Dimension and the Structure of Complexity Classes

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Dedicated to the memory of Alan L. Selman

Abstract

We prove three results on the dimension structure of complexity classes.

1. The Point-to-Set Principle, which has recently been used to prove several new theorems in fractal geometry, has resource-bounded instances. These instances characterize the resource-bounded dimension of a set X of languages in terms of the relativized resource-bounded dimensions of the individual elements of X, provided that the former resource bound is large enough to parameterize the latter. Thus for example, the dimension of a class X of languages in EXP is characterized in terms of the relativized p-dimensions of the individual elements of X.
2. Every language that is $\leq_m^P$-reducible to a p-selective set has p-dimension 0, and this fact holds relative to arbitrary oracles. Combined with a resource-bounded instance of the Point-to-Set Principle, this implies that if NP has positive dimension in EXP, then no quasipolynomial time selective language is $\leq_m^P$-hard for NP.
3. If the set of all disjoint pairs of NP languages has dimension 1 in the set of all disjoint pairs of EXP languages, then NP has positive dimension in EXP.

1 Introduction

Alan Selman was a pioneer and a leader in elucidating the structure of complexity classes. He initiated many of the most important concepts of structural complexity theory, he investigated them brilliantly, and he inspired generations of computer scientists to contribute to this endeavor.

Our objective in this paper is to show how resource-bounded dimension, which is a generalization of classical Hausdorff dimension, can extend Selman's research program in fruitful new directions. To this end, we present three new results, one bringing the Point-to-Set Principle into complexity classes, one on dimension and p-selective sets, and one on dimension and disjoint NP pairs. The rest of this introduction motivates and explains these three results.

Hausdorff dimension, developed in 1919 [16, 6], is a scheme for assigning a *dimension* $\dim_{\mathrm{H}}(E)$ to every subset E of a given metric space. Assume for a moment that this metric space is a Euclidean space $\mathbb{R}^n$. Then $\dim_{\mathrm{H}}(\mathbb{R}^n) = n$, and the Hausdorff dimension is monotone, i.e., $E \subseteq F$ implies that $\dim_{\mathrm{H}}(E) \leq \dim_{\mathrm{H}}(F)$. For integers $d = 0, \ldots, n$, subsets E of $\mathbb{R}^n$ that are intuitively d-dimensional have $\dim_{\mathrm{H}}(E) = d$. However, *every* real number $s \in [0, n]$ is the Hausdorff dimension

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of infinitely many (in fact, $2^{|\mathbb{R}|}$ many) subsets of $\mathbb{R}$. In general, $\dim_{\mathrm{H}}(E) < n$ implies that E is a Lebesgue measure 0 subset of $\mathbb{R}^n$. (The converse does not hold.) Hausdorff dimension can thus be regarded as a measure of the "sizes" of Lebesgue measure 0 subsets of $\mathbb{R}^n$. Hausdorff dimension has become a powerful tool for investigations in fractal geometry, probability theory, and other areas of mathematical analysis [6, 41, 36, 2].

We momentarily shift the focus of our discussion from Euclidean spaces $\mathbb{R}^n$ to another metric space, the *Cantor space* $\mathbf{C}$ consisting of all *decision problems*, which are equivalently regarded as subsets of $\{0,1\}^*$ or as infinite binary sequences. At the beginning of the present century, the first author proved a theorem characterizing Hausdorff dimension in $\mathbf{C}$ in terms betting strategies called *gales*, which are minor but convenient generalization of martingales. Based on this characterization, he introduced two related methods for *effectivizing* Hausdorff dimension, i.e., imposing computability or complexity constraints on these gales. The first of these methods [26], called *resource-bounded dimension* imposes Hausdorff dimension structure on complexity classes. For example this theory defines, for every subset X of $\mathbf{C}$, a quasipolynomial-time (i.e., $n^{\mathrm{polylog}\, n}$-time) dimension $\dim_{\mathrm{qp}}(X)$ in such a way that $\dim(X \mid \mathrm{EXP}) = \dim_{\mathrm{qp}}(X \cap \mathrm{EXP})$ is a coherent notion of the dimension of X within the complexity class $\mathrm{EXP} = \mathrm{TIME}(2^{\mathrm{polynomial}})$. The second method [27], *algorithmic dimension* (also called *constructive dimension* or *effective dimension*) has to date been more widely investigated, partly because of its interactions with algorithmic randomness (i.e., Martin-Löf randomness [35]) and partly because of its applications to classical fractal geometry [30, 31]. Algorithmic dimension plays a motivating role in this paper, but resource-bounded dimension is our main topic.

Several recent results in algorithmic fractal dimensions are based on the 2017 Point-to-Set Principle introduced by the first two authors [29]. This principle is a family of theorems, the first of which says that, for any set $E \subseteq \mathbb{R}^n$,

$$\dim_{\mathrm{H}}(E) = \min_{A \in \mathbf{C}} \sup_{x \in E} \dim^A(x), \tag{1.1}$$

where $\dim^A(x)$ is the algorithmic dimension of the individual point x relative to the oracle A. This theorem completely characterizes the *classical* Hausdorff dimensions of sets E in terms of the relativized algorithmic dimensions of their elements x. The term "classical" here does not mean "old," but rather refers to mathematical concepts and theorems that, like Hausdorff dimension, do not involve computability or logic in their formulations. Thus the left-hand side of (1.1) is classical, but the right-hand side, involving computability, is not. The characterization theorem (1.1) is called the *Point-to-Set Principle for Hausdorff dimension*, because it enables one to prove lower bounds on the Hausdorff dimensions of *sets* by reasoning about the relativized algorithmic dimensions of judiciously chosen individual *points* in those sets. The paper [29] also proved a second instance of the Point-to-Set Principle that characterizes another classical fractal dimension, the packing dimension [6], in a manner dual to (1.1). These instances of the Point-to-Set Principle have recently been used to prove several new theorems in classical fractal geometry [34, 33, 32, 20]. The authors also recently extended (1.1) and its dual from $\mathbb{R}^n$ to arbitrary separable metric spaces and to Hausdorff and packing dimensions with very general gauge families [20].

The above instances of the Point-to-Set Principle characterize classical fractal dimensions of sets in terms of the relativized algorithmic dimensions of the individual elements of those sets. In Section 4 below, we prove more general instances of the Point-to-Set Principle that characterize the *classical or perhaps somewhat effective* dimensions of sets in $\mathbf{C}$ in terms of the relativized *more effective* dimensions of the individual elements of those sets. One example of this says that, for every subset X of $\mathbf{C}$,

$$\dim_{\mathrm{H}}(X) = \min_{B \in \mathbf{C}} \sup_{A \in X} \dim_{\mathrm{p}}^B(A). \tag{1.2}$$

That is, we can replace the algorithmic dimension on the right-hand side of (1.1) by the more effective polynomial-time dimension. Another example characterizes the quasipolynomial-time dimension of each subset X of $\mathbf{C}$ by

$$\dim_{\mathrm{qp}}(X) = \min_{B \in \mathrm{EXP}} \sup_{A \in X} \dim_{\mathrm{p}}^{\langle B \rangle}(A), \tag{1.3}$$

i.e., in terms of the more effective polynomial-time dimensions of the individual elements A of X. (The "$\langle B \rangle$" refers to a technically restricted relativization of p-dimension to the oracle B explained in Section 4.). This implies that, for every subset X of $\mathbf{C}$ and every EXP-complete language C,

$$\dim(X \mid \mathrm{EXP}) = \sup_{A \in X \cap \mathrm{EXP}} \dim_{\mathrm{p}}^{\langle C \rangle}(A). \tag{1.4}$$

The instances (1.2), (1.3), and (1.4) are all special cases of Theorem 4.2 in Section 4.

