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ORIGINAL PAPER

A cyclotomic family of thin hypergeometric monodromy groups in $\mathbf{Sp}_4(\mathbb{R})$

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Abstract

We exhibit an infinite family of discrete subgroups of $\mathbf{Sp}_4(\mathbb{R})$ which have a number of remarkable properties. Our results are established by showing that each group plays ping-pong on an appropriate set of cones. The groups arise as the monodromy of hypergeometric differential equations with parameters $\left(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N}\right)$ at infinity and maximal unipotent monodromy at zero, for any integer $N \geq 4$. Additionally, we relate the cones used for ping-pong in $\mathbb{R}^4$ with crooked surfaces, which we then use to exhibit domains of discontinuity for the monodromy groups in the Lagrangian Grassmannian. These domains of discontinuity lead to uniformizations of variations of Hodge structure with Hodge numbers (1, 1, 1, 1).

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Contents

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1 Introduction

The monodromy of hypergeometric differential equations has been actively studied for a long time. A historical overview going back to the nineteenth century can be found in the book of Gray [20] and the more recent developments relevant to the present text started with the work of Beukers–Heckman [3] who analyzed the basic features of the monodromy groups of hypergeometric equations on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. In particular, they described the Zariski closure of the discrete groups which arise.

A more refined question about the monodromy group concerns its relation to the ambient arithmetic lattice (see the discussion after Theorem 1.1 for a description of the arithmetic lattice). The most interesting case is when the monodromy representation identifies the (orbifold) fundamental group of the base with the corresponding arithmetic lattice, and this leads to uniformization of algebraic manifolds by domains. A representative example is the congruence subgroup $\Gamma(2)$ of $\mathbf{SL}_2(\mathbb{Z})$. In a different direction, the representation can surject (with large kernel) onto a finite index subgroup of the arithmetic group. Results of this nature have been obtained recently by Singh–Venkataramana [29] (see also Detinko–Flannery–Hulpke [7]). Finally, the image of the representation[1] can be an infinite index subgroup of the lattice, which is called "thin". This is the case of interest to us.

Specifically, we exhibit an infinite family of hypergeometric parameters with monodromy a discrete subgroup in $\mathbf{Sp}_4(\mathbb{R})$. Furthermore, we explicitly describe domains of discontinuity for these groups in the Lagrangian Grassmannian of $\mathbb{R}^4$. These domains and associated groups lead to new, non-classical, uniformization results for variations of Hodge structure with Hodge numbers (1, 1, 1, 1), see [15]. Note that as our ambient dimension is fixed at 4, the matrix entries of the groups will necessarily lie in number fields of increasing size. This family of parameters initially emerged from numerical experiments on Lyapunov exponents of hypergeometric differential equations [18].

Let us note that previously, an infinite family of thin monodromy groups has been obtained by Fuchs, Meiri, and Sarnak [17] inside the indefinite orthogonal group $\mathbf{SO}_{1,n}(\mathbb{R})$, finitely many for each n, and arbitrarily large n. By different methods, Brav and Thomas [5] exhibited 7 parameters for which the monodromy group is thin in $\mathbf{Sp}_4(\mathbb{Z}) \subset \mathbf{Sp}_4(\mathbb{R})$. Our methods are closely related to those of Brav and Thomas and, we hope, shed some light on the underlying symplectic geometry used to construct the ping-pong cones.

[1] So far, all known thin examples appear also to be (essentially) injective.

Parameters for hypergeometric equations

We will consider rank 4 hypergeometric groups with maximal unipotent monodromy at zero (see Sect. 2 for more background on hypergeometric equations). This leads to the differential operator

$$D^4 - z(D+\alpha_1)(D+\alpha_2)(D+\alpha_3)(D+\alpha_4) \quad D = z\partial_z$$

which has regular singular points at $0, 1, \infty \in \mathbb{P}^1(\mathbb{C})$. The α-parameters that we consider are $\left(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N}\right)$ for $N \geq 4$. We let $\Gamma_N \subset \mathbf{Sp}_4(\mathbb{R})$ be the monodromy group of the equation, and $P\Gamma_N \subset \mathbf{PSp}_4(\mathbb{R})$ its image in the projectivized symplectic group. Let R be the monodromy at infinity and T the monodromy around 1.

Theorem 1.1 (Discrete monodromy) *The projective monodromy group is isomorphic, as an abstract group, to:*

$$P\Gamma_N \cong \left\langle R, T \mid R^N = 1 \right\rangle$$

Furthermore, it is a discrete subgroup of $\mathbf{PSp}_4(\mathbb{R})$.

The full monodromy group Γ_N *is isomorphic to* $P\Gamma_N$ *if* N *is odd, and is a* $\mathbb{Z}/2$ *central extension if* N *is even. It is also a discrete subgroup of* $\mathbf{Sp}_4(\mathbb{R})$.

Furthermore, denote by Y_N *the orbifold* $\mathbb{P}^1 \setminus \{0, 1\}$ *with an orbifold point of order* N *at infinity. Then the monodromy representation is an isomorphism of* $\pi_1^{orb}(Y_N)$ *and* $P\Gamma_N$.

Proof In Sect. 2.2 we construct a group generated by three reflections $\tilde{\Gamma}_N$ that contains Γ_N with index 2. The reflections are denoted A, B, C and Γ_N is mapped to $\tilde{\Gamma}_N$ via $R \mapsto BC$ and $T \mapsto AB$. In Theorem 3.2.5 we show that $\tilde{\Gamma}_N$ acts on a set of $2N$ cones in $\mathbb{P}(\mathbb{R}^4)$ in a (generalized) ping-pong manner. It follows that for its image in the projective group we have:

$$P\tilde{\Gamma}_N \cong \left\langle A, B, C \mid A^2 = B^2 = C^2 = 1, (BC)^N = 1 \right\rangle.$$

Then $\tilde{\Gamma}_N$ is either a $\mathbb{Z}/2$-extension of $P\tilde{\Gamma}_N$ or isomorphic to it, according to whether $R^N = -\mathbf{1}$ (N is even) or $R^N = \mathbf{1}$ (N is odd).

Discreteness follows from Theorem 1.2 below which shows that the groups have nonempty domains of discontinuity in the Lagrangian Grassmannian. □

Thinness

According to the customary definition, see e.g. the survey of Sarnak [28], a thin group is one which is of infinite index in an arithmetic lattice. Let us explain how to describe the arithmetic lattice in our context.

Fix a primitive $2N$-th root of unity $\zeta_{2N} \in \mathbb{C}$ (for example, $e^{\pi\sqrt{-1}/N}$ with the standard choice of $\sqrt{-1}$) and let $\mathbb{Q}(\zeta_{2N}) \subset \mathbb{C}$ be the corresponding extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_{2N}$. The matrices in Γ_N have entries in $\mathcal{O}_{2N}$, since the two generators can be taken to be the companion matrices associated to the polynomials with roots $\{e^{2\pi\sqrt{-1}\alpha_i}\}_i$ and $\{e^{2\pi\sqrt{-1}\beta_j}\}_j$ respectively, see e.g. [3, Thm. 3.5] for the explicit formula. The group Γ_N preserves a symplectic form with coefficients in the same ring (see [3, Thm. 4.3]). Note however that it is the roots of the characteristic polynomial that belong to $\mathcal{O}_{2N}$, so the coefficients of the characteristic polynomial and hence of the matrices lie in a smaller ring that we now describe.

Recall that the group of multiplicatively invertible residue classes $\mathrm{Gal}_N := (\mathbb{Z}/2N)^\times$ is naturally the Galois group of $\mathbb{Q}(\zeta_{2N})$ over $\mathbb{Q}$. With our choice of ζ_{2N}, we can identify the $(2N)$-th roots of 1 with rationals in $[0, 1)$ with denominator $2N$ by sending ζ_{2N} to $\frac{1}{2N}$. With this identification, the Galois action of Gal_N on $(2N)$-th roots of 1 corresponds to the multiplicative action on rationals in $[0, 1)$ mod $\mathbb{Z}$.

Then a subgroup $S_N \subseteq \mathrm{Gal}_N$ will stabilize (as a set, but not necessarily element-wise) our given four-tuple $\alpha_\bullet^{(0)} = \left(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N}\right)$. We also have the orbit of our four-tuple under this multiplicative action, with representatives (all mod 1) $\alpha_\bullet^{(i)}$, say a total of k distinct representatives. Note that the orbit can be identified with Gal_N / S_N and $k = \#\, \mathrm{Gal}_N / S_N$, and the Galois group of K_N over $\mathbb{Q}$ is Gal_N / S_N.

This defines a subfield $K_N \subset \mathbb{Q}(\zeta_{2N})^+$ such that the elements of K_N are fixed by S_N (note that $-1 \in (\mathbb{Z}/2N)^\times$ corresponds to complex conjugation and stabilizes our four-tuple $\alpha_\bullet^{(0)}$ so the subfield K_N is totally real). If we denote by $\mathcal{O}_{K_N}$ the ring of integers in K_N then our monodromy group Γ_N has entries in $\mathbf{Sp}_4(\mathcal{O}_{K_N})$ since the coefficients of the characteristic polynomials of the generators of Γ_N are stabilized by the group S_N.

Now the group $\mathbf{Sp}_4(\mathcal{O}_{K_N})$ is an arithmetic lattice in $\mathbf{Sp}_4(\mathbb{R})^k$ (a product of k copies of $\mathbf{Sp}_4(\mathbb{R})$) where $k = \mathrm{Gal}_N / S_N$ counts the distinct embeddings of K_N into $\mathbb{R}$ arising from the Galois action. The projection of $\Gamma_N \hookrightarrow \mathbf{Sp}_4(\mathcal{O}_{K_N}) \hookrightarrow \mathbf{Sp}_4(\mathbb{R})^k$ to any of the $\mathbf{Sp}_4(\mathbb{R})$-factors yields the Galois-conjugate local systems of our original one, corresponding to the distinct parameters $\alpha_\bullet^{(i)}$. The subgroup Γ_N is thin in $\mathbf{Sp}_4(\mathcal{O}_{K_N})$ since Γ_N is virtually free, and a higher rank arithmetic lattice cannot be virtually free for many reasons (a higher rank lattice has property T, satisfies Margulis super-rigidity, or for an argument based on cohomological dimension see [5, bottom of p. 334]).

The monodromy group Γ_N is visibly discrete in the product $\mathbf{Sp}_4(\mathbb{R})^k$ since it is contained in the discrete lattice there. But Theorem 1.1 implies that Γ_N is in fact discrete when projected to the factor corresponding to $\alpha_\bullet^{(0)}$. The situation is analogous to the classical constructions of Deligne–Mostow [9] of non-arithmetic lattices in $\mathbf{SU}(1, n)$, with the difference that our group $\mathbf{Sp}_4(\mathbb{R})$ is of higher rank.

Note that the Galois-conjugate monodromy representations yield groups which are abstractly isomorphic to the original one. However, the discreteness part of Theorem 1.1 has no reason to extend to the Galois-conjugate local systems (and we suspect it does not hold in general).

This situation is analogous to one arising in Teichmüller dynamics. Namely, Veech groups are contained in Hilbert modular groups, which themselves are arithmetic lattices in products of $\mathbf{SL}_2(\mathbb{R})$. However, a Veech group is thin inside a Hilbert modular group since the Veech group projects discretely to one of the $\mathbf{SL}_2(\mathbb{R})$-factors. The relation between these groups and Hodge theory was investigated by Möller [25], and in higher rank in Teichmüller dynamics in [13]. See also McMullen's recent investigation [26] in this direction, and Zorich's survey [32], or [16] for further background in Teichmüller dynamics.

Let us finally remark that the $\mathbb{R}$-Zariski density of Γ_N inside $\mathbf{Sp}_4(\mathbb{R})^k$ follows from the combination of the results of Beukers–Heckman [3], which establish Zariski density in each factor separately, and Goursat's lemma in group theory (combined with the fact that $\mathbf{Sp}_4$ is simple).

log-Anosov property and further consequences

Our method of proof, namely giving cones on which the group plays ping-pong, has many further consequences which are developed in [15]. Namely, the monodromy groups Γ_N are examples of Anosov representations introduced by Labourie [24] (see also [21] for further developments of the notion, as well as [19, 23]), except that the definition needs to be adapted in order to allow for unipotents. Such an extension has been provided by Kapovich and Leeb [22], Zhu [31], and in [15].

Let G_N be the reflection group generated by the triangle in hyperbolic space with a point of angle π/N in $\mathbb{H}^2$ and two points at infinity. The content of Theorem 1.1 is that G_N is mapped isomorphically onto the subgroup $P\tilde{\Gamma}_N \subset \mathbf{PSp}_4(\mathbb{R})$. In fact, the method of proof also implies:

(i) There exists a G_N-equivariant continuous (Hölder) map

$$\xi : \partial\mathbb{H}^2 \to \mathbb{P}(\mathbb{R}^4)$$

This is illustrated in Fig. 1 below.

(ii) There exists a nonempty open set $\Omega \subset \mathrm{LGr}(\mathbb{R}^4)$ in the Lagrangian Grassmannian on which $P\tilde{\Gamma}_N$ acts properly discontinuously.

(iii) The conjectured formula for the sum of Lyapunov exponents from [11] holds.

(iv) The group $P\tilde{\Gamma}_N$ acts properly discontinuously on $\mathrm{LGr}^{1,1}(\mathbb{C}^4)$, the Grassmannian of Lagrangians on which the indefinite hermitian pairing of signature $(2, 2)$ restricts to signature $(1, 1)$.

The third point on Lyapunov exponents was exactly the property observed numerically and conjectured in [18] which motivated the study of this family of parameters. Note that in weight 2, for variations of Hodge structure of K3 type, the formula for the sum of Lyapunov exponents was established in [14].

The reader familiar with Hodge theory will recognize that $\mathrm{LGr}^{1,1}(\mathbb{C}^4)$ is the target of a forgetful map from the Griffiths period domain of Hodge structures with Hodge numbers $(1, 1, 1, 1)$, given by forgetting the first term of the Hodge filtration. This is in contrast to Siegel space, which consists of Lagrangians for which the restricted hermitian pairing has signature $(2, 0)$. The action of a discrete group in $\mathbf{Sp}_4(\mathbb{R})$ has no apriori reason to act properly on $\mathrm{LGr}^{1,1}(\mathbb{C}^4)$, even though it always acts properly on Siegel space. It is also established in [15] that the quotient of the domain of discontinuity $\Omega \subset \mathrm{LGr}(\mathbb{R}^4)$ by $P\tilde{\Gamma}_N$ can be identified with a circle bundle over the orbifold $\mathbb{H}^2/G_N$, constructed using Hodge theory.

Crooked surfaces

Drumm [10] introduced crooked surfaces to construct fundamental domains for discrete groups acting on Minkowski space. These have found further applications in Lorenzian geometry, see e.g. [8] for some recent applications. Let us note that in contrast to hyperbolic or euclidean spaces, where totally geodesic hyperplanes are natural and effective tools for constructing fundamental domains of group actions, in higher rank situations such obvious choices are not available. Crooked surfaces have been effective in constructing fundamental domains in 3-dimensional homogeneous spaces with Lorentz metrics (or conformal classes thereof).

In Sect. 5 we connect the cones that are used to prove Theorem 1.1 to crooked surfaces. As it turns out, many properties of crooked surfaces can be conveniently expressed using

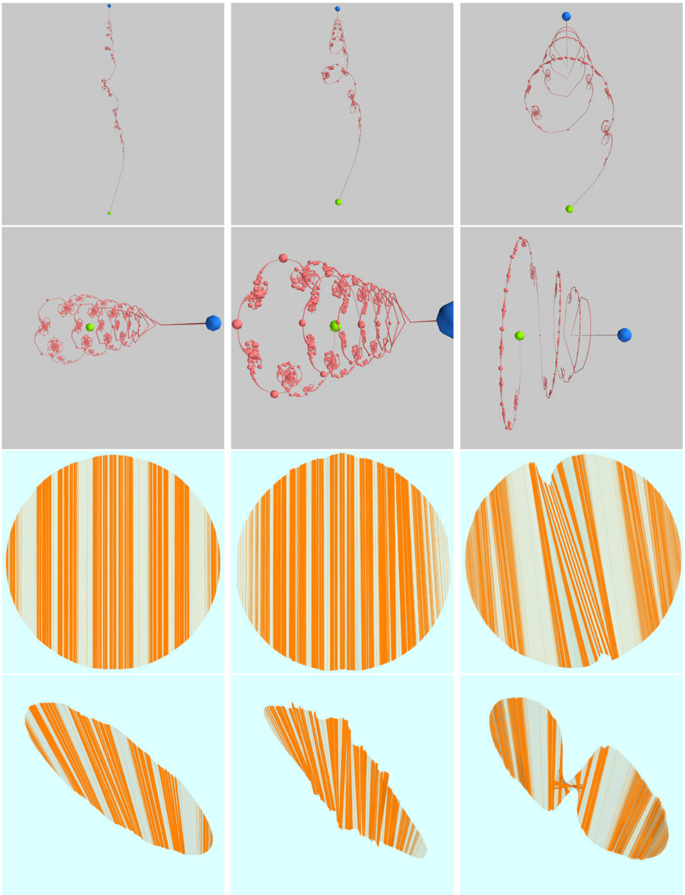

Fig. 1 Limit curve in projective space and associated limit set in the Lagrangian Grassmanian

cones, via the dictionary relating symplectic geometry in $\mathbb{R}^4$ to the causal geometry of the projectivized null vectors in $\mathbb{R}^{2,3}$, the latter being just the Lagrangian Grassmannian of $\mathbb{R}^4$. Most importantly for us, the criteria establishing disjointness of crooked surfaces developed in [2] are concisely expressed by the containment of cones established during the ping-pong argument. We need to further extend their criterion to allow the crooked surfaces to touch, see Sect. 5.2 for details. In particular, we can explicitly analyze the action of our groups on $\mathrm{LGr}(\mathbb{R}^4)$ and obtain:

Theorem 1.2 (Domain of discontinuity)
For each $N \geq 4$ there exists a nonempty open set $\Omega_N \subset \mathrm{LGr}(\mathbb{R}^4)$ on which Γ_N acts properly discontinuously.

See Theorem 5.3.9 and the discussion preceding it.

On Fig. 1

We include some numerical simulations demonstrating the limit curves and surfaces. The limit surface is the complement of the domain of discontinuity Ω_N from Theorem 1.2. The parameter N is fixed in each column and is equal to $N = 4, 5$ and 11 respectively. Each row gives a view of the limit curve in $\mathbb{P}(\mathbb{R}^4)$, and limit surface in $\mathrm{LGr}(\mathbb{R}^4)$, from roughly the same position. Only the part of the limit curve between the points stabilized by the maximally unipotent matrix, resp. rank 1 unipotent matrix, is displayed. The limit surface is intersected with a Euclidean sphere and displayed in a chart of $\mathrm{LGr}(\mathbb{R}^4)$ which is conformally equivalent to Minkowski space $\mathbb{R}^{2,1}$ (the chart is given as the complement of the nullcone of one Lagrangian).

Numerical experiments

Our work started from observations on the numerical behavior of the action of monodromy groups on the cones that can be accessed at https://gitlab.com/fougeroc/ping-pong. We have also used symbolic computation tools from SageMath [27] and our final worksheet can be found at https://gitlab.com/fougeroc/notebook-cyclotomic-family. This can be handy, but not logically necessary, for the reader who wants to follow our computations.

