HIGHER REGULARITY FOR SINGULAR KÄHLER-EINSTEIN METRICS

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Abstract. We study singular Kähler-Einstein metrics that are obtained as non-collapsed limits of polarized Kähler-Einstein manifolds. Our main result is that if the metric tangent cone at a point is locally isomorphic to the germ of the singularity, then the metric converges to the metric on its tangent cone at a polynomial rate on the level of Kähler potentials. When the tangent cone at the point has a smooth cross section, then the result implies polynomial convergence of the metric in the usual sense, generalizing a result due to Hein-Sun. We show that a similar result holds even in certain cases where the tangent cone is not locally isomorphic to the germ of the singularity. Finally we prove a rigidity result for complete $\partial\bar{\partial}$-exact Calabi-Yau metrics with maximal volume growth. This generalizes a result of Conlon-Hein, which applies to the case of asymptotically conical manifolds.

1. Introduction

Since the celebrated work of Yau [38] on the existence of Kähler-Einstein metrics there has been increasing interest in the understanding of singular Kähler-Einstein metrics. An early result in this direction is Kobayashi [27] on orbifold Kähler-Einstein metrics, while a definitive existence result for a large class of singularities was obtained by Eyssidieux-Guedj-Zeriahi [21]. These works focus on the case of non-positive Ricci curvature, however recently Li-Tian-Wang [28] extended Chen-Donaldson-Sun's solution [5, 6, 7, 8] of the Yau-Tian-Donaldson conjecture to general $\mathbf{Q}$-Fano varieties. As a result we now have several sources of singular Kähler-Einstein metrics on normal varieties.

For applications it is desirable to have control of the geometry of these singular metrics near the singularities, but so far little is known in general. The main progress in this direction is due to Hein-Sun [26], who showed that near a large class of smoothable isolated singularities that are locally isomorphic to a Calabi-Yau cone, the singular Calabi-Yau metric must be asymptotic in a strong sense to the Calabi-Yau cone metric. Recently an analogous result was shown by Datar-Fu-Song [17] in the case of isolated log canonical singularities using the bounded geometry method, and precise asymptotics were obtained shortly after by Fu-Hein-Jiang [22]. In more general settings the best results so far give some control of the Kähler potential, such as the work of Guedj-Guenancia-Zeriahi [24] showing continuity.

Our main result in this paper extends the work of Hein-Sun [26] to a large class of possibly non-isolated singularities. In order to state the result, let us suppose that (Z, p) is the non-collapsed pointed Gromov-Hausdorff limit of a sequence of complete polarized Kähler-Einstein manifolds (M_i, g_i, p_i), satisfying $\mathrm{Ric}(g_i) = \lambda_i g_i$ with $|\lambda_i| \leq 1$. The results of Donaldson-Sun [19, 20], Li-Xu [30] and Li-Wang-Xu [29] imply that Z is a normal complex variety admitting a singular Kähler-Einstein metric ω_Z, and the metric tangent cone Z_p at p is homeomorphic to a normal affine variety uniquely determined by the germ (Z, p). The tangent cone Z_p admits a singular Ricci flat cone metric ω_{Z_p}. Our first result is the following.

Theorem 1.1. *Suppose that the germ (Z_p, o) is biholomorphic to (Z, p), where o denotes the vertex of the cone Z_p. Then for some $r_0 > 0$ there exists a biholomorphism $\phi : B(o, r_0) \to U$ from the unit ball in Z_p to a neighborhood of $p \in Z$ with $\phi(o) = p$ satisfying the following. There are constants $C, \alpha > 0$ and functions u_r on $B(o, r)$ for $0 < r < r_0$, satisfying*

$$\phi^*\omega_Z = \omega_{Z_p} + \sqrt{-1}\partial\bar{\partial}u_r$$

on the smooth locus of Z_p, and

$$\sup_{B(o,r)} |u_r| \leq Cr^{2+\alpha}$$

for all $0 < r < r_0$.

Combining with [30][29], Theorem 1.1 implies that if the germ (Z, p) is biholomorphic to the germ $(C(Y), o)$ in a possibly singular Ricci flat Kähler cone $C(Y)$ with vertex o, then ω_Z is asymptotic to the cone metric $\omega_{C(Y)}$ in the sense of Theorem 1.1.

Hein-Sun [26] consider the case of singular Calabi-Yau metrics where the tangent cone Z_p has an isolated singularity at the vertex, and in addition is "strongly regular". Most likely the approach of Hein-Sun can be extended to the more general Kähler-Einstein setting, without the strongly regular assumption, by appealing to the more recent works [30, 29]. On the other hand their approach uses that the tangent cone Z_p has a smooth cross section in an essential way, since they rely on analysis in weighted Hölder spaces. The main novelty in our approach is that by working on the level of L^∞-bounds for the Kähler potential, we are able to treat tangent cones with arbitrary singular sets. We can then obtain estimates for derivatives of the metric away from the singular set, which in particular can be used to recover Hein-Sun's result in the setting of tangent cones with isolated singularities (see Corollary 4.3).

For an example where Theorem 1.1 applies, see for instance the example constructed by Cynk and van Straten [16, Theorem]. It is a smoothable Calabi-Yau threefold with canonical singularities, whose singular set is a double line with four pinch points. The germ at a general point of the line is $\mathbf{C} \times A_1$, so our Theorem 1.1 applies there. On the other hand, it is known that the pinch point singularity admits a Ricci flat Kähler cone metric (see e.g. the discussion in [33, p.60] and the references therein). It follows that Theorem 1.1 also applies at the pinch points.

When the germ of the tangent cone (Z_p, o) is not biholomorphic to (Z, p), then the situation is more complicated, and has not been considered before. A family of examples given in [36] (also Hein-Naber [25]), are the hypersurfaces $A_{p-1} \subset \mathbf{C}^{n+1}$ defined by

$$z^p + x_1^2 + \ldots + x_n^2 = 0,$$

where $p > 2\frac{n-1}{n-2}$. In [36] the second author constructed a Calabi-Yau metric $\omega_{A_{p-1}}$ on a neighborhood of $0 \in A_{p-1}$, with tangent cone given by $\mathbf{C} \times A_1$, where $A_1 \subset \mathbf{C}^n$ is defined by $x_1^2 + \ldots + x_n^2 = 0$ and is equipped with the Stenzel cone metric. Our result in this case is the following.

Theorem 1.2. *Suppose that, as above, (Z, p) is the pointed Gromov-Hausdorff limit of a non-collapsing sequence of polarized Kähler-Einstein manifolds, with singular Kähler-Einstein metric ω_Z. Suppose that the germ (Z, p) is isomorphic to the germ $(A_{p-1}, 0)$ at the origin. Then for some $r_0 > 0$ there is a biholomorphism $\phi : B(0, r_0) \to U \subset Z$, with $\phi(0) = p$, and constants $\Lambda, C, \alpha > 0$, such that*

$$\phi^*\omega_Z = \Lambda\omega_{A_{p-1}} + \sqrt{-1}\partial\bar{\partial}u_r$$

for some u_r defined on $B(0, r)$, and

$$\sup_{B(0,r)} |u_r| \leq Cr^{2+\alpha}$$

for all $r < r_0$.

In other words the singular Kähler-Einstein metric ω_Z converges to a suitable scaling of the model metric $\omega_{A_{p-1}}$ at a polynomial rate, at the level of potentials. Note that in contrast with Theorem 1.1, where the model metrics were cones, here the rescalings of $\omega_{A_{p-1}}$ are not isometric to each other. In general we expect that for more complicated singularities it is possible to have higher dimensional families of model metrics, similarly to how in [11] a two dimensional family of complete Ricci flat Kähler metrics was constructed on $\mathbf{C}^3$ with tangent cone $\mathbf{C} \times A_2$ at infinity.

Our last result is the following uniqueness theorem for solutions of the Monge-Ampère equation on complete manifolds.

Theorem 1.3. *Let (X, ω) be a $\partial\bar{\partial}$-exact Calabi-Yau manifold with maximal volume growth. Suppose that u is a smooth solution of the complex Monge-Ampère equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = \omega^n.$$

In addition suppose that u has subquadratic growth in the sense that $|u| \leq C(1 + r)^{2-\delta}$ for some $C, \delta > 0$, where r is the distance from a fixed point in X. Then $\partial\bar{\partial}u = 0$.

This result should be compared with the uniqueness result in Conlon-Hein [15, Theorem 3.1]. The main novelty is that in our result we do not need to assume that the tangent cone of X at infinity has an isolated singularity, which is implied by the asymptotically conical assumption of [15]. Note, however, that the $\partial\bar{\partial}$-exactness is not required in [15].

The main new technical ingredient in the proofs of these theorems is an estimate for solutions of the complex Monge-Ampère equation on non-collapsed balls in polarized Kähler manifolds with Ricci curvature bounds, or their Gromov-Hausdorff limits. This extends a related estimate from [37], where we considered balls that are Gromov-Hausdorff close to a metric cone of the form $\mathbf{C} \times C(Y)$, with smooth Y. Roughly speaking the result says that if a solution u of a Monge-Ampère equation with sufficiently small L^∞ norm concentrates near the (almost) singular set of such a ball, then the solution must decay by a definite amount when passing to a smaller ball. Together with the harmonic approximation for the small solution of the Monge-Ampère equation in the generic region, this implies the decay of the sup norm of the solution upon passing to smaller scales. We will discuss this estimate in Section 2 and we expect it to be of independent interest.

In Section 3 we define the notion of families of model metrics as well as a convergence result for the singular Kähler-Einstein metric ω_Z that can be approximated by these model metrics near the singularities. This unifies certain aspects of Theorems 1.1 and 1.2. We then prove these theorems by showing the existence of families of model metrics and the existence of approximations in the corresponding cases in Sections 4 and 5. Finally, in Section 6 we prove Theorem 1.3.

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2. Non-concentration

In this section we study the complex Monge-Ampère equation on a ball in a non-collapsed Gromov-Hausdorff limit of Kähler-Einstein manifolds. More precisely, let (Z, p) be the pointed Gromov-Hausdorff limit of a sequence of complete pointed Kähler manifolds (M_i, g_i, p_i). We assume that the (M_i, g_i) are polarized, i.e. the Kähler forms are given by the curvature of line bundles over the M_i, that the metrics are Einstein, i.e. $\mathrm{Ric}(g_i) = \lambda_i g_i$ for some $|\lambda_i| \leq 1$, and that the non-collapsing condition $\mathrm{vol}(B_{g_i}(p_i, 1)) > \nu > 0$ holds for a fixed $v > 0$. By the results of Donaldson-Sun [19, 20], $B(p, 2)$ is a normal algebraic variety, and the metric singular set coincides with the algebro-geometric singular set $\Sigma \subset B(p, 2)$. For $q \in B(p, 1)$ let us denote by

$r_h(q)$ the harmonic radius at q, setting $r_h(q) = 0$ for $q \in \Sigma$. We denote the limit metric on the regular part of Z by ω. The main result of this section is the following estimate for solutions of the complex Monge-Ampère equation on $B(p,1)$.

Theorem 2.1. *There is a constant $C = C(n,\nu)$, such that for all $\gamma > 0$ there exist $\kappa, \delta > 0$ depending on n, ν, γ with the following property. Suppose that we have smooth functions u, f on $B(p,1)\setminus\Sigma$, satisfying $|u|, |f| < \kappa$, and*

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^f \omega^n. \tag{2.1}$$

Then

$$\sup_{B(p,1/2)} |u| \leq C \left(\sup_{\{r_h > \delta\} \cap B(p,1)} |u| + \sup_{B(p,1)} |f| + \gamma \sup_{B(p,1)} |u| \right).$$

We prove this result by proving successively more general cases. We start with the following, which follows the approach of [37, Proposition 4.5].

Lemma 2.2. *There is a $C_1 = C_1(n,\nu)$ such that for any $\gamma \in (0,1)$ there are $\kappa, \delta, \epsilon > 0$ depending on n, ν, γ satisfying the following. Suppose that $|u|, |f| < \kappa$ satisfy (2.1), and in addition $\mathrm{Ric}(\omega) > -\epsilon\omega$ and*

$$d_{GH}(B(p, \epsilon^{-1}), B(o, \epsilon^{-1})) < \epsilon,$$

where o is the vertex of a cone that splits an isometric factor of $\mathbf{C}^k$ for some $k \geq 0$. Let us write $o \in \mathbf{C}^k \times C(Y)$. Then

$$\sup_{B(p,1/2)} |u| \leq C_1 \left(\sup_{B(p,1)\setminus N_\delta} |u| + \sup_{B(p,1)} |f| + \gamma \sup_{B(p,1)} |u| \right), \tag{2.2}$$

where N_δ denotes the points x at distance at most δ from $\mathbf{C}^k \times \{0\}$ under the Gromov-Hausdorff approximation.

In this result we do not assume, as we did in [37], that Y is smooth. In addition, note that on the right hand side of (2.2) the supremum of $|u|$ is taken on the set $B(p,1) \setminus N_\delta$ which is typically larger than the set $\{r_h > \delta\} \cap B(p,1)$ if Y has singularities.

Proof. We claim that by [37, Proposition 4.4] there exists a constant $D > 0$ depending on n, ν, and for any $\delta > 0$ there exists $C_\delta > 0$ depending on δ, n, ν satisfying the following. If ϵ is sufficiently small (depending on δ, n, ν), then there exists a Lipschitz function v on $B(p, 1-\delta/2)$ satisfying

(1) $|\sqrt{-1}\partial\bar{\partial}v|_\omega < C_\delta$ on $B(p, 1-\delta/2) \setminus \Sigma$.
(2) $v > D^{-1}\delta^{-1/2}$ on $\partial B(p, 1-\delta) \cap N_\delta$.
(3) $v > D^{-1}$ on $B(p, 1-\delta)$, and $v < D$ on $B(p, 1/2)$.
(4) On $B(p, 1-\delta/2) \setminus \Sigma$, v satisfies the differential inequality:

$$\sum_i \mu_i + \mu_{max} < -1/10,$$

where μ_i are the eigenvalues of $\sqrt{-1}\partial\bar{\partial}v$ relative to ω, and μ_{max} is the largest eigenvalue.

