Uniform error estimate of an asymptotic preserving scheme for the Lévy-Fokker-Planck equation *

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Abstract

We establish a uniform-in-scaling error estimate for the asymptotic preserving scheme proposed in [16] for the Lévy-Fokker-Planck (LFP) equation. The main difficulties stem from not only the interplay between the scaling and numerical parameters but also the slow decay of the tail of the equilibrium state. We tackle these problems by separating the parameter domain according to the relative size of the scaling ε: in the regime where ε is large, we design a weighted norm to mitigate the issue caused by the fat tail, while in the regime where ε is small, we prove a strong convergence of LFP towards its fractional diffusion limit with an explicit convergence rate. This method extends the traditional AP estimates to cases where uniform bounds are unavailable. Our result applies to any dimension and to the whole span of the fractional power.

1 Introduction

Consider the Lévy-Fokker-Planck (LFP) equation

$$\begin{cases}\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(vf) - (-\Delta_v)^s f := \mathcal{L}^s(f), \qquad s\in(0,1),\\ f(0,x,v) = f_{in}(x,v),\end{cases} \tag{1.1}$$

where $f(t,x,v):(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}^+$ is the distribution function of a large group of particles, which undergoes a free transport dynamics along with an interaction with the background, described by the Levy-Fokker-Planck operator. In contrast to the conventional Fokker-Planck operator, here we have $(-\Delta_v)^s$ in place of Δ_v, which models the Lévy processes at the microscopic level instead of Brownian motion. One way to understand $(-\Delta_v)^s$ is via the Fourier transform. Namely, for any $\phi(v)\in L^1(\mathbb{R}^d)$,

$$-(-\Delta_v)^s\phi := \mathcal{F}^{-1}(|k|^{2s}\mathcal{F}\phi)\,,$$

where

$$\hat\phi(k) := \mathcal{F}[\phi](k) = \int_{\mathbb{R}^d}\phi(v)e^{-iv\cdot k}\mathrm{d}v, \qquad \mathcal{F}^{-1}[\phi](v) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat\phi(k)e^{iv\cdot k}\mathrm{d}k$$

are the Fourier and inverse Fourier transforms. Formally, from this definition $-(-\Delta_v)^s$ reduces to Δ_v if $s=1$. Another way of defining this operator is through the integral representation:

$$(-\Delta_v)^s f(v) := C_{s,d}\ \text{P.V.}\int_{\mathbb{R}^d}\frac{f(v)-f(w)}{|v-w|^{d+2s}}\mathrm{d}w, \tag{1.2}$$

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where $P.V.$ denotes the Cauchy principal value and $C_{s,d} = \frac{4^s\Gamma(d/2+s)}{\pi^{d/2}|\Gamma(-s)|}$. In principle, the fractional Laplacian allows particles to make long jumps at the microscopic scale, which leads to the nonlocal effect as written in (1.2) at the mesoscopic scale.

In the small mean free path regime with a long time, equation (1.1) can be rescaled as

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \mathcal{L}^s(f^\varepsilon), \\ f^\varepsilon(0,x,v) = f_{in}(x,v). \end{cases} \tag{1.3}$$

Formally, f^ε converges to $\rho(t,x)\mathcal{M}(v)$ as $\varepsilon \to 0$, where $\mathcal{M}(v)$ is the unique normalized equilibrium of (1.3) (see [5]) with the properties

$$\mathcal{L}^s(\mathcal{M}) = 0, \quad \int_{\mathbb{R}^d} \mathcal{M}(v)\mathrm{d}v = 1, \quad \mathcal{M}(v) \sim \frac{C}{|v|^{d+2s}} \text{ as } |v| \to \infty. \tag{1.4}$$

Meanwhile, the limiting density $\rho(t,x)$ solves

$$\begin{cases} \partial_t \rho + (-\Delta_x)^s \rho = 0, \\ \rho(0,x) = \rho_{in}(x) := \int_{\mathbb{R}^d} f_{in}(x,v)\mathrm{d}v. \end{cases} \tag{1.5}$$

A weak convergence from f^ε to $\rho\mathcal{M}$ has been established in [2]. In Section 2, we will strengthen this result in two aspects: first, we will show that, this convergence is strong in $L^2_{x,v}$. Second, we will provide an explicit convergence rate in terms of ε. See Theorem 1 for details.

A notable difference between the anomalous scaling in (1.3) and classical diffusive scaling (i.e., $s = 1$ in (1.3)) is that the classical diffusion limit takes the form

$$\partial_t \rho + \nabla_x \cdot (D\nabla_x \rho) = 0,$$

where D is the diffusion matrix

$$D = \int v \otimes v \mathcal{M} \mathrm{d}v.$$

It is clear that the fat tail equilibrium (1.4) renders D unbounded and therefore necessitates the anomalous scaling. Similar scaling has also been investigated in the framework of linear Boltzmann equation

$$\varepsilon^{2s}\partial_t f + \varepsilon v \cdot \nabla_x f = \int \phi(v,v')(\mathcal{M}(v)f(v') - \mathcal{M}(v')f(v))\mathrm{d}v', \tag{1.6}$$

where $\mathcal{M}(v)$ is a given heavy tailed equilibrium satisfies (1.4), and the diffusion limit is again (1.5) (see [1, 13]).

From a computational perspective, it is desirable to design a method that performs uniformly in ε. An asymptotic preserving (AP) scheme would suffice this purpose [9]. However, due to the algebraic decay in the tail of the equilibrium, existing methods that work for exponential decay equilibrium (i.e., Maxwellian) cannot be applied. The reason is that these methods always reside in a truncated velocity domain as the tail information is negligible. By contrast, when the anomalous scaling is considered, the tail carries the most important information that cannot be ignored. For this reason, special treatment are needed to preserve the tail information along the dynamics. This is the main idea behind the works [3, 4, 14, 15], which all deal with the linear Boltzmann type equation in (1.6). Even though the LFP equation shares a similar equilibrium state and diffusion limit with the linear Boltzmann equation (1.6), its structural differences from (1.6) prohibit a direct application of methods developed for (1.6). There are two main obstacles. The first one is because of the differential operators involved in LFP. More specifically, the method used for (1.6) that directly compensates the heavy tail using mass conservation does not apply to LFP since it will violate the smoothness requirement for LFP's differential operators. Second, the Hilbert expansion, which is key to the design of many AP schemes [9, 14, 15], applies well to the linear Boltzmann equation. Unfortunately we do not know how to make it work for LFP.

