corollary 4.5, we have

$$\limsup_i E(M, T_{\lambda_i}, 1)^{-1} E(M, T_{\lambda_i}, \rho) \leq c(\mathbf{C}) \rho^{\gamma-1} \sup_{\mathbf{C} \cap B_{2\rho}} ||x|^{-\gamma} bz| \leq c(\mathbf{C}) \rho^{\gamma - 1 + 2\epsilon_1}$$

for all $\rho \leq 1/4$. Choose $\rho(\mathbf{C})$ sufficiently small so that $c(\mathbf{C})\rho^{\epsilon_1} \leq 1$, we deduce

$$E(M, T_{\lambda_i}, \rho) \leq \rho^{\gamma-1+\epsilon_1} E(M, T_{\lambda_i}, 1)$$

for all $i \gg 1$. This is a contradiction, and finishes the proof of Proposition 6.1. $\square$

Proposition 6.2. *There are constants $\epsilon_2(\mathbf{C}) > 0$, $c_2(\mathbf{C}) > 1$ so that the following holds. Let M be a complete minimal hypersurface in $\mathbb{R}^{n+l+1}$, and suppose that $R^{-1}M \to [\mathbf{C}]$ as varifolds as $R \to \infty$. Then there is a λ so that*

$$E(M, T_\lambda, LR) \geq c_2(\mathbf{C})^{-1} L^{\gamma-1+\epsilon_2} E(M, T_\lambda, R) \tag{61}$$

for all $L > 1$ and R sufficiently large (depending only on M). In particular, either $M = T_\lambda$, or there is a constant $C(M) > 0$ independent of R so that

$$E(M, T_\lambda, R) \geq R^{\gamma-1+\epsilon_2}/C(M) \quad \forall R \geq C(M). \tag{62}$$

Remark 6.3. From (30) and the scaling of E, if (62) holds for some λ then (62) holds for any λ', with a potentially larger $C(M, \lambda')$.

Proof. Fix $\epsilon_2 = \epsilon = \min\{\epsilon_0, \epsilon_1, 1\}/16$, $L_0 = \max\{R_0(\mathbf{C}, \epsilon), 1/\rho_1(\mathbf{C})\}$, let $\delta_0(\mathbf{C}, \epsilon, R = L_0)$ be as in Lemma 5.1, and let $\delta_1(\mathbf{C}, \eta = \delta_0, \rho = 1/L_0)$ be as in Proposition 6.1. By our hypothesis there is a radius R_* so that for all $R \geq R_*$ we have

$$E(M, \mathbf{C}, R) < \min\{\delta_0, \delta_1\}, \quad \theta_M(0, R) \leq (3/2)\theta_{\mathbf{C}}(0).$$

Apply Proposition 6.1 to $R_*^{-1}M$ to obtain a T_λ, with $R_*^{1-\gamma}|\lambda| < \delta_0$, so that

$$E(M, T_\lambda, R_*/L_0) L_0^{\gamma-1+\epsilon} \leq E(M, T_\lambda, R_*).$$

By our choice of ϵ, L_0, R_*, we can then apply Lemma 5.1 to $R_*^{-1}M$ to get

$$E(M, T_\lambda, R_*) L_0^{\gamma-1+\epsilon} \leq E(M, T_\lambda, L_0 R_*).$$

Now since $(R_* L_0)^{-1} T_\lambda = T_{(R_* L_0)^{\gamma-1}\lambda}$, we can apply Lemma 5.1 again to $(L_0 R_*)^{-1}M$ to get

$$E(M, T_\lambda, L_0 R_*) L_0^{\gamma-1+\epsilon} \leq E(M, T_\lambda, L_0^2 R_*).$$

We can iterate to obtain

$$E(M, T_\lambda, L_0^{k+l} R_*) \geq L_0^{(\gamma-1+\epsilon)l} E(M, T_\lambda, L_0^k R_*), \quad \forall k, l \in \{0, 1, 2, \ldots\}.$$

(61) then follows with $c_2 = L_0^{\max\{2,\epsilon\}} = L_0^2$. This completes the proof of Proposition 6.2. $\square$