In 1979, Alan Selman adapted Jockusch's computability-theoretic notion of semirecursive sets [19], creating the complexity-theoretic notion of p-selective sets [38]. Briefly, a decision problem $A \subseteq \{0,1\}^*$ is p-*selective*, and we write $A \in$ p-SEL, if there is a polynomial-time algorithm that, given an ordered pair (x, y) of strings $x, y \in \{0,1\}^*$, outputs a string $z \in \{x, y\}$ such that $\{x, y\} \backslash A \neq \emptyset \implies z \in A$. (We note that the terms"p-selective" and "P-selective" have both been widely used for this notion. In fact, both have been used in papers with Selman as an author.) Every set $A \in P$ is clearly p-selective, but there are uncountably many p-selective sets, so the converse does not hold. There is an extensive literature on p-selective sets and the related notions that they have spawned. We especially refer the reader to the books by Hemaspaandra and Torenvliet [17] and Zimand [43] and the references therein.

Selman [38] proved that no p-selective set can be $\leq_m^P$-hard for EXP and that, if P $\neq$ NP, then no p-selective set can be $\leq_m^P$-hard for NP. In order to extend the class of provably intractable problems, the first author [22] defined a language H to be *weakly* $\leq_m^P$*-hard* for EXP if $\mu(P_m(H) \mid \mathrm{EXP}) \neq 0$, i.e., if the set $P_m(H)$ of languages A such that $A \leq_m^P H$ does not have measure 0 in EXP in the sense of resource-bounded measure [23, 25, 43]. Buhrman and Longpré [3] and, independently, Wang [42] proved that $\mu(P_m(\text{p-SEL}) \mid \mathrm{EXP}) = 0$, where for a class $X \subseteq \mathbf{C}$, $P_m(X) = \bigcup_{H \in X}(P_m(H))$. It follows that no p-selective set can be weakly $\leq_m^P$-hard for EXP. (They in fact proved the stronger fact that this also holds for $\leq_{tt}^P$-reductions.) See [43] for a host of related results.

After the development of resource-bounded dimension [26], Ambos-Spies, Merkle, Reimann, and Stephan [1] defined a language H to be *partially* $\leq_m^P$*-hard* for EXP if $\dim(P_m(H) \mid \mathrm{EXP}) > 0$. It is clear that weak hardness implies partial hardness, and it was shown in [1] that the converse does not hold. In Section 5 we use Theorem 4.2 (i.e., the Point-to-Set Principle) to prove that $\dim(P_m(\text{qp-SEL}) \mid \mathrm{EXP}) = 0$, where the set qp-SEL of qp-selective sets is the obvious quasipolynomial-time analog of p-SEL. This implies that no qp-selective set can be partially $\leq_m^P$-hard for EXP and that, if $\dim(\mathrm{NP} \mid \mathrm{EXP}) > 0$, then no qp-selective set can be $\leq_m^P$-hard for NP.

In 1984, Even, Selman, and Yacobi [5] defined a *promise problem* to be an ordered pair (A, B) of disjoint languages. A solution of a promise problem (A, B) is an algorithm or other device that decides *any separator* of (A, B), i.e., any language S such that $A \subseteq S$ and $S \cap B = \emptyset$. Intuitively, we are *promised* that every input will be an element of $A \cup B$, so we are only required to correctly distinguish inputs in A from inputs in B.

A *disjoint* NP *pair* is a promise problem (A, B) with $A, B \in$ NP. Disjoint NP pairs were first investigated by Selman and collaborators to better understand public key cryptosystems [5, 13, 39, 18]. Razborov [37] later established a deep connection between disjoint NP pairs and propositional proof systems, associating with each propositional proof system a canonical disjoint

NP pair. Glaßer, Selman, Sengupta, and Zhang [10, 9, 11, 12] investigated this connection further, and it is now known that the degree structure of propositional proof systems under the natural notion of proof simulation is identical to the degree structure of disjoint NP pairs under reducibility of separators. See [8] for a survey of this and related results and [4] for more recent work.

In 2012, Fortnow, the first author, and the third author [7] investigated strong hypotheses involving the intractability of disjoint NP pairs. Among other things, this paper proved that

$$\mu(\text{disjNP} \mid \text{disjEXP}) \neq 0 \implies \mu(\text{NP} \mid \text{EXP}) \neq 0 \tag{1.5}$$

and that $\mu(\text{NP} \mid \text{EXP}) \neq 0$ implies the existence, for every k, of disjoint NP pairs that cannot be separated in 2^{n^k} time. (Here disjNP is the set of disjoint NP pairs, and disjEXP is the set of disjoint EXP pairs, the latter endowed with a natural measure.)

In Section 6, we prove a dimension-theoretic analog of (1.5), namely that

$$\dim(\text{disjNP} \mid \text{disjEXP}) = 1 \implies \dim(\text{NP} \mid \text{EXP}) > 0. \tag{1.6}$$

Our proof of (1.6) is somewhat simplified by the use of Theorem 4.2 (i.e., the Point-to-Set Principle).

2 Resource Bounds

We work in the Cantor space $\mathbf{C}$ consisting of all *decision problems* (i.e., *languages*) $A \subseteq \{0,1\}^*$. We identify each decision problem A with its characteristic sequence

$$[\![s_0 \in A]\!]\, [\![s_1 \in A]\!]\, [\![s_2 \in A]\!] \ldots,$$

where $s_0, s_1, s_2, \ldots$ is the standard enumeration of $\{0,1\}^*$ and

$$[\![\varphi]\!] = \textbf{if } \varphi \textbf{ then } 1 \textbf{ else } 0$$

is the Boolean value of a statement φ. We thus regard $\mathbf{C}$ as either the power set $\mathcal{P}(\{0,1\}^*)$ of $\{0,1\}^*$ or as the set $\{0,1\}^\omega$ of all infinite binary sequences, whichever is most convenient in a given context.

A *resource bound* in this paper is any one of several classes of functions from $\{0,1\}^*$ to $\{0,1\}^*$ that we now specify.

