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Recent progress on minimal hypersurfaces with cylindrical tangent cones

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Dedicated to László Székelyhidi on occasion of his 70th birthday.

Abstract. We survey some recent results on minimal hypersurfaces in $\mathbb{R}^{n+1}$ with cylindrical tangent cones. We discuss the question of the uniqueness of tangent cones, the behavior of certain minimal hypersurfaces with cylindrical tangent cones, and a Liouville type theorem for entire minimal hypersurfaces.

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1. Introduction

Let $M \subset \mathbf{R}^{n+1}$ be a minimal hypersurface, that is a stationary point of the area functional under compactly supported variations. At any smooth point M is locally given by a graph of a real analytic function, and an important problem is to understand the behavior of M near its singular points.

For simplicity let us suppose that $0 \in M$ is a singular point. A first step is to consider the infinitesimal behavior of M at 0, described by its tangent cones. These are defined to be subsequential limits of sequences of rescalings $\sigma_k M$ with $\sigma_k \to \infty$. There are several basic questions that one can ask:

- Q1 Is the tangent cone unique? Or does it depend on the subsequence used to define it?
- Q2 To what extend does the tangent cone describe the behavior of M nearby?

The study of singularities of minimal hypersurfaces and their tangent cones has a global counterpart, namely the study of entire minimal hypersurfaces $M \subset \mathbf{R}^{n+1}$ and their tangent cones at infinity. Similarly to the tangent cones at singularities, tangent cones at infinity are given by subsequential limits of $\sigma_k M$ as $\sigma_k \to 0$. Uniqueness is again an important question, and in addition

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one expects rigidity/classification results for entire hypersurfaces with given tangent cones:

Q3 Can we classify entire minimial hypersurfaces $M \subset \mathbf{R}^{n+1}$ with a given tangent cone at infinity?

The most completely understood situation regarding Q1 and Q2 is when M has a multiplicity one tangent cone C at 0, which is smooth away from the origin. In this case Allard–Almgren [1] and Simon [14] showed that the tangent cone C is unique, and M can be written as a graph over C near the origin. In particular it follows that M itself has an isolated singularity at 0 just like its tangent cone, and moreover many examples of such minimal hypersurfaces were constructed by Caffarelli–Hardt–Simon [4]. The classification question Q3 is more difficult already in this case. Simon–Solomon [20] showed that when C is a quadratic cone, i.e. a cone over a product of spheres, then up to translations the only minimal hypersurfaces with tangent cone C at infinity are C itself, and the leaves of the Hardt–Simon foliation [8] (see below for more details). For more general strictly minimizing cones C, Chan [5] showed that there is a family of minimal hypersurfaces with tangent cone C at infinity, parametrized by a space of "slowly decaying" Jacobi fields on C. It is not known if this construction gives rise to all such minimal hypersurfaces.

In this survey we will focus on what is perhaps the next simplest situation, namely when M has a multiplicity one cylindrical tangent cone $C \times \mathbf{R}$ at the origin, or at infinity. We will assume that C is smooth away from the origin and in addition is strictly stable and strictly minimizing in the sense of Hardt–Simon [8]—see Sect. 2 for more details. For such cylindrical tangent cones much less is known about the questions Q1–Q3.

Regarding Q1, for a large class of cones C Simon [18] showed that the corresponding cylindrical tangent cones $C \times \mathbf{R}$ are unique. In particular his result holds for the quadratic cones $C = C(S^p \times S^q)$ when $p + q > 6$. Uniqueness of the tangent cone $C(S^3 \times S^3) \times \mathbf{R}$ was shown in [23]. Note that Simon's uniqueness result in [18] applies for tangent cones $C \times \mathbf{R}^k$ with higher dimensional Euclidean factors, but this is still open for the Simons cone $C = C(S^3 \times S^3)$. Similarly, the uniqueness of cylindrical tangent cones involving the last remaining minimizing quadratic cone $C(S^2 \times S^4)$ is still open. We will discuss these results in more detail in Sect. 4.

To address Question Q2 a first natural step is to construct (non-product) examples with cylindrical tangent cones, in analogy with Caffarelli–Hardt–Simon's work [4] on the case of tangent cones with isolated singularities. One class of examples was obtained by Smale [21], where the singular sets are orbits of certain group actions on Euclidean space. More recently Simon [12] constructed stable minimal hypersurfaces with cylindrical tangent cones $C(S^p \times S^q) \times \mathbf{R}^k$, which moreover can have essentially arbitrary singular sets, however for this one must allow for a perturbation of the Euclidean metric on $\mathbf{R}^{n+1}$. In [22] we constructed area minimizing hypersurfaces in $\mathbf{R}^{n+1}$ with isolated singularities, but with cylindrical tangent cones. See Sect. 5 for more details.

The other aspect of Q2 is to what extent we can understand a general minimal surface in a neighborhood of a singularity with a given cylindrical tangent cone. In particular one would like to address what the singular set can look like. General rectifiability results for the singular set have been obtained by Simon [17, 19] and Naber–Valtorta [10], but we are far from understanding its finer structure. It is unknown if the singular set can be a closed interval for instance unless we are allowed to perturb the ambient Euclidean metric, as in Simon's result cited above. For the Euclidean ambient metric we showed that under a strong symmetry assumption, if the tangent cone is $C(S^p \times S^q) \times \mathbf{R}$, then either the minimal hypersurface equals its tangent cone, or it has an isolated singularity at the origin. We will discuss these results further in Sect. 6.

So far question Q3 has been explored the least in the context of cylindrical tangent cones, and for instance we do not have a classification of all minimal hypersurfaces even for the simplest tangent cones $C(S^p \times S^q) \times \mathbf{R}$ at infinity. To obtain a classification we can make additional assumptions. A very natural question is to classify entire minimal hypersurfaces M which lie on one side of an area minimizing cone V. Indeed by a recent result of Wang [25] there always exists a smooth minimizing hypersurface M_0 on one side of V and in analogy with Hardt–Simon's result [8], in the case where V has an isolated singularity, it is natural to expect that the scalings λM_0 for $\lambda > 0$ exhaust all minimal hypersurfaces with tangent cone V at infinity, lying on the same side. Simon [11] showed that this is indeed the case for cylindrical tangent cones $V = C \times \mathbf{R}^k$ under an additional assumption on the normal vector. In work in progress with Edelen [6] we remove this additional assumption. These results, along with some further questions will be presented in Sect. 7.

Typically, the study of minimal surfaces with a given tangent cone V requires some understanding of minimal surfaces M that are "close" to V on an annulus A. When V is smooth away from the origin, then on an annulus A one can write M as a graph over V, and reduce many questions to studying the corresponding linearized problem. When V has a non-isolated singular set $\Sigma \subset V$, as in the cylindrical case $V = C \times \mathbf{R}$, the linearized problem may no longer accurately model the behavior of the minimal surface M near Σ. For this reason we need a non-concentration estimate, which roughly speaking says that the behavior of M near the singular set Σ is negligible, and so the linearized problem over the smooth part of V still governs the behavior of M on the annulus A. In Sect. 3 we will discuss such a non-concentration result introduced in [23], which is at the heart of many of the results described above.

2. Preliminaries

For background on minimal hypersurfaces and varifolds we refer to Simon [13]. Throughout the paper, on $\mathbf{R}^n \times \mathbf{R}$ we use coordinates $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$. We write $r = |x|$ and $\rho = (r^2 + y^2)^{1/2}$.

We let $C \subset \mathbf{R}^n$ be a strictly minimizing and strictly stable cone in the sense of Hardt–Simon [8], which is smooth away from 0. Recall that by [8] there are minimal hypersurfaces H_-, H_+ contained in the two connected components of $\mathbf{R}^n \backslash C$, such that the scalings $\lambda H_-, \lambda H_+$ for $\lambda > 0$ together with C foliate $\mathbf{R}^n$. This foliation has important applications in barrier arguments. For instance the existence of the foliation implies that all leaves $\lambda H_\pm$ are area minimizing (not just minimal) hypersurfaces. At the end of this section we describe how leaves of this foliation can be used to construct useful barrier surfaces related to cylindrical tangent cones.

