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ORIGINAL ARTICLE

On Landis' Conjecture in the Plane for Potentials with Growth

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Abstract

We investigate the quantitative unique continuation properties of real-valued solutions to Schrödinger equations in the plane with potentials that exhibit growth at infinity. More precisely, for equations of the form $\Delta u - Vu = 0$ in $\mathbb{R}^2$, with $|V(z)| \lesssim |z|^N$ for some $N \geq 0$, we prove that real-valued solutions satisfy exponential decay estimates with a rate that depends explicitly on N. The case $N = 0$ corresponds to the Landis conjecture, which was proved for real-valued solutions in the plane in Logunov et al. (arXiv:2007.07034, 2020) As such, the results in this article may be interpreted as generalized Landis-type theorems. Our proof techniques rely heavily on the ideas presented in Logunov et al. (arXiv:2007.07034, 2020).

Keywords Landis conjecture · Unique continuation · Schrödinger equation

Mathematics Subject Classification (2010) 35B60 · 35J10

1 Introduction

In the late 1960s, E.M. Landis [16] conjectured that if u is a bounded solution to

$$\Delta u - Vu = 0 \quad \text{in } \mathbb{R}^n, \tag{1}$$

where V is a bounded function and u satisfies $|u(x)| \lesssim \exp(-c|x|^{1+})$, then $u \equiv 0$. This conjecture was later disproved by Meshkov [19] who constructed non-trivial $\mathbb{C}$-valued functions u and V that solve $\Delta u - Vu = 0$ in $\mathbb{R}^2$, where V is bounded and $|u(x)| \lesssim \exp(-c|x|^{4/3})$. Meshkov also proved a *qualitative unique continuation* result: If $\Delta u - Vu = 0$ in $\mathbb{R}^n$, where V is bounded and u satisfies a decay estimate of the form $|u(x)| \lesssim \exp(-c|x|^{4/3+})$, then necessarily $u \equiv 0$.

In their work on Anderson localization [2], Bourgain and Kenig established a quantitative version of Meshkov's result. As a first step in their proof, they used three-ball inequalities derived from Carleman estimates to establish *order of vanishing* estimates for local solutions to Schrödinger equations. Then, through a scaling argument, they proved a *quantitative*

Dedicated to Carlos Kenig on the occasion of his 70th birthday

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unique continuation result. More specifically, they showed that if u and V are bounded, and u is normalized so that $|u(0)| \geq 1$, then for sufficiently large values of R,

$$\inf_{|x_0|=R} \|u\|_{L^\infty(B(x_0,1))} \geq \exp\left(-CR^\beta \log R\right), \tag{2}$$

where $\beta = \frac{4}{3}$. Since $\frac{4}{3} > 1$, the constructions of Meshkov, in combination with the qualitative and quantitative unique continuation theorems just described, indicate that Landis' conjecture cannot be true for complex-valued solutions in $\mathbb{R}^2$. However, at the time, Landis' conjecture still remained open in the real-valued and higher-dimensional settings. In [15, Questions 1, 2], Kenig asked if the exponent could be reduced from $\frac{4}{3}$ down to 1 in the real-valued setting; and if the related order of vanishing estimate could be improved to match those of Donnelly–Fefferman from [10, 11].

In recent years, there has been a surge of activity surrounding Landis' conjecture in the real-valued planar setting. The breakthrough article [13] by Kenig, Silvestre and Wang proved a quantitative form of Landis' conjecture under the assumption that the zeroth-order term satisfies $V \geq 0$ a.e. Subsequent papers established analogous results in the settings with drift terms [14], variable coefficients [8], and singular lower-order terms [9, 14]. Then we showed that this theorem still holds when V_- exhibits rapid decay at infinity [4], and when V_- exhibits slow decay at infinity [7]. The work of Logunov, Malinnikova, Nadirashvili, and Nazarov [18] shows that Landis' conjecture holds in the real-valued planar setting. Their proof uses the nodal structure of the domain along with a domain reduction technique to eliminate any sign condition on the zeroth-order term. The techniques and ideas from [18] will be used extensively in this article.

In [5], I studied the quantitative unique continuation properties of solutions to more general elliptic equations of the form

$$\Delta u + W \cdot \nabla u + Vu = \lambda u \quad \text{in } \mathbb{R}^n,$$

where V and W exhibit pointwise decay at infinity, and $\lambda \in \mathbb{C}$. It was shown that if $|V(x)| \lesssim \langle x \rangle^{-N}$ and $|W(x)| \lesssim \langle x \rangle^{-P}$ for $N, P \geq 0$, then the quantitative estimate (2) holds with $\beta = \max\{1, \frac{4-2N}{3}, 2-2P\}$. These results complement those in [3], where analogous qualitative unique continuation theorems are established in the setting where $W \equiv 0$ and $N \in \mathbb{R}$. By building on the ideas of Meshkov from [19], the article [5] contains examples which prove that the estimates are sharp in certain settings, with further examples in [6]. These quantitative estimates were generalized in [17], where they proved analogous estimates for solutions to the corresponding equations with variable-coefficient leading terms.

This paper is concerned with proving quantitative unique continuation results for equations of the form (1), where $n = 2$, u and V are real-valued, and V exhibits growth at infinity. We build off of the techniques in [18] to establish quantitative versions of the results from [3] in the setting where V is real-valued and growing (denoted by $\varepsilon \leq 0$ in that article). We now give the precise statement of the main theorem.