To see this, recall that $B(p,1)$ is a ball in the pointed Gromov-Hausdorff limit of polarized Kähler-Einstein manifolds (M_i, p_i). Given $\epsilon > 0$ we have

$$d_{GH}(B(p_i, \epsilon^{-1}), B(o, \epsilon^{-1})) < \epsilon$$

for sufficiently large i, and so by [37, Proposition 4.4] we have functions v_i satisfying the properties (1) – (4) on $B(p_i, 1)$. While in [37] the property (4) is stated as $\sum \mu_i + \mu_{max} < 0$, from the proof the better bound $-1/10$ also follows (see Equation (4.3) and the inequality before it in [37]). Since v_i are constructed out of local Kähler potentials, we see that v_i and ∇v_i are uniformly bounded on $B(p_i, 1-\delta)$ and on compact sets away from the singular set of $B(p,1)$ (under Gromov-Hausdorff approximations) the functions v_i have uniform higher derivative estimates as well. We can therefore take a subsequential limit v of v_i on $B(p, 1-\delta)$, and conditions (1), (4) will follow from smooth convergence on the regular set. That the constants do not depend on the specific cone $C(Y)$, but only on n, ν, can be seen using a compactness argument.

Let us define

$$E = \sup_{B(p,1)\setminus N_\delta} |u| + \sup_{B(p,1)} |f| + \gamma \sup_{B(p,1)} |u| \leq 3\kappa,$$

and set $\delta \leq \gamma^2$. Define $\tilde{v} = DEv$. By (2), (3) above, on $\partial B(p, 1-\delta)$ we have $\tilde{v} > u$.

We claim that once κ is sufficiently small, then we have

$$\tilde{v} \geq u \text{ on } B(p, 1-\delta). \tag{2.3}$$

To see this, we argue as in [37], except we need to take care of the singular set Σ. Since Σ is a subvariety, there exists a plurisubharmonic function h on $B(p,1)$ such that $\Sigma = h^{-1}(-\infty)$. We will show (2.3) by showing that we have $\tilde{v} > u + \epsilon' h$ on $B(p, 1-\delta)$, for all $\epsilon' > 0$, and noting that $u, \tilde{v}$ are continuous. Suppose this is not the case. Write $B = B(p, 1-\delta)$ and for a fixed $\epsilon' > 0$ set

$$t_0 = \inf\{t > 0 \mid \tilde{v} + t > u + \epsilon' h \text{ on } B\}.$$

If $t_0 > 0$, then the graph of $\tilde{v} + t_0$ touches the graph of $u + \epsilon' h$ from above at some point $q \in B$. If $q \in \Sigma$, then $(u + \epsilon' h)(q) = -\infty$, so we must have $q \notin \Sigma$. At q we have

$$\sqrt{-1}\partial\bar{\partial}u(q) \leq \sqrt{-1}\partial\bar{\partial}u(q) + \epsilon'\sqrt{-1}\partial\bar{\partial}h(q) \leq \sqrt{-1}\partial\bar{\partial}\tilde{v}(q) \leq EDC_\delta\omega \tag{2.4}$$

by property (1) above and the fact that h is plurisubharmonic. Let λ_i be the eigenvalues of $\sqrt{-1}\partial\bar{\partial}u(q)$ relative to ω. From (2.4) we have $\lambda_i \leq C_\delta DE$. By (2.1), and using $|f| \leq E$, we have

$$e^{-E} \leq \prod_{i=1}^{n}(1 + \lambda_i) \leq e^{E}. \tag{2.5}$$

From (2.5) we have

$$1+\lambda_j \geq \frac{e^{-E}}{\prod_{i\neq j}(1+\lambda_i)} \geq e^{-E}(1+C_\delta E)^{-(n-1)} \geq 1 - C_{2,\delta}E \tag{2.6}$$

for some constant $C_{2,\delta} > 0$, once E is sufficiently small. On the other hand, if $\lambda_{max} < 0$ then (2.6) gives

$$\lambda_{max} \geq -E. \tag{2.7}$$

Finally, (2.5) together with the bounds for λ_i implies that

$$1 - E \leq e^{-E} \leq \prod_{i=1}^{n}(1+\lambda_i) \leq 1 + \sum_{i=1}^{n}\lambda_i + C_{3,\delta}E^2, \tag{2.8}$$

so (2.4) and (2.8) imply that

$$-2E - C_{3,\delta}E^2 \leq \sum_{i=1}^{n}\lambda_i + \lambda_{max} \leq DE\left(\sum_{i=1}^{n}\mu_i + \mu_{max}\right) \leq -\frac{DE}{10}.$$

The first inequality above uses (2.7). We can assume that $D > 30$. Since $E \leq 3\kappa$, by letting κ be sufficiently small, depending on δ, we get a contradiction. For such κ we have shown (2.3).

Using (2.3) and property (3) above, on $B(p, 1/2)$ we have

$$u \leq \tilde{v} \leq D^2E,$$

which implies the estimate from above for u required by (2.2). For the corresponding lower bound we can argue in a similar way, comparing u with $-\tilde{v}$ instead, to show that $u > -\tilde{v} + \epsilon' h$ on B for all $\epsilon' > 0$ once κ is sufficiently small. □

Next we have the following.

Lemma 2.3. *There is a $C_2 = C_2(n,\nu)$ such that for any $\gamma > 0$ there are $\kappa, \delta, \epsilon > 0$ depending on n, ν, γ satisfying the following. Suppose $|u|, |f| < \kappa$ satisfy (2.1), and $d_{GH}(B(p,\epsilon^{-1}), B(o,\epsilon^{-1})) < \epsilon$ for the vertex $o \in C(Y)$ in a cone. Then*

$$\sup_{B(p,1/2)}|u| \leq C_2\left(\sup_{\{r_h>\delta\}\cap B(p,1)}|u| + \sup_{B(p,1)}|f| + \gamma\sup_{B(p,1)}|u|\right). \tag{2.9}$$

Proof. We prove this by decreasing induction on the dimension of the Euclidean factor that splits off from the cone $C(Y)$, starting with $C(Y) = \mathbf{C}^n$. In this case, by Cheeger-Colding [2, Theorem 7.3], we have $r_h > r_0$ on $B(p,1)$ for a fixed $r_0 > 0$. The inequality (2.9) then holds if we choose $\delta < r_0$, and $C_2 > 1$.

Suppose now that the result holds whenever $B(p,\epsilon^{-1})$ is ϵ-close to a ball in a cone of the form $\mathbf{C}^j \times C(X)$ for $j \geq k+1$, and consider the case that

$$d_{GH}(B(p,\epsilon'^{-1}), B(o,\epsilon'^{-1})) < \epsilon', \tag{2.10}$$

where $o \in \mathbf{C}^k \times C(Y)$. By Lemma 2.2 there are $C_1(n,\nu)$ and $\kappa_1, \delta_1, \epsilon_1 > 0$ depending on γ, n, ν, such that if $|u|, |f| < \kappa_1$ and $\epsilon' < \epsilon_1$, then

$$\sup_{B(p,1/2)} |u| \leq C_1 \left(\sup_{B(p,1)\setminus N_{\delta_1}} |u| + \sup_{B(p,1)} |f| + \gamma \sup_{B(p,1)} |u| \right). \tag{2.11}$$

We will complete the proof by estimating $|u|$ outside of N_{δ_1} using the inductive hypothesis.

Given the $\epsilon > 0$ from the inductive hypothesis, there are $r, \epsilon_2 > 0$ depending on n, ν, γ with the following property. If $\epsilon' < \epsilon_2$ in (2.10), then for all $x \in B(p,1) \setminus N_{\delta_1}$ there is an $r_x > r$ such that

$$d_{GH}(B(x, \epsilon^{-1} r_x), B(o', \epsilon^{-1} r_x)) < \epsilon r_x,$$

for the origin $o' \subset \mathbf{C}^{k+1} \times C(Y')$ in a cone that splits off an isometric factor of $\mathbf{C}^{k+1}$. The reason for this is that if $x \in \mathbf{C}^k \times C(Y)$ does not lie in $\mathbf{C}^k \times \{0\}$, then the tangent cones at x split an additional Euclidean factor by Cheeger-Colding [1, Theorem 6.62] and Cheeger-Colding-Tian [3, Theorem 9.1].

At such a point $x \in B(p,1) \setminus N_{\delta_1}$ consider a ball $B(x, r_x)$ scaled up to unit size, which we denote by $B(x', 1)$. We can assume that r_x^{-1} is an integer, so the rescaled ball is also the limit of a sequence of polarized Kähler-Einstein manifolds. On the rescaled ball $B(x', 1)$ we have the equation

$$(\omega' + \sqrt{-1}\partial\bar{\partial} u')^n = e^{f'} \omega'^n,$$

where $\omega' = r_x^{-2}\omega$, $u' = r_x^{-2} u$ and $f' = f$. In particular

$$\sup_{B(x',1)} |u'| \leq r_x^{-2} \sup_{B(p,1)} |u|,$$
$$\sup_{B(x',1)} |f'| \leq \sup_{B(p,1)} |f|,$$

and

$$d_{GH}(B(x', \epsilon^{-1}), B(o', \epsilon^{-1})) < \epsilon.$$

We can now choose κ, δ, ϵ small enough, depending on n, ν, γ (recall that $r_x > r$ and r depends on n, ν, γ) so that the inductive hypothesis applies, and therefore

$$\sup_{B(x',1/2)} |u'| \leq C \left(\sup_{\{r_h' > \delta\} \cap B(x',1)} |u'| + \sup_{B(x',1)} |f'| + \gamma \sup_{B(x',1)} |u'| \right).$$

Here we are writing r_h' for the harmonic radius in the scaled up metric. We have $r_h' = r_x^{-1} r_h$. Scaling back down we have

$$\begin{aligned} |u(x)| &\leq C \left(\sup_{\{r_h > r_x \delta\} \cap B(x, r_x)} |u| + \sup_{B(x,r_x)} r_x^2 |f| + \gamma \sup_{B(x,r_x)} |u| \right) \\ &\leq C \left(\sup_{\{r_h > r\delta\} \cap B(p,1)} |u| + \sup_{B(p,1)} |f| + \gamma \sup_{B(p,1)} |u| \right). \end{aligned}$$

Since $x \in B(p,1) \setminus N_{\delta_1}$ was arbitrary, this inequality together with (2.11) implies the required result. □

Finally we can give the proof of Theorem 2.1.

Proof of Theorem 2.1. Given $\epsilon > 0$, by Cheeger-Colding [1] there exists a $\rho > 0$, depending on ϵ, n, ν, with the following property: for all $x \in B(p, 1/2)$ we have some $\rho_x > \rho$ such that

$$d_{GH}(B(x, \epsilon^{-1}\rho_x), B(o, \epsilon^{-1}\rho_x)) < \epsilon\rho_x,$$

for $o \in C(Y)$ in some metric cone $C(Y)$. We can then rescale the ball $B(x, \rho_x)$ to unit size, and if ϵ, κ, δ is chosen sufficiently small, then we can apply Lemma 2.3 to bound $|u(x)|$ similarly to the argument in the proof of Lemma 2.3. □

3. Decay estimate

The goal of this section is to prove a convergence result, Proposition 3.7 below, which contains some common features of Theorem 1.1 and Theorem 1.2. Let (Z, p) be the Gromov-Hausdorff limit of a non-collapsing sequence of polarized Kähler-Einstein manifolds of complex dimension n, and let $C(Y)$ be the tangent cone at p. We will define a family of model metrics in a neighborhood $\mathcal{U}$ of p in Z parametrized by small quadratic harmonic functions on $C(Y)$ which generate automorphisms of $C(Y)$, and prove an abstract decay estimate, Proposition 3.5 for the family. Throughout this section, as well as later on, we will denote by $\Psi(\epsilon)$ functions satisfying $\lim_{\epsilon\to 0} \Psi(\epsilon) = 0$.

We first recall some important properties of subquadratic harmonic functions on $C(Y)$. The following lemma combines results going back to Cheeger-Tian [4, Section 7], Conlon-Hein [15, Corollary 3.6] and Hein-Sun [26, Theorem 2.14] when $C(Y)$ has an isolated singularity:

Lemma 3.1. *Suppose $C(Y)$ is a metric tangent cone of a non-collapsed Gromov-Hausdorff limit of Kähler-Einstein manifolds. Let r denote the radial coordinate so that $r\partial_r$ is the homothetic vector field. Let J denote the complex structure. Suppose u is a harmonic function on $C(Y)$. Then we have the following:*

(1) *If u is s-homogeneous ($\nabla_{r\partial_r} u = su$) with $s < 2$, then u is pluriharmonic.*
(2) *If u is 2-homogeneous harmonic, then $u = u_1 + u_2$, where u_1 is pluriharmonic, and u_2 is $J(r\partial_r)$-invariant.*
(3) *The space of real holomorphic vector fields that commute with $r\partial_r$ can be written as $\mathfrak{p} \oplus J\mathfrak{p}$, where $\mathfrak{p}$ is spanned by $r\partial_r$ and vector fields of the form ∇u, where u is a $J(r\partial_r)$-invariant harmonic function homogeneous of degree 2. $J\mathfrak{p}$ consists of real holomorphic Killing vector fields.*

Proof. In our setting the singular set has Hausdorff codimension at least 4 [3]. To deal with the singular set we can use the cut-off functions for example in [10, Lemma 2.3]. (1) is proved in [10, Corollary 2.18]. For (2) and (3), see [9, Proposition 3.19] for more details. □

On $\mathcal{U}$, we consider a family of Calabi-Yau metrics on the regular set of $\mathcal{U}$ with tangent cone $C(Y)$ at p, satisfying properties that enable a decay estimate. To proceed, let H denote the space of quadratic harmonic functions h such that ∇h generates a biholomorphism which commutes with scaling. H as a vector space is equipped with the L^∞ norm on $B(0,1) \subset C(Y)$. For $h \in H$ let us denote this norm simply by $\|h\|$. In the following, we fix an embedding $F_\infty : B(0,1) \subset C(Y) \to \mathbf{C}^N$ whose components are given by polynomial growth holomorphic functions.

