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Calculus of Variations

The symmetric (log-)epiperimetric inequality and a decay-growth estimate

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Abstract
We introduce a symmetric (log-)epiperimetric inequality, generalizing the standard epiperimetric inequality, and we show that it implies a growth-decay for the associated energy: as the radius increases energy decays while negative and grows while positive. One can view the symmetric epiperimetric inequality as giving a log-convexity of energy, analogous to the 3-annulus lemma or frequency formula. We establish the symmetric epiperimetric inequality for some free-boundary problems and almost-minimizing currents, and give some applications including a "propagation of graphicality" estimate, uniqueness of blow-downs at infinity, and a local Liouville-type theorem.

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1 Introdution

We consider the decay-growth properties of certain functions u defined on $\mathbb{R}^n$. Typically when u solves (or almost solves) a second-order elliptic PDE, u will admit a Fourier-type decomposition and/or a monotone frequency function and/or a 3-annulus-type lemma. Any of these imply a decay-growth estimate for a suitably scaled $u_r := r^{-m}u(r\cdot)$ (or, often, the scale-invariant radial derivative $r\partial_r u_r$): there is a threshold r_0 so that, as the radius r increases, the norm $||u_r||_{L^2(\partial B_1)}$ will exhibit polynomial or logarithmic decay in r while $r \leq r_0$, and polynomial or logarithmic growth in r while $r \geq r_0$.

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In this paper we introduce a *symmetric (log-)epiperimetric inequality* for an energy functional $\mathcal{E} : H^1(B_1) \to \mathbb{R}$, which encompasses both the usual epiperimetric inequality and a new "reverse" epiperimetric inequality. We show that when u and $\mathcal{E}$ satisfy the symmetric epiperimetric inequality and a fairly universal monotonicity-type formula, then both $\mathcal{E}(u_r)$ and $r\partial_r u_r$ exhibit a decay-growth estimate as above. In fact we will characterize the threshold radius r_0 as the radius at which the associated energy $\mathcal{E}(u_r)$ changes sign.

As an application, we prove the symmetric epiperimetric inequality for some free-boundary-type variational problems and for almost-minimizing currents. The main consequence is a "quantitative uniqueness of tangent cone" which allows one to propagate graphicality or closeness near a smooth cone in regions where the energy stays close (see Theorems 3.1, 4.3, 5.2, and Example 1.2). This implies uniqueness of blow-ups and blow-downs, but is particularly useful in blow-up arguments to control transition regions between scales. For example, using our estimates we can prove some "local Liouville" theorems for minimizers of the Alt–Caffarelli functional and almost-minimizing currents (Theorems 3.5, 5.4).

Epiperimetric-type inequalities have already found extensive use in free-boundary type problems [4, 12, 25, 26, 28] and minimal surface theory [11, 20, 27, 30] to prove uniqueness of blow up and decay estimates near singularities. A key advantage of the symmetric epiperimetric inequality is that it can be applied in an annulus *without* a priori knowledge of the singular behavior (compare [9, Appendix 1]).

We observe here that many solutions $u \in H^1(B_R \subset \mathbb{R}^n)$ to variational problems involving an energy $\mathcal{E} : H^1(B_1) \to [-1, 1]$ satisfy the following two properties:

(1) *Monotonicity*: writing $u_r(x) = r^{-m}u(rx)$ and z_r for the m-homogenous extension of u_r, then for a.e. $r \in (\rho, R)$:

$$\frac{d}{dr}\mathcal{E}(u_r) \geq \frac{c_E}{r}(\mathcal{E}(z_r) - \mathcal{E}(u_r)) + \frac{c_E}{r}\int_{\partial B_1} r^2|\partial_r u_r|^2 d\mathcal{H}^{n-1} \tag{1.1}$$

for some fixed $c_E > 0, m \in (2 - n, \infty), \rho \in [0, R]$.

(2) *Symmetric log-epiperimetric inequality*: for a.e. $r \in (\rho, R)$:

$$\mathcal{E}(u_r) \leq \mathcal{E}(z_r) - \epsilon|\mathcal{E}(z_r)|^{1+\gamma}, \tag{1.2}$$

for some $\gamma \in [0, 1), \epsilon \in (0, 1]$.

When $\mathcal{E}(z_r) \geq 0$, then (1.2) reduces to the usual (log-)epiperimetric inequality. However, unlike standard (log-)epiperimetric, because of the absolute values, (1.2) can control $\mathcal{E}(u_r)$ even when $\mathcal{E}(z_r) \leq 0$. We call (1.2) "symmetric" because it does not care about the sign of $\mathcal{E}(z_r)$, or the direction of r (increasing or decreasing). For comparison, we might call (1.2) with $(-\mathcal{E}(z_r))$ in place of $|\mathcal{E}(z_r)|$ a "reverse" epiperimetric-type inequality.

We show in Theorem 2.2 that the above two properties guarantee a decay-growth estimate, which stated informally says the following.

Theorem 1.1 *In the notation of the above, as r increases from ρ to R, then $\mathcal{E}(u_r)$ is increasing, and both $\mathcal{E}(u_r)$ and $||r\partial_r u_r||_{L^2(\partial B_1)}$ polynomially/logarithmically decay while $\mathcal{E}(u_r) \leq 0$, and polynomially/logarithmically grow while $\mathcal{E}(u_r) \geq 0$. The nature of growth (polynomial or logarithmic) depends on whether $\gamma = 0$ or $\gamma > 0$ (resp.).*

If $\rho = 0$ (resp. $R = \infty$), then u_r has an $L^2(\partial B_1)$ limit as $r \to 0$ (resp. $r \to \infty$). If $\rho = 0$ and *$R = \infty$, then $\partial_r u_r \equiv 0$.*

Let us illustrate with an example. In many free-boundary-type problems the associated energy of interest is a modification of the following Weiss-type energy

$$W_0^m(u) = \int_{B_1} |Du|^2 - m \int_{\partial B_1} u^2, \quad u \in H^1(B_1). \tag{1.3}$$

It is a now-standard computation due to Weiss [29] that if $u_r(x) = r^{-m}u(rx)$, then $r \mapsto W_0^m(u_r)$ is absolutely continuous in $r \in (0, 1]$, and

$$\frac{d}{dr} W_0^m(u_r) = \frac{n + 2m - 2}{r}(W_0(z_r) - W_0(u_r)) + \frac{1}{r}\int_{\partial B_1} r^2 |\partial_r u_r|^2, \tag{1.4}$$

where $z_r(x) = |x|^m u_r(x/|x|)$ is the m-homogenous extension of u_r.

Example 1.2 We consider here the Alt–Caffarelli funtional J defined on an open set U by

$$J(u, U) = \int_U |Du|^2 dx + |\{u > 0\}|, \quad u \in H^1(U). \tag{1.5}$$

We say that a nonnegative function $u \in H^1(U)$ minimizes J if $J(u, U) \leq J(v, U)$ for all $u - v \in H_0^1(U)$, and $u \in H^1_{loc}(U)$ locally-minimizes J if $J(u, W) \leq J(v, W)$ for all $u - v \in H_0^1(W)$ for some $W \subset\subset U$. If $U = B_1$ we simply write $J(u)$.

The functional J was introduced by [1], who proved existence, compactness, and regularity properties of minimizers and their free-boundaries $\partial\{u > 0\} \cap U$. Minimizers u are known to be locally-Lipschitz, and (from additional work of [18, 29]) the free-boundary $\partial\{u > 0\} \cap U$ is known to be analytic away from a singular set of dimension at most $n - d^*$, for some critical dimension (as yet unknown) $d^* \in \{5, 6, 7\}$.

The natural scaling for J is 1-homogeneous, so let us write in this Example $u_r(x) = r^{-1}u(rx)$ and $\xi_r(x) = r^{-1}\xi(rx)$. It follows from the Weiss monotonicity (1.6) that any blow-up (or blow-down) u_0 is 1-homogenous. [1] proved a regularity theory that shows $\partial\{u > 0\}$ is regular at points where the blow-up is $u_0(x) = (x \cdot e)_+$ for some unit vector e. [12] used a log-epiperimetric inequality to prove uniqueness of blow-ups u_0 with isolated singularities (such as in dimension d^*).

Let us fix in this Example a 1-homogenous minimizer u_0 of J which is regular away from 0. We shall demonstrate the nature and use of the symmetric log-eperimetric inequality for minimizers near u_0, which is covered in more detail in Sect. 3. To study the behavior of u near a 1-homogenous u_0, the natural energy to consider is a normalized Weiss energy

$$\mathcal{E}(u) = W(u) - W(u_0), \quad W(u) = W_0^1(u) + |\{u > 0\} \cap B_1|,$$

where W_0^1 as in (1.3). The Weiss monotonicity (1.4) and the coarea formula imply that for any $u \in H^1(B_1)$, $r \mapsto \mathcal{E}(u_r)$ is absolutely continuous on $r \in (0, 1)$ and satisfies

$$\frac{d}{dr}\mathcal{E}(u_r) = \frac{n}{r}(\mathcal{E}(z_r) - \mathcal{E}(u_r)) + \frac{1}{r}\int_{\partial B_1} r^2 |\partial_r u_r|^2 \quad \text{for } a.e. r \in (0, 1), \tag{1.6}$$

where $z_r(x) = |x|u_r(x/|x|)$ is the 1-homogenous extension of u_r. Note in particular if $u \in H^1(B_1)$ minimizes J, then $r \mapsto \mathcal{E}(u_r)$ is increasing on $[0, 1]$.

Write $u_0(x = r\theta) = r\sigma(\theta)$, $\Omega = \{\sigma > 0\} \subset B_1$, and $C\Omega = \{u_0 > 0\}$, and $\nu(x)$ for the outer normal of $\Omega \subset B_1$. Given $\xi : \partial\Omega \to \mathbb{R}$, with $||\xi||_{C^0}$ sufficiently small depending only on Ω, we can define a new domain Ω_ξ as a perturbation of Ω, with boundary given by

$$\partial\Omega_\xi = \{\cos(\xi(\theta))\theta + \sin(\xi(\theta))\nu(\theta) : \theta \in \partial\Omega\}.$$

If $\xi : \Omega \cap A_{R,\rho} \to \mathbb{R}$, then we define conical graph

$$G_{C\Omega}(\xi) \cap A_{R,\rho} = \cup\{r\Omega_{\xi_r} : r \in (\rho, R)\}.$$

Here $A_{R,\rho}(x) = B_R(x) \setminus \overline{B_\rho(x)}$. Write λ_i^Ω for the Dirichlet eigenvalues of Ω, and ϕ_i^Ω for the corresponding $L^2(\Omega)$-ON eigenfunctions. We prove in Sect. 3 the following. Recall that $\mathcal{E}(u) = W(u) - W(u_0) \equiv W(u) - W(r\sigma)$.

Theorem 1.3 (Symmetric log-epi for Alt–Caffarelli near smooth cones) *There are constants $\delta(\sigma), \epsilon(\sigma) > 0$, $\gamma(\sigma) \in [0, 1)$, so that the following holds.*

Let $\xi \in C^{2,\alpha}(\partial\Omega)$, $z \in H^1(\partial B_1)$, be such that $\{z > 0\} = \Omega_\xi$ and

$$||z - \sigma||_{H^1(\partial B_1)} < \delta, \quad ||\xi||_{2,\alpha} < \delta. \tag{1.7}$$

Then there is an $h \in H^1(B_1)$ with $h|_{\partial B_1} = z$, so that

$$\mathcal{E}(h) - \mathcal{E}(rz) \leq -\epsilon|\mathcal{E}(rz)|^{1+\gamma}. \tag{1.8}$$

Theorem 1.3 implies that if $u \in H^1(B_1)$ minimizes J and satisfies

$$\sup_{r\in(\rho,1)} ||u_r - u_0||_{H^1(\partial B_1)} \leq \delta, \quad \text{and} \tag{1.9}$$

$$\partial\{u > 0\} \cap A_{1,\rho} = G_{C\Omega}(\xi) \cap A_{1,\rho}, \quad \sup_{r\in(\rho,1)} ||\xi_r||_{C^{2,\alpha}(\Omega)} \leq \delta, \tag{1.10}$$

for some $\rho \in [0, 1/2]$, then

$$\mathcal{E}(u_r) \leq \mathcal{E}(z_r) - \epsilon|\mathcal{E}(z_r)|^{1+\gamma} \quad \forall r \in (\rho, 1). \tag{1.11}$$

So for a.e. $r \in (\rho, 1)$, $\mathcal{E}(u_r)$ satisfies both a monotonicity formula (1.6) like (1.1) and a symmetric log-epiperimetric inequality (1.11) like (1.2). Our growth-decay Theorem 2.2 immediately implies:

Theorem 1.4 (Growth–Decay estimates near smooth cones) *There are $c(\sigma), \beta(\sigma) > 0$ so that if $u \in H^1(B_1)$ minimizes J and satisfies (1.9), (1.10) for some $\rho \in [0, 1/2]$ (and δ, γ as in Theorem 1.3), and $\rho' \in [\rho, 1]$ is chosen so that $\mathcal{E}(u_r) \leq 0$ for $r \in [\rho, \rho']$ and $\mathcal{E}(u_r) \geq 0$ for $r \in [\rho', 1]$, then we have the decay-growth estimates*

$$\int_{\max\{s,\rho'\}}^{\max\{r,\rho'\}} ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq \begin{cases} c([\mathcal{E}(1)_+]^{\frac{\gamma-1}{2}} + \delta\gamma \log(1/r)^{\frac{1-\gamma}{2\gamma}})^{-1} & \gamma > 0 \\ c\,[\mathcal{E}(1)_+]^{\frac{1}{2}} r^{\frac{\beta}{2}} & \gamma = 0 \end{cases}, \tag{1.12}$$

$$\int_{\min\{s,\rho'\}}^{\min\{r,\rho'\}} ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq \begin{cases} c([\mathcal{E}(\rho)_-]^{\frac{\gamma-1}{2}} + \delta\gamma \log(s/\rho)^{\frac{1-\gamma}{2\gamma}})^{-1} & \gamma > 0 \\ c\,[\mathcal{E}(\rho)_-]^{\frac{1}{2}} (\rho/s)^{\frac{\beta}{2}} & \gamma = 0 \end{cases}, \tag{1.13}$$

for every $\rho \leq s < r \leq 1$, and hence we have the Dini estimate

$$\int_\rho^1 ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq c([\mathcal{E}(\rho)_-]^{\frac{1-\gamma}{2}} + [\mathcal{E}(1)_+]^{\frac{1-\gamma}{2}}). \tag{1.14}$$

A key point is that no sign of $\mathcal{E}$ is assumed anywhere, and so no a priori assumptions on the singular set in B_ρ are required, but on the other hand one may have both decay and growth in r depending on the sign of $\mathcal{E}$. (1.12), (1.13) recover the uniqueness of blow-ups with isolated singularities as proven by [12], and also establish uniqueness of the same kinds of blow-downs (Corollary 3.3). Additionally, when combined with the ϵ-regularity of [1], (1.14) implies u stays close to u_0 while the energy $|\mathcal{E}|$ stays small. Precisely, in Theorem 3.1,

we show that if u satisfies (1.9), (1.10) with $\rho = 1/2$ and $\delta(\epsilon, \sigma)$ sufficiently small, and we have

$$|\mathcal{E}(u_r)| \leq \delta(\epsilon, \sigma) \quad \forall r \in (\rho''/2, 1)$$

for some $\rho'' \in [0, 1/2]$, then we can *deduce* (1.9), (1.10) holds with ρ'' in place of ρ and ϵ in place of δ. In Theorem 3.5 we show how this "propogation of closeness" can be used in blow-up arguments to prove local structure from global rigidity properties.