The largest resource bound is the set

$$\text{all} = \left\{ f \mid f : \{0,1\}^* \to \{0,1\}^* \right\}$$

we also use the resource bound

$$\text{comp} = \{f \in \text{all} \mid f \text{ is computable}\}.$$

As in [21, 24, 26], we define a hierarchy $G_0, G_1, G_2, \ldots$ of classes of *growth rates* $f : \mathbb{N} \to \mathbb{N}$ by the following recursion. (All logarithms in this paper are base-2.)

$$\begin{aligned} G_0 &= \{f \mid (\exists k)(\forall^\infty n) f(n) \leq kn\} \\ G_{i+1} &= 2^{G_i(\log n)} = \left\{ f \;\middle|\; (\exists g \in G_i)(\forall^\infty n) f(n) \leq 2^{g(\log n)} \right\}. \end{aligned}$$

Note that G_0 is the class of $O(n)$ growth rates and that G_1 is the class of polynomially bounded growth rates. For each $i \in \mathbb{N}$, define a canonical growth rate $\hat{g}_i \in G_i$ by $\hat{g}_0(n) = 2n$ and $\hat{g}_{i+1}(n) =$

$2^{\hat{g}_i(\log n)}$. It is easy to verify that each G_i is closed under composition, that each $f \in G_i$ is $o(\hat{g}_{i+1})$, and that each $\hat{g}_i$ is $o(2^n)$. Thus all growth rates in the G_i-hierarchy are subexponential.

Within the resource bound comp, we use the resource bounds

$$\mathrm{p}_i = \{f \in \mathrm{all} \mid f \text{ is computable in } G_i \text{ time}\} \quad (i \geq 1)$$

and

$$\mathrm{p}_i\mathrm{space} = \{f \in \mathrm{all} \mid f \text{ is computable in } G_i \text{ space}\} \quad (i \geq 1).$$

(The length of the output *is* included as part of the space used in computing f.) We write p for the polynomial-time resource bound p_1 and qp for the quasipolynomial-time resource bound p_2. Similarly the notations pspace and qpspace denote the space resource bounds $\mathrm{p}_1$space and $\mathrm{p}_2$space, respectively.

In this paper, a *resource bound* Γ or Δ is one of the classes all, comp, p_i $(i \geq 1)$, $\mathrm{p}_i$space $(i \geq 1)$ defined above. We will also use relativizations Δ^A or Δ^g of a resource bound Δ to oracles $A \subseteq \{0,1\}^*$ or function oracles $g : \{0,1\}^* \to \{0,1\}^*$.

A *constructor* is a function $\delta : \{0,1\}^* \to \{0,1\}^*$ such that $\delta(w)$ is a proper extension of w (i.e., w is a proper prefix of $\delta(w)$) for all $w \in \{0,1\}^*$. The *result* of a constructor δ is the unique sequence $R(\delta) \in \mathbf{C}$ such that $\delta^n(\lambda)$ is a prefix of $R(\delta)$ for all $n \in \mathbb{N}$. (Here $\delta^n(\lambda)$ is the n-fold application of δ to the empty string λ.)

The *result class* of a resource bound Δ is the class $R(\Delta)$ consisting of all languages $R(\delta)$ such that $\delta \in \Delta$ is a constructor. The following facts are easily verified.

1. $R(\mathrm{all}) = \mathbf{C}$.
2. $R(\mathrm{comp}) = \mathrm{DEC}$, the set of all decidable languages.
3. For all $i \geq 1$,
$$R(\mathrm{p}_i) = \mathrm{E}_i = \mathrm{TIME}(2^{G_{i-1}}).$$
In particular,
$$R(\mathrm{p}) = \mathrm{E} = \mathrm{TIME}(2^{\mathrm{linear}})$$
and
$$R(\mathrm{qp}) = \mathrm{EXP} = \mathrm{TIME}(2^{\mathrm{poly}}).$$
4. For all $i \geq 1$,
$$R(\mathrm{p}_i\mathrm{space}) = \mathrm{E}_i\,\mathrm{SPACE} = \mathrm{SPACE}(2^{G_{i-1}}).$$
In particular,
$$R(\mathrm{pspace}) = \mathrm{ESPACE} = \mathrm{SPACE}(2^{\mathrm{linear}})$$
and
$$R(\mathrm{qpspace}) = \mathrm{EXPSPACE} = \mathrm{SPACE}(2^{\mathrm{poly}}).$$

Many of our functions will be of the form $f : D \to [0,\infty)$, where D is a discrete domain such as $\{0,1\}^*$ or $\mathbb{N} \times \{0,1\}^*$ and $[0,\infty)$ is the set of nonnegative real numbers. If Δ is a resource bound, then such a function f is Δ-*computable* if there is a rational-valued function $\hat{f} : D \times \mathbb{N} \to \mathbb{Q} \cap [0,\infty)$ such that $|\hat{f}(r,x) - f(x)| \leq 2^{-r}$ for all $x \in D$ and $r \in \mathbb{N}$ and $\hat{f} \in \Delta$ (with r coded in unary and $\hat{f}(x,r)$ coded in binary).

We say that f is *lower semicomputable* if there is a computable function $\hat{f} : D \times \mathbb{N} \to \mathbb{Q} \cap [0,\infty)$ such that the following two conditions hold for all $x \in D$.

(i) For all $t \in \mathbb{N}$, $\hat{f}(x,t) \leq \hat{f}(x,t+1) \leq f(x)$.

(ii) $\lim_{t\to\infty} \hat{f}(x,t) = f(x)$.

3 Resource-Bounded Dimensions

This section briefly reviews the elements of resource-bounded dimension developed in [26].

Definition. *1. For $s \in [0,\infty)$, an* s-gale *is a function $d : \{0,1\}^* \to [0,\infty)$ such that, for all $w \in \{0,1\}^*$,*

$$d(w) = 2^{-s}[d(w0) + d(w1)].$$

2. A martingale *is a 1-gale.*

Observation 3.1 ([27])**.** *A function $d : \{0,1\}^* \to [0,\infty)$ is an s-gale if and only if the function $d' : \{0,1\}^* \to [0,\infty)$ defined by $d'(w) = 2^{(1-s)|w|}d(w)$ is a martingale.*

An s-gale d *succeeds* on a language $A \subseteq \{0,1\}^*$, and we write $A \in S^\infty[d]$, if

$$\limsup_{w \to A} d(w) = \infty,$$

where the limit superior is taken over successively longer prefixes of A.

Notation. *For $X \subseteq \mathbf{C}$, let $\mathcal{G}(X)$ be the set of all $s \in [0,\infty)$ such that there is an s-gale d for which $X \subseteq S^\infty[d]$.*

Readers unfamiliar with fractal geometry can safely use the following characterization as the definition of the *Hausdorff dimension* $\dim_{\mathrm{H}}(X)$ of each set $X \subseteq \mathbf{C}$.

Theorem 3.2 (gale characterization of Hausdorff dimension [26])**.** *For all $X \subseteq \mathbf{C}$,*

$$\dim_{\mathrm{H}}(X) = \inf \mathcal{G}(X).$$

Intuitively, an s-gale is a strategy for betting on the successive bits of languages $A \in \mathbf{C}$. The payoffs of these bets are fair if $s = 1$ and unfair if $s < 1$. Intuitively and roughly, Theorem 3.2 says that the Hausdorff dimension of X is the *most hostile betting environment* in which a gambler can succeed on every language $A \in X$.

Motivated by the above characterization of classical Hausdorff dimension, the first author defined resource-bounded dimensions and algorithmic dimensions as follows.