Related work, and generalizations

After a first version of this text was released, we learned from Fanny Kassel that together with Jean-Philippe Burelle, they have a forthcoming paper [4] which contains and generalizes some of the contents of our Sect. 5 on crooked surfaces. Specifically, the interpretation of crooked surfaces in the Einstein universe $\mathrm{Ein}^{1,2} \simeq \mathrm{LGr}(\mathbb{R}^4)$ in terms of projective simplices in $\mathbb{P}(\mathbb{R}^4)$, as well as the simpler interpretation of the disjointness criterion from [2], is also contained in their work and was known to them in 2018. Their work also contains an extension to higher dimensions. We arrived independently and unaware of their work at the results of Sect. 5.

2 Background on hypergeometric groups

Outline of section In Sect. 2.1 we recall some basic definitions regarding hypergeometric differential equations and their monodromy. Next, in Sect. 2.2 we define a $\mathbb{Z}/2$ extension of the monodromy group that is generated by reflections.

2.1 Notation

We recall here some standard facts on hypergeometric groups. See [3] or [30] for further background.

2.1.1 Setup

Fix two n-tuples of reals $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ subject to the normalizations $\alpha_i \in [0, 1)$ and $\beta_i \in (0, 1]$. Note that most classical normalizations, which involve explicit hypergeometric

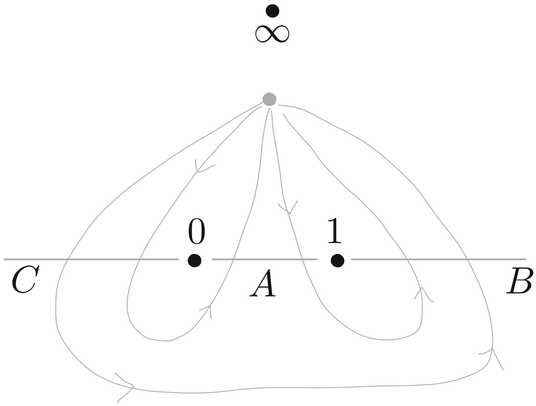

Fig. 2 The paths along which we parallel-transport solutions

functions, take $\beta_n = 1$, and in our case we will take $\beta_i = 1\ \forall i$ to ensure maximal unipotent monodromy at 0 (another popular normalization and different expressions for the differential operators are related to ours by $\beta_i \mapsto 1 - \beta_i$). Let also $a_i := \exp(2\pi\sqrt{-1}\alpha_i)$ and $b_i = \exp(2\pi\sqrt{-1}\beta_i)$ be (unit) complex numbers.

2.1.2 Differential operator and monodromy

Consider the differential operator

$$D_{\alpha,\beta} := \prod_{i=1}^{n}(D + \beta_i - 1) - z\prod_{i=1}^{n}(D + \alpha_i) \quad D := z\partial_z$$

In $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ its solutions form a local system $\mathbb{V}(\alpha, \beta)$ of rank n, which we will call the hypergeometric local system. Let g_0, g_1, g_∞ be the monodromy matrices of this local system, along paths as described in Fig. 2. Then their conjugacy classes are determined by the following conditions on the characteristic polynomials and ranks:

$$\det(t - g_\infty) = \prod_i (t - a_i)$$
$$\det(t - g_0^{-1}) = \prod_i (t - b_i)$$
$$\mathrm{rk}(g_1 - \mathbf{1}) = 1 \quad \det(g_1) = \exp\left(2\pi\sqrt{-1}\sum(\beta_i - \alpha_i)\right)$$

with the convention that whenever there are repeated roots, there is only one Jordan block.

2.1.3 Rigidity of the local system

Assuming that $\alpha_i - \beta_j \notin \mathbb{Z}$ for any i, j, the local system $\mathbb{V}(\alpha, \beta)$ is irreducible. Furthermore, any local system on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ which has the same conjugacy classes of monodromy matrices around the missing points is isomorphic to the hypergeometric local system. In particular, to verify that an explicit representation of the free group on two letters yields a hypergeometric local system, it suffices to consider the corresponding conjugacy classes of the monodromy matrices around the removed points.

2.1.4 Thin cyclotomic parameters

We will consider the family of hypergeometric groups with parameters

$$\beta_\bullet : \quad (1, 1, 1, 1)$$
$$\alpha_\bullet : \quad \left(\frac{N-3}{2N}, \frac{N-1}{2N}, \frac{N+1}{2N}, \frac{N+3}{2N}\right) \quad N \geq 3$$

Note that we have the linear equation $-\alpha_1 + 3\alpha_2 = 1$. When working with rotation matrices, we will make use of the parameters:

$$\begin{aligned} \mu_1 &= 2\pi\alpha_1 = \frac{N-3}{N}\pi \\ \mu_2 &= 2\pi\alpha_2 = \frac{N-1}{N}\pi \end{aligned} \tag{2.1.5}$$

2.2 Reflection structure

As stated in Sect. 2.1.3, in order to verify that a certain representation is the monodromy of a hypergeometric group, it suffices to consider the conjugacy classes of the corresponding matrices. In this section, we will enlarge (with index 2) our hypergeometric groups to groups generated by reflections. This structure arises because our parameters are real, hence the hypergeometric differential equation has a complex conjugation symmetry and its solutions can be Schwarz-reflected across the real axis.

2.2.1 Abbreviations

To keep formula sizes manageable, we will use the following abbreviations:

$$\begin{aligned} c_1 &:= \cos(\mu_1)\, s_1 := \sin(\mu_1) \\ c_2 &:= \cos(\mu_2)\, s_2 := \sin(\mu_2) \end{aligned} \tag{2.2.2}$$

where the parameters μ_1, μ_2 are introduced in Eq. (2.1.5). Their specific numerical values will not be relevant until we reach the calculations with rotated vectors in Sect. 4.

It will also be convenient to introduce the shorthands:

$$r_1 := \frac{2(c_1-1)^2}{s_1(c_1-c_2)} \qquad r_2 := \frac{2(c_2-1)^2}{s_2(c_2-c_1)} \tag{2.2.3}$$

2.2.4 The reflection matrices

With these preparations, define:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -r_1 & 1 & -r_1 & 0 \\ 0 & 0 & -1 & 0 \\ -r_2 & 0 & -r_2 & 1 \end{bmatrix} B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} C = \begin{bmatrix} -c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & -c_2 & s_2 \\ 0 & 0 & s_2 & c_2 \end{bmatrix}$$

Define also a symplectic pairing on $\mathbb{R}^4$ by the following matrix:

$$J = \begin{bmatrix} 0 & r_2 & 0 & 0 \\ -r_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_1 \\ 0 & 0 & -r_1 & 0 \end{bmatrix} \tag{2.2.5}$$

2.2.6 Properties of the reflection matrices

For $M \in \{A, B, C\}$, we have that

$$M^t J M = -J$$

or in other words, the above matrices satisfy $\langle Mv, Mw\rangle = -\langle v, w\rangle$ for vectors $v, w \in \mathbb{R}^4$ and $\langle v, w\rangle := v^t Jw$ the symplectic pairing. It also follows from the formulas that

$$A^2 = B^2 = C^2 = 1.$$

Let us verify that if we define the monodromy matrices of a local system using the above reflections, as described in Fig. 2, we obtain a hypergeometric group with parameters specified in Sect. 2.1.4.

It is immediate that the matrix BC is block-diagonal consisting of rotation matrices by angles μ_1, μ_2, so the conjugacy class at infinity is correct. It is immediate also that the matrix BA of monodromy around 1 is such that $BA - \mathbf{1} = B(A - B)$ is of rank 1. The only necessary calculation is that CA is a maximally unipotent matrix.

One could check it by a tedious and explicit calculation from the above formulas. A shortcut in computations is to use the vectors generating the cone $\mathcal{C}_0$ defined by Eq. (3.4.6), see also Eq. (3.2.3) for which vectors are v_i. Then two readily verified properties yield the result. First, one checks that each column vector is an eigenvector of A, with eigenvalue $(-1)^i$ for v_i. Next, one verifies that C satisfies $Cv_i = (-1)^i v_i + \sum_{j>i} c_i^j v_j$, i.e. C respects the filtration induced by the vectors v_i. It then follows that CA is a maximally unipotent matrix preserving the filtration induced by the cone vectors.

3 Cones and ping-pong

Outline of section We describe in Sect. 3.1 the hyperbolic triangle reflection groups that give the fundamental group of the orbifold which is the basis for our analysis. Next, in Sect. 3.2 we describe the abstract properties of the cones that are used for the ping-pong argument. Based on these abstract properties we explain in Sect. 3.3 how to reduce the proof of the ping-pong property to certain explicit calculations. Finally in Sect. 3.4 we give the explicit formula for the cone and verify that it has the properties that we used. This reduces the calculations to an explicit analysis in Sect. 4.

3.1 Triangle reflection groups

3.1.1 Setup

Fix an integer $N \geq 4$. We will be interested in the group given by the generators and relations:

$$G_N := \left\langle a, b, c \mid a^2 = b^2 = c^2 = 1, (bc)^N = 1 \right\rangle$$

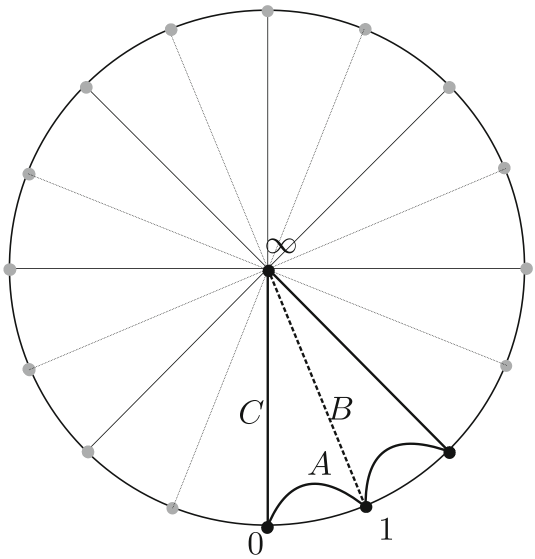

Fig. 3 Fundamental domain for the triangle group action

It is transparent that it acts on the hyperbolic plane such that a, b, c are reflections in geodesic sides of a hyperbolic triangle with two ideal vertices and one vertex of angle $\frac{\pi}{N}$; the sides for b and c form an angle of $\frac{\pi}{N}$ and the side for a connects the two points at infinity. An illustration in the disc model is provided in Fig. 3.

3.1.2 Linear and projective representation

Recall that our basic angles from Eq. (2.1.5) are:

$$(\mu_1, \mu_2) = \left(2\pi \frac{N-3}{2N}, 2\pi \frac{N-1}{2N}\right).$$

and that we defined the matrices A, B, C in Sect. 2.2.4. Define the group generated by them:

$$\tilde{\Gamma}_N := \langle A, B, C\rangle \subset \mathbf{GSp}_4(\mathbb{R})$$

They are in the general symplectic group, i.e. $\langle gv, gw\rangle = \chi(g)\langle v, w\rangle$ for a character $\chi : \mathbf{GSp}_4 \to \mathbb{G}_m$ and where $\langle -, -\rangle$ is the symplectic pairing.

Let $R := BC$ be the block rotation matrix by the corresponding angles μ_i. We then have the following basic dichotomy:

N is odd Then the order of R is N and $-\mathbf{1}$ is not in Γ_N.
N is even Then the order of R is $2N$ and $-\mathbf{1}$ is in Γ_N. Specifically $R^N = -\mathbf{1}$.

Indeed, we have that $gcd(N-1, 2N) = gcd(N-1, 2)$ which is 2 or 1, according to whether N is odd or even.

In both cases we will consider only the projective action of Γ_N, so let:

$$P\tilde{\Gamma}_N := \text{image of } \tilde{\Gamma}_N \subset \mathbf{PGSp}_4(\mathbb{R}).$$

It follows that independently of the parity of N we have the representation

$$\begin{aligned} G_N &\xrightarrow{P} \tilde{\Gamma}_N \curvearrowright \mathbb{P}(\mathbb{R}^4) \\ \{a, b, c\} &\mapsto \{A, B, C\} \end{aligned} \tag{3.1.4}$$

which will be our basic object of study.

3.2 The cones

The action of the reflections b, c on the boundary of hyperbolic space divides it into $2N$ circular arcs. We will associate a projective cone in $\mathbb{P}(\mathbb{R}^4)$ to one of the arcs, and propagate it to the remaining $2N-1$ arcs using the action of the matrices B, C.

3.2.1 Cones and projective cones

To specify a cone in $\mathbb{R}^4$ one can either give the vectors spanning it, or specify the equations of its faces. This, in particular, gives a duality between cones in a vector space and cones in its dual. Our cones will turn out to be self-dual when the vector space is identified with its dual via the symplectic pairing. Additional, our cones will be simplicial, i.e. have four faces and four extreme rays.

We will describe a (simplicial) cone $\mathcal{C}$ by specifying its four spanning vectors, and write $\mathcal{C} = [v_0|v_1|v_2|v_3]$ where v_i are column vectors in $\mathbb{R}^4$. So elements of $\mathcal{C}$ are of the form $\sum a_i v_i$ where $a_i \geq 0$. For the ping-pong argument we will consider the image of the cones in $\mathbb{P}(\mathbb{R}^4)$, but for calculations we will distinguish between a cone $\mathcal{C}$ and its negative $-\mathcal{C}$, spanned by $-v_i$. Given a cone $\mathcal{C}$ in $\mathbb{R}^4$ we denote by $\mathbb{P}\mathcal{C}$ its image in $\mathbb{P}(\mathbb{R}^4)$.

3.2.2 The ping-pong cones

Let us postpone the explicit definition of the cone vectors until Sect. 3.4 but use the following notation to describe some important properties. We start with a cone of the form

$$\mathcal{C}_0 := [v_0 \mid v_1 \mid v_2 \mid v_3] \tag{3.2.3}$$

The vectors $v_i \in \mathbb{R}^4$ will be eigenvectors with eigenvalues ± 1 for A, which will make it convenient later on to check that A maps other cones to the interior of $\mathcal{C}_0$ (see Sect. 3.3.7). Note that with this condition, the transformation A almost determines the cone $\mathcal{C}_0$, since its vertices are eigenvectors of A and the (± 1)-eigenspaces of A are two-dimensional.

Additionally, the vectors v_i will have the following properties:

(i) The vectors v_0, v_2 are fixed by B.
(ii) The vector v_3 satisfies $Cv_3 = -v_3$.

In particular $Bv_3 = B(-Cv_3) = (-R)v_3$.

We can then define the adjacent cone by reflection in B:

$$\mathcal{C}_0' := B\mathcal{C}_0 = [v_0 \mid Bv_1 \mid v_2 \mid (-R)v_3] \tag{3.2.4}$$

All the other cones are obtained by applying the rotation matrix to these basic cones, specifically:

$$\mathcal{C}_k := R^k \mathcal{C}_0 \quad \mathcal{C}_k' := R^k \mathcal{C}_0' \quad k = 1, \ldots, N-1.$$

Our main result then states:

Theorem 3.2.5 (Ping-pong property of cones) *Consider the projective cones* $\mathbb{P}\mathcal{C}_k, \mathbb{P}\mathcal{C}'_k$ *for* $k = 0, \ldots, N-1$ *defined above.*

(i) *The interiors of distinct cones are disjoint.*
(ii) *For any of the cones* $\mathbb{P}\mathcal{C}$ *except* $\mathcal{C}_0$*, we have that* $A \cdot \mathbb{P}\mathcal{C} \subset \mathbb{P}\mathcal{C}_0$ *where* A *is the matrix from Sect. 2.2.4.*

Therefore we have the isomorphism of groups:

$$P\tilde{\Gamma}_N \cong \left\langle a, b, c \mid a^2 = b^2 = c^2 = 1, (bc)^N = 1 \right\rangle.$$

3.2.6 Finding the cones

Our search for the ping-pong cone from Eq. (3.2.3) was guided by the existence of the limit curve displayed in Fig. 1, which we obtained during numerical simulations. Because the monodromy group has unipotent elements, and in particular a maximal unipotent, this heavily constraints the possible cones: some of the vertices of the cone must be given by fixed vectors of the unipotents. The remaining vectors were found using symmetry considerations and by considering the inequalities that their entries must satisfy.

3.3 Proof of the ping-pong property

In this section we reduce the proof of Theorem 3.2.5 to certain positivity properties that will be verified in Sect. 4.

3.2.1 Disjointness property

The dihedral group $\langle B, C\rangle$ acts freely and transitively on the set of projective cones $\{\mathbb{P}\mathcal{C}_i, \mathbb{P}\mathcal{C}'_i\}$. Therefore, to verify disjointness of any pair, it suffices to verify disjointness of $\mathbb{P}\mathcal{C}_0$ from any other cone in the list. Furthermore, because we work projectively, it suffices to show that the cones $(-R)^k\mathcal{C}_0$ and $(-R)^k\mathcal{C}'_0$ are disjoint from $\mathcal{C}_0$ and $-\mathcal{C}_0$ in $\mathbb{R}^4$.

3.2.2 Contraction property

Continuing to make use of the freedom to work projectively, for contraction it suffices to verify that $(-1)^k\mathcal{C}_k$ is mapped by A into $\mathcal{C}_0$, and similarly for $(-1)^k\mathcal{C}'_k$. Recall that we have

$$\begin{aligned} (-1)^k\mathcal{C}_k &= (-R)^k\mathcal{C}_0 = (-R)^k\,[v_0 \mid v_1 \mid v_2 \mid v_3] \\ (-1)^k\mathcal{C}'_k &= (-R)^k\mathcal{C}'_0 = (-R)^k\,[v_0 \mid Bv_1 \mid v_2 \mid (-R)v_3] \end{aligned} \tag{3.3.3}$$

So for contraction it suffices to verify that the vectors

$$(-R)^k v_0 \quad (-R)^k v_1 \quad (-R)^k v_2 \quad (-R)^k v_3 \quad (-R)^k Bv_1 \tag{3.3.4}$$

are mapped by A into the (closed) original cone $\mathcal{C}_0$, for any $k = 1, \ldots, N-1$. Similarly, for disjointness of cones it suffices to verify that the above vectors are themselves disjoint from the (closed) cone and its opposite (recall that we must check disjointness projectively). Let us emphasize here that we will verify this assertion for the cone $\mathcal{C}_0$ and vectors viewed in $\mathbb{R}^4$, not their projective versions.

Note that there is one exceptional case, namely for the vector v_3 we also have to consider $(-R)^N v_3$. However $(-R)^N = -\mathbf{1}$ independently of the parity of N, and $Av_3 = -v_3$, so the required positivity properties will follow straightforwardly in this endpoint case.

3.3.5 Verifying inclusion in a cone

For a simplicial cone $\mathcal{C}$ in $\mathbb{R}^4$, we will use the same letter for the matrix of its columns. In general, to certify that a column vector v belongs to $\mathcal{C}$ one must first compute a matrix $\check{\mathcal{C}}$, which defines the faces of $\mathcal{C}$, and check that $\check{\mathcal{C}}v$ has only non-negative entries. Conversely, if the result has at least one strictly negative entry, the vector is not in the cone.

Our cone $\mathcal{C}_0$ will have an additional self-duality property under the symplectic pairing. Specifically, we will find that $\mathcal{C}_0^t \cdot J \cdot \mathcal{C}_0$ is anti-diagonal, where J is the matrix of the symplectic pairing. Let now S be a diagonal matrix with the same signs on the diagonal as the anti-diagonal matrix $\mathcal{C}_0^t \cdot J \cdot \mathcal{C}_0$. We are thus lead to define the matrix:

$$M := S \cdot \mathcal{C}_0^t \cdot J \tag{3.3.6}$$

which gives the following certificate on a column vector $v \in \mathbb{R}^4$. Consider Mv: if all entries are non-negative then v belongs to $\mathcal{C}_0$, and if at least one entry is strictly negative then it is in the exterior. Furthermore, if one entry is strictly negative and one is strictly positive, then v is disjoint both from $\mathcal{C}$ and $-\mathcal{C}$, so disjoint projectively.