In [16], a new asymptotic preserving scheme was developed to alleviate the aforementioned difficulties. The main idea is based on a novel type of micro-macro decomposition, with a unique macro part that is inspired by the special choice of the test function in proving the weak convergence in [2]. More specifically, decompose f as

$$f(t,x,v) = \eta(t,x,v)\mathcal{M}(v) + g(t,x,v), \tag{1.7}$$

where $\eta(t,x,v)$ takes the form

$$\eta(t,x,v) = h(t, x+\varepsilon v) \tag{1.8}$$

for some function $h(t,x)$ and $\mathcal{M}$ is the equilibrium state satisfying

$$\partial_v(v\mathcal{M}) - (-\Delta_v)^s\mathcal{M} = 0, \qquad \int_{\mathbb{R}} \mathcal{M}(v)\mathrm{d}v = 1. \tag{1.9}$$

Note that although η depends on (x,v), intrinsically it lives on a lower dimensional manifold than f does. Direct computation using (1.8) gives

$$\varepsilon\partial_x\eta = \partial_v\eta, \qquad (-\Delta_v)^s\eta = \varepsilon^{2s}(-\Delta_x)^s\eta. \tag{1.10}$$

Inserting (1.7) into (1.3), we get

$$\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x(\eta\mathcal{M}+g) = \partial_v(v(\eta\mathcal{M}+g)) - (-\Delta_v)^s(\eta\mathcal{M}+g).$$

By (1.9) and (1.10), it simplifies to

$$\varepsilon^{2s}\partial_t(\eta\mathcal{M}+g) + \varepsilon v\partial_x g = \mathcal{L}^s(g) - \varepsilon^{2s}(-\Delta_x)^s\eta\mathcal{M} - I(\eta,\mathcal{M}), \tag{1.11}$$

where

$$\begin{aligned} I(p,q) &= (-\Delta_v)^s(p\,q) - q\,(-\Delta_v)^s p - p\,(-\Delta_v)^s q \\ &= C_{d,s}\int_{\mathbb{R}^d} \frac{(p(v)-p(w))(q(w)-q(v))}{|v-w|^{d+2s}}\mathrm{d}w. \end{aligned}$$

Splitting (1.11), we introduce the following macro-micro model

$$\begin{cases} \partial_t\eta = -(-\Delta_x)^s\eta, \\ \varepsilon^{2s}\partial_t g + \varepsilon v\partial_x g = \mathcal{L}^s(g) - I(\eta,\mathcal{M}), \end{cases} \tag{1.12}$$

with the initial condition given as

$$\eta_{in}(x,v) = \rho_{in}(x+\varepsilon v), \qquad g_{in}(x,v) = f_{in}(x,v) - \eta_{in}(x,v)\mathcal{M}(v). \tag{1.13}$$

One can recover f in (1.3) by solving (1.12) and using (1.7).

To numerically solving (1.12), we propose a semi-discrete scheme based on an operator splitting:

$$\begin{cases} \dfrac{1}{\Delta t}(\eta^{n+1} - \eta^n) = -(-\Delta_x)^s \eta^{n+1}, & (1.14a) \\ \dfrac{\varepsilon^{2s}}{\Delta t}(g^{n+\frac{1}{2}} - g^n) = \mathcal{L}^s(g^{n+\frac{1}{2}}) - \gamma g^{n+\frac{1}{2}} - I(\eta^{n+1}, \mathcal{M}), & (1.14b) \\ \dfrac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^{n+\frac{1}{2}}) + \varepsilon v \partial_x g^{n+1} = \gamma g^{n+1}, & (1.14c) \end{cases}$$

where γ is a positive constant. The choice of γ will be made explicit throughout the proof (see (4.20), (4.29) and (4.55)). The spatial derivative will be treated via the Fourier-based spectral method, and the velocity discretization will be done by using the mapped Chebyshev-polynomial-based pseudo-spectral method. Note that the scheme in (1.14) is slightly different from that in [16] as we treat η implicitly in (1.14a) and use η^{n+1} to obtain $g^{n+\frac{1}{2}}$ in (1.14c). This will not introduce extra computational cost as the Fourier-based spectral is method used for x .

The goal of this paper is to provide a rigorous foundation to the above scheme. In particular, we will show that the scheme is indeed asymptotic preserving. Hence the convergence of the scheme is uniform with respect to both ε and Δt. The general framework of AP schemes was laid out in [6] and the main idea is illustrated in Fig. 1. Denote by f^ε and f^0 the exact solutions to the

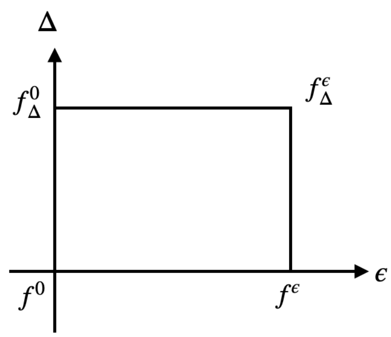

Figure 1: Illustration of AP schemes.

kinetic equation (e.g., (1.1)) and macroscopic limit (e.g, (1.5)), respectively. The same notation with subscript Δ represents the associated numerical solution. The key idea is that to establish a uniform bound for $\|f^\varepsilon_\Delta - f^\varepsilon\|$ by optimizing between two routes: one is the direct error estimate in the kinetic regimes satisfying

$$\|f^\varepsilon_\Delta - f^\varepsilon\| \le C\frac{\Delta^p}{\varepsilon}, \tag{1.15}$$

where p is the order of the adopted numerical discretization. The other is through the asymptotic relation

$$\|f^\varepsilon_\Delta - f^\varepsilon\| \le \left\|f^\varepsilon_\Delta - f^0_\Delta\right\| + \left\|f^0_\Delta - f^0\right\| + \left\|f^0 - f^\varepsilon\right\|, \tag{1.16}$$

where $\|f^0_\Delta - f^0\| \lesssim \Delta^p$ as there is no ε dependence in the equation, while $\|f^0 - f^\varepsilon\| \lesssim \varepsilon$ is guaranteed by the asymptotic relation at the continuum level. *If* the scheme is asymptotic preserving, then we have

$$\left\|f^\varepsilon_\Delta - f^0_\Delta\right\| \le C\varepsilon. \tag{1.17}$$

Consequently, (1.16) becomes

$$\|f^\varepsilon_\Delta - f^\varepsilon\| \le C(\Delta^p + \varepsilon)\,. \tag{1.18}$$

Finally, an optimization between (1.15) and (1.18) gives us the uniform estimate

$$\|f^\varepsilon_\Delta - f^\varepsilon\| \le C\Delta^{\frac{1}{2}} \,. \tag{1.19}$$

We emphasize that throughout the procedures above, C is a constant *independent* of both ε and Δ, a property that we view as a certain uniformity of the traditional AP estimates.

This framework, albeit universal, is not easy to carry out for specific problems. Oftentimes the bottleneck is to obtain the estimate (1.17), which can be as difficult as obtaining the uniform error estimate directly. For this reason, despite the large number of AP schemes designed, rigorous justification of the uniformly accuracy of these schemes is very limited. To our best knowledge, there are only a handful of works [6, 7, 8, 10, 11, 12] that provide a rigorous stability or error estimate, and they all are restricted to linear transport or relaxation type equations.

In this paper, we prove a uniform error estimate for the first time for the *fractional* kinetic equation. In our case, the strong non-locality, both in the Lévy Fokker Planck operator itself and in the slow algebraic decay of the equilibrium, poses tremendous difficulties in the proof as the basic energy estimate fails immediately. To solve this problem, we propose a novel domain decomposition as denoted in Fig. 2, which can be viewed as a generalization of the general framework described above. In particular, we divide the domain into two regimes—Regime I with $\varepsilon^{2s} \ge \Delta t^{2s\beta}$ and Regime II with $\varepsilon^{2s} < \Delta t^{2s\beta}$, where the constant $\beta \in (0, 1/4s)$ is specified in Theorem 2.