In Sect. 2 we prove an abstract decay-growth estimate, which shows how the monotonicity formula (1.1) and the symmetric epiperimetric inequality (1.2) imply a differential inequality for $\mathcal{E}$. In Sect. 4 we explain how to prove and apply the symmetric epiperimetric inequality to the obstacle problem, and in Sect. 5 we adapt the symmetric epiperimetric inequality to almost-minimizing currents. The setups and applications are similar to the Example above, but we want to highlight that we can additionally handle almost-minimizing currents defined only in an annulus (Sect. 5.7), similar to the situation considered in [19].

2 An abstract decay-growth theorem

In this section we prove that the symmetric (log-)epiperimetric inequality, combined with a monotonicity formula, implies a decay/growth estimate. We allow errors which polynomially grow or decay.

We shall work in $\mathbb{R}^n$. Fix $m \in (2 - n, \infty)$. Take $r_1 < r_3$, and $u \in H^1(B_{r_3})$. For $r \leq r_3$ define the m-homogenously rescaled $u_r(x) = r^{-m}u(rx) \in H^1(B_1)$, and for a.e. $r \in (0, r_3)$ define $z_r(x) = |x|^m u_r(x/|x|) \in H^1(B_1)$ to be the m-homogenous extension of u_r.

We shall assume, throughout this section, that the following assumptions hold.

Assumptions 2.1 There are constants $\Lambda_\pm \geq 0$, $\alpha, c_E, \epsilon \in (0, 1]$, $\gamma \in [0, 1)$, and an energy functional $\mathcal{E} : H^1(B_1) \to [-1, 1]$, so that the following hold:

(1) Almost-monotonicity: $r \mapsto \mathcal{E}(u_r)$ is a BV_{loc} function on (r_2, r_3), and satisfies

$$\frac{d}{dr}\mathcal{E}(u_r) \geq \frac{c_E}{r}(\mathcal{E}(z_r) - \mathcal{E}(u_r)) + \frac{c_E}{r}\int_{\partial B_1} r^2 |\partial_r u_r|^2 d\mathcal{H}^{n-1} - \Lambda_+ r^{\alpha-1} - \Lambda_- r^{-\alpha-1} \tag{2.1}$$

for a.e. $r \in (r_1, r_3)$.

(2) Symmetric (log-)epipimetric + almost-minimizing: for a.e. $r \in (r_2, r_3)$ we have

$$\mathcal{E}(u_r) \leq \mathcal{E}(z_r) - \epsilon|\mathcal{E}(z_r)|^{1+\gamma} + \Lambda_+ r^\alpha + \Lambda_- r^{-\alpha} \tag{2.2}$$

Our main Theorem is the following decay-growth estimate for $r\partial_r u_r$, which in turn (see Remark 2.3) controls the differences $||u_r - u_s||_{L^2(\partial B_1)}$.

Theorem 2.2 (Decay-growth of $r\partial_r u_r$) *Under the assumptions (2.1), (2.2), and provided* $\max\{\Lambda_+ r_3^\alpha, \Lambda_- r_1^{-\alpha}\} \leq 1$, *if we define*

$$G(r) = \mathcal{E}(u_r) + 3\alpha^{-1}\Lambda_+ r^\alpha - 3\alpha^{-1}\Lambda_- r^{-\alpha} \tag{2.3}$$

then the following holds.

(1) *G is increasing, and for a.e. $r \in (r_1, r_3)$ satisfies*

$$G' \geq (\delta/r)|G|^{1+\gamma}, \tag{2.4}$$

for $\delta = \delta(c_E, \epsilon, \gamma, \alpha)$.

(2) *We have the Dini estimate*

$$\int_{r_1}^{r_3} ||r\partial_r u_r||_{L^2(\partial B_1)} \frac{dr}{r} \leq c([G(r_1)_-]^{\frac{1-\gamma}{2}} + [G(r_3)_+]^{\frac{1-\gamma}{2}}), \tag{2.5}$$

where $v_+ = \max\{v, 0\}$, $v_- = -\min\{v, 0\}$, *and* $c = c(c_E, \epsilon, \gamma, \alpha)$.

(3) *If* $r_2 \in [r_1, r_3]$ *is chosen so that* $G \leq 0$ *on* $[r_1, r_2]$ *and* $G \geq 0$ *on* $[r_2, r_3]$, *then for any* $r_2 \leq s \leq r \leq r_3$ *we have*

$$\int_s^r ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq \begin{cases} c(G(r_3)^{-\gamma} + \delta\gamma \log(r_3/r))^{\frac{\gamma-1}{2\gamma}} & \gamma > 0 \\ cG(r_3)^{\frac{1}{2}}(r/r_3)^{\frac{\delta}{2}} & \gamma = 0 \end{cases}, \tag{2.6}$$

while for any $r_1 \leq s \leq r \leq r_2$ *we have*

$$\int_s^r ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq \begin{cases} c((-G(r_1))^{-\gamma} + \delta\gamma \log(s/r_1))^{\frac{\gamma-1}{2\gamma}} & \gamma > 0 \\ c(-G(r_1))^{\frac{1}{2}}(r_1/s)^{\frac{\delta}{2}} & \gamma = 0 \end{cases}, \tag{2.7}$$

where $c = c(c_E, \epsilon, \gamma, \alpha)$.

Remark 2.3 It is useful to note the elementary inequality:

$$||u_r - u_s||_{L^2(\partial B_1)} = \left(\int_{\partial B_1} \left| \int_s^r \partial_r u_r dr \right|^2 d\theta \right)^{1/2} \leq \int_s^r ||\partial_r u_r||_{L^2(\partial B_1)} dr. \tag{2.8}$$

Remark 2.4 Monotonicity (2.1) by itself only controls the *square* of the L^2 norm:

$$\int_{r_1}^{r_3} ||r\partial_r u_r||^2_{L^2(\partial B_1)} \frac{dr}{r} \leq c_E^{-1}(G(r_3) - G(r_1)),$$

which is insufficient to control the behavior of u across many scales.

Remark 2.5 If one allows the energy $\mathcal{E}(u_r)$ to take values outside $[-1, 1]$, then in place of (2.4) one has the ODE

$$G' \geq (\delta/r) \min\{|G|^{1+\gamma}, |G|\} \quad a.e. r \in (r_1, r_3). \tag{2.9}$$

Combining Theorem 2.2 and Remark 2.3, we obtain directly the following.

Corollary 2.6 (Uniqueness at 0 and ∞) *Under the same hypotheses as Theorem 2.2, if* $r_1 = 0$ *(and so, necessarily* $\Lambda_- = 0$*), then there is a limit* $u_0 \in L^2(\partial B_1)$ *so that*

$$||u_r - u_0||_{L^2(\partial B_1)} \leq \begin{cases} c \log(r_3/r)^{\frac{\gamma-1}{2\gamma}} & \gamma > 0 \\ c(r/r_3)^{\frac{\delta}{2}} & \gamma = 0 \end{cases} \quad \forall r < r_3. \tag{2.10}$$

where $c, \delta > 0$ *both depend on* $(c_E, \epsilon, \gamma, \alpha)$.

On the other hand if $r_3 = \infty$ *(and so* $\Lambda_+ = 0$*), then there is a limit* $u_\infty \in L^2(\partial B_1)$ *so that*

$$||u_r - u_\infty||_{L^2(\partial B_1)} \leq \begin{cases} c \log(r/r_1)^{\frac{\gamma-1}{2\gamma}} & \gamma > 0 \\ c(r_1/r)^{\frac{\delta}{2}} & \gamma = 0 \end{cases} \quad \forall r > r_1. \tag{2.11}$$

If both $r_1 = 0$ *and* $r_3 = \infty$, *then* $\partial_r u_r \equiv 0$.

Proof of Corollary 2.6 To prove the first statement, by Theorem 2.2 it suffices to show that $G(r) \geq 0$. Otherwise, if we had $G(r_2) < 0$, then for all $0 < r < r_2$ we would have either

$$-G(r_2) \leq ((-G(r))^{-\gamma} + \delta\gamma \log(r_2/r))^{-1/\gamma} \leq c(\delta, \gamma) \log(r_2/r)^{-1/\gamma}$$

or (recalling that $|\mathcal{E}(u_r)| \leq 1$)

$$-G(r_2) \leq (r/r_2)^{\delta}(-G(r)) \leq (1 + \Gamma r_2^{\alpha})(r/r_2)^{\delta},$$

which is a contradiction for sufficiently small r. The second statement follows by a similar argument. The third statement follows from the first two, and (2.1). □

Proof of Theorem 2.2 For shorthand we write $E(r) = \mathcal{E}(u_r)$, $F(r) = \mathcal{E}(z_r)$, $\lambda(r) = \Lambda_+ r^{\alpha} + \Lambda_- r^{-\alpha}$, and we remark that (2.2) implies

$$E - \lambda \leq F, \quad E' \geq (c_E/r)(F - E) - \lambda/r. \tag{2.12}$$

We first claim there is a $\delta'(c_E, \epsilon) > 0$ so that

$$E' \geq (\delta'/r)|E - \lambda|^{1+\gamma} - (2/r)\lambda \tag{2.13}$$

for a.e. $r \in (r_1, r_3)$. To prove this, we break into four cases.

(1) $F \geq 0$ and $E - \lambda \geq 0$: Then

$$\begin{aligned} E' &\geq (c_E/r)(F - E) - \lambda/r \geq (c_E/r)(\epsilon F^{1+\gamma} - \lambda) - \lambda/r \\ &\geq (c_E\epsilon/r)|E - \lambda|^{1+\gamma} - (2/r)\lambda. \end{aligned}$$

(2) $F \geq 0$ and $E - \Lambda r^{\alpha} \leq 0$: Then

$$\begin{aligned} E' &\geq (c_E/r)(-E) - \lambda \geq (c_E/r)|E - \lambda| - (2/r)\lambda \\ &\geq (c_E/3r)|E - \lambda|^{1+\gamma} - (2/r)\lambda. \end{aligned}$$

(3) $F \leq 0$ and $|F| \leq |E - \lambda|/2$: Since $\lambda \leq 0$, we have:

$$\begin{aligned} E' &\geq (c_E/r)(F - (E - \lambda)) - (2/r)\lambda \geq (c_E/2r)|E - \lambda| - (2/r)\lambda \\ &\geq (c_E/6r)|E - \lambda|^{1+\gamma} - (2/r)\lambda. \end{aligned}$$

(4) $F \leq 0$ and $|F| \geq |E - \lambda/2$: Then

$$E' \geq (c_E\epsilon/r)|F|^{1+\gamma} - (2/r)\lambda \geq (c_E\epsilon/4r)|E - \lambda|^{1+\gamma} - (2/r)\lambda.$$

This proves our claim.

Fix $\delta'' = \min\{\delta', 2^{-\gamma}(1 + 3/\alpha)^{-1-\gamma}\}$. Recalling that $\max\{\Lambda_+ r_3^{\alpha}, \Lambda_- r_1^{-\alpha}\} \leq 1$, $\gamma \in [0, 1)$, and $\Gamma_{\pm} = 3\Lambda_{\pm}/\alpha$, we compute:

$$\begin{aligned} G' &\geq (\delta''/r)|G - (1 + 3/\alpha)\Lambda_+ r^{\alpha} - (1 - 3/\alpha)\Lambda_- r^{-\alpha}|^{1+\gamma} + 2\Lambda_+ r^{\alpha-1} + 2\Lambda_- r^{-\alpha-1} \\ &\geq (\delta'' 2^{-\gamma}/r)|G|^{1+\gamma} - (\Lambda_+ r^{\alpha})^{1+\gamma}/r - (\Lambda_- r^{\alpha})^{1+\gamma}/r + 2\Lambda_+ r^{\alpha-1} + 2\Lambda_- r^{-\alpha-1} \\ &\geq (\delta'' 2^{-\gamma}/r)|G|^{1+\gamma}. \end{aligned}$$

In the second line we used the inequality $|a - b|^{1+\gamma} \geq 2^{-\gamma}|a|^{1+\gamma} - |b|^{1+\gamma}$. This proves (2.4), with $\delta = \delta''/2$.

We work towards proving (2.5). For shorthand write $||v|| = ||v||_{L^2(\partial B_1)}$. We first compute, noting that $G' = E' + (3/r)\lambda$,

$$\begin{aligned}\left(\int_s^r ||\partial_r u_r|| dr\right)^2 &\leq \log(r/s)\int_s^r \int_{\partial B_1} r|\partial_r u_r|^2 d\theta dr \\ &\leq c_E^{-1}\log(r/s)\int_s^r (E' + (2/r)\lambda) dr \\ &\leq c_E^{-1}\log(r/s)(G(r) - G(s)) \end{aligned} \tag{2.14}$$

Assume for the moment that $\gamma > 0$. For a.e. $r \in (r_2, r_3)$ we have $G' \geq (\delta/r)G^{1+\gamma}$. Therefore, for every $r_2 \leq s \leq r < r_3$ we have

$$(-1/\gamma)(G(r)^{-\gamma} - G(s)^{-\gamma}) \geq \delta \log(r/s),$$

and hence

$$G(s) \leq (G(r)^{-\gamma} + \delta\gamma \log(r/s))^{-1/\gamma}. \tag{2.15}$$

Combining (2.14), (2.15), we get

$$\int_s^r ||\partial_r u_r|| dr \leq c_E^{-1/2}\log(r/s)^{1/2}(G(r_3)^{-\gamma} + \delta\gamma\log(r_3/r))^{-1/2\gamma}$$

Let $s_i = e^{-e^i} r_3$. Choose $i \leq i'$ so that $s_{i'} \geq r \geq s_{i'+1}$ and $s_i \geq s \geq s_{i+1}$. Then we estimate

$$\begin{aligned}\int_s^r ||\partial_r u_r|| dr &\leq \sum_{j=i'}^{i} \int_{\max\{s_{j+1}, r_2\}}^{s_j} ||\partial_r u_r|| dr \\ &\leq c\sum_{j=i'}^{i} \log(s_j/s_{j+1})^{1/2}(G(r_3)^{-\gamma} + \delta\gamma\log(r_3/s_j))^{-1/2\gamma} \\ &\leq c\sum_{j=i'}^{i} e^{j/2}(G(r_3)^{-\gamma} + \delta\gamma e^j)^{-1/2\gamma} \\ &\leq c(G(r_3)^{-\gamma} + \delta\gamma e^{i'})^{(\gamma-1)/2\gamma} \\ &\leq c(c_E, \delta, \gamma)(G(r_3)^{-\gamma} + \delta\gamma\log(r_3/r))^{(\gamma-1)/2\gamma}. \end{aligned} \tag{2.16}$$

In the penultimate line we used the inequality

$$\sum_{j=i'}^{\infty} e^{j/2}(a + e^j)^{-\beta/2} \leq c(\beta)(a + e^{i'})^{(1-\beta)/2}$$

for any $\beta > 1$, $a \geq 0$, $i' \geq 0$.

