

Determinantal representations and the image of the principal minor map

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In this paper we explore determinantal representations of multiaffine polynomials and consequences for the image of various spaces of matrices under the principal minor map. We show that a real multiaffine polynomial has a definite Hermitian determinantal representation if and only if all of its so-called Rayleigh differences factor as Hermitian squares and use this characterization to conclude that the image of the space of Hermitian matrices under the principal minor map is cut out by the orbit of finitely many equations and inequalities under the action of $(\mathrm{SL}_2(\mathbb{R}))^n \rtimes S_n$. We also study such representations over more general fields with quadratic extensions. Factorizations of Rayleigh differences prove an effective tool for capturing subtle behavior of the principal minor map. In contrast to the Hermitian case, we give examples to show for any field $\mathbb{F}$, there is no finite set of equations whose orbit under $(\mathrm{SL}_2(\mathbb{F}))^n \rtimes S_n$ cuts out the image of $n \times n$ matrices over $\mathbb{F}$ under the principal minor map for every n .

1 Introduction

Given an $n \times n$ matrix A with entries in a field $\mathbb{F}$, let A_S denote the determinant of the submatrix of A indexed by the set S on the rows and columns. If we set $A_\emptyset = 1$, the principal minors of a matrix form a vector of length 2^n . The *principal minor map* is the map that assigns to each matrix the vector of its principal minors, namely

$$\varphi : \mathbb{F}^{n \times n} \longrightarrow \mathbb{F}^{2^n} \quad \text{given by} \quad A \longmapsto (A_S)_{S \subseteq [n]}.$$

One of the motivating goals of this paper is to characterize the image of this map. This problem dates back to the 19th century [39, 40]. In the cases $n = 2$ and $n = 3$, this image is characterized by $A_\emptyset = 1$ over $\mathbb{C}$. In the case $n = 4$, Lin and Sturmfels [35] give an explicit list of 65 polynomials that cut out the image and they conjectured that it is cut out by equations of degree 12 for any n .

The image of the space of real and complex symmetric matrices was studied by Holtz and Sturmfels [27], who show that the image is closed and invariant under an action of the group $\mathrm{SL}_2(\mathbb{R})^n \rtimes S_n$ and conjectured that the vanishing of polynomials in the orbit of the *hyperdeterminant* under this group cuts out the image of the principal minor map over $\mathbb{C}$. This conjecture was resolved by Oeding [41]. In [3], we build on techniques in [33] to generalize this result to hold over arbitrary unique factorization domain. Here we use similar techniques to characterize the image of Hermitian matrices.

The principal minor map problem appears in many different fields and applications, including statistical models, machine learning, combinatorics and matrix theory. One fundamental application is the study of determinantal point processes (DPP). These are probabilistic models that arise naturally in the study of random matrix theory [29] and machine learning [13, 21]. Symmetric DPPs have attracted a lot of attention as they reflect the repulsive behavior in modeling, see [1, 8, 18, 32, 45]. Non-symmetric kernels are also of interest for modeling both repulsive and attractive interactions [4, 12, 20]. Learning the parameters of such a model from data leads to the computation problem of reconstructing a matrix from the vector of its principal minors. Griffin and Tsatsomeros [22, 23] give a numerical algorithm that reconstructs a preimage of a matrix, if it exists, over $\mathbb{C}$. Rising, Kulesza and Taskarc [43] provide an efficient algorithm for reconstruction in the symmetric case.

In this paper, we study the principal minor map via determinantal representations of an associated multivariate polynomial. Explicitly, to each vector $\mathbf{a} = (a_S)_{S \subseteq [n]} \in \mathbb{F}^{2^n}$, we assign a multiaffine polynomial $f_{\mathbf{a}}$ where $f_{\mathbf{a}} = \sum_{S \subseteq [n]} a_S \prod_{i \in [n] \setminus S} x_i$. This transforms the problem of characterizing the image of the principal minor map to the problem of characterizing multiaffine polynomials with a *determinantal representation*, namely polynomials that can be written as $f = \det(\mathrm{diag}(x_1, \dots, x_n) + A)$ for some $n \times n$ matrix A . When the matrix

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A is symmetric (Hermitian), we call f a symmetric (Hermitian) multiaffine determinantal polynomial. In [3] we prove that the class of symmetric determinantal multiaffine polynomials is characterized by their Rayleigh differences being squares. The *Rayleigh difference* of a polynomial f with respect to $i, j \in [n]$ is defined to be

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} - f \frac{\partial^2 f}{\partial x_i \partial x_j}. \quad (1)$$

Here we explore the consequences of factorizations of Rayleigh differences for non-symmetric determinantal representations. As described below, the relationship is more subtle than in the symmetric case. While in [3] we were able to work over arbitrary unique factorization domains, here we mainly restrict ourselves to working over fields in order to deal with this increased complexity. While we do recover some of the results from [3], we do not recover them in full generality. In follow up work, the second author also uses factorizations of Rayleigh differences to recover the full fiber of the principal minor map [2].

Using factorizations of Rayleigh differences into Hermitian squares, we characterize Hermitian determinantal multiaffine polynomials over any field $\mathbb{K}$ with an automorphism of order two. In particular, this gives a characterization of Hermitian determinantal multiaffine polynomials over $\mathbb{C}$.

Main Result 1 (Corollary 5.4, Theorem 5.6). A real multiaffine polynomial f has a linear Hermitian determinantal representation if and only if all of its Rayleigh differences $\Delta_{ij}(f)$ factor as Hermitian squares. $\square$

One of the themes of this paper is that factorizations of Rayleigh differences can capture subtle behavior of the principal minor map.

Example (Example 4.8). For example the Rayleigh differences of the polynomial

$$f_{\mathbf{a}}(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 - x_1 x_2 - x_1 x_3 - x_1 x_4 - x_2 x_3 - x_2 x_4 - x_3 x_4 + 1$$

factor into Hermitian squares in multiple ways, e.g. $\Delta_{34}(f_{\mathbf{a}}) = (x_1 - \mathbf{i})(x_1 + \mathbf{i})(x_2 - \mathbf{i})(x_2 + \mathbf{i})$. These different choices of factorization capture some non-generic behavior in the fiber of the principal minor map and lead to three inequivalent determinantal representations of f :

$$\left\{ \begin{pmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & x_3 & \mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & x_4 \end{pmatrix}, \begin{pmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & -\mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & x_3 & \mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & x_4 \end{pmatrix}, \begin{pmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & x_3 & -\mathbf{i} \\ 1 & -\mathbf{i} & \mathbf{i} & x_4 \end{pmatrix} \right\}.$$

$\square$

It is natural to ask about the fibers of the principal minor map. In the symmetric case, the fibers were characterized by Engel and Schneider [19]. Given a matrix A , both its transpose and $D^{-1}AD$, for any diagonal matrix D , have the same principal minors as A . In 1984, Loewy and Hartfiel [25] and then Loewy [36] gave sufficient conditions on a matrix for its fiber under the principal minor map to be a single point, up to diagonal equivalence and taking transposes. As the example above shows, the fiber in general can be larger. Using the techniques of this paper, the second author establishes a converse to the theorem of Loewy in order to classify the fibers of the principal minor map [2].

Here we use the classical theory of determinantal representations to understand the principal minor map, including ideas from Dixon [16] on the construction of symmetric determinantal representations of plane curves. The study of symmetric and Hermitian determinantal representations is also closely related to the theory of hyperbolic and real stable polynomials, which are multivariable generalizations of real-rooted univariate polynomials. Hyperbolic and stable polynomials have found wide-spread applications in combinatorics [24, 37], convex analysis [6], operator theory [26, 38], probability [7], and theoretical computer science [5, 34, 44]. The question of which stable polynomials have definite Hermitian determinantal representations has implications in operator theory and the theory of convex optimization. See [46] for more. In general, the existence of definite Hermitian representations does not follow from the existence of general representations over $\mathbb{C}$. In this paper we show that this is not the case for multiaffine polynomials.

Main Result 2 (Theorem 6.4). If a multiaffine real stable polynomial f has a linear determinantal representation over $\mathbb{C}$, then it has a definite Hermitian determinantal representation. $\square$

From the classification above, we characterize the image of Hermitian matrices under the principal minor map by characterizing the set of real multiquadratic polynomials that factor as Hermitian squares. This leads to explicit equations and inequalities defining the image.

Main Result 3 (Corollary 7.14). The image of the set of $n \times n$ Hermitian matrices under the principal minor map is cut out by the orbit under $\mathrm{SL}_2(\mathbb{R})^n \rtimes S_n$ of two inequalities and three degree-12 equations defined by polynomials in $\mathbb{Q}[a_S : S \subseteq 4]$. $\square$

An explicit description of the image of general $n \times n$ matrices remains mysterious. Huang and Oeding [28] give description of the image in the special case where all principal minors of same size are equal (*the symmetrized principal minor assignment problem*) where they use the cycle sums in their approach. They provide a minimal parametrization of the respective varieties in the cases of symmetric, skew symmetric and square complex matrices. Kenyon and Pemantle [30] adjust the principal minor map by adding *the almost principal minors* to the vector in the image and they showed that the ideal of the variety in this case is generated by translations of a single relation using the rhombus tiling.

Arguably one of the most surprising results in this paper is that, unlike the case of symmetric and Hermitian matrices, one cannot hope to define the image general $n \times n$ of the principal minor map for n by the orbit of some fixed set of equations under the group $\mathrm{SL}_2(\mathbb{F})^n \rtimes S_n$. Using factorizations of Rayleigh differences, we found a family of examples that shows that such a finite description is impossible.

Main Result 4 (Theorem 8.1). For any field $\mathbb{F}$, there is *no finite set of equations* whose orbit under $\mathrm{SL}_2(\mathbb{F})^n \rtimes S_n$ cuts out the image of the principal minor map for all n . $\square$

Example (Example 8.1). For instance, we show that the coefficient vector of the polynomial

$$f = x_1(x_3x_4 + 1)(x_2x_5 + 1) + (x_2x_3 + 1)(x_4x_5 + 1) \quad (2)$$

does not belong to the image of the principal minor map but satisfies the orbit of equations vanishing on the principal minors of 4×4 matrices under the group $\mathrm{SL}_2(\mathbb{F})^5 \rtimes S_5$. $\square$

This paper is organized as follows. In Section 2 we introduce terminology and the basic properties of determinantal representations and the action of $\mathrm{SL}_2(\mathbb{F})^n \rtimes S_n$. In Section 3 we give a characterization of multiaffine determinantal polynomials involving the factoring of Rayleigh differences. For Hermitian determinantal representations, this condition simplifies and we give an algorithm for constructing such representations from a factorization, as described in Section 4 and Section 5. In Section 6 we give a characterization of multiaffine stable determinantal polynomials and prove Theorem 6.4. In Section 7 we translate these conditions into explicit equations and inequalities whose orbit under $\mathrm{SL}_2(\mathbb{R})^n \rtimes S_n$ cuts of the image of Hermitian matrices under the principal minor map. Finally, in Section 8 we conclude by presenting a family of examples that disproves the existing of such a finite description for the image of general $n \times n$ matrices under the principal minor map.

2 Background and notation

For a commutative ring R , we use $R[\mathbf{x}]$ to denote the polynomial ring $R[x_1, \dots, x_n]$ and for $f \in R[\mathbf{x}]$, we use $\deg_i(f)$ to denote the degree of f in the variable x_i . For $\mathbf{d} = (d_1, \dots, d_n)$ with $d_i \in \mathbb{Z}_{\geq 0}$, let $\mathbb{F}[\mathbf{x}]_{\leq \mathbf{d}}$ denote the set of polynomials with degree at most d_i in x_i for each $i = 1, \dots, n$. These form an R -module of rank $\prod_{i=1}^n (d_i + 1)$. When $d_1 = \dots = d_n = m$, we abbreviate $R[\mathbf{x}]_{\leq (m, \dots, m)}$ by $R[\mathbf{x}]_{\leq \mathbf{m}}$. Of particular interest are *multiaffine polynomials*, with degree ≤ 1 in each variable, and *multiquadratic polynomials*, with degree ≤ 2 in each variable. These are denoted by $R[\mathbf{x}]_{\leq 1} = R[\mathbf{x}]_{\mathrm{MA}}$ and $R[\mathbf{x}]_{\leq 2} = R[\mathbf{x}]_{\mathrm{MQ}}$, respectively.

We use $\mathrm{Mat}_n(\mathbb{F})$ to denote the set of $n \times n$ matrices with entries in $\mathbb{F}$.

2.1 The action of $\mathrm{SL}_2(R)^n \rtimes S_n$ and homogeneizations

The action of $\mathrm{SL}_2(R)^n$ on $R[\mathbf{x}]_{\leq \mathbf{d}}$ is defined as follows. Let $\gamma = (\gamma_i)_{i \in [n]}$ in $\mathrm{SL}_2(R)^n$ where $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Then for $f \in R[\mathbf{x}]_{\leq \mathbf{d}}$,

$$\gamma \cdot f = \prod_{i=1}^n (c_i x_i + d_i)^{d_i} \cdot f \left(\frac{a_1 x_1 + b_1}{c_1 x_1 + d_1}, \dots, \frac{a_n x_n + b_n}{c_n x_n + d_n} \right).$$

One way to interpret this action is with the multihomogeneization of f . Let $f^{\mathbf{d}-\mathrm{hom}}$ in $R[x_1, \dots, x_n, y_1, \dots, y_n]_{\mathbf{d}}$ denote the polynomial

$$f^{\mathbf{d}-\mathrm{hom}} = \prod_{i=1}^n y_i^{d_i} \cdot f(x_1/y_1, \dots, x_n/y_n).$$

The induced action of γ on $f^{\mathbf{d}-\text{hom}}$ is just a linear change of coordinates:

$$\gamma \cdot f^{\mathbf{d}-\text{hom}} = f^{\mathbf{d}-\text{hom}} \left(\gamma_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \gamma_n \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right).$$

Restricting to $y_1 = \dots y_n = 1$ gives back $\gamma \cdot f$.

We will also use the usual homogenization of a polynomial to some total degree d , using a single homogenizing variable y . That is, for $f = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha} \in R[\mathbf{x}]$ of total degree $d = \deg(f)$, its homogenization is

$$f^{\text{hom}} = y^d f(x_1/y, \dots, x_n/y) = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha} y^{d-|\alpha|} \in R[\mathbf{x}, y].$$

Suppose that $\mathbb{K}$ is a field with an automorphic involution $a \mapsto \bar{a}$ with fixed field $\mathbb{F}$. This extends to an involution on $\mathbb{K}[\mathbf{x}]$ by acting on the coefficients. We will say that a polynomial $q \in \mathbb{F}[\mathbf{x}]$ is a *Hermitian square* if $q = g\bar{g}$ for some $g \in \mathbb{K}[\mathbf{x}]$. To end this section, we remark that for $f \in \mathbb{F}[\mathbf{x}]$, the condition that $\Delta_{ij}(f)$ is a Hermitian square is robust to homogenization.

Proposition 2.1. *Suppose that $\mathbb{K}$ is a field with an automorphic involution $a \mapsto \bar{a}$ with fixed field $\mathbb{F}$. Let $f \in \mathbb{F}[\mathbf{x}]$. For $i, j \in [n]$, the polynomial $\Delta_{ij}(f)$ is a Hermitian square if and only if $\Delta_{ij}(f^{\text{hom}})$ is a Hermitian square. $\square$*

Proof. If $\Delta_{ij}(f^{\text{hom}})$ is a Hermitian square, then specializing to $y = 1$ gives a representation of $\Delta_{ij}(f)$ as a Hermitian square. For the converse, let $f \in \mathbb{F}[\mathbf{x}]$ with total degree d and suppose that $\Delta_{ij}f = g\bar{g}$ for some $g \in \mathbb{K}[\mathbf{x}]$. Let $m = \deg(g) = \deg(\bar{g})$. By definition, $\Delta_{ij}(f^{\text{hom}}) \in \mathbb{F}[\mathbf{x}, y]$ is homogeneous of degree $2d - 2$. Its restriction to $y = 1$ equals $\Delta_{ij}f$. Therefore $\Delta_{ij}(f^{\text{hom}})$ equals $y^{2d-2-2m}(\Delta_{ij}(f))^{\text{hom}}$, which is the Hermitian square $h\bar{h}$ where h is the homogeneization of g to total degree $d - 1$. $\blacksquare$

2.2 A description of the action of $\text{SL}_2(\mathbb{F})^n$ on determinantal polynomials and matrices

One can describe the rational action of $\text{SL}_2(\mathbb{F})^n$ on $\text{Mat}_n(\mathbb{F})$ via its action on multiaffine polynomials in $\mathbb{F}[x_1, \dots, x_n]$. Given a matrix $A \in \text{Mat}_n(\mathbb{F})$, consider the multiaffine polynomial $f = \det(\text{diag}(x_1, \dots, x_n) + A)$. For $\gamma = (\gamma_i)_{i \in [n]}$ in $\text{SL}_2(\mathbb{F})^n$ with $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $\gamma \cdot f$ is defined by:

$$\gamma \cdot f = \prod_{i=1}^n (c_i x_i + d_i) \cdot \det \left(\text{diag} \left(\frac{a_1 x_1 + b_1}{c_1 x_1 + d_1}, \dots, \frac{a_n x_n + b_n}{c_n x_n + d_n} \right) + A \right).$$

Let A_i denote the i th column of A and e_i the vector whose i th entry is one and zero otherwise. By using the factor $(c_i x_i + d_i)$ to scale the i th column, we see that

$$\gamma \cdot f = \det(C \text{diag}(x_1, \dots, x_n) + B)$$

where C is the matrix with i th column $C_i = (a_i e_i + c_i A_i)$ and B is the matrix with i th column $B_i = b_i e_i + d_i A_i$. When the matrix C is invertible, this gives

$$\gamma \cdot f = \det(C) \det(\text{diag}(x_1, \dots, x_n) + C^{-1}B).$$

Up to the scalar multiple $\det(C)$, the coefficients of $\gamma \cdot f$ are the principal minors of the matrix $C^{-1}B$. This defines a rational action of $\text{SL}_2(\mathbb{F})^n$ on $\mathbb{F} \times \text{Mat}_n(\mathbb{F})$. Namely, for $\gamma \in \text{SL}_2(\mathbb{F})^n$ and $(\alpha, A) \in \mathbb{F} \times \text{Mat}_n(\mathbb{F})$, this is given by

$$(\gamma, (\alpha, A)) \mapsto (\alpha^{(\gamma, A)}, A^{\gamma}) := (\alpha \det(C), C^{-1}B)$$

where the matrices B and C are defined as above. To avoid confusion, we reserve the notation $\gamma \cdot$ for the action of $\text{SL}_2(\mathbb{F})^n$ on the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$. We see that this rational action of $\text{SL}_2(\mathbb{F})^n$ on $\mathbb{F} \times \text{Mat}_n(\mathbb{F})$ is compatible its action on determinantal polynomials and that it commutes the action of $(\mathbb{F}^*)^n$ on $\text{Mat}_n(\mathbb{F})$ by diagonal conjugation. More precisely:

Proposition 2.2. *Let $\gamma \in \text{SL}_2(\mathbb{F})^n$ and $(\alpha, A) \in \mathbb{F} \times \text{Mat}_n(\mathbb{F})$. Let $f = \alpha \det(\text{diag}(x_1, \dots, x_n) + A)$ and suppose that the coefficient of $x_1 \cdots x_n$ in $\gamma \cdot f$ is nonzero. Then*

$$\gamma \cdot f = \alpha^{(\gamma, A)} \det(\text{diag}(x_1, \dots, x_n) + A^{\gamma}).$$

Moreover, for any invertible diagonal matrix D , $(\alpha^{(\gamma, D^{-1}AD)}, (D^{-1}AD)^{\gamma}) = (\alpha^{(\gamma, A)}, D^{-1}(A^{\gamma})D)$. $\square$

Proof. We take γ , A as above. The first claim follows from the arguments above. More specifically, the calculations above give that

$$\left(\text{diag} \left(\frac{a_1 x_1 + b_1}{c_1 x_1 + d_1}, \dots, \frac{a_n x_n + b_n}{c_n x_n + d_n} \right) + A \right) E = C \text{diag}(x_1, \dots, x_n) + B$$

where B and C are defined as above and E denotes the matrix $\text{diag}(c_1 x_1 + d_1, \dots, c_n x_n + d_n)$. Since diagonal matrices commute, we see that

$$\begin{aligned} \left(\text{diag} \left(\frac{a_1 x_1 + b_1}{c_1 x_1 + d_1}, \dots, \frac{a_n x_n + b_n}{c_n x_n + d_n} \right) + D^{-1} A D \right) E &= D^{-1} \left(\text{diag} \left(\frac{a_1 x_1 + b_1}{c_1 x_1 + d_1}, \dots, \frac{a_n x_n + b_n}{c_n x_n + d_n} \right) + A \right) E D \\ &= D^{-1} (C \text{diag}(x_1, \dots, x_n) + B) D \\ &= (D^{-1} C D) \text{diag}(x_1, \dots, x_n) + D^{-1} B D. \end{aligned}$$

This gives $(D^{-1} A D)^\gamma = (D^{-1} C D)^{-1} (D^{-1} B D) = D^{-1} C^{-1} B D = D^{-1} A^\gamma D$. Since the determinants of $C \text{diag}(x_1, \dots, x_n) + B$ and $D^{-1} (C \text{diag}(x_1, \dots, x_n) + B) D$ are equal, we see that the leading coefficient remains unchanged. That is, $\alpha^{(\gamma, D^{-1} A D)} = \alpha^{(\gamma, A)} = \alpha \det(C)$. ■

2.3 Resultants

For two univariate polynomials $a = \sum_{j=0}^d a_j t^j$ with $a_d \neq 0$ and $b = b_1 t + b_0$ with $b_1 \neq 0$, the resultant of a , b with respect to the variable t is defined to be

$$\text{Res}_t(a, b) = \sum_{j=0}^d a_j (-b_0)^j (b_1)^{d-j} = b_1^d \cdot (a|_{t=-b_0/b_1}). \quad (3)$$

This polynomial vanishes if and only if the univariate polynomials a and b have a common root, namely $t = -b_0/b_1$. See, for example, [15, §3.5] for more on resultants. We will focus on multiaffine polynomials and so focus on resultants in degree $d = 1$. For $k = 1, \dots, n$, define

$$\text{res}_{x_k}(g, h) = (g|_{x_k=0}) \cdot \frac{\partial}{\partial x_k} h - (h|_{x_k=0}) \cdot \frac{\partial}{\partial x_k} g.$$

In particular, if g and h both have degree one in x_k , this agrees with $\text{Res}_{x_k}(g, h)$. The benefit of this degree-dependent definition is that it is invariant under the action of $\text{SL}_2(R)$.

If $f \in R[\mathbf{x}]$ has degree ≤ 1 in both x_i and x_j , then

$$\Delta_{ij}(f) = \text{res}_{x_i} \left(\frac{\partial f}{\partial x_j}, f|_{x_j=0} \right) = \text{res}_{x_j} \left(\frac{\partial f}{\partial x_i}, f|_{x_i=0} \right). \quad (4)$$

Proposition 2.3. *If $f \in R[x_1, \dots, x_n]$ has degree one in each of x_i and x_j , then $\Delta_{ij}(f) = 0$ if and only if f factors into polynomial $g \cdot h$ with $g \in R[x_k : k \neq i]$ and $h \in R[x_k : k \neq j]$.* □

Proof. By assumption we can write $f = ax_i x_j + bx_i + cx_j + d$ for $a, b, c, d \in R[x_k : k \neq i, j]$. Then $\Delta_{ij}(f) = bc - ad$. If $\Delta_{ij}(f) = 0$, then $bc = ad$. Taking irreducible factorizations of both sides, we see that there is some factorization $b = b_1 b_2$ and $c = c_1 c_2$ for which $a = b_1 c_1$ and $d = b_2 c_2$. Then $f = (b_1 x_i + c_2)(c_1 x_j + b_2)$. Similarly, if $f = (b_1 x_i + c_2)(c_1 x_j + b_2)$ for some $b_1, b_2, c_1, c_2 \in R[x_k : k \neq i, j]$, then $\Delta_{ij}(f) = b_1 b_2 c_1 c_2 - b_1 b_2 c_1 c_2 = 0$. ■

Proposition 2.4. *Let $g \in R[\mathbf{x}]_{\leq \mathbf{d}}$ and $h \in R[\mathbf{x}]_{\leq \mathbf{e}}$ with $\mathbf{d}_k = \mathbf{e}_k = 1$. For $\gamma \in \text{SL}_2(R)^n$,*

$$\gamma \cdot \text{res}_{x_k}(g, h) = \text{res}_k(\gamma \cdot g, \gamma \cdot h),$$

where γ acts on $\text{res}_{x_k}(g, h)$ as polynomial of multidegree $\leq \mathbf{d} + \mathbf{e} - 2 \cdot \mathbf{1}_k$ with $\mathbf{1}_k$ is the vector with k th entry is 1 and zero otherwise. □

Proof. Write $g = g_1x_k + g_0$ and $h = h_1x_k + h_0$ where g_1, g_0, h_1, h_0 are polynomials in the polynomial ring $R[x_j : j \neq k]$. The resultant $\text{res}_{x_k}(g, h)$ is the determinant of the 2×2 matrix $\begin{pmatrix} h_1 & h_0 \\ g_1 & g_0 \end{pmatrix}$. Consider $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(R)$ acting on the j th coordinate. If $j = k$, then

$$\gamma \cdot g = g_1(ax_k + b) + g_0(cx_k + d), \quad \text{and} \quad \gamma \cdot h = h_1(ax_k + b) + h_0(cx_k + d).$$

Taking coefficients with respect to $\{1, x_k\}$, we see that the $\text{res}_{x_k}(\gamma \cdot g, \gamma \cdot h)$ equals

$$\det \begin{pmatrix} ah_1 + ch_0 & bh_1 + dh_0 \\ ag_1 + cg_0 & bg_1 + dg_0 \end{pmatrix} = \det \left(\begin{pmatrix} h_1 & h_0 \\ g_1 & g_0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \det \begin{pmatrix} h_1 & h_0 \\ g_1 & g_0 \end{pmatrix} = \text{res}_{x_k}(g, h).$$

Since γ acts on $R[\mathbf{x}]_{\leq \mathbf{d} + \mathbf{e} - 2\mathbf{1}_k}$ as the identity, this equals $\gamma \cdot \text{res}_{x_k}(g, h)$.

If $j \neq k$, then $\gamma \cdot g = (\gamma \cdot g_1)x_k + (\gamma \cdot g_0)$ and $\gamma \cdot h = (\gamma \cdot h_1)x_k + (\gamma \cdot h_0)$, where γ acts on g_1, g_0 and h_1, h_0 as elements of multidegree $\mathbf{d} - \mathbf{1}_k$ and $\mathbf{e} - \mathbf{1}_k$, respectively. It follows that

$$\text{res}_{x_k}(\gamma \cdot g, \gamma \cdot h) = \det \begin{pmatrix} \gamma \cdot h_1 & \gamma \cdot h_0 \\ \gamma \cdot g_1 & \gamma \cdot g_0 \end{pmatrix} = \gamma \cdot \text{res}_{x_k}(g, h).$$

■

From (4), this gives the following:

Corollary 2.5. Consider an element $\gamma \in \text{SL}_2(R)^n$ that acts by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the k -th coordinate and the identity in all others. For any $f \in R[\mathbf{x}]_{\leq \mathbf{1}}$,

$$\Delta_{ij}(\gamma \cdot f) = \begin{cases} \Delta_{ij}(f) & \text{if } k = i \text{ or } k = j \\ \gamma \cdot \Delta_{ij}(f) & \text{otherwise.} \end{cases}$$

□

3 Determinantal Representations and Rayleigh Differences

Let R be a unique factorization domain and denote by $\text{Mat}_n(R)$ the set of $n \times n$ matrices with entries in R .

Theorem 3.1. Let $f \in R[x_1, \dots, x_n]$ be multiaffine in the variables $x_1, \dots, x_n$ with its coefficient of $x_1 \cdots x_n$ equals one. Then $f = \det(\text{diag}(x_1, \dots, x_n) + A)$ for some $A \in \text{Mat}_n(R)$ if and only if for every $i \neq j \in [n]$, the polynomials $\Delta_{ij}(f)$ factor as the product $g_{ij} \cdot g_{ji}$ where

- (a) $g_{ij} \in R[x_k : k \neq i, j]$ is multiaffine in $x_1, \dots, x_n$ and
- (b) for every $k \in [n] \setminus \{i, j\}$, $\text{res}_{x_k}(g_{ij}, f) = g_{ik}g_{kj}$.

In this case, we can take g_{ij} to be the (i, j) th entry of $(\text{diag}(x_1, \dots, x_n) + A)^{\text{adj}}$ where M^{adj} represents the adjugate matrix of M . □

Proof of ($\Rightarrow$). This follows from a classical equality on the principal minors of an $n \times n$ matrix, used by Dodgson [17] as a method for computing determinants. This is also known as the Desnanot-Jacobi identity or more generally as Sylvester's determinantal identity. For any $i \neq j$ and $k \neq \ell$ in $[n]$,

$$(M^{\text{adj}})_{ik}(M^{\text{adj}})_{j\ell} - (M^{\text{adj}})_{i\ell}(M^{\text{adj}})_{jk} = \det(M) \cdot q \quad \text{where} \quad q = \det \begin{pmatrix} M & e_k & e_\ell \\ e_i^T & 0 & 0 \\ e_j^T & 0 & 0 \end{pmatrix}, \quad (5)$$

where e_i denotes the i coordinate unit vector in $\mathbb{R}^n$ and M^{adj} denotes the adjugate matrix of M . See also [33, Prop. 4.6]. For $M = \text{diag}(x_1, \dots, x_n) + A$,

$$\frac{\partial f}{\partial x_i} = (M^{\text{adj}})_{ii}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = \det \begin{pmatrix} M & e_i & e_j \\ e_i^T & 0 & 0 \\ e_j^T & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \Delta_{ij}(f) = (M^{\text{adj}})_{ij}(M^{\text{adj}})_{ji}. \quad (6)$$

The last equality follows from (5) with $k = i$ and $\ell = j$ after rearranging terms.

For every $i, j \in [n]$, let g_{ij} denote $(M^{\text{adj}})_{ij}$. Then $g_{ij} \in R[x_k : k \neq i, j]$ is multiaffine in $x_1, \dots, x_n$ and $\Delta_{ij}(f) = g_{ij}g_{ji}$. Under an appropriate choice of indices, [5] gives

$$g_{kk} \cdot g_{ij} - f \cdot q = g_{ik} \cdot g_{kj} \quad \text{where} \quad q = \det \begin{pmatrix} M & e_k & e_j \\ e_k^T & 0 & 0 \\ e_i^T & 0 & 0 \end{pmatrix}.$$

Note that $g_{kk} = \frac{\partial f}{\partial x_k}$ is the coefficient of x_k in f and q is the coefficient of x_k in g_{ij} . Therefore $g_{kk} \cdot g_{ij} - f \cdot q$ is the resultant of g_{ij} and f with respect to x_k . ■

Example 3.2. For $n \geq 5$, one cannot remove condition (b) from Theorem [3.1]. Consider

$$f = x_1x_2x_3x_4x_5 + x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 \\ + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_2x_3x_5 + x_3x_4x_5.$$

One can check that for every $i, j \in [5]$, $\Delta_{ij}(f)$ factors as the product of two multiaffine polynomials in $\mathbb{Q}[x_1, \dots, x_5]$. For example, $\Delta_{12}(f) = -x_3x_4x_5(x_4x_5 - x_3 + x_4 + x_5)$. Since there is an irreducible factor involving all three variables, there is only one possible factorization of $\Delta_{12}(f)$ as the product of two multiaffine polynomials $g_{12} \cdot g_{21}$, up to scalar multiples and switching the factors, namely $g_{12} = -x_3x_4x_5$ and $g_{21} = x_4x_5 - x_3 + x_4 + x_5$. Taking the resultant of g_{21} and f with respect to x_3 gives

$$\text{Res}_{x_3}(g_{21}, f) = (x_1x_5 + x_1 + x_5)(x_2x_4 + x_2 + x_4)(x_4x_5 + x_4 + x_5).$$

Each of the three quadratic factors are irreducible and so there is no way of writing this resultant as the product of two multiaffine polynomials. Therefore there is no choice of polynomials g_{23} and g_{31} satisfying the conditions in Theorem [3.1]. □

Lemma 3.3. Let $f \in R[x_1, \dots, x_n]$ be multiaffine in the variables $x_1, \dots, x_n$ and its coefficient of $x_1 \cdots x_n$ equals one. If $f = g \cdot h$ for some $g, h \in R[x_1, \dots, x_n]$, then g and h are multiaffine in disjoint subsets of the variables $x_1, \dots, x_n$ and we can take their leading coefficients in these variables to be one. Moreover, if the polynomials $\Delta_{ij}(f)$ have factorizations satisfying conditions (a) and (b) in Theorem [3.1] then so do $\Delta_{ij}(g)$ and $\Delta_{ij}(h)$. □

Proof. For any $i \in [n]$, the degree of f in x_i must be the sum of the degrees of g and h in x_i . Since this sum of nonnegative numbers is one for each $i \in [n]$, we see that for some subset $I \subseteq [n]$, g is multiaffine in $\{x_i : i \in I\}$, h is multiaffine in $\{x_j : j \in [n] \setminus I\}$, and $\deg_i(h) = \deg_j(g) = 0$ for any $i \in I$ and $j \notin I$.

The highest degree term in f with respect to the variables $x_1, \dots, x_n$, $\prod_{i=1}^n x_i$, is the product of the highest degree terms in g and h . Therefore after rescaling, we can assume that both g and h have leading coefficient in these variables equal to 1. For $i \in I$ and $j \notin I$, $\partial(g \cdot h)/\partial x_i = h \cdot \partial g/\partial x_i$ and $\partial(g \cdot h)/\partial x_j = g \cdot \partial h/\partial x_j$. From this, one can check that $\Delta_{ij}(gh)$ equals $h^2 \Delta_{ij}(g)$ for $i, j \in I$, $g^2 \Delta_{ij}(h)$ for $i, j \in [n] \setminus I$ and zero otherwise.

Suppose that for $i, j \in [n]$, $\Delta_{ij}(f) = m_{ij}m_{ji}$ with m_{ij} multiaffine in $x_1, \dots, x_n$ and $\text{res}_{x_k}(m_{ij}, f) = m_{ik}m_{kj}$ for every i, j, k . For $i, j \in I$, we see that $m_{ij}m_{ji} = h^2 \Delta_{ij}(g)$. Since m_{ij}, m_{ji} are multiaffine, they both must be divisible by h , leaving $\tilde{m}_{ij}\tilde{m}_{ji} = \Delta_{ij}(g)$, where $\tilde{m}_{ij}, \tilde{m}_{ji}$ are multiaffine in x_i for $i \in I$. Moreover, for k also in I ,

$$h^2 \tilde{m}_{ik}\tilde{m}_{kj} = m_{ik}m_{kj} = \text{Res}_{x_k}(m_{ij}, f) = \text{res}_{x_k}(\tilde{m}_{ij}h, gh) = h^2 \text{res}_{x_k}(\tilde{m}_{ij}, g)$$

showing that $\tilde{m}_{ik}\tilde{m}_{kj} = \text{res}_{x_k}(\tilde{m}_{ij}, g)$. The desired factorization for $\Delta_{ij}(h)$ with $i, j \in [n] \setminus I$ follows similarly. ■

Proof of ($\Leftarrow$). Suppose that f is irreducible and multiaffine of degree n . Let G denote the $n \times n$ matrix with (i, j) th entry g_{ij} for $i \neq j$ and $g_{ii} := \frac{\partial f}{\partial x_i}$ for $i = j$.

We claim that all of the 2×2 minors of G lie in $\langle f \rangle$. This is immediate for the symmetric minors, as $g_{ii}g_{jj} - g_{ij}g_{ji} = f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}$. Moreover, since $\frac{\partial f}{\partial x_1}$ is the coefficient of x_1 in f , the resultant $\text{res}_{x_1}(g_{ij}, f)$ has the form $\frac{\partial f}{\partial x_1}g_{ij} - qf$ for some q . This gives $g_{11}g_{ij} - g_{i1}g_{1j} = qf$. Finally, suppose that i, j, k, ℓ are all distinct. Then

$$g_{11}^2(g_{ij}g_{kl} - g_{il}g_{kj}) = (g_{11}g_{ij})(g_{11}g_{kl}) - (g_{11}g_{il})(g_{11}g_{kj}) \equiv g_{1i}g_{1j}g_{1k}g_{1l} - g_{1i}g_{1l}g_{1k}g_{1j} = 0 \pmod{\langle f \rangle}.$$

Since f is irreducible and $g_{11} = \partial f/\partial x_1$ has smaller degree, g_{11} is not a zero-divisor in $R[x_1, \dots, x_n]/\langle f \rangle$. Therefore the minor $g_{ij}g_{kl} - g_{il}g_{kj}$ belongs to $\langle f \rangle$.

From this it follows that f^{k-1} divides the $k \times k$ minors of G for every $2 \leq k \leq n$, see [42, Lemma 4.7]. In particular, f^{n-2} divides the entries of the adjugate matrix G^{adj} . Let

$$M = (1/f^{n-2}) \cdot G^{\text{adj}}. \quad (7)$$

Also f^{n-1} divides $\det(G)$, and since these both have degree $n(n-1)$, there must be some constant $\lambda \in R$ for which $\det(G) = \lambda \cdot f^{n-1}$.