Let us first recall some more details from the work of Hardt–Simon [8]. For an oriented hypersurface S in a Riemannian manifold N we denote by L_S the Jacobi operator on S, i.e. the linearization of the mean curvature operator on graphs over S. When S is a minimal hypersurface, then we have

$$L_S f = \Delta_S f + (|A_S|^2 + \mathrm{Ric}_N(\nu, \nu)) f,$$

where A_S is the second fundamental form of S, ν is the unit normal vector field to S and Ric_N is the Ricci tensor of N. We will have either $N = \mathbf{R}^k$ with $\mathrm{Ric}_N = 0$, or $N = S^{k-1}$ with $\mathrm{Ric}_N(\nu, \nu) = (k-2)$.

We denote by ϕ_i the i^{th} eigenfunction of $-L_\Sigma$ on the link $\Sigma = C \cap \partial B_1$, with eigenvalue λ_i. Corresponding to these there are homogeneous Jacobi fields $r^{-\gamma_i}\phi_i$ on C, where

$$\gamma_i^2 - (n-3)\gamma_i - (n-2+\lambda_i) = 0.$$

The strict stability condition implies that we can take $-\gamma_i > \frac{3-n}{2}$, and that there are no homogeneous Jacobi fields on C with degrees in the interval $(3-n+\gamma_1, -\gamma_1)$. We set $\gamma = \gamma_1$, and assume that $\phi_1 > 0$.

The assumptions that C is strictly minimizing and strictly stable imply that outside of a large ball, the surfaces $H_\pm$ are graphs of functions

$$\Psi_\pm = \pm r^{-\gamma}\phi_1 + v_\pm$$

over C. Here $v_\pm = O(r^{-\gamma-c})$ for some $c > 0$, and we will assume that c is small. We will use the following conventions for the orientations of H_-, C, H_+: on H_- the normal points towards C, on C it points towards H_+, and on H_+ it points away from C. This naturally extends to orientations of all the scalings $\lambda H_\pm$. Note that these orientations are consistent with our convention that outside of a large ball $H_\pm$ are the graphs of $\Psi_\pm$, where $\Psi_+ > 0$ and $\Psi_- < 0$. It will be convenient to combine the foliation of $\mathbf{R}^n$ into a single family of hypersurfaces $H(t)$ for $t \in \mathbf{R}$.

Definition 2.1. For $t \in \mathbf{R}$ we will write

$$H(t) = \begin{cases} |t|^{\frac{1}{\gamma+1}} H_+, & \text{for } t > 0, \\ C, & \text{for } t = 0, \\ |t|^{\frac{1}{\gamma+1}} H_-, & \text{for } t < 0. \end{cases}$$

Note that the scaling ensures that for a constant $C_1 > 0$ depending on the cone C, and for any $t \in \mathbf{R}$, the hypersurface $H(t)$ is the graph of the function

$$f_t(x) = |t|^{\frac{1}{\gamma+1}} \Psi_\pm \left(|t|^{-\frac{1}{\gamma+1}} x \right)$$

over C on the region $r > C_1 |t|^{1/(\gamma+1)}$, where the $\pm$ sign depends on the sign of t. The function f_t satisfies

$$|f_t - tr^{-\gamma}\phi_1| \leq C_1 |t|^{1+\frac{c}{\gamma+1}} r^{-\gamma-c}.$$

Thus, roughly speaking we can think of $H(t)$ as being the graph of $tr^{-\gamma}\phi_1$ over C, at least on the region where $|t| \ll r^{\gamma+1}$.

The link of $C \times \mathbf{R}$ is singular, with two singularities modeled on the cone C. We will only be interested in Jacobi fields u for which $r^{\gamma+\kappa}u$ is locally bounded away from the origin for a small $\kappa > 0$. Since there are no homogeneous Jacobi fields on C with growth rate in $(3 - n + \gamma, -\gamma)$, such u automatically satisfies that $r^\gamma u$ is locally bounded away from the origin, if κ is sufficiently small. Equivalently the Jacobi fields that we are interested in can be characterized as those that are in $W^{1,2}_{loc}$ away from the origin. This implies that the restrictions of u to the cross sections of the cone are in L^2 and can be decomposed according to the spectrum of the Jacobi operator on the (singular) link.

There are two basic results about such Jacobi fields that we will need. The first is an L^2 to L^∞ estimate.

Lemma 2.2. (See [23]) *Let u be a Jacobi field on $C \times \mathbf{R}$, such that $r^\gamma u$ is in L^∞ on $B_1(0)$. Then we have the estimate*

$$\sup_{B_{1/2}(0)} |r^\gamma u| \leq C \|u\|_{L^2(B_1)}.$$

The second is the following L^2 three annulus lemma, due to Simon [14, Lemma 2] (see also Lemma 3.3 in [15]). For a given $\rho_0 > 0$ let us use $\|u\|_{\rho_0,i}$ to denote the following L^2-norm on an annulus:

$$\|u\|^2_{\rho_0,i} = \int_{(C\times\mathbf{R})\cap(B_{\rho_0^i} \setminus B_{\rho_0^{i+1}})} |u|^2 \rho^{-n},$$

in terms of $n = \dim C \times \mathbf{R}$. Note that for a homogeneous degree zero function u the norm $\|u\|_{\rho_0,i}$ is independent of i.

Lemma 2.3. *Given $d \in \mathbf{R}$, there are small $\alpha_0' > \alpha_0 > 0$ and $\rho_0 > 0$ satisfying the following. Let u be a Jacobi field on the cone $C \times \mathbf{R}$, defined in the annulus $B_1 \setminus B_{\rho_0^3}$, such that $r^\gamma u \in L^\infty$. Then we have:*

(i) *If* $\|u\|_{\rho_0,1} \geq \rho_0^{d-\alpha_0}\|u\|_{\rho_0,0}$, *then* $\|u\|_{\rho_0,2} \geq \rho_0^{d-\alpha_0'}\|u\|_{\rho_0,1}$.

(ii) *If* $\|u\|_{\rho_0,1} \geq \rho_0^{-d-\alpha_0}\|u\|_{\rho_0,2}$, *then* $\|u\|_{\rho_0,0} \geq \rho_0^{-d-\alpha_0'}\|u\|_{\rho_0,1}$.

If in addition u *has no degree* d *component then the conclusion of either (i) or (ii) must hold.*

We now turn to the main existence result for barrier surfaces. These are the key ingredients in proving non-concentration estimates that we will discuss in the next section.

Proposition 2.4. *There is a large odd integer* p *and a constant* $Q > 0$ *depending on the cone* C, *with the following property. Let* $f : (a, b) \to \mathbf{R}$ *be a* C^3 *function, satisfying* $|f|_{C^3} \leq K$, *for some* $K > Q$. *Then for any* $\epsilon < Q^{-1}$ *there is an oriented hypersurface* X_ϵ *defined in the region where* $r < K^{-Q^2}$ *and* $y \in (a, b)$, *satisfying:*

(i) X_ϵ *is* C^2, *with negative mean curvature and no boundary in the region* $0 < r < K^{-Q}$, $y \in (a, b)$.

(ii) *At points of* X_ϵ *where* $r = 0$, *the tangent cone of* X_ϵ *is the graph of* $-\epsilon r$ *over* $C \times \mathbf{R}$.

(iii) *The* X_ϵ *vary continuously, and in each* y*-slice for* $y \in (a, b)$, X_ϵ *lies between the hypersurfaces*

$$H(\epsilon f(y)^p - \epsilon) \quad \text{and} \quad H(\epsilon f(y)^p + \epsilon).$$

In particular if V *is a stationary varifold in the region* $r < K^{-Q^2}$ *and the support of* V *intersects* X_ϵ, *then near the intersection point* V *cannot lie on the negative side of* X_ϵ.

For the proof see [22]. The basic idea is to first consider the hypersurface $\tilde{X}$, whose cross sections are given by $H(\epsilon f(y)^p)$ in $\mathbf{R}^n \times \{y\}$. The mean curvature of $\tilde{X}$ will not have the right sign, however we can take a graph over it to construct the required surface.

3. Non-concentration

In this section we discuss a basic non-concentration result for minimal hypersurfaces close to a cylinder $C \times \mathbf{R}$. To motivate the result, consider the following simple linear version for harmonic functions first.