Similarly, if instead $r_1 < s \leq r \leq r_2$, then we have

$$-G(r) \leq ((-G(s))^{-\gamma} + \delta\gamma\log(r/s))^{-1/\gamma}$$

and hence

$$\begin{aligned}\int_s^r ||\partial_r u_r|| dr &\leq c_E^{-1}\log(r/s)^{1/2}(-G(s))^{1/2} \\ &\leq c_E^{-1}\log(r/s)^{1/2}((-G(r_1))^{-\gamma} + \delta\gamma\log(s/r_1))^{-1/2\gamma}. \end{aligned}$$

Therefore, taking now $s_i = e^{e^i} r_1$ we can argue as above to get

$$\int_s^r ||\partial_r u_r|| dr \leq c(c_E, \gamma, \delta)((-G(r_1))^{-\gamma} + \delta\gamma \log(s/r_1))^{(\gamma-1)/2\gamma}. \tag{2.17}$$

Together, combining (2.16), (2.17) gives the Dini estimate (2.5), (2.6) in the case when $\gamma > 0$.

Assume now $\gamma = 0$. Then (2.4) becomes $G' \geq (\delta/r)|G|$. If $r_2 \leq s < r < r_3$, then we have

$$G(s) \leq (s/r)^\delta G(r),$$

and hence, arguing as before but with $s_i = e^{-i} r_3$, we have

$$\int_s^r ||\partial_r u_r|| dr \leq c(c_E, \delta) G(r_3)^{1/2} (r/r_3)^{\delta/2}.$$

If instead $r_1 < s < r \leq r_2$, then we have

$$-G(r) \leq (s/r)^\delta (-G(s)),$$

and (arguing as before but with $s_i = e^i r_1$)

$$\int_s^r ||\partial_r u_r|| dr \leq c(c_E, \delta)(-G(r_1))^{\delta/2} (r_1/s)^{\delta/2}.$$

This proves (2.5), (2.6) when $\gamma = 0$. □

3 Minimizers of the Alt–Caffarelli functional

We consider here the Alt–Caffarelli functional J as defined in Example 1.2. We shall use the same notation as Example 1.2, and in particular

(1HM) for the duration of this Section *we fix a* 1*-homogenous minimizer* $u_0(x = r\theta) = r\sigma(\theta)$ *which is regular away from* 0.

Recall that as outlined in Example 1.2 the key point is that if $u \in H^1(B_1)$ is a minimizer for J and satisfies (1.9), (1.10), then for all $r \in (\rho, 1)$ we have the monotonicity

$$\frac{d}{dr}\mathcal{E}(u_r) = \frac{n}{r}(\mathcal{E}(z_r) - \mathcal{E}(u_r)) + \frac{1}{r}\int_{\partial B_1} r^2 |\partial_r u_r|^2 \tag{3.1}$$

and (by Theorem 1.3) the symmetric log-epiperimetric inequality

$$\mathcal{E}(u_r) \leq \mathcal{E}(z_r) - \epsilon |\mathcal{E}(z_r)|^{1+\gamma} \quad \forall r \in (\rho, 1), \tag{3.2}$$

for $\mathcal{E}(u) = W(u) - W(u_0)$, $W(u) = W_0^1(u) + |\{u > 0\} \cap B_1|$, and $z_r(x) = |x| u_r(x/|x|)$ the 1-homogenous extension of u_r. (3.1), (3.2) together with Theorem 2.2 gives a decay-growth estimates (1.12), (1.13) and the Dini estimate

$$\int_\rho^1 ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq c([\mathcal{E}(\rho)_-]^{\frac{1-\gamma}{2}} + [\mathcal{E}(1)_+]^{\frac{1-\gamma}{2}}). \tag{3.3}$$

In Sect. 3.6 we prove the symmetric log-epiperimetric inequality (Theorem 1.3). In the remainder of this Section we highlight and prove several applications. The first and primary consequence is that regularity and graphicality propogate both outward and inward while the density remains close to constant.

Theorem 3.1 (Small density drop implies graphical + estimates) *Given $\eta > 0$, there are $\delta(u_0, \eta), \epsilon(u_0) > 0$, $\gamma(u_0) \in [0, 1)$ so that the following holds. Let $u \in H^1(B_1)$ minimize J, and suppose there is a $1/4 \geq \rho \geq 0$ so that*

$$\min\{||u - u_0||_{L^2(\partial B_1)}, ||u_{\rho/2} - u_0||_{L^2(\partial B_1)}\} \leq \delta, \tag{3.4}$$

and

$$W(u) \leq W(u_0) + \delta, \quad W(u_{\rho/2}) \geq W(u_0) - \delta. \tag{3.5}$$

Then we have

$$\sup_{r\in[\rho,1/2]} ||u_r - u_0||_{H^1(\partial B_1)} \leq \eta, \quad \sup_{r\in[2\rho,1/2]} ||u_r - u_0||_{H^1(A_{1,1/2})} \leq \eta, \tag{3.6}$$

and we can find a $\xi : C\Omega \cap A_{1/2,\rho} \to \mathbb{R}$ so that

$$\partial\{u > 0\} \cap A_{1/2,\rho} = G_{C\Omega}(\xi) \cap A_{1/2,\rho}, \quad \sup_{r\in[2\rho,1/2]} ||\xi_r||_{C^{2,\alpha}(C\Omega\cap A_{1,1/2})} \leq \eta. \tag{3.7}$$

Moreover, we have the Dini estimate

$$\int_\rho^{1/2} ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq c(u_0)\delta^{(1-\gamma)/2}. \tag{3.8}$$

Proof Assume that $||u - u_0||_{L^2(\partial B_1)} \leq \delta$. The other case is essentially the same. Let ρ_* be the least radius so that (3.6), (3.7), (3.8) hold with ρ_* in place of ρ. By Lemma 3.2, ensuring $\delta(\eta, u_0)$ is sufficiently small, we can assume that $\rho_* \leq \max\{\rho, 1/4\}$. Moreover, ensuring $\delta(\eta', u_0)$ is small, Lemma 3.2 implies we can assume that

$$||u_{1/2} - u_0||_{L^2(\partial B_1)} \leq \eta'. \tag{3.9}$$

Since there is no loss in generality in assuming $\eta(u_0)$ is small, by (3.1) and (3.2) we can apply Theorem 2.2 with $\mathcal{E}(v) = W(v) - W(u_0)$ and $\Lambda = 0$ to deduce that

$$\int_{\rho_*}^{1/2} ||\partial_r u_r||_{L^2(\partial B_1)} dr \leq c(u_0)\delta^{(1-\gamma)/2}.$$

From (3.9) and (2.8) we get

$$||u_r - u_0||_{L^2(\partial B_1)} \leq \eta' + c(u_0)\delta^{(1-\gamma)/2} \quad \forall \rho_* \leq r \leq 1/2.$$

If $\rho_* > \rho$, then provided $\eta'(\eta, u_0)$, $\delta(u_0, \eta', \eta)$ are small we can use Lemma 3.2 to deduce a contradiction. □

Lemma 3.2 *Given $\epsilon > 0$, there is a $\delta(u_0, \epsilon) > 0$ so that the following holds. Suppose $u \in H^1(B_1)$ minimizes J, and satisfies*

$$\min\{||u - u_0||_{L^2(\partial B_1)}, ||u_{1/8} - u_0||_{L^2(\partial B_1)}\} \leq \delta, \tag{3.10}$$

$$W(u) \leq W(u_0) + \delta, \quad W(u_{1/8}) \geq W(u_0) - \delta. \tag{3.11}$$

Then

$$\sup_{r\in[1/4,1/2]} ||u_r - u_0||_{H^1(\partial B_1)} \leq \epsilon, \quad ||u - u_0||_{H^1(A_{1/2,1/4})} \leq \eta, \tag{3.12}$$

$$\partial\{u > 0\} \cap A_{1/2,1/4} = G_{C\Omega}(\xi) \cap A_{1/2,1/4}, \quad ||\xi||_{C^{2,\alpha}(C\Omega\cap A_{1/2,1/4})} \leq \epsilon. \tag{3.13}$$

Proof Straightforward argument by contradiction, similar to [12, Lemma 4.1]. □

Combining Theorems 3.1, 2.2 gives directly uniqueness at at 0 and ∞. We state here only the uniqueness at ∞; uniqueness at 0 was first proven in [12].

Corollary 3.3 (Uniqueness at infinity) *Let $u \in H^1_{loc}(\mathbb{R}^n)$ be an entire minimizer of J. Suppose, for some $r_i \to \infty$, $u_{r_i} \to u_0$ in L^2_{loc}. Then there are constants $\gamma(u_0) \in [0,1)$, $\delta(u_0) > 0$, $C(u)$, so that*

$$||u_r - u_0||_{L^2(\partial B_1)} \leq \begin{cases} C \log(r)^{\frac{\gamma-1}{2\gamma}} & \gamma > 0 \\ Cr^{-\frac{\delta}{2}} & \gamma = 0 \end{cases} \quad \forall r > 1.$$

Here γ as in Theorem 1.3.

A less obvious corollary is the following one-sided perturbation theorem. [8] have classified entire minimizers to J which lie to one side of u_0 as fitting inside a Hardt–Simon-type foliation. Specifically, they have shown

Theorem 3.4 [8] *There exist regular, entire minimizers $\underline{u} \leq u_0 \leq \overline{u} \in H^1_{loc}(\mathbb{R}^n)$ to J, asymptotic to u_0 at infinity, with the property that if $u \in H^1_{loc}(\mathbb{R}^n)$ is an entire minimizer to J and $u \leq u_0$ (resp. $u \geq u_0$), then either $u = \underline{u}_{1/t}$ (resp. $u = \overline{u}_{1/t}$) for some $t > 0$, or $u = u_0$.*

By arguing analogously to [9], the following local version of Theorem 3.4 holds.

Theorem 3.5 (Local Liouville-type theorem) *Given $\epsilon > 0$, there is a $\delta(u_0, \epsilon) > 0$ so that the following holds. Let $u \in H^1(B_1)$ minimize J, and suppose that*

$$||u - u_0||_{L^2(B_1)} \leq \delta, \quad u \leq u_0. \tag{3.14}$$

Then either $u = u_0$, or there is a $0 < t \leq \epsilon$ so that

$$||u_r - \underline{u}_{r/t}||_{L^2(A_{1,1/2})} \leq \epsilon \quad \forall t < r \leq 1. \tag{3.15}$$

In particular, either $\partial\{u > 0\} = \partial\{u_0 > 0\} \cap B_1$, or $\partial\{u > 0\} \cap B_{1/2}$ is regular, and a small analytic perturbation of $\partial\{\underline{u}_{1/t} > 0\} \cap B_{1/2}$.

If one assumes $u \geq u_0$, then the same conclusion holds with $\overline{u}$ in place of $\underline{u}$.

Proof We show there is a $0 \leq t \leq \epsilon$ so that (3.15) holds, where we formally interpret $\underline{u}_{r/0} \equiv u_0$. If $t = 0$ then $u = u_0$ follows by the strong maximum principle ([13], or [8]). Suppose, towards a contradiction, this failed: there is a sequence $\delta_i \to 0$, $u_i \in H^1(B_1)$, so that (3.14) holds with u_i, δ_i in place of u, δ, but (3.15) fails every u_i and for every $0 \leq t \leq \epsilon$. By standard compactness for minimizers of J, we can assume that $u_i \to u_0$ in $H^1_{loc}(B_1) \cap C^0_{loc}(B_1)$.

For $\delta' > 0$ to be determined later, let ρ_i be the least radius so that

$$|W((u_i)_r) - W(u_0)| \leq \delta' \quad \forall \rho_i < r < 9/10.$$

By our convergence $u_i \to u_0$, we have $\rho_i \to 0$. By Theorem 3.1, provided we ensure $\delta'(u_0, \epsilon)$ sufficiently small, we have

$$||(u_i)_r - u_0||_{L^2(A_{1,1/2})} \leq \epsilon/2 \leq \epsilon \quad \forall 2\rho_i \leq r \leq 1. \tag{3.16}$$

If $\rho_i = 0$, then we obtain a contradiction, so we must have $\rho_i > 0$ for all i.

Define $u_i' = (u_i)_{\rho_i}$. Then for every $R > 1$ and $i >> 1$, we have

$$1 = \inf\{\rho : |W((u_i')_r) - W(u_0)| \leq \delta' \quad \forall \rho < r < R\}. \tag{3.17}$$

Passing to a subsequence, we can assume that $u_i' \to u'$ in $H^1_{loc}(\mathbb{R}^n) \cap C^0_{loc}(\mathbb{R}^n)$, for some minimizer u' of J. Since every $u_i \leq u_0$, we have $u' \leq u_0$, and therefore by Theorem 3.4 either $u' = u_0$ or $u' = \underline{u}_{1/t}$ for some $t > 0$.

If $u' = u_0$, then by our convergence $u_i' \to u'$ and the monotonicity of $r \mapsto W((u_i')_r)$, for $i >> 1$ we would have

$$|W((u_i')_r) - W(u_0)| \leq \delta'/2 \quad \forall r \in (1/2, 2),$$

which contradicts (3.17). Therefore $u' = \underline{u}_{1/t}$ for some $t > 0$.

From (3.17) we have $W(\underline{u}_{2/t}) \neq 0$, and so $t \leq t_0$ for some t_0 depending only on u_0. Therefore, since $\underline{u}$ is asymptotic to u_0 at infinity, there is an $R_0(u_0, \epsilon)$ so that

$$||\underline{u}_{r/t} - u_0||_{L^2(A_{1,1/2})} \leq \epsilon/2 \quad \forall r \geq R_0. \tag{3.18}$$

On the other hand, by our convergence $u_i' \to \underline{u}_{1/t}$, we have

$$||(u_i')_r - \underline{u}_{r/t}||_{L^2(A_{1,1/2})} \leq \epsilon \quad \forall t < r < 2R_0 \tag{3.19}$$

provided $i >> 1$.