Notation ([26, 27])**.** *Let Δ be a resource bound, and let $X \subseteq \mathbf{C}$.*

1. *$\mathcal{G}_\Delta(X)$ is the set of all $s \in [0,\infty)$ such that there is a Δ-computable s-gale d for which $X \subseteq S^\infty[d]$.*

2. *$\mathcal{G}_{\mathrm{alg}}(X)$ is the set of all $s \in [0,\infty)$ such that there is a lower semicomputable s-gale d for which $X \subseteq S^\infty[d]$.*

Definition ([26, 27])**.** *Let Δ be a resource bound, let $X \subseteq \mathbf{C}$, and let $A \in \mathbf{C}$.*

1. *The* Δ-dimension *of X is*

$$\dim_\Delta(X) = \inf \mathcal{G}_\Delta(X).$$

2. *The* Δ-dimension *of X* in *$R(\Delta)$ is*

$$\dim(X \mid R(\Delta)) = \dim_\Delta(X \cap R(\Delta)).$$

3. The Δ-dimension *of* A *is*

$$\dim_\Delta(A) = \dim_\Delta(\{A\}).$$

4. The algorithmic dimension *of* X *is*

$$\dim_{\mathrm{alg}}(X) = \inf \mathcal{G}_{\mathrm{alg}}(X).$$

5. The algorithmic dimension *of* A *is*

$$\dim(A) = \dim_{\mathrm{alg}}(\{A\}).$$

(Algorithmic dimension has also been called constructive dimension *and* effective dimension*.)*

The papers [26, 27] showed that the above-defined dimensions are coherent, well-behaved "versions" of Hausdorff dimension. All the defined dimensions lie in $[0,1]$, and all can take any real value in $[0,1]$. The dimensions 1., 2., and 4., have the crucial dimension properties that they are monotone in X and that they are *stable* in the sense that the dimension of $X \cup Y$ is the maximum of the dimensions of X and Y. Classical Hausdorff dimension (i.e., $\dim_{\mathrm{H}} = \dim_{\mathrm{all}}$) is also *countably stable*, meaning that

$$\dim_{\mathrm{H}}\left(\bigcup_{i\in I} X_i\right) = \sup_{i\in I} \dim_{\mathrm{H}}(X_i) \tag{3.1}$$

holds for all countable index sets I. The dimensions 1. and 2. are not countably stable for Δ smaller than all, but they are Δ-*countably stable* in that (3.1) holds if the countable union is "Δ-effective." The algorithmic dimension 4. is *absolutely stable* in the sense that (3.1) holds, regardless of whether I is countable. In particular, this implies that, for all $X \subseteq \mathbf{C}$,

$$\dim_{\mathrm{alg}}(X) = \sup_{A\in X} \dim(A). \tag{3.2}$$

As a consequence of (3.2), investigations of algorithmic dimension focus almost entirely on the dimensions $\dim(A)$ of *individual* languages (or, in other contexts, individual sequences or individual points in a metric space) A.

Turning to complexity classes, i.e., the cases where Δ is some resource bound p_i or $\mathrm{p}_i\mathrm{space}$, the dimension 2. is non-degenerate in the sense that $\dim(R(\Delta) \mid R(\Delta)) = 1$. If $X \subseteq R(\Delta)$ is finite or even "Δ-countable," then $\dim(X \mid R(\Delta)) = 0$. This implies for example that, for each fixed $k \in \mathbb{N}$,

$$\dim(\mathrm{TIME}(2^{kn}) \mid \mathrm{E}) = \dim(\mathrm{TIME}(2^{n^k}) \mid \mathrm{EXP}) = 0. \tag{3.3}$$

Finally, we mention interactions of dimensions with randomness. A language $A \in \mathbf{C}$ is Δ-*random* if no Δ-computable martingale succeeds on it [24]. A language $A \in \mathbf{C}$ is *algorithmically random* (or *Martin-Löf random* [35]) if no lower semicomputable martingale succeeds on it. Since a martingale is a 1-gale, this implies that $\dim_\Delta(A) = 1$ holds for every Δ-random language and $\dim(A) = 1$ holds for every algorithmically random language. In neither case does the converse hold.

4 The Point-to-Set Principle

As noted in the introduction, previous instances of the Point-to-Set Principle have characterized *classical* fractal dimensions of sets in terms of the relativized *algorithmic* dimensions of the elements of these sets. Here we make the Point-to-Set Principle more widely applicable by proving instances of it in which "classical" and "algorithmic" are replaced by resource bounds Δ and Γ, respectively, with Γ smaller ("more effective") than Δ.

To this end, we partially order our resource bounds by

$$\mathrm{p}_i < \mathrm{p}_{i+1} < \mathrm{comp},$$

$$\mathrm{p}_i\mathrm{space} < \mathrm{p}_{i+1}\mathrm{space} < \mathrm{comp},$$

and

$$\mathrm{p}_i \leq \mathrm{p}_i\mathrm{space}$$

for all $i \leq 1$ and

$$\mathrm{comp} < \mathrm{all}.$$

Aside from reflecting current knowledge about the inclusions among these classes, this ordering has the crucial property that, if Γ and Δ are resource bounds with $\Gamma < \Delta$, then Δ *parameterizes* Γ in the sense that there is a function $f \in \Delta$ such that

$$\Gamma = \{f_k \mid k \in \mathbb{N}\},$$

where each $f_k : \{0,1\}^* \to \{0,1\}^*$ is the k^{th} *slice* of f, defined by $f_k(x) = f(0^k 1x)$ for all $x \in \{0,1\}^*$. Moreover, this parameterization *relativizes* in the sense that, for each function oracle $g : \{0,1\}^* \to \{0,1\}^*$, there is a function $f^g \in \Delta^g$ such that

$$\Gamma^g = \{f_k^g \mid k \in \mathbb{N}\}.$$

Theorem 4.1. *If Γ and Δ are resource bounds with $\Gamma < \Delta$, then for each function oracle $g : \{0,1\}^* \to \{0,1\}^*$, there is a Δ^g-computable function d^g such that $\{d_k^g \mid k \in \mathbb{N}\}$ is the set of all martingales that are Γ^g-computable and satisfy $d_k^g(\lambda) \leq 1$.*

Proof. This is implicit in the proofs of the time and space hierarchy theorems [15, 40] (minus the "disagreement" step of the diagonalizations), together with the well-known fact that these proofs relativize. □

The following theorem is the main result of this section.

Theorem 4.2 (Point-to-Set Principle for Resource-Bounded Dimensions)**.** *If Γ and Δ are resource bounds with $\Gamma < \Delta$, then, for all $X \subseteq \mathbf{C}$,*

$$\dim_\Delta(X) = \min_{g \in \Delta} \sup_{A \in X} \dim_\Gamma^g(A). \tag{4.1}$$

Theorem 4.2 follows immediately from the following two lemmas, which we prove separately.