Theorem 1.1 *For some $N \geq 0$, $a_0 > 0$, let $V : \mathbb{R}^2 \to \mathbb{R}$ satisfy the growth condition*

$$|V(z)| \leq a_0 |z|^N. \tag{3}$$

Let u be a real-valued solution to (1) in $\mathbb{R}^2$ with the properties that

$$|u(0)| = 1$$

and for some $c_0 > 0$,

$$|u(z)| \leq \exp\left(c_0 |z|^{1+\frac{N}{2}}\right).$$

Then there exists constants $C_0 = C_0(a_0, c_0, N) > 0$ and $R_0 > 0$ so that whenever $|z_0| \geq R_0$, it holds that

$$\|u\|_{L^\infty(B(z_0,1))} \geq \exp\left(-C_0|z_0|^{1+\frac{N}{2}} \log^{\frac{3}{2}} |z_0|\right).$$

The results of [3] establish qualitative versions of (2) with $\beta = \frac{4+2N}{3} = \frac{4}{3}(1 + \frac{N}{2})$ under the assumption (3) in the complex-valued setting. Thus, as in the case of bounded V, this theorem shows that stronger bounds hold in the real-valued planar setting.

For some $\beta > 0$, $c \neq 0$, let

$$u(z) = \exp\left(c|z|^\beta\right).$$

A computation shows that $\Delta u - Vu = 0$, where $V(z) := c\beta^2(c|z|^\beta + 1)|z|^{\beta-2}$ satisfies $|V(z)| \lesssim |z|^{2\beta-2}$. By setting $\beta = 1 + \frac{N}{2}$, this example shows that the theorem is sharp whenever $N \geq 0$. Based on this example, it seems reasonable to assume that a version of Theorem 1.1 also holds for potentials that decay at infinity, i.e. for $N < 0$. To extend the arguments in this paper to decaying potentials, an iterative argument reminiscent of those in [4, 5, 17], or [7] may be needed. This approach was attempted in the preparation of this manuscript, but the exponent "got stuck" above 1 and a resolution to this issue was unclear at the time. In other words, modifications to the techniques of this paper do not appear to give such results for decaying potentials. In subsequent articles, we will study both singular potentials and potentials that exhibit decay at infinity.

To prove Theorem 1.1, we establish the following local result. Note that the $R_0 > 0$ here is the same universal constant as in Theorem 1.1.

Theorem 1.2 *Let u be a real-valued solution to $\Delta u - Vu = 0$ in $B(0, R) \subset \mathbb{R}^2$, where V is real-valued and $\|V\|_{L^\infty(B(0,R))} \leq a^2 R^{2\delta}$ for some $\delta \geq 0$, $a \geq 1$, $R > 0$. If $R \geq R_0$, $S \in \left[\frac{R}{4}, \frac{R}{2}\right]$, and there exists $M > 0$ so that*

$$\sup_{z \in B(0,R-S)} |u(z)| \geq e^{-M} \sup_{z \in B(0,R)} |u(z)|, \tag{4}$$

then there exists universal $C_1 > 0$ so that whenever $r \in \left(0, \frac{R}{2^{10}}\right)$, it holds that

$$\sup_{z \in B(0,r)} |u(z)| \geq \left(\frac{r}{R}\right)^{K(R,M)} \sup_{z \in B(0,R-S)} |u(z)|, \tag{5}$$

where $K(R, M) = C_1 \max\left\{aR^{1+\delta}\sqrt{\log R}, M + \frac{1}{\log R}\right\}$.

The proof of this theorem will be presented below in Section 3. As in [18], we reduce the problem to a question about harmonic functions. Those details are provided in Section 2.

Assuming that Theorem 1.2 holds, we present the proof of Theorem 1.1.

Proof of Theorem 1.1 Fix $z_0 \in \mathbb{R}^2$ with $|z_0| \geq \frac{R_0}{2}$. Set $R = 2|z_0| \geq R_0$ and $S = \frac{R}{2}$. Define

$$u_0(z) = u(z_0 + z) \quad \text{and} \quad V_0(z) = V(z_0 + z)$$

so that

$$\Delta u_0 + V_0 u_0 = 0 \quad \text{in} \;\; B(0, R).$$

Since $|z_0 + z| \leq \frac{3}{2}R$ for $z \in B(0, R)$, then

$$\|V_0\|_{L^\infty(B(0,R))} \leq a_0 \left(\frac{3}{2}R\right)^N = a_0 \left(\frac{3}{2}\right)^N R^N$$

and

$$\sup_{z\in B(0,R)} |u_0(z)| \le \exp\left[c_0\left(\frac{3}{2}\right)^{1+\frac{N}{2}} R^{1+\frac{N}{2}}\right].$$

As

$$\sup_{B(0,R-S)} |u_0| = \sup_{B(z_0,|z_0|)} |u| \ge |u(0)| = 1,$$

then Theorem 1.2 is applicable with $\delta = \frac{N}{2}$, $a = \max\left\{\sqrt{a_0}(\frac{3}{2})^{\frac{N}{2}}, 1\right\}$, and $M = c_0(\frac{3}{2})^{1+\frac{N}{2}} R^{1+\frac{N}{2}}$. Since

$$K(R,M) = C_1 \max\left\{aR^{1+\frac{N}{2}}\sqrt{\log R}, c_0\left(\frac{3}{2}\right)^{1+\frac{N}{2}} R^{1+\frac{N}{2}} + \frac{1}{\log R}\right\} \le c_1 R^{1+\frac{N}{2}}\sqrt{\log R},$$

where $c_1 = C_1\left[a + (\frac{3}{2})^{1+\frac{N}{2}} c_0\right]$, then

$$\begin{aligned}\sup_{B(z_0,r)} |u| = \sup_{B(0,r)} |u_0| &\ge \left(\frac{r}{R}\right)^{c_1 R^{1+N}\sqrt{\log R}} \sup_{B(0,R-S)} |u_0| \\ &= \left(\frac{r}{R}\right)^{c_1 R^{1+N}\sqrt{\log R}} \sup_{B(z_0,\frac{R}{2})} |u| \ge \left(\frac{r}{R}\right)^{c_1 R^{1+N}\sqrt{\log R}}.\end{aligned}$$