Defining $t_i = t\rho_i$, we deduce from (3.16), (3.18), (3.19) that

$$||(u_i)_r - \underline{u}_{r/t_i}||_{L^2(A_{1,1/2})} \leq \epsilon \quad \forall t_i < r \leq 1$$

when $i >> 1$. This is a contradiction, and finishes the proof of the Theorem. □

3.6 Proof of symmetric log-epiperimetric

We prove in this Section the symmetric log-epiperimetric inequality of Theorem 1.3 for Alt–Caffarelli functional (1.5). The proof is in principle a minor modification of the proof of the standard log-epiperimetric of [12]. However, we choose to give the full proof of Theorem 1.3 here, partly because there are subtleties involving the choice of constants which are non-obvious even in the original proof, and partly because we can give a more streamlined proof synthesizing both the standard and the "reverse" epiperimetric inequalities.

Theorem 1.3 is a largely direct consequence of the following two Lemmas, which deal with the "outer" variation and "inner" variation separately. The first is verbatim to [25], and the latter is our modified variant of [12].

Lemma 3.7 [25, Lemmas 2.5, 2.6] *Let Ω' be a fixed, Lipschitz domain in ∂B_1, and let ϕ_i be eigenfunctions of Ω', with eigenvalues λ_i satisfying $\lambda_i - (n-1) \geq \eta > 0$ for every $i \geq 2$ and some $\eta > 0$. There are numbers $\rho(n, \eta), \epsilon(n, \eta) \in (0, 1)$ so that the following holds.*

Let $z_+ \in H^1(\partial B_1)$ take the form $z_+ = \sum_{i=2}^{\infty} c_i \phi_i$. Define $h_+ = \sum_{i=2}^{\infty} c_i r^{\alpha_i} \phi_i$ to be the harmonic extension of z_+ to the cone over Ω', and let $\psi(r)$ be the harmonic function in $A_{1,\rho}$ such that $\psi(r = 1) = 1$, $\psi(r = \rho) = 0$. Then $\psi h_+ \in H^1(B_1)$ satisfies $\psi h_+|_{\partial B_1} = z_+$ and

$$W_0(\psi h_+) - W_0(r z_+) \leq -\epsilon W_0(r z_+) = -\epsilon |W_0(r z_+)|.$$

Lemma 3.8 *There are constants $\delta(\sigma), \epsilon(\sigma) > 0$, $\gamma(\sigma) \in [0, 1)$, so that the following holds.*

Take $\xi \in C^{2,\alpha}(\partial\Omega)$, $z_1 \in H^1(\partial B_1)$ such that

$$||z_1 - \sigma||_{L^2(\partial B_1)} \leq \delta, \quad ||\xi||_{2,\alpha} \leq \delta, \tag{3.20}$$

and assume additionally z_1 takes the form $z_1 = c_1 \phi_1^{\Omega_\xi}$. Then we can find an $h_1 \in H^1(B_1)$ satisfying $h_1|_{\partial B_1} = z_1$, so that

$$W(h_1) - W(r z_1) \leq -\epsilon |W(r z_1) - W(r\sigma)|^{1+\gamma}. \tag{3.21}$$

Proof of Theorem 1.3 given Lemmas 3.7, 3.8 Since $\lambda_1^{\Omega} = n-1$ and Ω is connected (see for instance [13, Theorem 2.3]), we have $\lambda_2^{\Omega} > n-1$. Therefore, provided $\delta(\sigma)$ is sufficiently small, we have $\lambda_2^{\Omega_\xi} - (n-1) \geq \epsilon_0(\sigma) > 0$.

Write

$$z = c_1 \phi_1^{\Omega_\xi} + \sum_{i \geq 2} c_i \phi_i^{\Omega_\xi} =: z_1 + z_+.$$

By our hypotheses and our previous discussion, we can apply Lemma 3.7 to z_+ to obtain an ψh_+, ρ, and apply Lemma 3.8 to z_1 to obtain an h_1.

Define h to be the competitor

$$h(x) = \begin{cases} r z_1 + \psi h_+ & \rho \leq |x| \leq 1 \\ \rho h_1(x/\rho) & 0 < |x| \leq \rho \end{cases}$$

By construction, $h \in H^1(B_1)$, and $h|_{\partial B_1} = z$ and $h|_{\partial B_\rho} = \rho z_1$. By orthogonality of the ϕ_i, the facts that $\{z > 0\} = \{z_1 > 0\}$ and $\{h_+ > 0\} \subset \{r z_1 > 0\}$, and the scaling of W, it's straightforward to verify that

$$\begin{aligned} &W(rz) = W(rz_1) + W_0(rz_+), \text{ and} \\ &W(h) - W(rz) \leq \rho^n (W(h_1) - W(rz_1)) + W_0(\psi h_+) - W_0(rz_+). \end{aligned}$$

Since by (1.7) $W_0(rz_+) \leq c(\sigma)$, we can therefore use Lemmas 3.7, 3.8 to estimate

$$\begin{aligned} W(h) - W(rz) &\leq -\rho^n \epsilon_1 |W(rz_1) - W(r\sigma)|^{1+\gamma} - \epsilon_2 |W_0(rz_+)| \\ &\leq -\rho^n \epsilon_1 |W(rz_1) - W(r\sigma)|^{1+\gamma} - (\epsilon_2/c) |W_0(rz_+)|^{1+\gamma} \\ &\leq -2^{-1-\gamma} \min\{\rho^n \epsilon_1, \epsilon_2/c(\sigma)\} |W(rz) - W(r\sigma)|^{1+\gamma}. \end{aligned}$$

This proves Theorem 1.3. □

Proof of Lemma 3.8 The proof is similar to [12], except to allow for $\mathcal{G}$ to be negative we must modify *both* positive and negative modes (i.e. $\xi_{\pm}^{\perp}$), and be slightly more careful in our treatment of the zero modes (i.e. ξ^T). We shall use heavily the notation from [12]. In this proof $\epsilon(\sigma), b(\sigma, \epsilon), \delta(\sigma, \epsilon)$ are small positive constants ≤ 1 which we shall choose as we go along, but can a posteriori be fixed. Letters $c(\sigma), c'(\sigma)$ represent large constants ≥ 1 which may increase from line to line.

Recall by [12, Lemma 3.6], if $v(r, \theta) \in H^1([0,1] \times \partial B_1)$, then $rv(r, \theta) \in H^1(B_1)$ satisfies

$$W(rv) = \int_0^1 W_S(v(r)) r^{n-1} dr + \int_0^1 \int_{\partial B_1} (\partial_r v)^2 r^{n+1} d\theta dr, \tag{3.22}$$

where $W_S : H^1(\partial B_1) \to \mathbb{R}$ is defined by

$$W_S(w) = \int_{\partial B_1} |\nabla w|^2 - (n-1) w^2 d\theta + \mathcal{H}^{n-1}(\{w > 0\}).$$

Our competitor h_1, like in [12], will take the form $h_1(r, \theta) = r\kappa(r) \phi_1^{\Omega_{g(r)}}$ for some suitable flows $\kappa(r)$, $g(r, \theta)$. To this end, we recall the functional introduced by [12] $\mathcal{G} : C^{2,\alpha}(\partial\Omega) \times \mathbb{R} \to \mathbb{R}$ given by

$$\mathcal{G}(\zeta, s) = (\kappa^2 + s^3)(\lambda_1^{\Omega_\zeta} - (n-1)) + \mathcal{H}^{n-1}(\Omega_\zeta) - \mathcal{H}^{n-1}(\Omega).$$

As shown in [12, Lemma 2.3], provided $||\zeta||_{2,\alpha} \leq \delta(\sigma)$ is sufficiently small, $\mathcal{G}$ is well-defined, analytic, satisfies

$$\mathcal{G}(0,0) = 0, \quad \delta\mathcal{G}(0,0) = 0, \quad \delta^2\mathcal{G}(0,0)[(\zeta,r),(\eta,s)] = \delta^2\mathcal{G}(0,0)[(\zeta,0),(\eta,0)].$$

The operator $\delta^2\mathcal{G}(0,0)[(\zeta,0),(\eta,0)]$ is a self-adjoint bilinear form on $H^{1/2}(\partial\Omega)$, with eigenvalues $\bar{\lambda}_i \to \infty$, and an $L^2(\partial\Omega)$-ON basis of smooth eigenfunctions ζ_i. Writing $K = \text{span}\{\zeta_i : \bar{\lambda}_i = 0\}$, then the kernel of $\delta^2\mathcal{G}(0,0)$ is $K \oplus \mathbb{R}$.

Write P_K for the $L^2(\partial\Omega)$-orthogonal projection to K, and $P_\pm$ for the projection onto $\text{span}\{\zeta_i : \pm\bar{\lambda}_i > 0\}$. Note that since $\delta^2\mathcal{G}(0,0)$ has finite index and kernel, we have the bounds

$$||P_K f||_{2,\alpha} + ||P_+ f||_{2,\alpha} + ||P_- f||_{2,\alpha} \leq c(\sigma)||f||_{2,\alpha}$$

for any $f \in C^{2,\alpha}(\partial\Omega)$.

Let $Y : (K \oplus \mathbb{R}) \cap U \to K^\perp$ be the Lyapunov-Schmidt map, defined in a suitable $(C^{2,\alpha}(\partial\Omega) \oplus \mathbb{R})$-neighborhood U of $(0,0)$. Y is analytic, and satisfies $Y(0,0) = 0$, $\delta Y(0,0) = 0$.

[12, (2.8), Section A.5] have shown that

$$|\delta^2\mathcal{G}(g,s)[(\zeta,0),(\zeta,0)] - \delta^2\mathcal{G}(0,0)[(\zeta,0),(\zeta,0)]| \leq (\omega(||g||_{2,\alpha}) + c(\sigma)s^3)||\xi||^2_{H^{1/2}}, \tag{3.23}$$

for some continuous, increasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$. There is no loss in assuming that $\omega(\tau) \geq \tau$. From [12, Section A] and [6, Lemma 2.8], we have

$$|\delta^2\mathcal{G}(0,0)[(\zeta,0),(\zeta,0)| \leq c(\sigma)||\zeta||^2_{H^{1/2}}, \tag{3.24}$$

while from [12, (A13), (A14)] we have the opposite inequality

$$||\zeta||^2_{H^{1/2}} \leq \frac{n}{\lambda_2^\Omega - (n-1)}\left(|\delta^2\mathcal{G}(0,0)[(\zeta,0),(\zeta,0)]| + \int_{\partial\Omega}\zeta^2 H_{\partial\Omega}d\theta\right). \tag{3.25}$$

As is well-known (see [13, Theorem 2.3]), since $r\sigma$ is a 1-homogenous minimizer of (1.5), $\lambda_1^\Omega = n-1$, Ω is connected, and $\sigma = \kappa\phi_1^\Omega$ for κ chosen so that $\partial_\nu\sigma = -1$. From [12, (2.7)], we have

$$||\phi_1^{\Omega_\zeta} - \phi_1^\Omega||_{L^2(\partial B_1)} \leq c(\sigma)||\zeta||_{L^2(\partial\Omega)},$$

and therefore, combined with (3.20), we deduce $|c_1 - \kappa| \leq c(\sigma)\delta$. Fix s_0 so that $c_1^2 = \kappa^2 + s_0^3$, and note that $|s_0| \leq c(\sigma)\delta^{1/3}$.

Expand

$$\xi = [P_K(\xi) + Y(\xi, s_0)] + \left[P_K^\perp(\xi) - Y(\xi, s_0)\right] =: \xi^T + \xi^\perp,$$

and write

$$\xi^\perp = P_+\xi^\perp + P_-\xi^\perp =: \xi_+^\perp + \xi_-^\perp.$$

Write $P_K\xi = \sum_{j=1}^m \mu_0^j\zeta_j$, for ζ_j spanning K, and write $\mu_0 = (\mu_0^1, \ldots, \mu_0^m) \in \mathbb{R}^m$. Define $G : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ by $G(\mu, s) = \mathcal{G}(\sum_{j=1}^m \mu^j\zeta_j + Y(\sum_{j=1}^m \mu^j\zeta_j, s))$. As shown in [12], there is a $\delta'(\sigma) > 0$ so that provided $|(\mu, s)| \leq \delta'$, G is well-defined, analytic, and (therefore) satisfies the Łojasiewicz-Simon inequality

$$|G(\mu,s)|^{1-\beta} \leq c(\sigma)|DG(\mu,s)|,$$

for some $\beta \in (0, 1/2]$. σ is integrable if and only if $\beta = 1/2$ if and only if $G \equiv 0$. Note also that $DG(\mu, s) = 0$ implies $G(\mu, s) = 0$.

Since $G(0, 0) = 0$, we have

$$|G(\mu, s)| \leq c(\sigma)|(\mu, s)|.$$

By construction we have $|\mu_0| \leq c(\sigma)||\xi||_{L^2(\partial B_1)} \leq c(\sigma)\delta$, and $|s_0| \leq c(\sigma)\delta^{1/3}$. Therefore, by ensuring $\delta(\sigma)$ is sufficiently small, we can assume

$$\max\{|(\mu_0, s_0)|, |G(\mu_0, s_0)|^{1-\beta}\} \leq c(\sigma)\delta^{1/6} \leq \min\{\delta'/4, \kappa^2/4\}. \tag{3.26}$$

If $G(\mu_0, s_0) = 0$ define $(\mu(t), s(t)) \equiv (\mu_0, s_0)$. Otherwise, let $(\mu(t), s(t))$ solve the ODE

$$(\mu'(t), s'(t)) = -\frac{DG(\mu, s)}{|DG(\mu, s)|}, \quad (\mu(0), s(0)) = (\mu_0, s_0).$$

The solutions $(\mu(t), s(t))$ exists smoothly on some positive, maximal time interval $[0, t_*)$, where either $t_* = \infty$, or $\lim_{t\to t_*} |(\mu(t), s(t)| = \delta'$, or $\lim_{t\to t_*} G(\mu(t), s(t)) = 0$. Since

$$|(\mu(t), s(t)) - (\mu_0, s_0)| \leq t, \quad |(\mu_0, s_0)| \leq \delta'/4,$$

we can assume that either $t_* \geq \delta'/2$ or $\lim_{t\to t_*} G(\mu(t), s(t)) = 0$. Note also that

$$\frac{d}{dt}G(\mu(t), s(t)) = -|DG(\mu(t), s(t))| < 0,$$

and so $G(\mu(t), s(t))$ is decreasing.

