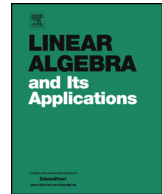




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Regular graphs of degree at most four that allow two distinct eigenvalues



Wayne Barrett^a, Shaun Fallat^{b,*}, Veronika Furst^c,
Shahla Nasserassr^d, Brendan Rooney^d, Michael Tait^e

^a Department of Mathematics, Brigham Young University, Provo, UT, United States of America

^b Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada

^c Department of Mathematics, Fort Lewis College, Durango, CO, United States of America

^d School of Mathematics and Statistics, Rochester Institute of Technology, Rochester, NY, United States of America

^e Department of Mathematics & Statistics, Villanova University, Villanova, PA, United States of America

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ABSTRACT

For an $n \times n$ matrix A , let $q(A)$ be the number of distinct eigenvalues of A . If G is a connected graph on n vertices, let $\mathcal{S}(G)$ be the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} = 0$ if and only if $\{i, j\}$ is not an edge of G . Let $q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}$. Studying $q(G)$ has become a fundamental sub-problem of the inverse eigenvalue problem for graphs, and characterizing the case for which $q(G) = 2$ has been especially difficult. This paper considers the problem of determining the regular graphs G that satisfy $q(G) = 2$. The resolution is straightforward if the degree of regularity is 1, 2, or 3. However, the 4-regular graphs with $q(G) = 2$ are much more difficult to characterize. A connected 4-regular graph has $q(G) = 2$ if and only if either G belongs to a specific infinite class of graphs, or else G is one of fifteen 4-regular graphs whose number of vertices ranges from

* Corresponding author.

E-mail addresses: wb@mathematics.byu.edu (W. Barrett), shaun.fallat@uregina.ca (S. Fallat), furst_v@fortlewis.edu (V. Furst), sxnsma@rit.edu (S. Nasserassr), brsma@rit.edu (B. Rooney), michael.tait@villanova.edu (M. Tait).

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5 to 16. This technical result gives rise to several intriguing questions.

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1. Introduction

For any connected graph G on n vertices, let $\mathcal{S}(G)$ denote the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ where $a_{ij} = 0$ if and only if $\{i, j\}$ is not an edge in G , and the entries a_{ii} can take any value. The *inverse eigenvalue problem* for a graph G asks to determine all possible spectra of matrices in $\mathcal{S}(G)$ [5,14].

This problem and several of its sub-problems have been studied extensively. One of these sub-problems is to consider all possible multiplicity lists of eigenvalues of matrices in $\mathcal{S}(G)$. If we look at the multiplicity lists of eigenvalues of all matrices in $\mathcal{S}(G)$ as lists of numbers, the shortest length among all these lists is the minimum number of distinct eigenvalues of matrices in $\mathcal{S}(G)$. This parameter is denoted by $q(G)$ and has been studied in [2,7,15–17]. In this paper, we investigate the problem of determining which regular connected graphs G have a matrix in $\mathcal{S}(G)$ with exactly two distinct eigenvalues, that is, with $q(G) = 2$.

The connected graphs G with $q(G) = n - 1$ or n have been characterized, see [7]. Graphs with $q(G) = 2$ are much harder to describe; for example, there is no forbidden subgraph characterization of graphs with $q(G) = 2$, as implied by Theorem 5.2 in [2]. It is known that $q(G) = 2$ if and only if there is an orthogonal matrix in $\mathcal{S}(G)$ [2], and so studying graphs G on n vertices with $q(G) = 2$ is equivalent to studying all possible zero patterns of $n \times n$ symmetric orthogonal matrices.

A graph G must have a sufficiently large number of edges to satisfy $q(G) = 2$. In [6], we showed that a connected graph G on n vertices with $q(G) = 2$ has at least $2n - 4$ edges. We also characterized the graphs for which equality is attained. This result immediately implies that the number of r -regular graphs with $r \in \{2, 3\}$ is finite, and in Section 2 we characterize these graphs. When considering 4-regular graphs with $q(G) = 2$, the difficulty increases significantly. Our main theorem (Theorem 2.4) characterizes all connected 4-regular graphs with $q(G) = 2$.

Throughout this paper, we only consider connected, simple, undirected graphs.

1.1. Preliminaries

One of the common ways to give a lower bound on $q(G)$ is to find a unique shortest path between vertices. This technique, specialized to the case $q(G) = 2$, is explained in the following lemma, which is a corollary of Theorem 3.2 in [2].

Lemma 1.1. *Let G be a connected graph with $q(G) = 2$. If xuy is a path of length 2, then either $x \sim y$ or there is another path xvy of length 2 between x and y .*

In [6], we used this lemma extensively in combination with a breadth-first search of a graph, and we use this strategy here as well. Given a fixed vertex v , we perform a breadth-first search from v . Denote the vertices at distance exactly i from v by $N_i(v)$, and call this set the i th *distance set* from v . We use $\epsilon(v)$ to denote the *eccentricity* of v , which is the maximum distance from v to any vertex in the graph. The distance sets from v partition the vertex set of G as

$$V(G) = \bigcup_{i=0}^{\epsilon(v)} N_i(v),$$

which we call the *distance partition* of G with respect to v . If $u \in N_i(v)$ and $w \in N_{i+1}(v)$ and $u \sim w$, we call u a *predecessor* of w and we call w a *successor* of u . A *terminal vertex* is a vertex with no successors.

Assume that G is a graph with $q(G) = 2$ and consider the distance partition from a vertex v . If a vertex u is in $N_i(v)$ for some $i \geq 2$, then by Lemma 1.1, it must have at least two predecessors, as otherwise there would be a unique shortest path of length 2 from u to a vertex in $N_{i-2}(v)$. If G is a 4-regular graph on n vertices, then there are $n - 5$ vertices in

$$\bigcup_{i=2}^{\epsilon(v)} N_i(v) = V(G) \setminus (\{v\} \cup N_1(v)),$$

and each of these vertices has at least two predecessors so there are at least $2(n - 5)$ edges incident to these vertices. With the 4 edges incident to v , this accounts for $2n - 6$ of the $2n$ edges of G . We call the remaining six edges *extra edges*, and throughout the paper we consider the possible locations of these six extra edges.

We use standard graph theory terminology and notations. Often we abbreviate an edge $\{u, v\} \in E(G)$ as uv for vertices $u, v \in V(G)$. For two graphs G and H with vertex sets $V(G)$ and $V(H)$, respectively, the Cartesian product $G \square H$ is the graph with vertex set $V(G) \times V(H)$ and (g_1, h_1) adjacent to (g_2, h_2) if either $g_1 = g_2$ and $h_1 h_2 \in E(H)$, or $h_1 = h_2$ and $g_1 g_2 \in E(G)$. The complete graph on n vertices, the complete bipartite graph on partite sets of sizes m and n , the cycle on n vertices, the path on n vertices, and the hypercube graph (the graph on 2^n vertices obtained by an n -fold Cartesian product of K_2 with itself) are denoted by K_n , $K_{m,n}$, C_n , P_n , Q_n , respectively. The circulant graph $G = C(n, \pm i, \pm j)$ is the graph with vertex set $V(C(n, \pm i, \pm j)) = \mathbb{Z}/n\mathbb{Z}$ that has edges $\{t, t \pm i\}$ and $\{t, t \pm j\}$ for all $t \in V(G)$.

Let G be a graph with $v \in V(G)$. A graph $\text{jdup}(G, v)$ is constructed from G by *joined duplicating* a vertex $v \in V(G)$ if $V(\text{jdup}(G, v)) = V(G) \cup \{u\}$ and $E(\text{jdup}(G, v)) = E(G) \cup \{uw : w \in \{v\} \cup N_1(v)\}$. From Lemma 2.9 in [16], $q(\text{jdup}(G, v)) \leq q(G)$ for any vertex v in a connected graph G .

Lemma 1.2. [Lemma 2.3, [1]] *Let G be a connected graph on n vertices with $q(G) = 2$. If S is an independent set of vertices, then $|S| \leq k$ where k is the least integer*

such that there is a matrix in $\mathcal{S}(G)$ with two distinct eigenvalues of multiplicities k and $n - k$.

Lemma 1.3. *Let*

$$M = \begin{bmatrix} C & B \\ B^T & D \end{bmatrix}$$

where C is an $m \times m$ symmetric matrix, $B = [b_1 \dots b_n]$ has no zero columns, and $D = \text{diag}(d_1, \dots, d_n)$. If M is orthogonal, then $n \leq m$, the vectors $b_1, \dots, b_n$ are pairwise orthogonal, and each b_i is a $(-d_i)$ -eigenvector for C .

Proof. Consider the last n columns of M . Since D is diagonal, these columns are pairwise orthogonal if and only if $n \leq m$ and the vectors $b_1, \dots, b_n$ are pairwise orthogonal.

Second, expanding $M^2 = I$ we have

$$M^2 = \begin{bmatrix} C^2 + BB^T & CB + BD \\ B^T C + DB^T & B^T B + D^2 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}.$$

From the $(1, 2)$ -block we see $CB + BD = 0$. Rearranging this equation we have

$$[Cb_1 \dots Cb_n] = -[d_1 b_1 \dots d_n b_n]$$

from which it follows that each b_i is a $(-d_i)$ -eigenvector for C . $\square$

As an illustration of Lemma 1.3, suppose G is a connected bipartite graph with bipartition $V = V_1 \cup V_2$. Further assume that $|V_1| = |V_2|$. It follows that if $q(G) = 2$, then there exists a matrix $M \in \mathcal{S}(G)$, where

$$M = \begin{bmatrix} C & B \\ B^T & D \end{bmatrix},$$

where $M^2 = I$, and both C and D are diagonal matrices. Applying Lemma 1.3 we have that B must be a matrix with orthogonal rows and columns. In fact, the converse also holds in this case. If such an orthogonal matrix B exists, then the matrix

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} I & B \\ B^T & -I \end{bmatrix}$$

is orthogonal and hence $q(G) = 2$.

2. Certain regular graphs that allow two distinct eigenvalues

From Theorem 3.1 in [6], if a connected graph G has fewer than $2n - 4$ edges, then $q(G) > 2$. This implies that there are only finitely many r -regular graphs with $r \leq 3$ that satisfy $q(G) = 2$. We describe them below.

Lemma 2.1. *Let $r \leq 3$. If G is a connected r -regular graph with $q(G) = 2$, then G has at most $8/(4 - r)$ vertices.*

Proof. In order for G to have $q(G) = 2$, G has to have at least $2n - 4$ edges. So we have the inequality

$$(r/2)n \geq 2n - 4.$$

Since $0 < r/2 < 2$, this simplifies to $n \leq 8/(4 - r)$. $\square$

Corollary 2.2. *If G is a connected r -regular graph with $q(G) = 2$ for some $r \leq 3$, then G is one of:*

- (1) K_2 ;
- (2) K_3 or C_4 ; or,
- (3) K_4 , $K_{3,3}$, $K_3 \square K_2$, or Q_3 .

Proof. The graph K_2 is the only connected 1-regular graph and has $q(K_2) = 2$. The only connected 2-regular graphs on $n \leq 4$ vertices are K_3 and C_4 and both have $q(G) = 2$. By Lemma 2.1, a 3-regular graph with $q(G) = 2$ has 4, 6 or 8 vertices. If $n = 4$, $G = K_4$ and $q(G) = 2$. If $n = 6$, the complement of G is 2-regular and so must be C_6 or $2K_3$. Thus G is either $K_3 \square K_2$ or $K_{3,3}$, respectively, both of which have $q(G) = 2$ from Corollaries 6.5 and 6.8 in [2]. If $n = 8$, G has $12 = 2(8) - 4$ edges. Thus by Theorem 3.1 from [6], $G \cong Q_3$. $\square$

We now proceed with the main purpose of this paper, to characterize the 4-regular graphs G with $q(G) = 2$. We begin by defining an infinite family of graphs called closed candles which are analogs to the single-ended and double-ended candles in [6]. For $k \geq 3$ the *closed candle*, H_k , is constructed from $2C_k$ as follows. Label the vertices of one C_k with the odd integers from 1 to $2k - 1$ and the other with the even integers from 2 to $2k$. Insert $2k$ additional edges between the two C_k 's according to the rule: i is adjacent to j , i odd, j even if $j - i = 3$, $j - i = -1$, or $j = 2$, $i = 2k - 1$, or $i = 1$, $j = 2k$. Thus H_k is a 4-regular graph. The graph H_{10} is shown in Fig. 1.

In the proof of Theorem 2.4, we will consider induced subgraphs that have the same structure as a closed candle. A *candle section* is a graph with vertices $u_1, \dots, u_t, v_1, \dots, v_t$ with edges $u_i u_{i+1}$, $v_i v_{i+1}$, $u_{i+1} v_i$, and $u_i v_{i+1}$ for $1 \leq i \leq t - 1$, see Fig. 2.

The following lemma gives a construction of orthogonal matrices for the closed candles.

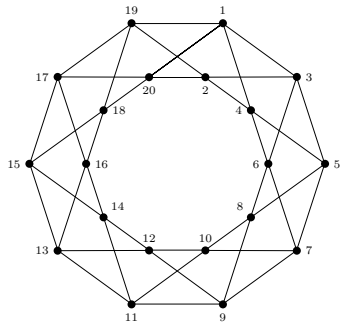


Fig. 1. Closed candle H_{10} .

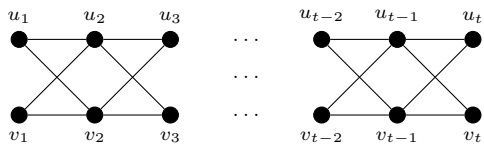


Fig. 2. Candle section.

Lemma 2.3. *For all $k \geq 3$, we have $q(H_k) = 2$.*

Proof. To see that the graphs H_k for $k \geq 3$ achieve two distinct eigenvalues, we construct orthogonal matrices for this family of graphs.

Let

$$R = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We consider two cases for k and construct the corresponding matrices W . Here all of the blocks are 2×2 . The matrix in each case is symmetric so the blocks below the diagonal blocks are transposes of the corresponding matrices.

Case 1: $n = 2k$ and $k \geq 4$ is even.

$$W_{ij} = \begin{cases} R, & \text{for } (i, j) = (1, 2), (3, 4), \dots, (k-3, k-2), (k-1, k) \\ J, & \text{for } (i, j) = (2, 3), (4, 5), \dots, (k-2, k-1) \\ J, & \text{for } (i, j) = (1, k) \\ O, & \text{otherwise} \end{cases}$$

Case 2: $n = 2k$ and $k \geq 3$ is odd.