We can see that $\lambda = 1$ by taking top degree terms. Since $\deg(f_i) = n-1$ and for all $i \neq j$, $\deg(g_{ij}) \leq n-2$, the leading degree term of $\det(G)$ comes uniquely from the product of the diagonals $f_1 \cdots f_n$ and is therefore $(\prod_{i=1}^n x_i)^{n-1}$. On the righthand side, the leading degree term of f^{n-1} is also $(\prod_{i=1}^n x_i)^{n-1}$, showing that $\lambda = 1$. Then

$$\det(M) = \frac{1}{f^{n(n-2)}} \cdot \det(G^{\text{adj}}) = \frac{1}{f^{n(n-2)}} \det(G)^{n-1} = \frac{1}{f^{n(n-2)}} f^{(n-1)^2} = f.$$

Note that the entries of M have degree $\leq (n-1)^2 - n(n-2) = 1$, so we can write M as $M_0 + \sum_{i=1}^n x_i M_i$ for some matrices $M_i \in R^{n \times n}$. We claim that $\sum_{i=1}^n x_i M_i = \text{diag}(x_1, \dots, x_n)$.

To see this, first note that a non-principal $(n-1) \times (n-1)$ minor of G involves at most $n-2$ elements from the diagonal of G and therefore has degree $\leq (n-2)(n-1) + (n-2) = n(n-2)$, since the off-diagonal entries of G have degree $\leq n-2$. Therefore the off diagonal entries of M have degree $\leq n(n-2) - n(n-2) = 0$.

Moreover in the expansion of any principal $(n-1) \times (n-1)$ minor of G , there is a *unique* term of degree $(n-1)^2$, namely the product of the leading terms of the diagonal elements, $\prod_{j \neq i} \text{LT}(g_{jj})$. We can therefore take the leading terms (7) to find that

$$\begin{aligned} \sum_{i=1}^n x_i M_i &= \frac{1}{(\text{LT}(f))^{n-2}} \cdot (\text{diag}(\text{LT}(g_{11}), \dots, \text{LT}(g_{nn})))^{\text{adj}} \\ &= \frac{1}{\left(\prod_{j=1}^n x_j\right)^{n-2}} \cdot \left(\prod_{j=1}^n x_j \cdot \text{diag}\left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right)\right)^{\text{adj}} \\ &= \text{diag}(x_1, \dots, x_n). \end{aligned}$$

Finally, for general f , we take a factorization of f into irreducible polynomials $f = \prod_{\ell} f_{\ell}$. By Lemma 3.3, for every i, j , $\Delta_{ij}(f_{\ell})$ has a factorization $m_{ij} m_{ji}$ so into multiaffine polynomials m_{ij} with $\text{Res}_{x_k}(m_{ij}, f) = m_{ik} m_{kj}$. By the arguments above, f_k has a determinantal representation of the correct form. Taking a block diagonal representation of these representations (and permuting the rows and columns if necessary to reorder $x_1, \dots, x_n$) gives a determinantal representation for f . ■

Remark 3.4. In Theorem 3.1 the matrix $G = (g_{ij})_{ij}$ and the corresponding determinantal representation $\text{diag}(x_1, \dots, x_n) + A$ of f satisfy

$$G = (\text{diag}(x_1, \dots, x_n) + A)^{\text{adj}} \text{ and } (\text{diag}(x_1, \dots, x_n) + A) = f^{2-n} G^{\text{adj}}.$$

□

4 Multiaffine algebra for constructing Hermitian factorizations

In this section, we develop an algorithm for constructing factorizations that satisfy the conditions in Theorem 3.1. To do this, we find it most convenient to work in the following level of generality throughout this section. Let S be a unique factorization domain with an automorphic involution $a \mapsto \bar{a}$. We use 0 and 1 to denote the additive and multiplicative identities of S . The map $S \rightarrow S$ given by $a \mapsto \bar{a}$ then must satisfy

$$\overline{\bar{a}} = a, \quad \bar{0} = 0, \quad \bar{1} = 1, \quad \overline{a+b} = \bar{a} + \bar{b} \text{ and } \overline{a \cdot b} = \bar{a} \cdot \bar{b}.$$

for all $a, b \in S$. Let R be the subring of elements fixed by this automorphism, that is $R = \{a \in S : \bar{a} = a\}$.

The example of interest is the ring $S = \mathbb{C}[x_{n+1}, \dots, x_m]$ of polynomials with complex coefficients with the involution given by complex conjugation. In this case the fixed ring is the subring of polynomials whose coefficients are real, $R = \mathbb{R}[x_{n+1}, \dots, x_m]$.

Assumptions 4.1. Let $f \in R[x_1, \dots, x_n]$ satisfy the following:

1. f is irreducible in $R[x_1, \dots, x_n]$,
2. f has degree ≤ 1 in each variable $x_1, \dots, x_n$,
3. the coefficient $\prod_{i=1}^n x_i$ in f is nonzero,
4. for every $1 \leq i < j \leq n$, $\Delta_{ij}(f)$ factors as $g_{ij} \bar{g}_{ij}$ in $S[x_1, \dots, x_n]$, and
5. for every $1 \leq i \leq n$, the partial derivative $\frac{\partial f}{\partial x_i}$ is irreducible in $R[x_1, \dots, x_n]$ up to a constant in R . That is, for any factorization $\frac{\partial f}{\partial x_i} = g \cdot h$ in $R[x_1, \dots, x_n]$, $g \in R$ or $h \in R$.

□

In what follows, we will build up tools to show that under these assumptions Algorithm 1 produces the desired representation of f . We first exploit some properties of multiaffine polynomials. For any disjoint subsets $S, T \subset [n]$, let

$$f_S^T = \prod_{i \in S} \partial_i \cdot f|_{\{x_j=0 : j \in T\}}.$$

Note that if f is multiaffine in $x_1, \dots, x_n$, then for any $1 \leq i < j \leq n$, we have

$$f = x_i f_i + f^i, \quad f_i = x_j f_{ij} + f_i^j, \quad \text{and} \quad f^i = x_j f_j^i + f^{ij}.$$

From this, one can check that the formula for $\Delta_{ij}f$ can be written without x_i, x_j :

$$\Delta_{ij}f = f_i^j \cdot f_j^i - f^{ij} \cdot f_{ij}$$

If in addition we assume that f and all its partial derivatives are irreducible, then $\Delta_{ij}(f)$ will have degree exactly 2 in each variable, as the following lemma shows.

Lemma 4.2. *If f satisfies Assumptions 4.1, then for all $1 \leq i, j \leq n$, $\Delta_{ij}(f)$ is quadratic in each variable x_k for $k \in [n] \setminus \{i, j\}$.* □

Proof. For $1 \leq i < j \leq n$, we write $\Delta_{ij}(f)$ as a quadratic polynomial in the variable x_k :

$$\Delta_{ij}(f) = f_i f_j - f_{ij} f = (f_{ik} x_k + f_i^k)(f_{jk} x_k + f_j^k) - (f_{ijk} x_k + f_{ij}^k)(f_k x_k + f^k),$$

which gives

$$\text{coeff}(\Delta_{ij}(f), x_k^2) = f_{ik} f_{jk} - f_{ijk} f_k = \Delta_{ij}(f_k).$$

If $\Delta_{ij}(f_k) = 0$, then by Proposition 2.3, f_k is reducible, contradicting Assumption 4.1(5). ■

We next use ring maps given by taking resultants with f . For any $i = 1, \dots, n$, define

$$\varphi_i : S[x_1, \dots, x_n] \rightarrow S[x_k : k \neq i] \quad \text{by} \quad \varphi_i(g) = \text{Res}_{x_i}(g, f),$$

as defined in (3). For instance if we restrict to polynomials $g = g_i x_j + g^i$ with degree one in x_i , then

$$\varphi_i(g) = -g_i f^i + g^i f_i.$$

More generally, $\varphi_i(g)$ is obtained from g by substituting $-f^i/f_i$ for x_i and multiplying the result by f_i^d where $d = \deg_i(g)$. First we will start by listing some of the properties of these maps.

Lemma 4.3. *If f satisfies Assumptions 4.1, then, for all $g, h \in S[\mathbf{x}]$, the maps $\varphi_1, \dots, \varphi_n$ satisfy the following:*

1. $\varphi_j(f_i) = \Delta_{ij}(f)$ for all $1 \leq i < j \leq n$,
2. $\varphi_j(\Delta_{ik}(f)) = \Delta_{ij}(f) \Delta_{jk}(f)$ for all distinct $1 \leq i, j, k \leq n$,
3. if $\deg_j(g) = 0$, then $\varphi_j(g \cdot h) = g \cdot \varphi_j(h)$,
4. if $\deg_j(g) > 0$ and $\deg_j(h) > 0$, then $\varphi_j(g \cdot h) = \varphi_j(g) \cdot \varphi_j(h)$,
5. if $\deg_i(g) = \deg_j(g) = 1$ with $sg_j \notin \langle f_j \rangle$ for all $s \in S$, then $\varphi_j \circ \varphi_i(g) = \Delta_{ij}f \cdot \varphi_j(g)$,
6. if $\deg_i(g) = 1$ and $\deg_j(g) = 0$, then $\varphi_j \circ \varphi_i(g) = \Delta_{ij}f \cdot \varphi_j(g) = \Delta_{ij}f \cdot g$, and
7. if $\deg_j(g) = 1$, then $\varphi_j(g) \equiv f_j \cdot g$ modulo $\langle f \rangle$.

□

Proof. We will prove (5), (6), and (7) and all the other properties follow similarly by direct computations. To prove property (5), we write $g = g_{ij} x_i x_j + g_i^j x_i + g_j^i x_j + g^{ij}$, then

$$\begin{aligned} \varphi_j \circ \varphi_i(g) &= \varphi_j(-g_{ij} x_j f^i - g_i^j f^i + g_j^i f_i x_j + g^{ij} f_i) \\ &= \varphi_j \left((-g_{ij} f_j^i + g_j^i f_{ij}) x_j^2 + (-g_{ij} f^{ij} + g_j^i f_i^j + g^{ij} f_{ij} - g_i^j f_j^i) x_j + (g^{ij} f_i^j - g_i^j f^{ij}) \right). \end{aligned}$$

If $\deg_j(g) = 1$ and $g_j \notin \langle f_j \rangle$, then we claim that the coefficient of x_j^2 in $\varphi_i(g)$ is nonzero. To see this, suppose to the contrary that $\text{coeff}(\varphi_i(g), x_j^2) = 0$, meaning $g_j^i f_{ij} = g_{ij} f_j^i$. By assumption, the polynomial $f_j = f_{ij} x_i + f_j^i$ is irreducible in $R[\mathbf{x}]$, up to a constant in R . In particular, the greatest common divisor $s = \gcd(f_{ij}, f_j^i)$ belongs

to S . Then $\frac{1}{s}f_j^i$ and $\frac{1}{s}f_{ij}$ are relatively prime and so the equation above gives that they divide g_j^i and g_{ij} respectively. From this we get that $sg_j \in \langle f_j \rangle$, contradicting Assumption 4.15. Therefore $\text{coeff}(\varphi_i(g), x_j^2) \neq 0$.

Applying the map φ_j to the expression above and simplifying then gives

$$\varphi_j \circ \varphi_i(g) = \Delta_{ij}(f)(-g_j f^j + g^j f_j) = \Delta_{ij}(f)(\varphi_j(g)).$$

Similarly, for (6), suppose that $\deg_i(g) = 1$ and $\deg_j(g) = 0$. The calculation of $\varphi_j \circ \varphi_i(g)$ above holds with $g_{ij} = g_j^i = 0$, giving

$$\varphi_j \circ \varphi_i(g) = \varphi_j \left((g^{ij} f_{ij} - g_j^i f_j^i) x_j + (g^{ij} f_i^j - g_i^j f^{ij}) \right) = (g^{ij} f_{ij} - g_j^i f_j^i)(-f^j) + (g^{ij} f_i^j - g_i^j f^{ij})(f_j) = \Delta_{ij}(f)g.$$

The second equality comes from the observation that, as in the argument above, the coefficient, $g^{ij} f_{ij} - g_j^i f_j^i$, of x_j in $\varphi_i(g)$ is nonzero. The last equality comes from substituting $-f^j = -f_i^j x_i - f^{ij}$ and $f_j = f_{ij} x_i + f_j^i$ and simplifying.

To prove (7), we write g as $g = g_j x_j + g^j$ and we use $f^j = f - f_j x_j$

$$\varphi_j(g) = -g_j f^j + g^j f_j = -g_j(f - f_j x_j) + g^j f_j = -g_j f + f_j g,$$

showing that $\varphi_j(g) \equiv f_j g$ modulo $\langle f \rangle$. ■

Lemma 4.4. *If f satisfies Assumptions 4.1 and $\Delta_{ij}(f) = p\bar{p}$ for some $1 \leq i < j \leq n$, then for every $k \in [n]$ with $k \neq i, j$, there is a factorization of each $\Delta_{ik}(f)$ and $\Delta_{jk}(f)$ into $q\bar{q}$ and $r\bar{r}$, respectively, such that $\varphi_k(p) = qr$. □*

Proof. Since $\Delta_{ik}(f)$ and $\Delta_{jk}(f)$ factor into two conjugates, we can write

$$\Delta_{ik}(f) = a_1 \cdots a_s \cdot \bar{a}_1 \cdots \bar{a}_s \quad \text{and} \quad \Delta_{jk}(f) = b_1 \cdots b_t \cdot \bar{b}_1 \cdots \bar{b}_t$$

where $a_1, \dots, a_s, b_1, \dots, b_t$ are irreducible in $S[x_1, \dots, x_m]$ and multiaffine in $x_1, \dots, x_n$. Then

$$\varphi_k(p)\varphi_k(\bar{p}) = \varphi_k(\Delta_{ij}(f)) = \Delta_{ik}(f)\Delta_{jk}(f) = a_1 \cdots a_s \cdot \bar{a}_1 \cdots \bar{a}_s \cdot b_1 \cdots b_t \cdot \bar{b}_1 \cdots \bar{b}_t.$$

After switching a_i with $\bar{a}_i$ and b_i with $\bar{b}_i$ if necessary, we get

$$\varphi_k(p) = a_1 \cdots a_s \cdot b_1 \cdots b_t = q \cdot r$$

where $q = a_1 \cdots a_s$ and $r = b_1 \cdots b_t$ are multiaffine polynomials such that $\Delta_{ik}(f) = q\bar{q}$ and $\Delta_{jk}(f) = r\bar{r}$ as desired. ■

Lemma 4.5. *If f satisfies Assumptions 4.1 and for some distinct $1 \leq i, j, k \leq n$, the factorizations $\Delta_{ij}(f) = p\bar{p}$, $\Delta_{ik}(f) = q\bar{q}$ and $\Delta_{jk}(f) = r\bar{r}$ satisfy $\varphi_k(p) = qr$, then*

$$\varphi_j(q) = p\bar{r} \quad \text{and} \quad \varphi_i(r) = p\bar{q}.$$

□

Proof. We will prove the first equality and the second holds similarly. First notice that $\deg_k(p) = 1$ and $\deg_j(p) = 0$. Also, $\deg_j(r) = \deg_k(r) = 0$. Then using the properties in Lemma 4.3 we get

$$\varphi_j(q)r = \varphi_j(qr) = \varphi_j \circ \varphi_k(p) = \Delta_{jk}(f) \cdot p.$$

Since $\Delta_{jk}(f) = r\bar{r}$, dividing the above equation by r gives the desired result. ■

The following algorithm gives the desired factorizations of $\Delta_{ij}(f)$ into $g_{ij}\bar{g}_{ij}$ that satisfy the hypothesis of Theorem 3.1 which will in turn give the desired Hermitian determinantal representation in Theorem 5.1

Algorithm 1 Compatible Hermitian factorizations of Rayleigh differences

Input: $f \in R[x_1, \dots, x_n]$ satisfying Assumptions [4.1](#)
Output: Polynomials $\{g_{jk} : 1 \leq j < k \leq n\}$ in $S[x_1, \dots, x_n]$

 Take $g_{12} \in \mathbb{K}[x_1, \dots, x_n]$ so that $\Delta_{12}f = g_{12} \cdot \overline{g_{12}}$.

for $k = 3, k \leq n, k++$ **do**
 $Q_0 := \gcd\{\Delta_{1k}(f), \varphi_k(g_{12}), \dots, \varphi_k(g_{1(k-1)})\}$

 Factor $\Delta_{1k}(f) = p_{k,1} \cdots p_{k,m_k} \cdot \overline{p_{k,1}} \cdots \overline{p_{k,m_k}}$ with $p_{k,j}$ irreducible for all j .

for $j = 1, j \leq m_k, j++$ **do**
if $p_{k,j} \overline{p_{k,j}}$ divides Q_{j-1} **then** $Q_j := Q_{j-1} / \overline{p_{k,j}}$
else $Q_j := Q_{j-1}$
 $g_{1k} := Q_{m_k}$
for $j = 2, j \leq k-1, j++$ **do**
 $g_{jk} := \varphi_k(g_{1j}) / g_{1k}$

Proposition 4.6. *The polynomials $\{g_{ik}\}_{1 \leq i < k \leq n}$ constructed in Algorithm [1](#) satisfy*

- (a) g_{1k} is multiaffine in $x_1, \dots, x_n$ for all $k > 1$,
- (b) $\varphi_k(g_{1i}) = g_{1k} \overline{g_{ik}}$ for all $1 < i < k$, and
- (c) $\Delta_{ik}(f) = g_{ik} \overline{g_{ik}}$ for all $1 \leq i < k$.

□

Proof. (a) This is immediate for $k = 2$. For $2 < k \leq n$, notice that $\Delta_{1k}(f)$ has degree two in $x_1, \dots, x_n$. Let $\ell \in [n] \setminus \{1, k\}$ and let $p_{k,j}, \overline{p_{k,j}}$ be the unique irreducible factors of $\Delta_{1k}(f)$ with degree one in x_ℓ . By construction, g_{1k} divides Q_j , which in turn divides $\Delta_{1k}(f) / \overline{p_{k,j}}$. Since this quotient only has degree one in x_ℓ , g_{1k} must have degree ≤ 1 in x_ℓ .

(b) follows directly from construction.

 (c) We proceed by induction on k . It is trivially true for $k = 2$. For the inductive step, we will prove the claim for $\Delta_{1k}(f)$ and the other cases follow. By construction, $g_{1k} \overline{g_{1k}}$ divides $\Delta_{1k}(f)$. To see this, note that for each $j = 1, \dots, m_k$ in Algorithm [1](#), we can take $q_j = p_{k,j}$ if $p_{k,j}$ divides g_{1k} and $q_j = \overline{p_{k,j}}$ otherwise. Then, by construction, g_{1k} divides $q = \prod_{j=1}^{m_k} q_j$ and $q \cdot \overline{q} = \Delta_{1k}(f)$, showing that $g_{1k} \cdot \overline{g_{1k}}$ divides $\Delta_{1k}(f)$.