If $G(\mu_0, s_0) = 0$, set $b = 0$ and $\eta(r) \equiv 0 \equiv b|G(\mu_0, s_0)|^{1-\beta}(1 - r)$.

If $G(\mu_0, s_0) < 0$, then $t_* \geq \delta'/2$. Let $\eta(r) = b|G(\mu_0, s_0)|^{1-\beta}(1 - r)$. By our choice of δ above we have $|\eta(r)| \leq \delta'/2$ for $r \in [0, 1]$, and so $\mu(\eta(r)), s(\mu(r))$ are well-defined on $[0, 1]$. We have the bounds

$$\begin{aligned} G(\mu(\eta(r)), s(\eta(r))) - G(\mu_0, s_0) &\leq -\int_0^{\eta(r)} |DG(\mu(t), s(t))|dt \\ &\leq -(1/c)\int_0^{\eta(r)} |G(\mu(t), s(t))|^{1-\beta}dt \\ &\leq -(1/c)|G(\mu_0, s_0)|^{1-\beta}\eta(r) \\ &\leq -(b/c)|G(\mu_0, s_0)|^{2-2\beta}(1 - r) \quad \forall r \in [0, 1]. \end{aligned}$$

If $G(\mu_0, s_0) > 0$, we break into two cases. Define

$$t_1 = \sup\{t \in [0, \delta'/2] : G(\mu(t), s(t)) \geq G(\mu_0, s_0)/2\}.$$

If $t_1 \geq bG(\mu_0, s_0)^{1-\beta}$, then define $\eta(r) = bG(\mu_0, s_0)^{1-\beta}(1 - r)$ as before, and estimate

$$\begin{aligned} G(\mu(\eta(r)), s(\eta(r))) - G(\mu_0, s_0) &\leq -(1/c)\int_0^{\eta(r)} |G(\mu(t), s(t))|^{1-\beta}dt \\ &\leq -(1/2c)|G(\mu_0, s_0)|^{1-\beta}\eta(r) \\ &\leq -(b/c)|G(\mu_0, s_0)|^{2-2\beta}(1 - r) \quad \forall r \in [0, 1]. \end{aligned}$$

If $t_1 < bG(\mu_0, s_0)^{1-\beta}$, then take $\eta(r) : [0, 1] \to \mathbb{R}$ to be a smooth, decreasing function satisfying

$$\eta|_{[0,1/2]} = t_1, \quad \eta(1) = 0, \quad |\eta'| \leq 3t_1.$$

In this case we estimate (recalling that $2-2\beta \geq 1$ and $b \leq 1$)

$$G(\mu(\eta(r)), s(\eta(r))) - G(\mu_0, s_0) = -G(\mu_0, s_0)/2$$
$$\leq -(b/2)|G(\mu_0, s_0)|^{2-2\beta}(1-r) \quad \forall r \in [0, 1/2].$$

Observe that, however we defined η, we have the bounds $|\eta'(r)| \leq 3b|G(\mu_0, s_0)|^{1-\beta}$ and

$$G(\mu(\eta(r)), s(\eta(r))) - G(\mu_0, s_0) \leq 0 \quad \forall r \in [0, 1], \tag{3.27}$$
$$G(\mu(\eta(r)), s(\eta(r))) - G(\mu_0, s_0) \leq -(b/c)|G(\mu_0, s_0)|^{2-2\beta}(1-r) \quad \forall r \in [0, 1/2]. \tag{3.28}$$

Moreover, by (3.26) and our definition of $\eta(r)$ we have the bounds

$$|(\mu(\eta(r)), s(\eta(r)))| \leq c(\sigma)\delta^{1/6} \quad \forall r \in [0, 1]. \tag{3.29}$$

Define $\eta_-(r) = 1 + a(1-r)\epsilon$. Then

$$\int_0^1 (\eta_-(r)^2 - 1) r^{n-1} dr = \frac{2a\epsilon}{n(n+1)} \pm c(n)a^2\epsilon^2 \geq 4\epsilon/n$$

provided $a(n)$ is chosen sufficiently large, and $\epsilon(n)$ sufficiently small. Fix a to be thus. Similarly, define $\eta_+(r) = 1 - a'(1-r)\epsilon$, and then

$$\int_0^1 (\eta_+(r)^2 - 1) r^{n-1} dr = \frac{-2a'\epsilon}{n(n+1)} \pm c(n)a'^2\epsilon^2 \leq -4\epsilon/n.$$

provided we choose and fix $a'(n)$ large, and ensure $\epsilon(n)$ is small.

We define our competitor as follows. First we define

$$g(r,\theta) = \left[\sum_j \mu^j(\eta(r))\zeta_j + Y\left(\sum_j \mu^j(\eta(r))\zeta_j, s(\eta(r))\right)\right] + \left[\eta_-(r)\xi_-^\perp + \eta_+(r)\xi_+^\perp\right]$$
$$=: g^T(r,\theta) + g^\perp(r,\theta),$$

and then set

$$h_1(x = r\theta) = r\kappa(r)\phi_1^{\Omega_{g(r)}}(\theta), \quad \text{for} \quad \kappa(r)^2 = \kappa^2 + s(\eta(r))^3.$$

From (3.22) and the form of h_1, we have

$$W(h_1) - W(rz) = \int_0^1 (\mathcal{G}(g(r), s(\eta(r))) - \mathcal{G}(\xi, s_0)) r^{n-1} dr \tag{3.30}$$
$$+ \int_0^1 \int_{\partial B_1} (\partial_r(h_1(r)/r))^2 r^{n+1} d\theta dr. \tag{3.31}$$

Provided $\delta(\sigma)$ is sufficiently small, we have

$$(\partial_r g)^2 \leq c(\sigma)|\eta'(r)|^2 + c(\sigma)\epsilon^2(|\xi_-^\perp(r,\theta)|^2 + |\xi_+^\perp(r,\theta)|^2),$$
$$\kappa'(r)^2 \leq c(\sigma)|\eta'(r)|^2$$

and therefore, using [12, (2.7)], we deduce

$$\begin{aligned}\int_{\partial B_1}(\partial_r(h_1(r)/r))^2 d\theta &\leq 2\kappa'(r)^2 + 2\kappa(r)^2\int_{\partial B_1}(\delta\phi_1^{g(r)}[\partial_r g(r)])^2 d\theta\\ &\leq c(\sigma)|\eta'(r)|^2 + c(\sigma)\kappa(r)^2||\partial_r g(r)||^2_{L^2(\partial B_1)}\\ &\leq c(\sigma)b^2|G(\mu_0,s_0)|^{2-2\beta} + c(\sigma)\epsilon^2||\xi^\perp||^2_{L^2(\partial B_1)}.\end{aligned} \tag{3.32}$$

We work towards estimating the first term in (3.31). Write

$$\begin{aligned}&\mathcal{G}(g(r),s(\eta(r))) - \mathcal{G}(\xi,s_0)\\ &\quad= [(\mathcal{G}(g(r),s(\eta(r))) - G(\mu(\eta(r)),s(\eta(r)))) - (\mathcal{G}(\xi,s_0) - G(\mu_0,s_0)]\\ &\qquad+ [G(\mu(\eta(r)),s(\eta(r))) - G(1)]\\ &\quad=: E^\perp(r) + E^T(r).\end{aligned}$$

Expand $\xi^\perp = \sum_i \alpha_i\zeta_i \equiv \sum_{\bar\lambda_i\neq 0}\alpha_i\zeta_i$.

By considering the Taylor expansion of

$$t \mapsto \mathcal{G}(g^T(r) + tg^\perp(r), s(\eta(r))),$$

we deduce there is a $\tau\in(0,1)$ so that, provided $\delta(\sigma)$ is sufficiently small:

$$\begin{aligned}&\mathcal{G}(g(r),s(\eta(r))) - G(\mu(\eta(r)),s(\eta(r)))\\ &\quad= \delta\mathcal{G}(g^T(r))[(g^\perp(r),0)] + \frac{1}{2}\delta^2\mathcal{G}(g^T(r)+\tau g^\perp(r))[(g^\perp(r),0),(g^\perp(r),0)]\\ &\quad= \frac{1}{2}\delta^2\mathcal{G}(g^T(r)+\tau g^\perp(r))[(g^\perp(r),0),(g^\perp(r),0)]\\ &\quad= \frac{1}{2}\delta^2\mathcal{G}(0)[(g^\perp(r),0),(g^\perp(r),0)] \pm (\omega(2||g(r)||_{2,\alpha}) + c(\sigma)s(\eta(r))^3)||g^\perp(r)||^2_{H^{1/2}}\\ &\quad= \eta_-(r)^2/2\sum_{\bar\lambda_i<0}\bar\lambda_i\alpha_i^2 + \eta_+(r)^2/2\sum_{\bar\lambda_i>0}\bar\lambda_i\alpha_i^2 \pm (\omega(c\delta^{1/6}) + c(\sigma)\delta^{1/2})||\xi^\perp||^2_{H^{1/2}}\\ &\quad= -\eta_-(r)^2/2\sum_{\bar\lambda_i<0}|\bar\lambda_i|\alpha_i^2 + \eta_+(r)^2/2\sum_{\bar\lambda_i>0}|\bar\lambda_i|\alpha_i^2 \pm 2\omega(\delta^{1/7})||\xi^\perp||^2_{H^{1/2}}.\end{aligned}$$

In the third line we used [12, (B4)], the fourth we used [12, (2.8), Section A5], and in the fifth line we used (3.29) to estimate

$$||g(r)||_{2,\alpha} \leq c(\sigma)|\mu(\eta(r))| + c(\sigma)||\xi||_{2,\alpha} \leq c(\sigma)\delta^{1/6}.$$

Similarly, again taking $\delta(\sigma)$ is small we can estimate

$$\mathcal{G}(\xi,s_0) - \mathcal{G}(\mu_0,s_0) = -(1/2)\sum_{\bar\lambda_i<0}|\bar\lambda_i|\alpha_i^2 + (1/2)\sum_{\bar\lambda_i>0}|\bar\lambda_i|\alpha_i^2 \pm 2\omega(\delta^{1/7})||\xi^\perp||^2_{H^{1/2}}. \tag{3.33}$$

Recalling our choice of $\eta_-(r)$, $\eta_+(r)$, we deduce

$$\int_0^1 E^\perp(r)r^{n-1}dr \leq -(2\epsilon/n)\sum_{\bar\lambda_i\neq 0}|\bar\lambda_i|\alpha_i^2 + c(\sigma)\omega(\delta^{1/7})||\xi^\perp||^2_{H^{1/2}}. \tag{3.34}$$

Combining (3.28), (3.32), (3.34), and ensuring $b(\sigma)$ is sufficiently small, we deduce

$$\begin{aligned} W(h_1) - W(rz) &\leq -(b/c)|G(\mu_0, s_0)|^{2-2\beta} + cb^2|G(\mu_0, s_0)|^{2-2\beta} \\ &\quad - (2\epsilon/n)\sum_{\bar{\lambda}_i \neq 0} |\bar{\lambda}_i|\alpha_i^2 + (c\omega(\delta^{1/7}) + c\epsilon^2)||\xi^{\perp}||^2_{H^{1/2}} \\ &\leq -(b/2c)|G(\mu_0, s_0)|^{2-2\beta} - (\epsilon/n)\sum_i |\bar{\lambda}_i|\alpha_i^2 \\ &\quad + (c\omega(\delta^{1/7}) + c\epsilon^2 - \epsilon/c)||\xi^{\perp}||^2_{H^{1/2}}, \end{aligned} \tag{3.35}$$

for $c = c(\sigma)$. In the second inequality we additionally used (3.25) to estimate

$$\begin{aligned} \sum_i |\bar{\lambda}_i|\alpha_i^2 &\geq \frac{1 + \min_{\bar{\lambda}_i \neq 0} |\bar{\lambda}_i|}{2}(|\delta^2 \mathcal{G}(0,0)[(\xi^{\perp}, 0), (\xi^{\perp}, 0)]| + ||\xi^{\perp}||^2_{L^2}) \\ &\geq (1/c(\sigma))||\xi^{\perp}||^2_{H^{1/2}}. \end{aligned}$$

Recall from (3.33) that we have (ensuring $\delta(\sigma)$ is small):

$$\begin{aligned} n(W(rz) - W(r\sigma)) &= \mathcal{G}(\xi, s_0) - \mathcal{G}(\mu_0, s_0) + \mathcal{G}(\mu_0, s_0) \\ &= (1/2)\sum_i \bar{\lambda}_i \alpha_i^2 + G(\mu_0, s_0) \pm 2\omega(\delta^{1/7})||\xi^{\perp}||^2_{H^{1/2}}. \end{aligned} \tag{3.36}$$

If $G(\mu_0, s_0) = 0$, then we can combine (3.35), (3.36), to estimate

$$W(h_1) - W(rz) \leq -\epsilon|W(rz) - W(r\sigma)| + (c\omega(\delta^{1/7}) + c\epsilon^2 - \epsilon/c)||\xi^{\perp}||^2_{H^{1/2}} \tag{3.37}$$

$$\leq -\epsilon|W(rz) - W(r\sigma)| \tag{3.38}$$

provided $\epsilon(\sigma)$ is small, and $\delta(\sigma, \epsilon)$ is small.

Suppose $G(\mu_0, s_0) \neq 0$. Using (3.24), (3.36), and the inequality $|a - b|^{1+\gamma} \geq 2^{-\gamma}|a|^{1+\gamma} - |b|^{1+\gamma}$ for any $a, b \in \mathbb{R}$, we get

$$\begin{aligned} |G(\mu_0, s_0)/n|^{1+\gamma} &\geq 2^{-\gamma}|W(rz) - W(r\sigma)|^{1+\gamma} \\ &\quad - \left|(1/2n)\sum_i \bar{\lambda}_i \alpha_i^2 \pm 2\omega(\delta^{1/7})||\xi^{\perp}||^2_{H^{1/2}}\right|^{1+\gamma} \end{aligned} \tag{3.39}$$

$$\geq 2^{-\gamma}|W(rz) - W(r\sigma)|^{1+\gamma} - c(\sigma)||\xi^{\perp}||^2_{H^{1/2}} \tag{3.40}$$

and therefore, writing $c' = c'(\sigma)$,

$$W(h_1) - W(rz) \leq (-b/c')|W(rz) - W(r\sigma)|^{1+\gamma} \tag{3.41}$$

$$+ (c'\omega(\delta^{1/7}) + c'b + c'\epsilon^2 - \epsilon/c')||\xi^{\perp}||^2_{H^{1/2}} \tag{3.42}$$

$$\leq (-b/c')|W(rz) - W(r\sigma)|^{1+\gamma} \tag{3.43}$$

provided $\epsilon(\sigma)$ is small, and $\delta(\sigma, \epsilon)$ is small, and $b(\sigma, \epsilon)$ is small. □

4 Obstacle and thin-obstacle problem

Similar results to the ones presented in the previous sections hold for almost minimizers of the obstacle problem and of the thin-obstacle problem. Since the modification for the thin-

obstacle problem are essentially already contained in [5], we present here only the statements and proofs for almost minimizers of the obstacle problem.