Proposition 3.1. *Let* $s \in (0, 1/2)$. *There is an* $r_0 = r_0(s, n) > 0$ *and a dimensional constant* $C = C(n)$ *with the following property. Let* $u : B \to \mathbf{R}$ *be a harmonic function on the unit ball* $B \subset \mathbf{R}^{n+1}$. *We write* $\mathbf{R}^{n+1} = \mathbf{R}^n_x \times \mathbf{R}$ *and* $r = |x|$ *as before. Then*

$$\sup_{0.5B} |u| \leq C \sup_{B \cap \{r > r_0\}} |u| + s \sup_B |u|.$$

The result says that a harmonic function on the unit ball B cannot concentrate in the cylindrical region $\frac{1}{2}B \cap \{r < r_0\}$, unless it has fast growth from $\frac{1}{2}B$ to B.

Proof. A straightforward proof follows from bounding $\sup_{0.5B} |u|$ in terms of the L^2 norm of u on B. Indeed, the L^2 to L^∞ estimate can itself be thought of as a non-concentration estimate, showing that the L^2 norm cannot concentrate on sets of small measure for instance. Instead we use a maximum principle argument that is closer to the arguments used for minimal hypersurfaces below.

Let us write $D = \sup_B |u|$ and $D_{r_0} = \sup_{B\cap\{r>r_0\}} |u|$. Write $z_{-1} = (0, -1)$, $z_1 = (0, 1) \in \mathbf{R}^n \times \mathbf{R}$, and define

$$V(z) = |z - z_{-1}|^{1-n} + |z - z_1|^{1-n},$$

so that V is harmonic on B, but blows up at $z_{\pm 1}$.

Let us define

$$t_0 = \inf\{t > 0 : tV \geq u \text{ on } B\}.$$

By the maximum principle, the graphs of t_0V and u must touch somewhere on the boundary ∂B. We consider two pieces of the boundary separately for some fixed small $r_0 > 0$:

- On $\{r > r_0\} \cap \partial B$ we have $V \geq 2^{1-n}$, while at the same time $u \leq D_{r_0}$. So on this piece of the boundary we have $u \leq 2^{n-1}D_{r_0}V$
- On $\{r < r_0\} \cap \partial B$, if r_0 is small enough, we have either $|z - z_{-1}| \leq 2r_0$ or $|z - z_1| \leq 2r_0$, and therefore $V \geq (2r_0)^{1-n}$. At the same time $u \leq D$, and so here we have $u \leq (2r_0)^{n-1}DV$.

In sum we have that $u \leq C(D_{r_0} + r_0^{n-1}D)V$ on ∂B, for a dimensional constant C. It follows that $t_0 \leq C(D_{r_0} + r_0^{n-1}D)$, and note that $u \leq t_0V$ on B.

On the half ball $0.5B$ we have $|z - z_{\pm 1}| \geq 1/2$, so $V \leq 2 \cdot 2^{n-1}$, and as a consequence

$$u \leq t_0 2^n \leq C'(D_{r_0} + r_0^{n-1}D) \quad \text{on } 0.5B,$$

for a different dimensional constant C'. We can bound u from below by arguing with $-u$ instead. Then we can choose r_0 sufficiently small depending on s, so that the required estimate holds for $\sup_{0.5B} |u|$. □

To derive an analogous non-concentration estimate for minimal hypersurfaces, we need to define a notion of distance analogous to the supremum norm of $|u|$. Since our barrier surfaces are defined in terms of the leaves $H(t)$ of the Hardt-Simon foliation, we define the distance in terms of them as well.

Definition 3.2. For any subsets $M, U \subset \mathbf{R}^{n+1}$ we define $D_{C\times\mathbf{R}}(M; U)$ to be the infimum of all $d > 0$ such that $M \cap U$ is contained in the region bounded between $H(\pm d) \times \mathbf{R}$.

In practice M will be a minimal hypersurface and U an open set, like a ball. We think of $D_{C\times\mathbf{R}}(M;U)$ as the distance of M from $C\times\mathbf{R}$ on the region U. By the discussion after Definition 2.1, if M is the graph of u over $C\times\mathbf{R}$ on the region U, then $D_{C\times\mathbf{R}}(M;U)$ is comparable to the sup norm $\sup_U |r^\gamma u|$.

In terms of this distance we have the following non-concentration estimate for minimal hypersurfaces close to a cylindrical cone $C\times\mathbf{R}$, analogous to Proposition 3.1.

Proposition 3.3. *Let $s\in(0,1/2)$. There is an $r_0=r_0(s,C)>0$ and a constant $A>0$ depending on the cone C with the following property. Suppose that $M\subset\mathbf{R}^n\times\mathbf{R}$ is a codimension one stationary integral varifold defined in the region $\{|y|<1\}$. Then*

$$D_{C\times\mathbf{R}}(M;\{|y|<1/2\})\le A\,D_{C\times\mathbf{R}}(M;\{r>r_0\}\cap\{|y|<1\}) + s\,D_{C\times\mathbf{R}}(M;\{|y|<1\}).$$

For the proof see [22]. The basic idea is similar to the proof of Proposition 3.1 above. The main difference is that instead of the comparison functions tV in that proof, here we use the barrier surfaces X_ϵ given in Proposition 2.4, using the function $f(y)=4(1+y)^{-1}+4(1-y)^{-1}$, which blows up at $y=\pm1$.

In applications it is important to have similar non-concentration estimates for a distance function D_{T_λ} from minimal surfaces T_λ that are perturbations of $C\times\mathbf{R}$ on an annular region, depending on a parameter λ. In different applications below different families will be required, the common feature being that in each cross section $\mathbf{R}^n\times\{y\}$ the surface T_λ is well approximated by a leaf of the Hardt–Simon foliation (depending on y). In these settings the definition of the distance D_{T_λ} is more complicated, based on a dichotomy: when M is sufficiently close to T_λ to be the graph of a function u over it, then we can define a sup-type norm in terms of u, while when M is relatively far from T_λ then we can define a distance in terms of the leaves $H(t)$ of the Hardt-Simon foliation as in Definition 3.2. For the detailed definition see for instance [22, Definition 11.7].

4. Uniqueness of tangent cones

In this section we will discuss Question Q1 from the Introduction, on the uniqueness of cylindrical tangent cones. The first such uniqueness result was shown by Simon [18]. Although his result holds for more general cones, for simplicity we will focus on the case of quadratic cones $C=C(S^p\times S^q)$.

4.1. Simon's uniqueness theorem

Theorem 4.1. [18] *Suppose that a minimal hypersurface has a multiplicity one cylindrical tangent cone $C(S^p \times S^q) \times \mathbf{R}$, where $p+q>6$. Then this tangent cone is unique.*

We briefly sketch a proof of this result based on the non-concentration estimate Proposition 3.3. Simon's proof is based on a different non-concentration estimate for an L^2 distance (see [18, Corollary 2.3]).

Let us denote by $\mathcal{V}$ the cone $C(S^p \times S^q) \times \mathbf{R}$ and all of its rotations in $\mathbf{R}^{n+1}$. It can be shown that all minimal cones sufficiently close to $C \times \mathbf{R}$ are in $\mathcal{V}$. In addition the minimal cones $C(S^p \times S^q) \times \mathbf{R}$ are integrable in a strong sense for $p+q>6$, namely all degree one Jacobi fields correspond to rotations (see Simon [18, Equation (17)]).

We can define the distance $D_V(M; U)$ for any $V \in \mathcal{V}$ analogously to Definition 3.2, and to simplify notation below, we write $D_V(M) = D_V(M; B_1 \backslash B_{\rho_0})$. Here ρ_0 is the constant appearing in Lemma 2.3 with $d=1$. We have the following geometric 3-annulus lemma.

Proposition 4.2. *There is an $L>0$ such that for sufficiently small $d>0$ we have the following. Suppose that $M \in \mathcal{M}$ is a minimal hypersurface in B_1. Suppose that $D_V(M) < d$ and $\alpha \in (\alpha_1, \alpha_2)$, for $V \in \mathcal{V}$. Then*

(i) *If $D_V(LM) \geq L^\alpha D_V(M)$, then $D_V(L^2 M) \geq L^\alpha D_V(LM)$.*
(ii) *If $D_V(LM) \geq L^\alpha D_V(L^2 M)$, then $D_V(M) \geq L^\alpha D_V(LM)$.*

The proof is by contradiction, similar to Simon [14, Lemma 2] for instance, with the non-concentration estimate, Proposition 3.3 playing an important role. For the proof see [23, Proposition 5.12].