7. One-sided decay and proof of main theorem

Proposition 7.1. *Let M be a complete minimal hypersurface in $\mathbb{R}^{n+l+1}$ lying to one side of $\mathbf{C}$, such that $\theta_M(\infty) < 2\theta_{\mathbf{C}}(0)$. Then for any $\epsilon > 0$, we have*

$$E(M, \mathbf{C}, LR) \leq c_3(\mathbf{C}, \epsilon) L^{\gamma-1+\epsilon} E(M, \mathbf{C}, R) \tag{63}$$

for all $L > 1$ and all R sufficiently large (depending only on M). In particular, there is a constant $C(M, \epsilon)$ independent of R so that

$$E(M, \mathbf{C}, R) \leq R^{\gamma-1+\epsilon} C(M, \epsilon) \quad \forall R \geq 1. \tag{64}$$

Proof. We first observe that by our hypotheses, the monotonicity formula, and Lemma 2.4 we must have $R^{-1}M \to [\mathbf{C}]$ as varifolds as $R \to \infty$, and in particular we have $E(M, \mathbf{C}, R) \to 0$ as $R \to \infty$.

If $M = \mathbf{C}$ then the Proposition trivially holds, so assume $M \neq \mathbf{C}$. Fix $0 < \epsilon < \epsilon_0/16$ (there is no loss in assuming ϵ is as small as we like), and let $L_0 = R_0(\mathbf{C}, \epsilon)$ as in Lemma 5.1. Suppose, towards a contradiction, there was a sequence $R_j \to \infty$ such that

$$E(M, \mathbf{C}, L_0 R_j) \geq L_0^{\gamma-1+\epsilon} E(M, \mathbf{C}, R_j).$$

Since $R_j^{-1}M \to \mathbf{C}$, we can fix an $R_* = R_j$ sufficiently large and apply Lemma 5.1 successively to $R_*^{-1}M$, $(L_0 R_*)^{-1}M$, etc. to deduce

$$E(M, \mathbf{C}, L_0^{k+l} R_*) \geq (L_0^k)^{\gamma-1+\epsilon} E(M, \mathbf{C}, L_0^k R_*) \quad \forall k, l \geq 0 \tag{65}$$

By iterating (65), we deduce

$$E(M, \mathbf{C}, LR) \geq c(\mathbf{C}, \epsilon)^{-1} L^{\gamma-1+\epsilon} E(M, \mathbf{C}, R) \tag{66}$$

for all $L > 1$, and all $R \geq R_*$.

Choose $R_i \to \infty$ so that

$$2E(M, \mathbf{C}, R_i) \geq \sup_{R \geq R_i} E(M, \mathbf{C}, R), \tag{67}$$

and consider the rescaled surfaces $M_i = R_i^{-1}M$. For $i \gg 1$, by Allard's theorem we can find an exhaustion U_i of $\mathbb{R}^{n+l+1} \setminus \{|x| = 0\}$ so that $M_i = \mathrm{graph}_{\mathbf{C}}(u_i)$. From (67), we have

$$2E(M_i, \mathbf{C}, 1) \geq \sup_{R \geq 1} E(M, \mathbf{C}, R).$$

Therefore by Corollary 4.5, after passing to a subsequence as necessary, the rescaled graphs $E(M_i, \mathbf{C}, 1)^{-1} u_i$ converge smoothly on compact subsets of $\mathbf{C} \setminus \{|x| = 0\}$ to a Jacobi field v satisfying

$$\sup_{\mathbf{C} \cap B_R} R^{\gamma-1} ||x|^{-\gamma} v| \leq c(\mathbf{C}) \quad \forall R \geq 1. \tag{68}$$

Moreover, since M and hence M_i all lie to one side of $\mathbf{C}$, after flipping orientation as necessary we can assume $v \geq 0$.

Lemma 2.7 implies $v(x = r\theta, y) = ar^\gamma \psi_1(\theta)$, where by (68) $|a| \leq c(\mathbf{C})$. From Corollary 4.5, for any $L > 1$ and any $i \gg 1$ we have

$$\frac{E(M_i, \mathbf{C}, L)}{E(M_i, \mathbf{C}, 1)} \leq c(\mathbf{C}) \sup_{\mathbf{C} \cap B_{2L}} L^{\gamma-1} ||x|^{-\gamma} v| \leq c(\mathbf{C}) |a| L^{\gamma-1} \leq c(\mathbf{C}) L^{\gamma-1},$$

and hence

$$E(M, \mathbf{C}, LR_i) \leq c(\mathbf{C}) L^{\gamma-1} E(M, \mathbf{C}, R_i) \tag{69}$$

for all $L > 1$ and i sufficiently large, depending on L.