Lemma 4.3. *If Γ, Δ, and X are as in Theorem 4.2 and $g \in \Delta$, then*

$$\dim_\Delta(X) \leq \sup_{A \in X} \dim_\Gamma^g(A). \tag{4.2}$$

Lemma 4.4. *If Γ, Δ, and X are as in Theorem 4.2, then there exists $g \in \Delta$ such that, for all $A \in X$,*

$$\dim_\Gamma^g(A) \leq \dim_\Delta(X). \tag{4.3}$$

Proof of Lemma 4.3. Let Γ, Δ, X, and g be as given, and let $s \in \mathbb{Q}$ satisfying

$$s > \sup_{A \in X} \dim_\Gamma^g(A). \tag{4.4}$$

It suffices to show that

$$\dim_\Delta(X) \leq s. \tag{4.5}$$

Since $\Gamma < \Delta$, Theorem 4.1 tells us that there is a Δ^g-computable function $d^g : \{0,1\}^* \to [0,\infty)$ such that the set $\{d_k^g \mid k \in \mathbb{N}\}$ of all slices of d^g is the set of all martingales that are Γ^g-computable and satisfy $d_k^g(\lambda) \leq 1$. In fact, since $g \in \Delta$, this function d^g is Δ-computable. Define the function $d^{g,s} : \{0,1\}^* \to [0,\infty)$ so that

$$d^{g,s}(0^k 1 x) = 2^{(s-1)|x|} d^g(0^k 1 x)$$

holds for all $k \in \mathbb{N}$ and $x \in \{0,1\}^*$. Then $d^{g,s}$ is Δ-computable, and Observation 3.1 tells us that $\{d_k^{g,s} \mid k \in \mathbb{N}\}$ is the set of all Γ^g-computable s-gales that satisfy $d_k^{g,s}(\lambda) \leq 1$. Define $d : \{0,1\}^* \to [0,\infty)$ by

$$d = \sum_{k=0}^{\infty} 2^{-k} d_k^{g,s}. \tag{4.6}$$

Then d is a Δ-computable s-gale, so to confirm (4.5) it suffices to show that

$$X \subseteq S^\infty[d]. \tag{4.7}$$

For this, let $A \in X$. Then, by (4.4), there is a Γ^g-computable s-gale $\tilde{d}$ such that $A \in S^\infty[\tilde{d}]$. Then there exists $k \in \mathbb{N}$ such that $d_k^{g,s} = \tilde{d}$, whence $A \in S^\infty[d_k^{g,s}]$. But then (4.6) tells us that

$$\limsup_{w \to A} d(w) \geq 2^{-k} \limsup_{w \to A} d_k^{g,s}(w) = \infty,$$

whence (4.7) holds. □

Proof of Lemma 4.4. Let Γ, Δ, and X be as given, and let $s \in \mathbb{Q}$ satisfy

$$s > \dim_\Delta(X). \tag{4.8}$$

If suffices to exhibit $g \in \Delta$ such that, for all $A \in X$,

$$\dim_\Gamma^g(A) \leq s. \tag{4.9}$$

By (4.8), there is a Δ-computable s-gale d such that

$$X \subseteq S^\infty[d]. \tag{4.10}$$

Let $g = \hat{d} \in \Delta$ testify to the Δ-computability of d as defined in Section 2. Then d is a Γ^g-computable s-gale, and (4.10) tells us that, for all $A \in X$, $A \in S^\infty[d]$, whence (4.9) holds. □

This completes the proof of Theorem 4.2. We now discuss some of its instances.

We first address a small technical issue regarding relativization. Instances of the Point-to-Set Principle are usually stated in terms of oracles in $\mathbf{C}$ rather than in terms of function oracles as in Theorem 4.2. These are equivalent for such large resource bounds as all and comp, but some care is required for smaller resource bounds. For example, the case $\Gamma = \mathrm{p}$, $\Delta = \mathrm{qp}$ of Theorem 4.2 says that, for all $X \subseteq C$,

$$\dim_{\mathrm{qp}}(X) = \min_{g \in \mathrm{qp}} \sup_{A \in X} \dim_{\mathrm{p}}^{g}(A). \tag{4.11}$$

On the right-hand side, we would like to replace "$g \in \mathrm{qp}$" by "$B \in R(\mathrm{qp})$," i.e., "$B \in \mathrm{EXP}$." However, this would *not* be equivalent to (4.11) and would in fact be false. The issue is that simulating an oracle query in the course of a computation of $d^B(w)$, where d is p-computable and $B \in \mathrm{EXP}$, could take $2^{|w|^k}$ time, which is *not* within the qp resource bound on the left-hand side of (4.11). We thus introduce the special notation $\dim_{\mathrm{p}}^{\langle B \rangle}(A)$ for the p-dimension of A relative to $B \in \mathbf{C}$, with the proviso that a relativized s-gale $d^{\langle B \rangle}$ upper bounding $\dim_{\mathrm{p}}^{\langle B \rangle}$ is, inf computing $d^{\langle B \rangle}(w)$, only allowed to submit queries of length $O(\log |w|)$ to the oracle B.

With the above proviso, the instance (4.11) of Theorem 4.2 says that, for all $X \subseteq \mathbf{C}$,

$$\dim_{\mathrm{qp}}(X) = \min_{B \in \mathrm{EXP}} \sup_{A \in X} \dim_{\mathrm{p}}^{\langle B \rangle}(A). \tag{4.12}$$

This implies that, for all $X \subseteq \mathbf{C}$,

$$\dim(X \mid \mathrm{EXP}) = \min_{B \in \mathrm{EXP}} \sup_{A \in X \cap \mathrm{EXP}} \dim_{\mathrm{p}}^{\langle B \rangle}(A). \tag{4.13}$$

The Point-to-Set Principle for Hausdorff dimension [29], stated in the context of $\mathbf{C}$, says that, for all $X \subseteq \mathbf{C}$,

$$\dim_{\mathrm{H}}(X) = \min_{B \in \mathbf{C}} \sup_{A \in X} \dim^{B}(A), \tag{4.14}$$

thus characterizing the classical Hausdorff dimension of X in terms of the relativized algorithmic dimensions of its individual elements. Since $\dim_{\mathrm{all}} = \dim_{\mathrm{H}}$, Theorem 4.2 tells us, for example, that we also have, for all $X \subseteq C$,

$$\dim_{\mathrm{H}}(X) = \min_{B \in \mathbf{C}} \sup_{A \in X} \dim_{\mathrm{p}}^{B}(A). \tag{4.15}$$

Note that we could use $\dim_{\mathrm{p}}^{\langle B \rangle}(A)$ on the right-hand side here, but it is unnecessary, because the resource bound all on the left-hand side of (4.15) is unrestricted.

5 Selectivity

Definition ([38]). *For any resource bound Δ, a language $A \subseteq \{0,1\}^*$ is* Δ-selective *if there is a* selector *function $f \in \Delta$ such that, for all pairs $a, b \in \{0,1\}^*$, we have $f(\langle a, b \rangle) \in \{a, b\}$ and*

$$a \in A \text{ or } b \in A \implies f(\langle a, b \rangle) \in A,$$

where $\langle \cdot, \cdot \rangle : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ is a standard pairing function.

Theorem 5.1. *If $A, B \in \mathbf{C}$ and $g : \{0,1\}^* \to \{0,1\}^*$ are such that B is p^g-selective and $A \leq_m^P B$, then $\dim_{\mathrm{p}}^{g}(A) = 0$.*

Proof. Let A, B, and g be as in the theorem statement. Let $f \in \mathrm{p}^g$ be a selector for A, let $h : \{0,1\}^* \to \{0,1\}^*$ be a $\leq_m^P$-reduction from A to B, and let $s > 0$. We will show that $\dim_{\mathrm{p}}^g(A) \leq s$ by constructing an s-gale that succeeds on A and is computable in polynomial time relative to g.