Setting $r = 1$ then shows that

$$\sup_{B(z_0,1)} |u| \ge \exp\left(-c_1 R^{1+\frac{N}{2}} \log^{\frac{3}{2}} R\right) \ge \exp\left(-C_0 |z_0|^{1+\frac{N}{2}} \log^{\frac{3}{2}} |z_0|\right),$$

where $C_0 = c_1 2^{1+\frac{N}{2}} \left(\frac{10}{9}\right)^{\frac{3}{2}}$. □

The remainder of the article is organized as follows. In Section 2, we present and prove a unique continuation theorem for harmonic functions in punctured domains. As in [18], this result for harmonic functions is essential to the proof of Theorem 1.2. We describe this reduction in Section 3, and explain how it implies the proof of Theorem 1.2. We use c, C to denote constants that may change from line to line, while constants with subscripts are fixed. Unless stated otherwise, all constants are universal.

2 Decay Properties of Harmonic Functions in Punctured Domains

In this section, we present and prove quantitative unique continuation results (in the form of three-ball inequalities) for harmonic functions in punctured domains. The next section shows how these results lead to the proof of Theorem 1.2. We begin with an application of the Harnack inequality.

Lemma 2.1 *Let $\{D_j\}$ be a finite collection of* 100*-separated unit disks in the plane. Assume that h is real-valued and harmonic in $\mathbb{R}^2 \setminus \cup D_j$ and that for each j, h does not change sign in $5D_j \setminus D_j$. There exists an absolute constant $C_H \ge 10$ for which*

1. $\max_{\delta 3D_j} |h| \le C_H \min_{\delta 3D_j} |h|$.
2. $\max_{\delta 3D_j} |\nabla h| \le C_H \min_{\delta 3D_j} |h|$.

Proof An application of the Harnack inequality shows that there exists $C_H > 0$ so that for every j

$$\max_{\delta 3D_j} |h| \leq \sup_{4D_j \setminus 2D_j} |h| \leq C_H \inf_{4D_j \setminus 2D_j} |h| \leq C_H \min_{\delta 3D_j} |h|.$$

For each $z \in \delta 3D_j$, since h does not change signs in $B(z, 2)$, an application of Cauchy's inequality as in [12, Lemma 1.11] shows that

$$|\nabla h(z)| \leq |h(z)|$$

and the conclusion follows. □

We now state and prove the main result of this section. The following is a slight modification of the result [18, Theorem 5.3].

Proposition 2.2 *Let $\{D_j\}$ be a finite collection of 100-separated unit disks in the plane for which $0 \notin \cup 3D_j$. For some $R \geq 2^{10}$, let h be a harmonic function in $B(0, R) \setminus \cup D_j$ with the property that for each j, h does not change sign in $(5D_j \setminus D_j) \cap B(0, R)$. Assume that for $S \in \left[\frac{R}{4}, \frac{R}{2}\right]$ and for some $M > 0$, it holds that*

$$\sup_{z \in B\left(0, R - \frac{S}{32}\right) \setminus \cup 3D_j} |h(z)| \geq e^{-M} \sup_{z \in B(0,R) \setminus \cup 3D_j} |h(z)|. \tag{6}$$

Then for every $r \in \left(0, \frac{R}{2^{10}}\right)$, we have

$$\sup_{z \in B(0,r) \setminus \cup 3D_j} |h(z)| \geq \left(\frac{16r}{R}\right)^{K(R,M)} \sup_{z \in B\left(0, R - \frac{S}{32}\right) \setminus \cup 3D_j} |h(z)|, \tag{7}$$

where $K(R, M) = \max\{6C_H R, C_2 M\}$, $C_H \geq 10$ is from Lemma 2.1, and $C_2 > 0$ is universal.

Remark 2.3 Since this statement, Proposition 2.2, appears to be very similar to [18, Theorem 5.3], we point out the main differences:

1. The domain on the left-hand side of (6) depends on S and is therefore variable.
2. The domain on the right-hand side of (7) matches that on the left-hand side of (6), while in [18, Theorem 5.3], the domain on the right-hand side of (7) matches that on the right-hand side of (6).
3. The power $K(R, M)$ here is given as a maximum of two values instead of a sum as in [18, Theorem 5.3].
4. There are differences in the assumed bounds on R and r and therefore constants are different.

Proof We may assume without loss of generality that

$$\sup_{z \in B\left(0, R - \frac{S}{32}\right) \setminus \cup 3D_j} |h(z)| = 1.$$

Set $k = \max\{2C_H R, \frac{C_2}{3} M\}$, where C_2 will be specified below. For the sake of contradiction, assume that

$$\sup_{z \in B(0,r) \setminus \cup 3D_j} |h(z)| \leq \left(\frac{16r}{R}\right)^{3k}. \tag{8}$$

Define the punctured annular region

$$\Omega := \left\{\frac{r}{2} < |z| < R - 1\right\} \setminus \cup 3D_j$$

and the function

$$f(z) = \frac{h_x - ih_y}{z^k}.$$

Observe that f is analytic in Ω and $|f(z)| = |\nabla h(z)||z|^{-k}$. We'll analyze the behavior of f over Ω. We begin with bounding h and ∇h over the innermost and outermost parts of the boundary of Ω (Figs. 1 and 2).

Let W_1 be the connected component of $\partial\Omega$ that intersects the inner circle $\{|z| = \frac{r}{2}\}$. If $z \in W_1$, then there are three cases to consider:

(a) $|z| \neq \frac{r}{2}$.
(b) $|z| = \frac{r}{2}$ and there exists j for which $z \in 4D_j \setminus 3D_j$.
(c) $|z| = \frac{r}{2}$ and $z \cap 4D_j$ is empty for all j.