Combining (66), (69) we get: if $L > 1$, then for all i large (depending on L, M) we have

$$c(\mathbf{C}, \epsilon)^{-1} E(M, \mathbf{C}, R_i) L^{\gamma-1+\epsilon} \leq E(M, \mathbf{C}, LR_i) \leq c(\mathbf{C}) L^{\gamma-1} E(M, \mathbf{C}, R_i). \tag{70}$$

Since $M \neq \mathbf{C}$, for all $i \gg 1$ $E(M, \mathbf{C}, R_i) \neq 0$, and so if we ensure $L(\mathbf{C}, \epsilon)$ is sufficiently large (70) will yield a contradiction. Therefore, recalling our initial contradiction hypothesis, we must have

$$E(M, \mathbf{C}, L_0 R) \leq L_0^{\gamma-1+\epsilon} E(M, \mathbf{C}, R) \tag{71}$$

for all $R \gg 1$. Iterating (71) gives (63) and (64). $\square$

Proof of Theorem 1.1. Assume that $M \neq T_\lambda$ for any λ. As in the proof of Proposition 7.1, we have $R^{-1}M \to [\mathbf{C}]$ as $R \to \infty$, and $E(M, \mathbf{C}, R) \to 0$ as $R \to \infty$. We can apply Proposition 6.2 and Remark 6.3 to find a constant $C(M)$ so that

$$E(M, \mathbf{C}, R) \geq R^{\gamma-1+\epsilon_2} C^{-1} \quad \forall R \geq C. \tag{72}$$

On the other hand, by Proposition 7.1, we can find another constant $C'(M)$ so that

$$E(M, \mathbf{C}, R) \leq R^{\gamma-1+\epsilon_2/2} C' \quad \forall R \geq 1,$$

which contradicts (72) when $R \gg 1$. $\square$

References

[1] Bombieri, E. and De Giorgi, E. and Giusti, E. *Minimal cones and the Bernstein problem*, Invent. Math., 7 (1969), 243–268.

[2] Chodosh, O. and Liokumovich, Y. and Spolaor, L. *Singular behavior and generic regularity of min-max minimal hypersurfaces*, Ars Inven. Anal., Paper No. 2 (2022), 27pp.

[3] Edelen, N. and Spolaor, L. *Regularity of minimal surfaces near quadratic cones*, 2019, arXiv:1910.00441.

[4] Edelen, N. *Degeneration of 7-dimensional minimal hypersurfaces which are stable or have bounded index*, Arch. Ration. Mech. Anal., (2021), to appear.

[5] Hardt, R. and Simon, L. *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math., 362 (1985), 102–129.

[6] Tom Ilmanen. *A strong maximum principle for singular minimal hypersurfaces.* Calc. Var. Partial Differential Equations, 4 (1996), 443–467.

[7] Li, Y. and Wang, Z. *Minimal hypersurfaces for generic metrics in dimension 8*, 2022, arXiv:2205.01047.

[8] Simon, L. *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, 3, 1983.

[9] Simon, L. *Uniqueness of some cylindrical tangent cones*, Comm. Anal. Geom., 2 (1994), no. 1, 1–33.

[10] Simon, L. *A Liouville-type theorem for stable minimal hypersurfaces*, Ars Inven. Anal., Paper No. 5 (2021), 35pp.

[11] Simon, L. *Stable minimal hypersurfaces in* $\mathbb{R}^{N+1+l}$ *with singular set an arbitrary closed* K *in* $\{0\} \times \mathbb{R}^l$, Ann. of Math. (2), (2021), to appear.

[12] Smale, N. *Generic regularity of homologically area minimizing hypersurfaces in eight-dimensional manifolds*, Comm. Anal. Geom., 1 (1993), no. 2, 217–228.

[13] Solomon, B. and White, B. *A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals*, Indiana Univ. Math. J., 38 (1989), no. 3, 683—691.

[14] Székelyhidi, G. *Uniqueness of certain cylindrical tangent cones*, 2020, arXiv:2012.02065.

[15] Székelyhidi, G. *Minimal hypersurfaces with cylindrical tangent cones*, 2021, arXiv:2107.14786.

[16] Wang, Z. *Deformations of singular minimal hypersurfaces I, isolated singularities*, 2020, arXiv:2011.00548.

[17] Wang, Z. *Mean convex smoothing of mean convex cones*, 2022, arXiv:2202.07851.

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
Email address: `nedelen@nd.edu`

Department of Mathematics, Northwestern University, Evanston, IL 60208
Email address: `gaborsz@northwestern.edu`