Let $k \in \mathbb{N}$ be sufficiently large so that

$$\frac{2^{ks}}{k+1} > 1. \tag{5.1}$$

We will consider blocks of k consecutive strings. For each $q \in \mathbb{N}$, define the directed graph G_q whose vertex set is $\{0, \ldots, k-1\}$ and edge set is

$$\{(i,j) \mid f(\langle h(s_{qk+i}), h(s_{qk+j})\rangle) = h(s_{qk+j})\}.$$

Notice that if $s_{qk+i} \in A$ and $s_{qk+j} \notin A$, then $h(s_{qk+i}) \in B$ and $h(s_{qk+j}) \notin B$. In this situation, the edge (i,j) cannot be present in G_q, and more generally there cannot be any path from i to j in G_q.

Let G'_q be the directed acyclic graph obtained by contracting each strongly connected component of G_q to a single vertex. Define a linear order $\prec_q$ on $\{0, \ldots, k-1\}$ by topologically sorting G'_q, breaking ties within each strongly connected component arbitrarily. In this order, $i \preceq_q j$ implies that there is a path from i to j in G_q.

Thus, if $i \preceq_q j$ and $s_{qk+i} \in A$, then $s_{qk+j} \in A$. Extending $\prec_q$ by defining $i \prec_q k$ for all $i \in \{0, \ldots, k-1\}$, it follows that

$$A \cap \{s_{qk}, \ldots, s_{qk+k-1}\} = \{s_{qk+j} \mid i \preceq_q j\} \tag{5.2}$$

for some $i \in \{0, \ldots, k\}$.

Define $d : \{0,1\}^* \to [0,\infty)$ and, for each $i \in \{0, \ldots, k\}$, $d_i : \{0,1\}^* \to [0,\infty)$ recursively as follows. For $w \in \{0,1\}^*$, let $qk + j = |w|$, where $j \in \{0, \ldots, k-1\}$.

- For all $i \in \{0, \ldots, k\}$, $d_i(\lambda) = d(\lambda) = 1$.
- $d(w) = \frac{1}{k+1} \sum_{i=0}^{k} d_i(w)$.
- For all $i \in \{0, \ldots, k\}$ and $j = 0$,

$$d_i(w0) = \begin{cases} 0 & \text{if } i \preceq_q j \\ 2^s d(w) & \text{otherwise,} \end{cases}$$

$$d_i(w1) = \begin{cases} 2^s d(w) & \text{if } i \preceq_q j \\ 0 & \text{otherwise,} \end{cases}$$

- For all $i \in \{0, \ldots, k\}$ and $j \in \{1, \ldots, k-1\}$,

$$d_i(w0) = \begin{cases} 0 & \text{if } i \preceq_q j \\ 2^s d_i(w) & \text{otherwise,} \end{cases}$$

$$d_i(w1) = \begin{cases} 2^s d_i(w) & \text{if } i \preceq_q j \\ 0 & \text{otherwise.} \end{cases}$$

Informally, each d_i represents a betting strategy, and d is an aggregate betting strategy that evenly re-allocates between the d_i after each block of k bits. Observe that d is an s-gale, although the individual d_i are not.

Now consider $d(A \upharpoonright n)$. If $n = 0$, then $d(A \upharpoonright n) = 1$. Otherwise, $n = qk + j$ for some $q \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$. Let $i \in \{0, \ldots, k\}$ be the value satisfying equation (5.2) for this q. Then

$$\begin{aligned} d(X \upharpoonright n) &\geq \frac{d_i(X \upharpoonright n)}{k+1} \\ &= \frac{2^{js} d(X \upharpoonright n-j)}{k+1} \\ &= \frac{2^{(qk+j)s}}{(k+1)^q} \\ &> \left(\frac{2^{ks}}{k+1}\right)^q. \end{aligned}$$

By inequality (5.1), this lower bound is monotonically increasing and unbounded, so

$$\liminf_{n\to\infty} d(A \upharpoonright n) = \infty.$$

Therefore the s-gale d succeeds on A. Furthermore, for all $w \in \{0,1\}^*$, the value $d(w)$ can be computed in polynomial time relative to g by:

- k calls to the polynomial-time reduction function h on inputs

  $$s_{qk}, \ldots, s_{qk+k-1},$$

  each of which has length $O(\log |w|)$;
- k^2 calls, for each ordered pairs from $\{s_{qk}, \ldots, s_{qk+k-1}\}$, to the selector function f, which runs in polynomial time relative to g; and
- standard graph algorithms on G_q, which has $k = O(1)$ vertices.

We conclude that $\dim_{\mathrm{p}}^g(A) < s$, and the theorem follows immediately. □

Lemma 5.2. *Let* qp' *be the set of all functions in* qp *whose output length is polynomially bounded. There is a function* $h \in \mathrm{qp}'$ *such that* $\mathrm{qp}' = \mathrm{p}^h$.

Proof. By standard techniques of clocking Turing machines and bounding their running times and output lengths, we can form an enumeration $M_0, M_1, M_2, \ldots$ of Turing machines such that qp' is exactly the set of functions computed by Turing machines in this list. Define $h : \{0,1\}^* \to \{0,1\}^*$ by

$$h(u) = \begin{cases} M_k(x) & \text{if } u = 0^k 1 x \\ \lambda & \text{if } u \text{ does not contain a 1.} \end{cases}$$

It is clear that $\mathrm{p}^h = \mathrm{qp}'$. □

Theorem 5.3. $\dim(P_m(\text{qp-SEL}) \mid \mathrm{EXP}) = 0$.

Proof. Let h be as in Lemma 5.2, and let $A \in P_m(\text{qp-SEL})$. Then there exists some language $B \in \mathbf{C}$ and function $g \in \mathrm{qp}' = \mathrm{p}^h$ such that $A \leq_m^P B$ and g is a selector for B, i.e., B is p^h-selective. By Theorem 5.1, then, $\dim_{\mathrm{p}}^h(A) = 0$. This holds for all $A \in P_m(\text{qp-SEL})$, so we can apply Theorem 4.2:

$$\begin{aligned} \dim_{\mathrm{qp}}(P_m(\text{qp-SEL})) &\leq \sup_{A \in P_m(\text{qp-SEL})} \dim_{\mathrm{p}}^h(A) \\ &= 0. \end{aligned}$$

Since $\dim(P_m(\text{qp-SEL}) \mid \text{EXP})$ is defined as

$$\dim_{\text{qp}}(P_m(\text{qp-SEL}) \cap \text{EXP}) \leq \dim_{\text{qp}}(P_m(\text{qp-SEL})),$$

this completes the proof. □

Corollary 5.4. *No* qp*-selective set is partially* $\leq_m^P$*-hard for* EXP*.*

Corollary 5.5. *If* $\dim(\text{NP} \mid \text{EXP}) > 0$*, then no* qp*-selective set is* $\leq_m^P$*-hard for* NP*.*

6 Disjoint NP Pairs

In this section we improve the results in [7] by proving that the dimension of disjNP in disjEXP is related to the dimension of NP inside EXP.