For any $B>0$ let us define the quantity

$$E_B(M) = \inf\{D_V(M) + D_V(L^B M) : V \in \mathcal{V}\},$$

for the value of L in the 3-annulus lemma above. Note that $E_B(M)$ can be used to control the distance between M and $L^B M$ on the annulus $B_1 \backslash B_{\rho_0}$ in the flat norm for instance (see [23, Lemma 6.2]). I.e. we have $d_{\mathcal{F}}(M, L^B M) \leq C E_B(M)$.

Uniqueness of the tangent cone follows from the following.

Proposition 4.3. *There is a $B>0$ with the following property. Suppose that M is a stationary integral varifold in B_1, such that the density of M at the origin equals that of $C(S^p \times S^q) \times \mathbf{R}$, and the area of M in B_1 is sufficiently close to that of $C(S^p \times S^q) \times \mathbf{R}$. Suppose $p+q>6$ as above. Then if the Hausdorff distance between M and $C(S^p \times S^q) \times \mathbf{R}$ is sufficiently small on the unit ball, we have*

$$E_B(L^B M) \leq \frac{1}{2} E_B(M).$$

Proof. We give a sketch of the proof. For the details see the proof of [23, Proposition 6.6], which applies in the more difficult non-integrable setting that we discuss below.

The proof is by contradiction. We suppose that we have a sequence M_i converging to $C(S^p \times S^q) \times \mathbf{R}$ on the unit ball B_1, satisfying the assumption on the density at the origin. We will show that the required conclusion holds for sufficiently large i, if B is chosen sufficiently large.

Let $\alpha_1' < \alpha_2'$ such that $\alpha_i' \in (\alpha_1, \alpha_2)$ for the constants in Proposition 4.2. First we claim that for large i there are $V_i \in \mathcal{V}$ such that one of the following two conditions holds:

(a) $D_{V_i}(L^{2B}M_i) \geq L^{\alpha_1' B} D_{V_i}(L^B M_i)$,
(b) $D_{V_i}(M_i) \geq L^{\alpha_2' B} D_{V_i}(L^B M_i)$.

This follows essentially by choosing V_i to be the "best fit" cone to $L^B M_i$ on the annulus $B_1 \backslash B_{\rho_0}$ and then using the last claim in Lemma 2.3, together with the non-concentration estimate. The integrability of the cone $C \times \mathbf{R}$ is crucial here, since it allows us to eliminate the degree 1 component of the Jacobi field that models the behavior of M_i relative to V_i by choosing V_i appropriately.

Next we suppose that condition (a) holds for sufficiently large i. Proposition 4.2 implies that for large i, we will have

$$D_{V_i}(L^{(k+1)B}M_i) \geq L^{\alpha_1' B} D_{V_i}(L^{kB} M_i),$$

for $k > 0$ as long as $L^{kB}M_i$ is still sufficiently close to V_i. Letting k_i be the largest value of k for which this still holds, we end up with a contradiction using the following: on the one hand the monotonicity formula for minimal surfaces implies that $L^{k_i B} M_i$ has to converge to a minimal cone as $i \to \infty$, but on the other hand the growth condition above (note that the rate of growth is independent of i) implies that this is not possible.

Finally, we can assume that condition (b) holds for all large i, while condition (a) fails. Let $d_i = D_{V_i}(L^B M_i)$. We have

$$\begin{aligned} D_{V_i}(M_i) &\geq L^{\alpha_2' B} D_{V_i}(L^B M_i) = L^{\alpha_2' B} d_i, \\ D_{V_i}(L^{2B} M_i) &\leq L^{\alpha_1' B} D_{V_i}(L^B M_i) = L^{\alpha_1' B} d_i. \end{aligned}$$

Using the second inequality we can estimate $E_B(L^B M_i)$ from above in terms of d_i: we have $E_B(L^B M_i) \leq C L^{\alpha_1' B} d_i$. Using the first inequality we can then argue that $E_B(M_i) \geq 2E_B(L^B M_i)$ once i is large enough. □

Uniqueness of the tangent cone follows by iterating this Proposition. For any $N > 0$, as long as $L^{kB}M$ stays sufficiently close to $C(S^p \times S^q) \times \mathbf{R}$ on the annulus $B_1 \backslash B_{\rho_0}$ for all $k \leq N$, we have the bound

$$\begin{aligned} d_{\mathcal{F}}(M, L^{(N+1)B}M) &\leq d_{\mathcal{F}}(M, L^B M) + \cdots + d_{\mathcal{F}}(L^{NB}M, L^{(N+1)B}M) \\ &\leq C(E_B(M) + \cdots + E_B(L^{NB}M)) \end{aligned}$$

$$\leq 2CE_B(M).$$

In particular if $E_B(M)$ is sufficiently small, then we can let N be arbitrarily large, and it follows that $L^{kB}M$ remains close to M on the annulus for all $k > 0$. This implies uniqueness of the tangent cone.

4.2. The tangent cone $C(S^3 \times S^3) \times \mathbf{R}$

The assumption that $p + q > 6$ in Simon's Theorem 4.1 implies that the corresponding cones $C(S^p \times S^q) \times \mathbf{R}$ are integrable, in the sense that all Jacobi fields that are homogeneous of degree one, and are locally in $W^{1,2}$, correspond to rotations in $\mathbf{R}^{n+1}$. There are two remaining minimizing cones of this type, $C(S^4 \times S^2) \times \mathbf{R}$ and $C(S^3 \times S^3) \times \mathbf{R}$. Neither of these is integrable, because they admit the degree one Jacobi field $\phi = y^3 r^{-2} - y$, where as above y is the coordinate on the $\mathbf{R}$ factor, while $r = |x|$ on the remaining $\mathbf{R}^n$ factor.

In this section we will discuss the following result.

Theorem 4.4. [23, Theorem 1.1] *Let M be an area-minimizing hypersurface in a neighborhood of $0 \in \mathbf{R}^9$, that admits $C \times \mathbf{R}$ as a multiplicity one tangent cone at the origin, where $C = C(S^3 \times S^3)$ is the Simons cone. Then $C \times \mathbf{R}$ is the unique tangent cone at 0.*

Note that uniqueness of the remaining quadratic cone $C(S^2 \times S^4) \times \mathbf{R}$ is still open.

For tangent cones with isolated singularities, uniqueness in the non-integrable case was shown by Simon [14] using his very influential infinite dimensional Łojasiewicz inequality. It seems to be difficult to extend this approach to cylindrical tangent cones, since in that case the cross section is singular and it is not clear whether a general Łojasiewicz inequality can still be expected to hold. Instead, the approach in [23] is based on constructing minimal surfaces T_δ modeled on the Jacobi field $\delta\phi$, and proving a Łojasiewicz type inequality only for this one-dimensional family.

4.2.1. The construction of T_δ. The first step in constructing the minimal surfaces T_δ is to focus on the link $\Sigma_0 = S^7 \cap (C(S^3 \times S^3) \times \mathbf{R})$. The minimal surface Σ_0 has two singular points, modeled on the Simons cone $C(S^3 \times S^3)$. The Jacobi field ϕ restricts to a Jacobi field, also denoted by ϕ, on Σ_0, which blows up at the rate of r^{-2} near the two singular points. Up to scaling this is the only Jacobi field that is also $O(4) \times O(4)$-invariant and has at worst $O(r^{-2})$ singularities. This Jacobi field spans the cokernel of the $O(4) \times O(4)$-invariant linearized operator, and so we can hope to find perturbations Σ_δ modeled on $\delta\phi$, which are minimal modulo the function ϕ.

It is convenient to choose a function ζ, compactly supported away from the singularities, such that ζ is also $O(4) \times O(4)$-invariant, it is an odd function of y, and $\int \zeta\phi = \int \phi^2$ on Σ_0. The following result shows that we can find smoothings of Σ_0 that are minimal modulo the function ζ.