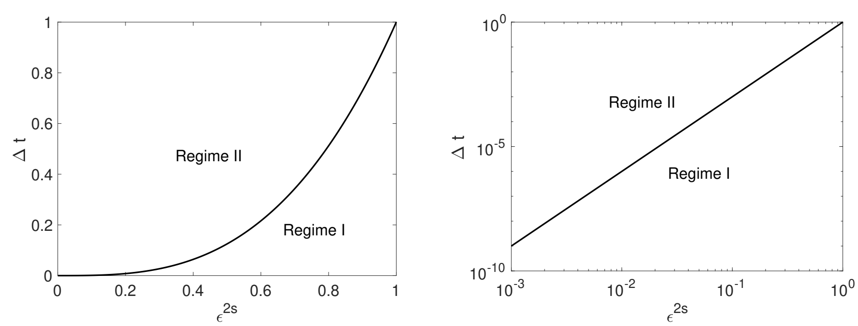

Figure 2: Two regimes separated by the relation between ε^{2s} and Δt. Left: linear scale. Right: log scale.

Denote $W^{k,p}$ and H^k as the usual Sobolev spaces. Introduce the notation $L^2_{\mathcal{M}^{-1}}$ such that

$$f \in L^2_{\mathcal{M}^{-1}} \Longleftrightarrow \|f\|^2_{\mathcal{M}^{-1}} = \int\int \frac{f^2}{\mathcal{M}} \mathrm{d}x\mathrm{d}v < \infty. \tag{1.20}$$

Leaving out some details, our main result is to establish the following error estimate

Main Theorem. *Suppose the initial data are smooth enough such that*

$$v f_{in} \in W^{1,1}_{x,v}, \qquad \rho_{in} \in H^2_x \cap W^{1,\infty}_x, \qquad g_{in},\, \partial_{x_j} g_{in},\, \partial_{v_j v_k} g_{in} \in L^2_{\mathcal{M}^{-1}}$$

for all $1 \le j, k \le d$*. Then outside of an initial layer, there exist constants* $C, b, \zeta > 0$ *independent of* ε *and* Δt *such that*

$$\| \langle v \rangle^{-b} (f^\varepsilon_\Delta - f^\varepsilon)\|_{\mathcal{M}^{-1}} \le C\Delta t^{\zeta}.$$

The specific definitions of b, ζ are prescribed in Theorem 2 and Theorem 3.

Compared with the general framework outlined in (1.15)–(1.19), the main novelty in our proof lies in the relaxation of the two key inequalities (1.15) and (1.17). In fact, owing to the slow decay in the equilibrium which may not even have a finite first moment, we cannot obtain estimates like (1.15) or (1.17) with a uniform constant C. Instead, we relax both upper bounds to a form of $\Delta t^a \varepsilon^b$, wherein one of a and b can be negative while the product as a whole is a quantity with positive power in Δt. This underpins the relation between a and b and leads to the division of the parameter domain. We expect that this generalization provides a new route for proving the uniform accuracy for asymptotic preserving schemes.

The rest of the paper is organized as follows. In Section 2, we prove the strong asymptotic limit from the kinetic to the macroscopic systems in the $L^2_{x,v}$-norm with an explicit convergence rate. This result will be used in obtaining the error estimate in Regime II. Section 3 consists of two technical lemmas on commutator estimates They will appear in the error estimate in Regime I. Finally, Section 4 is devoted to the proof of the main theorem.

2 Strong asymptotic limit

This section is devoted to establishing a strong convergence from (1.3) to (1.5) with an explicit convergence rate. Our main tool is the Fourier transform, through which solutions are derived explicitly. In particular, denote $\hat{f}(t,\xi,k)$ as the Fourier transform of $f(t,x,v)$, i.e.,

$$\hat{f}(t,\xi,k) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,x,v) e^{-i(x\cdot\xi + v\cdot k)} \mathrm{d}x\mathrm{d}v.$$

Then (1.3) rewrites as

$$\begin{cases} \varepsilon^{2s}\partial_t \hat{f} - \varepsilon\xi\cdot\nabla_k \hat{f} = -k\cdot\nabla_k \hat{f} - |k|^{2s}\hat{f}, \\ \hat{f}(0,\xi,k) = \hat{f}_{in}(\xi,k). \end{cases}$$

By the method of characteristics, its solution is

$$\hat{f}(t,\xi,k) = \hat{f}_{in}(\xi, \varepsilon\xi + e^{-\varepsilon^{-2s}t}(k-\varepsilon\xi))\, e^{-\int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w}. \tag{2.1}$$

Similarly, if we denote $\hat{\rho}(t,\xi)$ as the Fourier transform of $\rho(t,x)$, then from (1.5) we have

$$\hat{\rho}(t,\xi) = \hat{f}_{in}(\xi,0)e^{-|\xi|^{2s}t}.$$

Thus the equilibrium state in the Fourier space is

$$\hat{\rho}(t,\xi)\hat{\mathcal{M}}(k) = \hat{f}_{in}(\xi,0)e^{-|\xi|^{2s}t - \frac{1}{2s}|k|^{2s}}. \tag{2.2}$$

The difference of the solutions in (2.1) and (2.2) is as follows.

Theorem 1. *Let f and ρ be the solutions to* (1.3) *and* (1.5) *respectively. Let t_0 be a fixed constant greater than $C\varepsilon^{2s}$. Then for any $t \geq t_0$, we have*

$$\|f-\rho\mathcal{M}\|^2_{L^2_{x,v}} \leq C_{s,t_0}\varepsilon^{2s}\|\rho_{in}\|^2_{L^1} + C\left[\varepsilon^2\|v\cdot\nabla_x f_{in}\|^2_{L^1_{x,v}} + C_{t_0}\varepsilon^{4s}\|vf_{in}\|^2_{L^1_{x,v}}\right],$$

where C_{s,t_0}, C and C_{t_0} are three constants that do not depend on ε.

Some preparations for proving Theorem 1 are in place. Frist we prove an elementary lemma that will be used repeatedly.

Lemma 1. *For all $s \in (0,1)$ and $a, b > 0$, we have the following inequalities:*

$$(a+b)^s \le a^s + b^s; \tag{2.3}$$
$$|(a+b)^{2s} - (a^{2s} + b^{2s})| \le 2a^s b^s. \tag{2.4}$$

Proof. The first inequality follows from the concavity of the function $x \mapsto x^s$ with $x \ge 0$:

$$\left(\frac{a}{a+b}\right)^s + \left(\frac{b}{a+b}\right)^s \ge \left(\frac{a}{a+b} + \frac{b}{a+b}\right)^s = 1.$$

To prove (2.4), by symmetry we can assume that $a \ge b$. Denote $z = \frac{b}{a} \in [0,1]$. Then (2.4) is equivalent to

$$\left|(1+z)^{2s} - (1+z^{2s})\right| \le 2z^s,$$

or equivalently,

$$(1+z)^{2s} - (1+z^{2s}) \le 2z^s \quad \text{and} \quad (1+z^{2s}) - (1+z)^{2s} \le 2z^s. \tag{2.5}$$

The first inequality in (2.5) holds since by (2.3),

$$(1+z)^{2s} \le (1+z^s)^2 = (1+z^{2s}) + 2z^s.$$

The second inequality in (2.5) holds since

$$1 + z^{2s} - 2z^s = (1-z^s)^2 \le (1+z)^{2s}, \qquad z \in [0,1].$$

Thus (2.4) holds. □

By Parseval's identity, we will prove Theorem 1 in the Fourier space. Subtracting (2.2) from (2.1), we have

$$\begin{aligned}
&\hat{f}(t,\xi,k) - \hat{\rho}(t,\xi)\hat{\mathcal{M}}(k). \\
&= \hat{f}_{in}(\xi,0)[e^{-\int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w} - e^{-|\xi|^{2s}t - \frac{1}{2s}|k|^{2s}}] \\
&\qquad + [\hat{f}_{in}(\xi, \varepsilon\xi + e^{-\varepsilon^{-2s}t}(k-\varepsilon\xi)) - \hat{f}_{in}(\xi,0)]e^{-\int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w} \\
&=: \hat{f}_{in}(\xi,0)A_1 + A_2.
\end{aligned} \tag{2.6}$$

The following lemma quantifies the difference in the exponents in A_1.