First, note that when $k = 3$, the graph H_3 is the octahedron (the graph obtained from K_6 by deleting a perfect matching). By Corollary 6.9 in [8], we have $q(H_3) = q(G204) = 2$. Now if $k \geq 5$, we construct the matrices as follows.

$$W_{ij} = \begin{cases} J, & \text{for } (i, j) = (1, 2), (3, 4), \dots, (k-4, k-3) \\ R, & \text{for } (i, j) = (2, 3), (4, 5), \dots, (k-5, k-4), \text{ and } (k-2, k-1); \\ S, & \text{for } (i, j) = (1, k), (k-3, k-2) \\ S^T, & \text{for } (i, j) = (k-1, k) \\ O, & \text{otherwise} \end{cases}$$

Each row of W has Euclidean length 2. Since each of the 2×4 matrices $[J \ S]$, $[J \ R]$, $[S^T \ R]$, $[S \ S^T]$ has orthogonal rows, each pair of rows of W coming from the same block is orthogonal. For $i \neq j$, the (i, j) -block of W^2 is one of JR , JS , SR , S^2 or their transposes. In each of the cases the matrix is the zero matrix. Hence $W^T W = W^2 = 4I$. We conclude that $\frac{1}{2}W$ is an orthogonal matrix with whose graph is the closed candle on $n = 2k$ vertices. $\square$

For example, the matrices for $k = 4$ and $k = 5$ are respectively

$$\begin{bmatrix} O & R & O & J \\ R & O & J & O \\ O & J & O & R \\ J & O & R & O \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} O & J & O & O & S \\ J & O & S & O & O \\ O & S^T & O & R & O \\ O & O & R & O & S^T \\ S^T & O & O & S & O \end{bmatrix};$$

for $k = 6$ and $k = 7$, the matrices are respectively

$$\begin{bmatrix} O & R & O & O & O & J \\ R & O & J & O & O & O \\ O & J & O & R & O & O \\ O & O & R & O & J & O \\ O & O & O & J & O & R \\ J & O & O & O & R & O \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} O & J & O & O & O & O & S \\ J & O & R & O & O & O & O \\ O & R & O & J & O & O & O \\ O & O & J & O & S & O & O \\ O & O & O & S^T & O & R & O \\ O & O & O & O & R & O & S^T \\ S^T & O & O & O & O & S & O \end{bmatrix}.$$

Note that the closed candle H_k has independence number k if k is even, and $k-1$ if k is odd. By Lemma 1.2 when k is even the only achievable multiplicity list for two distinct eigenvalues is $[k, k]$; when k is odd, the only achievable multiplicity lists for two distinct eigenvalues are $[k, k]$ and $[k-1, k+1]$.

Lemma 2.3 provides an infinite family of 4-regular graphs with $q(G) = 2$. Our main theorem below characterizes all 4-regular graphs G for which $q(G) = 2$.

Theorem 2.4. *If G is a connected 4-regular graph with $q(G) = 2$, then G is either:*

- (1) K_5 ;
- (2) one of the graphs $R_{7,1}$, $R_{8,2}$, $R_{8,3}$, $R_{8,4}$, $R_{8,5}$, $R_{8,6}$ from Fig. 3;

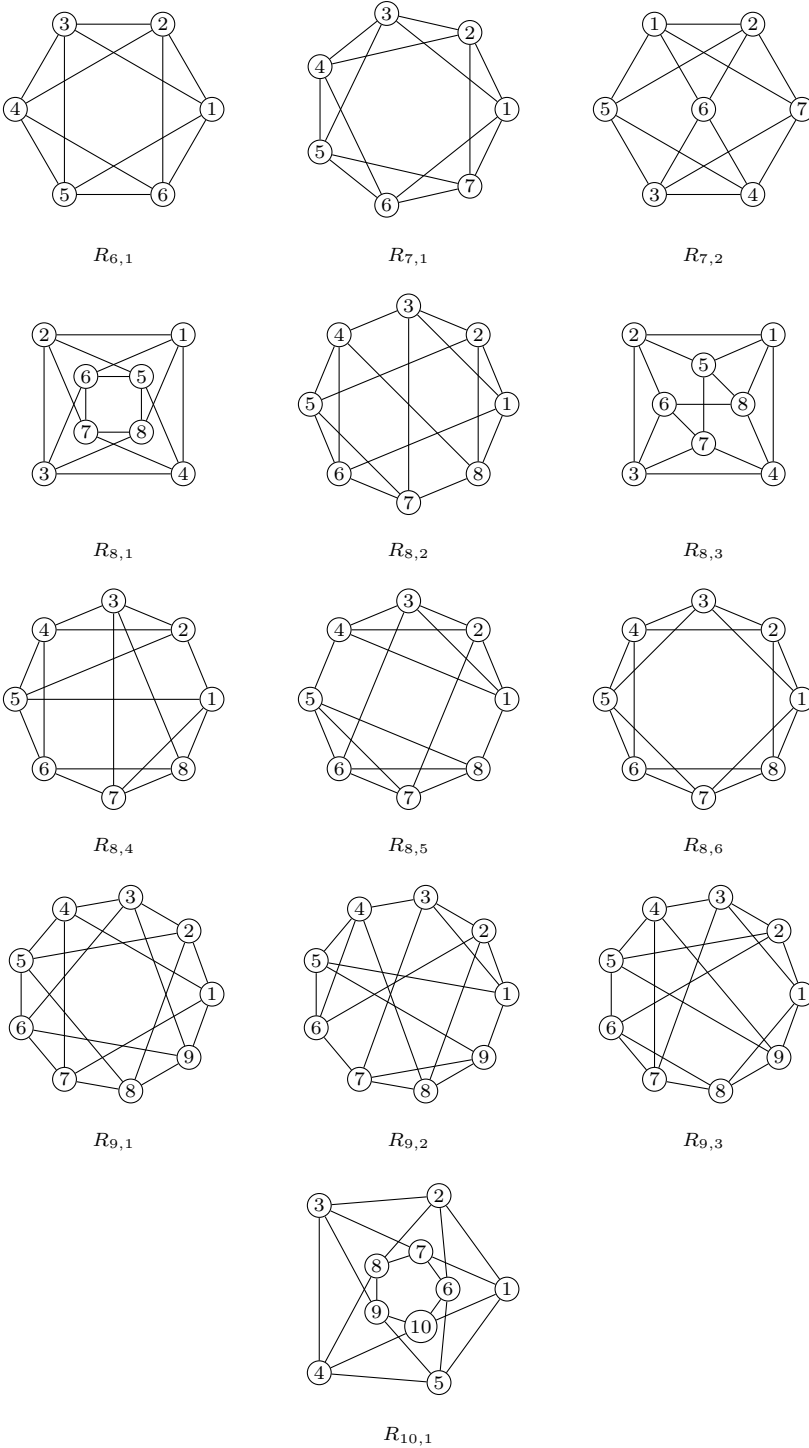


Fig. 3. Thirteen 4-regular graphs with diameter 2.

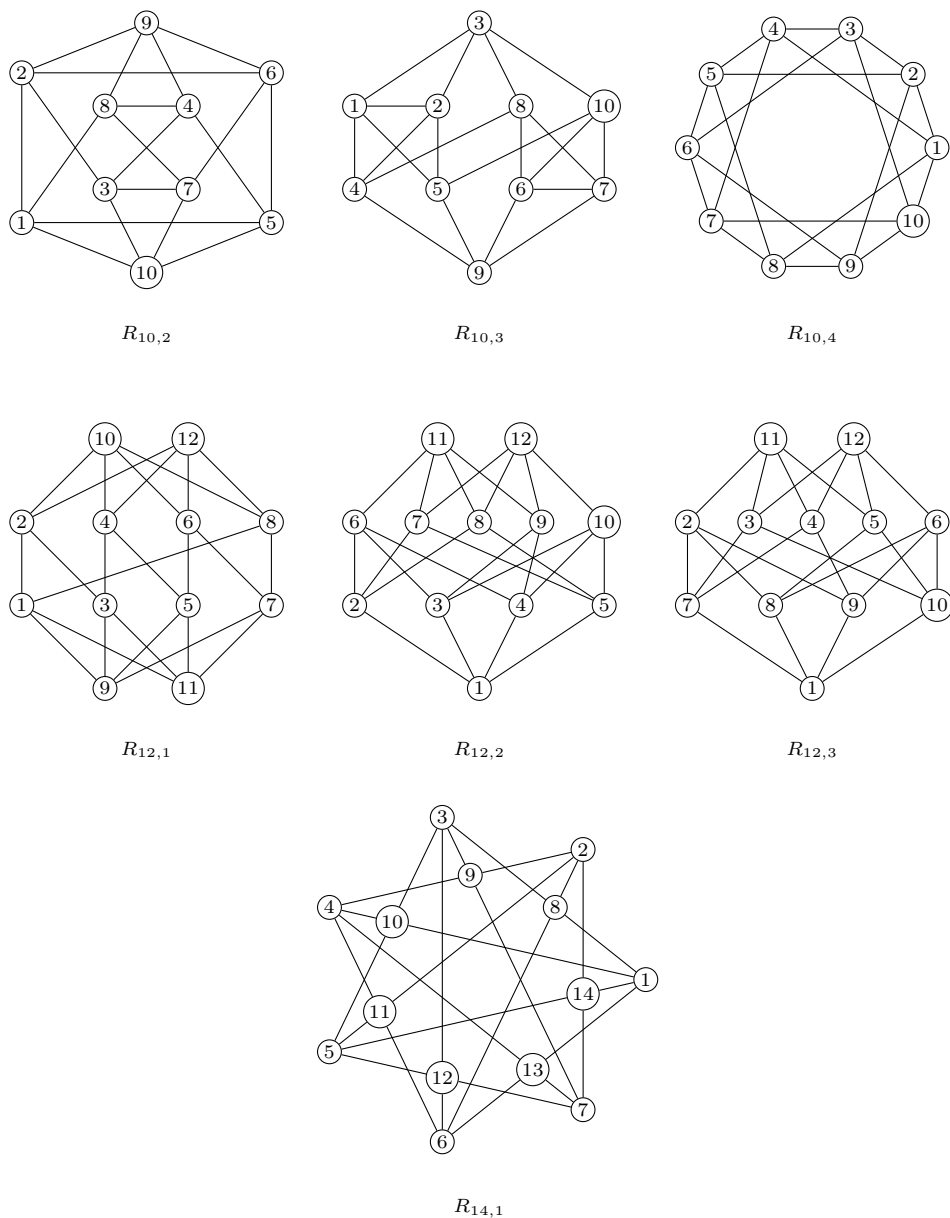


Fig. 4. Sporadic 4-regular graphs with diameter at least 3.

- (3) $K_3 \square C_4$, $K_{3,3} \square K_2$, one of the graphs $R_{10,2}$, $R_{10,3}$, $R_{10,4}$, $R_{12,3}$, $R_{14,1}$ from Fig. 4;
- (4) Q_4 ; or,
- (5) a closed candle H_k for some $k \geq 3$.

Notes: The graphs listed in items (1) through (4) of Theorem 2.4 have diameter 1 through 4 respectively. The graph $R_{7,1} \cong C(7, \pm 1, \pm 2)$, the graph $R_{8,5} \cong K_4 \square K_2$, and the

graph $R_{8,6} \cong C(8, \pm 1, \pm 2)$. The graph $R_{10,3}$ is the graph obtained from Q_3 by joined duplicating a pair of antipodal vertices (in [6] we referred to this graph as Q'_3). The graph $R_{10,4} \cong C(10, \pm 1, \pm 3)$, and the graph $R_{12,3} \cong C(12, \pm 1, \pm 3)$. The graph $R_{14,1}$ appears in [19] as S_{14} . Moreover $R_{14,1}$ is the Cayley graph for the dihedral group D_7 (with generators ρ and φ satisfying $\rho^2 = \varphi^7 = \varepsilon$ and $\rho\varphi = \varphi^6\rho$) with connection set $\{\rho, \varphi\rho, \varphi^2\rho, \varphi^4\rho\}$. It is also the point-block incidence graph of the non-trivial square $2 - (7, 4, 2)$ design and is distance regular with diameter 3, see [9]. The sporadic graphs $R_{6,1}$, $R_{8,1}$, and $R_{10,1}$ in Fig. 3 are H_3 , H_4 , and H_5 , respectively, so appear in item (5) of Theorem 2.4. Also note that the closed candles are all circulants. For $k \geq 3$, it can be verified that $H_k \cong C(2k, \pm 1, \pm(k-1))$.

The proof of Theorem 2.4 is split over Sections 3, 4, and 5.

3. Proof of Theorem 2.4 for 4-regular graphs with small diameter

In this section we prove items (1) and (2) of Theorem 2.4 for graphs with diameter at most 2. The only 4-regular graph with diameter 1 is K_5 , and $q(K_5) = 2$, so we focus on 4-regular graphs with diameter 2. We begin by enumerating the 4-regular graphs G with diameter 2 for which $q(G) = 2$ is not ruled out by Lemma 1.1.

Lemma 3.1. *If G is a connected 4-regular graph with diameter 2 such that $q(G) = 2$, then $6 \leq |V(G)| \leq 10$.*

Proof. Let G be a connected 4-regular graph with diameter 2. Consider an arbitrary vertex v in G and the distance partition of $V(G)$ from v . In order for G to have $q(G) = 2$, it must be the case that each $x \in N_2(v)$ has at least two neighbors in $N_1(v)$, otherwise there is a unique path of length 2 between x and v . Let X be the set of edges between $N_1(v)$ and $N_2(v)$. Then

$$2|N_2(v)| \leq |X| \leq 3|N_1(v)| = 12.$$

So $|N_2(v)| \leq 6$ and G has at most $1+4+6 = 11$ vertices. This establishes $6 \leq |V(G)| \leq 11$.

We now show the upper bound can be improved to 10. Consider a 4-regular graph G with 11 vertices and diameter 2. Using the notation above, we see $|N_2(v)| = 6$, and $|X| = 12$. So every vertex in $N_1(v)$ has exactly three neighbors in $N_2(v)$, and every vertex in $N_2(v)$ has exactly two neighbors in $N_1(v)$. In particular, this means the subgraph H of G induced by $N_2(v)$ is 2-regular. So we have two cases: either H is a 6-cycle, or H is the disjoint union of two 3-cycles.