Definition ([14, 28])**.** *For* $s \in [0, \infty)$ *and distribution* β *on alphabet* Σ*, a* β*-*s*-*gale *is a function* $d : \Sigma^* \to [0, \infty)$ *such that, for all* $w \in \Sigma^*$*,*

$$d(w) = \sum_{a \in \Sigma} d(wa)\beta(a)^s.$$

A β*-*s*-gale succeeds on a language* $A \subseteq \Sigma^*$*, and we write* $A \in S^\infty[d]$*, if*

$$\limsup_{w \to A} d(w) = \infty.$$

Let Δ *be a resource bound,* β *a distribution on alphabet* Σ*, and* $X \subseteq \mathcal{P}(\Sigma^*)$*. Then* $\mathcal{G}_{\Delta,\beta}(X)$ *denotes the set of all* $s \in [0, \infty)$ *such that there is a* Δ*-computable* β*-*s*-gale* d *for which* $X \subseteq S^\infty[d]$*, and the* Δ*-*β*-*dimension *of* X *is*

$$\dim_{\Delta,\beta}(X) = \inf \mathcal{G}_{\Delta,\beta}(X).$$

We code disjoint pairs as in [7], using the alphabet $\{0, 1, -1\}$. For a pair (A, B), 1 corresponds to A, -1 to B, and 0 to $(A \cup B)^c$.

We fix a probability distribution γ_0 on $\{0, 1, -1\}$ as $\gamma_0(0) = 1/4$, $\gamma_0(1) = \gamma_0(-1) = 3/8$, that is the natural distribution used in [7]. For disjoint pairs we write $\dim_\Delta(X)$ for $\dim_{\Delta,\gamma_0}(X)$. Theorem 4.2 extends routinely to this setting.

The main theorem of this section is the following

Theorem 6.1. *If* $\dim(\text{disjNP} \mid \text{disjEXP}) = 1$*, then* $\dim(\text{NP} \mid \text{EXP}) > 0$*.*

The proof of Theorem 6.1 is based on the following two results and Theorem 4.2.

Theorem 6.2. *Let* β *be a positive distribution on* $\{0, 1\}$*,* $X \subseteq \mathbf{C}$*, and* $g : \{0, 1\}^* \to \{0, 1\}^*$*. If* $\dim_{\text{p}}^g(X) = 0$*, then* $\dim_{\text{p},\beta}^g(X) < 1$*.*

Theorem 6.3. *Let* $\beta = (1/4, 3/4)$ *and* $g : \{0, 1\}^* \to \{0, 1\}^*$*. If* $\dim_{\text{p},\beta}^g(\text{NP}) < 1$*, then*

$$\dim_{\text{p}}^g(\text{disjNP}) < 1.$$

Theorem 6.2 is a consequence of the following lemma.

Lemma 6.4. *Let* $g : \{0, 1\}^* \to \{0, 1\}^*$*, let* s *be such that* $\dim_{\text{p}}^g(X) < s$*, and let* β *be a distribution on* $\{0, 1\}$*. If* $\max(\beta(0), \beta(1)) < 2^{-s}$*, then* $\dim_{\text{p},\beta}^g(X) < 1$*.*

Proof of Lemma 6.4. Let $s' > s$ and $t \in (0,1)$ be such that $\max(\beta(0), \beta(1)) < 2^{-s'/t}$. Let d be a p^g-computable s-gale. Define

$$d'(wb) = d'(w)\frac{d(wb)}{2^s d(w)}\frac{1}{\beta(b)^t}.$$

Then d' is a p^g-computable β-t-gale. Furthermore,

$$d'(w) \geq d(w)2^{-s|w|}\frac{1}{\beta(w)^t} > d(w)2^{-s|w|}2^{s'|w|},$$

and therefore $S^\infty[d] \subseteq S^\infty[d']$.

□

Theorem 6.3 is a consequence of the following lemma.

Lemma 6.5. *Let $g : \{0,1\}^* \to \{0,1\}^*$, γ a positive distribution on $\{0,1,-1\}$, β a distribution on $\{0,1\}$ with $\beta(0) = \gamma(0)$, and $X \subseteq \mathbf{C}$ a class that is closed under union. If $\dim^g_{\mathrm{p},\beta}(X) < 1$, then $\dim^g_{\mathrm{p},\gamma}(\mathrm{disj}X) < 1$.*

Proof of Lemma 6.5. If $\dim^g_{\mathrm{p},\beta}(X) < s < 1$ and d is a p^g-computable β-s gale succeeding on X, let $s' \in (0,1)$ with $\beta(1)^s \geq \gamma(1)^{s'} + \gamma(-1)^{s'}$ and $\beta(0)^s \geq \gamma(0)^{s'}$.

We define a p^g-computable γ-s' gale D by

$$\begin{aligned} D(w0) &= D(w)\frac{d(\overline{w}0)}{d(\overline{w})}\frac{\beta(0)^s}{\gamma(0)^{s'}} \\ D(w1) = D(w-1) &= D(w)\frac{d(\overline{w}1)}{d(\overline{w})}\frac{\beta(1)^s}{\gamma(1)^{s'} + \gamma(-1)^{s'}}, \end{aligned}$$

where

$$\begin{aligned} \overline{w}[i] &= 0 \quad \text{if} \quad w[i] = 0 \\ \overline{w}[i] &= 1 \quad \text{if} \quad w[i] = 1 \text{ or } w[i] = -1. \end{aligned}$$

That is, if w is a prefix of (A,B) then $\overline{w}$ is a prefix of $A \cup B$.

Notice that $D(w) \geq d(\overline{w})$ for every w.

Thus if $(A,B) \in \mathrm{disj}X$, then $A \cup B \in X$ and D succeeds on (A,B). □

Proof of Theorem 6.1. We prove the contrapositive. Suppose that $\dim(\mathrm{NP} \mid \mathrm{EXP}) = 0$. By Theorem 4.2, there is a $g \in \mathrm{qp}$ such that $\dim^g_{\mathrm{p}}(\mathrm{NP}) = 0$.

Let $\beta = (1/4, 3/4)$. By Theorem 6.2, $\dim^g_{\mathrm{p},\beta}(\mathrm{NP}) < 1$. By Theorem 6.3, $\dim^g_{\mathrm{p}}(\mathrm{disjNP}) < 1$.