Case (a): There exists j for which $z \in \partial 3D_j$ and $3D_j \cap \{|z| = \frac{r}{2}\}$ is non-empty. An application of Lemma 2.1 combined with the fact that $\partial 3D_j \cap B(0, r)$ is non-empty shows that

$$|h(z)|, \|\nabla h(z)| \le C_H \min_{\partial 3D_j} |h| \le C_H \sup_{B(0,r)\setminus \cup 3D_j} |h| \le C_H \left(\frac{16r}{R}\right)^{3k},$$

where the last inequality follows from (8).

Case (b): Since h does not change signs in $B(z, 1)$, then an application of [12, Lemma 1.11] shows that

$$|\nabla h(z)| \le 2|h(z)| \le 2 \sup_{B(0,r)\setminus \cup 3D_j} |h| \le 2\left(\frac{16r}{R}\right)^{3k},$$

where the second inequality uses that $z \notin \cup 3D_j$ and we have again applied (8).

Case (c): Let $d = \min\{1, \frac{r}{2}\}$ and observe that $B(z, d) \subset B(0, r) \setminus \cup 3D_j$, so an application of Cauchy's inequality, [12, Lemma 1.10], shows that

$$|\nabla h(z)| \le \frac{2}{d} \sup_{B(z,d)} |h| \le \frac{2}{d} \sup_{B(0,r)\setminus \cup 3D_j} |h| \le \frac{2}{d}\left(\frac{16r}{R}\right)^{3k}.$$

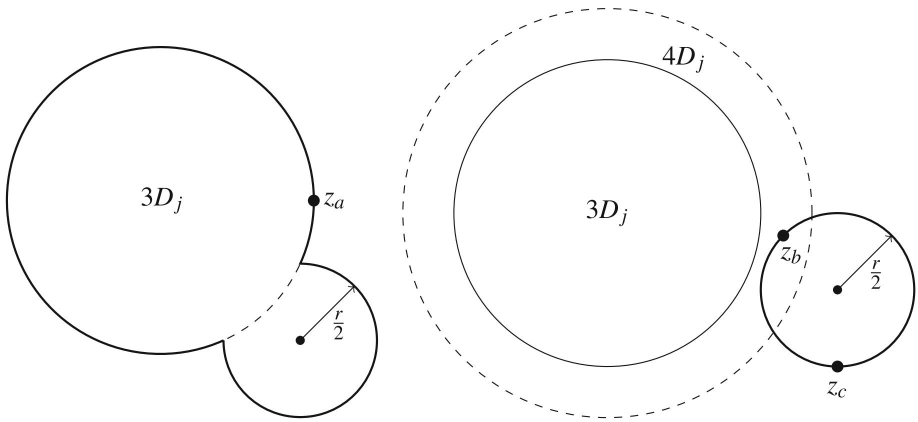

Fig. 1 Possible images of W_1 with cases (a), (b) and (c) illustrated by the points z_a, z_b, and z_c, respectively

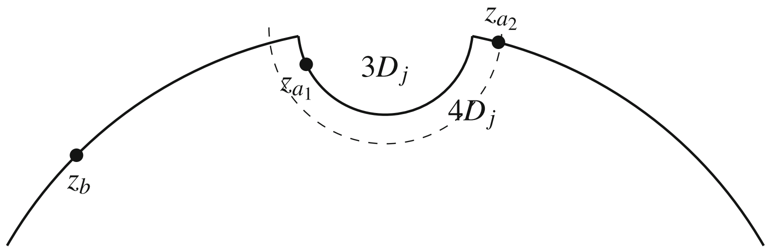

Fig. 2 A possible image of W_2 with case (a) illustrated by the points z_{a_1} and z_{a_2}, and case (b) illustrated by z_b

If $d = 1$, since $k > 1$, then $\frac{2}{d}\left(\frac{16r}{R}\right)^k = 2\left(\frac{16r}{R}\right)^k < \frac{32r}{R} < \frac{1}{2^5} < \frac{1}{2}$. On the other hand, if $d = \frac{r}{2}$, then $\frac{2}{d}\left(\frac{16r}{R}\right)^k = \frac{4}{r}\left(\frac{16r}{R}\right)^k = \frac{64}{R}\left(\frac{16r}{R}\right)^{k-1} < \frac{1}{2^4} < \frac{1}{2}$.

Since $k \geq 2C_H R \geq 2^{11} C_H$, then $2^{10k} \geq \max\{C_H, 2\} = C_H$ and $\left(\frac{r}{R}\right)^k \leq 2^{-10k} \leq \frac{1}{\max\{C_H,2\}}$. Therefore, by combining all three cases, we see that

$$\sup_{W_1} |h|, \ \sup_{W_1} |\nabla h| \leq \left(\frac{16r}{R}\right)^{2k}. \tag{9}$$

Let W_2 be the connected component of $\partial\Omega$ that intersects the outer circle $\{|z| = R - 1\}$ and note that $W_2 \subset \overline{B(0, R-1)} \setminus B(0, R-7)$. Now if $z \in W_2$, there are two cases to consider:

(a) there exists j for which $z \in 4D_j$.
(b) $|z| = R - 1$ and $z \cap 4D_j$ is empty for all j.