Case 1: $H \cong C_6$

Let the vertices of H be $x_1, x_2, x_3, x_4, x_5, x_6$ in cyclic order. Note that H is bipartite, and in H there is a unique shortest path of length 2 between any two vertices in the same partite set. Since there can be no unique shortest path of length 2 in G , the edges

of X must supply an additional path of length 2 between every pair of vertices in each partite set.

Let $N_1(v) = \{v_1, v_2, v_3, v_4\}$. Since $v_i v v_j$ is a path of length 2, in order for this path not to be unique there must be some x_k that is adjacent to both v_i and v_j . In particular, this means we cannot have some v_i whose neighbors are $\{x_1, x_3, x_5\}$ and some v_j whose neighbors are $\{x_2, x_4, x_6\}$. If there is no v_i whose neighbors are $\{x_1, x_3, x_5\}$, then $\{x_1, x_3, x_5\}$ together with the common neighbor of $\{x_1, x_3\}$, the common neighbor of $\{x_3, x_5\}$ and the common neighbor of $\{x_1, x_5\}$ form a 6-cycle. Without loss of generality, suppose this 6-cycle is $(v_1, x_1, v_2, x_3, v_3, x_5)$. Then, in order for each pair of vertices in $\{x_2, x_4, x_6\}$ to have a common neighbor, we must have v_4 adjacent to each of $\{x_2, x_4, x_6\}$. Thus the edges between $\{v_1, v_2, v_3\}$ and $\{x_2, x_4, x_6\}$ are a perfect matching.

We know v_1 is matched to one of x_2, x_4 , or x_6 . We also know that v_1 is already adjacent to x_1 and x_5 . Note that in H , x_1 is at distance 3 from x_4 . So if v_1 is matched to x_4 , then we have a unique shortest path of length 2, $x_1 v_1 x_4$. Similarly, x_5 is at distance 3 from x_2 in H , so if we match v_1 to x_2 , we get another unique shortest path of length 2. Thus v_1 must be matched to x_6 . A similar argument shows that v_2 must be matched to x_2 , and v_3 must be matched to x_4 . This accounts for all of the edges in X . But now we see that $v_1 x_1 x_2$ is a unique shortest path of length 2 in G .

Case 2: $H \cong 2C_3$

Let the vertices of H be x_1, x_2, x_3 and y_1, y_2, y_3 , where all of the x_i 's are adjacent, and all of the y_i 's are adjacent. Let $N_1(v) = \{v_1, v_2, v_3, v_4\}$. We know that x_1 has 2 neighbors in $N_1(v)$. Without loss of generality, suppose x_1 is adjacent to v_1 . Then $v_1 x_1 x_2$ and $v_1 x_1 x_3$ are paths of length 2. In order for them not to be unique shortest paths of length 2, we must have at least one of the edges $v_1 x_2$ and $v_1 x_3$. Suppose v_1 is adjacent to exactly one of x_2 and x_3 . Then v_1 is adjacent to exactly one of the y_i vertices, and we have a unique path $v_1 y_i y_j$ between v_1 and some y_j . Thus v_1 must be adjacent to both x_2 and x_3 . If v_2 is the other neighbor of x_1 in $N_1(v)$, then following the same argument as for v_1 , we must also have edges $v_2 x_2$ and $v_2 x_3$. This accounts for all edges in X with ends v_1 and v_2 , and all edges in X with ends x_1, x_2 , or x_3 . Thus the remaining edges in X are all possible edges between $\{v_3, v_4\}$ and $\{y_1, y_2, y_3\}$. But now $v_1 v v_3$ is a unique shortest path of length 2. $\square$

Since there are only 84 connected 4-regular graphs with order $6 \leq n \leq 10$ [20], we can generate the list of 4-regular graphs with diameter 2 for which $q(G) = 2$ is not ruled out by Lemma 1.1. We do this by:

- (1) using nauty's geng function [18] to generate all connected 4-regular graphs on n vertices for each $6 \leq n \leq 10$;
- (2) checking the diameter of the graphs generated in (1), and eliminating all with diameter at least 3; then

Table 1
The reduced list of thirteen 4-regular graphs with diameter 2.

Graph	q -value	Graph	q -value	Graph	q -value
$R_{6,1}$	2	$R_{8,3}$	2	$R_{9,1}$	> 2
$R_{7,1}$	2	$R_{8,4}$	2	$R_{9,2}$	> 2
$R_{7,2}$	3	$R_{8,5}$	2	$R_{9,3}$	3
$R_{8,1}$	2	$R_{8,6}$	2	$R_{10,1}$	2
$R_{8,2}$	2				

(3) checking the graphs remaining after (2) for any unique shortest paths connecting two vertices at distance 2 and eliminating those graphs.

At the end of this computation we are left with a set of thirteen 4-regular graphs of diameter 2 for which $q(G) = 2$ is not ruled out. Fig. 3 gives these thirteen graphs. Lemma 3.2 completes the proof of Theorem 2.4 for graphs with diameter at most 2.

Lemma 3.2. *Table 1 lists all thirteen candidate 4-regular graphs with diameter 2 (shown in Fig. 3) and includes their q -values (or a bound on their q -value).*

Proof. We treat each of the graphs in Table 1 separately, in the order they appear in the table.

$R_{6,1}$: The graph $R_{6,1}$ is a closed candle, $R_{6,1} \cong H_3$. Thus $q(R_{6,1}) = 2$ by Lemma 2.3.

$R_{7,1}$: The following matrix M is a matrix in $\mathcal{S}(R_{7,1} - \{1, 7\})$ (the graph $R_{7,1}$ with the edge $\{1, 7\}$ deleted),

$$M = \frac{1}{6} \begin{bmatrix} 3 & \sqrt{6} & -3 & 0 & 0 & 2\sqrt{3} & 0 \\ \sqrt{6} & 0 & \sqrt{6} & 2\sqrt{2} & 0 & 0 & -4 \\ -3 & \sqrt{6} & -1 & 2\sqrt{3} & 2\sqrt{2} & 0 & 0 \\ 0 & 2\sqrt{2} & 2\sqrt{3} & -3 & \sqrt{6} & 1 & 0 \\ 0 & 0 & 2\sqrt{2} & \sqrt{6} & -2 & \sqrt{6} & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 0 & 1 & \sqrt{6} & -3 & 2\sqrt{2} \\ 0 & -4 & 0 & 0 & 2\sqrt{3} & 2\sqrt{2} & 0 \end{bmatrix}.$$

The matrix M is orthogonal and has the Strong Spectral Property; see pages 10 and 11 in [7]. Thus $q(R_{7,1}) = 2$.

$R_{7,2}$: Note that $R_{7,2}$ can be constructed from $K_{4,3}$ by adding edges $\{1, 2\}$ and $\{3, 4\}$. Suppose $M \in \mathcal{S}(R_{7,2})$. We write M as

$$M = \begin{bmatrix} C & B \\ B^T & D \end{bmatrix}$$

where C is a 4×4 matrix, $B = [b_1 \ b_2 \ b_3]$ has no zero entries, and $D = \text{diag}(d_1, d_2, d_3)$. Moreover,

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

where each $C_i \in \mathcal{S}(K_2)$.

Assume M is an orthogonal matrix. Using Lemma 1.3, the columns of B are pairwise orthogonal and $Cb_i = -d_i b_i$ for $i = 1, 2, 3$. Partition each b_i into vectors $x_i, y_i \in \mathbb{R}^2$. Now

$$\begin{bmatrix} C_1 x_i \\ C_2 y_i \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = Cb_i = \begin{bmatrix} -d_i x_i \\ -d_i y_i \end{bmatrix},$$

so each x_i is a $(-d_i)$ -eigenvector for C_1 , and each y_i is a $(-d_i)$ -eigenvector for C_2 .

The matrices C_1 and C_2 are each 2×2 non-scalar symmetric matrices, so each has two distinct eigenvalues. Thus, d_1, d_2, d_3 cannot be all distinct. Without loss of generality, suppose $d_2 = d_3$. And since the dimension of the $(-d_2)$ -eigenspace of C_i is 1, the $(-d_2)$ -eigenvectors of C_i are scalar multiples of each other. That is, there exist $\alpha, \beta \neq 0$ so that $x_3 = \alpha x_2$ and $y_3 = \beta y_2$.

Now we consider the $(2, 2)$ -block of M^2 . We have

$$\begin{aligned} I_3 - D^2 &= B^T B \\ &= \begin{bmatrix} x_1^T & y_1^T \\ x_2^T & y_2^T \\ \alpha x_2^T & \beta y_2^T \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \alpha x_2 \\ y_1 & y_2 & \beta y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1^T x_1 + y_1^T y_1 & 0 & 0 \\ 0 & x_2^T x_2 + y_2^T y_2 & 0 \\ 0 & 0 & \alpha^2 x_2^T x_2 + \beta^2 y_2^T y_2 \end{bmatrix}. \end{aligned}$$

From

$$x_2^T x_2 + y_2^T y_2 = 1 - d_2^2 = \alpha^2 x_2^T x_2 + \beta^2 y_2^T y_2$$

we conclude

$$(\alpha^2 - 1)x_2^T x_2 + (\beta^2 - 1)y_2^T y_2 = 0. \quad (3.0.1)$$

Since the second and third columns of B are orthogonal we also have

$$\alpha x_2^T x_2 + \beta y_2^T y_2 = 0,$$

thus

$$x_2^T x_2 = -\frac{\beta}{\alpha} y_2^T y_2. \quad (3.0.2)$$

Substituting (3.0.2) into (3.0.1) we obtain

$$\begin{aligned} (\alpha^2 - 1) \left(-\frac{\beta}{\alpha} y_2^T y_2 \right) + (\beta^2 - 1) y_2^T y_2 &= 0 \Rightarrow -\alpha^2 \beta + \beta + \alpha \beta^2 - \alpha = 0 \\ &\Rightarrow (\beta - \alpha)(\alpha \beta + 1) = 0 \\ &\Rightarrow \beta = \alpha \quad \text{or} \quad \alpha \beta = -1. \end{aligned}$$

Since the columns of B are orthogonal, $\alpha \neq \beta$. We show that $\alpha \beta = -1$ leads to a contradiction which in turn implies that no orthogonal $M \in \mathcal{S}(R_{7,2})$ exists. Hence, $q(R_{7,2}) > 2$. Suppose $\alpha \beta = -1$; then

$$\begin{aligned} I_4 - \begin{bmatrix} C_1^2 & 0 \\ 0 & C_2^2 \end{bmatrix} &= I_4 - C^2 = BB^T = \begin{bmatrix} x_1 & x_2 & \alpha x_2 \\ y_1 & y_2 & \beta y_2 \end{bmatrix} \begin{bmatrix} x_1^T & y_1^T \\ x_2^T & y_2^T \\ \alpha x_2^T & \beta y_2^T \end{bmatrix} \\ &= \begin{bmatrix} x_1 x_1^T + (1 + \alpha^2) x_2 x_2^T & x_1 y_1^T + (1 + \alpha \beta) x_2 y_2^T \\ y_1 x_1^T + (1 + \alpha \beta) y_2 x_2^T & y_1 y_1^T + (1 + \beta^2) y_2 y_2^T \end{bmatrix} \\ &= \begin{bmatrix} x_1 x_1^T + (1 + \alpha^2) x_2 x_2^T & x_1 y_1^T \\ y_1 x_1^T & y_1 y_1^T + (1 + \beta^2) y_2 y_2^T \end{bmatrix}. \end{aligned}$$

So $x_1 y_1^T = 0$. This implies that either $x_1 = 0$ or $y_1 = 0$ which is a contradiction since the entries of B are nonzero.

To show $q(R_{7,2}) = 3$, we see that $R_{7,2}$ results from joined duplication of a vertex of G_{189} in [3] or [8] (i.e., the graph obtained from $R_{7,2}$ by contracting edge $\{3, 4\}$ is isomorphic to G_{189}). From Table 3 in [8] we find that $q(G_{189}) = 3$. Thus $q(R_{7,2}) = 3$.

$R_{8,1}$: The graph $R_{8,1}$ is a closed candle, $R_{8,1} \cong H_4$. Thus $q(R_{8,1}) = 2$ by Lemma 2.3.

$R_{8,2}$: The following matrix $M_{8,2}$ is a matrix in $\mathcal{S}(R_{8,2})$,

$$M_{8,2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & \beta & 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & 0 & 1 & 1 & 0 & \alpha \\ 0 & -1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & \alpha & 0 & -1 & -1 & 0 & \beta \\ -1 & 1 & 0 & \alpha & 0 & 0 & \beta & 0 \end{bmatrix},$$

where $\alpha = (\sqrt{5} + 1)/2$ and $\beta = (\sqrt{5} - 1)/2$. Since $M_{8,2}$ is orthogonal, $q(R_{8,2}) = 2$.

$R_{8,3}$: The following matrix $M_{8,3}$ is a matrix in $\mathcal{S}(R_{8,3})$,

$$M_{8,3} = \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & \sqrt{2} & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & \sqrt{2} & 0 & -2 & -2 & 0 & 0 \\ 0 & \sqrt{2} & -2 & \sqrt{2} & 0 & -\sqrt{2} & -\sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & -2 & 2 \\ \sqrt{2} & -2 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & -2 & -\sqrt{2} & 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & -\sqrt{2} & -2 & -2 & 1 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 2 & -1 & 2 & 0 & 1 \end{bmatrix}.$$

The matrix $M_{8,3}$ is orthogonal, so $q(R_{8,3}) = 2$.

$R_{8,4}$: Note that in our labeling of $R_{8,4}$, $N_1(7) = \{1, 3, 6, 8\}$ and $N_1(8) = \{1, 3, 6, 7\}$. Let G be the graph obtained from $R_{8,4}$ by contracting the edge $\{7, 8\}$ to the vertex (78) (and replacing every pair of multiple edges by a single edge). The following matrix M is a matrix in $\mathcal{S}(G)$,

$$M = \frac{1}{3} \begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{3} & 1 & -\sqrt{3} & 1 & 1 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & -\sqrt{3} \\ 0 & 1 & \sqrt{3} & 1 & 1 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 & 1 & 1 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} & -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

(here the vertices are ordered as $\{1, 2, 3, 4, 5, 6, (78)\}$). The matrix M is orthogonal, so $q(G) = 2$. Since $R_{8,4}$ is obtained from G by joined duplication of (78) , $q(R_{8,4}) = 2$.