Using Theorem 4.2 again, $\dim(\mathrm{disjNP} \mid \mathrm{disjEXP}) = \dim_{\mathrm{qp}}(\mathrm{disjNP}) < 1$. □

References

[1] Ambos-Spies, K., Merkle, W., Reimann, J., Stephan, F.: Hausdorff dimension in exponential time. In: Proceedings of the 16th Annual IEEE Conference on Computational Complexity, Chicago, Illinois, USA, June 18–21, 2001, pp. 210–217. IEEE Computer Society (2001)

[2] Bishop, C.J., Peres, Y.: Fractals in Probability and Analysis. Cambridge University Press (2017)

[3] Buhrman, H., Longpré, L.: Compressibility and resource bounded measure. In: Proceedings of the Thirteenth Symposium on Theoretical Aspects of Computer Science, pp. 13–24. Springer-Verlag, Berlin (1996)

[4] Dose, T., Glaßer, C.: NP-completeness, proof systems, and disjoint NP-pairs. In: C. Paul, M. Bläser (eds.) 37th International Symposium on Theoretical Aspects of Computer Science, STACS 2020, March 10–13, 2020, Montpellier, France, *LIPIcs*, vol. 154, pp. 9:1–9:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020)

[5] Even, S., Selman, A.L., Yacobi, Y.: The complexity of promise problems with applications to public-key cryptography. Information and Control **61**(2), 159–173 (1984)

[6] Falconer, K.: Fractal Geometry: Mathematical Foundations and Applications, 3rd edition. John Wiley & Sons (2014)

[7] Fortnow, L., Lutz, J.H., Mayordomo, E.: Inseparability and strong hypotheses for disjoint NP pairs. Theory of Computing Systems **51**, 229–247 (2012)

[8] Glaßer, C., Hughes, A., Selman, A.L., Wisiol, N.: Disjoint NP-pairs and propositional proof systems. SIGACT News **45**(4), 59–75 (2014)

[9] Glaßer, C., Selman, A.L., Sengupta, S.: Reductions between disjoint NP-pairs. Information and Computation **200**, 247–267 (2005)

[10] Glaßer, C., Selman, A.L., Sengupta, S., Zhang, L.: Disjoint NP-pairs. SIAM Journal on Computing **33**, 1369–1416 (2004)

[11] Glaßer, C., Selman, A.L., Zhang, L.: Canonical disjoint NP-pairs of propositional proof systems. Theoretical Computer Science **370**, 60–73 (2007)

[12] Glaßer, C., Selman, A.L., Zhang, L.: The informational content of canonical disjoint NP-pairs. Int. J. Found. Comput. Sci. **20**(3), 501–522 (2009)

[13] Grollmann, J., Selman, A.: Complexity measures for public-key cryptosystems. SIAM J. Comput. **11**, 309–335 (1988)

[14] Gu, X., Lutz, J., Mayordomo, E., Moser, P.: Dimension spectra of random subfractals of self-similar fractals. Annals of Pure and Applied Logic **165**, 1707–1726 (2014)

[15] Hartmanis, J., Stearns, R.: On the computational complexity of algorithms. Transactions of the American Mathematical Society **117**, 285–306 (1965)

[16] Hausdorff, F.: Dimension und äußeres Maß. Math. Ann. **79**, 157–179 (1919)

[17] Hemaspaandra, L.A., Torenvliet, L.: Theory of Semi-Feasible Algorithms. Springer-Verlag (2002)

[18] Homer, S., Selman, A.L.: Oracles for structural properties: The isomorphism problem and public-key cryptography. Journal of Computer and System Sciences **44**, 287–301 (1992)

[19] Jockusch, C.G.: Semirecursive sets and positive reducibility. Trans. Amer. Math. Soc. **131**, 420–436 (1968)

[20] Lutz, J., Lutz, N., Mayordomo, E.: Extending the reach of the point-to-set principle (2020). URL `https://arxiv.org/pdf/2004.07798.pdf`

[21] Lutz, J.H.: Resource-bounded category and measure in exponential complexity classes. Ph.D. thesis, California Institute of Technology (1987)

[22] Lutz, J.H.: Category and measure in complexity classes. SIAM Journal on Computing **19**, 1100–1131 (1990)

[23] Lutz, J.H.: Almost everywhere high nonuniform complexity. J. Comput. Syst. Sci. **44**(2), 220–258 (1992)

[24] Lutz, J.H.: Almost everywhere high nonuniform complexity. Journal of Computer and System Sciences **44**(2), 220–258 (1992)

[25] Lutz, J.H.: The quantitative structure of exponential time. In: L.A. Hemaspaandra, A.L. Selman (eds.) Complexity Theory Retrospective II, pp. 225–254. Springer-Verlag (1997)

[26] Lutz, J.H.: Dimension in complexity classes. SIAM J. Comput. **32**(5), 1236–1259 (2003)

[27] Lutz, J.H.: The dimensions of individual strings and sequences. Information and Computation **187**(1), 49–79 (2003)

[28] Lutz, J.H.: A divergence formula for randomness and dimension. Theoretical Computer Science **412**, 166–177 (2011)

[29] Lutz, J.H., Lutz, N.: Algorithmic information, plane Kakeya sets, and conditional dimension. In: Proceedings of the 34th Symposium on Theoretical Aspects of Computer Science, STACS 2017, March 8–11, 2017, Hannover, Germany, pp. 53:1–53:13 (2017)

[30] Lutz, J.H., Lutz, N.: Who asked us? How the theory of computing answers questions about analysis. In: D. Du, J. Wang (eds.) Complexity and Approximation: In Memory of Ker-I Ko, pp. 48–56. Springer (2020)

[31] Lutz, J.H., Mayordomo, E.: Algorithmic fractal dimensions in geometric measure theory. In: V. Brattka, P. Hertling (eds.) Handbook of Computability and Complexity in Analysis, pp. 271–302. Springer (2021)

[32] Lutz, N.: Fractal intersections and products via algorithmic dimension. ACM Trans. Comput. Theory **13**(3) (2021)

[33] Lutz, N., Stull, D.M.: Projection theorems using effective dimension. In: 43rd International Symposium on Mathematical Foundations of Computer Science, MFCS 2018, August 27-31, 2018, Liverpool, UK, pp. 71:1–71:15 (2018)

[34] Lutz, N., Stull, D.M.: Bounding the dimension of points on a line. Information and Computation **275** (2020)

[35] Martin-Löf, P.: The definition of random sequences. Information and Control **9**, 602–619 (1966)

[36] Mattila, P.: Fourier Analysis and Hausdorff Dimension. Cambridge Studies in Advanced Mathematics. Cambridge University Press (2015)

[37] Razborov, A.: On provably disjoint NP pairs. Tech. Rep. 94-006, ECCC (1994)

[38] Selman, A.: P-selective sets, tally languages, and the behavior of polynomial time reducibilities on NP. Mathematical Systems Theory **13**, 55–65 (1979)

[39] Selman, A.: Complexity issues in cryptography. In: Computational complexity theory (Atlanta, GA, 1988), *Proc. Sympos. Appl. Math.*, vol. 38, pp. 92–107. Amer. Math. Soc. (1989)

[40] Stearns, R.E., Hartmanis, J., Lewis, P.: Hierarchies of memory limited computations. In: Proc. 6th Annual Symp. on Switching Circuit Theory and Logical Design, pp. 179–190 (1965)

[41] Stein, E.M., Shakarchi, R.: Real Analysis: Measure Theory, Integration, and Hilbert Spaces. Princeton Lectures in Analysis. Princeton University Press (2005)

[42] Wang, Y.: Randomness and complexity. Ph.D. thesis, Department of Mathematics, University of Heidelberg (1996)

[43] Zimand, M.: Computational Complexity: A Quantitative Perspective. Elsevier, Amsterdam (2004)