Lemma 2. *Suppose t_0 is a fixed constant greater than $C\varepsilon^{2s}$. Then for any fixed k and ξ, there exists a constant C_{s,t_0} depending only on s and t_0, such that*

$$\left|\int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w - \left(|\xi|^{2s}t + \frac{|k|^{2s}}{2s}\right)\right| \le C_{s,t_0}\varepsilon^s\left(|\xi|^{2s} + |k|^{2s}\right), \quad \forall\ t \ge t_0.$$

Proof. First we reformulate the integral by decomposing ξ into the component in the direction of k and the one perpendicular to it: $\xi_k := \xi \cdot \frac{k}{|k|}$ and $\xi^\perp = \xi - \xi_k$. Then

$$
\begin{aligned}
J &:= \int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w \\
&= \int_0^{\varepsilon^{-2s}t} \Big[|e^{-w}|k| + (1-e^{-w})\varepsilon\xi_k|^2 + (1-e^{-w})^2\varepsilon^2|\xi^\perp|^2\Big]^s \mathrm{d}w \\
&= \int_0^{\varepsilon^{-2s}t} [\underbrace{e^{-2w}|k|^2 + 2e^{-w}(1-e^{-w})\varepsilon\,\xi_k\,|k|}_{a} + \underbrace{(1-e^{-w})\varepsilon^2|\xi|^2}_{b}]^s\mathrm{d}w \\
&= \int_0^{\varepsilon^{-2s}t} \left((a+b)^s - a^s - b^s + a^s + b^s\right)\mathrm{d}w.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left|J - \left(|\xi|^{2s}t + \frac{|k|^{2s}}{2s}\right)\right| &\le \int_0^{\varepsilon^{-2s}t} |(a+b)^s - a^s - b^s|\,\mathrm{d}w \\
&\quad + \left|\int_0^{\varepsilon^{-2s}t} a^s\mathrm{d}w - \frac{|k|^{2s}}{2s}\right| + \left|\int_0^{\varepsilon^{-2s}t} b^s\mathrm{d}w - |\xi|^{2s}t\right|.
\end{aligned}
$$

We show that each term is of order ϵ^s. By (2.4),

$$
\begin{aligned}
\int_0^{\varepsilon^{-2s}t} |(a+b)^s - a^s - b^s|\,\mathrm{d}w &\le 2\int_0^{\varepsilon^{-2s}t} a^{\frac{s}{2}} b^{\frac{s}{2}}\,\mathrm{d}w \\
&= 2\int_0^{\varepsilon^{-2s}t} [(e^{-w}|k|)^2 + 2e^{-w}(1-e^{-w})\varepsilon\xi_k|k|]^{\frac{s}{2}}(1-e^{-w})^s\varepsilon^s|\xi|^s\mathrm{d}w \\
&\le 2\int_0^{\varepsilon^{-2s}t} [(e^{-w}|k|)^s + (2e^{-w}(1-e^{-w})\varepsilon\xi_k|k|)^{\frac{s}{2}}](1-e^{-w})^s\varepsilon^s|\xi|^s\mathrm{d}w \\
&\le C\varepsilon^s[|k|^{2s} + |\xi|^{2s}],
\end{aligned} \tag{2.7}
$$

where the first inequality follows from (2.3). Next,

$$
\left|\int_0^{\varepsilon^{-2s}t} b^s\mathrm{d}w - |\xi|^{2s}t\right| = \varepsilon^{2s}|\xi|^{2s}\int_0^{-\varepsilon^{-2s}t} \left|(1-e^{-w})^{2s} - 1\right|\mathrm{d}w \le \frac{2}{s}\varepsilon^{2s}|\xi|^{2s}. \tag{2.8}
$$

Using (2.3) again, we have

$$
\begin{aligned}
\int_0^{\varepsilon^{-2s}t}(e^{-w}|k|)^{2s}\mathrm{d}w \le \int_0^{\varepsilon^{-2s}t} a^s\mathrm{d}w &\le \int_0^{\varepsilon^{-2s}t}(e^{-w}|k|)^{2s} + (2e^{-w}(1-e^{-w})\varepsilon\xi_k|k|)^s\mathrm{d}w \\
&\le \int_0^{\varepsilon^{-2s}t}(e^{-w}|k|)^{2s}\mathrm{d}w + C\varepsilon^s\left(|\xi_k|^{2s} + |k|^{2s}\right).
\end{aligned}
$$

Since

$$
\left|\int_0^{\varepsilon^{-2s}t}(e^{-w}|k|)^{2s}\mathrm{d}w - \frac{|k|^{2s}}{2s}\right| = \int_{\varepsilon^{-2s}t}^{\infty}(e^{-|w|}|k|)^{2s}\mathrm{d}w \le |k|^{2s}e^{-\frac{2st}{\varepsilon^{2s}}},
$$

we have, for $t \ge t_0$,

$$
\left|\int_0^{\varepsilon^{-2s}t} a^s\mathrm{d}w - \frac{|k|^{2s}}{2s}\right| \le C_{s,t_0}\varepsilon^s\left(|\xi|^{2s} + |k|^{2s}\right). \tag{2.9}
$$

The desired bound follows from (2.7), (2.8) and (2.9). $\square$

A corollary follows immediately from Lemma 2.

Corollary 1. *Consider $t > t_0$ where t_0 is a fixed constant larger than $C\varepsilon^{2s}$. Then for any fixed k and ξ, there exists a constant C_{s,t_0} that only depends on s and t_0 such that*

$$(1 - C_{s,t_0}\varepsilon^s)\left(|\xi|^{2s}t + \frac{1}{2s}|k|^{2s}\right) \leq \int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w \leq (1 + C_{s,t_0}\varepsilon^s)\left(|\xi|^{2s}t + \frac{1}{2s}|k|^{2s}\right).$$

Then for ε sufficiently small, we have

$$\int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w \geq \frac{1}{2}\left(|\xi|^{2s}t + \frac{1}{2s}|k|^{2s}\right). \tag{2.10}$$

In addition,

$$\left| e^{-\int_0^{\varepsilon^{-2s}t} |e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w} - e^{-|\xi|^{2s}t - \frac{1}{2s}|k|^{2s}} \right| \leq e^{-\frac{1}{2}\left(|\xi|^{2s}t+\frac{1}{2s}|k|^{2s}\right)} C_{s,t_0}\varepsilon^s[|\xi|^{2s} + |k|^{2s}].$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. By Lemma 2 and Corollary 1, the A_1-term in (2.6) satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\hat{f}_{in}(\xi,0)A_1|^2 \mathrm{d}k\mathrm{d}\xi \\
&\leq C_{s,t_0}\varepsilon^{2s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\hat{f}_{in}(\xi,0)|^2 e^{-(|\xi|^{2s}t + \frac{|k|^{2s}}{2s})}[|\xi|^{2s} + |k|^{2s}]^2 \mathrm{d}k\mathrm{d}\xi \leq C_{s,t_0}\varepsilon^{2s}\left\|\rho_{in}\right\|_{L^1}^2.
\end{aligned} \tag{2.11}$$