3.3.7 Contraction implies disjointness

By the discussion in the preceding paragraphs, our task is reduced to showing that for certain vectors v listed in Eq. (3.3.4), the vector MAv has all entries positive (to certify contraction) while Mv has two entries of opposite sign (to certify projective disjointness).

Our matrix M will have the further useful property that its rows are eigenvectors of A, with eigenvalues ± 1. Specifically the first and third rows are (right) eigenvectors with eigenvalue $+1$, and the second and fourth rows have eigenvalue -1. So if all entries of MAv are non-negative, and at least three entries are not zero, then Mv has two entries of opposite sign.

Note that the property of M to have rows which are (right) eigenvectors of A is equivalent, by the construction of M, to the property that the original cone $\mathcal{C}_0$ is spanned by eigenvectors of A, with the sign pattern of eigenvalues flipped since $A^t JA = -J$.

3.3.8 Summary

To sum up, we have reduced the proof of Theorem 3.2.5 to showing that vectors of the form $MA(-R)^k v$ have all entries non-negative (and some strictly positive) for $k = 1 \dots N-1$ and $v \in \{v_0, v_1, v_2, v_3, Bv_1\}$. We next exhibit in Sect. 3.4 the explicit vectors and matrices described above and verify that they satisfy the useful properties we stated. We then proceed to actually verify the required positivity properties in Sect. 4.

3.4 Explicit cones and properties

3.4.1 Further abbreviations

Besides the abbreviations c_i, s_i for cosines and sines from Sect. 2.2.1, the following quantities

$$cc_1 := 1 - c_1 \qquad cc_2 := 1 - c_2 \tag{3.4.2}$$

will prove useful. The somewhat clumsy notation cc_i is kept for compatibility with the symbolic computations that the reader can verify at [12]. Additionally it will prove useful to introduce the quantities:

$$\begin{aligned} L_1 &:= \frac{cc_1 \cdot cc_2 - 3(cc_2 - cc_1)}{-cc_1 \cdot cc_2 + 3(cc_1 + cc_2)} \\ L_2 &:= \frac{cc_1 \cdot cc_2 + 3(cc_2 - cc_1)}{-cc_1 \cdot cc_2 + 3(cc_1 + cc_2)} \end{aligned} \tag{3.4.3}$$

These quantities satisfy a number of useful identities which will be discussed below. The main ones, which characterize the L_i in terms of cc_i, are:

$$3(L_1 + L_2) = (L_1 + 1)cc_2 = (L_2 + 1)cc_1 \tag{3.4.4}$$

We will deduce some further properties of these quantities in Sect. 4.2.8

3.4.5 The cone

With these preparations, here is the cone:

$$\mathcal{C}_0 := \begin{bmatrix} 0 & -L_2 \cdot cc_1 & 0 & cc_1 \\ -L_1 \cdot \frac{cc_1}{s_1} & \frac{-cc_1^2}{s_1} & -\frac{cc_1}{s_1} & -\frac{cc_1^2}{s_1} \\ 0 & L_1 \cdot cc_2 & 0 & -cc_2 \\ L_2 \cdot \frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} & \frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} \end{bmatrix} \tag{3.4.6}$$

3.4.7 Self-duality of the cone

Recall that we introduced the matrix of the symplectic pairing J in Eq. (2.2.5). Then it is a direct algebraic verification that:

$$\mathcal{C}_0^t \cdot J \cdot \mathcal{C}_0 = \begin{bmatrix} 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{bmatrix} \quad \alpha := 2 \cdot \frac{L_1 - L_2}{cc_1 - cc_2} \cdot \frac{cc_1^2 cc_2^2}{s_1 s_2}$$

The verification of this is immediate, the only non-trivial algebraic manipulation is to show that the fourth entry in the second row, and the second entry in the fourth row, vanish. This ultimately follows from the identity $(L_1 + 1)cc_2 = (L_2 + 1)cc_1$ stated in Eq. (3.4.4). It is important to note that $\alpha > 0$, since $L_2 > L_1$ and $cc_1 > cc_2$, as will be established later in Sect. 4.2.8.

3.4.8 Contraction matrix

The identity for $\mathcal{C}_0^t J \mathcal{C}_0$ from the previous paragraph implies that the equations for the faces of the cone are given by taking the symplectic pairing against the spanning vectors (with appropriate signs). So if we denote by S' the diagonal matrix with entries $1, -1, 1, -1$ it follows that $S' \cdot \mathcal{C}_0^t \cdot J$ is the matrix that detects if a vector belongs, or not, to the cone $\mathcal{C}_0$. In fact, we could have used instead of S' any diagonal matrix with entries having the same pattern of signs. Let us therefore consider the matrix:

$$M := \tfrac{1}{\alpha'} \cdot S' \cdot \mathcal{C}_0^t \cdot J \quad \text{where} \quad \alpha' := \frac{2}{cc_2 - cc_1} \cdot \frac{cc_1^2 cc_2^2}{s_1 s_2} > 0$$

Computing it explicitly yields:

$$M=\begin{bmatrix}-\frac{L_1}{cc_1} & 0 & -\frac{L_2}{cc_2} & 0\\ 1 & -L_2\frac{s_1}{cc_1} & 1 & -L_1\frac{s_2}{cc_2}\\ -\frac{1}{cc_1} & 0 & -\frac{1}{cc_2} & 0\\ 1 & \frac{s_1}{cc_1} & 1 & \frac{s_2}{cc_2}\end{bmatrix} \tag{3.4.9}$$

3.4.10 Why contraction implies disjointness

Recall that in order to establish Theorem 3.2.5, we must verify that for certain vectors v the vector MAv has all entries non-negative (the contraction action of A) and that for the same vectors Mv has two entries of opposite sign.

We next observe that the matrices M and MA have the property that their second and fourth rows agree, while the first and third are negative of each other. Specifically, a direct computation yields:

$$MA=\begin{bmatrix}\frac{L_1}{cc_1} & 0 & \frac{L_2}{cc_2} & 0\\ 1 & -L_2\frac{s_1}{cc_1} & 1 & -L_1\frac{s_2}{cc_2}\\ \frac{1}{cc_1} & 0 & \frac{1}{cc_2} & 0\\ 1 & \frac{s_1}{cc_1} & 1 & \frac{s_2}{cc_2}\end{bmatrix} \tag{3.4.11}$$

In order to verify this identity, we just multiply the expression for M from Eq. (3.3.6) and for A from Sect. 2.2.4. The only identity that needs to be used (when computing the first and fourth entries of the second row) is that

$$1=\frac{L_1cc_2-L_2cc_1}{cc_1-cc_2}$$

which follows readily from Eq. (3.4.4).

From the formulas for the matrices M and MA, it is now clear that if MAv has all entries non-negative, and at least one entry in each of the pairs {first,third} and {second,fourth} is nonzero, then automatically Mv will have two entries of opposite sign. It follows that it suffices to consider vectors of the form MAv and establish that their entries are non-negative; it will be transparent from the calculations that the needed nonvanishing will also hold.

4 Computations with rotated vectors

This section contains only calculations necessary for the proof of Theorem 3.2.5. Assuming this last result, the reader can skip directly to Sect. 5 without any loss of continuity.

Outline of section In Sect. 4.1 we introduce powers of the rotation matrix which is used to transport the vectors of interest. Next we introduce some further notation and some background calculations in Sect. 4.2.

The bulk of the calculations is performed in the remaining sections. We tackle the vectors in increasing order of complexity, namely v_2, v_0, v_3, v_1, Bv_1. In addition to verifying directly our calculations, they are verified using symbolic calculations that are available at [12].

4.1 The vectors, their rotations, and contraction

Recall that the vectors we need to consider are v_0, v_1, v_2, v_3 and Bv_1, where the v_i span the cone $\mathcal{C}_0$. Here are the vectors:

$$[v_0|v_1|v_2|v_3|Bv_1] = \begin{bmatrix} 0 & -L_2cc_1 & 0 & cc_1 & -L_2cc_1 \\ -L_1\frac{cc_1}{s_1} & \frac{-cc_1^2}{s_1} & -\frac{cc_1}{s_1} & -\frac{cc_1^2}{s_1} & \frac{cc_1^2}{s_1} \\ 0 & L_1cc_2 & 0 & -cc_2 & L_1cc_2 \\ L_2\frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} & \frac{cc_2}{s_2} & \frac{cc_2^2}{s_2} & \frac{-cc_2^2}{s_2} \end{bmatrix} \tag{4.1.1}$$

4.1.2 Passing to θ_N-parameter

When facing trigonometric expressions, it will be convenient to express everything in terms of the basic angle

$$\theta_N = \frac{\pi}{N} \text{ so that } \mu_1 = \pi - 3\theta_N \text{ and } \mu_2 = \pi - \theta_N. \tag{4.1.3}$$

It is then immediate that:

$$\begin{aligned} \cos(\mu_1) &= \cos(\pi - 3\theta_N) = -\cos(3\theta_N) \\ \sin(\mu_1) &= \sin(\pi - 3\theta_N) = \sin(3\theta_N) \\ \cos(\mu_2) &= \cos(\pi - \theta_N) = -\cos(\theta_N) \\ \sin(\mu_2) &= \sin(\pi - \theta_N) = \sin(\theta_N) \end{aligned} \tag{4.1.4}$$

Recall next that the matrix $R = BC$ is block-diagonal, rotating in the first block by μ_1 and in the second by μ_2. Therefore we have:

$$(-R) = \begin{bmatrix} \cos(3\theta_N) & \sin(3\theta_N) & 0 & 0 \\ -\sin(3\theta_N) & \cos(3\theta_N) & 0 & 0 \\ 0 & 0 & \cos(\theta_N) & \sin(\theta_N) \\ 0 & 0 & -\sin(\theta_N) & \cos(\theta_N) \end{bmatrix}$$

so $(-R)$ is rotation by $-3\theta_N$ in the first block and by $-\theta_N$ in the second block. To abbreviate further the sines of multiple angles, we will use the notation:

$$\begin{aligned} cp_1 &:= \cos(3k\theta_N)\; sp_1 := -\sin(3k\theta_N) \\ cp_2 &:= \cos(k\theta_N) \;\; sp_2 := -\sin(k\theta_N) \end{aligned} \tag{4.1.5}$$

where cp and sp are meant to denote “cosine power” and “sine power”. These are precisely the entries of $(-R)^k$:

$$(-R)^k = \begin{bmatrix} cp_1 & -sp_1 & 0 & 0 \\ sp_1 & cp_1 & 0 & 0 \\ 0 & 0 & cp_2 & -sp_2 \\ 0 & 0 & sp_2 & cp_2 \end{bmatrix} \tag{4.1.6}$$

We have chosen to express $(-R)^k$ with the sign choices standard for a counterclockwise rotation matrix, but the reader should keep in mind that the signs of sines are as stated in Eq. (4.1.5).

Let us finally recall the matrix MA from Eq. (3.4.11):

$$MA = \begin{bmatrix} \frac{L_1}{cc_1} & 0 & \frac{L_2}{cc_2} & 0 \\ 1 & -L_2\frac{s_1}{cc_1} & 1 & -L_1\frac{s_2}{cc_2} \\ \frac{1}{cc_1} & 0 & \frac{1}{cc_2} & 0 \\ 1 & \frac{s_1}{cc_1} & 1 & \frac{s_2}{cc_2} \end{bmatrix} \tag{4.1.7}$$

Our task has been reduced to computing $MA(-R)^k v$, for $k = 1 \dots N-1$, and for each column vector v in Eq. (4.1.1).

This will take up the rest of this section, after some preliminary recollections from trigonometry in the next section.

4.2 Frequently used expressions

4.2.1 Some trigonometric formulas

Since $\mu_1 \equiv 3\mu_2 \mod 2\pi$ it will be useful to make use of angle-tripling formulas:

$$\begin{aligned} \sin(3\theta) &= \sin(\theta)\cdot\left(3-4\sin^2(\theta)\right) = \sin(\theta)\cdot(2\cos(2\theta)+1) \\ \cos(3\theta) &= \cos(\theta)\cdot\left(4\cos^2(\theta)-3\right) = \cos(\theta)\cdot(2\cos(2\theta)-1) \end{aligned} \tag{4.2.2}$$

Besides considering $\cos(3\theta)/\cos(\theta)$ and similarly for sine, we will frequently also use the following difference of cosines:

$$\begin{aligned} \cos(3\theta)-\cos(\theta) &= \cos(\theta)\cdot 2\cdot(\cos(2\theta)-1) \\ &= -4\cdot\cos(\theta)\cdot\sin^2(\theta) \\ &= -\sin(\theta)\cdot 2\cdot\sin(2\theta) \end{aligned} \tag{4.2.3}$$

and its analogue for sines:

$$\sin(3\theta)-\sin(\theta) = \sin(\theta)\cdot 2\cdot\cos(2\theta). \tag{4.2.4}$$

We'll also make use of the standard addition/subtraction formulas:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \\ \cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) \end{aligned} \tag{4.2.5}$$

4.2.6 Abbreviations

Recall that we introduced the algebraic expressions

$$\begin{aligned} cc_1 &:= 1-c_1 \quad \text{so that} \quad \frac{cc_1}{s_1} = \frac{\sin(\mu_1/2)}{\cos(\mu_1/2)} = \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} \\ cc_2 &:= 1-c_2 \quad \text{so that} \quad \frac{cc_2}{s_2} = \frac{\sin(\mu_2/2)}{\cos(\mu_2/2)} = \frac{\cos(\theta_N/2)}{\sin(\theta_N/2)} \end{aligned} \tag{4.2.7}$$

where we used the formulas $1-\cos(\theta) = 2\sin^2(\theta/2)$ and $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$.

4.2.8 The parameters L_i

To shorten notation, we introduced in Eq. (3.4.3) the constants L_1, L_2 which we will mostly use through the identities:

$$3(L_1 + L_2) = (L_1 + 1)cc_2 = (L_2 + 1)cc_1 = \frac{6 \cdot cc_1 \cdot cc_2}{-cc_1 \cdot cc_2 + 3(cc_1 + cc_2)} \tag{4.2.9}$$

which are verified directly from the definitions.

Let us record the basic inequalities which we use frequently:

$$cc_2 > cc_1 \quad \text{and} \quad L_2 > L_1$$

The first one is verified from the definitions of μ_i while the second one follows from the first, which we will also frequently write as $cc_2 - cc_1 > 0$.

The relations between L_1 and L_2 imply

$$\begin{aligned}(L_1 + 1)cc_2 = 3(L_1 + L_2) > 6L_1 \iff cc_2 > (6 - cc_2)L_1 \\ \iff L_1 < \frac{cc_2}{6 - cc_2}.\end{aligned}$$

As $cc_2 < 2$ we obtain $L_1 < \frac{1}{2}$. This also implies the following

$$3(L_1 + L_2) = (L_1 + 1)cc_2 < \frac{3}{2} \cdot 2 = 3.$$

Hence

$$L_1 + L_2 < 1. \tag{4.2.10}$$

4.2.11 The difference of cosines

Using the triple-angle formula we find:

$$\begin{aligned}c_1 - c_2 &= \cos(3\mu_2) - \cos(\mu_2) = 4\cos(\mu_2)\big[\cos(\mu_2)^2 - 1\big] \\ &= -4c_2 s_2^2 \\ &= 4(c_2 - 1)c_2(c_2 + 1) \\ &> 0\end{aligned}$$

The sign $c_1 - c_2 > 0$ follows from the above algebraic expressions and $c_2 < 0$, or by looking at the explicit values of μ_i.

4.2.12 The smallest and largest sines and cosines

Our rotation will range over $k = 1, \ldots, N - 1$. We have the elementary inequalities

$$0 < \sin(\mu_2) \leq \left|\sin\left(k\frac{N-1}{N}\pi\right)\right| \leq |\cos(\mu_2)| < 1$$

4.2.13 Frequently occurring differences of ratios

The following manipulation is used frequently, so we record it once here, using the properties of the ratios cc_i/s_i from Eq. (4.2.7) as well as the addition formula for sines Eq. (4.2.5):

$$\begin{aligned}\frac{cc_2}{s_2}-\frac{cc_1}{s_1}&=\frac{\cos(\theta_N/2)}{\sin(\theta_N/2)}-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\\&=\frac{\sin(\theta_N)}{\sin(\theta_N/2)\sin(3\theta_N/2)}\\&=\frac{2\cancel{\sin(\theta_N/2)}\cos(\theta_N/2)}{\cancel{\sin(\theta_N/2)}\sin(3\theta_N/2)}\\&=2\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)}\end{aligned}\tag{4.2.14}$$

We now give a list of useful inequalities appearing with powers of the rotation.

$$\begin{aligned}\frac{sp_1}{s_1}-\frac{sp_2}{s_2}&=\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot\left(\frac{-\sin(3k\theta_N)}{\sin(k\theta_N)}+\frac{\sin(3\theta_N)}{\sin(\theta_N)}\right)\\&=2\cdot\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot(\cos(2\theta_N)-\cos(2k\theta_N))\end{aligned}\tag{4.2.15}$$

In any event, this expression is always negative (non-positive) for $k=1,\ldots,N-1$. It vanishes precisely for $k=1,N-1$.

Similarly

$$\begin{aligned}\frac{sp_1}{s_1}+\frac{sp_2}{s_2}&=-\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot\left(\frac{\sin(3k\theta_N)}{\sin(k\theta_N)}+\frac{\sin(3\theta_N)}{\sin(\theta_N)}\right)\\&=-2\cdot\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\cdot(\cos(2k\theta_N)+\cos(2\theta_N)+1)\end{aligned}\tag{4.2.16}$$

which is always non-positive.

Next we have the combinations

$$\frac{cc_1}{s_1}sp_1\pm\frac{cc_2}{s_2}sp_2.$$

Recall that $cc_i=1-c_i$ and $c_1=-\cos(3\theta_N)$, $c_2=-\cos(\theta_N)$ and we'll use the basic identity

$$1+\cos(\theta)=2-2\sin^2(\theta/2)=2\cos^2(\theta/2)$$

to reduce to the consideration of

$$\begin{aligned}&raccc_1s_1sp_1-\frac{cc_2}{s_2}sp_2=-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cdot\sin(3k\theta_N)+\frac{\cos(\theta_N/2)}{\sin(\theta_N/2)}\cdot\sin(k\theta_N)\\&\sin(k\theta_N)\left[-\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cdot(2\cos(2k\theta_N)+1)+\frac{\cos(\theta_N/2)}{\sin(\theta_N/2)}\right]\\&=\sin(k\theta_N)\left[-2\cos(2k\theta_N)\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}+\frac{\sin(\theta_N)}{\sin(3\theta_N/2)\sin(\theta_N/2)}\right]\\&=\frac{2\sin(k\theta_N)}{\sin(3\theta_N/2)}\left[\cos\left(\frac{\theta_N}{2}\right)-\cos(2k\theta_N)\cos\left(\frac{3\theta_N}{2}\right)\right]\end{aligned}\tag{4.2.17}$$

This expression is manifestly non-negative, since the value $\cos(\theta_N/2)$ is larger that $\cos(3\theta_N/2)$ for all $N\geq 4$.

Let's do the same but with a sum:

$$\begin{aligned}
&\frac{cc_1}{s_1}sp_1 + \frac{cc_2}{s_2}sp_2 = -\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} \cdot \sin(3k\theta_N) - \frac{\cos(\theta_N/2)}{\sin(\theta_N/2)} \cdot \sin(k\theta_N) \\
&= \sin(k\theta_N)\left[-2\cos(2k\theta_N)\frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} - \frac{\sin(2\theta_N)}{\sin(3\theta_N/2)\sin(\theta_N/2)}\right] \\
&= -\frac{2\sin(k\theta_N)}{\sin(3\theta_N/2)}\left[\cos(2k\theta_N)\cos(3\theta_N/2) + 2\cos(\theta_N)\cos(\theta_N/2)\right] \qquad (4.2.18)
\end{aligned}$$

This is non-positive, in fact the term $\cos(\theta_N)\cos(\theta_N/2)$ is larger in absolute value than the other one $\cos(2k\theta_N)(\cos(3\theta_N/2))$.