$R_{8,5}$: Note that $R_{8,5} \cong K_4 \square K_2$. Thus by Corollary 6.8 in [2], we have $q(R_{8,5}) = 2$.

$R_{8,6}$: The following matrix $M_{8,6}$ is a matrix in $\mathcal{S}(R_{8,6})$,

$$M_{8,6} = \frac{1}{\sqrt{10}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} & -1 & 0 & 0 & 0 & -1 & -2 \\ \sqrt{2} & \sqrt{2} & -2 & 1 & 0 & 0 & 0 & 1 \\ -1 & -2 & -\sqrt{2} & \sqrt{2} & 1 & 0 & 0 & 0 \\ 0 & 1 & \sqrt{2} & \sqrt{2} & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -\sqrt{2} & \sqrt{2} & -1 & 0 \\ 0 & 0 & 0 & -1 & \sqrt{2} & \sqrt{2} & -2 & 1 \\ -1 & 0 & 0 & 0 & -1 & -2 & -\sqrt{2} & \sqrt{2} \\ -2 & 1 & 0 & 0 & 0 & 1 & \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

The matrix $M_{8,6}$ is orthogonal, so $q(R_{8,6}) = 2$.

$R_{9,1}$: Consider a matrix $M \in \mathcal{S}(R_{9,1})$ where we use variables for each edge and vertex. That is,

$$[M]_{ij} = \begin{cases} x_{ij} & \text{if } ij \in E(R_{9,1}), \\ x_{ii} & \text{if } i = j, \text{ and} \\ 0 & \text{if } ij \notin E(R_{9,1}). \end{cases}$$

Suppose M is an orthogonal matrix. Note that the edges of $R_{9,1}$ can be partitioned into the 9-cycle $(1, 2, 3, 4, 5, 6, 7, 8, 9, 1)$, and the 3-cycles $(1, 4, 7, 1)$, $(2, 5, 8, 2)$, and $(3, 6, 9, 3)$. Since the edge $\{1, 2\}$ does not lie in any triangle in $R_{9,1}$, we have

$$0 = [M^2]_{12} = x_{11}x_{12} + x_{12}x_{22}.$$

Since $x_{12} \neq 0$, we conclude that $x_{22} = -x_{11}$. Repeating this argument for each edge of the 9-cycle, we see that $x_{ii} = -x_{ii}$, or $x_{ii} = 0$ for all $1 \leq i \leq 9$.

Now consider the 3-cycle $(1, 4, 7, 1)$. Taking account of the walks of length 2 between 1 and 7, we have

$$0 = [M^2]_{17} = x_{11}x_{17} + x_{17}x_{77} + x_{14}x_{47} = x_{14}x_{47}.$$

But since $x_{14}, x_{47} \neq 0$, this is impossible. Thus there is no orthogonal matrix $M \in \mathcal{S}(R_{9,1})$, and we conclude $q(R_{9,1}) > 2$.

$R_{9,2}$: Consider a matrix $M \in \mathcal{S}(R_{9,2})$ where we use variables for each edge and vertex. That is,

$$[M]_{ij} = \begin{cases} x_{ij} & \text{if } ij \in E(R_{9,2}), \\ x_{ii} & \text{if } i = j, \text{ and} \\ 0 & \text{if } ij \notin E(R_{9,2}). \end{cases}$$

Suppose M is an orthogonal matrix. Note that $\{6, 7\} \in E(R_{9,2})$, but $\{6, 7\}$ is not included in any 3-cycle in $R_{9,2}$. Thus

$$0 = [M^2]_{67} = x_{66}x_{67} + x_{67}x_{77}.$$

Since the variable $x_{67} \neq 0$, this implies that $x_{66} = -x_{77}$. Similarly, we see that edges $\{2, 6\}$ and $\{3, 7\}$ are not included in any 3-cycles in $R_{9,2}$. Considering $[M^2]_{26}$ and $[M^2]_{37}$ we derive $x_{22} = -x_{66}$ and $x_{33} = -x_{77}$. Combining these three equations, we have $x_{22} = -x_{33}$. Now consider the paths of length 2 between vertices 2 and 3. We have

$$0 = [M^2]_{23} = x_{22}x_{23} + x_{23}x_{33} + x_{12}x_{13} = x_{12}x_{13}.$$

But since $x_{12}, x_{13} \neq 0$, this is impossible. Thus there is no orthogonal matrix $M \in \mathcal{S}(R_{9,2})$, and we conclude $q(R_{9,2}) > 2$.

$R_{9,3}$: Note that $\{1, 2, 3\}$, $\{4, 5, 9\}$, and $\{6, 7, 8\}$ all induce K_3 subgraphs in $R_{9,3}$. Moreover, $\{1, 8, 9\}$, $\{2, 5, 6\}$, and $\{3, 4, 7\}$ also induce K_3 subgraphs in $R_{9,3}$, and we see that $R_{9,3} \cong K_3 \square K_3$. We prove in Lemma 3.3 that $q(K_m \square K_n) = 3$ for all $m, n \geq 3$. This establishes $q(R_{9,3}) = 3$.

$R_{10,1}$: The graph $R_{10,1}$ is a closed candle, $R_{10,1} \cong H_5$. Thus $q(R_{10,1}) = 2$ by Lemma 2.3. $\square$

In order to complete the preceding proof, we establish the following lemma, which shows that the bound in Proposition 3.1 of [8] is sharp for complete graphs.

Lemma 3.3. *For $m, n \geq 3$, we have $q(K_m \square K_n) = 3$.*

Proof. Since $q(K_s) = 2$ for any $s \geq 2$ it follows from a basic application of Kronecker products that $q(K_m \square K_n) \leq 3$ (this inequality can also be deduced from Proposition 3.1 in [8]). It remains to verify that in fact $q(K_m \square K_n) \geq 3$.

If $q(K_m \square K_n) = 2$, then there exists a matrix $C \in \mathcal{S}(K_m \square K_n)$ that satisfies $C^2 = I$ and is given by

$$C = \begin{bmatrix} A_{11} & D_{12} & D_{13} & \cdots & D_{1m} \\ D_{12} & A_{22} & D_{23} & \cdots & D_{2m} \\ & & \ddots & & \\ D_{1m} & D_{2m} & D_{3m} & \cdots & A_{mm} \end{bmatrix},$$

where D_{ij} is an $n \times n$ diagonal matrix with nonzero diagonal entries for each $1 \leq i, j \leq m$ and $A_{ii} \in \mathcal{S}(K_n)$ for $1 \leq i \leq m$.

Let $[C^2]_{ij}$ denote the (i, j) block of the matrix C^2 partitioned conformally with C above. Then

$$\begin{aligned} [C^2]_{12} &= A_{11}D_{12} + D_{12}A_{22} + \sum_{j=3}^m D_{1j}D_{2j} = 0, \\ [C^2]_{13} &= A_{11}D_{13} + D_{13}A_{33} + \sum_{j \neq 1,3} D_{1j}D_{3j} = 0, \\ [C^2]_{23} &= A_{22}D_{23} + D_{23}A_{33} + \sum_{j \neq 2,3} D_{2j}D_{3j} = 0. \end{aligned}$$

Since D_{ij} is invertible, we have

$$A_{22} = -D_{12}^{-1}A_{11}D_{12} - D_{12}^{-1} \left(\sum_{j=3}^m D_{1j}D_{2j} \right), \quad (3.0.3)$$

$$A_{33} = -D_{13}^{-1}A_{11}D_{13} - D_{13}^{-1} \left(\sum_{j \neq 1,3} D_{1j}D_{3j} \right), \quad (3.0.4)$$

$$A_{22} = -D_{23}A_{33}D_{23}^{-1} - \left(\sum_{j \neq 2,3} D_{2j}D_{3j} \right) D_{23}^{-1}. \quad (3.0.5)$$

From (3.0.4) and (3.0.5) we have

$$A_{22} = D_{23}D_{13}^{-1}A_{11}D_{13}D_{23}^{-1} + D_{23}D_{13}^{-1} \left(\sum_{j \neq 1,3} D_{1j}D_{3j} \right) D_{23}^{-1} - \left(\sum_{j \neq 2,3} D_{2j}D_{3j} \right) D_{23}^{-1}. \quad (3.0.6)$$

Note that $[D_{12}]_{ij} = c_{i(n+j)}$, $[D_{13}]_{ij} = c_{i(2n+j)}$, and $[D_{23}]_{ij} = c_{(n+i)(2n+j)}$, and, by assumption, each such entry is nonzero when $i = j$. By a direct calculation we have

$$[D_{23}D_{13}^{-1}A_{11}D_{13}D_{23}^{-1}]_{ij} = \frac{c_{ij}c_{(n+i)(2n+i)}c_{j(2n+j)}}{c_{i(2n+i)}c_{(n+j)(2n+j)}}$$

and

$$[D_{12}^{-1}A_{11}D_{12}]_{ij} = \frac{c_{ij}c_{j(n+j)}}{c_{i(n+i)}}.$$

Using (3.0.3) and (3.0.6), and letting $(i, j) = (1, 2), (1, 3)$, and $(2, 3)$ (there are no contributions from the diagonal terms in (3.0.3) nor (3.0.6)) we have

$$\frac{c_{12}c_{(n+1)(2n+1)}c_{2(2n+2)}}{c_{1(2n+1)}c_{(n+2)(2n+2)}} = -\frac{c_{12}c_{2(n+2)}}{c_{1(n+1)}} \quad (3.0.7)$$

$$\frac{c_{13}c_{(n+1)(2n+1)}c_{3(2n+3)}}{c_{1(2n+1)}c_{(n+3)(2n+3)}} = -\frac{c_{13}c_{3(n+3)}}{c_{1(n+1)}} \quad (3.0.8)$$

$$\frac{c_{23}c_{(n+2)(2n+2)}c_{3(2n+3)}}{c_{2(2n+2)}c_{(n+3)(2n+3)}} = -\frac{c_{23}c_{3(n+3)}}{c_{2(n+2)}}. \quad (3.0.9)$$

Manipulating equations (3.0.7) and (3.0.8) produces the equation

$$\frac{c_{2(2n+2)}c_{(n+3)(2n+3)}}{c_{(n+2)(2n+2)}c_{3(2n+3)}} = \frac{c_{2(n+2)}}{c_{3(n+3)}}$$

or

$$c_{3(n+3)}c_{2(2n+2)}c_{(n+3)(2n+3)} = c_{(n+2)(2n+2)}c_{3(2n+3)}c_{2(n+2)}.$$

However from (3.0.9) we have

$$c_{(n+2)(2n+2)}c_{3(2n+3)}c_{2(n+2)} = -c_{2(2n+2)}c_{(n+3)(2n+3)}c_{3(n+3)},$$

which is a contradiction. $\square$

Table 2
Sporadic 4-Regular Graphs with diameter at least 3.

Graph	q -value	Graph	q -value
$R_{10,2}$	2	$R_{12,2}$	> 2
$R_{10,3}$	2	$R_{12,3}$	2
$R_{10,4}$	2	$R_{14,1}$	2
$R_{12,1}$	> 2		

This completes the verification of the q -values listed in Table 1, which completes the proof of Theorem 2.4 for graphs with diameter at most 2.

4. Sporadic 4-regular graphs and Theorem 2.4

In this section we establish whether or not $q(G) = 2$ for a collection of sporadic graphs. These graphs are all connected with diameter at least 3 and arise within the proofs in Section 5.

Lemma 4.1. *Table 2 lists candidate 4-regular graphs with diameter at least 3 (also shown in Fig. 4) and includes their q -values (or a bound on their q -value).*

Proof. $R_{10,2}$: To see that $q(R_{10,2}) = 2$, note that the following matrix $M_{10,2} \in \mathcal{S}(R_{10,2})$ is orthogonal,

$$M_{10,2} = \frac{1}{4} \begin{bmatrix} -\sqrt{2} & -2 & 0 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 & -2 \\ -2 & \sqrt{2} & 2 & 0 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 0 & 2 & -\sqrt{2} & -2 & 0 & 0 & \sqrt{2} & 0 & 0 & -2 \\ 0 & 0 & -2 & \sqrt{2} & 2 & 0 & 0 & -\sqrt{2} & -2 & 0 \\ \sqrt{2} & 0 & 0 & 2 & -\sqrt{2} & 2 & 0 & 0 & 0 & -2 \\ 0 & \sqrt{2} & 0 & 0 & 2 & \sqrt{2} & -2 & 0 & 2 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -2 & -\sqrt{2} & -2 & 0 & -2 \\ 2 & 0 & 0 & -\sqrt{2} & 0 & 0 & -2 & \sqrt{2} & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & 0 & 0 \end{bmatrix}.$$

$R_{10,3}$: Note that contracting edges $\{1, 2\}$ and $\{6, 7\}$ in $R_{10,3}$ gives a graph isomorphic to Q_3 . Since $R_{10,3}$ can be obtained from Q_3 by joined duplicating a pair of antipodal vertices, Corollary 3.3 from [3] implies that $q(R_{10,3}) \leq q(Q_3) = 2$. Thus $q(R_{10,3}) = 2$.

$R_{10,4}$: Theorem 5.3 in [4] shows that $K_{n,n}$ with a perfect matching deleted has q -value 2 for all $n \neq 1, 3$. Since $R_{10,4}$ is isomorphic to $K_{5,5}$ with a perfect matching deleted, $q(R_{10,4}) = 2$.

$R_{12,1}$: Following the example after Lemma 1.3, set $V_1 = \{1, 3, 5, 7, 10, 12\}$ and $V_2 = \{2, 4, 6, 8, 9, 11\}$. Assume that there exists an $M \in \mathcal{S}(R_{12,1})$ with $M^2 = I$ and M is of the form

$$M = \begin{bmatrix} C & B \\ B^T & D \end{bmatrix},$$

where C and D are diagonal matrices. Then by Lemma 1.3 we know that B must be an orthogonal matrix. Consider the last two rows of B (whose nonzero patterns are the same since vertices 10 and 12 have the same neighbors) and suppose they are equal to $u = [a, b, c, d, 0, 0]$ and $v = [x, y, z, w, 0, 0]$. Since both u and v are orthogonal to row 4 of B we may deduce that $[z, w] = \alpha[c, d]$ for some nonzero scalar α . Similarly since both u and v are orthogonal to row 2 and to row 3 we conclude that $[y, z] = \beta[b, c]$ and $[x, y] = \gamma[a, b]$, where β and γ are nonzero. Now it follows that $\alpha = \beta = \gamma$ and hence row 5 is a multiple of row 6 which contradicts the assumption that B is orthogonal. Hence $q(R_{12,1}) > 2$.