Case (a): Since h does not change sign in $B(z, 1)$, then an application of [12, Lemma 1.10] shows that

$$|\nabla h(z)| \leq 2|h(z)| \leq 2 \sup_{B(0,R)\setminus \cup 3D_j} |h| \leq 2e^M,$$

where we have applied (6).
Case (b): Since $B(z, 1) \subset B(0, R) \setminus \cup 3D_j$, then

$$|\nabla h(z)| \leq 2 \sup_{B(z,1)} |h| \leq 2 \sup_{B(0,R)\setminus \cup 3D_j} |h| \leq 2e^M.$$

By combining both cases, we see that

$$\sup_{W_2} |h| \leq e^M, \quad \sup_{W_2} |\nabla h| \leq 2e^M. \tag{10}$$

Now we'll use these estimates on h to understand the behavior of the function $f = \frac{h_x - i h_y}{z^k}$. Define the set $\Omega_1 \subset \Omega$ as

$$\Omega_1 := \left\{\frac{r}{2} < |z| < R - \frac{S}{32}\right\} \setminus \cup 3D_j.$$

Using containment, assumption (8), and our rescaling, we see that

$$\sup_{B(0,\frac{r}{2})\setminus \cup 3D_j} |h| \leq \sup_{B(0,r)\setminus \cup 3D_j} |h| \leq \left(\frac{r}{R}\right)^{3k} < 1 = \sup_{B\left(0,R-\frac{S}{32}\right)\setminus \cup 3D_j} |h|.$$

Therefore, there exists $z_0 \in \Omega_1$ for which $|h(z_0)| = 1$. By (9), we have

$$\sup_{W_1} |h| \leq \left(\frac{16r}{R}\right)^{2k} < \frac{1}{2},$$

so there exists $z_1 \in W_1$ for which $|h(z_1)| = \alpha < \frac{1}{2}$. Let Γ be a path in Ω_1 from z_0 to z_1 for which $\ell(\Gamma) \leq 4R$. If we assume that $|\nabla h(z)| < \frac{1}{8R}$ for all $z \in \Gamma$, then

$$\frac{1}{2} < |h(z_1) - h(z_0)| = \left|\int_\Gamma \nabla h(w) \cdot dw\right| \leq \int_\Gamma |\nabla h(w)||dw| < \frac{1}{8R}\ell(\Gamma) < \frac{1}{2},$$

which is impossible so it follows that

$$\sup_{\Omega} |\nabla h| \geq \sup_{\Omega_1} |\nabla h| \geq \frac{1}{8R}.$$

Therefore,

$$\sup_{\Omega} |f| \geq \sup_{\Omega_1} |f| \geq \sup_{\Omega_1} |\nabla h| \left(R - \frac{S}{32}\right)^{-k} \geq \frac{1}{8R}\left(R - \frac{S}{32}\right)^{-k}.$$

An application of (9) shows that

$$\max_{W_1} |f| \leq \max_{W_1} |\nabla h| \left(\frac{2}{r}\right)^k \leq \left(\frac{16r}{R}\right)^{2k}\left(\frac{2}{r}\right)^k = \left(\frac{2^9 r}{R}\right)^k R^{-k} < 2^{-k} R^{-k},$$

where we have used that $\frac{R}{r} > 2^{10}$. Since $k \geq 2C_H R \geq 20R$, then $2^{-k} < \frac{1}{8R}$. In particular, by combining the previous two inequalities, we deduce that

$$\max_{W_1} |f| < \frac{1}{8R}\left(R - \frac{S}{32}\right)^{-k} \leq \sup_{\Omega} |f|.$$

Similarly, an application of (10) shows that

$$\max_{W_2} |f| \leq \max_{W_2} |\nabla h| (R-7)^{-k} \leq 2e^M \left(\frac{R - \frac{S}{32}}{R-7}\right)^k \left(R - \frac{S}{32}\right)^{-k}.$$

Now

$$2e^M \left(\frac{R - \frac{S}{32}}{R-7}\right)^k < \frac{1}{8R} \iff k \log\left(\frac{R-7}{R - \frac{S}{32}}\right) > M + \log(16R). \tag{11}$$

Since $S \in \left[\frac{R}{4}, \frac{R}{2}\right]$ and $R \geq 2^{10}$, then

$$\log\left(\frac{R-7}{R-\frac{S}{32}}\right) = \log\left(\frac{1 - 7 \cdot R^{-1}}{1 - \frac{S}{32R}}\right) \geq \log\left(\frac{1 - 7 \cdot R^{-1}}{1 - \frac{1}{128}}\right)$$
$$\geq c_1 := \log\left(\frac{1 - 7 \cdot 2^{-10}}{1 - 2^{-7}}\right) > 2^{-10}.$$

Since $k \geq \frac{C_2}{3} M$, then

$$\frac{k}{2}\log\left(\frac{R-7}{R-\frac{S}{32}}\right) \geq \frac{C_2 c_1}{6} M,$$

while $k \geq 2C_H R \geq 20R$ implies that

$$\frac{k}{2} \log \left(\frac{R-7}{R-\frac{S}{32}} \right) > 2^{-10} 10R > \log(16R),$$

since $R \geq 2^{10}$. If we choose $C_2 = \frac{6}{c_1}$, then (11) holds. Thus, we see that

$$\max_{W_2} |f| < \sup_{\Omega} |f|.$$

Since f is a holomorphic function in Ω, then the maximum principle guarantees that $\sup_\Omega |f| = \sup_{\partial\Omega} |f|$. As shown above, the maximum does not occur on W_1 or W_2, so there must exist a disk $3D_j \subset \{\frac{r}{2} < |z| < R-1\}$ for which $\sup_\Omega |f| = \sup_{\partial 3D_j} |f|$.