Definition 3.2. Let $U \subset H$ be an open neighborhood of $0 \in H$. A family $\mathcal{F}$ of model Calabi-Yau metrics consists of a set of Calabi-Yau metrics ω_h on the regular set of $\mathcal{U}$, whose metric completion is homeomorphic to $\mathcal{U}$, parametrized by $h \in U$, with the following properties:

(1) For sequences $h_i \in U$ and $r_i \to 0$, set $B_i = B_{r_i^{-2}\omega_{h_i}}(p,1)$. Then there is a sequence of holomorphic maps $F_i : B_i \to \mathbf{C}^N$, and $\Psi(i^{-1})$-Gromov-Hausdorff approximations $f_i : B_i \to B(0,1)$ such that $|F_i - F_\infty \circ f_i| < \Psi(i^{-1})$.
(2) The volume form ω_h^n is independent of $h \in U$.
(3) For $h, k \in U$ and $r > 0$, we have $|d_{\omega_h} - d_{\omega_k}| \leq C(\|k\| + \|h\|)r$ on $B_{\omega_h}(p,r)$.
(4) For $h, k \in U$, on $B_{\omega_h}(p,2)$ we have $\omega_k = \omega_h + \sqrt{-1}\partial\bar{\partial}u$, and for every $r > 0$, we have $|u| \leq C\|h-k\|r^2$ on $B_{\omega_h}(p,r)$.
(5) Suppose that there are $r_i \to 0$ and sequences $h_i, k_i \in U$ such that $\|h_i\|, \|k_i\| \to 0$. Write $\omega_{k_i} = \omega_{h_i} + \sqrt{-1}\partial\bar{\partial}u_i$ as in (4). For any $\epsilon > 0$ and K a compact set in the regular set of $B(0,1) \subset C(Y)$, there exist compact sets $K_i \subset B_{r_i^{-2}\omega_{h_i}}(p,1)$ such that $K_i \to K$ in the Gromov-Hausdorff sense, and
$$|r_i^{-2}u_i - f_i^*(k_i - h_i)| \leq \epsilon\|k_i - h_i\|$$
on K for all sufficiently large i, where f_i is the Gromov-Hausdorff approximation in (1).

In each of the applications below (see Sections 4 and 5), the model metrics are isometric up to small scaling and biholomorphism. In general we expect to have higher dimensional families of model metrics, and we expect in such cases Definition 3.2 can be suitably adapted.

The following lemma shows that we have higher regularity of the solutions to the complex Monge-Ampère equation if the L^∞ norm is sufficiently small.

Lemma 3.3. *Suppose that $B(p,2)$ is a ball in a Kähler-Einstein manifold of complex dimension n, with metric ω satisfying* $\mathrm{Ric}(\omega) = c'\omega$*, such that* *in suitable coordinates z^i the components $\omega_{i\bar{j}}$ satisfy $|\partial^3(\delta_{i\bar{j}} - \omega_{i\bar{j}})| < \frac{1}{100}$ in terms of the Euclidean metric $\delta_{i\bar{j}}$. If $\epsilon > 0$ is sufficiently small, then we have the following.*

Suppose that $\eta = \omega + \sqrt{-1}\partial\bar{\partial}u$ is another Kähler-Einstein metric on $B(p,2)$ with $\mathrm{Ric}(\eta) = c\eta$ and $\eta^n = e^f\omega^n$, so that

$$|u|, |f|, |c|, |c'| < \epsilon.$$

There exist $C_k > 0$ depending on the dimension n and on k, such that

$$\|u\|_{C^{k,\alpha}(B(p,1))} < C_k\epsilon.$$

Proof. All the operators and norms below are taken with respect to ω, and the constants C_k may change from line to line. Note first that from elliptic regularity for the equation $\mathrm{Ric}(\omega) = c'\omega$, we obtain higher order estimates $|\partial^k\omega_{i\bar{j}}| < C_k$ for the components of ω. From the equation $\eta^n = e^f\omega^n$ and the Kähler-Einstein condition for ω and η, we have $c\eta = -\sqrt{-1}\partial\bar{\partial}f + c'\omega$, so the function $v = cu + f$ satisfies $\sqrt{-1}\partial\bar{\partial}v = (c' - c)\omega$. It follows that $\Delta v = (c' - c)n$. Using the Schauder estimates we then have $\|v\|_{C^k} < C_k\epsilon$ on the ball where $\{|z| < 1.9\}$.

We now rewrite the equation in a form so that Savin's small perturbation result [34] can be applied. Consider the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_0)^n = e^{v - cu_0}\omega^n$$

for u_0, with $u_0 = 0$ on the boundary of the ball $\{|z| < 1.9\}$ in our coordinates. Define

$$F : C_0^{2,\alpha} \times C^{2,\alpha} \times \mathbf{R} \to C^{0,\alpha}$$
$$(u_0, v, c) \mapsto \log\det\left(\frac{(\omega + \sqrt{-1}\partial\bar{\partial}u_0)^n}{\omega^n}\right) - v + cu_0,$$

where $C_0^{2,\alpha}, C^{2,\alpha}$ denote functions on the ball $\{|z| < 1.9\}$, with zero boundary values in the first case. Note that $F(0,0,0) = 0$, and the linearization at $(0,0,0)$ in the u_0 direction is $\Delta + c$. As long as c is sufficiently small, this operator is invertible. By the implicit function theorem, for sufficiently small $v \in C^{2,\alpha}$ and $c \in \mathbf{R}$ we can find u_0 that satisfies the equation, with $\|u_0\|_{C^{2,\alpha}} < \delta$, where $\delta > 0$ can be made as small as we like by choosing ϵ small.

To write our equation in a different form, let $h = u - u_0$. Then h satisfies

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_0 + \sqrt{-1}\partial\bar{\partial}h)^n = e^{-ch}e^{v - cu_0}\omega^n.$$

Thanks to the bounds for v and u_0, the above equation is uniformly elliptic, and $h = 0$ is a solution of it. By Savin's theorem [34], for any given $\delta > 0$ we have $\|h\|_{C^{2,\alpha}(B(p,1))} < \delta$ once h is sufficiently small in L^∞. It follows that if ϵ is chosen sufficiently small, then h and u_0, and therefore also u will satisfy $|u|_{C^2} < \delta$ on the ball $\{|z| < 1.8\}$.

Let us now write the equation $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^f\omega^n$ for u as

(3.1)
$$\left(n\omega^{n-1} + \binom{n}{2}\omega^{n-2}\wedge(\sqrt{-1}\partial\bar{\partial}u) + \cdots + (\sqrt{-1}\partial\bar{\partial}u)^{n-1}\right)\wedge\sqrt{-1}\partial\bar{\partial}u = (e^f - 1)\omega^n.$$

If δ is sufficiently small, then this can be written as a uniformly elliptic linear equation

$$Pu = e^f - 1,$$

where the coefficients of P (which depend on u) are bounded in C^k. Note that if $|f| < \epsilon$ for small ϵ, then $|e^f - 1| < 2\epsilon$. We can now use standard L^p and Schauder estimates, as well as bootstrapping using the estimates that we already have for $cu + f$, to obtain $|u|_{C^k} < C_k\epsilon$ on the smaller ball $\{|z| < 1.7\}$. □

We will need the following result, which allows us to estimate the difference between the distance functions of a model metric and a Gromov-Hausdorff limit. This will be used in the proof of Proposition 3.7 below, to ensure that along the iteration procedure the distance functions of the two metrics that we are comparing remain close to each other at smaller and smaller scales.

Lemma 3.4. *Let $\lambda > 0$. Then for all sufficiently small $\epsilon > 0$ and $r > 0$, the following holds. Let $\omega = \omega_h \in \mathcal{F}$ be a model metric with $\|h\| \le \epsilon$. Now, suppose η is another Kähler-Einstein metric on the regular set of $B_\omega(p, 2r)$ obtained as the non-collapsed Gromov-Hausdorff limit of polarized Kähler-Einstein manifolds, with the following properties:*

- $\mathrm{Ric}(\eta) = c\eta$ *with* $|c| < r^{-2}\epsilon$;
- $\eta^n = e^f\omega^n$ *with* $|f| < \epsilon$;
- $\omega = \eta + \sqrt{-1}\partial\bar{\partial}u$ *with* $|u| < r^2\epsilon$;
- $|d_\omega - d_\eta| < r/100$.

Then we have $|d_\omega - d_\eta| < \lambda r$ on $B_\omega(p, r)$.

Proof. We argue by contradiction, supposing that we have $\epsilon_i, r_i \to 0$ and corresponding η_i, f_i and u_i such that the result fails. Let us rescale the metrics by setting $\tilde{\omega}_i = r_i^{-2}\omega, \tilde{\eta}_i = r_i^{-2}\eta_i$. Set $A_i = B_{\tilde{\eta}_i}(0, 1)$ and $B_i = B_{\tilde{\omega}_i}(0, 2)$. By the assumption on $|d_{\omega_i} - d_{\eta_i}|$ we have the inclusions $\phi_i : A_i \subset B_i$. To get a contradiction, we will show that ϕ_i is a $\Psi(i^{-1})$-Gromov-Hausdorff approximation for sufficiently large i. Let us define $\psi_i = F_i \circ \phi_i$, where F_i are the maps in property (1) of Definition 3.2. Then $\psi_i : A_i \to \mathbf{C}^N$ are holomorphic maps. By property (1) of Definition 3.2, we have $|\psi_i| \le C$ for some constant $C > 0$ once i is sufficiently large. Then by the gradient estimate for holomorphic maps, we have $|\nabla\psi_i|_{\tilde{\eta}_i} \le C$ for a uniform constant $C > 0$. This implies that ψ_i are equicontinuous.

We claim that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in B_i$ and $|F_i(x) - F_i(y)| < \delta$, then $d_{\tilde{\omega}_i}(x, y) < \epsilon$. If this is not the case, then there exist $x_i, y_i \in B_i$ with $|F_i(x_i) - F_i(y_i)| \to 0$ but $d_{\tilde{\omega}_i}(x_i, y_i) \ge \epsilon$. By passing to a subsequence, we may assume that $x_i \to x$ and $y_i \to y$ for $x, y \in B(0,2)$ under the Gromov-Hausdorff convergence $B_i \to B(0,2)$. The maps F_i converge in the Gromov-Hausdorff sense to the fixed embedding $F_\infty : B(0,2) \subset C(Y) \to \mathbf{C}^N$. It follows that $F(x) = F(y)$ but $d_{C(Y)}(x,y) \geq \epsilon$, contradicting the fact that F is an embedding. This proves the claim.

It follows from the claim that the maps $\phi_i = F_i^{-1} \circ \psi_i$ form an equicontinuous family of maps from A_i to B_i. Thus there exists a subsequence of ϕ_i converging to a map $\phi_\infty : A \to B$ under the Gromov-Hausdorff convergence $A_i \to A$ and $B_i \to B$. Let us denote the singular Kähler-Einstein metrics on A and B by ω_A and ω_B, respectively. The proof can be concluded once we show that ϕ_∞ is an isometry onto its image. Since A is the metric completion of its regular set $\mathcal{R}_A$, it is enough to show that for $x, y \in \mathcal{R}_A$, $d(x,y) = d(\phi_\infty(x), \phi_\infty(y))$. Note that by property (1) in Definition 3.2 we have $B = B(0,2) \subset C(Y)$.

Let γ be a minimal geodesic connecting x, y. By Colding-Naber [14], γ lies entirely in $\mathcal{R}_A$. Let V be an open set containing γ such that the compact closure of V is contained in $\mathcal{R}_A$, and let $V_i \subset A_i$ be the corresponding open sets converging to V under the Gromov-Hausdorff convergence. On V_i we have uniform bounds of the geometry of $\tilde{\eta}_i$, so by Lemma 3.3, we have bounds $|\nabla^j(\tilde{\eta}_i - \phi_i^*\tilde{\omega}_i)| < C\epsilon_i$ on V_i for $j = 0, 1$. Letting $i \to \infty$, it follows that $\phi_\infty : V \to V'$ is an isomorphism onto its image, and $\phi_\infty^*\omega_B = \omega_A$. So we have $d_A(x,y) = \mathrm{length}_{\omega_A}(\gamma) = \mathrm{length}_{\omega_B}(\phi_\infty \circ \gamma) \geq d_B(\phi_\infty(x), \phi_\infty(y))$. To prove the opposite inequality, let us now suppose that $\tilde{\gamma}$ is a minimal geodesic connecting $\phi_\infty(x)$ and $\phi_\infty(y)$. Since $B = B(0,2)$ by property (1) Definition 3.2, by Colding-Naber [14] $\tilde{\gamma}$ is contained in an open set W with compact closure in the regular set of B. Let W_i be open sets in B_i corresponding to W under the Gromov-Hausdorff convergence $B_i \to B$, and let $\gamma_i \subset W_i$ be curves converging to $\tilde{\gamma}$, with endpoints $x_i \to x, y_i \to y$. Over W_i we have smooth convergence of the metrics $\tilde{\eta}_i \to \omega_A$ and $\tilde{\omega}_i \to \omega_B$ in the Gromov-Hausdorff sense. So we have $d_B(\phi_\infty(x), \phi_\infty(y)) = \mathrm{length}_{\omega_B}(\tilde{\gamma}) = \lim_{i\to\infty} \mathrm{length}_{\tilde{\omega}_i}(\gamma_i) = \lim_{i\to\infty} \mathrm{length}_{\tilde{\eta}_i}(\gamma_i) \geq \lim_{i\to\infty} d_{\tilde{\eta}_i}(x_i, y_i) = d_A(x,y)$. We have shown that ϕ_∞ is an isometry onto its image, so it follows that ϕ_i is a $\Psi(i^{-1})$-Gromov-Hausdorff approximation. □

The main result in this section is the following abstract decay estimate.