We bound A_2 as follows.

$$\begin{aligned}
\|A_2\|_{L^2_{k,\xi}}^2 &= \int\int \left| \frac{1}{\langle k\rangle}[\hat{f}_{in}(\xi, \varepsilon\xi + e^{-\varepsilon^{-2s}t}(k-\varepsilon\xi)) - \hat{f}_{in}(\xi,0)]\,\langle k\rangle\, e^{-\int_0^{\varepsilon^{-2s}t}|e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w} \right|^2 \mathrm{d}k\mathrm{d}\xi \\
&\leq \left\| \frac{1}{\langle k\rangle}[\hat{f}_{in}(\xi, \varepsilon\xi + e^{-\varepsilon^{-2s}t}(k-\varepsilon\xi)) - \hat{f}_{in}(\xi,0)] \right\|_{L^\infty_{k,\xi}}^2 \left\| \langle k\rangle\, e^{-\int_0^{\varepsilon^{-2s}t}|e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w} \right\|_{L^2_{k,\xi}}^2 \\
&\leq \left\| (\varepsilon\xi + C_{t_0}\varepsilon^{2s})\partial_k \hat{f}_{in} \right\|_{L^\infty_{k,\xi}}^2 \left\| \langle k\rangle\, e^{-\int_0^{\varepsilon^{-2s}t}|e^{-w}(k-\varepsilon\xi)+\varepsilon\xi|^{2s}\mathrm{d}w} \right\|_{L^2_{k,\xi}}^2 \\
&\leq \left[\varepsilon\left\|v\cdot\nabla_x f_{in}\right\|_{L^1_{x,v}} + C_{t_0}\varepsilon^{2s}\left\|v f_{in}\right\|_{L^1_{x,v}}\right]^2 \left\| \langle k\rangle\, e^{-\frac{1}{2}(|\xi|^{2s}t + \frac{|k|^{2s}}{2s})} \right\|_{L^2_{k,\xi}}^2 \\
&\leq C\left[\varepsilon^2\left\|v\cdot\nabla_x f_{in}\right\|_{L^1_{x,v}}^2 + C_{t_0}\varepsilon^{4s}\left\|v f_{in}\right\|_{L^1_{x,v}}^2\right],
\end{aligned} \tag{2.12}$$

where we have used that for ε sufficiently small and $t > t_0$, it holds that $e^{-\varepsilon^{-2s}t} \leq C_{t_0}\varepsilon^{2s}$. The second last inequality follows from (2.10). Combining (2.11) and (2.12) leads to the desired result. □

At the end of this section, we prove a useful property regarding the decay of the derivatives of the equilibrium state $\mathcal{M}(v)$.

Proposition 1. *Derivatives of the equilibrium state $\mathcal{M}(v)$ satisfies*

$$|\nabla_v^m \mathcal{M}(v)| \leq C_{d,m}|\mathcal{M}(v)|,$$

where $\nabla_v^m\mathcal{M} = \partial_{v_1}^{\alpha_1}\cdots\partial_{v_d}^{\alpha_d}\mathcal{M}(v)$, for $\alpha_i \geq 0$ and $\sum_{i=1}^d \alpha_i = m$ and the constant C_m only depends on m and d.

Proof. Let $\chi(k) \in C_c^\infty$ be a cutoff function with

$$\chi(k) = \begin{cases} 1 & \text{for} \quad k \in B(0,1), \\ 0 & \text{for} \quad |k| > 2. \end{cases}$$

Recall that the equilibrium $\mathcal{M}(v)$ has the following Fourier transform

$$\hat{\mathcal{M}}(k) = Ce^{-\frac{1}{2s}|k|^{2s}}, \quad s \in (0,1).$$

Thus its derivative in the Fourier space reads

$$\widehat{\nabla^m \mathcal{M}}(k) = C(ik)^m \chi(k)\, e^{-\frac{1}{2s}|k|^{2s}} + C(ik)^m (1-\chi(k))\, e^{-\frac{1}{2s}|k|^{2s}}, \quad (ik)^m = (ik_1)^{\alpha_1} \cdots (ik_d)^{\alpha_d}.$$

Taking the inverse Fourier transform, one arrives at

$$\begin{aligned} \nabla^m \mathcal{M}(v) &= \mathcal{F}^{-1}(\widehat{\nabla^m \mathcal{M}}(k)) \\ &= \mathcal{F}^{-1}(C(ik)^m \chi(k) e^{-\frac{1}{2s}|k|^{2s}}) + \mathcal{F}^{-1}(C(ik)^m (1-\chi(k)) e^{-\frac{1}{2s}|k|^{2s}}) \\ &=: M_1(v) + M_2(v). \end{aligned}$$

Since $C(ik)^m(1-\chi(k))e^{-\frac{1}{2s}|k|^{2s}}$ is a Schwartz class function, $M_2(v)$ decays faster than any polynomial. Thus $|M_2(v)| \le C_{d,m}\mathcal{M}(v)$. For M_1, let

$$\hat{g}(k) := C(ik)^m \chi(k), \quad g(v) = \mathcal{F}^{-1}(\hat{g}).$$

Then

$$\begin{aligned} M_1(v) = \mathcal{F}^{-1}(\hat{g}(k)\hat{\mathcal{M}}(k)) &= \int_{\mathbb{R}^d} g(v-w)\mathcal{M}(w)\mathrm{d}w \\ &= \int_{|w| \ge \frac{1}{2}|v|} g(v-w)\mathcal{M}(w)\mathrm{d}w + \int_{|w| < \frac{1}{2}|v|} g(v-w)\mathcal{M}(w)\mathrm{d}w. \end{aligned}$$

When $|w| \ge \frac{1}{2}|v|$, we have $\mathcal{M}(w) \le C\mathcal{M}(v)$. Thus

$$\int_{|w| \ge \frac{1}{2}|v|} g(v-w)\mathcal{M}(w)\mathrm{d}w \le C\mathcal{M}(v) \int_{\mathbb{R}^d} |g|\mathrm{d}w \le C_{d,m}\mathcal{M}(v).$$

When $|w| < \frac{1}{2}|v|$, we have $|v-w| > \frac{1}{2}|v|$. Since $\hat{g}$ is a Schwartz class function, in this subdomain $g(v-w) \le C_\ell \langle v \rangle^{-l}$ for any l. Therefore,

$$\int_{|w| < \frac{1}{2}|v|} g(v-w)\mathcal{M}(w)\mathrm{d}w \le C_\ell \langle v \rangle^{-l} \int_{\mathbb{R}^d} \mathcal{M}(v)\mathrm{d}v, \qquad \forall\, l.$$

The final result follows from combining the estimates for M_1 and M_2. □

3 Two technical lemmas

In this section we group some technical commutator estimates that will appear in the error analysis in the next section.