We give in the following two propositions simplified expression for some combinations of terms that appear several times in our computations.

Proposition 4.2.19 *The following formulas hold:*

$$\pm cp_1 - \frac{cc_1}{s_1}sp_1 = \frac{\sin(3(k \pm \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}$$

$$\pm cp_2 - \frac{cc_2}{s_2}sp_2 = \frac{\sin((k \pm \frac{1}{2})\theta_N)}{\sin(\theta_N/2)}$$

$$\pm(cp_1 - cp_2) - \frac{cc_1}{s_1}sp_1 + \frac{cc_2}{s_2}sp_2 = -4 \cdot \frac{\sin((k \pm \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k \pm 1)\theta_N)$$

Proof On one hand we have

$$\begin{aligned}
\pm cp_1 - \frac{cc_1}{s_1}sp_1 &= \pm\cos(3k\theta_N) + \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\sin(3k\theta_N) \\
&= \frac{\sin(3(k \pm \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}
\end{aligned}$$

and similarly for the second equality.

We now consider:

$$\begin{aligned}
&\frac{\sin(3(k \pm \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} - \frac{\sin((k \pm \frac{1}{2})\theta_N)}{\sin(\theta_N/2)} \\
&= \frac{\sin((k \pm \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\left(\frac{\sin(3(k \pm \frac{1}{2})\theta_N)}{\sin((k \pm \frac{1}{2})\theta_N)} - \frac{\sin(3\theta_N/2)}{\sin(\theta_N/2)}\right) \\
&= 2 \cdot \frac{\sin((k \pm \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\left(\cos((2k \pm 1)\theta_N) - \cos(\theta_N)\right) \\
&= -4 \cdot \frac{\sin((k \pm \frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k \pm 1)\theta_N)
\end{aligned}$$

□

The second proposition is for a similar expression where we invert the fractions.

Proposition 4.2.20 *The following formulas hold:*

$$cp_1 \pm \frac{s_1}{cc_1} sp_1 = \frac{\cos(3(k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)}$$

$$cp_2 \pm \frac{s_2}{cc_2} sp_2 = \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(\theta_N/2)}$$

$$cp_1 - cp_2 \pm \left(\frac{s_1}{cc_1} sp_1 - \frac{s_2}{cc_2} sp_2\right) = -4 \cdot \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k \pm 1)\theta_N)$$

Proof On one hand we have

$$\begin{aligned} cp_1 \pm \frac{s_1}{cc_1} sp_1 &= \cos(3k\theta_N) \mp \frac{\sin(3\theta_N/2)}{\cos(3\theta_N/2)} \sin(3k\theta_N) \\ &= \frac{\cos(3(k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \end{aligned}$$

and similarly for the second equality.

We now consider

$$\begin{aligned} &\frac{\cos(3(k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} - \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(\theta_N/2)} \\ &= \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \left(\frac{\cos(3(k \pm \frac{1}{2})\theta_N)}{\cos((k \pm \frac{1}{2})\theta_N)} - \frac{\cos(3\theta_N/2)}{\cos(\theta_N/2)}\right) \\ &= 2 \cdot \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \left(\cos((2k \pm 1)\theta_N) - \cos(\theta_N)\right) \\ &= -4 \cdot \frac{\cos((k \pm \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k \pm 1)\theta_N) \end{aligned}$$

□

We are now ready to proceed to the analysis of vectors.

4.3 Computing with v_2

The vector v_2 is the third column of the matrix M:

$$v_2 = \begin{bmatrix} 0 \\ -\frac{cc_1}{s_1} \\ 0 \\ \frac{cc_2}{s_2} \end{bmatrix} \text{ so } (-R)^k v_2 = \begin{bmatrix} \frac{cc_1}{s_1} sp_1 \\ -\frac{cc_1}{s_1} cp_1 \\ -\frac{cc_2}{s_2} sp_2 \\ \frac{cc_2}{s_2} cp_2 \end{bmatrix}$$

Applying the matrix MA yields:

$$MA(-R)^k v_2 = \begin{bmatrix} L_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot \frac{sp_2}{s_2} \\ L_2 \cdot cp_1 - L_1 \cdot cp_2 + cc_1 \cdot \frac{sp_1}{s_1} - cc_2 \cdot \frac{sp_2}{s_2} \\ \frac{sp_1}{s_1} - \frac{sp_2}{s_2} \\ -cp_1 + cp_2 + cc_1 \cdot \frac{sp_1}{s_1} - cc_2 \cdot \frac{sp_2}{s_2} \end{bmatrix}$$

4.3.1 The third entry of $MA(-R)^k v_2$

The third entry of the vector is:

$$\frac{sp_1}{s_1} - \frac{sp_2}{s_2} = \frac{-\sin(3k\theta_N)}{\sin(3\theta_N)} - \frac{-\sin(k\theta_N)}{\sin(\theta_N)}$$

We rewrite this as:

$$\frac{\sin(k\theta_N)}{\sin(3\theta_N)}\left(\frac{\sin(3\theta_N)}{\sin(\theta_N)} - \frac{\sin(3k\theta_N)}{\sin(k\theta_N)}\right)$$

The first factor is clearly positive for our range of k, and we rewrite the difference using the formula for sine of the triple angle (Eq. (4.2.2)) as:

$$\left(\frac{\sin(3\theta_N)}{\sin(\theta_N)} - \frac{\sin(3k\theta_N)}{\sin(k\theta_N)}\right) = 2\cos(2\theta_N) - 2\cos(2k\theta_N) \geq 0$$

and equality holds precisely for $k = 1, N-1$.

4.3.2 The fourth entry of $MA(-R)^k v_2$

By Proposition 4.2.19, the 4th entry of the vector is

$$-cp_1 + cp_2 + \frac{cc_1}{s_1}sp_1 - \frac{cc_2}{s_2}sp_2 = 4 \cdot \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k+1)\theta_N)$$

which is non-negative for $1 \leq k \leq N-1$.

4.3.3 The second entry of $MA(-R)^k v_2$

We have to show that

$$L_2 cp_1 - L_1 cp_2 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2} \geq 0$$

Observing the similarity to the fourth entry, we divide the claim into two parts. For $k \geq \frac{N}{2}$ we claim that the above quantity is clearly greater or equal to the fourth entry, hence non-negative. For $\frac{N}{2} \geq k$, we will replace $L_2 cp_1 - L_1 cp_2$ by $cp_1 - cp_2$, as detailed in Sect. 4.3.6 below, and proceed as with the fourth entry.

4.3.4 The case $N > k \geq N/2$

To check that the second entry is greater than the fourth entry reduces to the inequality

$$(L_2+1)cp_1 - (L_1+1)cp_2 \geq 0$$

We now rewrite this, taking into account the identities for L_i from Eq. (4.2.9):

$$(L_2+1)cp_1 - (L_1+1)cp_2 = (L_2+1)cc_1\frac{cp_1}{cc_1} - (L_1+1)cc_2\frac{cp_2}{cc_2}$$
$$= 3(L_1+L_2)\left(\frac{cp_1}{cc_1} - \frac{cp_2}{cc_2}\right)$$

The factor $3(L_1 + L_2)$ is positive, so we can drop it. We can also factor out $\frac{-cp_2}{cc_1}$, taking into account that $cp_2 = \cos(k\theta_N) \leq 0$ when $N \geq k \geq \frac{N}{2}$, to reduce to showing:

$$\frac{cc_1}{cc_2} - \frac{cp_1}{cp_2} \geq 0 \tag{4.3.5}$$

Now we rewrite everything in terms of θ_N and use the angle-tripling formula for cosines to express the terms as:

$$\frac{cc_1}{cc_2} = \frac{1 + \cos(3\theta_N)}{1 + \cos(\theta_N)} \quad \frac{cp_1}{cp_2} = 2\cos(2k\theta_N) - 1$$

We next reduce the expressions:

$$\begin{aligned} &\frac{1 + \cos(3\theta_N)}{1 + \cos(\theta_N)} \geq 2\cos(2k\theta_N) - 1 \iff \\ \iff\ & 1 + \cos(3\theta_N) \geq 2\cos(2k\theta_N) - 1 + 2\cos(\theta_N)\cos(2k\theta_N) - \cos(\theta_N) \\ \iff\ & 1 + \frac{\cos(3\theta_N) + \cos(\theta_N)}{2} \geq \cos(2k\theta_N) + \cos(\theta_N)\cos(2k\theta_N) \end{aligned}$$

The left-hand side above is independent of k while the right-hand side is monotonically increasing and achieves its maximum when $k = N - 1$ to reduce to

$$1 + \frac{\cos(3\theta_N) + \cos(\theta_N)}{2} \geq \cos(2\theta_N) + \cos(\theta_N)\cos(2\theta_N)$$

Finally we use again the angle-tripling formula to find $\cos(3\theta_N) + \cos(\theta_N) = 2\cos(\theta_N)\cos(2\theta_N)$ and reduce to

$$1 + \cancel{\cos(\theta_N)\cos(2\theta_N)} \geq \cos(2\theta_N) + \cancel{\cos(\theta_N)\cos(2\theta_N)}$$

which clearly holds.

4.3.6 The case $N/2 \geq k \geq 1$

In this case we claim that we have

$$L_2 cp_1 - L_1 cp_2 \geq cp_1 - cp_2.$$

Indeed, it is obvious in the case $k = N/2$ and otherwise it is equivalent to

$$\begin{aligned} &(1 - L_1)cp_2 \geq (1 - L_2)cp_1 \\ \iff\ & \frac{1 - L_1}{1 - L_2} \geq \frac{cp_1}{cp_2} = 2\cos(2k\theta_N) - 1 \end{aligned}$$

which is true since $L_1 < L_2$ implies that

$$\frac{1 - L_1}{1 - L_2} > 1 \geq 2\cos(2k\theta_N) - 1.$$

We now proceed as in the analysis of the fourth entry but this time subtracting $cp_2 - cp_1$, i.e.

$$L_2 cp_1 - L_1 cp_2 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2} \geq cp_1 - cp_2 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2}.$$

This last expression can be rewritten using Proposition 4.2.19

$$4 \cdot \frac{\sin((k-\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k-1)\theta_N).$$

Which is positive for $1 \le k \le N-1$.

4.3.7 The first entry of $MA(-R)^k v_2$

We have to show that the expression

$$L_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot \frac{sp_2}{s_2}$$

is non-negative. We factor out the term $-sp_2/s_1$ and it suffices to show that the resulting expression

$$L_2 \frac{s_1}{s_2} - L_1 \cdot \frac{sp_1}{sp_2}$$

is positive since $s_1 > 0$ and $-sp_2 = \sin(k\theta_N) > 0$. Both ratios sp_1/sp_2 and s_1/s_2 are ratios of a triple sine over a sine, so we rewrite them using Eq. (4.2.2) to find:

$$L_2(3-4s_2^2) - L_1(3-4sp_2^2)$$

We rewrite this, using a "polarization" identity:

$$Ax - By = \frac{A-B}{2}(x+y) + \frac{A+B}{2}(x-y). \tag{4.3.8}$$

After multiplying by 2, we get

$$(L_2 - L_1)\left[6 - 4(s_2^2 + sp_2^2)\right] + (L_2 + L_1)\left[sp_2^2 - s_2^2\right].$$

Now $L_2 - L_1 > 0$ and $s_2^2 = \sin^2(\theta_N) \le \frac{1}{2}$ since $N \ge 3$, so the first term is clearly positive. For the second one we observe that $sp_2^2 \ge s_2^2$ since this is saying that $\sin(k\theta_N) \ge \sin(\theta_N)$ for $k = 1, \ldots N-1$.

4.4 Computing with v_0

4.4.1 Setup

Recall that

$$v_0 = \begin{bmatrix} 0 \\ -L_1 \frac{cc_1}{s_1} \\ 0 \\ L_2 \frac{cc_2}{s_2} \end{bmatrix} \text{ so } (-R)^k v_0 = \begin{bmatrix} L_1 \frac{cc_1}{s_1} sp_1 \\ -L_1 \frac{cc_1}{s_1} cp_1 \\ -L_2 \frac{cc_2}{s_2} sp_2 \\ L_2 \frac{cc_2}{s_2} cp_2 \end{bmatrix}.$$

So the vector that we have to analyze, namely $MA(-R)^k v_0$ is:

$$\begin{bmatrix} L_1^2 \cdot \frac{sp_1}{s_1} - L_2^2 \cdot \frac{sp_2}{s_2} \\ L_1 L_2 \cdot cp_1 - L_1 L_2 \cdot cp_2 + L_1 \cdot cc_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot cc_2 \cdot \frac{sp_2}{s_2} \\ L_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot \frac{sp_2}{s_2} \\ -L_1 \cdot cp_1 + L_2 \cdot cp_2 + L_1 \cdot cc_1 \cdot \frac{sp_1}{s_1} - L_2 \cdot cc_2 \cdot \frac{sp_2}{s_2} \end{bmatrix} \tag{4.4.2}$$

4.4.3 The third entry of $MA(-R)^k v_0$

Notice that the third entry is the same as the first entry of v_2, which we already checked is positive in Sect. 4.3.7.

4.4.4 The first entry of $MA(-R)^k v_0$

The first entry is similar to the one of v_2 dealt with in Sect. 4.3.7, we use the same method to show positivity.

First factor out the positive term $-sp_2/s_1$ and use the angle tripling formula Eq. (4.2.2) to get

$$L_2^2(3-4s_2^2) - L_1^2(3-4sp_2^2)$$

Using the polarization identity Sect. 4.3.8 after multiplying by 2, we get

$$(L_2^2 - L_1^2)\left[6 - 4(s_2^2 + sp_2^2)\right] + (L_2^2 + L_1^2)\left[sp_2^2 - s_2^2.\right]$$

Now $L_2^2 - L_1^2 > 0$ and $s_2^2 = \sin^2(\theta_N) \leq \frac{1}{2}$ since $N \geq 3$, so the first term is positive. For the second one use again that $sp_2^2 \geq s_2^2$ for $k = 1, \ldots N-1$.

4.4.5 The fourth entry of $MA(-R)^k v_0$

Consider first the terms with a factor of L_1:

$$\begin{aligned} -\left(cp_1 - cc_1\frac{sp_1}{s_1}\right) &= -\left(\cos(3k\theta_N) + \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\sin(3k\theta_N)\right) \\ &= -\frac{\sin\left(3(k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}. \end{aligned}$$

Similarly consider those with a factor of L_2:

$$cp_2 - cc_2\frac{sp_2}{s_2} = \frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)}.$$

The fourth entry has thus the following expression

$$L_2\frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)} - L_1\frac{\sin\left(3(k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}$$

We factor out the positive term $\sin\left((k+\frac{1}{2})\theta_N\right)/\sin(3\theta_N/2)$,

$$\begin{aligned} &L_2\frac{\sin(3\theta_N/2)}{\sin(\theta_N/2)} - L_1\frac{\sin\left(3(k+\frac{1}{2})\theta_N\right)}{\sin\left((k+\frac{1}{2})\theta_N\right)} - \\ &= L_2\left(2\cos(\theta_N)+1\right) - L_1\left(2\cos\left((2k+1)\theta_N\right)+1\right) \\ &= (L_2 - L_1) + 2\left(L_2\cos(\theta_N) - L_1\cos\left((2k+1)\theta_N\right)\right). \end{aligned}$$

As $L_2 > L_1 > 0$ and $\cos(\theta_N) \geq \cos((2k+1)\theta_N)$ for all k this expression is positive.

4.4.6 The second entry of $MA(-R)^k v_0$

observing the similarity to the fourth entry, we divide the claim into two parts. For $k \geq \frac{N}{2}$ we claim that the above quantity is greater than or equal to the fourth entry, hence non-negative. For $\frac{N}{2} \geq k$, we use relations with L_1, L_2.

4.4.7 The case $N > k \geq N/2$

To check that the second entry is greater than the fourth entry reduces to the inequality

$$L_1(L_2+1)cp_1 - L_2(L_1+1)cp_2 \geq 0$$

Using identities of L_i from Eq. (4.2.9):

$$\begin{aligned} L_1(L_2+1)cp_1 - L_2(L_1+1)cp_2 &= (L_2+1)cc_1 L_1 \frac{cp_1}{cc_1} - (L_1+1)cc_2 L_2 \frac{cp_2}{cc_2} \\ &= 3\,(L_1+L_2)\left(L_1\frac{cp_1}{cc_1} - L_2\frac{cp_2}{cc_2}\right). \end{aligned}$$

The factors $3(L_1+L_2)$ and $\frac{-cp_2}{cc_1}$ are positive when $N \geq k \geq \frac{N}{2}$, then we are reduced to showing:

$$L_2\frac{cc_1}{cc_2} - L_1\frac{cp_1}{cp_2} \geq 0$$

As $L_2 > L_1 > 0$ this is implied by the inequality

$$\frac{cc_1}{cc_2} - \frac{cp_1}{cp_2} \geq 0$$

proved earlier, see Eq. (4.3.5).

4.4.8 The case $N/2 \geq k \geq 1$

Notice that $L_2 > L_1$, $1-L_1 > 1-L_2$ always, and $cp_2 \geq cp_1$ in this range since $cp_2 - cp_1 = 2\sin(k\theta_N)\sin(2k\theta_N)$. Multiplying these three inequalities, we get

$$L_2(1-L_1)cp_2 \geq L_1(1-L_2)cp_1$$

then

$$L_1L_2cp_1 - L_1L_2cp_2 \geq L_1cp_1 - L_2cp_2.$$

Using this inequality we reduce the positivity of the second entry in this range to the positivity of

$$L_1(cp_1 + cc_1\frac{sp_1}{s_1}) - L_2(cp_2 + cc_2\frac{sp_2}{s_2}). \tag{4.4.9}$$

Using Proposition 4.2.19, we can rewrite this expression as:

$$L_2\frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin\left(\theta_N/2\right)} - L_1\frac{\sin\left(3(k-\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}.$$

Factoring out the positive term $\sin((k-\frac{1}{2})\theta_N)/\sin(3\theta_N/2)$,

$$\begin{aligned}
&L_2\frac{\sin(3\theta_N/2)}{\sin(\theta_N/2)} - L_1\frac{\sin\left(3(k-\frac{1}{2})\theta_N\right)}{\sin\left((k-\frac{1}{2})\theta_N\right)}\\
&= L_2\left(2\cos(\theta_N)+1\right) - L_1\left(2\cos\left((2k-1)\theta_N\right)+1\right)\\
&= (L_2-L_1) + 2\left(L_2\cos(\theta_N) - L_1\cos\left((2k-1)\theta_N\right)\right).
\end{aligned}$$

As $L_2 > L_1 > 0$ and $\cos(\theta_N) \geq \cos((2k-1)\theta_N)$ for all k this expression is positive.