 Suppose for the sake of contradiction that $\Delta_{1k}(f) \neq g_{1k} \overline{g_{1k}}$. Then there is some irreducible factor p of $\Delta_{1k}(f)$ such that $p\overline{p}$ does not divide $g_{1k} \overline{g_{1k}}$. We claim that for every $1 < i < k$, either p or $\overline{p}$ divides $\varphi_k(g_{1i})$. By induction, for $1 < i < k$, $g_{1i} \overline{g_{1i}} = \Delta_{1i}(f)$. Applying φ_k gives

$$\varphi_k(g_{1i}) \cdot \varphi_k(\overline{g_{1i}}) = \varphi_k(\Delta_{1i}(f)) = \Delta_{1k}(f) \cdot \Delta_{ik}(f).$$

 Since p is irreducible and divides $\Delta_{1k}(f)$, it must divide either $\varphi_k(g_{1i})$ or $\varphi_k(\overline{g_{1i}}) = \overline{\varphi_k(g_{1i})}$. In the latter case, $\overline{p}$ divides $\varphi_k(g_{1i})$. Since neither p nor its conjugate divide g_{1k} , it follows from the construction that neither p nor $\overline{p}$ divide $Q_0 = \gcd\{\Delta_{1k}(f), \varphi_k(g_{12}), \dots, \varphi_k(g_{1(k-1)})\}$. Hence there exists distinct $2 \leq i, j < k$ such that neither p divide $\varphi_k(g_{1i})$ nor $\overline{p}$ divide $\varphi_k(g_{1j})$. By switching p and $\overline{p}$ if needed, we can assume $i < j$.

 By induction (on k), we know that $\Delta_{1i}(f) = g_{1i} \overline{g_{1i}}$, $\Delta_{1j}(f) = g_{1j} \overline{g_{1j}}$ and $\Delta_{ij}(f) = g_{ij} \overline{g_{ij}}$. Moreover, by (b), $\varphi_j(g_{1i}) = g_{1j} \overline{g_{ij}}$. Lemma [4.5](#) then implies that $\varphi_1(g_{ij}) = g_{1i} \overline{g_{1j}}$ and

$$\Delta_{1k}(f) \varphi_k(g_{ij}) = \varphi_k(\varphi_1(g_{ij})) = \varphi_k(g_{1i} \overline{g_{1j}}) = \varphi_k(g_{1i}) \varphi_k(\overline{g_{1j}}).$$

 Now neither $\varphi_k(g_{1i})$ nor $\varphi_k(\overline{g_{1j}}) = \overline{\varphi_k(g_{1j})}$ is divisible by p while p divides $\Delta_{1k}(f)$ and this gives the desired contradiction. Therefore $\Delta_{1k}(f) = g_{1k} \overline{g_{1k}}$.

 For $1 < i < k$, we calculate that

$$g_{ik} \cdot \overline{g_{ik}} = \frac{\varphi_k(\overline{g_{1i}})}{\overline{g_{1k}}} \cdot \frac{\varphi_k(g_{1i})}{g_{1k}} = \frac{\varphi_k(\overline{g_{1i}} g_{1i})}{\Delta_{1k}(f)} = \frac{\Delta_{1k}(f) \Delta_{ik}(f)}{\Delta_{1k}(f)} = \Delta_{ik}(f).$$

■

Corollary 4.7. *If $f \in R[x_1, \dots, x_n]$ satisfies Assumptions [4.1](#), then there exists a factorization of $\Delta_{ij}(f)$ into $g_{ij} g_{ji}$ such that $g_{ij} \in S[x_1, \dots, x_n]$, $g_{ji} = \overline{g_{ij}}$, and $\varphi_k(g_{ij}) = g_{ik} \overline{g_{kj}}$ for all distinct $1 \leq i, j, k \leq n$.* □

Proof. Let $\{g_{ij} : 1 \leq i < j \leq n\}$ be the polynomials given by Algorithm 1 and for $i < j$ let $g_{ji} = \overline{g_{ij}}$. By Proposition 4.6 $\Delta_{ij}(f) = g_{ij}\overline{g_{ij}} = g_{ji}g_{ij}$. Since $\Delta_{ij}(f)$ is quadratic in each variable $x_1, \dots, x_n$, then g_{ij} is multiaffine in $x_1, \dots, x_n$. We will show that $\varphi_k(g_{ij}) = g_{ik}g_{kj}$ for all distinct i, j, k . Assuming that $i < j < k$ and using Proposition 4.6 we get

$$\begin{aligned} \text{res}_{x_k}(g_{1i}, f) &= \varphi_k(g_{1i}) = g_{1k}\overline{g_{ki}} = g_{1k}g_{ki}, \text{ and} \\ \text{res}_{x_k}(g_{j1}, f) &= \varphi_k(g_{j1}) = \overline{\varphi_k(g_{1j})} = \overline{g_{1k}\overline{g_{kj}}} = g_{jk}g_{k1}. \end{aligned}$$

Multiplying the above two equations and using Properties 4.3 we get

$$\varphi_k(g_{1i}g_{j1}) = \Delta_{1k}(f)g_{ki}g_{jk}.$$

Using Proposition 4.6 again, we know that $\varphi_j(g_{1i}) = g_{1j}g_{ji}$ and Lemma 4.5 implies that $\varphi_1(g_{ji}) = g_{j1}g_{1i}$. Again using Properties 4.3 we find that

$$\Delta_{1k}(f)\varphi_k(g_{ji}) = \Delta_{1k}(f)g_{ki}g_{jk}.$$

Since f is irreducible, $\Delta_{1k}(f)$ is nonzero and thus $\varphi_k(g_{ij}) = \overline{\varphi_k(g_{ji})} = \overline{g_{ki}g_{jk}} = g_{ik}g_{kj}$. Using Lemma 4.5, we get that $\varphi_j(g_{ik}) = g_{ij}g_{jk}$ and $\varphi_i(g_{jk}) = g_{ji}g_{ik}$ as desired. ■

Example 4.8. ($n = 4$). Consider $f \in \mathbb{R}[x_1, x_2, x_3, x_4]$ given by

$$f(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4 + 1.$$

For any distinct $i, j, k, \ell \in [4]$, the Raleigh differences of f with respect to x_i and x_j are

$$\Delta_{ij}(f) = (x_k^2 + 1)(x_\ell^2 + 1) = (x_k - \mathbf{i})(x_k + \mathbf{i})(x_\ell + \mathbf{i})(x_\ell - \mathbf{i}).$$

Using Algorithm 1, we can choose g_{12} as any multiaffine factor of $\Delta_{12}(f)$ of degree two. There are four possibilities $(x_3 \pm \mathbf{i})(x_4 \pm \mathbf{i})$ and one can check that every choice works. We choose $g_{12} = (x_3 + \mathbf{i})(x_4 + \mathbf{i})$ and compute

$$\varphi_3(g_{12}) = \mathbf{i}(x_1 - \mathbf{i})(x_2 - \mathbf{i})(x_4 - \mathbf{i})(x_4 + \mathbf{i}).$$

To choose g_{13} we compute $\gcd(\Delta_{13}(f), \varphi_3(g_{12})) = (x_2 - \mathbf{i})(x_4 - \mathbf{i})(x_4 + \mathbf{i})$. Thus, up to scalar multiples, we have two choices for g_{13} , namely $(x_2 - \mathbf{i})(x_4 + \mathbf{i})$ or $(x_2 - \mathbf{i})(x_4 - \mathbf{i})$. We will choose the first option, giving

$$g_{23} = \overline{(\varphi_3(g_{12})/g_{13})} = -\mathbf{i}(x_1 + \mathbf{i})(x_4 + \mathbf{i}).$$

To find g_{14} , we compute the $\gcd(\Delta_{14}(f), \varphi_4(g_{12}), \varphi_4(g_{13})) = (x_2 - \mathbf{i})(x_3 - \mathbf{i})$ and we get g_{24} and g_{34} similarly. Taking $g_{ji} = \overline{g_{ij}}$ for $j > i$ and $g_{ii} = \partial f / \partial x_i$, the final matrix is $G =$

$$\begin{pmatrix} x_2x_3x_4 - x_2 - x_3 - x_4 & (x_3 + \mathbf{i})(x_4 + \mathbf{i}) & (x_2 - \mathbf{i})(x_4 + \mathbf{i}) & (x_2 - \mathbf{i})(x_3 - \mathbf{i}) \\ (x_3 - \mathbf{i})(x_4 - \mathbf{i}) & x_1x_3x_4 - x_1 - x_3 - x_4 & -\mathbf{i}(x_1 + \mathbf{i})(x_4 + \mathbf{i}) & -\mathbf{i}(x_1 + \mathbf{i})(x_3 - \mathbf{i}) \\ (x_2 + \mathbf{i})(x_4 - \mathbf{i}) & \mathbf{i}(x_1 - \mathbf{i})(x_4 - \mathbf{i}) & x_1x_2x_4 - x_1 - x_2 - x_4 & -\mathbf{i}(x_1 + \mathbf{i})(x_2 + \mathbf{i}) \\ (x_2 + \mathbf{i})(x_3 + \mathbf{i}) & \mathbf{i}(x_1 - \mathbf{i})(x_3 + \mathbf{i}) & \mathbf{i}(x_1 - \mathbf{i})(x_2 - \mathbf{i}) & x_1x_2x_3 - x_1 - x_2 - x_3 \end{pmatrix}.$$

Now we compute the adjugate matrix of G and divide its entries by f^2 we get

$$M = \frac{1}{f^2} G^{\text{adj}} = \begin{pmatrix} x_1 & -1 & -1 & -1 \\ -1 & x_2 & \mathbf{i} & \mathbf{i} \\ -1 & -\mathbf{i} & x_3 & \mathbf{i} \\ -1 & -\mathbf{i} & -\mathbf{i} & x_4 \end{pmatrix} = D^{-1} \begin{pmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & x_3 & \mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & x_4 \end{pmatrix} D$$

where $D = \text{diag}(-1, 1, 1, 1)$. Indeed one can check that $\det(M) = f$.

If we instead choose $g_{12} = (x_3 + \mathbf{i})(x_4 - \mathbf{i})$, then $\varphi_3(g_{12}) = \mathbf{i}(x_1 - \mathbf{i})(x_2 - \mathbf{i})(x_4 - \mathbf{i})^2$. In this case, $\gcd(\Delta_{13}(f), \varphi_3(g_{12})) = (x_2 - \mathbf{i})(x_4 - \mathbf{i})$, which must be g_{13} . The other entries are also determined. Running over all choices gives a total of six representations of f , up to diagonal equivalence, namely the following three matrices and their transposes:

$$\left\{ \begin{pmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & x_3 & \mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & x_4 \end{pmatrix}, \begin{pmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & -\mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & x_3 & \mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & x_4 \end{pmatrix}, \begin{pmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & x_3 & -\mathbf{i} \\ 1 & -\mathbf{i} & \mathbf{i} & x_4 \end{pmatrix} \right\}.$$

The six matrices A obtained by specializing $x_i = 0$ for all i all have principal minors given by the coefficients of f . Since f is symmetric under the action of the symmetric group, S_4 . One could also consider the orbit of these matrices under conjugation by permutation matrices $A \mapsto P^{-1}AP$. One can check that, up to diagonal equivalence, these six matrices belong to a single orbit under this action, as predicted by [28, Corollary 3.7] □

5 Hermitian determinantal representations

Let $\mathbb{K}$ be a field with an automorphism $a \mapsto \bar{a}$ of order two. Let $\mathbb{F}$ be the fixed field of this automorphism. We call a matrix $A \in \text{Mat}_n(\mathbb{K})$ Hermitian if $A = \bar{A}^T$.

5.1 Consequences of Algorithm 1

Theorem 5.1. *Let $f \in \mathbb{F}[x_1, \dots, x_m]$ be a polynomial of total degree $n \leq m$ that is multiaffine in $x_1, \dots, x_n$ and coefficient of $x_1 \cdots x_n$ equal to one. There exist Hermitian matrices $A_{n+1}, \dots, A_m, A_0$ so that*

$$f = \det \left(\text{diag}(x_1, \dots, x_n) + \sum_{j=n+1}^m x_j A_j + A_0 \right)$$

if and only if for all $i, j \in [n]$, $\Delta_{ij}(f)$ is a Hermitian square in $\mathbb{K}[x_1, \dots, x_m]$. $\square$

Lemma 5.2. *Let $\mathbb{F}$ be an infinite field and $f \in \mathbb{F}[x_1, \dots, x_m]$ be irreducible, multiaffine in the variables $x_1, \dots, x_n$ with $n \leq m$ and have coefficient of $x_1 \cdots x_n$ equal to 1. Let $R = \mathbb{F}[x_{n+1}, \dots, x_m]$. For a generic element $\gamma \in \text{SL}_2(\mathbb{F})^n$, the derivatives $\frac{\partial}{\partial x_j}(\gamma \cdot f)$ are irreducible in $R[x_1, \dots, x_n]$ up to a constant in R for all $j = 1, \dots, n$ and the coefficient of $\prod_{i=1}^n x_i$ is nonzero. $\square$*

As in Assumptions 4.15, we say that an element $f \in R[x_1, \dots, x_n]$ is irreducible up to a constant in R if for any factorization $f = g \cdot h$ with in $g, h \in R[x_1, \dots, x_n]$, $g \in R$ or $h \in R$.

Proof. Consider $\gamma_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{F})$ acting on x_j . Then $\partial_j(\gamma \cdot f) = af_j + cf^j$ where $f = f_j x_j + f^j$. Consider the set

$$\mathcal{X} = \{(a, c) \in \mathbb{F}^2 : af_j + cf^j \text{ is reducible in } R[x_1, \dots, x_n] \text{ up to a constant in } R\}.$$

Suppose that the multidegree of f in $x_1, \dots, x_m$ is given by $\mathbf{d} \in \mathbb{N}^m$. By assumption, $\mathbf{d}_i = 1$ for $i = 1, \dots, n$. Note that $\mathcal{X}$ is contained in the union $\bigcup_{\mathbf{e}} \mathcal{X}_{\mathbf{e}}$ where

$$\mathcal{X}_{\mathbf{e}} = \{(a, c) \in \mathbb{F}^2 : af_j + cf^j \in \mathbb{F}^{\text{alg}}[x_1, \dots, x_m]_{\leq \mathbf{e}} \cdot \mathbb{F}^{\text{alg}}[x_1, \dots, x_m]_{\leq \mathbf{d}-\mathbf{e}}\}$$

and the union is taken over all vectors $\mathbf{e} \in \mathbb{N}^m$ that are coordinate-wise $\leq \mathbf{d}$ with the property that $\mathbf{e}_i = 1$ and $\mathbf{e}_k = 0$ for some $i, k \in [n] \setminus \{j\}$. Here $\mathbb{F}^{\text{alg}}$ denotes the algebraic closure of $\mathbb{F}$. To see this, suppose $(a, c) \in \mathcal{X}$, meaning $af_j + cf^j = g \cdot h$ where $g, h \in R[x_1, \dots, x_n]$ are not elements of R . In particular, for some $i, k \in [n] \setminus \{j\}$, $\deg_i(g) > 0$ and $\deg_k(h) > 0$. Since $af_j + cf^j$ has degree at most one in each of x_i and x_k , it follows that $\deg_i(g) = \deg_k(h) = 1$ and $\deg_k(g) = \deg_i(h) = 0$. Taking $\mathbf{e} \in \mathbb{N}^m$ to be the multidegree of g gives $g \in \mathbb{F}[x_1, \dots, x_m]_{\leq \mathbf{e}}$ and $h \in \mathbb{F}[x_1, \dots, x_m]_{\leq \mathbf{d}-\mathbf{e}}$. Then $(a, c) \in \mathcal{X}_{\mathbf{e}}$.

By the projective elimination theorem, the image $\mathbb{F}^{\text{alg}}[x_1, \dots, x_m]_{\leq \mathbf{e}} \cdot \mathbb{F}^{\text{alg}}[x_1, \dots, x_m]_{\leq \mathbf{d}-\mathbf{e}}$ is Zariski-closed in the vectorspace $\mathbb{F}^{\text{alg}}[x_1, \dots, x_m]_{\leq \mathbf{d}}$. Intersecting with the $\mathbb{F}$ -subspace spanned by $\{f_j, f^j\}$ shows that $\mathcal{X}_{\mathbf{e}}$ and hence $\bigcup_{\mathbf{e}} \mathcal{X}_{\mathbf{e}}$ is Zariski-closed in $(\mathbb{F}^2)^2$. Therefore this union is either all of $\mathbb{F}^2$ or is an algebraic set of codimension ≥ 1 . Suppose, for the sake of contradiction, that it is all of $\mathbb{F}^2$. Then there exists some $\mathbf{e}$ for which $\mathcal{X}_{\mathbf{e}} = \mathbb{F}^2$. By assumption, there are $i, k \in [n]$ so that $\mathbf{e}_i = 1$ and $(\mathbf{d} - \mathbf{e})_k = 1$. By Proposition 2.3, it follows that for all $a, c \in \mathbb{F}$, $\Delta_{ik}(af_j + cf^j)$ is identically zero. For $c = 1$, this corresponds to the evaluation of $\Delta_{ik}(f)$ at $x_j = a$. It follows that the polynomial $\Delta_{ik}(f)$ is identically zero (see e.g. [3]). Proposition 2.3 then implies that f factors as the product of two elements in $R[x_1, \dots, x_n]$ neither of which belong to R . $\blacksquare$

Proof of Theorem 5.1 ($\Rightarrow$) By Theorem 3.1, for all $i \neq j$, $\Delta_{ij}(f)$ factors as $g_{ij}g_{ji}$ where g_{ij} the (i, j) th entry of the adjugate matrix $(\text{diag}(x_1, \dots, x_n) + A)^{\text{adj}}$. Since A is Hermitian, $g_{ji} = \bar{g}_{ij}$, showing that $\Delta_{ij}(f) = g_{ij}\bar{g}_{ij}$.

($\Leftarrow$) First assume that $\mathbb{F}$ is an infinite field and that f is irreducible in $\mathbb{F}[x_1, \dots, x_m]$. Let $R = \mathbb{F}[x_{n+1}, \dots, x_m]$. By Lemma 5.2 there exists a generic $\gamma \in \text{SL}_2(\mathbb{F})^n$ such that the partial derivatives of $\gamma \cdot f$ are all irreducible in $R[x_1, \dots, x_n]$ and the coefficient of $x_1 \cdots x_n$ in $\gamma \cdot f$ is nonzero. Then by Corollary 4.7, there exists $\{g_{ij}\}_{1 \leq i \neq j \leq n}$ with g_{ij} in $\mathbb{K}[x_\ell : \ell \neq i, j]$ satisfying $\Delta_{ij}(\gamma \cdot f) = g_{ij}g_{ji}$, $g_{ji} = \bar{g}_{ij}$, and $\text{res}_{x_k}(g_{ij}, \gamma \cdot f) = g_{ik}g_{kj}$ for all distinct $i, j, k \in [n]$. Acting by γ^{-1} on $\gamma \cdot f$ and using Proposition 2.4 we get $h_{ij} = \gamma^{-1} \cdot g_{ij}$ such that $\Delta_{ij}(f) = h_{ij}\bar{h}_{ij}$ and $\text{res}_{x_k}(h_{ij}, f) = h_{ik}h_{kj}$ and thus using Theorem 3.1 we get a determinantal representation of f over $\mathbb{K}$ and since $h_{ji} = \bar{h}_{ij}$, the matrix will be Hermitian.