Proposition 3.5. *There exist constants $C, \alpha, \lambda > 0$ (depending on the cone $C(Y)$) $\lambda < 1$ such that for $\epsilon, r > 0$ sufficiently small, we have the following. Fix a model metric ω_h with $\|h\| \leq \epsilon$. Let η be another metric on $B_{\omega_h}(p, 2r)$ obtained as the non-collapsed Gromov-Hausdorff limit of a sequence of polarized Kähler-Einstein manifolds. Suppose that $\eta = \omega_h + \sqrt{-1}\partial\bar{\partial}u$ on $B_{\omega_h}(p, 2r)$ satisfies $\eta^n = e^f \omega_h^n$, and for some $\kappa < \epsilon$ we have* $\mathrm{Ric}(\eta) = c\eta$ *for*

$|c| \leq r^{-2}\kappa$, *and*

$$|d_\eta - d_{\omega_h}| < \frac{r}{100},$$
$$|u| < r^2\kappa,$$
$$|\nabla f|_\eta < r^{-1}\kappa\epsilon,$$
$$f(p) = 0.$$

Then we can find another model metric ω_k *and a smooth function* u' *on* $B_{\omega_h}(p, r)$ *satisfying*

(1) $\omega_h + \sqrt{-1}\partial\bar{\partial}u = \omega_k + \sqrt{-1}\partial\bar{\partial}u'$,
(2) $\|k - h\| \leq C\kappa$.
(3) $\sup_{B_{\omega_k}(p,4\lambda r)} |u'| \leq \lambda^{2+\alpha} r^2\kappa$.

We remark that the advantage of working with a bound for the gradient $|\nabla f|_\eta$, rather than with the sup norm $|f|$, is that after scaling the gradient bound improves. At the same time, using the estimate for the distance function of η, the gradient bound together with the condition $f(p) = 0$ implies a corresponding bound $|f| < 4\kappa\epsilon$.

Proof. We argue by contradiction, so suppose there are $\epsilon_i, r_i \to 0, \kappa_i < \epsilon_i$ and corresponding h_i, η_i, u_i, f_i with $\|h_i\| \leq \kappa_i, |u_i| < r_i^2\kappa_i, |\nabla f|_{\eta_i} < \kappa_i\epsilon_i$ such that no suitable α, λ exist. We will show by passing to a limit that for large enough i, the statement actually holds for some α, λ, thus reaching a contradiction. The argument is similar to the proof of Proposition 4.1 in [37]. In the following $C > 0$ will denote a uniform constant, whose value may change from line to line.

Let us scale up the metrics by defining $\tilde{\eta}_i = r_i^{-2}\eta_i$, $\omega_i = r_i^{-2}\omega_{h_i}$ and $\tilde{u}_i = r_i^{-2}u_i$. By the gradient bound for f_i and the estimate for $|d_{\omega_{h_i}} - d_{\eta_i}|$ we see that $|f_i| < 2\kappa_i\epsilon_i$ on $B_{\omega_i}(p, 1.9)$. Note that $\tilde{u}_i$ satisfies

$$(\omega_i + \sqrt{-1}\partial\bar{\partial}\tilde{u}_i)^n = e^{f_i}\omega_i^n,$$

with $|\tilde{u}_i| \leq \kappa_i$ on $B_{\omega_i}(p, 1.9)$. By Lemma 3.4, we have

$$|d_{\tilde{\eta}_i} - d_{\omega_i}| < \Psi(i^{-1}) \tag{3.2}$$

on $B_{\omega_i}(p, 1)$ once i is sufficiently large. It follows from (3.2) and property (1) of Definition 3.2 that both $B_{\eta_i}(0, 1)$ and $B_{\omega_i}(0, 1)$ converge to $B(0, 1)$ in the Gromov-Hausdorff sense.

By Lemma 3.3, for all sufficiently large i we have $\|\tilde{u}_i\|_{C^{k,\alpha}(A)} \leq C_{k,A}\kappa_i$ on any compact subset A of the regular set of $B_{\omega_i}(p, 1)$. So by passing to a subsequence, $\kappa_i^{-1}\tilde{u}_i$ converges locally smoothly to a function h on the regular set, satisfying $|h| \leq 1$. On the other hand, writing the equation for $\tilde{u}_i$ in the form of Equation (3.1), we find that away from the singular set, h is a harmonic function on $B(0, 1)$ with respect to the cone metric $\omega_{C(Y)} = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}r^2$. Since $|h| \leq 1$ and the singular set has codimension at least four, h extends as a harmonic function across the singular set as well.

We can decompose h into a sum of homogeneous harmonic functions on the cone $C(Y)$, and we write $h = h^{\leq 2} + h^{>2}$, where $h^{\leq 2}$ collects the components with at most quadratic growth and $h^{>2}$ is the rest. By Lemma 3.1 we can further decompose $h^{\leq 2} = h_{ph} + h_{aut}$, where h_{ph} is pluriharmonic and $h_{aut} \in H$. Since h_{ph} is pluriharmonic, h_{ph} is the real part of a holomorphic function, which is a restriction of a holomorphic function on $\mathbf{C}^N$. Using the biholomorphisms in property (1) of Definition 3.2, it follows that h_{ph} also defines a pluriharmonic function $h_{ph,i}$ on the scaled-up ball $B_{\omega_i}(p, 1)$ and $h_{ph,i}$ converges uniformly in the Gromov-Hausdorff sense to h_{ph}.

We now write down the new potential. For this let us define $k_i = h_i + \kappa_i h_{aut} \in H$. For sufficiently large i we have $k_i \in U$. Consider the corresponding model metric ω_{k_i}. By property (4) of Definition 3.2 we have $\omega_{k_i} = \omega_{h_i} + \sqrt{-1}\partial\bar{\partial} v_i$ with

$$|v_i| \leq C\|k_i - h_i\| r^2 \leq C\kappa_i r^2 \tag{3.3}$$

on $B_{\omega_{k_i}}(0, r)$. Let us define $\tilde{\omega}_i = r_i^{-2}\omega_{k_i}$. By property (3) of Definition 3.2, we have

$$|d_{\tilde{\omega}_i} - d_{\omega_i}| \leq C\epsilon_i \tag{3.4}$$

on $B_{\tilde{\omega}_i}(p, 1)$.

Now we switch our reference metric from ω_{h_i} to ω_{k_i}. We have

$$\begin{aligned}
\eta_i &= \omega_{h_i} + \sqrt{-1}\partial\bar{\partial} u_i \\
&= \omega_{k_i} + \sqrt{-1}\partial\bar{\partial}(u_i - v_i - r_i^2\kappa_i h_{ph,i}) \\
&= \omega_{k_i} + \sqrt{-1}\partial\bar{\partial} u_i',
\end{aligned}$$

where we define $u_i' = u_i - v_i - r_i^2\kappa_i h_{ph,i}$. By the estimate (3.3) for v_i and the assumption of u_i it follows that on $B_{\omega_{h_i}}(p, 2r_i)$ we have

$$|u_i'| \leq C\kappa_i r_i^2. \tag{3.5}$$

By property (3) of Definition 3.2 it follows that the same estimate also holds on $B_{\omega_{k_i}}(p, r_i)$. Let us define $\tilde{u}_i' = r_i^{-2}u_i'$. Then $\kappa_i^{-1}\tilde{u}_i'$ converges to $h^{>2}$ over compact subsets of the regular set of $B_{\tilde{\omega}_i}(p, 0.8)$. To see this, let A be a compact subset of the regular set of $B_{\tilde{\omega}_i}(p, 0.8)$. Using the Gromov-Hausdorff approximations as in property (5) of Definition 3.2, we compute

$$\begin{aligned}
|\kappa_i^{-1}\tilde{u}_i' - h^{>2}| &\leq |\kappa_i^{-1}\tilde{u}_i - h| + |h - r_i^{-2}\kappa_i^{-1}v_i - h_{pi,i} - h^{>2}| \\
&\leq \Psi(i^{-1}) + |h_{ph} - h_{ph,i}| + |r_i^{-2}\kappa_i^{-1}v_i - h_{aut}| \\
&\leq \Psi(i^{-1}) + \kappa_i^{-1}|r_i^{-2}v_i - \kappa_i h_{aut}| \\
&\leq \Psi(i^{-1}) + \kappa_i^{-1}\Psi(i^{-1})|\kappa_i h_{aut}| \\
&\leq \Psi(i^{-1}).
\end{aligned}$$

The second inequality uses the fact that $\kappa_i^{-1}\tilde{u}_i$ converges to h, while the second to last inequality uses property (5) of Definition 3.2. We will show that $\tilde{u}_i'$ is much smaller than κ_i on a smaller ball, using that it is modeled on a harmonic function of growth rate strictly greater than 2. Away from the singular set this follows from the convergence $\kappa_i^{-1}\tilde{u}_i' \to h^{>2}$ as shown above. To extend this estimate across the singular set we need to apply the non-concentration result in the previous section.

Let us first make precise the required decay for $h^{>2}$. Define the normalized L^2 norm of a function f on a ball B by $\|f\|_B^2 = \mathrm{vol}(B)^{-1}\int_B f^2$. Since $h^{>2}$ has faster than quadratic growth, there is an $\alpha > 0$ depending only on the cone $C(Y)$ such that

$$\|h^{>2}\|_{B(0,16r)} \leq Cr^{2+2\alpha}\|h^{>2}\|_{B(0,1)}$$

for any small $r > 0$. By the mean value inequality for harmonic functions,

$$\sup_{B(0,8r)} |h^{>2}| \leq C\|h^{>2}\|_{B(0,16r)} \leq Cr^{2+2\alpha}.$$

We think of r as fixed, to be chosen below.

To apply the non-concentration result in the previous section, we need to work with respect to $\tilde{\eta}_i$ instead of $\tilde{\omega}_i$, since $\tilde{\omega}_i$ in general is not a Gromov-Hausdorff limit, while $\tilde{\eta}_i$ is. By property (3) of Definition 3.2 and the estimate (3.2), we see that for i sufficiently large, on $B_{\tilde{\eta}_i}(p,1)$ we have

$$|d_{\tilde{\eta}_i} - d_{\tilde{\omega}_i}| < r. \tag{3.6}$$

Let us now scale up by $(16r)^{-1}$, replacing $\tilde{\omega}_i$ by $(16r)^{-2}\tilde{\omega}_i$ and $\tilde{\eta}_i$ by $(16r)^{-2}\tilde{\eta}_i$. Define $U_i' = (16r)^{-2}r_i^{-2}u_i'$. From (3.5) we have $|U_i'| \leq C\kappa_i r^{-2}$ on $B_{\tilde{\omega}_i}(p,2)$. So by (3.6) we have $|U_i'| \leq C\kappa_i r^{-2}$ on $B_{\tilde{\eta}_i}(p,1)$. Let r_h denote the harmonic radius of $\tilde{\omega}_i$, and let $\delta > 0$, whose value is to be determined later. On $\{r_h > \delta\}$, U_i' converges smoothly to $(16r)^{-2}\kappa_i h^{>2}$. So on $\{r_h > \delta\} \cap B_{\tilde{\omega}_i}(p,2)$ we have

$$|U_i'| < Cr^{2\alpha}\kappa_i.$$

Let $\tilde{r}_h$ be the harmonic radius of the metric $\tilde{\eta}_i$. By Lemma 3.3, for i sufficiently large we have $\{\tilde{r}_h > 2\delta\} \subset \{r_h > \delta\}$. It follows that

$$\sup_{B_{\tilde{\eta}_i}(p,1)\cap\{\tilde{r}_h > 2\delta\}} |U_i'| \leq Cr^{2\alpha}\kappa_i.$$

Note that on $B_{\tilde{\eta}_i}(p,1)$, using property (2) of Definition 3.2 we see that U_i' satisfies the equation

$$(\tilde{\eta}_i - \sqrt{-1}\partial\bar{\partial}U_i')^n = e^{-f_i}\tilde{\eta}_i^n,$$

and we have $|f_i| < 2\kappa_i\epsilon_i$. We are now ready to apply the non-concentration theorem, Theorem 2.1. Given $\gamma > 0$, Theorem 2.1 implies that there exists $\delta > 0$ such that

$$\begin{aligned}\sup_{B_{\tilde{\eta}_i}(p,0.5)} |U_i'| &\leq C\left(\sup_{B_{\tilde{\eta}_i}(p,1)\cap\{\tilde{r}_h>2\delta\}} |U_i'| + \sup_{B_{\tilde{\eta}_i}(p,1)} |f_i| + \gamma \sup_{B_{\tilde{\eta}_i}(p,1)} |U_i'|\right) \\ &\leq C(\kappa_i r^{2\alpha} + \kappa_i\epsilon_i + \gamma\kappa_i r^{-2}).\end{aligned}$$

Choosing $\gamma = r^{2+2\alpha}$ and i sufficiently large so that also $\epsilon_i \leq r^{2\alpha}$, we then have

$$\sup_{B_{\tilde{\eta}_i}(p,0.5)} |U_i'| \leq C\kappa_i r^{2\alpha}.$$

Scaling back this estimate, we find that for sufficiently large i (depending on r), we have

$$\sup_{B_{\tilde{\eta}_i}(p,8r)} |U_i'| \leq C\kappa_i r^{2+2\alpha}.$$

By the distance estimates (3.2) and (3.4) it follows that

$$\sup_{B_{\tilde{\omega}_i}(p,4r)} |U_i'| \leq C\kappa_i r^{2+2\alpha}$$

once i is sufficiently large. We can now choose $r = \lambda$ small enough so that

$$\sup_{B_{\tilde{\omega}_i}(p,4\lambda)} |U_i'| \leq \kappa_i \lambda^{2+\alpha}.$$

Scaling down by r_i, we get

$$\sup_{B_{\omega_{k_i}}(p,4\lambda r_i)} |u_i'| \leq \kappa_i \lambda^{2+\alpha} r_i^2.$$

This gives the required contradiction. □

We can now state the abstract convergence result. To do so, we need the following definition. Recall that Z is a non-collapsed Gromov-Hausdorff limit of polarized Kähler-Einstein manifolds, ω_Z is the singular Kähler-Einstein metric on Z, $p \in Z$, and the tangent cone at p is $C(Y)$. Assume that $\mathcal{U}$ is a neighborhood of p, and on $\mathcal{U}$ there is a family $\mathcal{F}$ of model metrics.