$R_{12,2}$: Using a similar setup as in the case of $R_{12,1}$ set $V_1 = \{2, 3, 4, 5, 11, 12\}$ and $V_2 = \{1, 6, 7, 8, 9, 10\}$. Assume that there exists an $M \in \mathcal{S}(R_{12,2})$ with $M^2 = I$ and M is of the form

$$M = \begin{bmatrix} C & B \\ B^T & D \end{bmatrix},$$

where C and D are diagonal matrices. Then by Lemma 1.3 we know that B must be an orthogonal matrix. Consider rows 2 and 3 of B (whose nonzero patterns are the same since vertices 3 and 4 have the same neighbors). Following a similar argument as used for the graph $R_{12,1}$ (both rows 2 and 3 must be orthogonal to rows 1, 5, and 6) we can deduce that rows 2 and 3 must be multiples of one another which is a contradiction. Hence $q(R_{12,2}) > 2$.

$R_{12,3}$: To see that $q(R_{12,3}) = 2$, we present an orthogonal matrix $M_{12,3} \in \mathcal{S}(R_{12,3})$ in block form, as above. We take $V_1 = \{1, 2, 3, 4, 5, 6\}$ and $V_2 = \{7, 8, 9, 10, 11, 12\}$. Let

$$B_{12,3} = \begin{bmatrix} -\zeta & -\beta & \zeta & -\sqrt{5}/5 & 0 & 0 \\ \varepsilon & -\gamma & \beta & 0 & \delta & 0 \\ \sqrt{5}/5 & 0 & 0 & -\zeta & -\beta & -\zeta \\ \sqrt{5}/5 & 0 & \sqrt{5}/5 & 0 & -\varepsilon & \zeta \\ 0 & \delta & 0 & -\varepsilon & \gamma & \beta \\ 0 & -\varepsilon & -\zeta & -\sqrt{5}/5 & 0 & \sqrt{5}/5 \end{bmatrix},$$

where

$$\alpha = (\sqrt{5} + 1)/2, \quad \beta = \sqrt{2\sqrt{5}/5 - 4/5}, \quad \gamma = \sqrt{-7\sqrt{5}/10 + 17/10}, \quad \delta = \sqrt{5}/10 + 1/2,$$

$$\varepsilon = \alpha\beta, \quad \text{and} \quad \zeta = \alpha\gamma.$$

Thus

$$M_{12,3} = \begin{bmatrix} 0 & B_{12,3} \\ B_{12,3}^T & 0 \end{bmatrix}$$

has the desired properties.

$R_{14,1}$: To see that $q(R_{14,1}) = 2$, we provide an orthogonal matrix $M_{14,1} \in \mathcal{S}(R_{14,1})$ in block form. We take $V_1 = \{1, 2, 3, 4, 5, 6, 7\}$ and $V_2 = \{8, 9, 10, 11, 12, 13, 14\}$, and let

$$B_{14,1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}.$$

Then

$$M_{14,1} = \begin{bmatrix} 0 & B_{14,1} \\ B_{14,1}^T & 0 \end{bmatrix}$$

has the desired properties. We note that $M_{14,1}$ was presented in [19]. $\square$

5. Proof of Theorem 2.4 for 4-regular graphs with large diameter

In this section we complete the proof of items (3), (4) and (5) of Theorem 2.4 for graphs with diameter at least 3. Recall from Section 1.1, in the distance partition from any vertex v , every $u \in N_i(v)$ has at least two neighbors in $N_{i-1}(v)$ for all $i \geq 2$. These edges account for $2n - 2 - \deg(v) = 2n - 6$ of the edges of G . We consider the possible locations of the six extra edges not accounted for by these predecessors. Throughout the proofs in this section, once all of the edges incident with a vertex have been accounted for, we say that the vertex is *full*.

Lemma 5.1. *Let G be a connected 4-regular graph with diameter at least 3 and let v be a vertex for which $\epsilon(v) \geq 3$. Suppose that every vertex in $N_2(v)$ and $N_3(v)$ has exactly two predecessors and that $G[N_1(v)] \cup G[N_2(v)]$ contains at most three edges. Then $q(G) > 2$.*

Proof. Assume $q(G) = 2$.

Case 1: $N_1(v)$ is an independent set.

Each pair of vertices of $N_1(v)$ shares v as a common neighbor, so each pair must share a common neighbor in $N_2(v)$ as well. There are $\binom{4}{2} = 6$ such pairings, each resulting in a distinct vertex in $N_2(v)$ with two predecessors in $N_1(v)$ and accounting for all the edges between $N_1(v)$ and $N_2(v)$, also implying that $|N_2(v)| = 6$.

Suppose each vertex in $N_2(v)$ has a neighbor in $N_2(v)$. Then there are three edges in $G[N_2(v)]$ and $G[N_2(v)] = 3K_2$. Let xy be an edge in $N_2(v)$ and let a, b be the predecessors of x . Then a, b are not both predecessors of y . Suppose that b and y are not adjacent. Then bxy is a unique shortest path of length 2 between b and y .

Otherwise some vertex, say $u \in N_2(v)$, has no neighbor in $N_2(v)$. Vertex u shares a common neighbor in $N_1(v)$ with four other vertices in $N_2(v)$, call them x_1, x_2, x_3, x_4 . To avoid unique paths from u to x_i for $i = 1, 2, 3, 4$, u and x_i must have a common successor. These common successors are distinct since each has just 2 predecessors. But then $\deg(u) \geq 6$.

Case 2: $G[N_1(v)]$ contains exactly one edge.

This case follows the same logic as Case 1, with the exception that there are five pairings of nonadjacent vertices in $N_1(v)$ that share v as a common neighbor so each of these pairs must have a common neighbor in $N_2(v)$. As before, this accounts for all the edges between $N_1(v)$ and $N_2(v)$, so there are exactly five vertices in $N_2(v)$. Since $G[N_2(v)]$ contains at most two edges, some vertex u in $N_2(v)$ has no neighbors in $N_2(v)$. Vertex u shares a neighbor in $N_1(v)$ with at least three other vertices in $N_2(v)$. Arguing similarly to Case 1, $\deg(u) \geq 5$.

Case 3: $G[N_1(v)]$ contains exactly two edges.

Suppose the two edges do not share a vertex. Then the four pairs of nonadjacent vertices in $N_1(v)$ that have v as a common neighbor must each have a common neighbor in $N_2(v)$, accounting for all the edges between $N_1(v)$ and $N_2(v)$ and resulting in exactly four vertices in $N_2(v)$. Since there is at most one edge in $N_2(v)$, there is a vertex u in $N_2(v)$ that has no neighbor in $N_2(v)$. Let a be a neighbor of u in $N_1(v)$ with ab one of the two edges in $G[N_1(v)]$. Then uab is a unique shortest path of length 2 in G .

Otherwise the two edges in $G[N_1(v)]$ share a vertex, say c . Then c has exactly one neighbor w in $N_2(v)$. Let the non-neighbor of c in $N_1(v)$ be d . Since c and d have v as a common neighbor, the other predecessor of w in $N_1(v)$ must be d . To prevent a unique shortest path of length 2 between w and the neighbors of c in $N_1(v)$, there must be an edge connecting w to a vertex, $y \in N_2(v)$ whose two predecessors are the two neighbors of c in $N_1(v)$. But this creates a unique shortest path dwy of length 2. Note that in this case we did not use the hypothesis that each vertex in $N_3(v)$ has two predecessors. This will be used later in the proof of Lemma 5.3.

Case 4: $G[N_1(v)]$ contains exactly three edges.

If $G[N_1(v)] \cong P_4$, then the two endpoints of the P_4 must have a common neighbor w in $N_2(v)$. Then w has a unique shortest path of length 2 to both non-end vertices of P_4 .

If $G[N_1(v)] \cong C_3 \cup K_1$, the same argument as in Case 3 applies, with c being any vertex in the C_3 . Note again, that in these first two possibilities that the hypothesis that each vertex in $N_3(v)$ has two predecessors was not used. This will be used at the end of the proof of Lemma 5.3.

Suppose $G[N_1(v)] \cong K_{1,3}$. Then each pair of nonadjacent vertices in $N_1(v)$ has a common neighbor in $N_1(v)$, in addition to the common neighbor v . But in order for each vertex in $N_1(v)$ to have degree 4, there must be three vertices in $N_2(v)$, each adjacent to a different pair of leaves of the $K_{1,3}$. Each pair of vertices in $N_2(v)$ has one common neighbor in $N_1(v)$, and since $N_2(v)$ is an independent set, this implies that each pair of vertices in $N_2(v)$ must have a common neighbor in $N_3(v)$. These neighbors are distinct because each vertex in $N_3(v)$ has two predecessors. Note that all three vertices in $N_2(v)$ are full. Let a be a leaf vertex of the $K_{1,3}$; then there exist two paths of length 2, asx and aty for some $s, t \in N_2(v)$ and $x, y \in N_3(v)$, that are the unique ax and ay paths of length 2. $\square$

Corollary 5.2. *Let G be a connected 4-regular graph with diameter at least 3. If v is a vertex with $\epsilon(v) \geq 3$ and for which there is no vertex in the distance partition from v with 3 or 4 predecessors, then $q(G) > 2$.*

Proof. Let $N_d(v)$ be the furthest distance set of v . Since every vertex in $N_d(v)$ has exactly two predecessors, we know that $G[N_d(v)]$ is 2-regular. So at least three of the six extra edges must be in $N_d(v)$, and the maximum number of edges that could appear within the subgraphs $G[N_i(v)]$ for $1 \leq i \leq d-1$ is three. So the hypotheses of Lemma 5.1 are satisfied and $q(G) > 2$. $\square$

Lemma 5.3. *Let G be a connected 4-regular graph with diameter at least 3 and let v be a vertex for which $\epsilon(v) \geq 3$. If in the distance partition from v no vertex has four predecessors, then either $G \cong R_{10,3}$, $G \cong K_3 \square C_4$, or $q(G) > 2$.*

Proof. Suppose $q(G) = 2$. We assume G has a vertex with three predecessors as otherwise the result follows from Corollary 5.2. We begin by establishing three claims.

Claim #1. *Any vertex with three predecessors cannot have a successor.*

Proof of Claim #1. Suppose some vertex w in $N_i(v)$ has three predecessors in $N_{i-1}(v)$ and one successor z in $N_{i+1}(v)$. Note that i must be at least 2. Since z has at most three predecessors in $N_i(v)$, there must be a neighbor x of z in $N_{i+1}(v)$ or $N_{i+2}(v)$. But then there is a unique shortest path of length 2 between w and x , as w is full.

Observe that the proof of Claim #1 also implies that if G is a graph and $v \in V(G)$ for which some vertex in the distance partition from v has four predecessors, any vertex $w \in N_i(v)$ with three predecessors cannot have a successor z unless z is a vertex with four predecessors. We will use this in Case 4 of the proof of Lemma 5.4.

Claim #2. *Vertices with three predecessors occur in pairs, with each pair the endpoints of a path in some distance set $N_i(v)$.*

Proof of Claim #2. Suppose $w_1 \in N_i(v)$ has three predecessors. Since it can have no successors but has degree 4, it must have a neighbor $w_2 \in N_i(v)$. So w_1 is one end of a path $w_1 w_2 \dots w_k$ in $N_i(v)$, where $w_2, \dots, w_{k-1}$ have two predecessors each, and hence is full, and w_{k-1} is the only neighbor of w_k in $N_i(v)$. If w_k has a successor y , then $y w_k w_{k-1}$ is a unique shortest path of length 2 between y and w_{k-1} . So w_k must have three predecessors.

Claim #3. *A pair of vertices with three predecessors can occur only in $N_i(v)$ for $i \geq 3$.*

Proof of Claim #3. Let w be a vertex with three predecessors. Suppose $w \in N_2(v)$. Note that w cannot have a successor. Let z be another vertex in $N_2(v)$ with three predecessors, and no successor. Then there are at least two vertices in $N_1(v)$ that are common predecessors of w and z . Since $\epsilon(v) \geq 3$, there must be a vertex $x \in N_2(v)$ that has a successor $y \in N_3(v)$. So x cannot have three predecessors and therefore must have two. If a predecessor of x is a common predecessor of w and z , then there is a unique shortest path of length 2 between y and this predecessor of x . So w and z must have exactly two shared predecessors, and the two predecessors of x each has exactly one of $\{w, z\}$ as a successor. This implies that, via $N_1(v)$, there is a path of length 2 between x and w (and between x and z). Since these paths cannot be unique shortest paths, and x cannot be adjacent to w (as it would then be on the wz path in $N_2(v)$, and hence have degree at least 5), x must have a common neighbor $t \neq z$ in $N_2(v)$ with w . But then yxt is a unique shortest path of length 2 between y and t .

Let S represent the set of vertices with three predecessors. We have shown that no vertex in S can have a successor and all vertices in S must occur in pairs in the distance sets $N_i(v)$ for $i \geq 3$. A pair of such vertices requires a minimum of three of the six additional edges. It follows that $|S| = 2$ or $|S| = 4$, leaving at most three or zero edges that could appear in the subgraphs $G[N_i(v)]$ for $1 \leq i \leq \epsilon(v) - 1$, respectively. Let j represent the smallest value of i for which $N_i(v)$ contains an element of S . If $j > 3$, then Lemma 5.1 implies that $q(G) > 2$, so we only have to consider $j = 3$.

Case 1: $N_1(v)$ is an independent set.

By the argument in Case 1 of Lemma 5.1, $N_2(v)$ consists of six vertices, such that each vertex shares one common neighbor in $N_1(v)$ with each of four other vertices in $N_2(v)$. Consider $u \in N_2(v)$, and denote by w the unique vertex in $N_2(v)$ with which u does not share a neighbor in $N_1(v)$.

Case 1(a): $N_2(v)$ is also an independent set.