4.1 Results for almost minimizers of the obstacle problem

In this section we follow the notations and arguments of [5]. Let B_1 be the unit ball in $\mathbb{R}^n$ and consider the functional

$$\mathcal{F}_{\text{OB}}(u, W) := \frac{1}{2}\int_W |\nabla u|^2\, dx + \int_W u\, dx,$$

where we will drop the dependence on the set if $W = B_1$, and the set of admissible functions

$$\mathcal{K}_{\text{OB}} := \left\{u \in H^1(B_1)\ \ u \geq 0 \text{ in } B_1\right\}.$$

Definition 4.1.1 A function $u \in \mathcal{K}_{\text{OB}}$ is a (Λ, α, r_0) *almost minimizer of the obstacle problem in* B_1 if

$$\mathcal{F}_{\text{OB}}(u, W) \leq (1 + \Lambda\, r^\alpha)\, \mathcal{F}_{\text{OB}}(v, W) \qquad \forall v \in \mathcal{K}_{\text{OB}} \tag{4.1}$$

for all $W \subset\subset B_r(x) \subset U$ with $0 < r < r_0$.

The relevant energy is given by the Weiss' boundary adjusted energy

$$\mathcal{E}(u) = W_0^2(u) + \int_{B_1} \max\{u, 0\} - \int_{B_1} \max\{\sigma, 0\},$$

where σ is a 2-homogeneous global minimizer of the energy. In particular we will assume σ to belong to the following class:

$$\mathcal{B} := \{Q_A : \mathbb{R}^d \to \mathbb{R} :\ Q_A(x) = x \cdot Ax,\ \ A \text{ symmetric non-negative with } \operatorname{tr} A = 1/4\}.$$

It is a simple computation to see that $\mathcal{E}$ satisfies the almost monotonicity 2.1 (1) when u is rescaled as $u_r(x) = u(rx)/r^2$ (see for instance [5]); for more details on the properties of the almost-minimizers of the obstacle and the thin-obstacle problems we refer for instance to [2, 15–17]. So in order to apply Theorem 2.2 we only need to check the symmetric epiperimetric inequality. This follows as a minor modification of [5, Proposition 3.1], which we outline in the next section for the reader's convenience.

Proposition 4.2 (Symmetric epiperimetric inequality for the obstacle problem) *Let* $\sigma \in \mathcal{B}$. *There are constants* $\delta(\sigma), E(\sigma) > 0$, $\gamma(\sigma) \in [0, 1)$, *so that the following holds.*

Let $z \in H^1(\partial B_1) \cap \mathcal{K}$ *be such that*

$$\|z - \eta\|_{L^2(\partial B_1)} \leq \delta \qquad \text{and} \qquad |\mathcal{F}_{\text{OB}}(rz) - \mathcal{F}_{\text{OB}}(\sigma)| \leq E.$$

Then there is an $h \in H^1(B_1)$ *with* $h|_{\partial B_1} = z$, *so that*

$$\mathcal{E}(h) - \mathcal{E}(r^2 z) \leq -\epsilon |\mathcal{E}(r^2 z)|^{1+\gamma}. \tag{4.2}$$

Using Proposition 4.2 and the monotonicity of $\mathcal{E}$, we can apply once again Theorem 2.2 to the energy $\mathcal{E}(u_r) = W_2(u_r)$ to obtain the following uniqueness of blow-ups and blow-down at singular points. We remark that the uniqueness for blow ups with logarithmic decay had already been established in [4].

Corollary 4.3 (Uniqueness of blow-downs for almost minimizers of obstacle problem) *Let $u \in H^1_{loc}(\mathbb{R}^n)$ be an entire almost minimizer of $\mathcal{F}_{\text{OB}}$. Suppose, for some $r_i \to \infty$, $u_{r_i} \to \sigma \in \mathcal{B}$ in L^2_{loc}. Then there are constants $\gamma(\sigma) \in [0, 1)$, $\delta(\sigma) > 0$, $C(\sigma)$, so that*

$$||u_r - u_0||_{L^2(\partial B_1)} \leq \begin{cases} C \log(r)^{\frac{\gamma-1}{2\gamma}} & \gamma > 0 \\ C r^{-\frac{\delta}{2}} & \gamma = 0 \end{cases} \quad \forall r > 1.$$

Here γ as in Proposition 4.2.

4.4 Proof of the symmetric epiperimetric inequality Proposition 4.5

The symmetric epiperimetric inequality of Propostion 4.2 follows from the following general proposition, analogous to [5, Proposition 3.1]. We fix $\mathcal{H} = L^2(\partial B_1)$, $\mathcal{K} \subset \mathcal{H}$ a convex cone, $\mathcal{W} = H^2(\partial B_1)$.

Proposition 4.5 *Let $\mathcal{G}$ be a functional satisfying assumptions (SL) and (FL) of Proposition 3.1 in [5], while assumption (ŁS)[1] is replaced by*

(ŁS') $\mathcal{F}$ has the constrained Łojasiewicz property, that is there are constants $\gamma \in]0, 1/2]$, $C_L > 0$, $\delta_L > 0$ and $E_L > 0$, depending on $\psi \in \mathcal{W} \cap \mathcal{K}$ a critical point of $\mathcal{F}$ in $\mathcal{K}$, such that for every critical point $\varphi \in \mathcal{S}$ the following inequality holds

$$\left|\mathcal{F}(u) - \mathcal{F}(\psi)\right|^{1-\gamma} \leq C_L \|\nabla \mathcal{F}(u)\|_{\mathcal{K}}, \tag{4.3}$$

for every $u \in \mathcal{K} \cap \mathcal{W}$ such that $\|u - \varphi\| \leq \delta_L$ and $|\mathcal{F}(u) - \mathcal{F}(\varphi)| \leq E_L$.

Then there are constants $\delta_0 > 0$ and $E > 0$, depending only on the dimension and ψ, such that: if $c \in H^1(\partial B_1) \cap \mathcal{K}$ satisfies

$$\|c - \psi\|_{L^2(\partial B_1)} \leq \delta_0 \quad \text{and} \quad |\mathcal{F}(c) - \mathcal{F}(\psi)| \leq E,$$

then there exists a function $h = h(r, \theta) \in H^1(B_1)$ satisfying $h(r, \cdot) \in \mathcal{K}$, for every $r \in (0, 1]$, and

$$\mathcal{G}(h) - \mathcal{G}(\phi) \leq (\mathcal{G}(z) - \mathcal{G}(\phi)) - \epsilon \, |\mathcal{G}(z) - \mathcal{G}(\phi)|^{2-2\gamma} \tag{4.4}$$

where $\phi(r, \theta) := r^k \psi(\theta)$, $z(r, \theta) := r^k c(\theta)$, $\epsilon > 0$ is a universal constant and $\gamma > 0$ is the exponent from (ŁS).

Proof The proof is the same as that of [5, Proposition 3.1], where one replaces the choice of ϵ_2 in equation (3.5) with

$$|\mathcal{F}(u_0) - \mathcal{F}(\psi)| \leq 2\left|\mathcal{F}(u(t)) - \mathcal{F}(\psi)\right| \quad \text{for every} \quad 0 < t \leq \epsilon_2, \tag{4.5}$$

and in the last string of inequalities in page 16 one uses (4.3) to replace $\mathcal{F}(u(t)) - \mathcal{F}(\psi)$ with $|\mathcal{F}(u(t)) - \mathcal{F}(\psi)|$. □

Proof of Proposition 4.2 The proof follows if we can verify the assumptions of Proposition 4.5. To this aim, we observe that properties (SL) and (FL) hold when we set

$$\mathcal{G} := \mathcal{F}_{\text{OB}} \quad \text{and} \quad \mathcal{F}(\phi) := \int_{\partial B_1} \left(|\nabla_\theta \phi|^2 - 2d\,\phi^2\right) d\mathcal{H}^{n-1},$$

[1] Property (ŁS) is the same as (ŁS') without the absolute value in the left hand side of (4.3).

as explained in the proof of [5, Theorem 1.10]. So we only need to show that property (ŁS') holds for $\mathcal{F}$. Even though this property is stronger than property (ŁS), its proof is similar to that of [5, Proposition 4.4]. So following the reasoning there we let $\varphi \in \mathcal{S}$ be such that

$$\|u - \varphi\|_2 \leq 2\delta.$$

Notice that $u - \varphi$ can be uniquely decomposed in Fourier series as $u - \varphi = Q_- + Q_0 + \eta$, where Q_- contains only lower eigenmodes (corresponding to eigenvalues $< 2d$), Q_0 is a $2d$-eigenfunction and

$$\eta(x) = \sum_{\{j:\lambda_j > 2d\}} c_j \phi_j(x),$$

which contains only higher eigenmodes (corresponding to eigenvalues $> 2d$). Thus,

$$u = Q_- + Q_0 + \varphi + \eta \quad \text{and} \quad \|Q_-\|_2, \|Q_0\|_2, \|\eta\|_2 \leq 2\delta.$$

We now consider

$$M := \max_{x \in \mathcal{M}} \{-2Q_-(x) - Q_0(x) - \varphi(x)\}$$

and suppose that the maximum is realized at a point $x_M \in \mathcal{M}$. Notice that since $Q_- + Q_0$ is a finite sum of (smooth) eigenfunctions, there is a constant $C > 0$ (depending onnly on the dimension) such that $\|Q_- + Q_0\|_{L^\infty} \leq C\delta$. Thus, if $M > 0$, then $x_M \in \{\varphi < C\delta\}$ and $M \leq C\delta$. We now choose δ such that $10C\delta < c_d := (2d)^{-\frac{1}{2}}$ and we claim that the function

$$\tilde{u} = 2Q_- + Q_0 + \varphi + \frac{2M}{c_d}\big(c_d - \varphi\big)$$

is non-negative. Indeed, it is sufficient to consider the following two cases:

- on the set $\{\varphi \geq 2C\delta\}$, we have that

$$\tilde{u} = \left(2Q_- + Q_0 + \frac{1}{2}\varphi\right) + 2M + \varphi\left(\frac{1}{2} - \frac{2M}{c_d}\right) \geq 0,$$

since each of the three terms is non-negative;

- on the set $\{\varphi \leq 2C\delta\}$, we have that

$$\tilde{u} \geq 2Q_- + Q_0 + \varphi + \frac{2M}{c_d}\big(c_d - 2C\delta\big) \geq 2Q_- + Q_0 + \varphi + M \geq 0.$$

Next, using the fact that $c_d - \varphi$ is a $2d$-eigenfunction (notice that the integral of $c_d - \varphi$ on ∂B_1 vanishes, due to the fact that $2d > 0$), we calculate

$$\begin{aligned}
-(\tilde{u} - u) \cdot \nabla \mathcal{F}(u) &= \int_{\partial B_1} \big(-\Delta u - 2du + 1\big)\left(-Q_- + \eta - \frac{2M}{c_d}(c_d - \varphi)\right) d\mathcal{H}^{d-1} \\
&= \int_{\partial B_1} \big(-\Delta u - 2du + 1\big)\,(\eta - Q_-)\, d\mathcal{H}^{d-1} \\
&= \int_{\partial B_1} \big(-\Delta(Q_- + Q_0 + \eta) - 2d(Q_- + Q_0 + \eta)\big)\,(\eta - Q_-)\, d\mathcal{H}^{d-1} \\
&= \int_{\partial B_1} \big(|\nabla \eta|^2 - \lambda \eta^2\big)\, d\mathcal{H}^{d-1} - \int_{\partial B_1} \big(|Q_- \eta|^2 - \lambda Q_-^2\big)\, d\mathcal{H}^{d-1} \\
&= 2\left(\mathcal{F}(\eta) - \mathcal{F}(Q_-)\right).
\end{aligned}$$

Notice that since the set of eigenvalues is discrete, there is a (spectral gap) constant $G(d) > 0$ such that $|\lambda_j - 2d| \geq G(\lambda)$, whenever $|\lambda_j - 2d| > 0$. In particular, we have the inequalities

$$2\mathcal{F}(\eta) = \sum_{j:\lambda_j > 2d} c_j^2(\lambda_j - 2d) \geq G(\lambda) \sum_{j:\lambda_j > \lambda} c_j^2 = G(\lambda)\|\eta\|_2^2$$
$$-2\mathcal{F}(Q_-) = -\sum_{j:\lambda_j < 2d} c_j^2(\lambda_j - 2d) \geq G(\lambda) \sum_{j:\lambda_j > 2d} c_j^2 = G(\lambda)\|Q_-\|_2^2$$

Thus, we get by definition of $\|\cdot\|_{\mathcal{K}}$,

$$\begin{aligned}\|\nabla\mathcal{F}(u)\|_{\mathcal{K}} &\geq \frac{-(\tilde{u}-u)\cdot\nabla\mathcal{F}(u)}{\|u-\tilde{u}\|_2} \geq \frac{2(\mathcal{F}(\eta)-\mathcal{F}(Q_-))}{\frac{2M}{c_\lambda}\|c_\lambda - \varphi\|_2 + \|\eta\|_2 + \|Q_-\|_2} \\ &\geq \frac{2(\mathcal{F}(\eta)-\mathcal{F}(Q_-))}{M\frac{2}{c_d}(\|c_d\|_2 + \|\varphi\|_2) + (2G(\lambda)\mathcal{F}(\eta))^{\frac{1}{2}} + (-2G(\lambda)\mathcal{F}(Q_-))^{\frac{1}{2}}} \\ &\geq C\frac{\mathcal{F}(\eta)-\mathcal{F}(Q_-)}{M + 3\,(\mathcal{F}(\eta)-\mathcal{F}(Q_-))^{\frac{1}{2}}},\end{aligned} \tag{4.6}$$

where C is a constant depending only on the dimension.

Now, in order to get (4.3), it only remains to estimate M and put together (4.6) and (4.7). We notice that:

- $2Q_- + Q_0 + \varphi$ is a (finite) linear combination of (orthonormal and smooth) eigenfunctions corresponding to eigenvalues $\leq 2d$;
- the L^2 norm of $2Q_- + Q_0 + \varphi$ is bounded by a universal constant.