Now suppose that f is reducible. Let g be an irreducible factor of f with $\deg_i(g) = 1$ for $i \in I \subseteq [n]$ and $\deg_j(g) = 0$ for $j \in [n] \setminus I$ where the coefficient of $\prod_{i \in I} x_i$ equals one. By Lemma 3.3, for every $i, j \in I$, $\Delta_{ij}(g)$ is a Hermitian square. Therefore g has a determinantal representation of the correct form, $g = \det(\text{diag}(x_i : i \in$

$I) + \sum_{j=n+1}^m x_j A_{ij} + A_{i0}$). Taking a block diagonal representation of these representations (and permuting the rows and columns to reorder $x_1, \dots, x_n$) gives a determinantal representation for f .

Now suppose that $\mathbb{F}$ is a finite field. Consider the transcendental extension of $\mathbb{F}$ to $\mathbb{F}(t)$ and of $\mathbb{K}$ to $\mathbb{K}(t)$. Then by the arguments above, $f = \det(\text{diag}(x_1, \dots, x_n) + A(t))$ for some Hermitian matrix $A(t) \in \text{Mat}_n(\mathbb{K}(t))$. The (i, j) th entry of $A(t)$ can be written as $a_{ij} = \frac{p_{ij}}{q_{ij}}$ where $p_{ij}, q_{ij} \in \mathbb{K}[t]$ are relatively prime and the polynomial q_{ij} is nonzero. Specializing to $t = 0$ will give a determinantal representation of f over $\mathbb{K}$. To do this, we need to check that $q_{ij}(0)$ is nonzero for all i, j . If $a_{ij} = 0$, then we can take $p_{ij} = 0$ and $q_{ij} = 1$. Suppose that for some $i, j \in [n]$, p_{ij} is nonzero and $q_{ij}(0) = 0$. Then t divides q_{ij} and so also divides $\overline{q_{ij}}$. Notice that

$$a_{ii} = \text{coeff} \left(f, \prod_{k \neq i} x_k \right) \quad \text{and} \quad a_{ij}a_{ji} = a_{ii}a_{jj} - \text{coeff} \left(f, \prod_{k \neq i, j} x_k \right)$$

are both in $\mathbb{F}$ and hence $p_{ij}\overline{p_{ij}} = r q_{ij}\overline{q_{ij}}$ for some $r \in \mathbb{F}^*$. We get the desired contradiction by noticing that t^2 divides the right-hand side of the equation, while it does not divide the left-hand side since p_{ij} and q_{ij} are relatively prime. Therefore we can specialize both sides of the equation $f = \det(\text{diag}(x_1, \dots, x_n) + A(t))$ to $t = 0$, which gives a Hermitian determinantal representation of f . ■

Example 5.3. Consider the polynomial $f = x_1x_2x_3 + x_1 + x_2 + x_3 + 1$ over the field $\mathbb{F} = \mathbb{F}_2$. The Rayleigh difference $\Delta_{12}(f) = x_3^2 + x_3 + 1$ does not factor in $\mathbb{F}_2[x_3]$, showing that the coefficient vector of f is not in the image of $\text{Mat}_3(\mathbb{F}_2)$ under the principal minor map.

Consider the quadratic extension $\mathbb{K} = \mathbb{F}_2[\alpha]/\langle \alpha^2 + \alpha + 1 \rangle$. The map $\alpha \mapsto 1 + \alpha$ extends to an automorphic involution on $\mathbb{K}$ that fixes $\mathbb{F}_2$. Over $\mathbb{K}$, the Rayleigh differences factor into multiaffine polynomials, namely $\Delta_{ij}(f) = (x_k + \alpha)(x_k + 1 + \alpha)$, for distinct i, j, k . As then guaranteed by Theorem 5.1, f has a Hermitian determinantal representation over $\mathbb{K}$:

$$f = \det \begin{pmatrix} x_1 & 1 + \alpha & 1 + \alpha \\ \alpha & x_2 & 1 + \alpha \\ \alpha & \alpha & x_3 \end{pmatrix}.$$

□

Corollary 5.4. Let $f \in \mathbb{R}[x_1, \dots, x_m]$ be a polynomial of total degree $n \leq m$ that is multiaffine in $x_1, \dots, x_n$ and coefficient of $x_1 \cdots x_n$ equals to one. There exist Hermitian matrices $A_{n+1}, \dots, A_m, A_0$ so that

$$f = \det \left(\text{diag}(x_1, \dots, x_n) + \sum_{j=n+1}^m x_j A_j + A_0 \right)$$

if and only if for all $i, j \in [n]$, $\Delta_{ij}(f)$ factors as $g_{ij}\overline{g_{ij}}$ for $g_{ij} \in \mathbb{C}[x_1, \dots, x_m]$. □

This provides a partial converse to [33, Corollary 4.3], which states that if some power of a polynomial f has a definite determinantal representation, then for all i, j , the Rayleigh difference $\Delta_{ij}(f)$ is a sum of squares. In particular, Hermitian representations of f give real symmetric determinantal representations of f^2 . We might hope for the following.

Conjecture 5.5. If $f \in \mathbb{R}[x_1, \dots, x_m]$ is multiaffine in $x_1, \dots, x_n$ with $n \leq m$, and coefficient of $x_1 \cdots x_n$ is nonzero, then some power of f has a definite real symmetric determinantal representation if and only if for all i, j , $\Delta_{ij}(f)$ is a sum of squares in $\mathbb{R}[x_1, \dots, x_m]$. □

5.2 Other multiaffine determinantal representations

In this section we restrict ourselves to fields and consider the set of multiaffine determinantal polynomials of the form

$$f(\mathbf{x}) = \lambda \det(V \text{diag}(x_1, \dots, x_m) V^* + W) = \lambda \det \left(\sum_{i=1}^m x_i v_i v_i^* + W \right) \quad (8)$$

for some $\lambda \in \mathbb{F}$, some matrix $V = (v_1, \dots, v_m) \in \mathbb{K}^{n \times m}$ and some $n \times n$ Hermitian matrix W . Note that when we take V to be the $n \times n$ identity matrix and $\lambda = 1$, this is the principal minor polynomial f_W . When $n < m$, the coefficient of $x_1 \cdots x_m$ in f is necessarily zero.

Theorem 5.6. A polynomial $f \in \mathbb{F}[\mathbf{x}]_{\text{MA}}$ has a determinantal representation [8] if and only if for all $i, j \in [n]$, $\Delta_{ij}f$ is a Hermitian square in $\mathbb{K}[\mathbf{x}]$. Moreover, one can always take a representation of size $n = \deg(f)$. □

Proof. ($\Rightarrow$) Without loss of generality, we show that $\Delta_{12}(f)$ is a Hermitian square. First suppose v_1 and v_2 are linearly dependent, i.e. let $v_1 = \alpha v_2$ for some $\alpha \in \mathbb{K}$. Then we get $v_1 v_1^* = \alpha \bar{\alpha} v_2 v_2^*$ and $f(x_1, \dots, x_m) = f(0, \alpha \bar{\alpha} x_1 + x_2, x_3, \dots, x_m)$. Taking partial derivatives shows that $\frac{\partial f}{\partial x_1} = \alpha \bar{\alpha} \frac{\partial f}{\partial x_2}$ and that $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$. Then $\Delta_{12}(f) = (\alpha \frac{\partial f}{\partial x_2})(\bar{\alpha} \frac{\partial f}{\partial x_2})$ is a Hermitian square.

If v_1 and v_2 are linearly independent, then there is an invertible matrix $U \in \mathbb{K}^{n \times n}$ with $Uv_1 = e_1$ and $Uv_2 = e_2$. Then

$$|\det(U)|^2 f = \lambda \det \left(U \left(\sum_{i=1}^m x_i v_i v_i^* + W \right) U^* \right) = \lambda \det \left(\text{diag}(x_1, x_2, \mathbf{0}) + \sum_{i=3}^m x_i \tilde{v}_i \tilde{v}_i^* + \tilde{W} \right).$$

where $\tilde{v}_i = Uv_i$ and $\tilde{W} = UWU^*$. These matrices are still Hermitian and so by equation (6), $\Delta_{12}(f)$ is Hermitian square.

($\Leftarrow$) Let $d = \deg(f)$. We can assume, without loss of generality, that the coefficient of $x_1 \cdots x_d$ in f is nonzero. Moreover since the set of polynomials of the form (8) is invariant under scaling, we can assume that this coefficient equals one. By Theorem 5.1 there are Hermitian matrices $A_0, A_{d+1}, \dots, A_m$ so that

$$f = \det \left(\text{diag}(x_1, \dots, x_d) + \sum_{j=d+1}^m x_j A_j + A_0 \right).$$

We take $W = A_0$. By Lemma 5.7 below, for every $k = n+1, \dots, m$, the rank of A_k equals the degree of f in x_k , which is one. It remains to show that the matrix A_k has the form $v_k v_k^*$ for some $v_k \in \mathbb{K}^d$.

By homogenizing and specializing variables to zero, it suffices to consider polynomials of the form $f = \det(\text{diag}(x_1, \dots, x_d) + x_{d+1}A)$ where $A \in \mathbb{K}^{n \times n}$ is Hermitian and rank-one. Then f can be written as $\prod_{i=1}^d x_i + \sum_{j=1}^d A_{jj} \prod_{i \in [d] \setminus \{j\}} x_i$, where A_{jj} is the j th entry of A . Then for $j = 1, \dots, d$,

$$\Delta_{j(d+1)}(f) = f_j^{d+1} f_{d+1}^j - f_{j(d+1)} f^{j(d+1)} = \left(\prod_{i=1}^d x_i \right) \left(A_{jj} \prod_{i \in [d] \setminus \{j\}} x_i \right) = A_{jj} \left(\prod_{i \in [d] \setminus \{j\}} x_i \right)^2.$$

By assumption, $\Delta_{j(d+1)}(f)$ is a Hermitian square, and so we see that $A_{jj} = \alpha_j \bar{\alpha}_j$ for some $\alpha_j \in \mathbb{K}$. Since A has rank one, we can write it as $\lambda u u^*$ for some $\lambda \in \mathbb{F}^*$ and $u \in \mathbb{K}^n$. If $u_j \neq 0$, then $\lambda u_j \bar{u}_j = \alpha_j \bar{\alpha}_j$, meaning that $\lambda = \beta \bar{\beta}$ for $\beta = \alpha_j / u_j$. It follows that $A = v v^*$ for $v = \beta u$. ■

Lemma 5.7. *If $f = \det(\text{diag}(x_1, \dots, x_n) + \sum_{j=n+1}^m x_j A_j + A_0)$ where $A_j \in \mathbb{K}^{n \times n}$ are Hermitian. Then the rank of A_j equals the degree of f in the variable x_j .* □

Proof. The bound $\deg_j(f) \leq \text{rank}(A_j)$ follows from the Laplace expansion of the determinant. To see equality, it suffices to take $j = m = n+1$ and $A_0 = 0$. Let f_A be the polynomial $f_A = \det(\text{diag}(x_1, \dots, x_n) + A)$ where $A \in \mathbb{K}^{n \times n}$ is Hermitian. Then $f = \sum_{S \subseteq [n]} A_S \mathbf{x}^{[n] \setminus S} y^{|S|}$ equals the homogenization of f_A . From this we see that the degree of f in the variable y equals the size of the largest nonzero *principal* minor of A . By the so-called Principal Minor Theorem [31, Strong PMT 2.9], this coincides with the size of the largest nonzero minor of A , i.e. $\text{rank}(A)$. Therefore for a general polynomial $f = \det(\text{diag}(x_1, \dots, x_n) + \sum_{j=n+1}^m x_j A_j + x_0 A_0)$, the restriction to $x_k = 0$ for $k \in \{n+1, \dots, m\} \setminus \{j\}$ and $x_0 = 0$ has degree $\text{rank}(A_j)$ in x_j , showing that $\deg_j(f) \geq \text{rank}(A_j)$ ■

This immediately gives the invariance of the set of determinantal polynomials.

Corollary 5.8. *The set of polynomials in $\mathbb{F}[\mathbf{x}]_{\text{MA}}$ with a determinantal representation (8) is invariant under the action of $\text{SL}_2(\mathbb{F})^n \rtimes S_n$.* □

Proof. By Corollary 2.5, for any $\gamma \in \text{SL}_2(\mathbb{F})^n$, $\Delta_{ij}(\gamma \cdot f) = \gamma \cdot \Delta_{ij}(f)$. If $\Delta_{ij}(f)$ is a Hermitian square $g\bar{g}$ with $g \in \mathbb{K}[\mathbf{x}]$ then so is $\Delta_{ij}(\gamma \cdot f) = (\gamma \cdot g)(\overline{\gamma \cdot g})$. ■

6 Determinantal Stable Polynomials

In this section we consider polynomials over $\mathbb{R}$ and $\mathbb{C}$ and show that any real stable multiaffine polynomial with a complex linear determinantal representation has a definite Hermitian determinantal representation (Theorem 6.4). Moreover, if the original polynomial is irreducible, then the matrix is diagonally similar to a Hermitian one (Theorem 6.6).

We build up to the proofs of these statements with a series of useful lemmas.

Lemma 6.1. *Let $f \in \mathbb{R}[x_1, \dots, x_m]$ be multiaffine in the variables $x_1, \dots, x_n$ for some $n \leq m$ with coefficient of $x_1 \cdots x_n$ equals to one. If f is irreducible, then for a generic element $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$, $\partial^S(\gamma \cdot f)$ is irreducible for every $S \subset [n]$.* $\square$

Proof. For each $S \subset [n]$, the set of $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$ for which $\partial^S(\gamma \cdot f)$ is irreducible is Zariski-open. Therefore it suffices to show that this set is nonempty for each $S \subset [n]$. Then the intersection of these nonempty, Zariski-open sets will be nonempty and Zariski open.

We will proceed by induction on $|S|$. For $|S| = 0$, this is immediate, so suppose that $|S| \geq 1$ and let $i \in S$. Note that $\partial^S(f) = \partial_i(\partial^{S \setminus \{i\}} f)$. By induction, for generic $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$, $\partial^{S \setminus \{i\}}(\gamma \cdot f)$ is irreducible. Moreover, its coefficient of $\prod_{j \in ([n] \setminus S) \cup \{i\}} x_j$ is nonzero. Therefore, up to a scalar multiple, $\partial^{S \setminus \{i\}}(\gamma \cdot f)$ satisfies the hypothesis of Lemma 5.2, and hence for generic $\tilde{\gamma} \in \mathrm{SL}_2(\mathbb{R})$ acting on the i th coordinate,

$$\partial_i(\tilde{\gamma} \cdot \partial^{S \setminus \{i\}}(\gamma \cdot f)) = \partial^S(\tilde{\gamma} \cdot \gamma \cdot f)$$

is irreducible. Here we use that $\tilde{\gamma}$ commutes with the differential operator $\partial^{S \setminus \{i\}}$, since $\tilde{\gamma}$ acts as the identity in the coordinates indexed by elements of $S \setminus \{i\}$. It follows that for a generic element $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$, $\partial^S(\gamma \cdot f)$ is irreducible. $\blacksquare$

Lemma 6.2. *If $g = ax_1^2 + bx_1 + c$ is nonnegative on $\mathbb{R}^m$ where $a, b, c \in \mathbb{R}[x_2, \dots, x_m]$, then the polynomial a is nonnegative on $\mathbb{R}^{m-1}$.* $\square$

Proof. Fix $\mathbf{p} \in \mathbb{R}^{m-1}$ and consider the specialization $g(x_1, \mathbf{p}) = a(\mathbf{p})x_1^2 + b(\mathbf{p})x_1 + c(\mathbf{p})$ in $\mathbb{R}[x_1]$. Since g is globally nonnegative on $\mathbb{R}^m$, $g(x_1, \mathbf{p})$ is nonnegative on $\mathbb{R}$ and so its leading coefficient $a(\mathbf{p})$ must be nonnegative. $\blacksquare$

Lemma 6.3. *Suppose $g, h \in \mathbb{C}[x_1, \dots, x_m]$ are multiaffine in $x_1, \dots, x_n$ and $\partial^{[n]}g$ and $\partial^{[n]}h$ are nonzero polynomials in $x_{n+1}, \dots, x_m$ of total degree at most one. If the product $g \cdot h$ has real coefficients and is nonnegative as a function on $\mathbb{R}^m$, then h is a positive scalar multiple of $\bar{g}$, i.e. $h = \lambda \bar{g}$ for some $\lambda \in \mathbb{R}_{>0}$.* $\square$

Proof. ($n = 0$) Let $g = a + ib$ and $h = c + id$ for some $a, b, c, d \in \mathbb{R}[x_1, \dots, x_m]$. Since $g \cdot h \in \mathbb{R}[x_1, \dots, x_m]$, we see that $ad = -bc$. Note that if $b = 0$, then $d = 0$ and so both g and h are real. In order for $g \cdot h$ to be nonnegative on $\mathbb{R}^n$, we must have $h = \lambda \cdot g$ for some $\lambda \in \mathbb{R}_{>0}$. The case $d = 0$ follows similarly.

Otherwise, since g and h are linear and thus irreducible, either $a = \lambda b$ and $c = -\lambda d$ or $a = \lambda c$ and $b = -\lambda d$ for some nonzero $\lambda \in \mathbb{R}$. In the first case, $g = (\lambda + i)b$ and $h = (-\lambda + i)d = (\lambda - i)(-d)$ and thus $g \cdot h = (\lambda^2 + 1)(-b \cdot d) \geq 0$ on $\mathbb{R}^n$. Thus $-d = \mu b$ for some $\mu \in \mathbb{R}_{>0}$. It follows that $h = (\lambda - i)(\mu b) = \mu \bar{g}$. The second case gives $g = \lambda h$. Since $g \cdot h = \lambda h \cdot \bar{g}$ is nonnegative on $\mathbb{R}_{>0}^n$, we conclude $\lambda > 0$, as desired.