Let y_1, y_2 be the successors of u in $N_3(v)$ and let x_1, x_2, x_3, x_4 be the vertices in $N_2(v)$ that share a unique common predecessor with u . Since there cannot be a unique shortest

path of length 2 between u and x_i for any i , each x_i must be adjacent to y_1 or y_2 . Since neither y_1 nor y_2 has four predecessors, y_1 must be adjacent to two of the x_i and y_2 to the other two. Then $y_1, y_2 \in S$. Neither y_1 nor y_2 is adjacent to w , so repeating the same argument, there are successors $z_1, z_2 \in S$ of w such that z_1 is adjacent to two of the x_i and z_2 to the other two. This accounts for all edges between $N_2(v)$ and $N_3(v)$, so $N_3(v) = \{y_1, y_2, z_1, z_2\} = S$. Note that each x_i is adjacent to exactly one of y_1, y_2 and one of z_1, z_2 . Since none of y_1, y_2, z_1, z_2 has a successor by Claim #1, there are exactly two edges in $G[N_3(v)]$ that do not share an endpoint. If $y_1 y_2$ is not an edge, then $y_1 u y_2$ is the unique shortest path of length 2. So $y_1 y_2$ is an edge, and $z_1 z_2$ is an edge. Without loss of generality, suppose that x_1 is adjacent to y_1 . Then x_1 and y_2 are not adjacent and $x_1 y_1 y_2$ is a unique shortest path of length 2.

Case 1(b): $N_2(v)$ is not independent, so $|S| = 2$.

Suppose there is an edge ut in $G[N_2(v)]$, not sharing an endpoint with any other edge in $G[N_2(v)]$. Then there is a shortest path of length 2 through $N_1(v)$ from u to at least three other vertices in $N_2(v)$. So for each of these three vertices there must be a shortest path of length 2 through $N_3(v)$. Since $\deg(u) = 4$, u can only have one successor, so there is a single vertex $x \in N_3(v)$ on all three paths. But then x has four predecessors.

Now suppose ut and rt are edges in $G[N_2(v)]$. Recall that w is the vertex in $N_2(v)$ that shares no predecessors with u . Let y and z be the remaining vertices in $N_2(v)$. To avoid a unique shortest path of length 2 through $N_1(v)$ from u to y or z , there must be a vertex x in $N_3(v)$ whose predecessors are u, y , and z . Then tux is a unique shortest path of length 2.

Therefore Case 1 cannot occur, and $N_1(v)$ is never an independent set.

Case 2: $G[N_1(v)]$ contains exactly one edge, so $|S| = 2$.

We follow the proof of Case 2 of Lemma 5.1 but make more careful note of the location of the edges in $N_2(v)$. Let a and b represent the adjacent vertices in $N_1(v)$. Let u be the vertex in $N_2(v)$ whose two predecessors are $N_1(v) \setminus \{a, b\}$. Each of the four vertices in $N_2(v) \setminus \{u\}$ has exactly one of a or b as a predecessor. Note that there are four shortest paths of length 2 between these vertices and the other vertex in $\{a, b\}$. Since these paths cannot be unique, there must be two independent edges in $N_2(v)$ that connect vertices with no common predecessor in $\{a, b\}$. Note that if two adjacent vertices in $N_2(v)$ do not share a predecessor in $N_1(v) \setminus \{a, b\}$, then there are unique shortest paths of length 2 between the vertices $N_1(v) \setminus \{a, b\}$ and the neighbors of their successors in $N_2(v)$. So the independent edges between the vertices of $N_2(v) \setminus \{u\}$ connect vertices so that each pair has one common predecessor in $N_1(v) \setminus \{a, b\}$ and has one non-common predecessor in $\{a, b\}$.

Each endpoint of the independent edges in $N_2(v)$ has exactly one successor, and u must have two successors. It follows that there are six edges between the vertices in $N_2(v)$ and the vertices in $N_3(v)$. Since each of the two vertices in $S \cap N_3(v)$ has three

predecessors, there can be no other vertices in $N_3(v)$ and $S = N_3(v)$. From Claim #1, these two vertices must be adjacent to each other. Each of the four vertices in $N_2(v) \setminus \{u\}$ is adjacent to one of the vertices in S . The two predecessors other than u of each vertex in S must be a pair of nonadjacent vertices in $N_2(v) \setminus \{u\}$ to avoid unique paths of length 2 between vertices in $N_2(v) \setminus \{u\}$ and their non-neighbor in $N_3(v)$. Additionally, the two successors of a (respectively, b) must have a common successor in S , otherwise there is a unique shortest path of length 2 between a (respectively, b) and a vertex in S . This implies that the predecessors of a vertex in S are u and two nonadjacent vertices in $N_2(v)$ that share a predecessor in $\{a, b\}$. Thus $G \cong K_3 \square C_4$.

Case 3: $G[N_1(v)]$ contains exactly two edges, so $|S| = 2$.

Then $q(G) > 2$ by Lemma 5.1, Case 3 and the observation at the end of its proof.

Case 4: $G[N_1(v)]$ contains exactly three edges, so $|S| = 2$.

If $G[N_1(v)] \cong P_4$ or $G[N_1(v)] \cong C_3 \cup K_1$, then by Case 4 of Lemma 5.1 and the observations within its proof, there is a pair of nonadjacent vertices with a unique path of length 2 between them. So we assume $G[N_1(v)] \cong K_{1,3}$. As in that proof, $N_2(v)$ consists of three vertices, each being the common successor of two leaves of the $K_{1,3}$. Since $j = 3$, there are two adjacent vertices of S in $N_3(v)$, each with three predecessors. Since $N_2(v)$ is an independent set, each of its three vertices has two successors, so there are six edges between $N_2(v)$ and $N_3(v)$. It can now be verified that $G \cong R_{10,3}$. For instance, let v correspond to vertex 1 in the drawing of $R_{10,3}$ in Fig. 4.

In all of the above cases, we either reach a contradiction to $q(G) = 2$, or G is isomorphic to either $R_{10,3}$ or $K_3 \square C_4$. Lemma 4.1 establishes $q(R_{10,3}) = 2$. Since $K_3 \square C_4 \cong K_3 \square K_2 \square K_2$, Corollary 6.8 in [2] implies $q(K_3 \square C_4) = 2$. This completes the proof. $\square$

Lemma 5.4. *If G is a connected 4-regular graph with $q(G) = 2$ and diameter at least 3, then G is either*

- (1) $K_3 \square C_4$, $K_{3,3} \square K_2$, one of the graphs $R_{10,2}$, $R_{10,3}$, $R_{10,4}$, $R_{12,3}$, or $R_{14,1}$ given in Fig. 4;
- (2) Q_4 ; or,
- (3) a closed candle H_k for some $k \geq 6$.

Proof. Recall that a full vertex is a vertex whose incident edges have all been accounted for.

Let v be a vertex of G with $\epsilon(v) \geq 3$. If no vertex in the distance partition of v has four predecessors, then $G \cong R_{10,3}$ or $G \cong K_3 \square C_4$ by Lemma 5.3. For the remainder of this proof, we assume that the distance partition of v contains at least one vertex with four predecessors. Let $d = \epsilon(v)$ be the index of the farthest distance set. Each vertex

with four predecessors uses two of the six extra edges, so $N_d(v)$ may contain no more than three vertices with four predecessors.

Case 1: Exactly three vertices in $N_d(v)$, say z_1, z_2 , and z_3 , have four predecessors.

All six of the extra edges are used by z_1, z_2 , and z_3 . By Case 1 of Lemma 5.1, $N_2(v)$ contains exactly six vertices, each being the common successor of a different pair of vertices in $N_1(v)$, and $d = 3$. Since $N_2(v)$ has exactly six vertices with two successors each, $N_3(v) = \{z_1, z_2, z_3\}$ and each vertex in $N_2(v)$ has two neighbors in $\{z_1, z_2, z_3\}$.

Let $N_2(v) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, and recall from Case 1 of Lemma 5.1 that each vertex in $N_2(v)$ shares a common predecessor in $N_1(v)$ with exactly four other vertices in $N_2(v)$. This partitions $N_2(v)$ into three pairs, say $\{y_1, y_2\}$, $\{y_3, y_4\}$, and $\{y_5, y_6\}$, where each pair does not share a predecessor in $N_1(v)$. Since each y_i has two successors among $\{z_1, z_2, z_3\}$, each of these pairs must share at least one successor in $N_3(v)$. Suppose y_1 and y_2 share z_1 as a common successor. To avoid $y_1 z_1 y_2$ being a unique path of length 2, y_1 and y_2 must also share their second successor, say z_2 . Similarly, each of $\{y_3, y_4\}$ and $\{y_5, y_6\}$ has two shared successors among $\{z_1, z_2, z_3\}$. Without loss of generality, we may assume that the successors of $\{y_3, y_4\}$ are $\{z_1, z_3\}$ and the successors of $\{y_5, y_6\}$ are $\{z_2, z_3\}$. Thus $G \cong R_{14,1}$, and Lemma 4.1 completes the proof.

Case 2: Exactly two vertices in $N_d(v)$, say z_1 and z_2 , have four predecessors.

In this case there remain two extra edges. We treat the possibilities for these edges in the following four subcases.

Case 2(a): z_1 and z_2 share all their predecessors in $N_{d-1}(v)$.

Denote the common predecessors of z_1 and z_2 by $\{y_1, y_2, y_3, y_4\}$. Suppose $d > 3$. Let $\{x_1, x_2\} \subseteq N_{d-2}(v)$ denote the predecessors of y_1 . If x_1 has three predecessors, then y_1 is its unique successor, and $x_1 y_1 z_1$ is a unique shortest path of length 2 between x_1 and z_1 . So we may let $\{w_1, w_2\} \subseteq N_{d-3}(v)$ denote the predecessors of x_1 . Since y_1 is full and $w_1 x_1 y_1$ is a shortest path of length 2, w_1 must be adjacent to x_2 . Similarly, w_2 is adjacent to x_2 . We repeat this argument, with x_1 replacing y_1 , and then with w_1 replacing x_1 , forming successive induced candle sections from $\{x_1, x_2\}$ until we reach, say, $\{a_1, a_2\} \subseteq N_1(v)$. Since both x_1 and x_2 are just one edge short of being full, there must exist a vertex, say y_4 , whose predecessors are not x_1 or x_2 . Denote the predecessors of y_4 by $\{x_3, x_4\}$. By the same argument, starting with y_4 we see successive induced candle sections from $\{x_3, x_4\}$ until $\{a_3, a_4\} \subseteq N_1(v)$. Recall that $\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$. A vertex in $\{w_1, w_2\} \cap \{w_3, w_4\}$ would have to have four successors, so $\{w_1, w_2\} \cap \{w_3, w_4\} = \emptyset$. Continuing, the two candle sections must be disjoint, including $\{a_1, a_2\} \cap \{a_3, a_4\} = \emptyset$. It follows that $N_1(v) = \{a_1, a_2, a_3, a_4\}$.

Suppose the two predecessors of y_2 are $x_i \in \{x_1, x_2\}$ and $x_j \in \{x_3, x_4\}$. Then x_i, x_j , and y_2 are all full, and $x_i y_2 x_j$ is a unique shortest path of length 2 between x_i and x_j . So we assume without loss of generality that the predecessors of y_2 are $\{x_1, x_2\}$, and of y_3 are $\{x_3, x_4\}$.

Each vertex in $N_1(v)$ is one edge short of being full, and any pair of these vertices shares v as a common neighbor. Suppose a_1 is adjacent to one of $\{a_3, a_4\}$, say a_3 . Then a_1 and a_3 are both full, and a_1va_4 is a unique shortest path of length 2 between a_1 and a_4 . So a_1 , a_3 , and a_4 must share a common successor $u \in N_2(v)$. Similarly, a_2 , a_3 , and a_4 must share a common successor, which must be u since a_3 and a_4 are full. Therefore, u has four predecessors, using up the remaining two extra edges, and all vertices listed so far are full. It follows that G cannot contain any additional vertices, and $G \cong H_k$ for some even $k \geq 6$ (as the number of vertices accounted for in the proof is divisible by 4). Recall that $q(H_k) = 2$ by Lemma 2.3.

Suppose $d = 3$ and again write $N_1(v) = \{a_1, a_2, a_3, a_4\}$. The exact number of edges that connect $\{y_1, y_2, y_3, y_4\}$ to vertices in $N_1(v)$ is eight. Furthermore, since z_1 and z_2 are full, each vertex in $N_1(v)$ must have either zero or at least two successors in $\{y_1, y_2, y_3, y_4\}$, to prevent a unique shortest path of length 2 between $N_1(v)$ and $N_3(v)$. If, say a_1 , has no successor in $\{y_1, y_2, y_3, y_4\}$, then it must be adjacent to each of $\{a_2, a_3, a_4\}$. However, to account for the edges between $N_1(v)$ and $N_2(v)$, some a_i must have three successors. This is impossible as $\deg(a_i) = 4$ for all i . So each a_i has two successors in $\{y_1, y_2, y_3, y_4\}$, accounting for eight edges. These eight edges and $\{y_1, y_2, y_3, y_4\} \cup \{a_1, a_2, a_3, a_4\}$ form $2C_4$ or C_8 . In the first case, it follows as in the previous paragraph that $G \cong H_6$.

Suppose the eight edges connecting the vertices $\{y_1, y_2, y_3, y_4\}$ to $\{a_1, a_2, a_3, a_4\}$ form a C_8 . Then there are two pairs of vertices in $N_1(v)$ that share v as a predecessor but share no successor in $\{y_1, y_2, y_3, y_4\}$. Without loss of generality, assume these pairs are $\{a_1, a_3\}$ and $\{a_2, a_4\}$. Suppose the two remaining extra edges are contained in $N_1(v)$. Then the two non-isomorphic ways to place these edges are $\{a_1a_2, a_3a_4\}$ or $\{a_1a_3, a_2a_4\}$. In the first case, a_1va_3 is a unique shortest path of length 2. If a_1a_3 and a_2a_4 are edges, and if the common successor of a_1 and a_4 is, say y_1 , then $y_1a_1a_3$ is a unique shortest path of length 2 between y_1 and a_3 . Therefore $N_1(v)$ cannot contain two edges and there is at least one (and at most two) additional vertex (vertices) in $N_2(v)$.

In either case, a vertex $y_5 \notin \{y_1, y_2, y_3, y_4\}$ cannot have a successor, since there would be a unique shortest path of length 2 between this successor and the predecessors of y_5 in $N_1(v)$. So y_5 must be a terminal vertex. Because we have only two extra edges remaining, y_5 must have at most one neighbor in $N_2(v)$ and hence must have at least three predecessors. If there were a second terminal vertex $y_6 \in N_2(v)$, then there would be at least six edges from $\{y_5, y_6\}$ to $N_1(v)$, contradicting that all four vertices in $N_1(v)$ are only one edge short of being full. The only possibility then is for $N_2(v)$ to contain the one additional vertex, y_5 , with all four vertices in $N_1(v)$ as its predecessors, using up the two remaining extra edges. Then all vertices are full, and in this case $G \cong R_{12,1}$, which satisfies $q(R_{12,1}) > 2$, by Lemma 4.1, and leads to a contradiction.