Proposition 4.5. *There exist smooth hypersurfaces Σ_δ for sufficiently small $\delta \neq 0$ such that their mean curvature is given by $m(\Sigma_\delta) = h(\delta)\zeta$ and*

$$\begin{aligned} h(\delta) &= c\delta^{4/3} + O(|\delta|^{4/3+\kappa}), \\ h'(\delta) &= \frac{4}{3}c\delta^{1/3} + O(|\delta|^{1/3+\kappa}). \end{aligned} \tag{4.1}$$

Here $c < 0$ and $\kappa > 0$.

To construct Σ_δ we first construct an approximate solution $\tilde{\Sigma}_\delta$ by gluing together the graph of $\delta\phi$ with scaled down copies $\pm\delta^{1/3}H$ of the Hardt–Simon smoothing of $C(S^3 \times C^3)$. The reason why this works is that to leading order $\pm\delta^{1/3}H$ is the graph of $\pm\delta r^{-2}$ over $C(S^3 \times S^3)$, which matches the leading order behavior of $\delta\phi$ at the singular points. We then construct Σ_δ as a graph over $\tilde{\Sigma}_\delta$.

Remark 4.6. Note that if Σ_0 were smooth, then it would follow from real analyticity of the mean curvature operator that we could solve the equation $m(\Sigma_\delta) = h(\delta)\zeta$, and the resulting $h(\delta)$ would be real analytic. In particular either $h(\delta)$ would have finite order of vanishing, or it would vanish identically. This kind of statement is at the heart of Simon's Łojasiewicz inequality.

In our singular setting we need to work with specific features of our problem to derive the estimate (4.1) with a nonzero coefficient c. In particular a key ingredient is a refined asymptotic expansion of H as a graph over $C(S^3 \times C^3)$. It turns out (see [23, Proposition 3.3]) that H is asymptotically the graph of

$$r^{-2} + br^{-3} + O(r^{-8})$$

over $C(S^3 \times S^3)$, where $b < 0$.

An expansion of this type is the main missing ingredient in proving the uniqueness result for the cylindrical cone $C(S^2 \times S^4) \times \mathbf{R}$. In principle it is possible that in the case of $C(S^2 \times S^4) \times \mathbf{R}$, or another cylindrical cone, the analogous function $h(\delta)$ vanishes to infinite order, but is not identically zero. This would be somewhat analogous to the situation exploited by White [26] to construct examples of harmonic maps with non-unique tangent maps.

Given the smoothings Σ_δ of the link, we construct minimal perturbations T_δ of the cone $V_0 = C(S^3 \times S^3) \times \mathbf{R}$, modeled on the Jacobi field $\delta\phi$. Let us denote by $V_\delta = C(\Sigma_\delta)$ the cone over Σ_δ. We try to construct a minimal graph T_δ over V_δ. Note that the mean curvature of V_δ satisfies $m(V_\delta) = h(\delta)\zeta\rho^{-2}$, where ζ is extended as a degree one homogeneous function, and $\rho = (r^2+y^2)^{1/2}$ as before.

To first order we need to take the graph of u over V_δ such that $L_{V_\delta}(u) = -h(\delta)\zeta\rho^{-2}$. This in turn is closely related to solving the equation $L_{V_0}(v) =$

$-h(\delta)\phi\rho^{-2}$ on the cone V_0. Using that ϕ is in the kernel of the linearized operator L_{Σ_0} on the link, we have

$$L_{V_0}(c_0\phi\ln\rho) = \phi\rho^{-2}$$

for a suitable constant c_0. It follows from these considerations that to leading order we need to consider the graph of $u_\delta = -c_0 h(\delta)\phi_\delta \ln\rho$ over V_δ, where ϕ_δ generates the family Σ_δ and is extended as a degree one function to V_δ. It is shown in [23, Proposition 4.5] that indeed there is a minimal hypersurface T_δ for small δ, which to leading order is the graph of u_δ over V_δ. Because of the $\ln\rho$ term, this T_δ is only defined on an annular region where $|\ln\rho| < |\delta|^{-\kappa}$ for some small $\kappa > 0$.

4.2.2. Proving Theorem 4.4. The proof of Theorem 4.4 follows from a decay estimate similar to Proposition 4.3. First, we define

$$E_B(M) = \inf\{D_W(M) + D_W(L^B M) : W \in \mathcal{W}\},$$

as before. Here $\mathcal{W}$ consists of all rotations of the cone V_0 as before, and in addition it contains rotations of cones W_δ modeled on the Jacobi fields $\delta\phi$ (see [23, Remark 4.6] for the definition). The cones W_δ are perturbations of the V_δ defined above, but to leading order their mean curvature is given by $m(W_\delta) = h(\delta)\phi_\delta\rho^{-2}$. The idea is that we would like to use ϕ instead of ζ in the contructions above, and ϕ_δ is the natural extension of the singular function ϕ to the smoothings Σ_δ. Geometrically the W_δ can be viewed as interchangeable with V_δ.

The definition of the distance D_W is much more subtle than before, since the cross sections of the hypersurfaces $W \in \mathcal{W}$ are not leaves of the Hardt–Simon foliation. We define the distance $D_{T_\delta}(M)$ analogously. The corresponding non-concentration estimate, relative to the surfaces T_δ, has the same form as Proposition 3.3, although the proof is substantially more complicated (see [23, Proposition 5.6]).

A final complication in trying to mimic the proof of Proposition 4.3 is that the minimal surfaces T_δ are only defined on annular regions of the form $|\ln\rho| < |\delta|^{-\kappa}$, rather than on the entire unit ball. This is an issue when we try to iterate the three annulus lemma, as in case (a) of the proof of Proposition 4.3 for instance, since the quantity $D_{L^{kB}T_{\delta_i}}(L^{kB}M_i)$ only makes sense if $L^{kB}T_{\delta_i}$ is still defined over the annulus $B_1 \backslash B_{\rho_0}$. This in turn is only true as long as $|\ln L^{kB}| < |\delta_i|^{-\kappa}$, i.e. $kB < |\delta_i|^{-\kappa}$ for a small $\kappa > 0$. Using this one can show that in case (a) we get a contradiction for large i as in the proof of Proposition 4.3, as long as $D_{L^B T_{\delta_i}}(L^B M_i) > \epsilon|h(\delta_i)|$, and ϵ can be chosen as small as we like if i is chosen larger.

On the other hand, if $D_{L^B T_{\delta_i}}(L^B M_i) \le \epsilon|h(\delta_i)|$, then we can show that

$$\mathcal{A}(L^B M_i)^\theta - \mathcal{A}(2L^B M_i)^\theta \ge C^{-1}|h(\delta_i)|$$

for large i, and a suitable $\theta > 0$. Here $\mathcal{A}$ is the excess

$$\mathcal{A}(M) = \mathrm{Area}(M \cap B_{1/2}) - \mathrm{Area}(V_0 \cap B_{1/2}).$$

This estimate can be viewed as a Łojasiewicz type inequality, and it relies on the fact that $h(\delta)$ has a finite order of vanishing in (4.1). Using this it follows that

$$E_B(L^B M_i) < CB(\mathcal{A}(L^B M_i)^\theta - \mathcal{A}(2L^B M_i)^\theta).$$

In conclusion we obtain the following decay estimate, analogous to Proposition 4.3.

Proposition 4.7. (See [23]) *There are $\theta, C, B > 0$ with the following property. Let M be an area minimizing hypersurface in B_1, with density equal to that of the cone $C \times \mathbf{R}$ at the origin. If the Hausdorff distance from M to $C \times \mathbf{R}$ on B_1 is sufficiently small, then one of the following holds for the quantity E_B defined above:*

(i) $E_B(L^B M) \leq \frac{1}{2} E_B(M)$.
(ii) $E_B(L^B M) \leq C\Big(\mathcal{A}(L^B M)^\theta - \mathcal{A}(2L^B M)^\theta\Big)$.

The uniqueness result, Theorem 4.4 follows from this decay estimate in a similar way as in the previous section.

5. Local construction of minimal hypersurfaces

In this section we discuss the construction of minimal hypersurfaces in a neighborhood of $0 \in \mathbf{R}^n \times \mathbf{R}$ with an isolated singularity at the origin, and tangent cone $C \times \mathbf{R}$. In the context of Question Q2 in the introduction, this result says that when the tangent cone is a cylindrical cone, then the singular set of the minimal surface can look very different from that of its tangent cone. The construction itself is analogous to the construction of singular Calabi–Yau metrics with isolated singularities in [24] (see also Hein–Naber [9]).