($n \geq 1$) Now suppose $n \geq 1$ and write $g = g_n x_n + g^n$ and $h = \bar{h}_n x_n + h^n$. Since $g \cdot h$ is real and nonnegative, so is its coefficient $g_n^2, g_n \cdot h_n$. In particular, g_n, h_n satisfy the hypothesis of the theorem and so by induction, $h_n = \lambda \bar{g}_n$ for some $\lambda \in \mathbb{R}_{>0}$. Moreover, for every $\mathbf{a} \in \mathbb{R}^{m-1}$ with $g_n(\mathbf{a}) \neq 0$, the roots (in x_n) of the specialization of $g \cdot h$ at $\mathbf{x} = \mathbf{a}$ come in complex conjugate pairs. It follows that $-h^n/h_n = -\bar{g}^n/\bar{g}_n$ as rational functions in $\mathbb{C}(x_k : k \neq n)$. Together with $h_n = \lambda \bar{g}_n$, this gives that $h = \lambda \bar{g}$. Moreover, since $g \cdot h = \lambda \cdot g \cdot \bar{g}$ is nonnegative on $\mathbb{R}^n$, we see that $\lambda > 0$. $\blacksquare$

Theorem 6.4. *Let $f \in \mathbb{R}[x_1, \dots, x_m]$ be stable and complex determinantal, i.e.*

$$f = \det \left(\mathrm{diag}(x_1, \dots, x_n) + \sum_{j=n+1}^m A_j x_j + A_0 \right)$$

for some $n \times n$ complex matrices A_j . Then there exists Hermitian matrices $B_0, B_{n+1}, \dots, B_m$ for which

$$f = \det \left(\mathrm{diag}(x_1, \dots, x_n) + \sum_{j=n+1}^m B_j x_j + B_0 \right).$$

$\square$

Proof. First suppose f is irreducible. By Lemma 6.1, there is $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$, such that $\partial^S(\gamma \cdot f)$ is irreducible for all $S \subset [n]$. By Proposition 2.2, we can replace f by $\gamma \cdot f$, and thereby assume that all the coefficients of $\prod_{k \in [n] \setminus \{i,j\}} x_k^2$ in the polynomials $\Delta_{ij}(\gamma \cdot f)$ are non-zero. To see this, notice that by induction on n , we can prove that

$$\mathrm{coeff} \left(\Delta_{ij}(f), \prod_{k \in [n] \setminus \{i,j\}} x_k^2 \right) = \Delta_{ij} \left(\partial^{[n] \setminus \{i,j\}}(f) \right).$$

If this coefficient is zero, then Lemma 3.3 implies that $\partial^{[n] \setminus \{i,j\}}(f)$ is reducible.

Let $i < j \in [n]$. Since f is determinantal, by Theorem 3.1, the polynomial $\Delta_{ij}(f)$ factors as $g_{ij} \cdot g_{ji}$ where g_{ij}, g_{ji} are multiaffine in $\{x_k : k \in [n] \setminus \{i,j\}\}$ and have total degree $\leq n-1$. In particular, the coefficient of $\prod_{k \in [n] \setminus \{i,j\}} x_k$ in both g_{ij} and g_{ji} has degree ≤ 1 in $x_{n+1}, \dots, x_m$. By the arguments above we can assume this coefficient is nonzero. Since f is real stable, $\Delta_{ij}(f)$ is also globally nonnegative on $\mathbb{R}^n$ [9]. Therefore by Lemma 6.3, $g_{ji} = \lambda \overline{g_{ij}}$ for some g_{ij} . It follows that $\Delta_{ij}(f)$ factors as a Hermitian square $h_{ij} \cdot \overline{h_{ij}}$ where $h_{ij} = \sqrt{\lambda} g_{ij}$. Theorem 5.1 then gives the desired Hermitian determinantal representation.

Now suppose f is reducible, say $f = f_1 \cdots f_r$ where each factor f_k is irreducible and multiaffine in the variables x_i for $i \in I_k \subset [n]$. Each factor is stable. Moreover, by Lemma 3.3, $\Delta_{ij}(f_k)$ is either zero or factors as a product of two polynomials that are multiaffine in $\{x_\ell : \ell \in I_k\}$ and with total degree $\leq |I_k| - 1$. Since f_k is irreducible, the arguments above show that for every $i, j \in I_k$, $\Delta_{ij}(f_k)$ is a Hermitian square, from which it follows that $\Delta_{ij}(f) = \Delta_{ij}(f_k) \cdot \prod_{\ell \neq k} f_\ell^2$ is a Hermitian square. Theorem 5.1 then gives the desired Hermitian determinantal representation. ■

Remark 6.5. Theorem 6.4 cannot hold for arbitrary real stable polynomials. For example, consider f to be the basis generating polynomial of the Vámos matroid, defined in [10]. It was shown by Wagner and Wei [47] that f is stable. By the theory of matrix factorizations, some power f^r of f has a complex linear determinantal representation (see [46, §3.3]). This power is necessarily stable, but as shown by Brändén [10], f^r does not have a definite Hermitian determinantal representation. □

When f is reducible, one can easily construct determinantal representations of f that are not Hermitian by taking block upper triangular representations. For example, $x_1 x_2$ equals $\det \begin{pmatrix} x_1 & 1 \\ 0 & x_2 \end{pmatrix}$. However, when f is irreducible and real stable, we see that all complex linear determinantal representations are Hermitian, up to conjugation by diagonal matrices.

Theorem 6.6. Let $f \in \mathbb{R}[x_1, \dots, x_m]$ be stable, irreducible, and complex determinantal, i.e.

$$f = \det \left(\mathrm{diag}(x_1, \dots, x_n) + \sum_{j=n+1}^m A_j x_j + A_0 \right)$$

for some $n \times n$ complex matrices A_j . Then there exists a real diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $D^{-1} A_j D$ is Hermitian for all j . □

Proof. By Lemma 6.1, there exists $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$, such that $\partial^S(\gamma \cdot f)$ is irreducible for all $S \subset [n]$. By Proposition 2.2, we can replace f by $\gamma \cdot f$, and thereby assume that all the coefficients of $\prod_{k \in [n] \setminus \{i,j\}} x_k^2$ in the polynomials $\Delta_{ij}(\gamma \cdot f)$ are non-zero, as in the proof of Theorem 6.4.

Let $A(x) = \sum_{k=n+1}^m A_k x_k + A_0$ and let $a_{ij} \in \mathbb{C}[x_{n+1}, \dots, x_m]$ denote the (i, j) th entry of $A(x)$. Then the coefficient of $\prod_{k \in [n] \setminus \{i,j\}} x_k^2$ in $\Delta_{ij} f$ is $a_{ij} a_{ji}$. Since f is stable, the polynomial $\Delta_{ij}(f)$ is nonnegative on $\mathbb{R}^m$. Then by Lemma 6.2, it follows that the coefficient $a_{ij} a_{ji}$ of $\prod_{k \in [n] \setminus \{i,j\}} x_k^2$ in $\Delta_{ij}(f)$ is nonnegative on $\mathbb{R}^{n-m}$. By Lemma 6.3, we can conclude that for each $1 \leq i < j \leq n$, there is some $\lambda_{ij} \in \mathbb{R}_{>0}$ such that $a_{ij} = \lambda_{ij} \overline{a_{ji}}$.

We claim that the scalars λ_{ij} satisfy $\lambda_{ij} = \lambda_{ik} \lambda_{kj}$ for all $1 \leq i < k < j \leq n$. For simplicity, we show this for $i = 1, k = 2, j = 3$ and the proof in general is virtually identical. By the arguments above, the starting determinantal representation of f has the form

$$\mathrm{diag}(x_1, \dots, x_n) + A(x) = \begin{pmatrix} x_1 + a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \lambda_{12} \overline{a_{12}} & x_2 + a_{22} & a_{23} & \cdots & \\ \lambda_{13} \overline{a_{13}} & \lambda_{23} \overline{a_{23}} & x_3 + a_{33} & & \\ \vdots & \vdots & & \ddots & \\ \lambda_{1n} \overline{a_{1n}} & & & & x_n + a_{nn} \end{pmatrix}.$$

Recall that by [6], the polynomial $\Delta_{ij}(f)$ factors as $(M^{\text{adj}})_{ij}(M^{\text{adj}})_{ji}$ where $M = \text{diag}(x_1, \dots, x_n) + A(x)$. These polynomials are affine in x_k for $k \in [n] \setminus \{i, j\}$. In particular,

$$\begin{aligned} g &:= \partial^{[n] \setminus \{1,2,3\}}(M^{\text{adj}})_{13} = a_{12}a_{23} - a_{13}(x_2 + a_{22}), \text{ and} \\ h &:= \partial^{[n] \setminus \{1,2,3\}}(M^{\text{adj}})_{31} = \lambda_{12}\lambda_{23}\overline{a_{12}a_{23}} - \lambda_{13}\overline{a_{13}}(x_2 + a_{22}). \end{aligned}$$

These polynomials satisfy the hypotheses of Lemma [6.3], and so there is some $\mu \in \mathbb{R}_{>0}$ for which $h = \mu\bar{g}$. Since a_{ij} is nonzero for all i, j and a_{22} is invariant under conjugation, we see that $\lambda_{12}\lambda_{23} = \mu = \lambda_{13}$. More generally $\lambda_{ij} = \lambda_{ik}\lambda_{kj}$ for any $i < k < j$.

Now define $D = \text{diag}(1, \sqrt{\lambda_{12}}, \dots, \sqrt{\lambda_{1n}})$. For $i < j$, $\lambda_{1j} = \lambda_{1i}\lambda_{ij}$ we calculate the (i, j) th and (j, i) th entries of $D^{-1}A(x)D$ as

$$(D^{-1}A(x)D)_{ij} = \frac{\sqrt{\lambda_{1j}}}{\sqrt{\lambda_{1i}}}a_{ij} = \sqrt{\lambda_{ij}}a_{ij} \quad \text{and} \quad (D^{-1}A(x)D)_{ji} = \frac{\sqrt{\lambda_{1i}}}{\sqrt{\lambda_{1j}}}\lambda_{ij}\overline{a_{ij}} = \sqrt{\lambda_{ij}}\overline{a_{ij}}.$$

■

7 Defining the set of factoring multiquadratic polynomials and the image of the principal minor map

In this section we give a complete characterization of the image of the principal minor map of Hermitian matrices using the characterization of Hermitian multiaffine determinantal polynomials from Section [5] and the characterization of multiquadratic polynomials that are Hermitian squares. This set is invariant under the action of $\text{SL}_2(R)^n \rtimes S_n$ and we derive the defining equations and numerical conditions as the orbit of a finite set under the action of this group, where R is a unique factorization domain. In this section, we will restrict to rings and fields of characteristic $\neq 2$.

Lemma 7.1. *Let $g = ax^2 + bx + c \in R[x]$ where $\text{char}(R) \neq 2$. The polynomial g factors into two linear factors in $R[x]$ if and only if its discriminant $\text{Discr}_x(g)$ is a square in R .* □

Proof. ($\Rightarrow$) If g factors, then it has a root in the fraction field of R . By the quadratic formula, this implies that the discriminant is a square in $\text{frac}(R)$, and hence in R .

($\Leftarrow$) Suppose that $b^2 - 4ac = q^2$ for some $q \in R$. We can rewrite this as $(b - q)(b + q) = 4ac$. Since R is a unique factorization domain, there is some choice of factorization of $a = a_1a_2$ and $c = c_1c_2$ so that $b - q = 2a_1c_1$ and $b + q = 2a_2c_2$. If $a = 0$, then g factors as $1 \cdot (bx + c)$, so we can assume $a \neq 0$. We can then write g as

$$g = a \left(x - \frac{-b+q}{2a} \right) \left(x - \frac{-b-q}{2a} \right) = a_1a_2 \left(x + \frac{c_1}{a_2} \right) \left(x + \frac{c_2}{a_1} \right) = (a_2x + c_1)(a_1x + c_2).$$

■

This lemma does not hold over rings of characteristic two. See [14, Section 2.4, Exercise 6] for further discussion. Note that for $g \in R[x, y]_{\text{MQ}}$, $\text{Discr}_x(g)$ is a polynomial of degree 4 in y whose coefficients are quadratic in the coefficients of g .

Lemma 7.2. *Let $h(x) = \sum_{i=0}^4 b_i x^i \in R[x]_4$ a univariate quartic where $\text{char}(R) \neq 2$. Then h is a square in $R[x]$ if and only if b_0, b_4 and $h(1) = \sum_j b_j$ are squares in R and the point $(b_0, b_1, b_2, b_3, b_4)$ satisfies*

$$\begin{aligned} b_4b_1^2 - b_3^2b_0 &= 0, & b_3^3 - 4b_4b_3b_2 + 8b_4^2b_1 &= 0, & b_1^3 - 4b_0b_1b_2 + 8b_0^2b_3 &= 0 \\ b_2b_3^2 - 4b_2^2b_4 + 2b_1b_3b_4 + 16b_0b_4^2 &= 0, & \text{and} & & b_1^2b_2 - 4b_0b_2^2 + 2b_0b_1b_3 + 16b_0^2b_4 &= 0. \end{aligned} \tag{9}$$

□

Proof. ($\Rightarrow$) If $h(x)$ is a square in $R[x]$, then $h(x) = \sum_{i=0}^4 b_i x^i = (\alpha x^2 + \beta x + \delta)^2$ for some $\alpha, \beta, \delta \in R$. We see that $b_4 = \alpha^2$, $b_0 = \delta^2$, and $\sum_{i=0}^4 b_i = (\alpha + \beta + \delta)^2$ are all squares in R . Each of the coefficients b_i is a polynomial in α, β, δ and one can quickly check that all the cubics in [9] vanish identically on this parametrization.

($\Leftarrow$) Let $b_4 = \alpha^2$, $b_0 = \delta^2$, and $\sum_j b_j = \lambda^2$ for some $\alpha, \delta, \lambda \in R$. From $b_0b_3^2 = b_1^2b_4$, we see that $\delta b_3 = \pm b_1\alpha$, and replacing α with $-\alpha$ if necessary, we can take $\delta b_3 = b_1\alpha$.

If b_3 is nonzero, we see from the second equation that b_4 , and hence α , must also be nonzero. Define $\beta = b_3/(2\alpha) \in \text{frac}(R)$. It follows immediately that $b_0 = \delta^2$, $b_1 = 2\delta\beta$, $b_3 = 2\beta\alpha$, and $b_4 = \alpha^2$. If $b_3 \neq 0$, the second equation implies that

$$b_2 = \frac{1}{4b_3b_4}(b_3^3 + 8b_1b_4^2) = \frac{1}{8\beta\alpha^3}(8\beta^3\alpha^3 + 16\delta\beta\alpha^4) = \beta^2 + 2\delta\alpha,$$

from which we conclude that $(\alpha x^2 + \beta x + \delta)^2 = h(x)$. Similarly, if b_1 is nonzero then so are b_0 and δ . We can define $\beta = b_1/(2\delta)$ and use $4b_0b_1b_2 = b_1^3 + 8b_0^2b_3$ to conclude that $(\alpha x^2 + \beta x + \delta)^2 = h(x)$. In either case, evaluating at $x = 1$ gives that $\alpha + \beta + \delta = \pm\lambda$, and $\beta = \pm\lambda - \alpha - \delta \in R$.

If $b_1 = b_3 = 0$, the equations simplify to $4b_4(b_2^2 - 4b_0b_2) = 0$ and $4b_0(b_2^2 - 4b_0b_2) = 0$. If b_0 or b_4 is nonzero, then $b_2 = \pm\delta\alpha$ and $h(x)$ is $(\alpha x^2 \pm \delta)^2$. Otherwise $b_0 = b_1 = b_3 = b_4 = 0$, in which case $b_2 = \lambda^2$ and $h(x) = (\lambda x)^2$. ■

Corollary 7.3. *When $\text{char}(R) \neq 2$, a quartic $h(x) = \sum_{j=0}^4 b_j x^j$ is a square in $R[x]$ if and only if for all γ in $\text{SL}_2(\{0, \pm 1\})$, $(\gamma \cdot h)_{x=0}$ is a square in R and $B_x(\gamma \cdot h) = C_x(\gamma \cdot h) = D_x(\gamma \cdot h) = 0$, where*

$$B_x(h) = b_4b_1^2 - b_3^2b_0, \quad C_x(h) = b_1^3 - 4b_0b_1b_2 + 8b_0^2b_3, \quad \text{and} \quad D_x(h) = b_1^2b_2 - 4b_0b_2^2 + 2b_0b_1b_3 + 16b_0^2b_4.$$

□

Proof. It suffices to show that we can recover the conditions in Lemma 7.2, which we can do this with three elements of $\text{SL}_2(\{0, \pm 1\})$: the identity, $\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, representing the fractional linear transformations $x \mapsto -1/x$ and $x \mapsto x + 1$, respectively. Note that $(\gamma_1 \cdot h)(0) = b_4$ and $(\gamma_2 \cdot h)(0) = h(1) = \sum_j b_j$, so from (i), we recover that all of these are squares in R . The element γ_1 induces the transposition $b_k \mapsto (-1)^k b_{4-k}$ for each k . One can quickly check that we recover the two missing cubics from this action of γ_1 . ■

Remark 7.4. The ideal generated by the five cubics in Lemma 7.2 is not saturated with respect to the ideal $\langle b_0, \dots, b_4 \rangle$. Its saturation is minimally generated by these five cubics together with $-b_1b_3^2 + 4b_1b_2b_4 - 8b_0b_3b_4$ and $-b_1^2b_3 + 4b_0b_2b_3 - 8b_0b_1b_4$. □

Note that the coefficients of $\text{Discr}_y(\gamma \cdot g)$ have degree two in the coefficients c_α of g , and so the polynomials listed in (ii) above have degree six. For example,

$$B_x(\text{Discr}_y(\gamma \cdot g)) = 4(c_{01}c_{11} - 2(c_{10}c_{02} + c_{00}c_{12}))^2(c_{21}^2 - 4c_{20}c_{22}) - 4(c_{01}^2 - 4c_{00}c_{02})(c_{11}c_{21} - 2(c_{20}c_{12} + c_{10}c_{22}))^2.$$

Theorem 7.5. *Let R be a unique factorization domain with $\text{char}(R) \neq 2$ and $|R| \geq 13$. A polynomial $g = \sum_{\alpha \in \{0,1,2\}^n} c_\alpha \mathbf{x}^\alpha \in R[\mathbf{x}]$ is the product of multiaffine polynomials if and only if for all $\gamma \in \text{SL}_2(R)^n \rtimes S_n$,*

- (i) $\text{Discr}_{x_1}(\gamma \cdot g)|_{x_2=\dots=x_n=0} = \gamma \cdot (c_{10}^2 - 4c_{00}c_{20})$ is a square in R ,
- (ii) the sextic polynomials in $\mathbf{c}$ given by specializing $B_{x_2}(\text{Discr}_{x_1}(\gamma \cdot g))$, $C_{x_2}(\text{Discr}_{x_1}(\gamma \cdot g))$ and $D_{x_2}(\text{Discr}_{x_1}(\gamma \cdot g))$ to $x_3 = \dots = x_n = 0$ are all zero.