Lemma 3. *For any function $\bar{g}(x,v)$ smooth enough, the commutator*

$$[\langle v\rangle^p, (-\Delta_v)^s]\bar{g} = \int \frac{\langle w\rangle^p - \langle v\rangle^p}{|v-w|^{d+2s}}\bar{g}(w)\mathrm{d}w \tag{3.1}$$

admits the following estimate for $-2s < p < \frac{d}{2} + 2s$:

$$\|[\langle v\rangle^p, (-\Delta_v)^s]\bar{g}\|_{L^2_{x,v}} \leq C\| \langle v\rangle^p \bar{g}\|_{L^2_{x,v}} + \| \langle v\rangle^{p-1} |\nabla_v \bar{g}|\|_{L^2_{x,v}}.$$

Proof. We divide the integral into four parts and estimate them separately:

$$\begin{aligned}&[\langle v\rangle^p, (-\Delta_v)^s]\bar{g}\\ &= \int_{|v-w|>1, |w|>\frac{1}{2}|v|} \cdot \mathrm{d}w + \int_{|v-w|>1, |w|<\frac{1}{2}|v|} \cdot \mathrm{d}w + \int_{|v-w|\leq 1, |w|>\frac{1}{2}|v|} \cdot \mathrm{d}w + \int_{|v-w|\leq 1, |w|<\frac{1}{2}|v|} \cdot \mathrm{d}w.\end{aligned}$$

When $|v-w| > 1$ and $|w| > \frac{1}{2}|v|$, we have

$$\begin{aligned}&\left\| \int_{|v-w|>1, |w|>\frac{1}{2}|v|} \frac{|\langle w\rangle^p - \langle v\rangle^p|}{|v-w|^{d+2s}} |\bar{g}(w)|\mathrm{d}w \right\|_{L^2_{x,v}}\\ &\qquad \leq \begin{cases} C\left\| \int_{|v-w|>1, |w|>\frac{1}{2}|v|} \frac{\langle v\rangle^p|\bar{g}(w)|}{|v-w|^{d+2s}}\mathrm{d}w \right\|_{L^2_{x,v}} & \text{if } \ p<0, \\ C\left\| \int_{|v-w|>1, |w|>\frac{1}{2}|v|} \frac{\langle w\rangle^p|\bar{g}(w)|}{|v-w|^{d+2s}}\mathrm{d}w \right\|_{L^2_{x,v}} & \text{if } \ p\geq 0. \end{cases}\end{aligned} \tag{3.2}$$

If $p < 0$, then (3.2) can be estimated as follows. If $\frac{1}{2}|v| \leq |w| \leq 2|v|$, then

$$\begin{aligned}&\left\| \int_{|v-w|>1, \frac{1}{2}|v|\leq|w|\leq 2|v|} \frac{|\langle w\rangle^p - \langle v\rangle^p|}{|v-w|^{d+2s}} |\bar{g}(w)|\mathrm{d}w \right\|_{L^2_{x,v}}\\ &\qquad \leq C\left\| (\langle v\rangle^p \bar{g}(v)) * \left(\mathbf{1}_{|v|>1}\frac{1}{|v|^{d+2s}}\right)\right\|_{L^2_{x,v}} \leq C\,\|\langle v\rangle^p \bar{g}\|_{L^2_{x,v}}, \quad p<0.\end{aligned} \tag{3.3}$$

where we have used Young's inequality and the integrability of $\frac{1}{|v|^{d+2s}}$ for $|v| > 1$ in the last inequality. When $|w| \geq 2|v|$, we have

$$\begin{aligned}&\left\| \int_{|v-w|>1, |w|\geq 2|v|} \frac{|\langle w\rangle^p - \langle v\rangle^p|}{|v-w|^{d+2s}} |\bar{g}(w)|\mathrm{d}w \right\|_{L^2_{x,v}} \leq C\left\| \int_{|v-w|>1, |w|\geq 2|v|} \frac{\langle v\rangle^p}{|v-w|^{d+2s}} |\bar{g}(w)|\mathrm{d}w \right\|_{L^2_{x,v}}\\ &\leq C\left\| \int_{|v-w|>1, |w|\geq 2|v|} \frac{\langle v\rangle^p}{|v-w|^{d+2s+p}} \langle w\rangle^p |\bar{g}(w)|\mathrm{d}w \right\|_{L^2_{x,v}}\\ &\leq C\left\| (\langle v\rangle^p \bar{g}(v)) * \left(\mathbf{1}_{|v|>1}\frac{1}{|v|^{d+2s+p}}\right)\right\|_{L^2_{x,v}}\\ &\leq C\,\|\langle v\rangle^p \bar{g}(v)\|_{L^2_{x,v}} \left\| \left(\mathbf{1}_{|v|>1}\frac{1}{|v|^{d+2s+p}}\right)\right\|_{L^1_v}\\ &\leq C\,\|\langle v\rangle^p \bar{g}(v)\|_{L^2_{x,v}}, \qquad \text{for} \quad d+2s+p>d \quad \text{or} \quad p>-2s,\end{aligned}$$

where we have used Young's inequality again. The lower bound $p > -2s$ is to guarantee the integrability of $|v|^{-d-2s-p}$. If $p \geq 0$, then the estimate (3.2) takes the following form:

$$\begin{aligned} &\left\| \int_{|v-w|>1, |w|>\frac{1}{2}|v|} \frac{|\langle w\rangle^p - \langle v\rangle^p|}{|v-w|^{d+2s}} |\bar{g}(w)| \mathrm{d}w \right\|_{L^2_{x,v}} \\ &\quad \leq C \left\| (\langle v\rangle^p \bar{g}) * \left(\mathbf{1}_{|v|>1} \frac{1}{|v|^{d+2s}} \right)] \right\|_{L^2_{x,v}} \leq C \left\| \langle v\rangle^p \bar{g} \right\|_{L^2_{x,v}}, \quad p \geq 0. \end{aligned}$$

Similarly, if $|v-w| > 1$ and $|w| < \frac{1}{2}|v|$, then

$$\left\| \int_{|v-w|>1, |w|<\frac{1}{2}|v|} \frac{|\langle v\rangle^p - \langle w\rangle^p|}{|v-w|^{d+2s}} |\bar{g}(w)| \mathrm{d}w \right\|_{L^2_{x,v}} \leq \begin{cases} C \left\| \int_{|v-w|>1} \frac{\langle w\rangle^p |\bar{g}(w)|}{|v-w|^{d+2s}} \mathrm{d}w \right\|_{L^2_{x,v}} & \text{if } \ p \leq 0, \\ C \left\| \int_{|v-w|>1} \frac{\langle v\rangle^p |\bar{g}(w)|}{|v-w|^{d+2s}} \mathrm{d}w \right\|_{L^2_{x,v}} & \text{if } \ p > 0. \end{cases}$$

The case when $p \leq 0$ can be estimated exactly in the same way as (3.3), which gives

$$\left\| \int_{|v-w|>1, |w|<\frac{1}{2}|v|} \frac{|\langle w\rangle^p - \langle v\rangle^p|}{|v-w|^{d+2s}} |\bar{g}(w)| \mathrm{d}w \right\|_{L^2_{x,v}} \leq C \left\| \langle v\rangle^p \bar{g} \right\|_{L^2_{x,v}}, \quad p \leq 0.$$