The starting point of the construction is to find a suitable Jacobi field on $C \times \mathbf{R}$, which will describe the leading order deviation of our surfaces from their tangent cone at the origin. Suppose that ℓ is an integer such that $\ell - \gamma > 1$. Here γ is as in Sect. 2. Then $C \times \mathbf{R}$ admits a homogeneous Jacobi field of degree $\ell - \gamma$ of the form

$$u_\ell = (y^\ell r^{-\gamma} + a_1 y^{\ell-2} r^{2-\gamma} + \dots a_{\lfloor \ell/2 \rfloor} y^{\ell - 2\lfloor \ell/2 \rfloor} r^{2\lfloor \ell/2 \rfloor - \gamma})\phi_1, \tag{5.1}$$

where, as before, ϕ_1 is the first eigenfunction of $-L_\Sigma$ on the link Σ of C, and $r^{-\gamma}\phi_1$ is the corresponding Jacobi field on C. The a_i are suitable constants uniquely determined by the condition that $L_{C\times\mathbf{R}} u_\ell = 0$.

We can consider the graph of u_ℓ over $C \times \mathbf{R}$ on a region where $|u_\ell| \ll r$, i.e. where $|y|^\ell \ll r^{\gamma+1}$. At the same time, on the region where r is much smaller, we can glue in suitable scaled copies of the Hardt–Simon smoothings $H_\pm$ in the slices of the form $\mathbf{R}^n \times \{y\}$. This uses that to leading order $H_\pm$ is the graph of $r^{-\gamma}\phi_1$ over C. Interpolating between the two regions using cutoff functions we obtain a hypersurface X whose mean curvature will be almost zero in a suitable weighted space. We then show that in a possibly smaller neighborhood of 0 we can find a minimal graph over X. The fact that many of these minimal hypersurfaces are area minimizing in a neighborhood of the origin follows by using barrier surfaces constructed in Proposition 2.4. For the details see [22].

Theorem 5.1. *There exist minimal hypersurfaces in a neighborhood of* $0 \in \mathbf{R}^n \times \mathbf{R}$, *modeled on the Jacobi field* u_ℓ, *that are smooth away from the origin, and have tangent cone* $C \times \mathbf{R}$ *as their unique (multiplicity one) tangent cone at the origin. If the integer* ℓ *in the construction is sufficiently large, then the minimal hypersurface that we construct is area minimizing in a neighborhood of the origin.*

The first step is to write down suitable approximate solutions. Let us define the number $a = \frac{\ell}{1+\gamma}$, and let $\beta \in (1, a)$. We define X in the ball $\{\rho \leq A^{-1}\}$ for a sufficiently large A, in the following three pieces:

- On the region where $r \geq 2|y|^\beta$ we let X be the graph of u_ℓ over $C \times \mathbf{R}$. It is convenient to deal separately with the region where $r \geq |y|$, where we have $|r^{-1}u_\ell| = O(r^{\ell-\gamma-1})$ as $r \to 0$. Since $\ell - \gamma > 1$, it makes sense to consider the graph of u_ℓ on a sufficiently small neighborhood of 0. At the same time, on the region $2|y|^\beta \leq r \leq |y|$ we have $|r^{-1}u_\ell| = O(|y|^\ell r^{-\gamma-1})$ as $r \to 0$. Since $\ell > \beta(\gamma+1)$, it makes sense to consider the graph of u_ℓ on this region too, once r is sufficiently small.
- On the region where $r \leq |y|^\beta$, we define X to be the surface $H(y^\ell)$ in the slice $\mathbf{R}^n \times \{y\}$. Note that by Definition 2.1 we have $H(y^\ell) = |y|^a H_\pm$.
- On the intermediate region $|y|^\beta \leq r \leq 2|y|^\beta$ we interpolate between the two definitions above, using a cutoff function. More precisely, let $\chi : \mathbf{R} \to [0,1]$ be a standard cutoff function, with $\chi(t) = 1$ for $t < 1$ and $\chi(t) = 0$ for $t > 2$. In addition recall that $H_\pm$ is the graph of $\pm r^{-\gamma}\phi_1 + v_\pm$ over C, outside of a large ball. On the region $|y|^\beta \leq r \leq 2|y|^\beta$ we let X be the graph of
$$u_\ell + \chi\left(\frac{|x|}{|y|^\beta}\right)\left[y^\ell r^{-\gamma}\phi_1 - u_\ell + |y|^a v_\pm(|y|^{-a}x)\right]$$
over $C \times \mathbf{R}$, where the choice of $v_\pm$ depends on the sign of y^ℓ. Note that this definition matches up with the definitions of X in the two regions above.

The following result shows that the mean curvature of X is small in a suitable weighted space.

Proposition 5.2. *Let* $\beta \in (1, a)$. *Suppose that* $\delta > \ell - \gamma$ *is sufficiently close to* $\ell - \gamma$, *and* $\tau \leq -\gamma$. *Then there exists a* $\kappa > 0$ *such that on the punctured ball*

$\{0 < \rho < A^{-1}\}$ for sufficiently large A, we have the estimates

$$|m_X| + r|\nabla m_X| < A^{-\kappa}\rho^{\delta-\tau}r^{\tau-2}.$$

Here m_X denotes the mean curvature of X, and ∇m_X is the derivative of m_X on X.

For the proof of this result, see [22]. The basic idea is that every point $(x, y) \in X$ has a neighborhood U of radius comparable to $R = |x|$, such that when we rescale by R^{-1}, the resulting surface $R^{-1}U$ can be viewed as a graph over either $H \times \mathbf{R}$ or $C \times \mathbf{R}$. We then estimate the mean curvatures of these graphs and scale back to obtain the required estimate.

Next we define suitable weighted spaces, in which we can analyze the linearized operator L_X of the mean curvature of graphs over X. We consider locally $C^{k,\alpha}$-functions on $X \cap \rho^{-1}(0, A_0^{-1})$ for a fixed large A_0, and define their weighted $C^{k,\alpha}_{\delta,\tau}$-norm by

$$\|f\|_{C^{k,\alpha}_{\delta,\tau}} = \sup_{R,S>0} R^{-\tau}S^{\tau-\delta}\|f\|_{C^{k,\alpha}_{R^{-2}g_X}(\Omega_{R,S})}.$$

Here $\Omega_{R,S} \subset X \cap \rho^{-1}(0, A_0^{-1})$ is the region where $\rho \in (S, 2S)$ and $r \in (R, 2R)$. The metric g_X denotes the induced metric on X, and the subscript $R^{-2}g_X$ indicates that we measure the usual Hölder norm using this rescaled metric. The metric $R^{-2}g_X$ has bounded geometry on $\Omega_{R,S}$, with bounds independent of R, S, which implies that the Jacobi operator of X defines a bounded linear map

$$L_X : C^{2,\alpha}_{\delta,\tau} \to C^{0,\alpha}_{\delta-2,\tau-2}.$$

At the same time the estimate in Proposition 5.2 implies that

$$\|m_X\|_{C^{2,\alpha}_{\delta,\tau}(\rho^{-1}(0,A^{-1}])} \leq CA^{-\kappa}$$

for $A > 2A_0$.

Given the approximate solution X, the main ingredient for constructing a minimal graph over X is the following result on inverting the Jacobi operator.

Proposition 5.3. *Let $\tau \in (3 - n + \gamma, -\gamma)$, and suppose that δ avoids a discrete set of indicial roots. Then for sufficiently large $A > 0$, the Jacobi operator*

$$L_X : C^{2,\alpha}_{\delta,\tau}(X \cap \rho^{-1}(0, A^{-1}]) \to C^{0,\alpha}_{\delta-2,\tau-2}(X \cap \rho^{-1}(0, A^{-1}])$$

is surjective, with a right inverse P bounded independently of A.

The proof of this result is based on first constructing an approximate right inverse $\tilde{P}$. Given a function u on $X \cap \rho^{-1}(0, A_0^{-1})$, we first use cutoff functions to decompose u into pieces supported on regions of X that are well approximated by either $C \times \mathbf{R}$ or $H \times \mathbf{R}$. We then analyze the linearized operator on these model pieces, and patch together local inverses using further cutoff functions. For the details, see [22], as well as [24] for the analogous result in the context of Calabi–Yau metrics.