Definition 3.6. We say that ω_Z can be approximated by $\mathcal{F}$ if the following holds. Fix any $0 < \kappa < \epsilon$. Then for all $r > 0$ sufficiently small, there exist $\Lambda > 0$ and an embedding $F : B_\omega(p, 2r) \subset \mathcal{U} \to Z$ from the ball with respect to $\omega = \omega_0 \in \mathcal{F}$ such that $F(p) = 0$ with the following properties. Let $\eta = \Lambda F^*\omega_Z$. Then on $B_\omega(p, 2r)$, the following hold:

(1) $\mathrm{Ric}(\eta) = c\eta$ with $|c| < r^{-2}\epsilon$.
(2) $\eta^n = e^f \omega^n$ and $\eta = \omega + \sqrt{-1}\partial\bar{\partial}u$, with

$$|u| < r^2\kappa, \quad f(p) = 0, \quad |\nabla f|_\eta < r^{-1}\kappa\epsilon.$$

(3) $|d_\eta - d_\omega| < r/100$.

Proposition 3.7. *Suppose that at $p \in Z$, ω_Z can be approximated by a family of $\mathcal{F}$ of model metrics in a neighborhood $\mathcal{U} \subset Z$ of p. Then for some $r_0 > 0$, there is a model metric $\omega \in \mathcal{F}$ and a holomorphic embedding $F : B_\omega(p, r_0) \to Z$, with $F(p) = p$, and constants $\Lambda, C, \alpha > 0$, such that*

$$\Lambda F^*\omega_Z = \omega + \sqrt{-1}\partial\bar{\partial}u_r$$

for some u_r defined on $B_\omega(p,r)$, and

$$\sup_{B_\omega(p,r)} |u_r| \leq Cr^{2+\alpha}$$

for all $r < r_0$.

Proof. We iterate the decay estimate, Proposition 3.5, as well as the distance estimate, Lemma 3.4. Let C, α and λ be the constants in Proposition 3.5, and let ϵ, r be sufficiently small so that both Lemma 3.4 and Proposition 3.5 hold. At the initial stage we let $\kappa < C^{-1}(1-\lambda^\alpha)\epsilon$. Exactly how small ϵ should be will be clear later. By letting r be smaller if necessary (depending on κ, ϵ), we have the corresponding approximation $F : B_\omega(p, 4r) \to Z$, where $\omega = \omega_0 \in \mathcal{F}$, with constant $\Lambda > 0$. Write $\eta = \Lambda F^*\omega_Z$. Then Lemma 3.4 implies that we have $|d_\eta - d_\omega| < \lambda r/200$ on the ball $B_\omega(p, 2r)$. We write $h_0 = 0$.

Applying Proposition 3.5 we have a model metric $\omega_1 = \omega_{h_1}$, with $\|h_1\| \leq C\kappa \leq \epsilon$, and a function u_1 on $B_\omega(p,r)$ such that $\eta = \omega_{h_1} + \sqrt{-1}\partial\bar{\partial}u_1$, and

$$\sup_{B_\omega(p,4\lambda r)} |u_1| \leq \lambda^{2+\alpha} r^2 \kappa.$$

By property (3) of Definition 3.2, it follows that

$$\sup_{B_{\omega_1}(p,2\lambda r)} |u_1| \leq \lambda^{2+\alpha} r^2 \kappa.$$

Also by property (3) of Definition 3.2, on $B_{\omega_1}(p, 2\lambda r)$ we have

$$|d_{\omega_0} - d_{\omega_1}| \leq C_1(\|h_1\| + \|h_0\|)\lambda r \leq 2C_1\epsilon\lambda r \leq \frac{\lambda r}{200}$$

if we choose ϵ to be sufficiently small. Consequently, on $B_{\omega_1}(p, 2\lambda r)$ we have

$$|d_\eta - d_{\omega_1}| \leq |d_\eta - d_\omega| + |d_\omega - d_{\omega_1}| \leq \frac{\lambda r}{100}.$$

The metrics η and ω_1 now satisfy the conditions of Lemma 3.4 and Proposition 3.5, with r replaced by λr and κ by $\lambda^\alpha\kappa$. We can iterate this construction and we obtain a sequence of model metrics $\omega_i = \omega_{h_i}$ with $\|h_{i+1} - h_i\| \leq C(\lambda^\alpha)^i\kappa$ such that on $B_{\omega_i}(p, 2\lambda^i r)$ we have $\eta = \omega_i + \sqrt{-1}\partial\bar{\partial}u_i$ with

$$\sup_{B(p,2\lambda^i r)} |u_i| \leq (\lambda^i)^{2+\alpha}\kappa r^2.$$

The harmonic functions h_i converge to a harmonic function k satisfying $\|k\| \leq \epsilon$, so $k \in U$ if ϵ is chosen small enough. Let $\tilde{\omega} = \omega_k$ be the corresponding model metric. By property (4) of Definition 3.2, there exists v_i on $B_{\tilde{\omega}}(0,1)$ such that

$$\omega_i - \tilde{\omega} = \sqrt{-1}\partial\bar{\partial}v_i,$$

with

$$\sup_{B_{\tilde{\omega}}(p,\lambda^i r)} |v_i| \leq C_2\|k - h_i\|(\lambda^i r)^2 \leq C_3(\lambda^i)^{2+\alpha} r^2\kappa.$$

So on $B_{\tilde{\omega}}(0, \lambda^i r)$ we have

$$\eta = \omega_i + \sqrt{-1}\partial\bar{\partial}u_i = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}(u_i + v_i) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{u}_i,$$

where $\tilde{u}_i = u_i + v_i$. Then $\tilde{u}_i$ satisfies

$$\sup_{B_{\tilde{\omega}}(0,\lambda^i r)} |\tilde{u}_i| \le (1 + C_3)\kappa(\lambda^i)^{2+\alpha} r^2 \le C'(\lambda^i r)^{2+\alpha},$$

where $C' = (1 + C_3)\kappa r^{-\alpha}$, and so $\tilde{\omega}$ and $\tilde{u}_i$ are as required. □

4. K-polystable singularities

Suppose, as above, that (Z, p) is the non-collapsed pointed Gromov-Hausdorff limit of a sequence of polarized Kähler-Einstein manifolds, with its singular Kähler-Einstein metric denoted by ω_Z. Let $C(Y)$ denote the metric tangent cone to Z at p. In this section we assume that the germ (Z, p) is isomorphic to the germ $(C(Y), o)$, where o is the vertex of the cone $C(Y)$. In particular this means that the affine variety $C(Y)$, equipped with the homothetic vector field ξ induced by the cone structure defines a K-polystable Fano cone singularity $(C(Y), \xi)$ in the terminology of Li-Wang-Xu [29].

In this section we prove our first main result, Theorem 1.1, by reducing it to Proposition 3.7. For this we need to construct a family $\mathcal{F}$ of model metrics on $C(Y)$ and then show that the Gromov-Hausdorff limit ω_Z can be approximated by $\mathcal{F}$.

The construction of $\mathcal{F}$ is fairly simple, since the model space $C(Y)$ is already a cone. Let H denote the space of quadratic harmonic functions h such that ∇h generates a biholomorphism which commutes with scaling (see Lemma 3.1). For $h \in H$, let $\phi(t)$ be the one-parameter group of biholomorphisms generated by $\frac{1}{2}\nabla h$. By the gradient estimate and h being homogeneous with quadratic growth, we have

$$\sup_{B_{C(Y)}(0,r)} |\nabla h|_{\omega_{C(Y)}} \le C \sup_{B_{C(Y)}(0,2r)} |h| r^{-1} \le C\|h\| r \tag{4.1}$$

for all $r > 0$. Let $g = \phi(1)$ and define $\omega_h = g^*\omega_{C(Y)}$.

Lemma 4.1. *There exists a neighborhood $0 \in U \subset H$ such that $\mathcal{F} = \{\omega_h \mid h \in U\}$ is a family of model metrics.*

Proof. For simplicity let us write $\omega = \omega_{C(Y)}$. We verify the properties in Definition 3.2. Property (1) is automatic since $C(Y)$ is a cone itself. Property (2) is satisfied since the automorphism g is generated by ∇h for a harmonic function h.

Let us consider property (3). Let $x, y \in B(0, r)$ be regular points. By differentiating $d_\omega(0, \phi(t)x)$ and using (4.1), we see that

$$d_\omega(0, \phi(t)x) \le e^{C\|h\|t} d_\omega(0, x). \tag{4.2}$$

Similarly, by differentiating $d_\omega(x, \phi(t)x)$ and using (4.1) and (4.2) we see that

$$d_\omega(x, gx) \le C(e^{C\|h\|} - 1) d_\omega(0, x) \le C\|h\| d_\omega(0, x). \tag{4.3}$$

For $x, y \in B_\omega(0, r)$, the triangle inequality together with (4.3) gives

$$|d_\omega(gx, gy) - d_\omega(x, y)| \le |d_\omega(x, gx) + d_\omega(y, gy)| \le C\|h\|(d_\omega(0, x) + d_\omega(0, y)).$$

This proves property (3).

To see property (4), recall that ω as a cone metric is given by $\omega = \sqrt{-1}\partial\bar{\partial}(r^2/2)$, where r is the distance to the vertex 0. Differentiating $\phi(t)^* r^2$ and using (4.2), we get

$$|g^* r^2 - r^2| \le C\|h\|r^2. \tag{4.4}$$

Now, let g_h and g_k denote the automorphisms generated by h and k, respectively. Define

$$u = g_k^* r^2 - g_h^* r^2 = g_k^*(r^2 - g^* r^2),$$

where $g = g_h g_k^{-1}$. By standard Lie theory, for sufficiently small h, k, we have $g = g_{\tilde{h}}$ for some $\tilde{h} \in H$ with $\tilde{h} = h - k + O(\|h - k\|\|h\|)$. Then (4.2) and (4.4) together imply that

$$|u| = |g_k^*(r^2 - g^* r^2)| \le C\|h - k\| g_k^* r^2 \le C\|h - k\| r^2$$

once h, k are sufficiently small. Since $\omega_k = \omega_h + \sqrt{-1}\partial\bar{\partial}u$, this proves property (4) of Definition 3.2 for a sufficiently small neighborhood U of $0 \in H$.

Finally, let us prove (5). Fix K a compact set in the regular set of $B(0, 1)$. Let $r_i \to 0$ and $h_i, k_i \in H$ such that $\|h_i\|, \|k_i\| \to 0$. Let K_i be compact sets in the regular set of $B_{r_i^{-2}\omega_{h_i}}(0, 1)$ converging to K in the Gromov-Hausdorff sense. Since ω_{h_i} is a cone metric, we may work as if $r_i = 1$. Thus on $B_{\omega_{h_i}}(0, 1)$ we can simply take $K_i = g_i^{-1}K$. To simplify the notations we suppress the subscript i in what follows. Let $\phi(t)$ and $\psi(t)$ be the flows of ∇h and ∇k, respectively, and set $g_h = \phi(1)$ and $g_k = \psi(1)$. Then we have $\omega_k = \omega_h + \sqrt{-1}\partial\bar{\partial}u$ with $u = g_k^*(r^2/2) - g_h^*(r^2/2)$. If $\|h\|, \|k\|$ are sufficiently small (depending on K), then we can expand $\psi(t)^* r^2$ and $\phi(t)^* r^2$ as power series in t for $t \in [0, 1]$, whose coefficients depend on $\nabla h, \nabla k$ and the derivatives of r^2. As a consequence we have an estimate of the form

$$|g_h^* r^2 - r^2 - \frac{1}{2}\nabla h(r^2)| \le C|\nabla h|^2_{\omega_h} \le C\|h\|^2 \tag{4.5}$$

on K, where the last inequality follows from (4.1). Note that since h is homogeneous with degree two, we have $\nabla h(r^2) = 4h$.