Case 2(b): z_1 and z_2 share exactly three of their four predecessors in $N_{d-1}(v)$.

Denote the predecessors of z_1 by $\{y_1, y_2, y_3, y_4\}$ and of z_2 by $\{y_2, y_3, y_4, y_5\}$. Then y_1 shares one common successor with y_2 , y_3 , and y_4 . Given that these three vertices have

two predecessors each, they are full, as is their other successor z_2 . So y_1 cannot be adjacent to any of them, nor can it have a second common successor with any of them. Therefore, y_1 must have a common predecessor with each of y_2, y_3 , and y_4 . In particular, let x_1 be a common predecessor of y_1 and y_2 . Then the path $x_1y_2z_2$ implies x_1 must be adjacent to y_3, y_4 , or y_5 , which implies $x_1 \in N_1(v)$ and $d = 3$. Denote the three remaining vertices in $N_1(v)$ by $\{x_2, x_3, x_4\}$. Moreover, the fullness of z_1, y_2, y_3 , and y_4 implies that any neighbor of y_1 other than z_1 must be its predecessor (to avoid a unique shortest path of length 2 between z_1 and this neighbor of y_1), and hence y_1 has three predecessors in $N_1(v)$. Similarly, y_5 has three predecessors in $N_1(v)$, using up the two remaining extra edges.

We make two observations. First, since there are exactly twelve edges connecting $\{y_1, y_2, y_3, y_4, y_5\}$ to vertices in $N_1(v)$, any successor of x_1, x_2, x_3 , and x_4 must be in $\{y_1, y_2, y_3, y_4, y_5\}$, and

$$V(G) = \{z_1, z_2, y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, x_4, v\}.$$

Second, y_1 and y_5 share at least two predecessors.

Suppose y_1 and y_5 share all three predecessors, say $\{x_1, x_2, x_3\}$. Then the three successors of x_4 must be y_2, y_3 , and y_4 . Finally, the remaining predecessor of y_3 and y_4 must be, without loss of generality, x_2 and x_3 , respectively. Hence all vertices are full, and $G \cong K_{3,3} \square K_2$. Corollaries 6.5 and 6.8 of [2] show that $q(K_{3,3} \square K_2) = 2$.

Now suppose y_1 and y_5 share two predecessors, say $\{x_1, x_2\}$. Without loss of generality, let x_3 and x_4 be the third predecessors of y_1 and y_5 , respectively. Either x_1 and x_2 share their third successor as well or they do not. Suppose x_1 and x_2 share all three successors, $\{y_1, y_2, y_5\}$. Then y_1, y_2 , and y_5 are all full, and the four remaining edges between $N_2(v)$ and $N_1(v)$ are determined. In this case, $G \cong R_{12,2}$, contradicting $q(G) = 2$ by Lemma 4.1. Therefore, we must have that x_1 and x_2 do not share a third common successor. Since the third successor of x_1 is y_2 , let the third successor of x_2 be y_3 . Again, the four remaining edges between $N_2(v)$ and $N_1(v)$ are determined, up to one choice, which yields graphs that are isomorphic. In this case, it can be verified that $G \cong R_{12,3} \cong C(12, \pm 1, \pm 3)$, and Lemma 4.1 implies $q(R_{12,3}) = 2$.

Case 2(c): z_1 and z_2 share exactly two of their four predecessors in $N_{d-1}(v)$.

Suppose the predecessors of z_1 are $\{y_1, y_2, y_3, y_4\}$ and the predecessors of z_2 are $\{y_3, y_4, y_5, y_6\}$. With two predecessors, y_3 is full (as is y_4). To resolve the unique shortest paths of length 2 through z_1 or z_2 , y_3 must share a predecessor with each of y_1, y_2, y_5 , and y_6 . It follows that the predecessors of y_3 have three successors each, and therefore $d = 3$.

Since there are at least twelve edges from $\{y_1, y_2, y_3, y_4, y_5, y_6\}$ to $N_1(v)$, and at most twelve edges from $N_1(v)$ to $N_2(v)$, we must have: $N_2(v) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, each vertex in $N_2(v)$ has exactly two predecessors in $N_1(v)$, and each vertex in $N_1(v)$ has exactly three successors in $N_2(v)$. Note that y_3 and y_4 are full while y_1, y_2, y_5 , and y_6 are each one edge short of full. We label $N_1(v) = \{x_1, x_2, x_3, x_4\}$.

To prevent unique shortest paths of length 2 between x_i and z_j , each x_i must be adjacent to at least one vertex in $\{y_3, y_4\}$. It follows that each x_i must have exactly one neighbor in $\{y_3, y_4\}$. Assume one successor of x_1 is y_3 . To avoid unique paths of length 2 between x_1 and both z_1 and z_2 , x_1 must have at least one more successor in $\{y_1, y_2\}$ and at least one more successor in $\{y_5, y_6\}$. Assume, without loss of generality, that the successors of x_1 are $\{y_1, y_3, y_5\}$. Suppose y_1 has a successor, z_3 , in addition to z_1 . Then z_3 must be adjacent to y_5 (to prevent $x_1y_1z_3$ being a unique shortest path of length 2 between x_1 and z_3) and z_3 must be adjacent to y_6 (to prevent $z_3y_5z_2$ being a unique shortest path of length 2 between z_3 and z_2). This leaves all vertices in $N_2(v)$ except y_2 full, but any successor of y_2 distinct from z_1 , z_2 , and z_3 would create a unique shortest path of length 2 between this successor and a predecessor of y_2 . So the successor of y_2 must be z_3 , contradicting the assumption of Case 2.

To become full, y_1 must have exactly one neighbor in $\{y_2, y_5, y_6\}$. If y_1 is adjacent to y_2 , then $x_1y_1y_2$ is a unique shortest path of length 2 between the full vertices x_1 and y_2 . But if y_1 is adjacent to y_5 or y_6 , then $z_1y_1y_5$ or $z_1y_1y_6$ is a unique shortest path of length 2 between z_1 and y_5 or y_6 , respectively, which would contradict $q(G) = 2$.

Case 2(d): z_1 and z_2 share exactly one, or none, of their four predecessors in $N_{d-1}(v)$. Here $N_{d-1}(v)$ must contain seven or eight vertices, at least six of which, after accounting for two predecessors, are exactly one edge short of being full. If any such $y \in N_{d-1}(v)$ has another successor $z_3 \in N_d(v)$, then z_3 must have a neighbor $z_4 \in N_d(v) \setminus \{z_1, z_2\}$. Since y , z_1 , and z_2 are full, yz_3z_4 is a unique shortest path of length 2 between y and z_4 . In order for the vertices in $N_{d-1}(v)$ to become full, at least six vertices require incidence with one of the two remaining extra edges, which is impossible.

Case 3: Exactly one vertex, say z , in $N_d(v)$ has four predecessors.

Four extra edges remain. Denote the predecessors of z by $\{y_1, y_2, y_3, y_4\} \subseteq N_{d-1}(v)$.

Case 3(a): Each of y_1 , y_2 , y_3 , and y_4 has three predecessors.

All remaining extra edges are used, and $N_d(v) = \{z\}$ and $N_{d-1}(v) = \{y_1, y_2, y_3, y_4\}$. Suppose $d = 3$. Each vertex in $N_1(v)$ has three successors in $N_2(v)$ (to account for the twelve edges connecting $N_2(v)$ to $N_1(v)$), and each y_i has one vertex in $N_1(v)$ as its non-neighbor. If $y_i \neq y_j$ have the same non-neighbor, then that vertex cannot have three successors. So the four vertices in $N_2(v)$ are in bijection with the four vertices in $N_1(v)$, via non-neighbors, determining the graph as $G \cong R_{10,4}$. Lemma 4.1 shows $q(R_{10,4}) = 2$.

Suppose $d > 3$. Each pair of vertices in $\{y_1, y_2, y_3, y_4\}$ shares z as a common successor, and each y_i is full with three predecessors, so each pair must share a common predecessor in $N_{d-2}(v)$. The vertices in $N_{d-2}(v)$ must have two predecessors each and therefore can have at most two successors in $N_{d-1}(v)$. This forces $N_{d-2}(v)$ to contain six vertices $\{x_1, x_2, x_3, x_4, x_5, x_6\}$, each being the common predecessor of a different pair of vertices in $\{y_1, y_2, y_3, y_4\}$, exhausting all edges between $\{y_1, y_2, y_3, y_4\}$ and $\{x_1, x_2, x_3, x_4, x_5, x_6\}$.

Consider y_1 and, without loss of generality, let $\{x_1, x_2, x_3\}$ denote its three predecessors. Then x_4 , x_5 , and x_6 each must have their two successors in $\{y_2, y_3, y_4\}$. We claim

that x_4 , x_5 , and x_6 must have a common predecessor in $N_{d-3}(v)$. Note that if x_4 , x_5 and x_6 have a common predecessor, this predecessor must lie in $N_1(v)$, and $d = 4$. Indeed, if any two of $\{x_4, x_5, x_6\}$ share a predecessor, then all three must share a predecessor, in order to prevent a unique shortest path of length 2 between this predecessor and one of $\{y_2, y_3, y_4\}$. However, if x_4 , x_5 , and x_6 have three distinct predecessors in $N_{d-3}(v)$ that are not predecessors of either of the other two, then each of these three predecessors must have two successors in $\{x_1, x_2, x_3\}$, thereby causing x_1 , x_2 , and x_3 to be full; the fourth vertex of $N_{d-3}(v) = N_1(v)$ must then have x_4 , x_5 , and x_6 as its three successors.

Let w_1 be a shared predecessor of $\{x_4, x_5, x_6\}$. Then there are two paths of length 2 between w_1 and each of $\{y_2, y_3, y_4\}$ and no paths of length 2 between w_1 and y_1 . Repeat this procedure for y_2 , y_3 , and y_4 , each time finding a common predecessor w_i of

$$\{x_1, x_2, x_3, x_4, x_5, x_6\} \setminus \{\text{predecessors of } y_i\}$$

for $i = 2, 3, 4$. The choice of edges between $\{y_1, y_2, y_3, y_4\}$ and $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ therefore determines the edges between $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\{w_1, w_2, w_3, w_4\}$. However, since all pairs of vertices in $\{y_1, y_2, y_3, y_4\}$ share a predecessor among the independent set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ of vertices, all choices yield isomorphic graphs. All vertices are now full, so we have listed all elements of each distance set, and $G \cong Q_4$. By [2] Corollary 6.9, $q(Q_4) = 2$.

Case 3(b): There exists a vertex in $N_{d-1}(v)$, say y_1 , that does not have three predecessors. The edges incident with z account for two of the extra edges. If y_1 has another successor $z_2 \in N_d(v)$, then z_2 must have a neighbor $z_3 \in N_d(v) \setminus \{z, z_2\}$; since y_1 (with two predecessors) and z are both full, $y_1 z_2 z_3$ is a unique shortest path of length 2 between y_1 and z_3 . So y_1 must have a neighbor $u \in N_{d-1}(v)$, using up an extra edge.

Suppose $u \notin \{y_2, y_3, y_4\}$. Then u must be adjacent to another vertex in $\{y_1, y_2, y_3, y_4\}$, say y_2 , to avoid a unique shortest path of length 2 between itself and z . This leaves two remaining extra edges. Let x_1 and x_2 denote the two predecessors of u . Suppose x_1 has two predecessors. Then it cannot be adjacent to both y_1 and y_2 . So there must exist $x_3 \in N_{d-2}(v)$ (possibly $x_3 = x_2$) such that x_1 is adjacent to x_3 , and x_3 is adjacent to y_1 and y_2 . But then x_3 can have only one predecessor, so we must have $d = 3$.

Let $N_1(v) = \{x_1, x_2, x_3, x_4\}$. If x_1 and x_2 are not incident with an edge of $G[N_1(v)]$, then x_1 and x_2 both must be adjacent to $\{u, y_1, y_2\}$ (to avoid uniqueness of the shortest paths between each x_1 , x_2 and each y_1 , y_2 through u); but then y_1 and y_2 are full, and $y_1 z y_3$ is a unique shortest path of length 2 between y_1 and y_3 . Hence $G[N_1(v)]$ must contain an edge, using a third extra edge. It follows that there are exactly ten edges connecting $N_1(v)$ to $N_2(v)$. Then no vertex in $N_2(v)$ can have three predecessors, which implies $G[N_2(v)]$ must contain another edge, the last of the extra edges. Moreover, this edge must connect y_3 and y_4 . Note this implies $N_1(v)$ cannot be an independent set of vertices.

Suppose x_2 is incident with the edge in $G[N_1(v)]$, and x_1 is not. Then x_1 must be adjacent to $\{u, y_1, y_2\}$. And since x_1 shares v as a common predecessor with x_3 and x_4 ,

and since u is full while y_1 and y_2 are one edge short of being full, we must have, without loss of generality, y_1 adjacent to x_3 and y_2 adjacent to x_4 . Since x_1 and y_1 are full, the uniqueness of the path x_2y_1 can only be avoided if x_2 is adjacent to x_3 . Similarly, the uniqueness of the path x_2y_2 can only be avoided if x_2 is adjacent to x_4 . But $G[N_1(v)]$ can only have one edge.

So the edge in $G[N_1(v)]$ must be x_1x_2 . Then x_1 has only one more successor in $N_2(v)$ which must simultaneously resolve v being the unique common neighbor of the pairs $\{x_1, x_3\}$ and $\{x_1, x_4\}$. This successor must then have three predecessors, which is impossible. Therefore $u \in \{y_2, y_3, y_4\}$.

Without loss of generality, suppose $u = y_2$. As at the start of this case, $\{y_1, y_2, y_3, y_4\}$ can have no more successors in $N_d(v)$. And y_1 and y_2 cannot be incident to another edge in $G[N_{d-1}(v)]$, in order to allow for each to have two predecessors. To prevent y_1 and y_3 having z as a unique common neighbor, they must therefore share a common predecessor, say $x_1 \in N_{d-2}(v)$. If x_1 has two predecessors, then it is full, and $x_1y_1y_2$ is a unique shortest path of length 2 between x_1 and y_2 . So x_1 must have only one predecessor, which implies $d = 3$.