Let us write $m_X(u)$ for the mean curvature of the graph of u over X, and define the non-linear part Q_X of the mean curvature operator by

$$m_X(u) = m_X + L_X(u) + Q_X(u).$$

Given the right inverse P constructed above, the problem of finding a minimal graph over X, i.e. solving $m_X(u) = 0$, can be written as

$$u = P(-m_X - Q_X(u)).$$

The solution u is then given by the contraction mapping theorem, once the parameter A is sufficiently large. For the details see [22]. We remark that the construction of the surfaces T_δ in the proof of Theorem 4.4 involves very similar

6. Highly symmetric hypersurfaces

In the previous section we showed that there are minimal hypersurfaces with isolated singularities, but with cylindrical tangent cones. In this section we show a converse of this, for highly symmetric minimal surfaces, addressing another aspect of Question Q2 from the Introduction. We restrict ourselves to minimizing cones $C = C(S^p \times S^q)$ for $p+q > 6$, and to codimension one stationary integral varifolds M in a neighborhood of the origin $0 \in \mathbf{R}^n \times \mathbf{R}$ that are invariant under the action of the group $O(p+1) \times O(q+1)$ on $\mathbf{R}^n = \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$.

In Sect. 5, given any integer ℓ such that $\ell - \gamma > 1$, we constructed a minimal hypersurface modeled on a Jacobi field u_ℓ, defined in (5.1). The construction can be performed in an $O(p+1) \times O(q+1)$-invariant way. Our goal is to show that in fact all such highly symmetric minimal surfaces with tangent cone $C \times \mathbf{R}$ at the origin are graphs over the hypersurfaces that we constructed previously. More precisely we have the following.

Theorem 6.1. *Let M be a stationary integral varifold in a neighborhood of the origin in $\mathbf{R}^n \times \mathbf{R}$, with tangent cone $C \times \mathbf{R}$ at the origin (with multiplicity one), where $C = C(S^p \times S^q)$, with $p+q > 6$. Suppose that M is invariant under the action of $G = O(p+1) \times O(q+1)$ on $\mathbf{R}^n$. Then either $M = C \times \mathbf{R}$ in a neighborhood of the origin, or M is a graph over one of the surfaces constructed in Sect. 5 near the origin and so it has an isolated singularity at the origin.*

Note that in this result it is essential that we use the Euclidean ambient metric on $\mathbf{R}^{n+1}$. Indeed, Simon's construction [12] can be done in an invariant way, and leads to minimal hypersurfaces with tangent cone $C \times \mathbf{R}$ at the origin, and essentially arbitrary singular set nearby, as long as we allow the ambient metric to be a perturbation of the Euclidean one.

The idea of the proof of Theorem 6.1 is to show that under the assumptions the minimal surface M is modeled to leading order on one of the Jacobi fields

u_ℓ on $C \times \mathbf{R}$ from before, unless $M = C \times \mathbf{R}$. At the same time we have already constructed minimal surfaces T that are modeled on the u_ℓ to leading order. We then show that M must approach one of these models T at a sufficiently fast rate as $\rho \to 0$ to ensure that M is actually graphical over T near the origin.

The first step in this approach is to show that indeed M is modeled on a non-zero Jacobi field over $C \times \mathbf{R}$, unless $M = C \times \mathbf{R}$. For this we need the following strong unique continuation result.

Theorem 6.2. *Suppose that M is an n-dimensional stationary integral varifold in a neighborhood of the origin $0 \in \mathbf{R}^n \times \mathbf{R}$, which admits $C \times \mathbf{R}$ as a (multiplicity one) tangent cone at the origin. Suppose that for all $k > 0$ there is a constant C_k such that for all $\rho < 1$ we have*

$$\int_{M \cap B_\rho(0)} d^2 < C_k \rho^k,$$

i.e. the L^2-distance from M to $C \times \mathbf{R}$ on the ball $B_\rho(0)$ vanishes to infinite order as $\rho \to 0$. Then $M = C \times \mathbf{R}$.

Sketch of proof. The proof relies on a similar idea as the monotonicity of frequency used by Almgren [2] and Garofalo–Lin [7], although the details are quite different, since we are not able to define a suitable frequency function in our setting. To explain the basic idea, let us denote by $d(M, \rho)$ some measure of the distance between M and $C \times \mathbf{R}$ on the ball $B_\rho(0)$. Suppose that $d(M, \rho)$ is defined in a scale invariant way, so that $d(M, \rho) = d(\rho^{-1}M, 1)$. In practice $d(M, \rho)$ is a scaled L^2-distance, "regularized" by adding a small multiple of an L^∞-type distance. It is possible that one could also use the L^2-distance itself by relying on the non-concentration result due to Simon [18, Corollary 2.3] in the arguments below.

Suppose that $\lambda > 0$. Let us say that the three-annulus property holds for the pair (M, λ), if $d(M, e^{-\lambda}) \geq \frac{1}{2} d(M, 1)$ implies $d(M, e^{-2\lambda}) \geq \frac{1}{2} d(M, e^{-\lambda})$. Note that if for some $\lambda > 0$ the three-annulus property holds for $(e^{k\lambda}M, \lambda)$ for all $k \geq 0$, and in addition $d(M, e^{-\lambda}) \geq \frac{1}{2} d(M, 1)$, then iterating the three-annulus property we find that $d(M, e^{-k\lambda}) \geq 2^{-k} d(M, 1)$. This should imply that M cannot approach its tangent cone at infinite order for any reasonable definition of the distance d.

From a quantitative version of the three-annulus lemma, Proposition 3.3 (see [22, Proposition 4.3]), together with a contradiction argument, one expects that perturbing λ slightly if necessary, the three-annulus property holds for (M, λ), whenever M is sufficiently close to $C \times \mathbf{R}$, say whenever $d(M, 1) < E(\lambda)$, for a function E converging to zero as $\lambda \to 0$. A precise version of this statement, [22, Proposition 4.4], says that for a suitable definition of d, we can choose $E(\lambda) = \lambda^Q$ for some $Q > 0$.

Given this, we can conclude as follows. Assuming that $\rho^{-1}M$ is sufficiently close to C for all $\rho < 1$, the three-annulus property will hold for the pairs $(\rho^{-1}M, \lambda_0)$, for some $\lambda_0 > 0$. If for a given $\rho_0 \in (0,1)$ we had $d(M, e^{-\lambda_0}\rho_0) \geq \frac{1}{2}d(M, \rho_0)$, then M would not approach its tangent cone at infinite order. Therefore for all $\rho < 1$ we must have

$$d(M, e^{-\lambda_0}\rho) < \frac{1}{2}d(M, \rho).$$

Then for all $\rho < e^{-\lambda_0}$ the three-annulus property holds for $(\rho^{-1}M, \lambda_1)$ for some $\lambda_1 < s\lambda_0$, with $s < 1$ depending only on the number Q above. Iterating this, it follows that if M approaches its tangent cone at infinite order, then we have $d(M, \rho_0) = 0$ for $\rho_0 = e^{-\lambda_0(1+s+s^2+\dots)}$, leading to Theorem 6.2. □

Let M be as in the statement of Theorem 6.1, and suppose that M is not equal to $C \times \mathbf{R}$ in a neighborhood of the origin. The strong unique continuation result implies that by a rescaling process we can extract a non-zero Jacobi field U on $C \times \mathbf{R}$, corresponding to the leading order behavior of M at the origin. We define the degree of M to be d, if

$$U = U_d + O(\rho^{c+\gamma}r^{-\gamma})$$

for some $c > d$ as $\rho \to 0$, and U_d is a non-zero degree d homogeneous Jacobi field.

Since we are considering M that are $O(p+1) \times O(q+1)$-invariant, the Jacobi field U is of the form

$$U = \sum_{k,\ell \geq 0} a_{k,\ell} r^{2k-\gamma} y^\ell,$$

i.e.

$$U = \lambda u_\ell + O(\rho^{c+\gamma}r^{-\gamma})$$

for some $\lambda \neq 0$ and $c > \ell - \gamma$, where u_ℓ is the function in (5.1). We have $U_d = \lambda u_\ell$, so that the degree of M is $d = \ell - \gamma$.