Considering only the disks D_j for which $3D_j \subset \{\frac{r}{2} < |z| < R-1\}$, define $z_j \in \partial 3D_j$ to be the point that is closest to the origin, i.e. has the smallest modulus. Then set $m_j = \min_{\partial 3D_j} |h|$. Define j_0 to be the index for which

$$m_{j_0} |z_{j_0}|^{-k} = \max_j m_j |z_j|^{-k} \tag{12}$$

and let j_1 be the index for which

$$\sup_\Omega |f| = \sup_{\partial 3D_{j_1}} |f|.$$

For any $z \in \Omega$, an application of Lemma 2.1 shows that

$$|\nabla h(z)||z|^{-k} \leq |z_{j_1}|^{-k} \sup_{\partial 3D_{j_1}} |\nabla h(z)| \leq C_H m_{j_1} |z_{j_1}|^{-k} \leq C_H m_{j_0} |z_{j_0}|^{-k}, \tag{13}$$

so we see that

$$\sup_{z\in\Omega} |\nabla h(z)| \leq C_H m_{j_0} \left(\frac{|z|}{|z_{j_0}|} \right)^k.$$

Since $\sup_\Omega |\nabla h| \geq \frac{1}{8R}$ and $z \in \Omega$ is arbitrary,

$$\frac{1}{8R} \leq C_H m_{j_0} \left(\frac{|z|}{|z_{j_0}|} \right)^k \leq C_H m_{j_0} \left(\frac{2R}{r} \right)^k$$

and then

$$|h(z_{j_0})| \geq m_{j_0} \geq \frac{1}{8C_H R} \left(\frac{r}{2R} \right)^k. \tag{14}$$

Define $s = \inf\{\tau \leq 1 : tz_{j_0} \in \Omega \text{ for all } t \in (\tau, 1)\}$ so that the straight line path defined by $\gamma(t) = tz_{j_0}$ for $s < t < 1$ is contained in Ω while $sz_{t_0} \in \partial\Omega$. In particular, we may integrate ∇h along γ to get

$$h(z_{j_0}) - h(sz_{j_0}) = \int_\gamma \nabla h(z) \cdot dz = \int_s^1 \nabla h(tz_{j_0}) \cdot z_{j_0} dt.$$

Applications of (13) and (14) show that

$$\begin{aligned} |h(sz_{j_0})| &\geq |h(z_{j_0})| - \left| \int_s^1 \nabla h(tz_{j_0}) \cdot z_{j_0} dt \right| \geq |h(z_{j_0})| - |z_{j_0}| \left| \int_s^1 C_H m_{j_0} t^k dt \right| \\ &\geq m_{j_0} - \frac{m_{j_0} C_H R}{k+1} = m_{j_0} \left(1 - \frac{C_H R}{k+1} \right) > \frac{m_{j_0}}{2}, \end{aligned} \tag{15}$$

where the last inequality uses that $k+1 > 2C_H R$. Since $k \geq 2C_H R$, then $2^k > 16C_H R$ and it follows that

$$\left(\frac{R}{r}\right)^k \geq 2^{10k} > 16C_H R \cdot 2^{9k}.$$

Combining (15) with (14) shows that

$$|h(sz_{j_0})| > \frac{m_{j_0}}{2} \geq \frac{1}{16C_H R}\left(\frac{r}{2R}\right)^k > 2^{9k}\left(\frac{r}{R}\right)^k\left(\frac{r}{2R}\right)^k = \left(\frac{16r}{R}\right)^{2k}.$$

By comparing this bound with (9), we conclude that $sz_{j_0} \notin W_1$ so it must hold that $sz_{j_0} \in \partial 3D_{j_2}$ for some $3D_{j_2} \subset \{\frac{r}{2} < |z| < R-1\}$. Then Lemma 2.1, that $|z_{j_2}| \leq |sz_{j_0}|$, and (15) show that

$$m_{j_2}|z_{j_2}|^{-k} \geq \frac{\sup_{\partial 3D_{j_2}}|h|}{C_H}|z_{j_2}|^{-k} \geq \frac{|h(sz_{j_0})|}{C_H}|sz_{j_0}|^{-k} \geq \frac{s^{-k}}{2C_H}m_{j_0}|z_{j_0}|^{-k}. \tag{16}$$

Since $z_{j_0} \in \partial 3D_{j_0}$ and $sz_{j_0} \in \partial 3D_{j_2}$ where $j_0 \neq j_2$, and the balls $\{D_j\}$ are of unit radius and 100-separated, then $|z_{j_0} - sz_{j_0}| \geq 96$. After rearrangement, we see that $s^{-k} \geq (1-\frac{96}{R})^{-k}$. Since $10 \leq C_H$, $2C_H R \leq k$, and $\frac{96}{R} < -\log(1-\frac{96}{R})$, then

$$\log(2C_H) < 96 \cdot 2C_H \leq \frac{96}{R}k < -k\log\left(1-\frac{96}{R}\right) \leq -k\log s,$$

from which it follows that $s^{-k} > 2C_H$. We then conclude from (16) that $m_{j_2}|z_{j_2}|^{-k} > m_{j_0}|z_{j_0}|^{-k}$ which contradicts (12) and gives the desired contradiction. In other words, (8) fails to hold and we see that

$$\sup_{z \in B(0,r)\setminus \cup 3D_j}|h(z)| > \left(\frac{16r}{R}\right)^{3k} = \left(\frac{16r}{R}\right)^{3k} \sup_{z \in B\left(0, R-\frac{S}{32}\right)\setminus \cup 3D_j}|h(z)|,$$

which implies (7) by our choice of k. □

3 The Proof of Theorem 1.2

In this section, we show how Theorem 1.2 follows from Proposition 2.2. This reduction is very similar to that described in [18] with rescaling changes to account for the size of V.

Let $u : B(0,R) \subset \mathbb{R}^2 \to \mathbb{R}$ be a solution to

$$\Delta u - Vu = 0 \quad \text{in } B(0,R),$$

where for some $a \geq 1$, $\delta \geq 0$,

$$\|V\|_{L^\infty(B(0,R))} \leq a^2 R^{2\delta}.$$

Let F_0 denote the nodal set of u, i.e.

$$F_0 = \{z \in \mathbb{R}^2 : u(z) = 0\}.$$

Define $z_0 \in \overline{B(0, R-S)}$ to satisfy

$$|u(z_0)| = \sup_{B(0,R-S)}|u|. \tag{17}$$

For $\rho > 0$ to be specified below and c_s a universal constant, there exists a set $F_1 \subset B(0, R)$ which consists of a collection of $c_s\rho$-separated closed disks of radius ρ which are also $c_s\rho$-separated from 0, z_0, F_0, and $\partial B(0, R)$. Moreover, the set $F_0 \cup F_1 \cup \partial B(0, R)$ is a $10c_s\rho$-net in $B(0, R)$. A more detailed description of this process is given in [18, §2, Act I].