A more involved estimate is needed for $p > 0$. Note that in this region, $|v| \geq \frac{2}{3}$ and $|v-w| \geq \frac{1}{2}|v|$. Thus we have

$$\begin{aligned} &\left\| \int_{|v-w|>1, |w|<\frac{1}{2}|v|} \frac{\langle v\rangle^p |\bar{g}(w)|}{|v-w|^{d+2s}} \mathrm{d}w \right\|_{L^2_{x,v}} \leq C \left\| \int_{|v|>\frac{2}{3}, |w|<\frac{1}{2}|v|} \frac{\langle v\rangle^p |\bar{g}(w)|}{|v|^{d+2s}} \mathrm{d}w \right\|_{L^2_{x,v}} \\ &= C \left\{ \iint_{|v|>\frac{2}{3}} \langle v\rangle^{2p-2d-4s} \left[\int_{|w|<\frac{1}{2}|v|} |\bar{g}(w)| \mathrm{d}w \right]^2 \mathrm{d}v \mathrm{d}x \right\}^{\frac{1}{2}} \\ &\leq C \left\{ \iint_{|v|>\frac{2}{3}} \langle v\rangle^{2p-2d-4s} \left[\int_{|w|<\frac{1}{2}|v|} \langle w\rangle^{-2p} \mathrm{d}w \right] \left\| \langle w\rangle^p \bar{g}(w) \right\|^2_{L^2_{x,v}} \mathrm{d}v \mathrm{d}x \right\}^{\frac{1}{2}} \\ &= C \left\{ \int_{|v|>\frac{2}{3}} \langle v\rangle^{2p-2d-4s} \left[\int_{|w|<\frac{1}{2}|v|} \langle w\rangle^{-2p} \mathrm{d}w \right] \mathrm{d}v \right\}^{\frac{1}{2}} \left\| \langle w\rangle^p \bar{g}(w) \right\|_{L^2_{x,v}} \\ &\leq \begin{cases} \text{if } p < \frac{d}{2} & C \left\{ \int_{|v|>\frac{2}{3}} \langle v\rangle^{2p-2d-4s} \langle v\rangle^{d-2p} \mathrm{d}v \right\}^{\frac{1}{2}} \left\| \langle w\rangle^p \bar{g}(w) \right\|_{L^2_{x,v}}, \\ \text{if } p > \frac{d}{2} & \qquad C \left\{ \int_{|v|>\frac{2}{3}} \langle v\rangle^{2p-2d-4s} \mathrm{d}v \right\}^{\frac{1}{2}} \left\| \langle w\rangle^p \bar{g}(w) \right\|_{L^2_{x,v}}, \\ \text{if } p = \frac{d}{2} & C \left\{ \int_{|v|>\frac{2}{3}} \langle v\rangle^{2p-2d-4s} \log \langle v\rangle \mathrm{d}v \right\}^{\frac{1}{2}} \left\| \langle w\rangle^p \bar{g}(w) \right\|_{L^2_{x,v}}. \end{cases} \end{aligned} \tag{3.4}$$

The integrals in (3.4) converge since $p < \frac{d}{2} + 2s$. Combining (3.3)–(3.4), we have

$$\left\| \int_{|v-w|>1} \frac{|\langle w\rangle^p - \langle v\rangle^p|}{|v-w|^{d+2s}} |\bar{g}(w)| \mathrm{d}w \right\|_{L^2_{x,v}} \leq C \| \langle v\rangle^p \bar{g} \|_{L^2_{x,v}}, \qquad -2s < p < \frac{d}{2} + 2s. \tag{3.5}$$

When $|v-w| \leq 1$, we consider $s < \frac{1}{2}$ and $s \geq \frac{1}{2}$ separately. For $s < \frac{1}{2}$, since $|v-w| \leq 1$, we have

$$|\langle w\rangle^p - \langle v\rangle^p| \leq C \langle w\rangle^p. \tag{3.6}$$

Thus

$$\left\|\int_{|v-w|\le 1}\frac{|\langle w\rangle^p-\langle v\rangle^p|}{|v-w|^{d+2s}}|\bar g(w)|\mathrm{d}w\right\|_{L^2_{x,v}}\le C\left\|\int_{|v-w|\le 1|}\frac{\langle w\rangle^{p-1}}{|v-w|^{2s+d-1}}|\bar g(w)|\mathrm{d}w\right\|_{L^2_{x,v}}$$
$$\le C\left\|(\langle v\rangle^{p-1}\bar g)*\frac{\mathbf{1}_{|v|<1}}{|v|^{2s+d-1}}\right\|_{L^2_{x,v}}\le C\left\|\langle v\rangle^{p-1}\bar g\right\|_{L^2_{x,v}},\qquad s<\frac12. \tag{3.7}$$

For $s\ge\frac12$, we rewrite (3.1) in the region $|v-w|\le 1$ as

$$\int_{|v-w|\le 1}\frac{\langle w\rangle^p-\langle v\rangle^p}{|v-w|^{d+2s}}\bar g(w)\mathrm{d}w=\int_{|v-w|\le 1}\frac{\langle w\rangle^p-\langle v\rangle^p}{|v-w|^{d+2s}}[\bar g(w)-\bar g(v)]\mathrm{d}w+\int_{|v-w|\le 1}\frac{\langle w\rangle^p-\langle v\rangle^p}{|v-w|^{d+2s}}\bar g(v)\mathrm{d}w$$
$$=: I_1+I_2.$$

We start with the estimate of I_2. Write

$$\langle w\rangle^p=\langle v\rangle^p+p\langle v\rangle^{p-2}v\cdot(w-v)+\frac12 D:(w-v)\otimes(w-v),$$

where D is the Hessian matrix

$$D=p(p-2)\langle\zeta\rangle^{p-4}\zeta\otimes\zeta,\quad \text{where } |\zeta|\in\lfloor|v|,|w|\rceil.$$

If $\frac12|v|<|w|<2|v|$, then $|D|\le C\langle v\rangle^{p-2}$. If $|w|\le\frac12|v|$ or $|w|>2|v|$, then both $|v|$ and $|w|$ are bounded and it holds that $|D|\le C\langle v\rangle^{p-2}$. Altogether, we have

$$\|I_2\|_{L^2_{x,v}}\le C\left\|\langle v\rangle^{p-2}\bar g\right\|_{L^2_{x,v}}. \tag{3.8}$$

Now we estimate I_1. Expand $\bar g(w)$ as

$$\bar g(w)=\bar g(v)+\int_0^1\nabla_v\bar g(tw+(1-t)v)\cdot(w-v)\mathrm{d}t.$$

Then

$$\|I_1\|^2_{L^2_{x,v}}\le\left\|\int_{|v-w|\le 1}\int_0^1|\nabla_v\bar g(tw+(1-t)v)|\frac{\langle\zeta\rangle^{p-1}}{|v-w|^{d+2s-2}}\mathrm{d}t\mathrm{d}w\right\|^2_{L^2_{x,v}}$$
$$\le C\iint\int_0^1\left[\int_{|v-w|\le 1}\frac{\langle\zeta\rangle^{p-1}}{|v-w|^{d+2s-2}}|\nabla_v\bar g(tw+(1-t)v)|\mathrm{d}w\right]^2\mathrm{d}t\mathrm{d}v\mathrm{d}x,$$

where $|\zeta|$ is between $|v|$ and $|w|$ and the inequalities follow from Cauchy-Schwartz. Let

$$z=tw+(1-t)v\quad\Rightarrow\quad t^d\mathrm{d}w=\mathrm{d}z,\quad |v-z|=t|v-w|.$$