We denote by $T_{\pm 1}$ the minimal surfaces constructed in Theorem 5.1, modeled on the Jacobi fields $\pm u_\ell$, and define

$$T_\lambda = \lambda^{(1-(\ell-\gamma))^{-1}} T_1, \quad T_{-\lambda} = \lambda^{(1-(\ell-\gamma))^{-1}} T_{-1} \quad \text{for } \lambda > 0.$$

We let $T_0 = C \times \mathbf{R}$. For sufficiently small $|\lambda|$, the surface T_λ is defined in $B_2(0)$ and to leading order we can think of T_λ as the graph of λu_ℓ over $C \times \mathbf{R}$, at least away from the singular ray.

To prove Theorem 6.1 the strategy is to show that under the assumptions in the theorem M will decay towards T_λ for a suitable $\lambda \neq 0$ at a rate faster than the degree $\ell - \gamma$. In a sufficiently small neighborhood of the origin this will imply that M is actually a graph over T_λ, and in particular it has an isolated singularity at the origin. The proof has similarities with the proof of Theorem 4.1, the difference being that instead of proving decay towards the tangent cone, we now need to prove decay towards one of the surfaces T_λ, determining the next leading order behavior M beyond the tangent cone. Here the fact that u_ℓ spans the space of $O(p+1)\times O(q+1)$-invariant Jacobi fields on $C\times\mathbf{R}$ plays the role of integrability of the tangent cone. In order to use this approach to prove a result similar to Theorem 6.1 without symmetry assumptions, we would need to construct hypersurfaces like T modeled on more general Jacobi fields. This leads to significant new difficulties if T is expected to still have a non-isolated singular set.

7. Liouville type theorems

Finally, we consider Question Q3 from the introduction, i.e. the question of classifying entire minimal surfaces in $\mathbf{R}^{n+1}$ with a given tangent cone at infinity. The simplest result of this kind follows directly from the monotonicity formula: if the tangent cone C of M at infinity is a (multiplicity one) hyperplane, then M is a translate of C. This is a Liouville type rigidity result, which is closely related to the regularity of minimal surfaces that are sufficiently close to a hyperplane in a ball.

The first classification result beyond this is the following.

Theorem 7.1. (Simon–Solomon [20]) *Let M be a minimal hypersurface in $\mathbf{R}^{n+1}$ with tangent cone at infinity given by a quadratic cone $C=C(S^p\times S^q)$. Then up to translations and scalings M is either equal to one of the Hardt–Simon smoothings of C, or to C itself.*

The basic input in this classification result is that for such quadratic cones C we have a good understanding of the space $\mathcal{J}_{\leq 1}$ of Jacobi fields on C with degree at most 1. As shown in [20] this space is spanned by the following:

(i) $r^{-\gamma}$, for a certain $0<\gamma<\frac{n-2}{2}$ as in Sect. 2.

(ii) The functions $x\mapsto z\cdot\nu(x)$, where $z\in\mathbf{R}^{n+1}, x\in C$ and ν is the unit normal vector to C. These Jacobi fields are homogeneous of degree 0.

(iii) The functions $x\mapsto Ax\cdot\nu(x)$ for $A\in\mathfrak{so}(n+1)$. These are homogeneous of degree 1.

In addition, these Jacobi fields each have geometric meaning in terms of deformations of the cone C through minimal surfaces: (i) corresponds to the Hardt–Simon foliation; (ii) to translations; (iii) to rotations.

Given this, we can give a rough sketch of Simon–Solomon's result.

Sketch of proof of Theorem 7.1. Suppose that M is a minimal hypersurface with tangent cone $C=C(S^p\times S^q)$ at infinity. The fact that the degree one Jacobi fields all correspond to rotations means that C is integrable. It follows from Allard–Almgren [1] that M converges to C at a polynomial rate. In particular, near infinity M is modeled on the graph of a Jacobi field u over C

with degree less than 1. Replacing C by a translate $C+z$, we can write M as a graph over $C+z$ near infinity, modeled on a Jacobi field u over C that has no degree 0 component. Similarly, replacing C by a leaf cH of the Hardt–Simon foliation (possibly $c=0$), we can write M as a graph over $cH+z$ near infinity, modeled on a Jacobi field u on C with no degree 0 or degree $-\gamma$ components. It follows that M converges to $cH+z$ at a rate faster than $r^{-\gamma}$. Replacing M with $M-z$ we can assume that $z=0$.

The fact that M approaches cH at a rate faster than $r^{-\gamma}$ implies that for any $t>0$, M is contained between the surfaces $(c\pm t)H$ near infinity. We can now argue using the maximum principle together with the Hardt-Simon foliation to show that M must equal one of the foliates, which is then cH. □

The generalization of Simon–Salamon's result to general minimizing cones C with isolated singularities was taken up by Chan [5]. She showed that there are minimal hypersurfaces asymptotic to C corresponding to the space of Jacobi fields of degree less than one on C, however it is not known whether her construction exhausts all such minimal hypersurfaces. The missing ingredient is to understand whether the minimal hypersurfaces constructed by Chan vary continuously—this statement in the case of quadratic cones is clear since we have an explicit understanding of the corresponding hypersurfaces.

Let us now consider the case of cylindrical tangent cones $C\times\mathbf{R}$ at infinity. In analogy with the cases discussed above, one expects that minimal hypersurfaces asymptotic to $C\times\mathbf{R}$ can be understood in terms of Jacobi fields on $C\times\mathbf{R}$ of degree at most 1. A difficulty is that already in the simplest case where C is a quadratic cone, there will be many more Jacobi fields than before. For example for $C(S^3\times\S^3)\times\mathbf{R}$, the Jacobi fields of degree less than 1 include

$$r^{-2}, yr^{-2}, y^2r^{-2}-\frac{1}{3},$$

as well as the Jacobi fields corresponding to translations. As above, it is important to understand what minimal surfaces would correspond to these Jacobi fields and their linear combinations. The fastest decaying Jacobi field r^{-2} corresponds to the Hardt–Simon smoothings $cH\times\mathbf{R}$ as before. The next, yr^{-2} corresponds to the minimal graphs constructed by Bombieri–De Giorgi–Giusti [3]. We do not know, however, what minimal surfaces correspond to $y^2y^{-2}-\frac{1}{3}$ and combinations of this with faster decaying Jacobi fields.

It is natural to impose additional conditions on M to obtain results. One important question is to classify minimal surfaces M that lie on one side of a minizing cone. For cylindrical tangent cones we have the following result, proved under additional hypotheses by Simon [11].

Theorem 7.2. [6] *Let $C\subset\mathbf{R}^n$ be a strictly minimizing and strictly stable cone in the sense of Hardt–Simon [8]. Let $M\subset\mathbf{R}^{n+k}$ be a minimal surface with tangent cone $C\times\mathbf{R}^k$ at infinity, which lies on one side of $C\times\mathbf{R}^k$. Then M coincides with a foliate $cH\times\mathbf{R}^k$ of the Hardt–Simon foliation.*

The proof is roughly along the lines of the sketch above for Theorem 7.1, crucially using a non-concentration estimate like Proposition 3.3 to relate the behavior of minimal surfaces to Jacobi fields. The point of the additional hypothesis of lying on one side of $C \times \mathbf{R}^k$ is that in this case M must be modeled on a Jacobi field u on $C \times \mathbf{R}^k$ near infinity that has a sign. But there is only a one-dimensional space of such Jacobi fields, corresponding to the first eigenfunction of the Jacobi operator on the link of C, and we understand the corresponding minimal surfaces well: they are precisely the Hardt–Simon smoothings $cH \times \mathbf{R}^k$.

In the case of quadratic cones $C(S^p \times S^q) \times \mathbf{R}$ we also understand a slightly larger space of Jacobi fields: the ones with degree at most $-\gamma + 1$, where γ is as in (i) above. Namely these are the Jacobi fields spanned by $r^{-\gamma}$ and $yr^{-\gamma}$. Using this, we expect that techniques similar to the proof of Theorem 7.2 can be used to show the following.

Conjecture 7.3. *Let $M \subset \mathbf{R}^{n+1}$ be a minimal graph, whose tangent cone at infinity is $C(S^p \times S^q) \times \mathbf{R}$. Then up to translation and scaling, M is the minimal graph constructed by Bombier–De Giorgi–Giusti* [3] *in the case of $p = q = 3$ and by Simon* [16] *more generally.*

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