Suppose that $G[N_2(v)]$ contains no edges other than y_1y_2 ; then y_3 and y_4 must have three predecessors each. Denote the other vertices in $N_1(v)$ by x_2, x_3 , and x_4 . Now, y_3 and y_4 must share at least two of their predecessors in $N_1(v)$, but x_1 cannot be this shared predecessor, since that would leave $x_1y_1y_2$ as the unique shortest path of length 2 between x_1 and y_2 . So assume the two shared predecessors of y_3 and y_4 are $\{x_3, x_4\}$. Neither x_3 nor x_4 can be adjacent to y_1 or y_2 , as these adjacencies would result in a unique shortest path of length 2 between one of $\{x_3, x_4\}$ and one of $\{y_1, y_2\}$. With exactly one remaining extra edge, this edge must be x_3x_4 to ensure that each of x_3 and x_4 has four neighbors. It now follows that x_1 is adjacent to y_2 , and hence the three remaining edges are x_2y_1 , x_2y_2 , and x_2y_4 . After connecting each x_i to v , all vertices are full, and $G \cong R_{10,3}$ (to see this let v be vertex 3 in the drawing of $R_{10,3}$ in Fig. 4). Applying Lemma 4.1 establishes $q(R_{10,3}) = 2$.

Finally, suppose $G[N_2(v)]$ contains another edge, which can only be y_3y_4 , as otherwise y_1 or y_2 would not have enough predecessors. So each y_i has exactly two predecessors. Then there are eight edges between $N_2(v)$ and $N_1(v)$, and each vertex in $N_1(v)$ must have exactly two successors. Each pair $\{y_1, y_3\}$, $\{y_1, y_4\}$, $\{y_2, y_3\}$, and $\{y_2, y_4\}$ must share a single predecessor in $N_1(v)$, causing each y_i to be full. Denote these unique common predecessors by x_1, x_2, x_3 , and x_4 , respectively. To simultaneously avoid the unique shortest paths of length 2 from x_1 to y_2 and y_4 , x_1 must be adjacent to x_4 . Similarly, x_2 must be adjacent to x_3 . This uses up the remaining extra edges. After connecting each x_i to v , all vertices are full, and $G \cong R_{10,2}$. Using Lemma 4.1 establishes $q(R_{10,2}) = 2$.

Case 4: $N_d(v)$ contains no vertices with four predecessors.

Since the distance partition from v contains at least one vertex with four predecessors, let b denote a vertex with four predecessors. This accounts for two of the six extra edges. By the proofs of Lemmas 5.1 and 5.3, $N_d(v)$ must contain a pair of vertices with three

predecessors each or a cycle of vertices with two predecessors each. Furthermore, each edge in $N_d(v)$ is an extra edge and any vertex in $N_d(v)$ with three predecessors accounts for an extra edge, and so $G[N_d(v)]$ must be isomorphic to one of K_2 , P_3 , C_3 , or C_4 . Each of these options requires either three or four of the remaining extra edges. Therefore, there is at most one extra edge unaccounted for. This implies b is the unique vertex with four predecessors.

We first claim that $b \in N_2(v)$. Indeed, suppose $b \in N_i(v)$ for some $2 < i < d$, and let $\{a_1, a_2, a_3, a_4\} \subseteq N_{i-1}(v)$ be the four predecessors of b . Then each a_i has at least two predecessors. With at most one of the extra edges left, we may assume, without loss of generality, that a_1 has a successor $b_1 \in N_i(v)$. Now, b_1 cannot be a terminal vertex, as that would require more extra edges than are currently unaccounted for, so let $c_1 \in N_{i+1}(v)$ be a successor of b_1 . Then $a_1 b_1 c_1$ is a unique shortest path of length 2. So $b \in N_2(v)$.

We next claim that there are no vertices with three predecessors in $N_i(v)$ for any $i < d$. As observed at the end of Claim #1 in the proof of Lemma 5.3, any successor of a vertex with three predecessors must be a vertex with four predecessors. Since $b \in N_2(v)$, any vertex with three predecessors in $N_i(v)$ for $i \geq 2$ cannot have a successor. It follows that such a vertex requires an edge in $G[N_i(v)]$, which would require the use of at least three extra edges. A similar argument shows that any vertex in $N_i(v)$ for $2 \leq i < d - 1$ cannot have a unique successor in $N_{i+1}(v)$. If there were a vertex x with two predecessors and a unique successor y , it must be adjacent to another vertex in $N_i(v)$. Then y must either be adjacent to a vertex in $N_{i+2}(v)$ or $N_{i+1}(v)$ yielding either a unique shortest path or too many extra edges. So, with the exception of b , every vertex in $N_i(v)$ for $2 \leq i < d$ has exactly two predecessors, and every vertex in $N_i(v)$ for $2 \leq i < d - 1$ has exactly two successors.

We have $\{a_1, a_2, a_3, a_4\} = N_1(v)$, and b is a common successor of these four vertices. With only one remaining extra edge, each a_i must have another successor in $N_2(v)$. Let $b_1 \neq b$ be a successor of a_1 in $N_2(v)$, let a_2 be the second predecessor of b_1 , and let $c_1 \in N_3(v)$ be a successor of b_1 . Note that this implies a_1 and a_2 are not adjacent. To avoid unique paths of length 2 between c_1 and each of a_1 and a_2 , the second predecessor of c_1 , say b_2 , must have a_1 and a_2 as its predecessors. Continuing in this way through $N_i(v)$ for all $3 \leq i \leq d - 2$, we see successive induced candle sections starting with $\{a_1, a_2\}$, and ending say at $\{y_1, y_2\} \subseteq N_{d-1}(v)$. Let z_1 be the common successor of $\{y_1, y_2\}$ in $N_d(v)$. This argument is similar to the argument we used in Case 2(a). Starting again from a_3 , we see successive induced candle sections starting with $\{a_3, a_4\}$ and ending at say $\{y_3, y_4\} \subseteq N_{d-1}(v)$ whose common successor is $z_2 \in N_d(v)$.

Note that all vertices in $N_i(v)$ for $i < d - 1$ are full, which implies $N_{d-1}(v) = \{y_1, y_2, y_3, y_4\}$. Each $N_i(v)$ is independent for $i < d - 1$, and we claim that $N_{d-1}(v)$ is independent as well. To see this, $y_i y_j$ an edge for $i \in \{1, 2\}$ and $j \in \{3, 4\}$ would create a unique shortest path of length 2 from $N_{d-1}(v)$ to $N_{d-2}(v)$. If $y_1 y_2$ is an edge, then y_1 and y_2 are full and hence $y_1 z_1 z$ is a unique shortest path for any neighbor z of z_1 in $N_d(v)$. The same observation shows $y_3 y_4$ cannot be an edge. Hence the last remaining

extra edge must be in $G[N_d(v)]$. This implies that $G[N_d(v)]$ is either isomorphic to P_3 or C_4 . Note that if some vertex in $N_d(v)$ has three predecessors, then without loss of generality, it is adjacent to y_1, y_2 , and y_3 . This results in a unique shortest path of length 2 through y_3 . Thus $G[N_d(v)] \cong C_4$.

We now claim that any successor of y_1 or y_2 must be a successor of both. Indeed, if z is a successor of y_1 , then the paths zy_1x_1 and zy_1x_2 force z to be adjacent to y_2 as well. Similarly, any successor of y_3 or y_4 must be a successor of both. Hence we have four vertices in $N_d(v)$ forming a cycle C_4 where each vertex is adjacent to either $\{y_1, y_2\}$ or $\{y_3, y_4\}$. This implies that the vertices around the cycle must alternate between being adjacent to $\{y_1, y_2\}$ and $\{y_3, y_4\}$ and so $G \cong H_k$ for some odd $k \geq 7$ (as the size of $V(G) \setminus \{b, v\}$ is divisible by 4). Applying Lemma 2.3 establishes $q(H_k) = 2$. $\square$

This completes the proof of Theorem 2.4.

6. Further observations and related problems

Corollary 2.2 and Theorem 2.4 give a complete characterization of the r -regular graphs that admit two distinct eigenvalues for $r \leq 4$. For $r = 4$, the closed candles H_k for $k \geq 3$ give the only infinite family of 4-regular graphs G with $q(G) = 2$. From these graphs we can construct an infinite family of r -regular graphs G with $q(G) = 2$ for all $r \geq 5$.

From Corollary 6.8 in [2], if G is any graph with $q(G) = 2$, then $q(G \square K_2) = 2$. Define $H_k^d = H_k \square Q_d$ for any $d \geq 0$ and $k \geq 3$. Now from Lemma 2.3 and [2, Cor. 6.9] we immediately have the following result.

Proposition 6.1. *For all $k \geq 3$ and $d \geq 0$, H_k^d is a $(4 + d)$ -regular graph with $q(H_k^d) = 2$.*

When $r \geq 5$, the number of edges in an r -regular graph grows faster than the lower bound $|E(G)| \geq 2n - 4$. So, characterizing r -regular graphs with two distinct eigenvalues for $r \geq 5$ could be difficult using the methods in this paper. This motivates questions about existence and structure of such graphs. That is, are there other infinite families of r -regular graphs G with $q(G) = 2$, and if so, can they be constructed with a method other than the Cartesian product used in Proposition 6.1?

Problem 6.2. Determine whether or not an infinite family of 5-regular graphs G with $q(G) = 2$ disjoint from the family $\{H_k^1 : k \geq 3\}$ exists.

This paper builds on the work in [6], where we established a lower bound on the number of edges a graph on n vertices must have in order to have $q(G) = 2$, and characterized the graphs that meet the bound with equality. The graphs that meet the bound $|E(G)| \geq 2n - 4$ are Q_3 and an infinite family of graphs called *double-ended candles*. The lower bound is improved to $|E(G)| \geq 2n - 3$ if G has an odd number of vertices, and the graphs that meet this improved bound are exactly the infinite family

of graphs called *single-ended candles*. These graphs, along with the closed candles H_k and some of the sporadic graphs in Theorem 2.4, appear in a set of papers focusing on a different problem.

McKee and Smyth [19], Taylor [21], and Greaves [12,13] all consider the following problem: for which graphs G is there a matrix *compatible* with G whose eigenvalues are contained in the interval $[-2, 2]$? For this problem, “compatible” means that the matrix M has exactly the same zero pattern as $A(G)$ off of the diagonal entries, and the entries of M are restricted to lie in the ring of integers for some quadratic field (McKee and Smyth consider matrices with integer entries. Taylor and Greaves consider matrices whose entries lie in the ring of integers for an imaginary quadratic extension of $\mathbb{Q}$). It follows from interlacing that if G has a compatible matrix whose eigenvalues are contained in $[-2, 2]$, then so do all of its induced subgraphs. This motivates the characterization of the maximal graphs G for which there is a matrix compatible with G whose eigenvalues are contained in $[-2, 2]$. Each of the papers [19], [12], [13], and [21] gives a characterization for the respective rings under consideration. The graphs that appear in these characterizations are a collection of small sporadic graphs, and a few infinite families: the single-ended candles, the double-ended candles, the closed candles, and the graphs obtained from a candle section by adding an edge between each pair of nearest degree 2 vertices.

We are currently unable to explain the overlap between the graphs in our characterizations of graphs with $2n - 4$, $2n - 3$, or $2n$ edges that admit two distinct eigenvalues, and the graphs in the characterizations in [19], [12], [13], and [21]. It seems reasonable to expect that the maximal graphs whose compatible matrices have all of their eigenvalues contained in $[-2, 2]$ should have all of their eigenvalues contained in $\{-2, 2\}$ and hence have q -value 2. But we do not have a proof of this claim. A complete explanation of this coincidence may help the investigation of r -regular graphs G with $q(G) = 2$ for $r \geq 5$.

Problem 6.3. For a given integer $l \geq 3$, determine whether the maximal graphs that admit an integer (or similarly constrained) matrix whose eigenvalues are contained in $[-l, l]$ have $q(G) = 2$.

A regular graph G that is neither complete nor empty is *strongly regular* if there exist constants a and c so that any two adjacent vertices in G have a common neighbors, and any two non-adjacent vertices have c common neighbors. The definition implies that the diameter of a connected strongly regular graph is at most 2. If G is a connected regular graph, then $A(G)$ has three distinct eigenvalues if and only if G is a strongly regular graph (see, e.g., Lemma 10.2.1 in [11]). So $q(G) \in \{2, 3\}$ for a strongly regular graph G . In [10], Furst and Grotts show that $L(K_n)$, the line graph of K_n , has $q(L(K_n)) = 2$ for all $n \geq 3$ (note that $L(K_n)$ is a strongly regular graph for all $n \geq 3$).

A connected graph G is *distance regular* if there are integers b_i and c_i for all $i \geq 0$ so that for any two vertices u and v at distance i in G ,

$$|N_1(v) \cap N_{i+1}(u)| = b_i \quad \text{and} \quad |N_1(v) \cap N_{i-1}(u)| = c_i.$$

(See [9] for an extensive treatment of distance-regular graphs.) Distance-regular graphs with diameter 2 are strongly regular. Since Corollary 2.2 and Theorem 2.4 give a complete characterization of the r -regular graphs that admit two distinct eigenvalues for $r \leq 4$, they also determine the distance-regular graphs of valency at most 4 that admit two distinct eigenvalues. Excluding the complete graphs and complete multipartite graphs in the list, these are Q_3 , $R_{10,4}$, $R_{14,1}$, and Q_4 . Note that these graphs all have diameter at least 3, and thus are not strongly regular. The closed candles H_3 and H_4 are complete multipartite graphs, and all H_k for $k \geq 5$ are not distance regular.

Problem 6.4. Determine the distance-regular graphs G with $q(G) = 2$.

We finish with two observations about the 4-regular graphs G with $q(G) = 2$, and problems they raise. With the exception of K_5 and $R_{7,1}$, all of the 4-regular graphs that appear in Theorem 2.4 have even order. In particular, if $|V(G)| \geq 8$, then G must have even order.

Problem 6.5. Determine whether for every integer k , there is some value n_k so that every k -regular graph G with $|V(G)| \geq n_k$ and $q(G) = 2$ has even order.

The graphs H_k are all circulant graphs, as are some of the sporadic graphs listed in Theorem 2.4. More broadly, many of the graphs in our characterization are vertex transitive. It would be interesting to investigate the highly symmetric graphs that admit a matrix with two distinct eigenvalues.

Problem 6.6. Determine the circulant graphs G that have $q(G) = 2$. What can be said about the automorphism groups of regular graphs with $q(G) = 2$?

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

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