□

Proof. We can express $g = \sum_{\beta \in \{0,1,2\}^2} g_\beta x_1^{\beta_1} x_2^{\beta_2}$ where $g_\beta \in R[x_3, \dots, x_n]_{\leq 2}$. The polynomial $B_{x_2}(\text{Discr}_{x_1}(g))$ has degree six in the coefficients g_β and so degree ≤ 12 in each variable x_j .

Consider $I \subset R$ with $|I| = 13$. For $\lambda_1 = \lambda_2 = 0$ and $\lambda_3, \dots, \lambda_n \in I$, consider the element $\gamma = \left(\begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix} \right)_j$ in $\text{SL}_2(R)^n$. For any polynomial $F \in R[\mathbf{x}]$ the evaluation of $\gamma \cdot F$ at $\mathbf{x} = 0$ equals the evaluation of F at $\mathbf{x} = (\lambda_1, \dots, \lambda_n)$. In particular, (ii) implies that the polynomials $B_{x_2}(\text{Discr}_{x_1}(\gamma \cdot g))$, $C_{x_2}(\text{Discr}_{x_1}(\gamma \cdot g))$ and $D_{x_2}(\text{Discr}_{x_1}(\gamma \cdot g))$ vanish at the point $\mathbf{x} = (\lambda_1, \dots, \lambda_n)$ for every choice of $\lambda_j \in I$. Since these polynomials have degree ≤ 12 in each variable x_j and $|I| \geq 13$, it follows that each of these polynomials is identically zero, using [3, Lemma 4.1].

We can now proceed by induction on n . The $n = 1$ case is the content of Lemma 7.1 together with the observation that the discriminant is invariant under the action of $\text{SL}_2(R)$, so we suppose $n \geq 2$. Let $h = \text{Discr}_{x_1}(g) \in S[x_2]$ where $S = R[x_3, \dots, x_n]$. By induction, for every $\gamma \in \text{SL}_2(R)$ acting on the variable x_2 , $(\gamma \cdot g)|_{x_2=0}$ factors into multiaffine polynomials and so $(\gamma \cdot h)|_{x_2=0} = \text{Discr}_{x_1}((\gamma \cdot g)|_{x_2=0})$ is a square in $S = R[x_3, \dots, x_n]$.

By Corollary 7.3, it follows that $\text{Discr}_{x_1}(g)$ is a square in $S[x_2]$. Then by Lemma 7.1, g factors into linear factors in x_1 in the ring $S[x_1, x_2] = R[\mathbf{x}]$. Using the action of S_n , we see that every irreducible factor of g must have degree ≤ 1 in each variable. ■

Remark 7.6. For every choice of $i \neq j \in [n]$ and $\lambda \in I^{n-3}$ we obtain three equations by evaluating $B_{x_j}(\text{Discr}_{x_i}(g))$, $C_{x_j}(\text{Discr}_{x_i}(g))$ and $D_{x_j}(\text{Discr}_{x_i}(g))$ at the point λ , along with additional two polynomials from the two missing analogous polynomials in Lemma 7.2 which can be recovered from the SL_2 -action on x_j . This gives a total of $5n(n-1)13^{n-3}$ sextic equations in the coefficients of g . $\square$

Lemma 7.7. Let S be a unique factorization domain with $\text{char}(S) \neq 2$ and an automorphic involution $a \rightarrow \bar{a}$ and let R be the fixed ring under this involution. The quadratic polynomial $g = ax^2 + bx + c \in R[x]$ is a Hermitian square in $S[x]$ if and only if a and c are Hermitian squares in S and the discriminant $\text{Discr}_x(g) = q^2$ with $q \in S[x]$ and $\bar{q} = -q$. $\square$

Proof. ($\Rightarrow$) If g factors into two conjugates $(sx + t)(\bar{s}x + \bar{t})$, then $a = s\bar{s}$ and $c = t\bar{t}$ and

$$\text{Discr}_x(g) = b^2 - 4ac = (s\bar{t} + t\bar{s})^2 - 4s\bar{s}t\bar{t} = (s\bar{t} - t\bar{s})^2$$

which satisfies the desired property.

($\Leftarrow$) Assume that $b^2 - 4ac = q^2$ such that $\bar{q} = -q$. If $a = 0$, then $b = \pm q$ and thus $\bar{b} = -b$. Since $b \in R$, then $b = 0$ and $g = c$ is a Hermitian square as desired. If $a \neq 0$, then $(b - q)(b + q) = (b - q)(b - \bar{q}) = 4ac = 4s\bar{s}t\bar{t}$, where $a = s\bar{s}$ and $c = t\bar{t}$. Thus, after relabeling if needed, we may assume that $b - \bar{q} = 2st$. Thus, we can write g as

$$g = a \left(x - \frac{-b + \bar{q}}{2a} \right) \left(x - \frac{-b + q}{2a} \right) = s\bar{s} \left(x + \frac{t}{\bar{s}} \right) \left(x + \frac{\bar{t}}{s} \right) = (sx + t)(\bar{s}x + \bar{t}).$$

■

Theorem 7.8. Let S be a unique factorization domain with $\text{char}(S) \neq 2$ and an automorphic involution $a \mapsto \bar{a}$. Let R be the fixed ring of this automorphism with $|R| \geq 13$. The polynomial $g = \sum_{\alpha \in \{0,1,2\}^n} c_\alpha \mathbf{x}^\alpha$ in $R[\mathbf{x}]_{\text{MQ}}$ is a Hermitian square if and only if $(\gamma \cdot g)|_{x_3=\dots=x_n=0}$ is a Hermitian square in $S[x_1, x_2]$ for all $\gamma \in \text{SL}_2(R)^n \rtimes S_n$. $\square$

Proof. If for all $\gamma \in \text{SL}_2(R)^n \rtimes S_n$, the polynomial $(\gamma \cdot g)|_{x_3=\dots=x_n=0}$ is a Hermitian square in $S[x_1, x_2]$, then by Lemma 7.1, $\text{Discr}_{x_1}(\gamma \cdot g)|_{x_3=\dots=x_n=0}$ is a square in $S[x_2]$. Using Corollary 7.3 we see that the two conditions of Theorem 7.5 are satisfied and hence we deduce that g is a product of multiaffine polynomials in $S[\mathbf{x}]$. To prove that g is a Hermitian square, we will proceed by induction on n . The case $n = 2$ is trivially satisfied. For the inductive step, write g as $g = p_2x_1^2 + p_1x_1 + p_0$ for some $p_2, p_1, p_0 \in \tilde{R} = R[x_2, \dots, x_n]$. By induction we see that p_2 and p_0 are both Hermitian squares and as g is a product of multiaffine polynomials, then by Lemma 7.1 we see that $\text{Discr}_{x_1}(g) = p_1^2 - 4p_2p_0 = q^2$ for some $q \in S[x_2, \dots, x_n]$. Since $p_1^2 - 4p_2p_0 \in \tilde{R}$, then $q^2 \in \tilde{R}$ and so $q = -\bar{q}$ or $q = \bar{q}$. In the former case, Lemma 7.7 implies that g is a Hermitian square and we are done. Otherwise we get $(\gamma \cdot q)|_{\mathbf{x}=0} = \overline{(\gamma \cdot q)|_{\mathbf{x}=0}}$ for all $\gamma \in \text{SL}_2(R)^{n-1}$. Notice that by induction on the other hand, $(\gamma \cdot g)|_{\mathbf{x}=0}$ is a Hermitian square and hence

$$\text{Discr}_{x_1}((\gamma \cdot g)|_{\mathbf{x}=0}) = (\gamma \cdot (p_1^2 - 4p_2p_0))|_{\mathbf{x}=0} = (\gamma \cdot q)|_{\mathbf{x}=0}^2 \text{ with } (\gamma \cdot q)|_{\mathbf{x}=0} = -\overline{(\gamma \cdot q)|_{\mathbf{x}=0}}.$$

Thus we conclude that $(\gamma \cdot q)|_{\mathbf{x}=0} = 0$ for all $\gamma \in \text{SL}_2(R)^{n-1}$. Consider $\gamma = (\gamma_i)_{2 \leq i \leq n}$ where $\gamma_i = \begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix}$ for $\lambda_i \in R$. Notice that $\gamma \cdot q|_{(x_2=\dots=x_n=0)} = q|_{(x_2=\lambda_2, \dots, x_n=\lambda_n)} = 0$. Since $|R| \geq 3$, [3, Lemma 4.1] implies that $q \equiv 0$ and thus $q = -\bar{q}$ and we apply Lemma 7.7 again to deduce that g is a Hermitian square. $\blacksquare$

Assumptions 7.9. We take $\mathbb{F}$ to be a field of $\text{char}(\mathbb{F}) \neq 2$ with $|\mathbb{F}| \geq 13$ and $\mathbb{K}$ to be a degree-two extension field. Let δ denote the square root of the discriminant of the minimal polynomial of this field extension. Then $\mathbb{K} = \mathbb{F}(\delta)$ and the involution $\delta \rightarrow \bar{\delta} = -\delta$ extends to an automorphism of $\mathbb{K}$ with fixed field $\mathbb{F}$. $\square$

Remark 7.10. In the field $\mathbb{K}$, $\bar{q} = -q$ is equivalent to requiring $q = \delta r$ for some $r \in \mathbb{F}$. $\square$

Lemma 7.11. Let $\mathbb{F}, \mathbb{K}$ satisfy Assumptions 7.9 and let $g = \sum_{\alpha \in \{0,1,2\}^2} c_\alpha \mathbf{x}^\alpha \in \mathbb{F}[x_1, x_2]_{\text{MQ}}$. The polynomial g is a Hermitian square in $\mathbb{K}[x_1, x_2]$ if and only if for all $\gamma \in \text{SL}_2(\{0, \pm 1\})^2 \rtimes S_2$

- (i) $\gamma \cdot c_{(0,0)}$ is a Hermitian square in $\mathbb{K}$, and
- (ii) $\frac{1}{\delta^2} \text{Discr}_{x_1}(\gamma \cdot g)$ is a square in $\mathbb{F}[x_2]$.

□

Proof. Write g as $g = p_2x_1^2 + p_1x_1 + p_0$ where p_2, p_1 and p_0 are quadratics in $\mathbb{F}[x_2]$. Using Lemma 7.7, we see that g is a product of two conjugate factors if and only if p_2 and p_0 are product of two conjugates in $\mathbb{K}[x_2]$ and $\text{Disc}_{x_1}g = \bar{q}^2$ where $\bar{q} = -q$ for some $q \in \mathbb{K}[x_2]$. Notice that by Remark 7.10, this condition is equivalent to $q = \delta r$ where $r \in \mathbb{F}[x_2]$ and thus requiring that $\frac{1}{\delta^2}\text{Disc}_{x_1}g$ is a square in $\mathbb{F}[x_2]$. Using Lemma 7.7, p_2 and p_0 are conjugates if and only if $c_{(i,j)}$ is a product of two conjugates for $i, j \in \{0, 2\}$ and $\frac{1}{\delta^2}\text{Disc}_{x_2}(\gamma \cdot g)|_{x_1=0}$ is a square for $\gamma \in \text{SL}_2(\mathbb{F})$ and this gives the desired equivalence. ■

Theorem 7.12. Let $\mathbb{F}, \mathbb{K}$ satisfy Assumptions 7.9. A polynomial $g = \sum_{\alpha \in \{0,1,2\}^n} c_\alpha \mathbf{x}^\alpha \in \mathbb{F}[\mathbf{x}]$ is a Hermitian square in $\mathbb{K}[\mathbf{x}]$ if and only if for all $\gamma \in \text{SL}_2(\mathbb{F})^n \rtimes S_n$,

- (i) $(\gamma \cdot c_0)$ is a Hermitian square in $\mathbb{K}$,
- (ii) $\frac{1}{\delta^2}\text{Disc}_{x_1}(\gamma \cdot g)|_{x_2=\dots=x_n=0} = \gamma \cdot \left(\frac{1}{\delta^2}(c_{10}^2 - 4c_{00}c_{20})\right)$ is a square in $\mathbb{F}$, and
- (iii) the sextic polynomials in $\mathbf{c}$ given by specializing $B_{x_2}(\text{Disc}_{x_1}(\gamma \cdot g))$, $C_{x_2}(\text{Disc}_{x_1}(\gamma \cdot g))$ and $D_{x_2}(\text{Disc}_{x_1}(\gamma \cdot g))$ to $x_3 = \dots = x_n = 0$ are all zero.

□

Proof. Using Lemma 7.8, g is a Hermitian square in $\mathbb{K}[\mathbf{x}]$ if and only if for all $\gamma \in \text{SL}_2(\mathbb{F})^n \rtimes S_n$, the restriction $(\gamma \cdot g)|_{x_3=\dots=x_n=0}$ is a Hermitian square in $\mathbb{K}[x_1, x_2]$. By Lemma 7.11, this is equivalent to $\gamma \cdot c_0$ factoring as a product of two conjugates and $\frac{1}{\delta^2}\text{Disc}_{x_1}(\gamma \cdot g)$ being a square in $\mathbb{F}[x_2]$. By Corollary 7.3, this discriminant being a square is equivalent to conditions (ii) and (iii). ■

Now we are ready to give a complete characterization of the image of the principal minor map of Hermitian matrices using the characterization of Hermitian multiaffine determinantal polynomials from Section 5 and the characterization of multiquadratic polynomials that are Hermitian squares.

Recall that to each element $\mathbf{a} = (a_S)_{S \subseteq [n]}$ in $\mathbb{F}^{2^n}$ we associate the multiaffine polynomial

$$f_{\mathbf{a}} = \sum_{S \subseteq [n]} a_S \mathbf{x}^{[n] \setminus S}.$$

For $n = 3$, the discriminant of the Rayleigh difference $\Delta_{12}(f)$ with respect to x_3 is Cayley's $2 \times 2 \times 2$ hyperdeterminant

$$\begin{aligned} \text{HypDet}(\mathbf{a}) &= (a_1a_{23} + a_2a_{13} - a_3a_{12} - a_0a_{123})^2 - 4(a_1a_2 - a_0a_{12})(a_{13}a_{23} - a_3a_{123}) \\ &= a_0^2a_{123}^2 + a_1^2a_{23}^2 + a_2^2a_{13}^2 + a_3^2a_{12}^2 - 2a_0a_1a_{23}a_{123} - 2a_0a_2a_{13}a_{123} - 2a_0a_3a_{12}a_{123} \\ &\quad - 2a_1a_2a_{13}a_{23} - 2a_1a_3a_{12}a_{23} - 2a_2a_3a_{12}a_{13} + 4a_0a_{23}a_{13}a_{12} + 4a_{123}a_1a_2a_3. \end{aligned}$$

This quartic polynomial therefore appears in the arithmetic conditions on the image of the principal minor map.

Theorem 7.13. Let $\mathbb{F}, \mathbb{K}$ satisfy Assumptions 7.9 and let $\mathbf{a} = (a_S)_{S \subseteq [n]} \in \mathbb{F}^{2^n}$ with $a_0 = 1$. There exists a Hermitian matrix over $\mathbb{K}$ with principal minors $\mathbf{a}$ if and only if for every $\gamma \in \text{SL}_2(\mathbb{F})^n \rtimes S_n$,

- (i) $\gamma \cdot (a_1a_2 - a_0a_{12})$ is a Hermitian square in $\mathbb{K}$,
- (ii) $\frac{1}{\delta^2}\text{HypDet}(\gamma \cdot \mathbf{a})$ is a square in $\mathbb{F}$, and
- (iii) $\gamma \cdot \mathbf{a}$ satisfies the degree-12 polynomials given by specializing $B_{x_4}(\text{Disc}_{x_3}(\gamma \cdot \Delta_{12}f_{\mathbf{a}}))$, $C_{x_4}(\text{Disc}_{x_3}(\gamma \cdot \Delta_{12}f_{\mathbf{a}}))$ and $D_{x_4}(\text{Disc}_{x_3}(\gamma \cdot \Delta_{12}f_{\mathbf{a}}))$ to $x_5 = \dots = x_n = 0$.

□

Here the operators B_x, C_x, D_x are defined in Corollary 7.3.

Proof of Theorem 7.13. By Theorem 5.1 with $n = m$, $\mathbf{a} = (a_S)_{S \subseteq [n]} \in \mathbb{F}^{2^n}$ is in the image of the principal minor map if and only if $\Delta_{ij}(f_{\mathbf{a}})$ is a Hermitian square for all $i, j \in [n]$, which according to Theorem 7.8, is satisfied if and only if for all $\gamma \in \text{SL}_2(\mathbb{F})^n \rtimes S_n$, $\gamma \cdot \Delta_{34}(f_{\mathbf{a}})|_{x_5=\dots=x_n=0}$ is a Hermitian square in $\mathbb{K}[x_1, x_2]$. This is equivalent to the three hypotheses of Theorem 7.12, which in turn are equivalent to the three hypotheses of the theorem. ■

Taking $\mathbb{K} = \mathbb{C}$ with the action complex conjugation then gives the following.

Corollary 7.14. Let $\mathbf{a} = (a_S)_{S \subseteq [n]} \in \mathbb{R}^{2^n}$ with $a_0 = 1$. There exists a Hermitian matrix over $\mathbb{C}$ with principal minors $\mathbf{a}$ if and only if for every $\gamma \in \text{SL}_2(\mathbb{R})^n \rtimes S_n$,

- (i) $\gamma \cdot (a_1a_2 - a_0a_{12}) \geq 0$,
- (ii) $\text{HypDet}(\gamma \cdot \mathbf{a}) \leq 0$, and
- (iii) $\gamma \cdot \mathbf{a}$ satisfies the three degree-12 equations given by restricting $B_{x_4}(\text{Disc}_{x_3}(\Delta_{12}f_{\gamma \cdot \mathbf{a}}))$, $C_{x_4}(\text{Disc}_{x_3}(\Delta_{12}f_{\gamma \cdot \mathbf{a}}))$ and $D_{x_4}(\text{Disc}_{x_3}(\Delta_{12}f_{\gamma \cdot \mathbf{a}}))$ to $x_5 = \dots = x_n = 0$.

□

8 A family of counterexamples

Let $\mathbb{F}$ be a field and for $n \geq 2$, consider the multiaffine polynomial $f_{2n+1} \in \mathbb{F}[x_1, \dots, x_{2n+1}]$ given by

$$f_{2n+1} = x_1 \cdot \prod_{j=1}^n (x_{2j+1}x_{2j+2} + 1) + \prod_{j=1}^n (x_{2j}x_{2j+1} + 1) \quad (10)$$

where we take $x_{2n+2} = x_2$. We show that this polynomial is not determinantal, i.e. its vector of coefficients do not belong to the image of the principal minor map, but is determinantal after specializing any one variable:

Theorem 8.1. *There is no finite set of equations whose orbit under $\mathrm{SL}_2(\mathbb{F})^n \rtimes S_n$ set-theoretically cuts out the image of the principal minor map for all n .* $\square$

Let $I_n \subset \mathbb{F}[a_S : S \subseteq [n]]$ be the homogeneous ideal of polynomials vanishing on the image of $n \times n$ matrices under the principal minor map in $\mathbb{P}^{2^n-1}(\mathbb{F})$. There is a natural inclusion of I_n into $\mathbb{F}[a_S : S \subseteq [n+1]]$.