4.5 Computing with v_3

4.5.1 Setup

We have that

$$v_3 = \begin{bmatrix} cc_1 \\ -\frac{cc_1^2}{s_1} \\ -cc_2 \\ \frac{cc_2^2}{s_2} \end{bmatrix} \text{ so } (-R)^k v_3 = \begin{bmatrix} cc_1\cdot cp_1 + \frac{cc_1^2}{s_1}\cdot sp_1 \\ -\frac{cc_1^2}{s_1}\cdot cp_1 + cc_1\cdot sp_1 \\ -cc_2\cdot cp_2 - \frac{cc_2^2}{s_2}\cdot sp_2 \\ \frac{cc_2^2}{s_2}\cdot cp_2 - cc_2\cdot sp_2 \end{bmatrix}$$

Using the identity

$$cc_i^2 + s_i^2 = 2cc_i$$

for $i \in \{0, 1\}$ and identity (3.4.4), we then proceed to compute $MA(-R)^k v_3$ to be:

$$\begin{bmatrix} L_1\cdot cc_1\cdot\frac{sp_1}{s_1} - L_2\cdot cc_2\cdot\frac{sp_2}{s_2} + L_1 cp_1 - L_2 cp_2 \\ 3\,(L_1+L_2)(cp_1-cp_2) - (L_2+1)\,s_1 sp_1 + (L_1+1)\,s_2 sp_2 + 2\left(cc_1\cdot\frac{sp_1}{s_1} - cc_2\cdot\frac{sp_2}{s_2}\right) \\ cp_1 - cp_2 + cc_1\cdot\frac{sp_1}{s_1} - cc_2\cdot\frac{sp_2}{s_2} \\ 2\left(cc_1\cdot\frac{sp_1}{s_1} - cc_2\cdot\frac{sp_2}{s_2}\right) \end{bmatrix}$$

4.5.2 The third entry of $MA(-R)^k v_3$

The third entry

$$cp_1 - cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}$$

was already proved to be positive in Sect. 4.3.6, using Proposition 4.2.19.

4.5.3 The first entry of $MA(-R)^k v_3$

The first entry of v_3 is the same as Eq. (4.4.9) proved to be positive in Sect. 4.4.8.

4.5.4 The fourth entry of $MA(-R)^k v_3$

Factoring out 2, we recognize the last two terms of the fourth entry of v_2. As in Sect. 4.3.2 we use the triple-angle formula for sines:

$$\begin{aligned}\frac{cc_1}{s_1}sp_1 - \frac{cc_2}{s_2}sp_2 &= (-sp_2)\left(\frac{cc_2}{s_2} - \frac{cc_1}{s_1}\frac{sp_1}{sp_2}\right)\\ &= (-sp_2)\left(\frac{cc_2}{s_2} - \frac{cc_1}{s_1} - 2\frac{cc_1}{s_1}\cos(2k\theta_N)\right)\end{aligned}$$

now using Sect. 4.2.14,

$$\begin{aligned}&= 2(-sp_2)\left(\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)} - \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}\cos(2k\theta_N)\right)\\ &= 2\sin(k\theta_N)\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)}\big(1 - \frac{\cos(3\theta_N/2)}{\cos(\theta_N/2)}\cos(2k\theta_N)\big)\\ &= 2\sin(k\theta_N)\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)}\big(1 - (2\cos(\theta_N) - 1)\cos(2k\theta_N)\big)\end{aligned}$$

For all $1 \le k < N$, $2\sin(k\theta_N)\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)} > 0$. Moreover, as $N > 3$, we have $0 < 2\cos(\theta_N) - 1 < 1$ hence $1 - (2\cos(\theta_N) - 1)\cos(2k\theta_N) > 0$.

4.5.5 The second entry of $MA(-R)^k v_3$

As for the second entry of the previous vectors we divide the claim into two parts. For $k \ge \frac{N}{2}$ we claim that the above quantity is greater than or equal to the fourth entry, hence non-negative. For $\frac{N}{2} \ge k$ we use other relations.

4.5.6 The case $N > k > N/2$

We show in this case that

$$3(L_1 + L_2)(cp_1 - cp_2) - (L_2 + 1)s_1 sp_1 + (L_1 + 1)s_2 sp_2 \ge 0.$$

Using Eq. (4.2.9), we can factor out $3(L_1 + L_2)$ to get an equivalent inequality

$$cp_1 - cp_2 - \frac{s_1}{cc_1}sp_1 + \frac{s_2}{cc_2}sp_2 \ge 0.$$

By Proposition 4.2.20, this latter expression is equal to

$$-4 \cdot \frac{\cos((k - \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k-1)\theta_N).$$

As $k > \frac{N}{2}$, $\cos((k - \frac{1}{2})\theta_N) < 0$ and moreover $k < N$ then $\sin(k\theta_N)$ and $\sin((k-1)\theta_N)$ are non-negative, thus the inequality is satisfied.

4.5.7 The case $1 \leq k \leq N/2$

First we use the fact $3(L_1 + L_2) \leq 3$ established in Eq. (4.2.10). Moreover for $1 \leq k \leq N/2$, $\cos((k - \frac{1}{2})\theta_N) > 0$ thus according to the previous formula

$$cp_1 - cp_2 - \frac{s_1}{cc_1} sp_1 + \frac{s_2}{cc_2} sp_2 \leq 0.$$

Hence the second entry in this case is not smaller than

$$3 \cdot \left(cp_1 - cp_2 - \frac{s_1}{cc_1} sp_1 + \frac{s_2}{cc_2} sp_2 \right) + 2 \cdot \left(cc_1 \cdot \frac{sp_1}{s_1} - cc_2 \cdot \frac{sp_2}{s_2} \right).$$

Using the trigonometric computations of Proposition 4.2.20 and Eq. (4.2.17) we can rewrite this expression as

$$-12 \cdot \frac{\cos((k - \frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k-1)\theta_N)$$
$$+4 \sin(k\theta_N) \left(\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)} - \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} \cos(2k\theta_N) \right).$$

Factoring out the positive term $4 \cdot \frac{\sin(k\theta_N)}{\cos(3\theta/2) \cdot \sin(3\theta/2)}$ we are reduced to show positivity for

$$\begin{aligned}
&-3 \sin(3\theta_N/2) \cdot \cos((k - \frac{1}{2})\theta_N) \cdot \sin((k-1)\theta_N) \\
&+\cos(3\theta_N/2) \cdot \left(\cos(\theta_N/2) - \cos(3\theta_N/2)\cos(2k\theta_N) \right) \\
&= -\frac{3}{2} \sin(3\theta_N/2) \cdot \left(\sin((2k - \frac{3}{2})\theta_N) - \sin(\theta_N/2) \right) \\
&+\cos(3\theta_N/2) \cdot \left(\cos(\theta_N/2) - \cos(3\theta_N/2)\cos(2k\theta_N) \right)
\end{aligned}$$

Hence we are reduced to showing that

$$\begin{aligned}
&\frac{3}{2} \sin(3\theta_N/2) \cdot \sin((2k - \frac{3}{2})\theta_N) + \cos(3\theta_N/2)^2 \cos(2k\theta_N) \\
&\leq \frac{3}{2} \sin(3\theta_N/2) \cdot \sin(\theta_N/2) + \cos(3\theta_N/2)\cos(\theta_N/2).
\end{aligned}$$

In the following we show that the expression

$$\frac{3}{2} \sin(3\theta_N/2) \cdot \sin((2k - \frac{3}{2})\theta_N) + \cos(3\theta_N/2)^2 \cos(2k\theta_N)$$

takes its maximal value at $k = 1$.

Proposition 4.5.8 *Let a, b, ϵ be three positive parameters. Then there exists a unique $x_M \in [0, \pi)$ such that the map*

$$f(x) = a \sin(x - \epsilon) + b \cos(x)$$

is extremal at x_M. Moreover $\tan(x_M) = \frac{a \cos(\epsilon)}{b - a \sin(\epsilon)}$ and $f(x_M - x) = f(x)$.

Proof We compute the derivative

$$f'(x) = a \cos(x - \epsilon) - b \sin(x).$$

We look for zeros of this map,

$$\begin{aligned} f'(x) = 0 &\iff a\cos(x-\epsilon) = b\sin(x) \\ \iff a\cos(x)\cos(\epsilon) &= (b - a\sin(\epsilon))\sin(x) \\ &\iff \tan(x) = \frac{a\cos(\epsilon)}{b - a\sin(\epsilon)}. \end{aligned}$$

The symmetry property comes from the fact that for all a, b, ϵ there exists A, D such that $f(x) = A\cos(x - D)$. □

We apply the proposition in the setting where $a = \frac{3}{2}\sin(3\theta_N/2)$, $b = \cos(3\theta_N/2)^2$, $\epsilon = 3\theta_N/2$. Then

$$\frac{a\cos(\epsilon)}{b - a\sin(\epsilon)} = \frac{3\sin(\epsilon)\cos(\epsilon)}{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2}.$$

Notice that $\tan(2\epsilon) = \frac{2\sin(\epsilon)\cos(\epsilon)}{\cos(\epsilon)^2 - \sin^2(\epsilon)}$. Hence

$$\begin{aligned} \tan(x_M) \le \tan(2\epsilon) &\iff \frac{3\sin(\epsilon)\cos(\epsilon)}{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2} \le \frac{2\sin(\epsilon)\cos(\epsilon)}{\cos(\epsilon)^2 - \sin^2(\epsilon)} \\ &\iff \frac{3}{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2} \le \frac{2}{\cos(\epsilon)^2 - \sin^2(\epsilon)} \\ &\iff \frac{2\cos(\epsilon)^2 - 3\sin(\epsilon)^2}{3} \ge \frac{\cos(\epsilon)^2 - \sin^2(\epsilon)}{2} \\ &\iff \frac{2\cos(\epsilon)^2}{6} \ge \frac{\sin^2(\epsilon)}{2} \iff \tan(\epsilon)^2 \le \frac{2}{3} \end{aligned}$$

which is true for $N \ge 7$. Hence for any $N > 6$, $x_M < 3\theta_N$. Notice moreover that $f(0) = \frac{3}{2}\sin(\epsilon)^2 + \cos(\epsilon)^2 \ge 0$ hence the map is maximal at x_M. As we are only considering even multiples $2k\theta_N$, by the symmetry property, the map

$$\frac{3}{2}\sin(3\theta_N/2)\sin((2k - \frac{3}{2})\theta_N) + \cos(3\theta_N/2)^2\cos(2k\theta_N)$$

is maximal for $k = 1$ with value

$$\begin{aligned} &\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) + \cos(3\theta_N/2)^2\cos(2\theta_N) \\ \le &\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) + \cos(3\theta_N/2)\cos(\theta_N/2). \end{aligned}$$

For the cases $N = 4, 5, 6$ one only has to check the inequality for $k = 2$. This can be done directly.

4.6 Computing with v_1

4.6.1 Setup

Recall that the vector v_1 is the second one in the cone, so we have

$$v_1 = \begin{bmatrix} -L_2 cc_1 \\ -\frac{cc_1^2}{s_1} \\ L_1 cc_2 \\ \frac{cc_2^2}{c_2} \end{bmatrix} \text{ so } (-R)^k v_1 = \begin{bmatrix} -L_2 \cdot cc_1 \cdot cp_1 + \frac{cc_1^2}{s_1} \cdot sp_1 \\ -L_2 \cdot cc_1 \cdot sp_1 - \frac{cc_1^2}{s_1} \cdot cp_1 \\ L_1 \cdot cc_2 \cdot cp_2 - \frac{cc_2^2}{c_2} \cdot sp_2 \\ L_1 \cdot cc_2 \cdot sp_2 + \frac{cc_2^2}{c_2} \cdot cp_2 \end{bmatrix}$$

Computing now $MA(-R)^k v_1$ we find:

$$\begin{bmatrix} L_1 L_2 (cp_2 - cp_1) + L_1 cc_1 \frac{sp_1}{s_1} - L_2 cc_2 \frac{sp_2}{s_2} \\ L_2^2 sp_1 s_1 - L_1^2 sp_2 s_2 + cc_1^2 \frac{sp_1}{s_1} - cc_2^2 \frac{sp_2}{s_2} \\ L_1 cp_2 - L_2 cp_1 + cc_1 \frac{sp_1}{s_1} - cc_2 \frac{sp_2}{s_2} \\ -(L_2+1) cc_1 cp_1 + (L_1+1) cc_2 cp_2 - L_2 sp_1 s_1 + L_1 sp_2 s_2 + cc_1^2 \frac{sp_1}{s_1} - cc_2^2 \frac{sp_2}{s_2} \end{bmatrix}$$

4.6.2 The first entry of $MA(-R)^k v_1$

Notice that it is similar to the second entry for v_0. As in Sect. 4.4.6 we split the proof into two cases. The proof is very similar but exchanges the two cases.

4.6.3 The case $N/2 \le k < N$

Notice that as $L_2 > L_1$, $1 - L_1 > 1 - L_2$ and $-cp_2 \ge -cp_1$ on this range since $cp_2 - cp_1 = 2\sin(k\theta_N)\sin(2k\theta_N) \le 0$. Multiplying these three inequalities, we get

$$-L_2(1 - L_1)cp_2 \ge -L_1(1 - L_2)cp_1$$

then

$$L_1 L_2 cp_2 - L_1 L_2 cp_1 \ge L_2 cp_2 - L_1 cp_1.$$

Hence the entry is not larger than

$$L_1 \left(cc_1 \frac{sp_1}{s_1} - cp_1 \right) - L_2 \left(cc_2 \frac{sp_2}{s_2} - cp_2 \right)$$

which is exactly the fourth coordinate of v_0, proved to be positive in Sect. 4.4.5.

4.6.4 The case $1 \le k \le N/2$

We compare the first entry with

$$L_1 (cp_1 + cc_1 \frac{sp_1}{s_1}) - L_2 (cp_2 + cc_2 \frac{sp_2}{s_2}).$$

which was proved to be positive in Sect. 4.4.8. Subtracting the above expression from the entry of interest, it suffices to show that

$$L_2(L_1 + 1)cp_2 - L_1(L_2 + 1)cp_1 \ge 0.$$

Using identities for L_i from Eq. (4.2.9) and factoring out $3(L_1+L_2)$ we reduce to

$$L_2\frac{cp_2}{cc_2} - L_1\frac{cp_1}{cc_1} \geq 0.$$

The factor $\frac{cp_2}{cc_1}$ is positive in this range of k, so we reduced to showing:

$$L_2\frac{cc_1}{cc_2} - L_1\frac{cp_1}{cp_2} \geq 0$$

This was done in Sect. 4.4.7.

4.6.5 The third entry of $MA(-R)^k v_1$

This entry is equal to

$$\begin{aligned}
&L_1 cp_2 - L_2 cp_1 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} = \\
&\quad = (L_1+1)cp_2 - (L_2+1)cp_1 + cp_1 - cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} \\
&\quad = 3(L_1+L_2)\left(\frac{cp_2}{cc_2} - \frac{cp_1}{cc_1}\right) + \left(cp_1 + cc_1\frac{sp_1}{s_1}\right) - \left(cp_2 + cc_2\frac{sp_2}{s_2}\right)
\end{aligned}$$

Using the formulas from Sect. 4.4.8 we get

$$= 3(L_1+L_2)\frac{cp_2}{cc_1}\left(\frac{cc_1}{cc_2} - \frac{cp_1}{cp_2}\right) - \frac{\sin\left(3(k-\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)} + \frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)}$$

with the angle tripling formula

$$\begin{aligned}
&= 6(L_1+L_2)\frac{cp_2}{cc_1}\left(\cos(2\theta_N) - \cos(2k\theta_N)\right) \\
&\quad + 2\cdot\frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\left(\cos(\theta_N) - \cos\left((2k-1)\theta_N\right)\right).
\end{aligned}$$

Using the difference of cosines formula, and factoring out 4, we get

$$\begin{aligned}
&3(L_1+L_2)\frac{cp_2}{cc_1}\sin\left((k+1)\theta_N\right)\sin\left((k-1)\theta_N\right) \\
&\quad + \frac{\sin\left((k-\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\sin\left(k\theta_N\right)\sin\left((k-1)\theta_N\right).
\end{aligned}$$

Factoring out $\frac{\sin((k-1)\theta_N)}{cc_1\sin(3\theta_N/2)}$, we get:

$$\begin{aligned}
&3(L_1+L_2)\cdot cp_2\cdot\sin\left((k+1)\theta_N\right)\cdot\sin(3\theta_N/2) \\
&+2\cdot\sin\left((k-\frac{1}{2})\theta_N\right)\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2.
\end{aligned}$$

For $1 \leq k \leq N/2$ this expression is clearly positive. Let us deal with $N/2 < k < N$. In this case, the first term of the sum is negative and the second positive. Using the fact that $L_1+L_2 < 1$ and $\sin\left((k-\frac{1}{2})\theta_N\right) > \sin\left((k+1)\theta_N\right)$ on this domain we have the following lower bound

$$\sin\left((k+1)\theta_N\right)\left(3\cdot\cos(k\theta_N)\cdot\sin(3\theta_N/2) + 2\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2\right).$$

Factoring out the sine factor and using the fact that for $N \geq 5$, $\cos(3\theta_N/2) \geq \frac{1}{2}$ we observe that

$$\begin{aligned}
&3 \cdot \cos(k\theta_N) \cdot \sin(3\theta_N/2) + 2 \cdot \sin(k\theta_N) \cdot \cos(3\theta_N/2)^2 \\
&\geq 3 \cdot \cos(k\theta_N) \cdot \sin(3\theta_N/2) + \sin(k\theta_N) \cdot \cos(3\theta_N/2) \\
&\geq 3\sin((k+\frac{3}{2})\theta_N) - 2\cos(3\theta_N/2) \cdot \sin(k\theta_N) \\
&\geq 3\sin((k+\frac{3}{2})\theta_N) - \sin(k\theta_N) \\
&\geq 3\sin(k\theta_N) - \frac{3}{2}\theta_N - \sin(k\theta_N) \\
&\geq 2\sin(k\theta_N) - \frac{3}{2}\theta_N.
\end{aligned}$$

This last expression is minimal for $k = N-1$. We are then reduced to showing that $\sin(\theta_N) \geq \frac{3}{4}\theta_N$. This is true for $N > 4$ since $\cos(x) \geq \frac{3}{4}$ for all $x \in [0, \frac{\pi}{5}]$ and can be checked directly for $N = 4$.

4.6.6 The fourth entry of $MA(-R)^k v_1$

With the identity

$$cc_i^2 + s_i^2 = 2cc_i$$

for $i \in \{0, 1\}$, we can rewrite the fourth coordinate as

$$\begin{aligned}
&-(L_2+1)cc_1cp_1 + (L_1+1)cc_2cp_2 - (L_2+1)sp_1s_1 + (L_1+1)sp_2s_2 \\
&+2\left(cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}\right) \\
&\quad = 3(L_1+L_2)\left(-cp_1 + cp_2 - \frac{s_1}{cc_1}sp_1 + \frac{s_2}{cc_2}sp_2\right) + 2\left(cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}\right).
\end{aligned}$$

4.6.7 The case $1 \leq k \leq N/2$

Notice that the second term of the sum is exactly the fourth coordinate for v_3 proved to be positive in Sect. 4.5.4.

The first term of the sum can be rewritten, using Proposition 4.2.20, as

$$4 \cdot 3(L_1+L_2)\frac{\cos\left((k+\frac{1}{2})\theta_N\right)}{\cos(3\theta_N/2)}\sin\left((k+1)\theta_N\right)\sin(k\theta_N)$$

which is positive for $1 \leq k < N/2$.