Define $\Omega = B(0, R) \setminus (F_0 \cup F_1)$ and $\Omega_1 = B(0, R) \setminus F_1$. As shown in [18, §3.1], there exists a constant c_P (depending on c_s) so that Ω has Poincare constant bounded above by $c_P\rho^2$. In particular, since $c_P\rho^2\|V\|_{L^\infty(B(0,R))} \leq c_P\rho^2 a^2 R^{2\delta}$, then by choosing $\rho \ll 1$, we can apply [18, Lemma 3.2]. For $\varepsilon \ll 1$ to be defined later on, let

$$\rho = \varepsilon a^{-1} R^{-\delta}. \tag{18}$$

An application of the arguments in [18, §3.2] then shows that there exists $\varphi : \Omega \to \mathbb{R}$ with the properties that

$$\begin{aligned} &\Delta\varphi - V\varphi = 0 \quad \text{in } \Omega \\ &\varphi - 1 \in W_0^{1,2}(\Omega) \\ &\|\varphi - 1\|_\infty \leq c_b(\rho a R^\delta)^2 = c_b\varepsilon^2, \end{aligned} \tag{19}$$

where c_b is a universal constant depending on c_P, and we have used (18). By extending φ to equal 1 across $F_0 \cup F_1$, it is then shown in [18, Lemma 4.1] that $f := \frac{u}{\varphi} \in W^{1,2}_{\text{loc}}(B(0, R))$ is a weak solution to the divergence-form equation

$$\operatorname{div}(\varphi^2\nabla f) = 0 \quad \text{in } \Omega_1.$$

We then introduce the Beltrami coefficient μ, defined as follows:

$$\mu = \begin{cases} \frac{1-\varphi^2}{1+\varphi^2}\frac{f_x+if_y}{f_x-if_y} & \text{in } \Omega_1 \text{ when } \nabla f \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|\mu| \lesssim \varepsilon^2$, then as shown in [1], there exists a K-quasiconformal homeomorphism of the complex plane where $K \leq 1+C_K\varepsilon^2$, where C_K depends on c_b. That is, there exists some $w \in W^{1,2}_{\text{loc}}$ which satisfies the Beltrami equation $\frac{\partial w}{\partial \bar{z}} = \mu\frac{\partial w}{\partial z}$. In fact, an application of the Riemann uniformization theorem shows that there exists a K-quasiconformal homeomorphism g of $B(0, R)$ onto itself with $g(0) = 0$. Moreover, the function $h := f \circ g^{-1}$ is harmonic in $g(\Omega_1)$.

Mori's Theorem implies that

$$\frac{1}{16}\left|\frac{z_1 - z_2}{R}\right|^K \leq \frac{|g(z_1) - g(z_2)|}{R} \leq \frac{1}{16}\left|\frac{z_1 - z_2}{R}\right|^{\frac{1}{K}}.$$

Thus, if we set

$$\varepsilon = \frac{c_e}{\sqrt{\log R}} \tag{20}$$

for some $c_e > 0$, then $K \in \left[1, 1 + \frac{C_K c_e^2}{\log R}\right]$ and $R \simeq R^K \simeq R^{\frac{1}{K}}$. By appropriately choosing our (universal) constants c_s and c_e, it can be shown that h is harmonic in $B(0, R) \setminus \cup D_j$, where each D_j is a disk of radius 32ρ. Moreover, the disks are 3200ρ-separated from each other, 0, and $g(z_0)$, while h does not change sign in any of the annuli $100D_j \setminus D_j$.

Since $g : B(0, R) \to B(0, R)$, then we may rescale the map to get

$$\tilde{g} := \frac{g}{32\rho} : B(0, R) \to B\left(0, \frac{R}{32\rho}\right)$$

which is onto with $\tilde{g}(0) = 0$. Using (18) and (20), set

$$\widetilde{R} = \frac{R}{32\rho} = \frac{R}{32\frac{c_e}{\sqrt{\log R}}a^{-1}R^{-\delta}} = \frac{a}{32c_e}R^{1+\delta}\sqrt{\log R} =: C_3 a R^{1+\delta}\sqrt{\log R},$$

where we introduce $C_3 = \frac{1}{32c_e}$. From here, we see that $\tilde{h} := f \circ \tilde{g}^{-1}$ is harmonic in $\tilde{g}(\Omega_1)$. In particular, $\tilde{h}$ is harmonic in $B(0, \widetilde{R}) \setminus \cup \widetilde{D}_j$, where now the $\widetilde{D}_j$ are unit disks that are 100-separated from each other, from 0, and from $\tilde{g}(z_0)$. Moreover, $\tilde{h}$ doesn't change signs on any annuli $5\widetilde{D}_j \setminus \widetilde{D}_j$.