Now, if h, k are sufficiently small, we have $\tilde{h} \in H$ as above. Using (4.5), we compute

$$\begin{aligned} |g_k^* r^2 - g_h^* r^2 - 2(k - h)| &\le g_k^*|r^2 - g_{\tilde{h}}^* r^2 + 2\tilde{h}| + 2|(h - k) - g_k^*\tilde{h}| \\ &\le C\|\tilde{h}\|^2 + C\|h - k\|\|h\| \\ &\le \epsilon\|h - k\| \end{aligned}$$

for any $\epsilon > 0$ once h, k are sufficiently small. This proves (5). ∎

It remains to show that ω_Z can be approximated by $\mathcal{F}$. As in Donaldson-Sun [20], we let $\lambda = 1/\sqrt{2}$, and let (Z_i, p_i) denote (Z, p) scaled up by a factor of λ^{-i}, which is still a pointed Gromov-Hausdorff limit of polarized Kähler-Einstein manifolds. Let B_i denote the unit ball around p_i, i.e. the ball $B(p, \lambda^i)$ scaled up to unit size. Let us denote the unit ball in $C(Y)$ by B, and let $F_\infty : B \to \mathbf{C}^N$ be an embedding given by an L^2-orthonormal set of homogeneous functions. Using this embedding we will also view $C(Y) \subset \mathbf{C}^N$. Since $C(Y)$ is the tangent cone at p, we have $B_i \to B$ in the Gromov-Hausdorff sense. We choose distance functions on the disjoint unions $B_i \sqcup B$ realizing the Gromov-Hausdorff convergence.

Proposition 4.2. *For sufficiently large i we have holomorphic maps $F_i : B_i \to \mathbf{C}^N$ satisfying the following properties, where $\Psi(i^{-1})$ denotes a function converging to zero as $i \to \infty$.*

1. *Under the Gromov-Hausdorff approximations between B_i and B we have $|F_i - F_\infty| < \Psi(i^{-1})$, and the image $F_i(B_i) \subset C(Y)$.*
2. *Let $\omega_i = (F_i^{-1})^*(\lambda^{-2i}\omega_Z)$ denote the metric on the image $F_i(B_i)$ induced by $\lambda^{-2i}\omega_Z$. Then we have $\mathrm{Ric}(\omega_i) = c_i\omega_i$ for some $|c_i| < \Psi(i^{-1})$, and the distance functions $d_{\omega_i}, d_{\omega_{C(Y)}}$ satisfy $|d_{\omega_i} - d_{\omega_{C(Y)}}| < \Psi(i^{-1})$.*
3. *We have $\omega_i^n = e^{f_i}\omega_{C(Y)}^n$ and $\omega_i = \omega_{C(Y)} + \sqrt{-1}\partial\bar{\partial}u_i$ with $f_i(0) = 0$ and $|\nabla f_i|_{\omega_i}, |u_i| < \Psi(i^{-1})$.*

In particular ω_Z can be approximated by $\mathcal{F}$ in the sense of Definition 3.6.

Proof. Let $\mathcal{O}_p$ be the ring of germs of holomorphic functions on Z at p. As in Donaldson-Sun [20], for $f \in \mathcal{O}_p$ we can define

$$d_{KE}(f) = \lim_{r\to 0} \frac{\sup_{B(p,r)} \log |f|}{\log r}.$$

By Li-Xu [30, Theorem 1.4], d_{KE} is the unique K-semistable valuation in $\mathrm{Val}_{Z,p}$. On the other hand, $C(Y)$ admits a Ricci-flat Kähler cone metric, and so the homothetic scaling on $C(Y)$ gives rise to a K-polystable valuation by Li-Wang-Xu [29, Corollary A.4], which in particular is K-semistable. It follows that these two valuations coincide.

The coordinate ring $R(C(Y))$ is a sum of the homogeneous pieces

$$R(C(Y)) = \bigoplus_{k\geq 0} R_{d_k}(C(Y)),$$

where R_{d_k} is the degree d_k piece under the homothetic action. Let us suppose that $R(C(Y))$ is generated by the functions of degree less than D, and let $k_0 = \max\{k \geq 0 \,|\, d_k < D\}$. We have a subspace $P \subset \mathcal{O}_p$, and an adapted sequence of bases for P as in [20, Section 3.2], which for sufficiently large i define holomorphic embeddings $F_i : B_i \to \mathbf{C}^N$. Under the Gromov-Hausdorff convergence $B_i \to B \subset C(Y)$, the maps F_i converge to an embedding $B \to \mathbf{C}^N$ using an L^2-orthonormal set of homogeneous functions in $R(C(Y))$ and up to modifying our maps by unitary transformations we can assume that this embedding of B coincides with our embedding F_∞. We will denote the L^2-norm of functions on B_i by $\|\cdot\|_i$.

Recall that the adapted sequence of bases are bases $\{G_i^1, \ldots, G_i^m\}$ of P satisfying the following:

- The L^2 norm on B_i satisfies $\|G_i^a\|_i = 1$, and $\langle G_i^a, G_i^b\rangle_i \to 0$ as $i \to \infty$.
- We have $G_i^a = \mu_{ia}^{-1} G_{i-1}^a + p_i^a$, with $\|p_i^a\|_i \to 0$ as $i \to \infty$.
- $\mu_{ia} \to \lambda^{d_a}$ as $i \to \infty$.

For each a, i we can write

$$G_i^a = g_i^a + k_i^a,$$

where g_i^a is homogeneous of degree d_a and k_i^a has strictly greater degree. There exists an $\epsilon > 0$ such that for all a, i we have $d(k_i^a) > d_a + \epsilon$. Let us also decompose $p_i^a = (p_i^a)_{d_a} + (p_i^a)_{>d_a}$ into the homogeneous degree d_a piece, and the remainder. We then have

$$G_i^a = \mu_{ia}^{-1}(g_{i-1}^a + k_{i-1}^a) + p_i^a,$$

and so

$$\begin{aligned} g_i^a &= \mu_{ia}^{-1} g_{i-1}^a + (p_i^a)_{d_a}, \\ k_i^a &= \mu_{ia}^{-1} k_{i-1}^a + (p_i^a)_{>d_a}. \end{aligned}$$

Since $d(k_{i-1}^a) > d_a + \epsilon$ and $\mu_{ia} \to \lambda^{d_a}$, for sufficiently large i we have

$$\|\mu_{ia}^{-1} k_{i-1}^a\|_i \leq \mu_{ia}^{-1}\lambda^{d_a+\epsilon/2}\|k_{i-1}^a\|_{i-1} \leq \lambda^{\epsilon/4}\|k_{i-1}^a\|_{i-1}.$$

It follows that $\|k_i^a\| \to 0$ as $i \to \infty$, and so if we define the functions $\tilde{F}_i$ to have components g_i^a, then $\sup_{B_i} |F_i - \tilde{F}_i| \to 0$. We claim that for sufficiently large i, $\tilde{F}_i$ is also an embedding. To see this, suppose that $\tilde{F}_i$ is not an injection. Since there are no compact subvarieties in B_i, $\tilde{F}_i$ is finite. By our assumption, for a generic regular point $q \in C(Y)$, there are at least two preimages $p_{1,i}, p_{2,i}$ under $\tilde{F}_i$. Note that from the convergence $\tilde{F}_i \to F_\infty$, we also have $p_{1,i}, p_{2,i} \to p$ for some $p \in C(Y)$ in the Gromov-Hausdorff sense. F_∞ is a biholomorphism on a neighborhood of p and therefore so is $\tilde{F}_i$ for large enough i.

We claim that further modifying the $\tilde{F}_i$ by elements in $GL(N)$ converging to the identity, and commuting with the homothetic action on $C(Y)$, we can assume that $\tilde{F}_i(B_i) \subset C(Y) \subset \mathbf{C}^N$. To see this, recall that the homothetic action on $C(Y)$ generates the algebraic action of a complex torus T on $C(Y)$, which we can assume is given by a linear action on $\mathbf{C}^N$. By our construction each $F_i(B_i)$ lies in the image $g_i C(Y)$ of the cone by a matrix $g_i \in GL(N)^T$ commuting with T. We need to show that there are elements $h_i \in GL(N)^T$ converging to the identity such that $h_i g_i C(Y) = C(Y)$. Since $C(Y)$ admits a Ricci flat Kähler cone metric, the group of linear automorphisms of $C(Y)$ commuting with T is reductive (see Donaldson-Sun [20]). Using this, we can apply the variant of Luna's slice theorem shown in Donaldson [18, Proof of Proposition 1] to the multigraded Hilbert scheme. In this Hilbert scheme we have $g_i C(Y) \to C(Y)$, and therefore there are some $h_i \to 1$ in $GL(N)^T$ such that $h_i g_i C(Y)$ lie in the slice at $C(Y)$. The orbit of $C(Y)$ can only meet the slice at finitely many points near $C(Y)$, therefore for sufficiently large i we have $h_i g_i C(Y) = C(Y)$. Replacing the F_i by $h_i \circ \tilde{F}_i$ we now have embeddings F_i of the B_i, satisfying Condition (1) in the statement of the Proposition.

Regarding Condition (2), the estimate for the Ricci curvature is immediate since by construction $\mathrm{Ric}(\omega_i) = c\lambda^{2i}\omega_i$ for some $|c| \leq 1$. The estimate $|d_{\omega_i} - d_{\omega_{C(Y)}}| < \Psi(i^{-1})$ follows from the estimate $|F_i - F_\infty| < \Psi(i^{-1})$. More precisely, for any $\epsilon > 0$ we need to show that for sufficiently large i we have $|d_{\omega_i} - d_{\omega_{C(Y)}}| < \epsilon$. Let $x, y \in F_i(B_i)$, so that $x = F_i(x_i), y = F_i(y_i)$. We can find $x_i', y_i' \in B$ such that under the Gromov-Hausdorff approximations we have $d(x_i, x_i'), d(y_i, y_i') < \Psi(i^{-1})$, and then $|x - F_\infty(x_i')|, |y - F_\infty(y_i')| < \Psi(i^{-1})$ by Condition (1). At the same time we also have points $x', y' \in B$ such that $x = F_\infty(x'), y = F_\infty(y')$. Our goal is to show that $|d_B(x', y') - d_{B_i}(x_i, y_i)| < \epsilon$ if i is sufficiently large (independent of x, y), where we are emphasizing that we are taking the distance with respect to the $\omega_{C(Y)}$ and ω_i metrics by writing d_B, d_{B_i}. Using the metrics on $B_i \sqcup B$ realizing the Gromov-Hausdorff convergence, we have

$$\begin{aligned} |d_B(x', y') - d_{B_i}(x_i, y_i)| &\leq d_B(x_i', x') + d_B(y_i', y') + d(x_i, x_i') + d(y_i, y_i') \\ &\leq d_B(x_i', x') + d_B(y_i', y') + \Psi(i^{-1}). \end{aligned}$$

Finally to see that $d_B(x_i', x')$ is small for large i, we can use that $|F_\infty(x_i') - F_\infty(x')| < \Psi(i^{-1})$ and the fact that F_∞^{-1} is uniformly continuous. It follows that for sufficiently large i we have $d_B(x_i', x') < \epsilon/2$, and the same holds for $d_B(y_i', y')$. Combining these results, we get $|d_B(x', y') - d_{B_i}(x_i, y_i)| < \epsilon$ for large i as required.

Since $\omega_{C(Y)}$ is a cone metric, it admits the Kähler potential $\psi = \frac{1}{2} d_{C(Y)}(o, \cdot)^2$. At the same time, using [31, Proposition 3.1], we can find Kähler potentials ϕ_i for ω_i on $F_i(B_i)$, such that

$$|\phi_i - \frac{1}{2} d_{\omega_i}(o, \cdot)^2| < \Psi(i^{-1}).$$

Using the estimate for the distance functions in Condition (2) we find that $\omega_i = \omega_{C(Y)} + \sqrt{-1}\partial\bar{\partial} u_i$, where

$$|u_i| = |\phi_i - \psi| = \frac{1}{2} |d_{\omega_i}(o, \cdot)^2 - d_{\omega_{C(Y)}}(o, \cdot)^2| < \Psi(i^{-1}).$$

Since ω_i satisfies $\mathrm{Ric}(\omega_i) = c_i\omega_i$ and $\mathrm{Ric}(\omega_{C(Y)}) = 0$, we have $\omega_i^n = e^{f_i}\omega_{C(Y)}^n$ for some f_i satisfying $c_i\omega_i = -\sqrt{-1}\partial\bar{\partial} f_i$ on the regular part of B, i.e.

$$\sqrt{-1}\partial\bar{\partial}(f_i + c_i\phi_i) = 0.$$

By Grauert-Remmert [23] the pluriharmonic function $f_i + c_i\phi_i$ extends to a pluriharmonic function across the (codimension 2) singular set of B. By

Colding's volume convergence theorem [12] we have

$$\int_B \omega_i^n = \int_B \omega_{C(Y)}^n + \Psi(i^{-1}),$$

and so since $c_i \to 0$ as $i \to \infty$ while ϕ_i is uniformly bounded, we have

$$\int_B e^{f_i + c_i\phi_i} \omega_{C(Y)}^n = \int_B \omega_{C(Y)}^n + \Psi(i^{-1}).$$

In particular we have a uniform bound for the L^1 norm of the plurisubharmonic function $e^{f_i+c_i\phi_i}$ on B, with respect to $\omega_{C(Y)}^n$, and so by the mean value inequality on a slightly smaller ball we have a uniform upper bound $e^{f_i+c_i\phi_i} < 1 + \Psi(i^{-1})$, or in other words $f_i + c_i\phi_i < \Psi(i^{-1})$. Similarly, we have a uniform upper bound for the integral of $e^{-f_i-c_i\phi_i}$ with respect to ω_i^n, and so we have $-f_i - c_i\phi_i < \Psi(i^{-1})$ on a slightly smaller ball. This implies that $|f_i + c_i\phi_i| < \Psi(i^{-1})$, and since $c_i \to 0$, we obtain $|f_i| < \Psi(i^{-1})$. We can arrange that $f_i(0) = 0$ by composing the F_i by an element of $GL(N)$ close to the identity, inducing a homothetic scaling on the cone $C(Y)$. Finally, using the equation $nc_i = -\Delta_{\omega_i} f_i$ together with the gradient estimate for harmonic functions implies $|\nabla f_i|_{\omega_i} < \Psi(i^{-1})$ as required in condition (3). □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.2, ω_Z can be approximated by $\mathcal{F}$. We can set $\Lambda = 1$ in Definition 3.6 because the model metric ω_0 is a cone metric. Applying Proposition 3.7, we see that there exist a model metric $\omega_h \in \mathcal{F}$, $r_0 > 0$, and a holomorphic map $F : B_{\omega_h}(0, r_0) \to Z$ with $F(0) = p$ and constants $C, \alpha > 0$, such that

$$F^*\omega_Z = \omega_h + \sqrt{-1}\partial\bar{\partial}u_r \tag{4.6}$$

for some u_r defined on $B(0, r)$ and

$$\sup_{B_{\omega_h}(0,r)} |u_r| \le Cr^{2+\alpha}$$

for all $r < r_0$. By construction (see Lemma 4.1), $\omega_h = g^*\omega_{C(Y)}$. Thus (4.6) becomes

$$(F \circ g^{-1})^*\omega_Z = \omega_{C(Y)} + \sqrt{-1}\partial\bar{\partial}(u_r \circ g^{-1}).$$

Since $B(0, r/2) \subset B_{\omega_h}(0, r)$, we have

$$\sup_{B(0,r/2)} |u_r \circ g^{-1}| \le Cr^{2+\alpha}. \tag{4.7}$$

This completes the proof. □

In the case of tangent cones with isolated singularities we have the following corollary, generalizing Hein-Sun [26, Theorem 1.4].