4.6.8 The case $N/2 \leq k < N$

In this case

$$3(L_1+L_2)\left(-cp_1 + cp_2 - \frac{s_1}{cc_1}sp_1 + \frac{s_2}{cc_2}sp_2\right) \geq 3\left(-cp_1 + cp_2 - \frac{s_1}{cc_1}sp_1 + \frac{s_2}{cc_2}sp_2\right)$$

Using the previous computation and Sect. 4.5.4, we are reduced to showing positivity of

$$4 \cdot 3 \frac{\cos\left((k+\frac{1}{2})\theta_N\right)}{\cos(3\theta_N/2)} \sin\left((k+1)\theta_N\right) \sin(k\theta_N)$$
$$+2 \cdot 2 \sin(k\theta_N) \left(\frac{\cos(\theta_N/2)}{\sin(3\theta_N/2)} - \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)} \cos(2k\theta_N) \right).$$

Factoring $\frac{4\sin(k\theta_N)}{\sin(3\theta_N/2)\cos(3\theta_N/2)}$ out yields:

$$3\sin(3\theta_N/2)\cos\left((k+\frac{1}{2})\theta_N\right)\sin((k+1)\theta_N)$$
$$+\cos(3\theta_N/2)\left(\cos(\theta_N/2) - \cos(3\theta_N/2)\cos(2k\theta_N)\right).$$

Using the relation

$$2\cos((k+\frac{1}{2})\theta_N)\sin((k+1)\theta_N) = \sin\left((2k+\frac{3}{2})\theta_N\right) + \sin(\theta_N/2),$$

we can rewrite

$$\frac{3}{2}\sin(3\theta_N/2)\sin\left((2k+\frac{3}{2})\theta_N\right) - \cos(3\theta_N/2)^2\cos(2k\theta_N)$$
$$+\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) + \cos(3\theta_N/2)\cos(\theta_N/2)$$

which positivity is equivalent to

$$\frac{3}{2}\sin(3\theta_N/2)\cos\left((2k+\frac{3}{2})\theta_N\right) - \cos(3\theta_N/2)^2\cos(2k\theta_N)$$
$$\geq -\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) - \cos(3\theta_N/2)\cos(\theta_N/2).$$

We now need to understand the minimal value of the function

$$\frac{3}{2}\sin(3\theta_N/2)\sin\left((2k+\frac{3}{2})\theta_N\right) - \cos(3\theta_N/2)^2\cos(2k\theta_N) =$$
$$a\sin(x-\epsilon) + b\cos(x) = f(x)$$

where $a = \frac{3}{2}\sin(3\theta_N/2)$, $b = -\cos(3\theta_N/2)^2$ and $\epsilon = 3\theta_N/2$. Then according to Proposition 4.5.8, if x_M is a point on which the function is extremal, then

$$\tan(x_M) = \frac{a\cos(\epsilon)}{b - a\sin(\epsilon)}$$
$$= \frac{3\sin(\epsilon)\cos(\epsilon)}{-2\cos(\epsilon)^2 - 3\sin(\epsilon)^2}.$$

Hence

$$\tan(-x_M) \leq \tan(2\epsilon) \iff \frac{3\sin(\epsilon)\cos(\epsilon)}{2\cos(\epsilon)^2 + 3\sin(\epsilon)^2} \leq \frac{2\sin(\epsilon)\cos(\epsilon)}{\cos(\epsilon)^2 - \sin(\epsilon)^2}$$
$$\iff \frac{3}{2\cos(\epsilon)^2 + 3\sin(\epsilon)^2} \leq \frac{2}{\cos(\epsilon)^2 - \sin(\epsilon)^2}$$
$$\iff \frac{2\cos(\epsilon)^2 + 3\sin(\epsilon)^2}{3} \geq \frac{\cos(\epsilon)^2 - \sin(\epsilon)^2}{2}$$
$$\iff 4\cos(\epsilon)^2 + 6\sin(\epsilon)^2 \geq 3\cos(\epsilon)^2 - 3\sin(\epsilon)^2$$

which is true. Hence $x_M \in (\pi - 2\epsilon, \pi)$ and moreover as $f(0) = -\frac{3}{2}\sin(\epsilon)^2 - \cos(\epsilon)^2 < 0$, x_M is the maximum of f and its minimum is at $x_m = x_M + \pi \in (2\pi - 2\epsilon, 2\pi)$. Thus the above expression takes its minimum value for $k = N-1$ and is bounded from below by

$$\frac{3}{2}\sin(3\theta_N/2)\cos\left((2(N-1)+\frac{3}{2})\theta_N\right) - \cos(3\theta_N/2)^2\cos\left(2(N-1)\theta_N\right)$$
$$\frac{3}{2}\sin(3\theta_N/2)\cos\left(\theta_N/2\right) - \cos(3\theta_N/2)^2\cos\left(\theta_N\right)$$
$$\geq -\frac{3}{2}\sin(3\theta_N/2)\sin(\theta_N/2) - \cos(3\theta_N/2)\cos(\theta_N/2).$$

The last inequality follows from the fact that $\cos(3\theta_N/2)^2\cos(\theta_N) \leq \cos(3\theta_N/2)\cos(\theta_N/2)$.

4.6.9 The second entry of $MA(-R)^k v_1$

$$\begin{aligned} &L_2^2 sp_1 s_1 - L_1^2 sp_2 s_2 + cc_1^2\frac{sp_1}{s_1} - cc_2^2\frac{sp_2}{s_2} \\ &= (L_2^2 s_1 + \frac{cc_1^2}{s_1})sp_1 - (L_1^2 s_2 + \frac{cc_2^2}{s_2})sp_2. \end{aligned}$$

Factoring out $\sin(k\theta_N)$ we get

$$-(L_2^2 s_1 + \frac{cc_1^2}{s_1})\,(2\cos(2k\theta_N)+1) + (L_1^2 s_2 + \frac{cc_2^2}{s_2}).$$

As cosine is decreasing on $[0, \pi]$, and increasing on $[\pi, 2\pi]$, we only have to check that

$$-(L_2^2 s_1 + \frac{cc_1^2}{s_1})\,(2\cos(2\theta_N)+1) + (L_1^2 s_2 + \frac{cc_2^2}{s_2}) \geq 0$$

and

$$-(L_2^2 s_1 + \frac{cc_1^2}{s_1})\,(2\cos(2(N-1)\theta_N)+1) + (L_1^2 s_2 + \frac{cc_2^2}{s_2}) \geq 0.$$

These two inequalities are equivalent to

$$\begin{aligned} &\frac{L_1^2 s_2 + \frac{cc_2^2}{s_2}}{L_2^2 s_1 + \frac{cc_1^2}{s_1}} \geq 2\cos(2\theta_N)+1 \\ \iff\ &\frac{s_1}{s_2}\cdot\frac{L_1^2 s_2^2 + cc_2^2}{L_2^2 s_1^2 + cc_1^2} \geq 2\cos(2\theta_N)+1 \\ \iff\ &(2\cos(3\theta_N/2)+1)\,\frac{L_1^2 s_2^2 + cc_2^2}{L_2^2 s_1^2 + cc_1^2} \geq 2\cos(2\theta_N)+1. \end{aligned}$$

Again as cosine is decreasing on $[0, \pi]$, we only need to show

$$\begin{aligned} &L_1^2 s_2^2 + cc_2^2 \geq L_2^2 s_1^2 + cc_1^2 \\ \iff\ &(cc_1 + cc_2)(cc_2 - cc_1) \geq (L_2 s_1 + L_1 s_2)(L_2 s_2 - L_1 s_1) \end{aligned}$$

Notice that $(L_2s_1 + L_1s_2)(L_2s_2 - L_1s_1) \leq s_2$ and $(cc_1 + cc_2)(cc_2 - cc_1) = 2(cc_1 + cc_2)\sin(\theta_N)\sin(2\theta_N)$. Hence the inequality follows from the fact that $2(cc_1 + cc_2) \geq 1$.

4.7 Computing with Bv_1

4.7.1 Setup

We finally have to handle the vector Bv_1, namely

$$Bv_1 = \begin{bmatrix} L_2 \cdot cc_1 \\ -\frac{cc_1^2}{s_1} \\ -L_1 \cdot cc_2 \\ \frac{cc_2^2}{s_2} \end{bmatrix} \text{ so } (-R)^k Bv_1 = \begin{bmatrix} L_2cc_1 \cdot cp_1 + \frac{cc_1^2}{s_1} sp_1 \\ L_2cc_1 \cdot sp_1 - \frac{cc_1^2}{s_1} \cdot cp_1 \\ -L_1cc_2 \cdot cp_2 - \frac{cc_2^2}{s_2} \cdot sp_2 \\ -L_1cc_2 \cdot sp_2 + \frac{cc_2^2}{s_2} \cdot cp_2 \end{bmatrix}$$

Computing now $MA(-R)^k Bv_1$ we find:

$$\begin{bmatrix} L_1L_2(cp_1 - cp_2) + L_1cc_1\frac{sp_1}{s_1} - L_2cc_2\frac{sp_2}{s_2} \\ 2L_2cc_1cp_1 - 2L_1cc_2cp_2 + L_1^2 sp_2s_2 - L_2^2 sp_1s_1 + cc_1^2\frac{sp_1}{s_1} - cc_2^2\frac{sp_2}{s_2} \\ L_2cp_1 - L_1cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} \\ (L_2 - 1)cc_1cp_1 - (L_1 - 1)cc_2cp_2 + L_2sp_1s_1 - L_1sp_2s_2 + cc_1^2\frac{sp_1}{s_1} - cc_2^2\frac{sp_2}{s_2} \end{bmatrix}$$

4.7.2 The first entry of $MA(-R)^k Bv_1$

Notice that it corresponds to the second entry for v_0 proved to be positive in Sect. 4.4.6.

4.7.3 The third entry of $M_{test}(-R)^k Bv_1$

This entry is very similar to the one in Sect. 4.6.5 and we follow the same scheme of proof.

$$\begin{aligned} &L_2cp_1 - L_1cp_2 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} \\ &\quad = (L_2 + 1)cp_1 - (L_1 + 1)cp_2 + cp_2 - cp_1 + cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2} \\ &\quad = 3(L_1 + L_2)\left(\frac{cp_1}{cc_1} - \frac{cp_2}{cc_2}\right) - \left(cp_1 - cc_1\frac{sp_1}{s_1}\right) + \left(cp_2 - cc_2\frac{sp_2}{s_2}\right) \end{aligned}$$

Using formulas from Sect. 4.4.5 we get

$$= 3(L_1 + L_2)\frac{cp_2}{cc_1}\left(\frac{cp_1}{cp_2} - \frac{cc_1}{cc_2}\right) - \frac{\sin\left(3(k + \frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)} + \frac{\sin\left((k + \frac{1}{2})\theta_N\right)}{\sin(\theta_N/2)}$$

with the angle tripling formula

$$\begin{aligned} &= -6(L_1 + L_2)\frac{cp_2}{cc_1}\left(\cos(2\theta_N) - \cos(2k\theta_N)\right) \\ &\quad + 2 \cdot \frac{\sin\left((k + \frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\left(\cos(\theta_N) - \cos\left((2k + 1)\theta_N\right)\right). \end{aligned}$$

Using the difference of cosines formula, and factoring out 4, we get

$$-3(L_1+L_2)\frac{cp_2}{cc_1}\sin\left((k+1)\theta_N\right)\sin\left((k-1)\theta_N\right)$$
$$+\frac{\sin\left((k+\frac{1}{2})\theta_N\right)}{\sin(3\theta_N/2)}\sin\left(k\theta_N\right)\sin\left((k-1)\theta_N\right).$$

Factoring be $\frac{\sin((k-1)\theta_N)}{cc_1\sin(3\theta_N/2)}$,

$$-3(L_1+L_2)\cdot cp_2\cdot\sin\left((k+1)\theta_N\right)\cdot\sin(3\theta_N/2)$$
$$+2\cdot\sin\left((k+\frac{1}{2})\theta_N\right)\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2.$$

For $N/2\leq k<N$ this expression is clearly positive. Let us deal with $1\leq k<N/2$. In this case, the first term of the sum is negative and the second positive. Using the fact that $L_1+L_2<1$ and $\sin\left((k+1)\theta_N\right)<\sin\left((k+\frac{1}{2})\theta_N\right)$ on this domain we have the following lower bound

$$\sin\left((k+\frac{1}{2})\theta_N\right)\left(-3\cdot\cos(k\theta_N)\cdot\sin(3\theta_N/2)+2\cdot\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)^2\right).$$

Factoring out the first and using the fact that for $N\geq 5$, $\cos(3\theta_N/2)\geq\frac{1}{2}$ we observe that

$$\begin{aligned}-3\cdot\cos(k\theta_N)\cdot\sin(3\theta_N/2)+\sin\left(k\theta_N\right)\cdot\cos(3\theta_N/2)\\ \geq 3\sin((k-\frac{3}{2})\theta_N)-2\cos(3\theta_N/2)\cdot\sin(k\theta_N)\\ \geq 3\sin((k-\frac{3}{2})\theta_N)-\sin(k\theta_N)\\ \geq 3\sin(k\theta_N)-\frac{3}{2}\theta_N-\sin(k\theta_N)\\ \geq 2\sin(k\theta_N)-\frac{3}{2}\theta_N.\end{aligned}$$

This last expression is minimal for $k=1$. We are then reduced to showing that $\sin(\theta_N)\geq\frac{3}{4}\theta_N$. This was proved to be true for all $N\geq 4$ in Sect. 4.6.5.

4.7.4 The second entry of $MA(-R)^kBv_1$

Let us give an alternative expression for the second entry.

$$\begin{aligned}2L_2cc_1cp_1-2L_1cc_2cp_2+L_1^2sp_2s_2-L_2^2sp_1s_1+cc_1^2\frac{sp_1}{s_1}-cc_2^2\frac{sp_2}{s_2}\\ =3(L_1+L_2)(cp_1-cp_2)-2cc_1cp_1+2cc_2cp_2\\ +(L_1^2+1)sp_2s_2-(L_2^2+1)sp_1s_1+2cc_1\frac{sp_1}{s_1}-2cc_2\frac{sp_2}{s_2}\\ =3(L_1+L_2)(cp_1-cp_2)\\ +(L_1^2+1)sp_2s_2-(L_2^2+1)sp_1s_1+2cc_1\left(\frac{sp_1}{s_1}-cp_1\right)-2cc_2\left(\frac{sp_2}{s_2}-cp_2\right)\end{aligned}$$

4.7.5 The case $1 \leq k \leq N/2$

Notice that on this domain, we have $cp_1 < cp_2$ and $cc_2cp_2 - cc_1cp_1 > 0$. Thus we are reduced to showing positivity for

$$3(L_1 + L_2)(cp_1 - cp_2) + (L_1^2 + 1)sp_2s_2 - (L_2^2 + 1)sp_1s_1 + 2cc_1\frac{sp_1}{s_1} - 2cc_2\frac{sp_2}{s_2}.$$

Factoring $\sin(k\theta_N)$ out, we get

$$\begin{aligned}
&- 6(L_1 + L_2)\sin(2k\theta_N) + \left(2\frac{cc_2}{s_2} - (L_1^2 + 1)s_2\right) \\
&\quad - \left(2\frac{cc_1}{s_1} - (L_2^2 + 1)s_1\right)(2\cos(2k\theta_N) + 1) \\
&= \left(2\frac{cc_2}{s_2} - (L_1^2 + 1)s_2\right) - \left(2\frac{cc_1}{s_1} - (L_2^2 + 1)s_1\right) \\
&\quad - 2\left(\left(2\frac{cc_1}{s_1} - (L_2^2 + 1)s_1\right)\cos(2k\theta_N) + 3(L_1 + L_2)\sin(2k\theta_N)\right)
\end{aligned}$$

Let us determine the maximal value of

$$\left(2\frac{cc_1}{s_1} - (L_2^2 + 1)s_1\right)\cos(2k\theta_N) + 3(L_1 + L_2)\sin(2k\theta_N) = f(2k\theta_N).$$

There exists a unique $x_M \in [0, \pi)$ such that $f(x_M)$ is maximal, since $L_1 + L_2 > 0$, and it satisfies

$$\tan(x_M) = \frac{3(L_1 + L_2)s_1}{2cc_1 - (L_2^2 + 1)s_1} \leq \frac{3s_1}{2 - 2s_1^2} = \frac{3}{2c_1}\tan(\theta_N).$$

For $N \geq 5$, $\cos(3\pi/2N) \geq \frac{1}{2}$, hence

$$\tan(x_M) \leq 3\tan(\theta_N).$$

By convexity of the tangent function on $[0, \frac{\pi}{2})$ this is less than $\tan(2\theta_N)$. This implies that the maximum of the entry is reached at $k = 1$. Hence the minimal value of the entry on this domain is also reached at $k = 1$ which evaluated on the first expression gives

$$\begin{aligned}
&3(L_1 + L_2)(c_1 - c_2) - L_1^2s_2^2 + L_2^2s_1^2 - cc_1^2 + cc_2^2 \\
&= 3(L_1 + L_2)(c_1 - c_2) + (cc_1 + cc_2)(c_2 - c_1) + L_2^2s_1^2 - L_1^2s_2^2 \\
&\geq (cc_1 + cc_2 - 3)(c_2 - c_1).
\end{aligned}$$

This last expression is positive, since $\cos(3\theta_N) + \cos(\theta_N) = 2\cos(\theta_N)(\cos(2\theta_N) + 2)$ which is increasing with N and equal to $\frac{4}{\sqrt{2}} > 1$ for $N = 4$.

4.7.6 The case $N/2 < k < N$

We compute the difference with the fourth entry

$$-L_2(L_2 + 1)sp_1s_1 + L_1(L_1 + 1)sp_2s_2 + 3(L_1 + L_2)(cp_1 - cp_2)$$

Factoring $\sin(k\theta_N)$ out, we get

$$L_2(L_2+1)s_1(2\cos(2k\theta_N)+1) - L_1(L_1+1)s_2 - 6(L_1+L_2)\sin(2k\theta_N).$$

Notice that this expression evaluated at $k=0$ is

$$3L_2(L_2+1)s_1 - L_1(L_1+1)s_2 \geq 2L_2(L_2+1)s_1 > 0.$$

And at $k = \frac{N}{2} + \frac{1}{2}$,

$$\begin{aligned} L_2(L_2+1)s_1(-2c_2+1) - L_1(L_1+1)s_2 + 6(L_1+L_2)s_2 \\ \geq 6(L_1+L_2)s_2 - L_2(L_2+1)s_1 - L_1(L_1+1)s_2 \\ \geq 6(L_1+L_2)s_2 - 2s_1(L_1+L_2). \end{aligned}$$

Where we have assumed that N is large enough for L_1 to be positive. Otherwise we have the lower bound

$$6(L_1+L_2)s_2 - L_2(L_2+1)s_1 \geq 6s_2 - 2s_1.$$

Hence positivity of the expression follows from the inequality $3s_2 \geq s_1$ *i.e.* $2\cos(2\theta_N)+1 \leq 3$.

4.7.7 The fourth entry of $MA(-R)^k Bv_1$

We rewrite it as:

$$\begin{aligned} &(L_2-1)cc_1cp_1 - (L_1-1)cc_2cp_2 + L_2sp_1s_1 - L_1sp_2s_2 + cc_1^2\frac{sp_1}{s_1} - cc_2^2\frac{sp_2}{s_2} \\ &= (1-L_1)cc_2cp_2 - (1-L_2)cc_1cp_1 + (1-L_1)sp_2s_2 - (1-L_2)sp_1s_1 \\ &\quad + 2\left(cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}\right) \\ &= 3(L_1+L_2)\left(cp_1 - cp_2 + sp_1\frac{s_1}{cc_1} - sp_2\frac{s_2}{cc_2}\right) \\ &\quad -2(cc_1cp_1 - cc_2cp_2 + sp_1s_1 - sp_2s_2) + 2\left(cc_1\frac{sp_1}{s_1} - cc_2\frac{sp_2}{s_2}\right) \end{aligned}$$

4.7.8 The case $1 \leq k \leq N/2$

We have proved in Sect. 4.7.5 positivity of

$$3(L_1+L_2)(cp_1-cp_2) + (L_1^2+1)sp_2s_2 - (L_2^2+1)sp_1s_1 + 2cc_1\frac{sp_1}{s_1} - 2cc_2\frac{sp_2}{s_2}.$$

We compute the difference of the fourth entry with this expression

$$2(cc_2cp_2 - cc_1cp_1) + (L_2^2+L_2+1)sp_1s_1 - (L_1^2+L_1+1)sp_2s_2.$$

As $cc_2cp_2 > cc_1cp_1$ on the domain, we are reduced to showing

$$(L_2^2+L_2+1)sp_1s_1 \geq (L_1^2+L_1+1)sp_2s_2$$

which is true since $s_1 > s_2 > 0$, $sp_1 = (2\cos(2k\theta_N)+1)sp_2 \geq sp_2 \geq 0$ on the domain, $L_1 < L_2$ and $L_1^2 + L_1 + 1 \geq 0$. This last inequality follows from the fact that $L_2 > \frac{1}{2}$, $3(L_1+L_2) = (L_2+1)cc_2 > 0$ and finaly $L_1 + \frac{1}{2} > 0$.