As a consequence, there is a universal (Lipschitz) constant L, depending only on the dimension, such that $\|\nabla(2Q_- + Q_0 + \varphi)\|_{L^\infty(\mathcal{M})} \leq L$. Thus, since the negative part $\psi := -\inf\{(2Q_- + Q_0 + \varphi), 0\}$ is such that $\sup\psi = M$ is small enough (bounded by a dimensional constant, as already mentioned above), we get that there is a constant C (depending on the dimension) such that

$$\|\psi\|^2_{L^2(\partial B_1)} \geq CL^{-d}M^{d+2} = CL^{-d}\|\psi\|^{d+2}_{L^\infty(\partial B_1)}.$$

Since $u \geq 0$ on ∂B_1, we have that $\psi \leq \eta - Q_-$ and so,

$$M^{d+2} \leq C^{-1}L^d\left(\|\eta\|_2^2 + \|Q_-\|_2^2\right) \leq \frac{2L^d}{CG(\lambda)}\left(\mathcal{F}(\eta) - \mathcal{F}(Q_-)\right),$$

which, together with (4.6) and (4.7), implies that, if we set $\gamma = \frac{1}{d+2}$, then

$$\|\nabla\mathcal{F}(u)\|_{\mathcal{K}} \geq C\ \left(\mathcal{F}(\eta) - \mathcal{F}(Q_-)\right)^{1-\gamma}.$$

Using that, by orthogonality, it holds

$$|\mathcal{F}(u) - \mathcal{F}(\varphi)| = |\mathcal{F}(\eta) + \mathcal{F}(Q_-)| \leq \mathcal{F}(\eta) - \mathcal{F}(Q_-), \tag{4.7}$$

we conclude (4.3). □

5 Almost-minimizing currents

We are interested here in the regularity properties of almost minimizing currents. Given $\alpha > 0$, $\Lambda \geq 0$, and $r_0 > 0$, an integral n-current T in an open subset $U \subset \mathbb{R}^{n+k}$ is called

(Λ, α, r_0)-almost minimizing in U if

$$||T||(W) \leq ||T + \partial R||(W) + \Lambda r^{n+\alpha} \tag{5.1}$$

for $x \in \text{spt}T$, $W \subset\subset B_r(x) \subset U$ with $r < r_0$, and all integral $(n+1)$-currents R supported in W. The same definition carries over in the obvious way to integral n-currents in a Riemannian manifold N^{n+k}.

Analogous to varifolds with bounded mean-curvature, there is a monotonicity formula (5.2), an Allard-type regularity theorem [3], and a compactness theorem (Lemma 5.6) for almost-minimizing currents. Recall that the density ratios of T are given by

$$\theta_T(x, r) := \frac{||T||(B_r(x))}{\omega_n r^n}.$$

Following the computations of [7], we have the inequality

$$\left[\theta_T(x, r) + \frac{n\Lambda}{\omega_n \alpha} r^\alpha\right] - \left[\theta_T(x, s) + \frac{n\Lambda}{\omega_n \alpha} s^\alpha\right] \geq \frac{1}{2\omega_n} \int_{B_r(x)\setminus B_s(x)} \frac{|\pi_T^\perp(y - x)|^2}{|y - x|^{n+2}} d||T||(y) \tag{5.2}$$

for all $B_s(x) \subset B_r(x) \subset U$. In particular $r \mapsto \theta_T(x, r) + \frac{n\Lambda}{\omega_n \alpha} r^\alpha$ is increasing.

We remark that Λ scales like $r^{-\alpha}$, in the following sense: if $(\eta_{x,\rho})_\sharp T$ is the pushforward of T under the translation/dilation map $\eta_{x,\rho}(y) := (y - x)/\rho$, then $(\eta_{x,\rho})_\sharp T$ is $(\Lambda\rho^\alpha, \alpha, r_0/\rho)$-almost minimizing in $\eta_{x,\rho}(U)$. So for the purposes of regularity there is often no loss of generality in assuming Λ is small.

Let us also remark that the general compactness/regularity theory of almost-minimizing currents in N^{n+k} is essentially the same as in $\mathbb{R}^{n+k}$. For, if g is a C^2 Riemannian metric on $B_1 \subset \mathbb{R}^{n+k}$, satisfying $|g - g_{eucl}|_{C^2} \leq \delta$, then (provided $\delta(n)$ is sufficiently small) we have $(1 - \delta)|x - y| \leq d_g(x, y) \leq (1 + \delta)|x - y|$, and $B_{(1-\delta)r}(x) \subset B_r^g(x) \subset B_{(1+\delta)r}(x)$, and for every $B_r(x) \subset B_{1-10\delta}$ there is a normal change of coordinates $\phi_x : B^g_{(1+\delta)r}(x) \to B^g_{(1+\delta)r}(x)$, in which the metric satisfies $|(\phi_x^* g)(z) - g_{eucl}| \leq c(n)\delta|z - x|^2$. By working in these normal coordinates (different for each ball one is considering in the almost-minimizing condition (5.1)), and by using the monotonicity (5.2), it is straightforward to verify that if T is (Λ, α, r_0)-almost-minimizing in (B_1, g), then T is $(\Lambda + C\delta, \min\{\alpha, 2\}, (1 - \delta)r_0)$-almost-minimizing in $(B_{1-10\delta}, g_{eucl})$, for C a constant depending on n, Λ, $||T||(B_1)$.

A log-epiperimetric inequality for smooth, minimizing cones was established by [11], who used it to prove uniqueness of smooth, multiplicity-one tangent cones for almost-minimizing currents. We outline here the symmetric log-epiperimetric inequality for smooth cones – the details are essentially the same as in Sect. 3.

Take $\mathbf{C}^n \subset \mathbb{R}^{n+k}$ a minimal cone with smooth cross section Σ. If $u : \Sigma \to \Sigma^\perp$ or $h : \mathbf{C} \cap A_{R,\rho} \to \mathbf{C}^\perp$, define the spherical graphing functions

$$G_\Sigma(u) = \left\{\frac{\theta + u(\theta)}{|\theta + u(\theta)|} : \theta \in \Sigma\right\}, \tag{5.3}$$

$$G_{\mathbf{C}}(h) \cap A_{R,\rho} = \left\{|x|\frac{x + h(x)}{|x + h(x)|} : x \in \mathbf{C} \cap A_{R,\rho}\right\}. \tag{5.4}$$

For ease of notation write $G_{\mathbf{C}}(h) = G_{\mathbf{C}}(h) \cap A_{1,0}$. Recall that we defined $A_{R,\rho}(x) = B_R(x)\setminus\overline{B_\rho}(x)$, and $A_{R,0}(x) = B_R(x)\setminus\{0\}$.

By a straightforward adaption of the proof in Sect. 3, one can prove the following symmetric log-epiperimetric inequality. We remark that the epiperimetric inequality as stated in [11] used the $C^{1,\alpha}$ norm, but it is not hard to see that it suffices to consider only the C^1 norm.

Theorem 5.1 (Symmetric log-epiperimetric inequality for smooth minimal cones) *Let* $\mathbf{C}^n \subset \mathbb{R}^{n+k}$ *(for* $n \geq 2$*) be a smooth, minimal cone, with cross section* $\Sigma = \mathbf{C} \cap \partial B_1$*. There exist* $\epsilon(\mathbf{C}, k)$, $\delta(\mathbf{C}, k)$ *positive, and a* $\gamma(\mathbf{C}, k) \in [0, 1)$ *so that the following holds.*

Let $u : \Sigma \to \Sigma^{\perp}$ *be such that* $|u|_{C^1} < \delta$*. Then there is a function* $h \in C^1(\mathbf{C} \cap B_1 \setminus \{0\}, \mathbf{C}^{\perp})$ *satisfying*

$$h|_{\mathbf{C}\cap\partial B_1} = u, \quad |x|^{-1}|h| + |\nabla h| \leq c(\Sigma, k)|u|_{C^1}^{(1+\gamma)/2}, \tag{5.5}$$

so that

$$\mathcal{H}^n(G_{\mathbf{C}}(h)) \leq \mathcal{H}^n(G_{\mathbf{C}}(z)) - \epsilon|\mathcal{H}^n(G_{\mathbf{C}}(z)) - \mathcal{H}^n(G_{\mathbf{C}}(0))|^{1+\gamma}, \tag{5.6}$$

where $z(x) = |x|u(x/|x|)$ *is the* 1*-homogenous extension of* u*. If* $\mathbf{C}$ *is integrable,*[2] *then one can take* $\gamma = 0$*.*

Let T be a $(\Lambda, \alpha, 1)$-almost-minimizing n-current in $B_1 \subset \mathbb{R}^{n+k}$ satisfying $\partial T = 0$ and

$$T \llcorner A_{1,\rho} = [G_{\mathbf{C}}(u) \cap A_{1,\rho}], \quad |x|^{-1}|u| + |\nabla u| \leq \delta,$$

for $\mathbf{C}$ a smooth minimizing cone in $\mathbb{R}^{n+k}$ and some $\rho \in [0, 1/2]$. Provided $\delta(\mathbf{C}, k)$ is sufficiently small, the coarea formula, the computation [7, (2.5)], and the Hardt–Simon inequality (see e.g. [23, (11)]) imply that

$$\frac{d}{dr}\theta_T(0, r) \geq \frac{n}{2r}(\theta_{T_r}(0, r) - \theta_T(0, r)) + \frac{1}{4\omega_n r}\int_{\Sigma} r^2|\partial_r u_r|^2 - \Lambda r^{\alpha-1} \quad \forall r \in (\rho, 1), \tag{5.7}$$

where T_r is the cone over $\partial(T \llcorner B_r)$, and $u_r(x) = r^{-1}u(rx)$.

On the other hand, from Theorem 5.1 we get the symmetric epiperimetric inequality

$$\theta_T(0, r) \leq \theta_{T_r}(0, r) - \epsilon|\theta_{T_r}(0, r) - \theta_{\mathbf{C}}(0)|^{1+\gamma} + \Lambda r^{\alpha} \quad \forall r \in (\rho, 1), \tag{5.8}$$

where T_r denotes the cone over $\partial(T \llcorner B_r)$.

Ensuring Λ, $|\theta_T(0, 1) - \theta_{\mathbf{C}}(0)|$, $|\theta_T(0, \rho) - \theta_{\mathbf{C}}(0)|$ are sufficiently small (depending only on n, α), we have by (3.1) that

$$|\theta_T(0, r) - \theta_{\mathbf{C}}(0)| \leq 1, \quad \forall r \in (\rho, 1).$$

One can then prove an exact analogue of Theorem 2.2, with Σ in place of ∂B_1 and $\theta_T(0, r) - \theta_{\mathbf{C}}(0)$ in place of $\mathcal{E}(u_r)$ and $\theta_{T_r}(0, r) - \theta_{\mathbf{C}}(0)$ in place of $\mathcal{E}(z_r)$, to deduce the function

$$G(r) = \theta_T(0, r) - \theta_{\mathbf{C}}(0) + 3\alpha^{-1}\Lambda r^{\alpha}$$

satisfies $G' \geq (\delta'/r)|G|^{1+\gamma}$ as in (2.4), and to deduce the Dini estimate

$$\begin{aligned}\int_{\mathbf{C}\cap A_{1,\rho}} &|\partial_r(u/r)|r^{1-n} \\ &\leq c(\mathbf{C}, \alpha)(|\theta_T(0, 1) - \theta_{\mathbf{C}}(0)|^{(1-\gamma)/2} \\ &\quad + |\theta_T(0, \rho) - \theta_{\mathbf{C}}(0)|^{(1-\gamma)/2} + \Lambda^{(1-\gamma)/2}).\end{aligned} \tag{5.9}$$

A direct application is the following analogue of Theorem 3.1 for almost-minimizing currents, which says that graphicality propagates as long as the density stays close to the density of the cone (c.f. [21, Theorem 13.1]).

[2] As in [11], integrable means that every 1-homogenous Jacobi field on $\mathbf{C}$ can be realized by a 1-parameter family of smooth, minimal cones.

Theorem 5.2 *Let $\mathbf{C}^n \subset \mathbb{R}^{n+k}$ be a smooth, minimal cone. Given $\epsilon, \alpha > 0$, there are $\Lambda(\mathbf{C}, \epsilon, \alpha, k)$, $\delta(\mathbf{C}, \epsilon, \alpha, k)$ positive so that the following holds.*

Let T be $(\Lambda, \alpha, 1)$-almost-minimizing in $B_1 \subset \mathbb{R}^{n+k}$ with $\partial T = 0$, and let $\rho \geq 0$, such that

$$\mathrm{spt}T \cap A_{1,1/2} = G_{\mathbf{C}}(u) \cap A_{1,1/2}, \quad |u|_{C^1} \leq \delta \tag{5.10}$$

$$\theta_T(0,1) \leq \theta_{\mathbf{C}}(0) + \delta, \quad \theta_T(0,\rho/2) \geq \theta_{\mathbf{C}}(0) - \delta. \tag{5.11}$$

Then $\mathrm{spt}T \cap A_{1,\rho} = G_{\mathbf{C}}(u) \cap A_{1,\rho}$, with $|x|^{-1}|u| + |\nabla u| \leq \epsilon$. Moreover, u admits the Dini-type estimate

$$\int_{\mathbf{C}\cap A_{1,\rho}} |\partial_r(u/r)| r^{1-n} \leq \epsilon. \tag{5.12}$$

Proof Same as Theorem 3.1, using the Dini estimate (5.9), and Lemma 5.3 in place of Lemma 3.2. □

Lemma 5.3 *Let $\mathbf{C}^n \subset \mathbb{R}^{n+k}$ be a smooth minimal cone. Given $\epsilon, \alpha > 0$, there are $\Lambda(\mathbf{C}, \epsilon, \alpha, k)$, $\delta(\mathbf{C}, \epsilon, \alpha, k)$ positive so that the following holds.*

Let T be $(\Lambda, \alpha, 1)$-minimizing integral n-current in T with $\partial T = 0$, and suppose that

$$\mathrm{spt}T \cap A_{1,1/2} = G_{\mathbf{C}}(u) \cap A_{1,1/2}, \quad |u|_{C^1} \leq \delta, \tag{5.13}$$

$$\theta_T(0,1) \leq \theta_{\mathbf{C}}(0) + \delta, \quad \theta_T(0,1/8) \geq \theta_{\mathbf{C}}(0) - \delta. \tag{5.14}$$

Then $\mathrm{spt}T \cap A_{1,1/4} = G_{\mathbf{C}}(u) \cap A_{1,1/4}$, with $|u|_{C^1} \leq \epsilon$.