Corollary 4.3. *Suppose that in the setting of Theorem 1.1 the tangent cone $C(Y)$ has an isolated singularity at the origin. Then the metric $\phi^*\omega_Z$ satisfies*

$$\sup_{B(o,r)\setminus B(o,r/2)} |\nabla^j_{\omega_{C(Y)}}(\phi^*\omega_Z - \omega_{C(Y)})|_{\omega_{C(Y)}} \leq C_j r^{\alpha-j},$$

for all $r < r_0$, constants C_j, and the α from Theorem 1.1.

Proof. This follows from rescaling the estimate (4.7) by a factor of r^{-1}, and then applying Lemma 3.3. □

5. The unstable case

In this section we prove Theorem 1.2. Suppose that Z is the Gromov-Hausdorff limit of a non-collapsing sequence of polarized Kähler-Einstein manifold. Let $p \in Z$, and suppose $C(Y)$ is the tangent cone at p. Unlike the previous section, we deal with an example for which the germ (Z, p) is not isomorphic to the germ $(C(Y), o)$, where o is the vertex of the cone. Assume that

$$C(Y) = \mathbf{C} \times \{f(x) = x_1^2 + x_2^2 + \cdots + x_n^2 = 0\} \subset \mathbf{C}^{n+1}.$$

This is equipped with the Calabi-Yau cone metric

$$\omega_{C(Y)} = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}(|z|^2 + r^2),$$

where $r^2 = |x|^{2\frac{n-2}{n-1}}$ is the distance squared of the Stenzel metric [35]. Recall that the homothetic action on the coordinates x_i has weights $w_i = \frac{n-1}{n-2}$, and f is homogeneous with degree $d = 2\frac{n-1}{n-2}$. We assume that the germ (Z, p) is isomorphic to the isolated singularity

$$X = \{z^p + x_1^2 + \cdots + x_n^2 = 0\} \subset \mathbf{C}^{n+1}$$

for a fixed integer $p > d$. The effect is that the $\mathbf{C}^*$ action extending the homothetic action on $C(Y)$ degenerates X to $C(Y)$. By [36, Theorem 2], there exists a Calabi-Yau metric ω on a neighborhood of the singular point 0, whose tangent cone at 0 is $C(Y)$.

As in the previous section, we will prove Theorem 1.2 by showing that there exists a family $\mathcal{F}$ of model metrics built from applying automorphisms and scalings to ω, and that the singular Kähler-Einstein metric ω_Z on Z can be approximated by $\mathcal{F}$ near p. We have the following lemma, characterizing the space H of quadratic harmonic functions whose gradients generate automorphisms of $C(Y)$ that commute with scaling.

Lemma 5.1. *Let H be the space of quadratic harmonic functions, whose gradients generate automorphisms of $C(Y)$ that commute with scaling. Then H is spanned by*

$$(n-1)|z|^2 - |x|^{2\frac{n-2}{n-1}}$$

and

$$|x|^{-\frac{2}{n-1}} a_{jk} x_j \bar{x}_k,$$

where $(a_{jk}) \in \sqrt{-1}\mathfrak{o}(n, \mathbf{R})$. *For* $h \in H$ *there exist a holomorphic vector field* V *on* $\mathbf{C}^{n+1}$ *preserving the hypersurfaces* $X_c = \{cz^p + x_1^2 + \cdots x_n^2 = 0\} \subset \mathbf{C}^{n+1}$, *and a constant* β *such that* $L_V \Omega = n\beta\Omega$, *where* $\Omega = (1/x_1) dz \wedge dx_2 \cdots \wedge dx_n$ *is the holomorphic volume form on* X_c, *and*

$$V\left(\frac{|z|^2 + r^2}{2}\right) - \beta\left(\frac{|z|^2 + r^2}{2}\right) = h. \tag{5.1}$$

In addition we have $|\beta| \leq C\|h\|$ *and*

$$\sup_{B_{\omega_c}(0,r)} |V|_{\omega_c} \leq C\|h\| r,$$

i.e. V *has at most linear growth. Here* $B(o,1) \subset C(Y)$ *is the unit ball and* $\omega_c = |s|^{-2} F_c^* \omega$ *is the rescaled metric on* X_c *with* $F_c : X_c \to X$ *given by* $F_c(z, x) = (sz, s^{\frac{n-1}{n-2}} x)$, *and* $s^{p-2\frac{n-1}{n-2}} = c$.

Proof. The first part follows from [37, Lemma 2.2] using Fourier transform in the $\mathbf{C}$ direction or [9, Subsection 3.4.1] using Lemma 3.1 (3). For the holomorphic vector fields, it is very similar to the proof of [37, Lemma 2.3]. The only difference is that when $h = |z|^2 - \frac{1}{n-1}|x|^{2\frac{n-2}{n-1}}$, we consider the real holomorphic vector field

$$V = \mathrm{Re}\left(\frac{1}{p} z\partial_z + \frac{1}{2} x_i \partial_{x_i}\right).$$

Then V preserves the hypersurfaces $cz^p + x_1^2 + \cdots x_n^2 = 0$, we have

$$V(|z|^2 + r^2) - \left(\frac{2 + np - 2p}{2np}\right)(|z|^2 + r^2) = \left(\frac{2n - 2 - np + 2p}{2np}\right) h_{aut},$$

and we have

$$L_V \Omega = \left(\frac{2 + np - 2p}{2p}\right)\Omega.$$

The estimate for $|V|_{\omega_c}$ is analogous to [37, Proposition 2.1 (2)], using the construction of ω. □

We now construct the family of model metrics. Let $h \in H$. Then by Lemma 5.1 there exists a vector field V on $\mathbf{C}^{n+1}$ and a constant $\beta > 0$ (both depending on h) satisfying the required properties. Let $\phi(t)$ be the one-parameter group of biholomorphisms of X generated by V. Set $g_h = \phi(1)$ and define $\omega_h = e^{-\beta} g_h^* \omega$.

Lemma 5.2. *There exists a neighborhood* $0 \in U \subset H$ *such that* $\mathcal{F} = \{\omega_h \mid h \in U\}$ *is a family of model metrics in the sense of Definition 3.2.*

Proof. This is similar to the proof of Lemma 4.1. By the construction of ω, for $r_i \to 0$, $(r_i \cdot) : B_\omega(0,1) \to \mathbf{C}^{n+1}$ is a holomorphic map which is a $\Psi(i^{-1})$-Gromov-Hausdorff approximation in the sense of property (1) of Definition 3.2, where $\cdot$ denotes the homothetic scaling. Let $h_i \in H$ be a bounded sequence, and consider the corresponding model metrics $\omega_{h_i} = e^{-\beta_i} g_i^* \omega$. Since $r_i' = r_i e^{\beta_i/2} \to 0$ as β_i are bounded (Lemma 5.1), it follows that $F_i = (r_i' \cdot) \circ g_i : B_{r_i^{-2}\omega_{h_i}}(0,1) \to \mathbf{C}^{n+1}$ is also a $\Psi(i^{-1})$-Gromov-Hausdorff approximation. This establishes property (1) for any bounded neighborhood U of $0 \in H$.

Property (2) follows from Lemma 5.1. Property (3) is entirely similar to the proof of Lemma 4.1. For the rest, recall that $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$, where we have

$$\sup_{B_\omega(0,r)} |\varphi| \leq Cr^2,$$

which follows from the construction in [36, Section 8]. Since $\Delta_\omega \varphi = n$, we can apply the gradient estimate in annuli to get

$$\sup_{B_\omega(0,r)} |\nabla \varphi| \leq Cr$$

for all $r > 0$. Differentiating $\phi(t)^*\varphi$ and using the bounds in Lemma 5.1, we have

$$|g_h^* \varphi - \varphi| \leq C\|h\| r^2$$

for all $r > 0$. It follows that $|e^{-\beta} g_h^* \varphi - \varphi| \leq C\|h\| r^2$. Now let $k \in H$ be another quadratic harmonic function, and let W, γ be the corresponding vector field and constant given in (5.1) of Lemma 5.1. First we note that the vector fields given by (5.1) form a Lie subalgebra. Thus by standard Lie theory, for sufficiently small h, k, $g_{\tilde{h}} = g_h g_k^{-1}$ for some $\tilde{h} \in H$, with $\tilde{h} = h - k + O(\|h - k\|\|h\|)$. Let $\tilde{V}$ and $\tilde{\beta}$ be the vector field and the constant associated to $\tilde{h}$ in (5.1). We then have

$$\begin{aligned}
|e^{-\gamma} g_k^* \varphi - e^{-\beta} g_h| &\leq e^{-\gamma} g_k^* |\varphi - e^{-(\beta-\gamma)} g_{\tilde{h}}^* \varphi| \\
&\leq e^{-\gamma} g_k^* (|\varphi - e^{-\tilde{\beta}} g_{\tilde{h}}^* \varphi| + |e^{-\tilde{\beta}} - e^{-(\beta-\gamma)}||g_{\tilde{h}}^* \varphi|) \\
&\leq e^{-\gamma} g_k^* (C\|\tilde{h}\| r^2 + C\|\tilde{h}\||g_{\tilde{h}}^* \varphi|) \\
&\leq C\|h - k\| r^2.
\end{aligned}$$

This proves property (4) for some small neighborhood U.

Finally, let us prove (5). Let $r_i \to 0$ and $h_i, k_i \in H$ with $\|h_i\|, \|k_i\| \to 0$. Let V_i, W_i be the corresponding vector fields for h_i, k_i defined in Lemma 5.1. Let $\phi_i(t), \psi_i(t)$ be the flows of V_i, W_i, respectively. Set $g_{h_i} = \phi(1)$ and $g_{k_i} = \psi(1)$. Then the model metrics are given by $\omega_{h_i} = e^{-\beta_i} g_{h_i}^* \omega$ and $\omega_{k_i} = e^{-\gamma_i} g_{k_i}^* \omega$, with $|\beta_i| \leq C\|h_i\|$ and $|\gamma_i| \leq C\|k_i\|$. Fix a compact set K in the regular set of $B(0,1)$, and let $K_i \subset B_{r_i^{-2}\omega_{h_i}}(0,1)$ be compact sets converging to K in the Gromov-Hausdorff sense. By (5.1) in Lemma 5.1, we have

$$V_i(r^2/2) - \beta_i(r^2/2) = h_i$$

and the analogous equation for W_i, γ_i, k_i. Here we denote the cone metric as $\omega_{C(Y)} = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}r^2$. Since $\varphi_i = r_i^{-2}\varphi$ on K_i converges to $r^2/2$ in C^∞ on K, it follows that under the Gromov-Hausdorff approximation,

$$|V_i\varphi_i - \beta_i\varphi_i - h_i| \le \Psi(i^{-1})\|h_i\|.$$

Using power series expansion as in Lemma 4.1 and the above inequality, it follows that

$$|e^{-\beta_i}g_{h_i}^*\varphi_i - \varphi_i - h_i| \le O(\|h_i\|^2) + \Psi(i^{-1})\|h_i\| \le \Psi(i^{-1})\|h_i\|.$$

Now, let $\tilde{h}_i \in H$ with vector field $\tilde{V}_i$ and constant $\tilde{\beta}_i$ such that $g_{\tilde{h}_i} = g_{h_i}g_{k_i}^{-1}$ and $\tilde{h}_i = h_i - k_i + O(\|h_i - k_i\|\|k_i\|)$. Then we have

$$\begin{aligned} |e^{-\beta_i}g_{k_i}^*\varphi_i - e^{-\gamma_i}g_{h_i}^*\varphi_i - (k_i - h_i)| &\le e^{-\beta_i}g_{k_i}^*|\varphi_i - e^{-(\gamma_i-\beta_i)}g_{\tilde{h}_i}^*\varphi_i + \tilde{h}_i| \\ &\quad + |e^{-\beta_i}g_{k_i}^*\tilde{h}_i + (k_i - h_i)| \\ &\le \Psi(i^{-1})\|\tilde{h}_i\| + C\|h_i - k_i\|\|k_i\| \\ &\le \Psi(i^{-1})\|h_i - k_i\|. \end{aligned}$$

Setting $u_i = e^{-\beta_i}g_{k_i}^*\varphi_i - e^{-\gamma_i}g_{h_i}^*\varphi_i$, this conclude the proof of (5). □

Now we turn to showing that ω_Z can be approximated by $\mathcal{F}$. As in the previous section, let $\lambda = 1/\sqrt{2}$, and let (Z_i, p_i) denote (Z, p) scaled up by a factor of λ^{-i}. Let B_i denote the unit ball centered at p_i, i.e. the ball $B(p, \lambda^i)$ scaled up to unit size. Let F_∞ denote the inclusion of $C(Y)$ in $\mathbf{C}^{n+1}$. Note that the components of F_∞ consist of L^2 orthonormal homogeneous functions z, x_i. Let $B \subset C(Y)$ be the unit ball centered at 0.