4.7.9 The case $N/2 < k < N$

First notice that, by Proposition 4.2.20,

$$cp_1 - cp_2 + sp_1\frac{s_1}{cc_1} - sp_2\frac{s_2}{cc_2} = -4 \cdot \frac{\cos((k+\frac{1}{2})\theta_N)}{\cos(3\theta_N/2)} \cdot \sin(k\theta_N) \cdot \sin((k+1)\theta_N)$$

which is non negative on the domain. Thus we are reduced to showing positivity of

$$\frac{cc_1}{s_1}sp_1 - cc_1cp_1 - sp_1s_1 - \frac{cc_2}{s_2}sp_2 + cc_2cp_2 + sp_2s_2.$$

Now, notice that

$$\begin{aligned}\frac{cc_1}{s_1}sp_1 - cc_1cp_1 - sp_1s_1 &= \frac{\cos(3\theta_N/2)}{\sin(3\theta_N/2)}(-\sin(3k\theta_N)) - (1+\cos(3\theta_N))\,cp_1 - sp_1s_1 \\ &= \frac{1}{\sin(3\theta_N/2)}\big(\cos(3\theta_N/2)(-\sin(3k\theta_N)) - \cos(3k\theta_N)\sin(3\theta_N/2)\big) \\ &\quad - \cos(3\theta_N)\cos(3k\theta_N) + \sin(3k\theta_N)\sin(3\theta_N) \\ &= \frac{-\sin(3(k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)} - \cos(3(k+1)\theta_N).\end{aligned}$$

Similarly, we have

$$\frac{cc_2}{s_2}sp_2 - cc_2cp_2 - sp_2s_2 = \frac{-\sin((k+\frac{1}{2})\theta_N)}{\sin(\theta_N/2)} - \cos((k+1)\theta_N).$$

Subtracting the two terms, we get

$$\begin{aligned}&\frac{-\sin(3(k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2} + \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(\theta_N/2)} - \cos(3(k+1)\theta_N) + \cos((k+1)\theta_N) \\ &= \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}(2\cos(\theta_N) - 2\cos((2k+1)\theta_N)) - \cos(3(k+1)\theta_N) + \cos((k+1)\theta_N)\end{aligned}$$

Using the formula for sum of cosines,

$$\begin{aligned}&= 4 \cdot \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\sin(k\theta_N)\sin((k+1)\theta_N) + 2 \cdot \sin((k+1)\theta_N)\sin(2(k+1)\theta_N) \\ &= 4 \cdot \frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\sin(k\theta_N)\sin((k+1)\theta_N) + 4 \cdot \sin((k+1)\theta_N)^2\cos((k+1)\theta_N) \\ &= 4 \cdot \sin((k+1)\theta_N)\left(\frac{\sin((k+\frac{1}{2})\theta_N)}{\sin(3\theta_N/2)}\sin(k\theta_N) + \sin((k+1)\theta_N)\cos((k+1)\theta_N)\right).\end{aligned}$$

The sine function is decreasing on this domain, hence we bound from below by

$$4\sin((k+1)\theta_N)^2\left(\frac{\sin(k\theta_N)}{\sin(3\theta_N/2)} + \cos((k+1)\theta_N)\right).$$

We are thus reduced to showing positivity for

$$\sin(k\theta_N) + \sin(3\theta_N/2)\cos((k+1)\theta_N)$$

in the range $k > \frac{N}{2}$ and $k + 1 < N$. As this last expression is decreasing on the domain, we only have to check it for $k = N - 2$, i.e.

$$\sin(2\theta_N) - \sin(3\theta_N/2)\cos(\theta_N) \geq 0.$$

This is implied by the fact that $\sin(2\theta_N) \geq \sin(3\theta_N/2)$.

5 Symplectic geometry and causality

Outline of section Introduced by Drumm in [10], crooked surfaces are used in Lorenzian geometry to produce fundamental domains for group actions, see e.g. [8]. The basic definitions and constructions are introduced in Sect. 5.1. In Sect. 5.2 we will reinterpret some of the disjointness criteria for crooked surfaces from [2] using cones. We extend their analysis to situations when crooked surfaces can touch, a geometric situation that occurs in our case. Finally, in Sect. 5.3, we will construct a domain of discontinuity for the action of Γ_N on $\mathrm{LGr}(V)$. The construction will be in two stages: first an open set Ω° built directly from the definition of crooked surfaces, then a larger domain Ω where we have added some sets where the crooked surfaces "touch", but on which the action is nonetheless properly discontinuous.

General conventions

To lighten the notation, when it is clear from the context a nonzero element in a vector space and the induced line in the projectivization will carry the same notation. Given their structure, it seems natural to us to call the objects in this section "winged surfaces" instead of "crooked surfaces". We will continue to use the term "crooked surface" but denote them by $\mathcal{WS}$, to denote that they consist of two wings and a stem.

5.1 Crooked surfaces

5.1.1 Symplectic conventions

Let V be a real 4-dimensional symplectic vector space. Fix a basis e_1, f_1, e_2, f_2 such that the symplectic pairing satisfies

$$\langle e_1, f_1\rangle = \langle e_2, f_2\rangle = 1$$

Fix also an anti-symplectic involution A given by the formula:

$$Ae_i = -e_i \quad Af_i = +f_i$$

Let us note for convenience of reference that our basis is related to the one used in [2, §5] by:

$$\begin{aligned} e_1 &= u_+ \; f_1 = -u_- \\ e_2 &= v_+ \; f_2 = v_- \end{aligned} \tag{5.1.2}$$

Note in particular the minus sign in front of u_-, which we hope minimizes the number of further negative signs later.

Given two vectors $v, v' \in V$, not proportional, we will denote by $L_{vv'}$ their 2-dimensional span or its projectivization. Typically we will consider the case when this is a Lagrangian.

5.1.3 The cone

In analogy with our constructions in previous sections, we will consider the cone

$$\mathcal{C} := \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}f_1 + \mathbb{R}_{\geq 0}f_2$$

We regard this as a projective cone $\mathcal{C} \subset \mathbb{P}(V_{\mathbb{R}})$, and later will denote by $\mathring{\mathcal{C}}$ the interior of the cone, also in projective space. The projective cone is a tetrahedron with four of the edges contained in the Lagrangian planes (projective lines):

$$L_{e_1e_2} := \operatorname{span}(e_1, e_2)\ L_{e_1f_2} := \operatorname{span}(e_1, f_2)$$
$$L_{f_1f_2} := \operatorname{span}(f_1, f_2)\ L_{e_2f_1} := \operatorname{span}(e_2, f_1)$$

The remaining two edges are contained in the projectivization of subspaces orthogonal for the symplectic form, and on which the symplectic form is non-degenerate:

$$S_1 = \operatorname{span}(e_1, f_1) \quad S_2 = \operatorname{span}(e_2, f_2)$$

We will associate a crooked surface to a basis e_1, e_2, f_1, f_2 as above. Equivalently, and more intrinsically, a crooked surface is associated to an anti-symplectic involution A of a 4-dimensional symplectic vector space, together with a choice of basis e_1, e_2 of the (-1)-eigenspace of A (the dual basis f_1, f_2 of the $(+1)$-eigenspace is then uniquely determined). A graphical rendering of a crooked surface is available in [2, Fig. 2].

5.1.4 Indefinite inner product conventions

Let now $W := \Lambda^2_0 V$ denote the subspace of the second exterior power which wedges to zero against $e_1 \wedge f_1 + e_2 \wedge f_2$ (or equivalently is in the kernel of the symplectic form). Then W is equipped with a nondegenerate quadratic form of signature $(2, 3)$ given by taking the wedge product of elements and using the trivialization of $\Lambda^4 V$ by the volume form induced from the symplectic pairing. Given these sign conventions, we will say that a subspace is time-like if it is positive definite, and space-like if it is negative definite.

We will use the explicit basis of W given by

$$e_1 \wedge e_2 \quad e_1 \wedge f_2 \quad e_1 \wedge f_1 - e_2 \wedge f_2 \quad f_1 \wedge e_2 \quad f_1 \wedge f_2$$

In this basis the induced involution (which actually preserves the inner product) has eigenvalue $+1$ on $e_1 \wedge e_2$ and $f_1 \wedge f_2$, and eigenvalue -1 on the remaining three basis vectors.

5.1.5 The Lagrangian Grassmannian

Recall next that the Lagrangian Grassmannian $\operatorname{LGr}(V)$ is equal to the quadric of null vectors in $\mathbb{P}(W)$:

$$\operatorname{LGr}(V) = \{[w] \in \mathbb{P}(W) \colon w^2 = 0\} \subset \mathbb{P}(W).$$

It is equipped with a conformal class of Lorenzian metrics of signature $(1, 2)$ and is frequently also called an Einstein universe and denoted $\operatorname{Ein}^{1,2}$.

5.1.6 Photons

Associated to a nonzero vector $v \in V$ (rather, the corresponding point in $\mathbb{P}(V)$) there is a "photon" of Lagrangians:

$$\phi(v) := \{L \in \mathrm{LGr}(V) \colon v \in L\} \subset \mathrm{LGr}(V)$$

The photon can also be identified as

$$\phi(v) = \mathbb{P}\left(v \wedge v^{\perp}\right) \cong \mathbb{P}\left(v^{\perp}/v\right)$$

where $v^{\perp} \subset V$ denotes the symplectic-orthogonal to v.

5.1.7 The wings of the crooked surface

We can now define the crooked surface. It consists of two "wings" and a "stem" (the stem also decomposes into two pieces, see Sect. 5.1.9 below).

Consider the "interval of lines"

$$[e_1, e_2] := \{se_1 + te_2 \colon s, t \geq 0, s + t = 1\} \subset \mathbb{P}(V) \tag{5.1.8}$$

which is one of the boundary edges of the cone $\mathcal{C}$. Then the e-wing is defined as:

$$\mathcal{W}_e := \phi([e_1, e_2]) = \bigcup_{l \in [e_1, e_2]} \phi(l).$$

Analogously define the f-wing:

$$\mathcal{W}_f := \phi([f_1, f_2]).$$

5.1.9 The stem of the crooked surface

Consider the "Einstein torus" consisting of Lagrangians spanned by one vector in each of S_1, S_2:

$$\mathrm{Ein}^{1,1}(S_1, S_2) = \{L = w_1 \wedge w_2 \colon w_i \in S_i\} \tag{5.1.10}$$

Then the stem is defined as:

$$\mathcal{S} := \{L \in \mathrm{Ein}^{1,1}(S_1, S_2) \colon |\mathrm{Maslov}(L_{e_1 f_2}, L, L_{e_2 f_1})| = 2\}$$

Note that we can further decompose the stem as $\mathcal{S} = \mathcal{S}^+ \coprod \mathcal{S}^-$ according to the sign of the Maslov index (see Sect. 5.1.12 below for the definition of the Maslov index).

Set now the crooked surface to be

$$\mathcal{WS} := \mathcal{W}_e \coprod \mathcal{S} \coprod \mathcal{W}_f \tag{5.1.11}$$

Observe that according to the definitions, the wings are relatively closed subsets, while the stem is a relatively open set in $\mathcal{WS}$.

5.1.12 Making the stem explicit

Recall that the Maslov index of a Lagrangian, relative to two specified transverse Lagrangians, is defined as the index of the quadratic form obtained from the symplectic form using the direct sum decomposition provided by the two transverse Lagrangians. In the case at hand $V = L_{e_1 f_2} \oplus L_{e_2 f_1}$ and the quadratic form, denoted Q_{12}, comes out to be

$$\begin{aligned} Q_{12}(\alpha_1 e_1 + \alpha_2 e_2 + \beta_1 f_1 + \beta_2 f_2) &:= \langle \alpha_1 e_1 + \beta_2 f_2, \alpha_2 e_2 + \beta_1 f_1 \rangle \\ &= \alpha_1 \cdot \beta_1 - \alpha_2 \cdot \beta_2 \end{aligned}$$

If the Lagrangian L is spanned by $w_i \in S_i$ with coordinates

$$w_1 = \alpha_1 e_1 + \beta_1 f_1 \quad w_2 = \alpha_2 e_2 + \beta_2 f_2$$

then we observe that w_1 and w_2 are orthogonal with respect to Q_{12} and $Q_{12}(w_i) = (-1)^{i+1} \alpha_i \cdot \beta_i$. So for the Lagrangian to belong to the stem both products have to be of opposite sign. We thus have:

$$\begin{array}{lcc} L = w_1 \wedge w_2 \text{ belongs to:} & \mathcal{S}^+ & \mathcal{S}^- \\ w_1 = \alpha_1 e_1 + \beta_1 f_1 & \alpha_1 \cdot \beta_1 > 0 & \alpha_1 \cdot \beta_1 < 0 \\ w_2 = \alpha_2 e_2 + \beta_2 f_2 & \alpha_2 \cdot \beta_2 < 0 & \alpha_2 \cdot \beta_2 > 0 \end{array}$$

5.1.13 Action of reflection

The anti-symplectic involution A preserves the crooked surface as a set. Furthermore it exchanges the two components of the stem: $A\mathcal{S}^{\pm} = \mathcal{S}^{\mp}$ and fixes as a set each photon on the wings. On individual photons on the wings, it fixes two points and exchanges the two complementary regions. Explicitly, on the photon $\phi(s \cdot e_1 + t \cdot e_2)$ belonging to the e-wing, the fixed points are the two Lagrangians $e_1 \wedge e_2$ and $(se_1 + te_2) \wedge (-tf_1 + sf_2)$. The formula for photons on the f-wing is analogous.

5.2 Disjointness of crooked surfaces

In this section, we proceed to study the geometric configurations that crooked surfaces, photons, and Lagrangians, can be in. First, some of the results from [2] can be reinterpreted using the cones that we introduced earlier. This gives transparent geometric conditions for when photons, or crooked surfaces, are disjoint. We then further refine our analysis to situations when crooked surfaces can "touch", an inevitable situation when facing groups with unipotent elements.

Proposition 5.2.1 (Disjointness of photon from crooked surface) *Consider a vector $v \in V$ with coordinates*

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \beta_1 f_1 + \beta_2 f_2.$$

The following are equivalent:

(i) *The photon $\phi(v) \in \mathrm{LGr}(V)$ is disjoint from the crooked surface $\mathcal{WS}$.*
(ii) *The following inequalities hold:*

$$\alpha_1 \cdot \alpha_2 > 0 \quad \textit{and} \quad \beta_1 \cdot \beta_2 > 0.$$

(iii) *The (projectivized) vector v is either in the interior of the cone $\mathring{\mathcal{C}}$ or in the interior of the reflected cone $A\mathring{\mathcal{C}}$.*

Proof The equivalence of (i) and (ii) is simply a restatement of [2, Lemma 9]. The geometric interpretation of (ii) with cones in (iii) follows directly. Indeed the cone $\mathcal{C}$ corresponds to vectors with all coordinates of the same sign, while $A\mathcal{C}$ to those vectors where the pair $(\alpha_1 : \alpha_2)$ has the same sign, and so does $(\beta_1 : \beta_2)$, but the signs of the two pairs are opposite. □

We now deduce the following criterion:

Proposition 5.2.2 (Disjointness of crooked surfaces) *Let $\mathcal{WS}$, $\mathcal{WS}'$ be two crooked surfaces, with corresponding vectors e_i, f_i, e_i', f_i', cones $\mathcal{C}, \mathcal{C}'$ and anti-symplectic involutions A, A'. The following are equivalent:*

(i) *The crooked surfaces $\mathcal{WS}$ and $\mathcal{WS}'$ are disjoint.*
(ii) *The photons $\phi(e_1), \phi(e_2), \phi(f_1), \phi(f_2)$ are disjoint from $\mathcal{WS}'$ and also the photons $\phi(e_1'), \phi(e_2'), \phi(f_1'), \phi(f_2')$ are disjoint from $\mathcal{WS}$.*
(iii) *The following vectors are contained in the interiors of the cones:*

$$e_1, e_2, f_1, f_2 \in \mathring{\mathcal{C}}' \cup A' \cdot \mathring{\mathcal{C}}' \text{ and } e_1', e_2', f_1', f_2' \in \mathring{\mathcal{C}} \cup A \cdot \mathring{\mathcal{C}}.$$

Proof Again, the equivalence of (i) and (ii) is the content of [2, Thm. 10]. The geometric interpretation in (iii) follows from Proposition 5.2.1 applied to each photon individually. □

Let us also recall the basic facts on the topology of the crooked surface:

Proposition 5.2.3 (Connectivity and topology of crooked surface) *Given a crooked surface $\mathcal{WS}$:*

(i) *It is homeomorphic to a Klein bottle: $\mathcal{WS} \approx \mathbb{K}^2$.*
(ii) *Its complement $\mathrm{LGr}(V) \setminus \mathcal{WS}$ has two connected components. The components can be labeled according to the cones $\mathcal{C}$ and $A\mathcal{C}$, corresponding to the photons which are contained in one component or the other. The components are exchanged by the anti-symplectic involution A.*

Proof That crooked surfaces are homeomorphic to Klein bottles is [1, Thm. 8.3.1]. That a crooked surface disconnects $\mathrm{LGr}(V)$ is proved in [6, Thm 3.16], and that the anti-symplectic involution exchanges the two components follows immediately as well. □

5.2.4 Position of a Lagrangian

Let us now list the possibilities for the position of a Lagrangian relative to the crooked surface, when viewing the picture in $\mathbb{P}(V)$. Regard the Lagrangian as projectivized in $\mathbb{P}(V)$, thus yielding a line. If the Lagrangian intersects the interior of either $\mathcal{C}$ or $A\mathcal{C}$ then it is clearly in the corresponding component in $\mathrm{LGr}(V)$, since it lies on a photon entirely contained in such a component. But the Lagrangian could intersect also just the boundary, say the boundary of $\mathcal{C}$ for simplicity. If it intersects a vertex, or more generally one of the edges $[e_1, e_2]$ or $[f_1, f_2]$ then it clearly lies on the respective wing. If it intersects one of the edges $[e_1, f_1]$ or $[e_2, f_2]$, the assumption that it doesn't go through the interior of $\mathcal{C}$ implies that L must belong to one of pieces of the stem $\mathcal{S}^{\pm}$. Finally, suppose L intersects the edge $[e_1, f_2]$ (for $[e_2, f_1]$ the analysis is similar), say in $v = \alpha_1 e_1 + \beta_2 f_2$ with $\alpha_1, \beta_2 > 0$. Then its orthogonal complement is spanned by $e_1, f_2, \beta_2 f_1 + \alpha_1 f_1$, and unless L is the span of e_1, f_2, it is immediate that some linear combination of vectors in L lies in the interior of $\mathcal{C}$, placing L in the interior of the respective component.

In our geometric applications, a "touching" of crooked surfaces occurs, because the cones can intersect along edges or faces. This situation is handled in Proposition 5.2.6 below, and we need some preliminaries on Einstein tori.