For $r \ll 1$, since $g(B(0,r))$ contains a disk of radius r_0, where

$$r_0 \geq \frac{R}{16}\left(\frac{r}{R}\right)^K \geq \frac{R}{16}\left(\frac{r}{R}\right)^2,$$

then $\tilde{g}(B(0,r)) \supset B(0, \tilde{r})$, where $\tilde{r} = \frac{r_0}{32\rho}$ so that

$$\frac{16\tilde{r}}{\widetilde{R}} \geq \left(\frac{r}{R}\right)^2. \tag{21}$$

Since $\tilde{g}(0) = 0$, then for $r \ll 1$, it holds that $B(0, \tilde{r}) \setminus \cup 3\widetilde{D}_j = B(0, \tilde{r})$ and then

$$\sup_{B(0,\tilde{r})\setminus\cup 3\widetilde{D}_j} |\tilde{h}| = \sup_{B(0,\tilde{r})} |\tilde{h}| \leq \sup_{\tilde{g}(B(0,r))} |f \circ \tilde{g}^{-1}| = \sup_{B(0,r)} |f|. \tag{22}$$

Since $u = \varphi f$, then the bound on φ from (19) implies that

$$(1 - c_b\varepsilon^2)|f(z)| \leq |u(z)| \leq (1 + c_b\varepsilon^2)|f(z)|. \tag{23}$$

As z_0 is as given by (17), then for any $z_1 \in \overline{B(0, R-S)}$, it follows that

$$|u(z_1)| \leq |u(z_0)| \leq (1 + c_b\varepsilon^2)|f(z_0)|.$$

Since $z_0 \in \overline{B(0, R-S) \cap \Omega}$, then the distortion estimate and the separation of $\tilde{g}(z_0)$ from $\cup 3\widetilde{D}_j$ implies that $\tilde{g}(z_0) \in B\left(0, \widetilde{R} - \frac{\widetilde{S}}{32}\right) \setminus \bigcup 3\widetilde{D}_j$, where we introduce

$$\widetilde{S} := \frac{S}{32\rho} = C_3 a S R^{\delta}\sqrt{\log R}.$$

Combining these observations shows that,

$$\frac{1}{1 + c_b\varepsilon^2}\sup_{B(0,R-S)} |u| \leq |f(z_0)| = |\tilde{h} \circ \tilde{g}(z_0)| \leq \sup_{B\left(0,\widetilde{R}-\frac{\widetilde{S}}{32}\right)\setminus\cup 3\widetilde{D}_j} |\tilde{h}|. \tag{24}$$

Moreover,

$$\sup_{B(0,\widetilde{R})\setminus\cup 3\widetilde{D}_j} |\tilde{h}| \leq \sup_{B(0,\widetilde{R})} |\tilde{h}| = \sup_{\tilde{g}(B(0,R))} |f \circ \tilde{g}^{-1}| = \sup_{B(0,R)} |f|. \tag{25}$$

Subsequent applications of (24), the assumption (4) from Theorem 1.2, (23), and (25) then show that

$$\begin{aligned}\sup_{B\left(0,\widetilde{R}-\frac{\widetilde{S}}{32}\right)\setminus\cup 3\widetilde{D}_j} |\tilde{h}| &\geq \frac{1}{1 + c_b\varepsilon^2}\sup_{B(0,R-S)} |u| \geq \frac{1}{1 + c_b\varepsilon^2}e^{-M}\sup_{B(0,R)} |u| \\ &\geq \frac{1 - c_b\varepsilon^2}{1 + c_b\varepsilon^2}e^{-M}\sup_{B(0,R)} |f| \geq e^{-\left(M + \frac{c_d}{\log R}\right)}\sup_{B(0,\widetilde{R})\setminus\cup 3\widetilde{D}_j} |\tilde{h}|,\end{aligned}$$

where c_d depends on c_b and c_e as in (20).

Set $R_0 = \max\{2^{10}, \exp(C_3^{-2})\}$. Since $a \geq 1$, then $R \geq R_0$ implies that $C_3 a R^\delta \sqrt{\log R} \geq a R^\delta \geq 1$ and $R \geq 2^{10}$, so that $\widetilde{R} = R C_3 a R^\delta \sqrt{\log R} \geq 2^{10}$ as well. As $S R^{-1} = \widetilde{S} \widetilde{R}^{-1}$, then the hypotheses of Proposition 2.2 hold with h, $\{D_j\}$, r, R, S, and M replaced by $\tilde{h}$, $\{\widetilde{D}_j\}$, $\tilde{r}$, $\widetilde{R}$, $\widetilde{S}$, $\widetilde{M} := M + \frac{c_d}{\log R}$, respectively.

Applications of (23), (22), the conclusion (7) from Proposition 2.2, (21), and (24) show that

$$\frac{1}{1 - c_b \varepsilon^2} \sup_{B(0,r)} |u| \geq \sup_{B(0,r)} |f| \geq \sup_{B(0,\tilde{r}) \setminus \cup 3\widetilde{D}_j} |\tilde{h}| \geq \left(\frac{16\tilde{r}}{\widetilde{R}}\right)^{K(\widetilde{R},\widetilde{M})} \sup_{B\left(0,\widetilde{R} - \frac{\widetilde{S}}{32}\right) \setminus \cup 3\widetilde{D}_j} |\tilde{h}|$$

$$\geq \frac{1}{1 + c_b \varepsilon^2} \left(\frac{r}{R}\right)^{\widetilde{K}(R,M)} \sup_{B(0,R-S)} |u|,$$

where we have introduced

$$\widetilde{K}(R, M) = 2K(\widetilde{R}, \widetilde{M}) = 2\max\left\{6C_H \widetilde{R}, C_2 \widetilde{M}\right\}$$
$$= \max\left\{12 C_H C_3 a R^{1+\delta} \sqrt{\log R}, 2C_2\left(M + \frac{c_d}{\log R}\right)\right\}.$$

Since $\frac{1 - c_b \varepsilon^2}{1 + c_b \varepsilon^2} \geq e^{-\frac{c_d}{\log R}}$ and $e < 4 \leq (2^{10})^{\frac{1}{5}} \leq \left(\frac{R}{r}\right)^{\frac{1}{5}}$, then with universal C_1 and

$$K(R, M) := C_1 \max\left\{a R^{1+\delta} \sqrt{\log R}, \left(M + \frac{1}{\log R}\right)\right\}$$

we deduce that

$$\sup_{B(0,r)} |u| \geq \left(\frac{r}{R}\right)^{K(R,M)} \sup_{B(0,R-S)} |u|$$

and the conclusion described by (5) has been shown.

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