Proposition 5.3. *For sufficiently large i we have holomorphic maps $F_i : B_i \to \mathbf{C}^{n+1}$ with the following properties, where $\Psi(i^{-1})$ denotes a function converging to zero as $i \to \infty$.*

1. *On the ball B_i the map F_i gives a $\Psi(i^{-1})$-Gromov-Hausdorff approximation to the embedding $F_\infty : B \to \mathbf{C}^{n+1}$. Moreover, the image $F_i(B_i) \subset \{a_i z^p + x_1^2 + \cdots + x_n^2 = 0\}$ for some $a_i > 0$ with $F_i(p_i) = 0$*
2. *There exist a subsequence $F_{j(i)}$ of F_i and a sequence of scalings $g_i : (z, \mathbf{x}) \mapsto (m_i z, m_i^{(n-1)/(n-2)}\mathbf{x})$ with $C_n^{-1} < m_i < C_n$ for some dimensional constant $C_n > 0$, such the image of the map $F_i' = g_i \circ F_{j(i)}$ lies in $X_i = \{(\lambda^i)^{p-2\frac{n-1}{n-2}}z + x_1^2 + \cdots + x_n^2 = 0\} \subset \mathbf{C}^{n+1}$. X_i is equipped with the "unknown metric" $\eta_i = (F_i'^{-1})^*(m_i^{-2i}\lambda^{-2j(i)}\omega_Z)$ as well as the model metric $\omega_i = \lambda^{-2i}G_i^*\omega$, where $G_i : (z, \mathbf{x}) \mapsto (\lambda^i z, (\lambda^i)^{\frac{n-1}{n-2}}\mathbf{x})$.*
3. *We have $\mathrm{Ric}(\eta_i) = c_i\eta_i$ for some $|c_i| < \Psi(i^{-1})$, and the distance functions d_{η_i}, d_{ω_i} satisfy $|d_{\eta_i} - d_{\omega_i}| < \Psi(i^{-1})$.*

(4) *We have* $\eta_i^n = e^{f_i}\omega_i^n$ *and* $\eta_i = \omega_i + \sqrt{-1}\partial\bar{\partial}u_i$*, with* $f_i(0) = 0$ *and* $|\nabla f_i|_{\eta_i}, |u_i| < \Psi(i^{-1})$.

In particular ω_Z *can be approximated by* $\mathcal{F}$ *in the sense of Definition 3.6.*

Proof. Identifying the germ of (Z, p) with the germ of $(X, 0)$, we can assume that $B_i \subset X$. Write $R = \mathcal{O}_{X,0}$, and let v be the valuation of $(X, 0)$ associated to ω. By the construction of ω, the associated graded ring R_v is isomorphic to $R(C(Y))$, which is a Ricci-flat Kähler cone. So by Li-Xu [30, Theorem 1.3] and Li-Wang-Xu [29, Corollary A.4], we have $d_{KE} = v$, where d_{KE} is the valuation given by ω_Z.

For (1), we will focus on the case when $n = 3$. For $n > 3$, the argument is the same, with the simplification that the function z^2 has higher degree than x_i. As in Proposition 4.2, we have a subspace $P \subset \mathcal{O}_{Z,p}$ and an adapted sequence $\{G_i^a\}_i$ of bases for P, which for sufficiently large i define holomorphic embeddings $F_i : B_i \to \mathbf{C}^N$. F_i converges in the Gromov-Hausdorff sense to F_∞, which up to a unitary rotation is given by $(1, z, z^2, \mathbf{x})$, the components of which form an orthonormal basis for the corresponding space in $R(C(Y))$ (we assume $n = 3$). Here $\mathbf{x} = (x_1, x_2, x_3)$. From this we see that $N = 6$. Note that since we have the isomorphism of germs, $\mathcal{O}_{Z,p}$ is also generated by $S = \{1, z, z^2, \mathbf{x}\}$. We can decompose G_i^a as $G_i^a = g_i^a + k_i^a$, where g_i^a is a linear combination of elements in S with degree equal to d_a and k_i^a has degree $> d_a$. As in the proof of Proposition 4.2 we have $\sup_{B_i} |G_i^a - g_i^a| \to 0$ as $i \to \infty$.

Define $\tilde{F}_i = (g_i^a)$. We can write $\tilde{F}_i = (c_i, z_i, w_i, \mathbf{x}_i)$, where

$$\begin{aligned} z_i &= d_i z, \\ w_i &= W_i^T \mathbf{x} + b_i z^2, \\ \mathbf{x}_i &= A_i \mathbf{x} + z^2 V_i, \end{aligned}$$

and b_i, c_i, d_i are scalars, V_i, W_i are vectors and A_i is a matrix. Using the fact that $\sup_{B_i} |G_i^a - g_i^a| \to 0$ as $i \to \infty$, we deduce that $d_i - \lambda^{-1}d_{i-1} \to 0$, $b_i - \lambda^{-2}b_{i-1} \to 0$, $A_i - \lambda^{-2}A_{i-1} \to 0$, $W_i - \lambda^{-2}W_{i-1} \to 0$, and $V_i - \lambda^{-2}V_{i-1} \to 0$.

On the other hand, writing the equation for X in terms of $z_i, \mathbf{x}_i$ and using the above convergence result for V_i, we must have $V_i = 0$ for all sufficiently large i and

$$d_i^{-p}\|A_i\|^2, |(\|A_i\|^{-1}A_i)^T(\|A_i\|^{-1}A_i) - Id| \le \Psi(i^{-1}).$$

In particular, by modifying A_i by matrices of the form $Id + \Psi(i^{-1})$, we may assume that $\|A_i\|^{-1}A_i \in O(3)$. We now drop the first and the third components of $\tilde{F}_i$ and obtain embeddings $F_i = (z_i, \mathbf{x}_i)$ into $\mathbf{C}^4$, whose image is given by $d_i^{-p}\|A_i\|^2 z^p + \mathbf{x}_i^T\mathbf{x}_i = 0$. Set $a_i = d_i^{-p}\|A_i\|^2$. Then by the above convergence results we have $a_i/a_{i-1} \to \lambda^{p-4} < 1$. By applying scalings $(z, \mathbf{x}) \to (cz, c^{\frac{n-1}{n-2}}\mathbf{x})$ with some $|c| = 1$, we can assume that $a_i > 0$. So we have proved (1).

To prove (2), we argue as in the proof of [37, Theorem 1.1]. Since $a_i/a_{i-1} \to \lambda^{p-2\frac{n-1}{n-2}}$, for sufficiently large i, we can find $j(i)$ such that $C_n^{-1}a_{j(i)} < (\lambda^i)^{p-2\frac{n-1}{n-2}} < C_n a_{j(i)}$ for a dimensional constant $C_n > 0$. We can therefore find $m_i \in (C_n^{-p+2\frac{n-1}{n-2}}, C_n^{p-2\frac{n-1}{n-2}})$ such that

$$m_i^{p-2\frac{n-1}{n-2}} a_{j(i)} = (\lambda^i)^{p-2\frac{n-1}{n-2}}.$$

This proves (2). The rest follows verbatim the proof of Proposition 4.2. □

Proof Theorem 1.2. Proposition 5.3 shows that ω_Z can be approximated by $\mathcal{F}$ constructed in Lemma 5.2. The rest of the proof is very similar to the proof of Theorem 1.1, so we omit it. □

6. Uniqueness of Calabi-Yau metrics under small perturbation

In this section we prove Theorem 1.3, which says that polynomially subquadratic perturbation of a $\partial\bar{\partial}$-exact Calabi-Yau metric with maximal volume growth must be trivial. Recall that X is said to have maximal volume growth if there exists $v > 0$ such that for all $p \in X$ and $r > 0$, we have $\mathrm{Vol}(B(p, r)) \geq vr^{2n}$. It was proved in [32] that tangent cones at infinity of a Calabi-Yau manifold with maximal volume growth is an affine variety. It was also observed in [37, Section 3.1] that Donaldson-Sun theory extends to the $\partial\bar{\partial}$-exact case. In particular the tangent cone at infinity is unique. To prove Theorem 1.3, we need the following decay estimate. For the following, let $o \in X$ be a fixed point, and write $B(o, r)$ for the r-ball in X with respect to the rescaled metric $c^2\omega$, where $0 < c \ll 1$.

Lemma 6.1. *For any $\alpha > 0$ sufficiently small, there exists a constant $\lambda_0 > 0$ such that if $\lambda < \lambda_0$ and $\epsilon > 0$ is sufficiently small (depending on λ), then we have the following. Suppose that*

$$d_{GH}(B(o, \epsilon^{-1}), B(0, \epsilon^{-1})) < \epsilon,$$

where $B(0, \epsilon^{-1})$ is the corresponding ball in the tangent cone $C(Y)$. Suppose u is a smooth function on $B(o, 1)$ with $\sup_{B(o,1)} |u| < \epsilon$ satisfying

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = \omega^n.$$

Then we can find a smooth function u' on $B(o, 1/2)$ such that

(1) $\partial\bar{\partial}(u - u') = 0$,
(2) $\sup_{B(o,\lambda)} |u'| \leq \lambda^{2-\alpha} \sup_{B(o,1)} |u|$.

Proof. The proof is very similar to the proof of [37, Proposition 4.1], so we omit it. We note that the decay rate in (2) is slower than quadratic. Thus for this result we only need to subtract "subquadratic" harmonic functions from u and automorphisms of the cone do not enter the argument. The $\partial\bar{\partial}$-exactness is required to apply Theorem 2.1, and to embed the manifold X as an affine variety in $\mathbf{C}^N$. This in turn is required to employ the fact that subquadratic harmonic functions on the cone extend to pluriharmonic functions on the manifold. □

Proof of Theorem 1.3. We scale down the metric. Let $\omega_i = 2^{-2i}\omega$, and let $u_i = 2^{-2i}u$. Denote $B(o_i, 1)$ the unit ball with respect to the scaled-down metric ω_i. Let i_0 be large enough so that

$$\sup_{B(o_i,1)} |u_i| \leq 2^{-2i}C(1 + 2^{-i})^{2-\delta} \leq C'2^{-i\delta} < \epsilon,$$

and that

$$d_{GH}(B(o_i, \epsilon^{-1}), B(o, \epsilon^{-1})) < \epsilon$$

for $i > i_0$, where ϵ is given in Lemma 6.1. Let $\alpha > 0$ be sufficiently small as in Lemma 6.1. In particular we also want $\alpha < \delta$. Then we can apply Lemma 6.1. We may set $\lambda = 2^{-m}$, where $m > 0$ an sufficiently large integer. Let $i = i_0 + km$, where $k > 0$ is an integer. Then by Lemma 6.1, there exists a smooth function u' on $B(o_i, 1/2)$ such that $\partial\bar{\partial}(u_i - u') = 0$ and $\sup_{B(o_i,\lambda)} |u'| \leq \lambda^{2-\alpha} \sup_{B(o,1)} |u_i|$. Set $u'_{i-1} = \lambda^{-2}u'$. Note that

$$B(o_i, \lambda) = B(o_{i-m}, 1) = B(o_{i_0+(k-1)m}, 1).$$

So we have

$$\sup_{B(o_{i_0+(k-1)m},1)} |u'_{i-1}| \leq \lambda^{-\alpha} \sup_{B(o_i,1)} |u_i| \leq 2^{m\alpha-km\delta}C'2^{-i_0\delta} < \epsilon.$$

We can then iterate this process k times. In the end, we have a function u'_{i_0} on $B(o_{i_0}, 1)$ with

$$\sup_{B(o_{i_0},1)} |u'_{i_0}| \leq 2^{km(\alpha-\delta)}C'2^{-i_0\delta}$$

Rescaling back, we now have a smooth function $v_k = 2^{2i_0}u'_{i_0}$ satisfying

$$(\omega + \sqrt{-1}\partial\bar{\partial}v_k)^n = \omega^n$$

on $B(o, 2^{i_0})$ such that $\partial\bar{\partial}(u - v_k) = 0$ and

$$\sup_{B(o,2^{i_0})} |v_k| \leq 2^{km(\alpha-\delta)}C'.$$

By Lemma 3.3, up to passing to a subsequence v_k converges uniformly in C^∞ to 0 as $k \to \infty$. It follows that $\partial\bar{\partial}u = 0$ on $B(o, 2^{i_0})$. We can then increase i_0 and conclude that $\partial\bar{\partial}u = 0$ on X. □

We remark that the $\partial\bar{\partial}$-exactness condition is not required when the tangent cone at infinity has a smooth link (and hence is unique by Colding-Minicozzi [13]). In this case one can show Lemma 6.1 using the existence of adapted sequences of bases for harmonic functions with polynomial growth (see for example [9, 4.2.2]) and the maximum principle for the complex Monge-Ampère equation. While the setup in this case is closer to the asymptotically conical case considered in Conlon-Hein [15], this approach has the advantage that the polynomial convergence to the tangent cone at infinity is not required. It would be interesting to know if a version of the $\partial\bar{\partial}$ lemma holds in the setting of maximal volume growth, which would enable us to prove results on the level of metrics similar to [15, Theorem 3.1] as opposed to potentials.

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