5.2.5 Einstein tori

Recall that associated to a symplectic-orthogonal splitting $V = S_1 \oplus S_2$ we defined an Einstein torus $\mathrm{Ein}^{1,1}(S_1, S_2)$ in Eq. (5.1.10). As a real projective algebraic manifold it is naturally isomorphic to a product of two projective lines $\mathbb{P}(S_1) \times \mathbb{P}(S_2)$, since it is also a quadric in the projectivization of a space of signature (2, 2) (see also [1, §5.3]). Note that the Einstein torus embedded in $\mathbb{P}(W)$ is given as the intersection of the orthogonal complement of a negative-definite vector with the null quadric (i.e. $\mathrm{LGr}(V)$). Furthermore the torus is equipped with a natural conformal class of Lorentz metric, for which the light rays are fibers of the projection to one coordinate $\mathbb{P}^1$-factor.

Suppose given now two Einstein tori $E, E' \subset \mathrm{LGr}(V)$. Then the intersection $E \cap E'$ viewed as a subset of $E = \mathbb{P}^1 \times \mathbb{P}^1$ is a (1, 1)-curve, i.e. cut out by a homogeneous equation of bi-degree (1, 1) in each of the homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. Three possibilities can occur for $E \cap E'$ inside E (see [2, §3]): it can be a timelike curve, it can be a spacelike curve, or it can be the union of two intersecting light rays. In the first two cases the intersection projects isomorphically to any of the $\mathbb{P}^1$-factors, in the last case each light ray projects isomorphically to a corresponding $\mathbb{P}^1$-factor.

Below is a criterion for when crooked surfaces can touch. Recall that if p, q are nonzero vectors then $[p, q]$ denotes the closed projective interval of their positive linear combinations (see Eq. (5.1.8)) and we will denote by (p, q) the open interval where both coefficients are strictly positive.

Proposition 5.2.6 (Tangency of crooked surfaces) *Let $\mathcal{C}, \mathcal{C}'$ be cones determining crooked surfaces $\mathcal{WS}, \mathcal{WS}'$, with notation as in Proposition 5.2.2.*

(i) *Suppose that $[e_1, e_2] = [e_1', e_2']$ and f_1', f_2' belong to the interior of $\mathcal{C}$. Then*

$$\mathcal{WS} \cap \mathcal{WS}' = \mathcal{W}_e \text{ which also equals } \mathcal{W}_e'$$

i.e. the surfaces intersect along the e-wing but nowhere else.

(ii) *Suppose that we have:*

$$\begin{aligned} f_1' &= f_1 \qquad f_2' = f_2 + e_2 + \tfrac{1}{2} f_1 \\ e_2' &= e_2 + f_1 \; e_1' = e_1 + f_2 + \tfrac{1}{2} e_2 + \tfrac{1}{6} f_1 \end{aligned}$$

Then:

$$\mathcal{WS} \cap \mathcal{WS}' = \phi(f_1) \text{ which also equals } \phi(f_1')$$

i.e. the surfaces intersect along a photon on their f-wings but nowhere else.

The formulas in case (ii) arise when the cone $\mathcal{C}'$ is the image of $\mathcal{C}$ under a maximally unipotent symplectic matrix, which preserves the flag $f_1 \subseteq L_{f_1 e_2} \subseteq f_1^{\perp} \subseteq V$. Case (i) arises when $\mathcal{C}'$ is the image of $\mathcal{C}$ under a rank 1 symplectic unipotent matrix.

Proof For both cases, it is immediate that the stated sets are in the intersection. We must check that no intersections occur elsewhere.

Consider case (i). First, let us see that $\mathcal{W}_f$ is disjoint from $\mathcal{WS}'$, and similarly for $\mathcal{W}'_f$ and $\mathcal{WS}$. Indeed the vectors in V spanning the photons are assumed in the interior of a cone, and so are the segments connecting them, so the disjointness of an f-wing from the (other) crooked surface follows by Proposition 5.2.1.

To see that the stems $\mathcal{S}, \mathcal{S}'$ don't intersect either, let E, E' be the Einstein tori containing them. By the discussion in Sect. 5.2.5 we see that $E \cap E' = \phi(e_1) \cup \phi(e_2)$. Indeed the two photons are clearly in the intersection (they give the joining places of the e-wings to the stem), and since the tori are distinct they account for all the intersection points. Since the stems are in the complement of these "joining" photons, their disjointness follows.

Consider now case (ii). The photons in the wing $\mathcal{W}'_e$ don't intersect $\mathcal{WS}$ in the range $[e'_1, e'_2)$ by the criterion of Proposition 5.2.1. The photon $\phi(e'_2)$ intersects $\mathcal{WS}$ at the Lagrangian $L_{e_2 f_1}$ and note except for this one point of intersection, this photon lies in the component of $\mathrm{LGr}(V) \setminus \mathcal{WS}$ corresponding to $\mathcal{C}$ (by an arbitrarily small perturbation it can be "pushed" to be entirely in the interior of that component, by pushing the associated vector to the interior of the cone and using Proposition 5.2.1).

The photons in the wing $\mathcal{W}'_f$ intersect $\mathcal{WS}$ at the Lagrangian $L_{f'_1 f'_2} = L_{f_1 f'_2} \in \phi(f_1) \subset \mathcal{W}_f$, but nowhere else and lie, except for this one point of intersection, in the component corresponding to $\mathcal{C}$ in $\mathrm{LGr}(V) \setminus \mathcal{WS}$ (again, a small push takes them to the interior).

Finally, the Einstein tori E, E' containing the stems intersect in two photons $E \cap E' = \phi(f_1) \cup \phi(v)$ where $v = f_2 + \frac{1}{2} e_2$ is the point of intersection between the line $f'_2 e'_2$ and the segment $[f_2, e_2]$. Let us check that the piece of the photon $\phi(v)$ that belongs to the stem $\mathcal{S}'$ is not in the stem $\mathcal{S}$. Indeed, that piece consists of Lagrangians of the form $L_{vv'}$ where $v' = \alpha'_1 e'_1 + \beta'_1 f'_1$ with $\alpha'_1 \cdot \beta'_1 > 0$, since v lies outside the segment $[f'_2, e'_2]$. This Lagrangian will intersect the subspace S_1 spanned by e_1, f_1 at the point $\alpha'_1(e_1 + \frac{1}{6} f_1) + \beta'_1 f_1$, which can be checked directly from the formulas for v, e'_1, f'_1. But this intersection point has both coordinates positive with respect to e_1, f_1, and since v also has both coordinates positive with respect to e_2, f_2, it follows that this Lagrangian is not in $\mathcal{S}$ (see Sect. 5.1.12). □

Corollary 5.2.7 (Cutting along crooked surfaces) *Suppose that two crooked surfaces $\mathcal{WS}, \mathcal{WS}'$ are either in the configuration of Proposition 5.2.2, i.e. disjoint, or in one of the configurations in Proposition 5.2.6. Then $\mathcal{WS}'$ is entirely contained in one component of $\mathrm{LGr}(V) \setminus \mathcal{WS}$ in the first case, or contained in a component except for a set of photons along which in intersects $\mathcal{WS}$ in the second case.*

5.3 Domain of discontinuity

5.3.1 Setup

We now apply the preceding formalism of crooked surfaces to analyze the domains of discontinuity for the groups $\tilde{\Gamma}_N$ from Theorem 3.2.5, using the cones constructed in Sect. 3.

Let $I := \{0, 0', \ldots, (N-1), (N-1)'\}$ denote the indexing set for the cones, with the convention that the cone denoted $\mathcal{C}'_j$ in Sect. 3.2.2 will now be denoted $\mathcal{C}_{j'}$. Given an index $i \in I$, we have a cone $\mathcal{C}_i \subset \mathbb{P}(V)$, and an associated reflection $A_i \in \mathbf{GSp}(V)$. Explicitly, associated to $\mathcal{C}_0$ is the reflection A, to $\mathcal{C}_i := R^i \mathcal{C}_0$ the reflection $R^i A R^{-i}$, and to $\mathcal{C}_{i'} = (R^i B)\mathcal{C}_0$ the reflection $(R^i B) A (B R^{-i})$.

In the Lagrangian Grassmannian we then obtain a crooked surface $\mathcal{WS}_i \subset \text{LGr}$, and its complement decomposes into two open sets

$$\text{LGr}(V) \setminus \mathcal{WS}_i = \mathcal{L}_{i,s} \coprod \mathcal{L}_{i,b}$$

where the "small" open set $\mathcal{L}_{i,s}$ is associated to the component determined by the cone $\mathcal{C}_i$, and the "big" open set $\mathcal{L}_{i,b}$ is associated to the component determined by the cone $A_i \cdot \mathcal{C}_i$. The reflection A_i preserves $\mathcal{WS}_i$ as a set, and exchanges the two components $\mathcal{L}_{i,\bullet}$. We will denote by $\overline{\mathcal{L}}$ the closure of a component (so $\overline{\mathcal{L}} = \mathcal{L} \coprod \mathcal{WS}$).

5.3.2 Left and right adjacency

Recall that the indexing set I for the cones is cyclically ordered. Let then $r(i)$, resp. $l(i)$, denote the right, resp. left, neighbors of the element i. We will also use the composition of reflections:

$$T_{i,r} := A_i A_{r(i)} \quad T_{i,l} := A_i A_{l(i)}$$

which for adjacent vertices satisfy $T_{i,r} \cdot T_{r(i),l} = 1$ and $T_{i,l} \cdot T_{l(i),r} = 1$. Note that each $T_{i,l/r}$ is a unipotent transformation taking the cone $\mathcal{C}_i$ to itself, and one of the matrices is a rank 1 unipotent while the other is maximally unipotent.

5.3.3 Finite approximations to limit set and domain of discontinuity

We can now combine the calculations with containments of cones from Theorem 3.2.5 with the disjointness/touching criteria from Propositions 5.2.2 and 5.2.6. It follows that when $i \neq j$, we have that $\mathcal{WS}_j \subset \overline{\mathcal{L}_{i,b}}$ and more generally $\overline{\mathcal{L}_{j,s}} \subset \overline{\mathcal{L}_{i,b}}$.

Let us define

$$\Lambda_1 := \bigcup_{i\in I} \overline{\mathcal{L}_{i,s}} \quad \text{and} \quad \Omega_1 := \bigcap_{i\in I} \mathcal{L}_{i,b} = \text{LGr}(V) \setminus \Lambda_1$$

These provide a first approximation to the domain of discontinuity Ω and limit set Λ. We can define Λ_n and $\Omega_n := \text{LGr}(V) \setminus \Lambda_n$ recursively, or in a more direct manner:

$$\begin{aligned} \Lambda_n &:= \{x \in \text{LGr}(V) \colon \exists i_1, \ldots, i_n \in I \text{ s.t. } i_l \neq i_{l+1} \\ &\qquad \text{and } x_{i_1} \in \overline{\mathcal{L}_{i_1,s}} \text{ s.t. } x = A_{i_n} \cdots A_{i_2} \cdot x_{i_1}\} \\ \Omega_n &:= \{x \in \text{LGr}(V) \colon \forall i_1, \ldots, i_n \in I \text{ s.t. } i_l \neq i_{l+1} \\ &\qquad \text{we have that } A_{i_2} \cdots A_{i_n} \cdot x \notin \overline{\mathcal{L}_{i_1,s}}\} \end{aligned} \tag{5.3.4}$$

It is immediate from the definitions that $\Omega_n = \text{LGr}(V) \setminus \Lambda_n$, and that Λ_n is closed (resp. Ω_n is open). Let us point out that the sequence $i_1, \ldots, i_n$ which certifies that $x \in \Lambda_n$ need not be uniquely associated to x.

We also have that $\Lambda_{n+1} \subset \Lambda_n$ since if $x \in \Lambda_{n+1}$ with $x = A_{i_{n+1}} \cdots A_{i_2} \cdot x_{i_1}$ then we can also use $x_{i_2} := A_{i_2} x_{i_1}$ and the last n terms of the sequence, to see that $x \in \Lambda_n$, since $A_{i_2} \cdot \overline{\mathcal{L}_{i_1,s}} \subset \overline{\mathcal{L}_{i_2,s}}$. Similarly note that $\Omega_{n+1} \supset \Omega_n$, since if $x \notin \Omega_{n+1}$ then there exists a sequence $i_1, \ldots, i_{n+1}$ with $A_{i_2} A_{i_3} \cdots A_{i_{n+1}} x \in \overline{\mathcal{L}_{i_1,s}}$, but then $A_{i_3} \cdots A_{i_{n+1}} x \in A_{i_2} \overline{\mathcal{L}_{i_1,s}} \subset \overline{\mathcal{L}_{i_2,s}}$ showing that $x \notin \Omega_n$ either.

5.3.5 A preliminary domain of discontinuity

We can define now the sets

$$\Lambda^\circ := \bigcap_{n\geq 1} \Lambda_n \text{ and its complement } \Omega^\circ := \bigcup_{n\geq 1} \Omega_n.$$

By construction Λ° is closed and Ω° is open, and both sets are $\tilde{\Gamma}_N$-invariant. We will see in Sect. 5.3.8 below that Ω° can be slightly enlarged to a bigger $\tilde{\Gamma}_N$-invariant set, while its complement Λ° can be slightly shrunk.

5.3.6 Boundary of the fundamental domain

If we denote by $\overline{\Omega}_1^{rel} \subset \Omega^\circ$ the relative closure of the first domain Ω_1, then it is immediate to check from the properties of the action, and the definitions, that the $\tilde{\Gamma}_N$-orbit of any point intersects $\overline{\Omega}_1^{rel}$.

Let us further analyze the boundary of this fundamental domain. We have the following set-theoretic calculations:

$$\begin{aligned}\Omega_2 \setminus \left(\Omega_1 \cup \bigcup_{i\in I} A_i \cdot \Omega_1\right) &= \Omega_1 \cap \left(\Lambda_1 \cap \bigcap_{i\in I} A_i \cdot \Lambda_1\right)\\ &= \Omega_2 \cap \bigcap_{i\in I} \mathcal{WS}_i.\end{aligned}$$

Consider now a point $x \in \Omega_2 \cap \mathcal{WS}_{i_0}$. The assumption $x \in \Omega_2$ is equivalent to the statement that for any $i_1 \neq i_2$ we have that $A_{i_2}x \notin \overline{\mathcal{L}_{i_1,s}}$. Observe that if i_2 is not adjacent to i_0 then this is automatic since the reflection A_{i_2} will map $\mathcal{L}_{i_0,s}$ strictly inside $\mathcal{L}_{i_2,s}$, and so the same will remain true of the boundary $\mathcal{WS}_{i_0}$.

So we have to consider the cases $i_2 \in \{r(i_0), l(i_0)\}$. By an analogous reasoning, if $i_1 \neq i_0$ then we have $A_{i_2}\mathcal{WS}_{i_0} \cap \overline{\mathcal{L}_{i_1,s}} = \emptyset$, so we have to consider only the case $i_1 = i_0$. A point $x \in \Omega_2 \cap \mathcal{WS}_{i_0}$ is characterized by

$$A_{r(i_0)}x \notin \overline{\mathcal{L}_{i_0,s}} \text{ and } A_{l(i_0)}x \notin \overline{\mathcal{L}_{i_0,s}}$$

which, by applying A_{i_0} to both sides, and using the notation from Sect. 5.3.2, is equivalent to

$$T_{i_0,r}x \notin \overline{\mathcal{L}_{i_0,b}} \text{ and } T_{i_0,l}x \notin \overline{\mathcal{L}_{i_0,b}}.$$

In other words we have

$$\Omega_2 \cap \mathcal{WS}_{i_0} = \mathcal{WS}_{i_0} \setminus \left(T_{i_0,r} \cdot \mathcal{WS}_{i_0} \cup T_{i_0,l} \cdot \mathcal{WS}_{i_0}\right)$$

so we must eliminate the intersections of the original crooked surface with its translates by two unipotent transformations. These are precisely the sets described in Proposition 5.2.6: one intersection is along a full wing of the surfaces, while another is along a single photon.

5.3.7 Action of reflections on a wing

It follows from the previous analysis that the sets Λ_n will contain full wings of adjacent crooked surfaces, along which the sources “touch” in the sense of Proposition 5.2.6. Let us analyze now the dynamics of the two reflections which fix, as a set, the particular wing. Up to conjugacy, the model is that of the group generated by the matrices A, B from Sect. 2.2.4. Their product $AB = (A - B)B + 1$ is a rank 1 unipotent matrix, since $A - B$ is visibly a rank 1 matrix.

Recall also that the cone is given in Eq. (3.4.6) and its column vectors are (up to scaling) what we called e_1, f_1, e_2, f_2. Then the geometry is as follows. The group generated by A, B preserves as a set each photon $\phi(v)$ for $v = \alpha_1 e_1 + \alpha_2 e_2$. The two reflections fix the Lagrangian $L_{e_1 e_2}$, which lies on each of the photons in question. Identifying $\phi(v) = \mathbb{P}(v^\perp / v) \cong \mathbb{P}^1(\mathbb{R})$, and removing "the point at infinity" $L_{e_1 e_2}$, the action of A, B then becomes that of two Euclidean reflections on $\mathbb{R}$. Under increasingly longer words in A, B the orbit of a point approaches the point at infinity $L_{e_1 e_2}$.

The above description holds except for a photon $\phi(v_0)$, where v_0 is the image of $AB - \mathbf{1}$. The action of the group generated by A, B is trivial on this photon, and the vector v_0 is the "attractor" for the projective action of large powers of AB. Let us call $\phi(v_0) \subset \mathcal{W}_e$ the "attractor photon" on the corresponding e-wing.

5.3.8 Enlarging the domain of discontinuity

We can now enlarge our open set Ω° to a larger domain of discontinuity, as follows. Enlarge Ω_2 by adding, for each index $i \in I$, the complement in the e-wing of the "attractor photon". Then, take the image of Ω_2 under the group $\tilde{\Gamma}_N$ and call the resulting set Ω, with complement Λ.

This construction is equivalent to removing from the limit set Λ_2 the e-wings, except for the attractor photons, and then taking successive images and intersecting as in Sect. 5.3.3.

Let us finally remark that the limit set Λ intersects each crooked surface $\mathcal{WS}_i$ in two photons only, namely the f-vertex photon which is the attractor for the maximally unipotent matrix, and another photon attractor for the rank 1 unipotent transformation on the e-wing.

Theorem 5.3.9 (Proper discontinuity) *The action of* $\tilde{\Gamma}_N$ *on the open sets* Ω° *and* Ω *in* $\mathrm{LGr}(V)$ *is properly discontinuous.*

Proof It suffices to restrict our attention to the finite index subgroup of $\tilde{\Gamma}_N$ generated by the reflections A_i for $i \in I$. The proper discontinuity for the action on Ω° follows from its construction in Sect. 5.3.3 and the mapping properties of the reflections A_i for the surfaces $\mathcal{WS}_i$ and their configurations. The orbit of a point $x \in \Omega^\circ$, and a sufficiently small open set containing it, can be traced combinatorially through the open sets $\Omega_n \subset \Omega^\circ$ just like the corresponding orbit of a point in the hyperbolic plane for the corresponding action of the reflection group there.

The only new points in Ω are those that belong to initial e-wings, plus their images. As explained in Sect. 5.3.7, the action of the dihedral group preserving that wing is properly discontinuous on the complement of the fixed photon, and if we apply a reflection not in the dihedral group, the point goes to the interior of domains $\mathcal{L}_{i,s}$ and its subsequent orbit does not return to the initial neighborhood, again by the mapping properties of the cones and corresponding regions $\mathcal{L}_{i,s/b}$ and reflections A_i. □

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