Theorem 8.2. *The coefficient vector of the polynomial f_{2n+1} belongs to the variety of polynomials in the orbit $(\mathrm{SL}_2(\mathbb{F})^{2n+1} \rtimes S_{2n+1}) \cdot I_{2n}$ but not the variety of I_{2n+1} .* $\square$

The proof of this theorem relies on the fact that the coefficient of any generic specialization of f_{2n+1} lies in the image of the principal minor map, up to scaling. One key observation is that the Rayleigh differences of f_{2n+1} do not all factor as the product of *two* multiaffine polynomials, but do have such factorizations after specializing any one variable. We show this explicitly by writing down the determinantal representations of these specializations.

Lemma 8.3. *The rational function $\frac{1}{1+x_1}f_{2n+1}$ can be written as $\det(\mathrm{diag}(x_2, \dots, x_{2n+1}) + A)$ where for $2 \leq i, j \leq 2n+1$,*

$$A_{ij} = \begin{cases} 1/(1+x_1) & \text{if } i \text{ is odd, } j \text{ is even, and } i > j, \\ -x_1/(1+x_1) & \text{if } i \text{ is odd, } j \text{ is even, and } i < j, \\ -1 & \text{if } i \text{ is even, } j = i+1, \\ 1 & \text{if } i \text{ is even, } j = i-1, \\ -x_1 & \text{if } i = 2, j = 2n+1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Proof. Let D denote the determinant of the matrix $M = \det(\mathrm{diag}(x_2, \dots, x_{2n+1}) + A)$. By definition, D is a polynomial in $\frac{1}{1+x_1}, x_1, x_2, \dots, x_n$. Moreover the entries for which $x_1 + 1$ appears in the denominator form a square submatrix whose rows correspond to odd indices and whose columns correspond to even ones. It has the form

$$\frac{1}{1+x_1} \begin{pmatrix} 1 & -x_1 & -x_1 & \dots & -x_1 \\ 1 & 1 & -x_1 & \dots & -x_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & -x_1 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} = \frac{1}{1+x_1} J - U$$

where J is the all ones matrix and U is an upper triangular matrix with $U_{ij} = 1$ for $i < j$ and $U_{ij} = 0$ otherwise. Since J has rank one, the exponent of $1+x_1$ appearing in the denominator of *any* minor of this matrix is at most one. This also shows that despite the many appearances of x_1 in numerator of this matrix, it does not appear in the numerator of any minor. There is only one other entry in M containing x_1 , and so the determinant D can be written as $(x_1 + 1)^{-1}p_1 + p_2$ where p_1 and p_2 are multiaffine in $x_1, \dots, x_{2n+1}$. Moreover, the only term in the Laplace expansion of the determinant of M avoiding this submatrix is the product of the diagonal $\prod_{j=2}^{2n+1} x_j$. Therefore we can write D as $(x_1 + 1)^{-1}p$ where p is multiaffine in $x_1, \dots, x_{2n+1}$. To show that $p = f_{2n+1}$ it suffices to show that they have the same specialization at $x_1 = 0$ and the same coefficient of x_1 .

When we specialize x_1 to zero, M becomes a block upper-triangular matrix with diagonal blocks of the form $\begin{pmatrix} x_{2j} & -1 \\ 1 & x_{2j+1} \end{pmatrix}$. Its determinant agrees with the specialization of $\frac{1}{1+x_1}f_{2n+1}$ to $x_1 = 0$.

$$\begin{pmatrix} x_2 & -1 & 0 & 0 & 0 & -x_1 \\ \frac{1}{x_1+1} & x_3 & -\frac{x_1}{x_1+1} & 0 & -\frac{x_1}{x_1+1} & 0 \\ 0 & 1 & x_4 & -1 & 0 & 0 \\ \frac{1}{x_1+1} & 0 & \frac{1}{x_1+1} & x_5 & -\frac{x_1}{x_1+1} & 0 \\ 0 & 0 & 0 & 1 & x_6 & -1 \\ \frac{1}{x_1+1} & 0 & \frac{1}{x_1+1} & 0 & \frac{1}{x_1+1} & x_7 \end{pmatrix} \sim \begin{pmatrix} x_2 & 0 & 0 & -1 & 0 & -x_1 \\ 0 & x_4 & 0 & 1 & -1 & 0 \\ 0 & 0 & x_6 & 0 & 1 & -1 \\ \frac{1}{x_1+1} & -\frac{x_1}{x_1+1} & -\frac{x_1}{x_1+1} & x_3 & 0 & 0 \\ \frac{1}{x_1+1} & \frac{1}{x_1+1} & -\frac{x_1}{x_1+1} & 0 & x_5 & 0 \\ \frac{1}{x_1+1} & \frac{1}{x_1+1} & \frac{1}{x_1+1} & 0 & 0 & x_7 \end{pmatrix}$$

 Fig. 1: The matrix A in Lemma 8.3 for $2n + 1 = 7$.

Consider the rational function g obtained by inverting x_1 in $\frac{1}{1+x_1}f_{2n+1}$, which is

$$\frac{x_1}{1+x_1}f_{2n+1}(x_1^{-1}, x_2, \dots, x_n) = \frac{1}{1+x_1} \cdot \left(\prod_{j=1}^n (x_{2j+1}x_{2j+2} + 1) + x_1 \cdot \prod_{j=1}^n (x_{2j}x_{2j+1} + 1) \right).$$

Let M' be the matrix obtained from M by replacing x_1 by x_1^{-1} and then multiplying the column indexed by 2 by x_1^{-1} and the row indexed by 2 by x_1 . The entries are now rational functions in x_1 with only $1+x_1$ appearing in the denominator. After specializing M' to $x_1 = 0$ and cyclic shifting the rows and columns by one, we find another block upper triangular matrix with diagonal blocks of the form $\begin{pmatrix} x_{2j+1} & -1 \\ 1 & x_{2j+2} \end{pmatrix}$ for $j = 1, \dots, n-1$ and $\begin{pmatrix} x_{2n+1} & 1 \\ -1 & x_2 \end{pmatrix}$. Therefore the determinant of M' restricted to $x_1 = 0$ is given by $\prod_{j=1}^n (x_{2j+1}x_{2j+2} + 1)$.

By definition, the determinant of M' equals $D(x_1^{-1}, x_2, \dots, x_n) = \frac{x_1}{1+x_1}p(x_1^{-1}, \dots, x_n)$. Restricting to $x_1 = 0$ gives the coefficient of x_1 in p , which must be $\prod_{j=1}^n (x_{2j+1}x_{2j+2} + 1)$. Therefore p coincides with f_{2n+1} . ■

Lemma 8.4. *For every $m = 2, \dots, 2n+1$, the coefficients of $\frac{1}{x_m}f_{2n+1}$ are the principal minors of a $2n \times 2n$ matrix with entries in $\{0, \pm 1, x_m^{\pm 1}\}$. In particular, the rational function $\frac{1}{x_{2n+1}}f_{2n+1}$ can be written as $\det(\text{diag}(x_1, \dots, x_{2n}) + B)$ where for $1 \leq i, j \leq 2n$,*

$$B_{ij} = \begin{cases} 1 & \text{if } j = i+1 \text{ and } i > 1 \text{ or } (i, j) = (1, 1) \text{ or } (i, j) = (2, 1), \\ -1 & \text{if } i \text{ is even and } j = i-1 \text{ or } (i, j) = (2n, 1), \\ x_{2n+1} & \text{if } i \text{ odd, } i \geq 3, \text{ and } j = 1, \\ 1/x_{2n+1} & \text{if } i \in \{1, 2\} \text{ and } j \text{ is even, and} \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof. Let $M = \det(\text{diag}(x_1, \dots, x_{2n}) + B)$ and let D denote its determinant. As in the proof of Lemma 8.3, the entries of M with x_{2n+1} appearing in the denominator appear in a submatrix of rank-one. The entries with x_{2n+1} appearing in the numerator are contained in the first column. Moreover, in the Laplace expansion of the determinant, the only terms avoiding the submatrix of entries x_{2n+1}^{-1} must include the $(1, 1)$ and $(2, 2)$ entries, and so will not involve any entries with x_{2n+1} . It follows that D can be written as $x_{2n+1}^{-1}p$ where p is multiaffine in $x_1, \dots, x_{2n+1}$. Therefore it suffices to check that f_{2n+1} and p have the same restriction to $x_1 = 0$ and same coefficient of x_1 .

We see that the coefficient of x_1 in D is the determinant of the matrix M after removing the first row and column. This minor is a block matrix with one block of the form $(x_2 + 1/x_{2n+1})$ and the rest of the form $\begin{pmatrix} x_{2j+1} & 1 \\ -1 & x_{j+2} \end{pmatrix}$. Therefore the coefficient of x_1 in p and f_{2n+1} agree.

The specialization of M to $x_1 = 0$ is a matrix has the form $\begin{pmatrix} 1 & b^T \\ c & A \end{pmatrix}$. Using Schur complements, we see that the determinant equals the determinant of $A - cb^T$. One can check that the matrix $A - cb^T$ is a block-lower triangular matrix with diagonal blocks $\begin{pmatrix} x_{2j} & 1 \\ -1 & x_{2j+1} \end{pmatrix}$ for $j = 1, \dots, n-1$ and $x_{2n} + 1/x_{2n+1}$. This shows that the restriction of p to $x_1 = 0$ agrees with that of f_{2n+1} .

For the corresponding statement with arbitrary $m \neq 1$, we use the symmetries of f_{2n+1} under the action of a dihedral group of order n with the cyclic action $j \mapsto j+2$ (identifying $2n+j = j$ for $j \geq 2$) and reflection $n+1-j \leftrightarrow n+2-j$. There is some element of this group that moves m to $2n+1$, and we can take the image of the representation above. ■

$$\begin{pmatrix} x_1 + 1 & \frac{1}{x_7} & 0 & \frac{1}{x_7} & 0 & \frac{1}{x_7} \\ 1 & x_2 + \frac{1}{x_7} & 1 & \frac{1}{x_7} & 0 & \frac{1}{x_7} \\ x_7 & 0 & x_3 & 1 & 0 & 1 \\ 0 & 0 & -1 & x_4 & 1 & 0 \\ x_7 & 0 & 0 & 0 & x_5 & 1 \\ -1 & 0 & 0 & 0 & -1 & x_6 \end{pmatrix} \quad \begin{pmatrix} x_2 & 1 & 0 & 0 & 0 \\ -1 & x_3 & 0 & 0 & 0 \\ 0 & -1 & x_4 & 1 & 0 \\ -1 & 0 & -1 & x_5 & 0 \\ \frac{1}{x_7} & 0 & \frac{1}{x_7} & -1 & x_6 + \frac{1}{x_7} \end{pmatrix}$$

Fig. 2: The matrices B (left) and $A - cb^T$ (right) in Lemma 8.4 for $2n + 1 = 7$.

To show that f_{2n+1} does not belong to I_{2n+1} , we will use the following:

Lemma 8.5. *The set of polynomials*

$$\mathcal{F}_n = \{f \in \mathbb{F}[\mathbf{x}]_{\text{MA}} : \text{for all } i, j \in [n], \Delta_{ij}(f) = g_{ij} \cdot h_{ij} \text{ for some } g_{ij}, h_{ij} \in \mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MA}}\}$$

is Zariski closed in $\mathbb{F}[\mathbf{x}]_{\text{MA}} \cong \mathbb{F}^{2^{[n]}}$, where $\mathbb{F}^{\text{alg}}$ denotes the algebraic closure of $\mathbb{F}$. $\square$

Proof. The set of multiquadratic polynomials in $\mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MQ}}$ that factor as the product of two multiaffine polynomials is the image of $\mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MA}} \times \mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MA}}$ under $(g, h) \mapsto g \cdot h$. Since this map is bilinear, it follows from the projective elimination theorem that the set $\{q \in \mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MQ}} : q = g \cdot h \text{ for some } g, h \in \mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MA}}\}$ is Zariski-closed in $\mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MQ}}$.

Pulling back by the map Δ_{ij} , it follows that for each $i, j \in [n]$, the set of polynomials $f \in \mathbb{F}^{\text{alg}}[\mathbf{x}]_{\text{MA}}$ for which $\Delta_{ij}(f)$ factors as the product of two multiaffine polynomials is Zariski-closed, as is their intersection over all $i, j \in [n]$. It follows that its intersection with $\mathbb{F}[\mathbf{x}]_{\text{MA}}$ is Zariski-closed in $\mathbb{F}[\mathbf{x}]_{\text{MA}}$. $\blacksquare$

Theorem 3.1 implies that the image of $\mathbb{F}^{n \times n}$ under the principal minor map is a subset of the variety $\mathcal{F}_n$, although as Example 3.2 shows, this containment can be strict. In order to show that f_{2n+1} does not belong to the variety of I_{2n+1} , it suffices to show that f_{2n+1} does not belong to $\mathcal{F}_{2n+1}$.

Recall that for $f = \sum_{S \subseteq [n]} a_S \mathbf{x}^{[n] \setminus S}$, the coefficient vector of f is defined to be

$$\text{coeff}(f) = (a_S)_{S \subseteq [n]} \in \mathbb{F}^{2^{[n]}}.$$

Proof of Theorem 8.2. For convenience, let $f = f_{2n+1}$. Let $P \in I_{2n}$ be a homogenous polynomial vanishing on the image of $\mathbb{F}^{2n \times 2n}$ under the principal minor map. Let Q denote the image of P under inclusion into $\mathbb{F}[a_S : S \subseteq [2n+1]]$. Note that Q only involves a_S with $2n+1 \notin S$. Since our indexing of coefficients is inclusion reversing, we see that the evaluation of Q at the coefficient vector of f depends only on coefficients of monomials containing x_{2n+1} . In particular, its evaluation at the coefficient vector of f equals the evaluation of P at the coefficient vector of derivative of f with respect to x_{2n+1} , i.e.

$$Q(\text{coeff}(f)) = P(\text{coeff}(\partial f / \partial x_{2n+1})). \quad (11)$$

If $\mathbb{F}$ is finite, it suffices to replace it with any infinite field extension, such as $\mathbb{F}(t)$ or $\mathbb{F}^{\text{alg}}$. Let $(\gamma, \pi) \in \text{SL}_2(\mathbb{F})^{2n+1} \rtimes S_{2n+1}$, with γ generic, where $\mathbb{F}^{\text{alg}}$ denote the (necessarily infinite) algebraic closure of $\mathbb{F}$. We can write (γ, π) as the composition of elements $(\hat{\gamma}, \hat{\pi})$ in $\text{SL}_2(\mathbb{F})^{2n} \rtimes S_{2n}$ and (γ_{2n+1}, σ) , where $\gamma_{2n+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{F})$ acts on x_{2n+1} and σ is the transposition $\sigma = (m(2n+1)) \in S_{2n+1}$. Then

$$(\gamma_{2n+1}, \sigma) \cdot f = (cx_{2n+1} + d)f \left(x_1, \dots, x_{m-1}, \frac{ax_{2n+1} + b}{cx_{2n+1} + d}, x_{m+1}, \dots, x_{2n}, x_m \right).$$

By the genericity of γ , $c \neq 0$ and

$$\frac{\partial}{\partial x_{2n+1}}((\gamma_{2n+1}, \sigma) \cdot f) = cf|_{\{x_m = a/c, x_{2n+1} = x_m\}}.$$

Call this polynomial g . The coefficient λ of $\prod_{i=1}^{2n} x_i$ in g is $a + c$ for $m = 1$ and a for $m > 1$. In either case, we can assume it is nonzero by the genericity of γ .

By Lemma 8.3 for $m = 1$ and Lemma 8.4 for $m > 1$, the polynomial $\frac{1}{\lambda}g$ is determinantal and its coefficient vector belongs to the image of the principal minor map. Since the image of the principal minor map is invariant under the action of $(\text{SL}_2(\mathbb{F})^{2n} \rtimes S_{2n})$, by (11),

$$0 = P(\text{coeff}(g)) = P(\text{coeff}((\hat{\gamma}, \hat{\pi}) \cdot g)) = Q(\text{coeff}((\gamma, \pi) \cdot f)).$$

This shows that the coefficient vector of f belongs to the variety of $(\mathrm{SL}_2(\mathbb{F})^{2n+1} \rtimes S_{2n+1}) \cdot I_{2n}$.

On the other hand, we calculate that

$$\Delta_{12}(f) = (x_3 - x_{2n+1}) \prod_{i=3}^{2n} (x_i x_{i+1} + 1).$$

These form a cycle of length $2n - 1$ of irreducible bivariate factors, which cannot be factored as the product of two multiaffine polynomials. It follows that f does not belong to the variety $\mathcal{F}_{2n+1}$ from Lemma 8.5, which, by Theorem 3.1, contains the variety of I_{2n+1} . ■

Example 8.1 ($2n + 1 = 5$). Consider $f_5 = x_1(x_3x_4 + 1)(x_2x_5 + 1) + (x_2x_3 + 1)(x_4x_5 + 1)$. By Lemmas 8.3 and 8.4, f_5 is determinantal over $\mathbb{F}(x_j)[x_i : i \in [5] \setminus \{j\}]$ for each $j = 1, \dots, 5$. For example, for $j = 1$ and 5,

$$f_5 = (x_1 + 1) \det \begin{pmatrix} x_2 & -1 & 0 & -x_1 \\ \frac{1}{x_1+1} & x_3 & -\frac{x_1}{x_1+1} & 0 \\ 0 & 1 & x_4 & -1 \\ \frac{1}{x_1+1} & 0 & \frac{1}{x_1+1} & x_5 \end{pmatrix} = x_5 \det \begin{pmatrix} x_1 + 1 & \frac{1}{x_5} & 0 & \frac{1}{x_5} \\ 1 & x_2 + \frac{1}{x_5} & 1 & \frac{1}{x_5} \\ x_5 & 0 & x_3 & 1 \\ -1 & 0 & -1 & x_4 \end{pmatrix}.$$

However $\Delta_{12}(f_5) = (x_3 - x_5)(x_3x_4 + 1)(x_4x_5 + 1)$, which cannot be written as a product of two multiaffine polynomials in $\mathbb{F}[x_3, x_4, x_5]$. Therefore f_5 is not determinantal over $\mathbb{F}[x_1, \dots, x_5]$. □

The polynomial f_{2n+1} shows that the orbit of the ideal I_{2n} under $(\mathrm{SL}_2(\mathbb{F})^{2n+1} \rtimes S_{2n+1})$ is not enough to cut out the set of polynomials $f \in \mathbb{F}[x_1, \dots, x_{2n+1}]$ all of whose Rayleigh differences factor as the product of two multiaffine polynomials. As Example 3.2 shows, even this is not enough to cut out the image of the principal minor map. This leaves the question of what conditions cut out the image of the principal minor map for arbitrary n wide open.

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