Proof Follows by a straightforward contradiction argument, using Lemma 5.6, [3] and taking $\Lambda, \delta \to 0$. □

Since the other ingredients involved (monotonicity, compactness, partial regularity) already have direct analogues for the case of almost-minimizers, most of the results in [9] carry over to almost-minimizers. For example, we have:

Theorem 5.4 (Almost-minimizers near Simons' cones) *Let $\mathbf{C}^n \subset \mathbb{R}^{n+1}$ be a minimizing quadratic hypercone, and let $\{S_\lambda\}_\lambda$ be the associated Hardt–Simon foliation (see [10] for notation). Given $\epsilon, \alpha > 0$, there is a $\delta(\mathbf{C}, \epsilon, \alpha) > 0$ so that the following holds.*

Let T be a $(\delta, \alpha, 1)$-almost-minimizing n-current in B_1 with $\partial T = 0$, and suppose that

$$d_H(\mathrm{spt}T \cap B_1, \mathbf{C} \cap B_1) \leq \delta, \quad (1/2)\theta_{\mathbf{C}}(0) \leq \theta_T(0,1/2), \quad \theta_T(0,1) \leq (3/2)\theta_{\mathbf{C}}(0). \tag{5.15}$$

Then we can find an $a \in \mathbb{R}^{n+1}$, $\lambda \in \mathbb{R}$, $q \in SO(n+1)$, satisfying

$$|a| + |q - Id| + |\lambda| \leq \epsilon, \tag{5.16}$$

and a C^1 function $u : (a + q(S_\lambda)) \cap B_{1/2}(a) \to S_\lambda^\perp$, so that

$$\mathrm{spt}T \cap B_{1/4} = \mathrm{graph}_{a+q(S_\lambda)}(u) \cap B_{1/4}, \quad |x-a|^{-1}|u| + |\nabla u| \leq \epsilon. \tag{5.17}$$

If $\mathbf{C}^n \subset \mathbb{R}^{n+1}$ is a general smooth (away from 0) area-minimizing hypercone, and one additionally assumes $\mathrm{spt}T$ lies to one side of $\mathbf{C}$, then the same conclusion holds with $a = 0$, $q = Id$. Moreover, if T is in fact mass-minimizing, then either $\mathrm{spt}T = \mathbf{C} \cap B_1$ or $\lambda \neq 0$.

Proof Same as in [9, Theorem 2.8], except with Theorem 5.2 in place of [9, Theorem 13.1], and Lemma 5.6, [3] in place of Allard's varifold compactness, regularity. If spt$T$ lies to one side of $\mathbf{C}$, then since T is assumed to be (almost-)minimizing, one can use the Liouville theorem of [14] in place of [24], and thereby assume only that $\mathbf{C}$ is minimizing. The very last statment, that spt$T = \mathbf{C}$ when T is minimizing and $\lambda = 0$, follows from (5.17) and the strong maximum principle [22]. □

Theorem 5.5 (Finite diffeotypes of almost-minimizers) *Let (N^{n+1}, g) be a closed Riemannian manifold of dimension $n + 1 \leq 8$. Given any $\Lambda, \Gamma \geq 0$, $\alpha, r > 0$, there is a constant $C(N, g, \Lambda, \alpha, r, \Gamma)$ so if T is any (Λ, α, r)-almost-minimizing n-current in (N, g), with $\partial T = 0$ and having mass $||T||(N) \leq \Gamma$, then* sing$T$ *is discrete (if $n = 7$)/ empty (if $n \leq 6$),* reg$T$ *fits into one of C diffeomorphism classes, and* spt$T$ *fits into one of C bi-Lipschitz equivalence classes.*

Proof Same as [9, Theorem 2.4]. □

The following compactness theorem for almost-minimizing currents should be standard, but we were not able to find a reference.

Lemma 5.6 *Let T_i be a sequence of (Λ, α, r_0)-almost-minimizing n-currents in $U \subset \mathbb{R}^{n+k}$. Suppose that*

$$\sup_i ||T_i||(W) + ||\partial T_i||(W) < \infty \quad \forall W \subset\subset U.$$

Then after passing to a subsequence, we can find a (Λ, α, r_0)-almost-minimizing n-current T so that $T_i \to T$ as currents and $||T_i|| \to ||T||$ as Radon measures.

Proof The proof that T is (Λ, α, r_0)-almost-minimizing is the same as the proof for T_i, T being mass-minimizing (see e.g. [21, Chapter 7, Theorem 2.4]). We highlight here how to show convergence $||T_i|| \to ||T||$. Passing to a subsequence i', we can assume that $||T_{i'}|| \to \lambda$, for some Radon measure λ. Lower-semi-continuity of mass implies $||T||(W) \leq \lambda(W)$ for all $W \subset\subset U$.

Given $x \in \text{spt}T \cap U$, then by the monotonicity formula (5.2) we have $\theta_T(B_r(x)) \geq 1/2$ for all r sufficiently small. Arguing as in [21], we have for a.e. r small the inequality

$$\lambda(B_r(x)) \leq ||T||(B_r(x)) + \Lambda r^{n+\alpha} \leq (1 + c(n)\Lambda r^{\alpha})||T||(B_r(x)),$$

which in turn implies

$$1 = \lim_{r \to 0} \frac{||T||(\overline{B_r(x)})}{\lambda(\overline{B_r(x)})}$$

for λ-a.e. x. Since $||T|| << \lambda$, the Radon-Nikodyn theorem implies $\lambda = ||T||$. □

5.7 Currents in an annulus

Lastly we comment on integral n-currents T which may only be defined in an annulus. The point of this section is that actually *no knowledge* of T is required inside a small ball to get good decay-growth estimates in an annulus, provided T still satisfies some reasonable minimization-type condition. Situations of this kind arise in obstacle minimizing problems like in [19], and indeed our exposition here gives an alternate approach [19]'s "mesoscale flatness" theory. Most of the computations and technical details in this section are very similar

to or the same as the previous section, so here we will give more a brief sketch of the main ideas and formulas.

We take T to be an integral n-currents in $A_{1,\rho} \subset \mathbb{R}^{n+k}$, for some $\rho \in (0, 1/2]$, with $\partial T = 0$ in $A_{1,\rho}$ and which are $(\Lambda, \alpha, 1)$-almost-minimizing in $A_{1,\rho}$. T need only be defined in $A_{1,\rho}$, and we shall interpret $||T||(B_r) \equiv ||T||(A_{r,\rho})$. In particular $\theta_T(0, r) \equiv \frac{||T||(A_{r,\rho})}{\omega_n r^n}$.

We shall additionally assume T satisfies the following "global" almost-minimizing property:

$$\begin{cases} ||T||(B_r) \leq ||S||(B_r) + \Lambda r^{n+\alpha} \text{ for any integral } n\text{-current } S \text{ in } B_1 \\ \text{with } \partial S = 0 \text{ in } B_1 \text{ and } \mathrm{spt}(S - T) \subset \overline{A_{r,\rho}} \end{cases} \tag{5.18}$$

The comparison principle (5.18) arises naturally in obstacle problems like in [19].

By standard slicing theory [21, Section 6.4], for a.e. $r \in (\rho, 1)$ the cone T_r over $\partial(T \llcorner A_{r,\rho})$ (taking the boundary in $A_{1,\rho}$) is an integral n-current in B_r with $\partial T_r = \partial(T \llcorner A_{r,\rho})$. By the deformation lemma [21, Section 6.5], we can find an integral n-current S_r supported in ∂B_ρ with $\partial S_r = \partial(T_r \llcorner B_\rho)$ and $||S_r||(\partial B_\rho) \leq c_0(n, k)||T_r||(B_\rho) = c_0 \rho^n r^{-n} ||T_r||(B_r)$.

From assumption (5.18) we deduce that for a.e. $r \in (M\rho, 1)$ we have

$$||T||(B_r) \leq (1 + c_0 \rho^n r^{-n})||T_r||(B_r) + \Lambda r^{n+\alpha} \leq 2||T_r||(B_r) + \Lambda r^{n+\alpha}$$

provided we ensure $M(n, k)$ is suficiently large. By repeating a similar computation to (5.2), we deduce that

$$\left[\frac{\theta_T(0, r)}{1 + c_0 \rho^n r^{-n}} + \frac{n\Lambda}{\omega_n \alpha} r^\alpha\right] - \left[\frac{\theta_T(0, s)}{1 + c_0 \rho^n s^{-n}} + \frac{n\Lambda}{\omega_n \alpha} s^\alpha\right] \geq \frac{1}{4\omega_n} \int_{A_{r,s}} \frac{|\pi_T^\perp(x)|^2}{|x|^{n+2}} d||T||(x).$$

for all $M\rho < s < r < 1$. In particular, given any $\epsilon > 0$ then provided $M(n, k, \alpha, \epsilon)^{-1}$, $\Lambda(n, k, \alpha, \epsilon)$ are sufficiently small, we have

$$\theta_T(0, M\rho) - \epsilon \leq \theta_T(0, r) \leq \theta_T(0, 1) + \epsilon \quad \forall r \in (M\rho, 1). \tag{5.19}$$

Suppose T additionally satisfies

$$T \llcorner A_{1,M\rho} = [G_{\mathbf{C}}(u) \cap A_{1,M\rho}], \quad |x|^{-1}|u| + |\nabla u| < \delta.$$

Now for every $r \in (M\rho, 1)$, we note that $||T_r||(B_r) \leq c_1(\mathbf{C}, k) r^n$ provided $\delta(\mathbf{C}, k)$ is sufficiently small, and therefore we can argue as in the previous section to deduce the monotonicity

$$\begin{aligned} \frac{d}{dr}\theta_T(0, r) &\geq \frac{n}{2r}(\theta_{T_r}(0, r) - \theta_T(0, r)) + \frac{1}{4\omega_n r} \int_\Sigma r^2 |\partial_r u_r|^2 - c_1 \rho^n r^{-n-1} - \Lambda r^{\alpha-1} \\ &\geq \frac{n}{2r}(\theta_{T_r}(0, r) - \theta_T(0, r)) + \frac{1}{4\omega_n r} \int_\Sigma r^2 |\partial_r u_r|^2 - c_1 M^{\alpha-n} \rho^\alpha r^{-\alpha-1} - \Lambda r^{\alpha-1} \end{aligned}$$

and epiperimetric inequality

$$\theta_T(0, r) \leq \theta_{T_r}(0, r) - \epsilon |\theta_{T_r}(0, r) - \theta_{\mathbf{C}}(0)|^{1+\gamma} + c_1 M^{\alpha-n} \rho^\alpha r^{-\alpha} + \Lambda r^\alpha.$$

Noting that $c_1 M^{\alpha-n} \rho^\alpha r^{-\alpha-1} \leq c_1 M^{-n}$, then provided $M^{-1}(\mathbf{C}, \alpha, k)$, $\Lambda(\mathbf{C}, \alpha, k)$ are sufficiently small and $\theta_T(0, 1) \leq \theta_{\mathbf{C}}(0) + 1/2$, $\theta_T(0, M\rho) \geq \theta_{\mathbf{C}}(0) - 1/2$, we can prove the analogue of Theorem 2.2 with $\partial B_1 \equiv \Sigma$, $\mathcal{E}(u_r) \equiv \theta_{T,\rho}(0, r) - \theta_{\mathbf{C}}(0)$, $\mathcal{E}(z_r) \equiv \theta_{T_r,\rho}(0, r) - \theta_{\mathbf{C}}(0)$, $\Lambda_+ = \Lambda$, $\Lambda_- = c_1 M^{\alpha-n} \rho^\alpha$, to deduce that the function

$$G(r) = \theta_T(0, r) - \theta_{\mathbf{C}}(0) + 3\alpha^{-1} \Lambda r^\alpha - 3\alpha^{-1} c_1 M^{\alpha-n} \rho^\alpha r^{-\alpha}$$

satisfies the ODE $G' \geq (\delta'/r)|G|^{1+\gamma}$ as in (2.4), and to deduce the Dini estimate

$$\begin{aligned}\int_{\mathbf{C}\cap A_{1,M\rho}} |\partial_r(u/r)|r^{1-n} \leq c(\mathbf{C},k,\alpha)(&|\theta_{T,\rho}(0,M\rho)-\theta_{\mathbf{C}}(0)|^{(1-\gamma)/2}\\ &+|\theta_{T,\rho}(0,1)-\theta_{\mathbf{C}}(0)|^{(1-\gamma)/2})\\ &+c(\mathbf{C},k,\alpha)(\Lambda^{(1-\gamma)/2}+M^{-n(1-\gamma)/2}).\end{aligned} \tag{5.20}$$

We obtain the following variant of Theorem 5.2. Theorem 5.8 below is a "minimizing" version of [9, Theorem 6.3], and gives an alternate approach to the "mesoscale flatness" of [19, Theorem 1.9].

Theorem 5.8 *Let $\mathbf{C}^n \subset \mathbb{R}^{n+k}$ be a smooth minimal cone. Given $\epsilon, \alpha > 0$ there is a $\delta(\mathbf{C}, k, \epsilon, \alpha)$ positive so that the following holds. Let $\rho \geq 0$, and let T be $(\delta, \alpha, 1)$-almost-minimizing in $A_{1,\delta\rho}$ with $\partial T = 0$, and satisfying the global comparison property (5.18), and for which*

$$\begin{aligned} T\llcorner A_{1,1/2} &= [G_{\mathbf{C}}(u)\cap A_{1,1/2}], \quad |u|_{C^1} \leq \delta\\ \theta_T(0,1) &\leq \theta_{\mathbf{C}}(0)+\delta, \quad \theta_T(0,\rho/2) \geq \theta_{\mathbf{C}}(0)-\delta.\end{aligned}$$

Then $T\llcorner A_{1,\rho} = [G_{\mathbf{C}}(u)\cap A_{1,\rho}]$ with $|x|^{-1}|u| + |\nabla u| \leq \epsilon$. Morevoer, u admits the Dini estimate

$$\int_{\mathbf{C}\cap A_{1,\rho}} |\partial_r(u/r)|r^{1-n} \leq \epsilon.$$

Proof Same as Theorem 5.2, using Lemma 5.9, (5.19), (5.20). □

Lemma 5.9 *Let $\mathbf{C}^n \subset \mathbb{R}^{n+k}$ be a smooth minimal cone. Given $\epsilon, \alpha > 0$, there is a $\delta(\mathbf{C}, \epsilon, \alpha, k)$ positive so that the following holds. Let T be $(\delta, \alpha, 1)$-almost-minimizing integral n-current in $A_{1,\delta}$ with $\partial T = 0$, and suppose that*

$$\begin{aligned} T\llcorner A_{1,1/2} &= [G_{\mathbf{C}}(u)\cap A_{1,1/2}], \quad |u|_{C^1} \leq \delta,\\ \theta_T(0,1) &\leq \theta_{\mathbf{C}}(0)+\delta, \quad \theta_T(0,1/8) \geq \theta_{\mathbf{C}}(0)-\delta.\end{aligned}$$

Then

$$T\llcorner A_{1,1/4} = [G_{\mathbf{C}}(u)\cap A_{1,1/4}], \quad |u|_{C^1} \leq \epsilon.$$

Proof Same as Lemma 5.3. □

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