Note that (3.6) holds when $|v-w|\le 1$. Thus

$$\|I_1\|^2_{L^2_{x,v}}\le C\iint\int_0^1\left[\int_{|v-z|\le t}\frac{\langle z\rangle^{p-1}}{|v-z|^{d+2s-2}}|\nabla_z\bar g(z)|t^{d+2s-2}t^{-d}\mathrm{d}z\right]^2\mathrm{d}t\mathrm{d}v\mathrm{d}x$$
$$=C\int_0^1\left\|\int_{|v-z|\le t}\frac{\langle z\rangle^{p-1}}{|v-z|^{d+2s-2}}|\nabla_z\bar g(z)|t^{d+2s-2}t^{-d}\mathrm{d}z\right\|^2_{L^2_{x,v}}\mathrm{d}t$$
$$\le C\int_0^1\left\|\mathbf{1}_{|z|\le t}\frac{1}{|z|^{d+2s-2}}\right\|^2_{L^1_{x,v}}\left\|\langle z\rangle^{p-1}|\nabla_z\bar g(z)|\right\|^2_{L^2_{x,v}}t^{4s-4}\mathrm{d}t$$
$$=C\left\|\langle z\rangle^{p-1}|\nabla_z\bar g(z)|\right\|^2_{L^2_{x,v}}. \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\left\| \int_{|v-w|\le 1} \frac{|\langle w\rangle^p - \langle v\rangle^p|}{|v-w|^{d+2s}} |\bar g(w)| \mathrm{d}w \right\|_{L^2_{x,v}} \le C \| \langle v\rangle^{p-2} \bar g\|_{L^2_{x,v}} + \| \langle v\rangle^{p-1} |\nabla_v \bar g|\|_{L^2_{x,v}} \quad s \ge \frac{1}{2}. \tag{3.10}$$

The final result follows from (3.5), (3.7) and (3.10). □

The second lemma in this section is a commutator estimate for weights and the operator $\mathcal{L}^s$.

Lemma 4. *For $0 \le b \le d+2s$, if u satisfies*

$$|u| \le \frac{C}{\varepsilon^{2s}} \mathcal{M}, \tag{3.11}$$

then the commutator

$$\begin{aligned}[(1+\delta\langle v\rangle)^{-b}, \mathcal{L}^s]u := &- \Big[\nabla_v \cdot (v(1+\delta\langle v\rangle)^{-b} u) - (1+\delta\langle v\rangle)^{-b} \nabla_v \cdot (vu)\Big] \\ &+ \Big[(-\Delta_v)^s((1+\delta\langle v\rangle)^{-b}u) - (1+\delta\langle v\rangle)^{-b}(-\Delta_v)^s u\Big]\end{aligned}$$

satisfies the bound

$$\left| \iint (1+\delta\langle v\rangle)^{-b} u [(1+\delta\langle v\rangle)^{-b}, \mathcal{L}^s] u \frac{1}{\mathcal{M}} \mathrm{d}v\mathrm{d}x \right| \le C\delta^{\frac{2s}{d+4}} \|(1+\delta\langle v\rangle)^{-b} u\|^2_{\mathcal{M}^{-1}} + C\varepsilon^{-4s}\delta^{\min\{1,2s\}}.$$

Proof. Denote

$$\begin{aligned} T_1 u &= \nabla_v \cdot (v(1+\delta\langle v\rangle)^{-b}u) - (1+\delta\langle v\rangle)^{-b}\nabla_v \cdot (vu), \\ T_2 u &= (-\Delta_v)^s((1+\delta\langle v\rangle)^{-b}u) - (1+\delta\langle v\rangle)^{-b}(-\Delta_v)^s u. \end{aligned}$$

Then $[(1+\delta\langle v\rangle)^{-b}, \mathcal{L}^s]u = -T_1 u + T_2 u$. Note that

$$T_1 u = vu \cdot \nabla_v[(1+\delta\langle v\rangle)^{-b}] = -bu(1+\delta\langle v\rangle)^{-b-1}\delta \frac{|v|^2}{\langle v\rangle}.$$

Then from the assumption (3.11), we have

$$\begin{aligned} &\left| \int (1+\delta\langle v\rangle)^{-b} u T_1 u \frac{1}{\mathcal{M}} \mathrm{d}v \right| \le Cb\varepsilon^{-4s} \int \frac{\delta\langle v\rangle}{1+\delta\langle v\rangle} \mathcal{M}\mathrm{d}v \\ &= Cb\varepsilon^{-4s} \int_{|v|<1} \frac{\delta\langle v\rangle}{1+\delta\langle v\rangle}\mathcal{M}\mathrm{d}v + Cb\varepsilon^{-4s}\int_{|v|\ge 1} \frac{\delta\langle v\rangle}{1+\delta\langle v\rangle}\mathcal{M}\mathrm{d}v \le Cb\varepsilon^{-4s}(\delta^{2s}+\delta). \end{aligned}$$

It is immediate that

$$Cb\varepsilon^{-4s}\int_{|v|<1} \frac{\delta\langle v\rangle}{1+\delta\langle v\rangle}\mathcal{M}\mathrm{d}v \le Cb\varepsilon^{-4s}\delta \int_{|v|<1} \langle v\rangle \mathcal{M}\mathrm{d}v \le Cb\varepsilon^{-4s}\delta.$$

Moreover,

$$\begin{aligned} &Cb\varepsilon^{-4s}\int_{|v|\ge 1} \frac{\delta\langle v\rangle}{1+\delta\langle v\rangle}\mathcal{M}\mathrm{d}v \\ &= Cb\varepsilon^{-4s}\int_{1\le|v|\le\frac{1}{\delta}} \frac{\delta\langle v\rangle}{1+\delta\langle v\rangle}\mathcal{M}\mathrm{d}v + Cb\varepsilon^{-4s}\int_{|v|\ge\frac{1}{\delta}} \frac{\delta\langle v\rangle}{1+\delta\langle v\rangle}\mathcal{M}\mathrm{d}v \\ &\le Cb\varepsilon^{-4s}\delta\int_{1\le|v|\le\frac{1}{\delta}} \langle v\rangle \mathcal{M}\mathrm{d}v + Cb\varepsilon^{-4s}\int_{|v|\ge\frac{1}{\delta}} \mathcal{M}\mathrm{d}v \le Cb\varepsilon^{-4s}\delta^{2s}. \end{aligned}$$

Therefore,

$$\left|\int (1+\delta\langle v\rangle)^{-b} u T_1 u \frac{1}{\mathcal{M}} \mathrm{d}v\right| \le Cb\varepsilon^{-4s}(\delta^{2s}+\delta). \tag{3.12}$$

To estimate $T_2 u$, we first rewrite it as

$$T_2 u = C_{s,d} \int u(w) \frac{(1+\delta\langle v\rangle)^{-b} - (1+\delta\langle w\rangle)^{-b}}{|v-w|^{d+2s}} \mathrm{d}w.$$

Hence,

$$\begin{aligned}
&\int (1+\delta\langle v\rangle)^{-b} u T_2 u \frac{1}{\mathcal{M}(v)} \mathrm{d}v \\
&= C_{s,d} \int\int \frac{(1+\delta\langle v\rangle)^{-b} - (1+\delta\langle w\rangle)^{-b}}{|v-w|^{d+2s}} u(w) \frac{u(v)}{(1+\delta\langle v\rangle)^{b}} \frac{1}{\mathcal{M}(v)} \mathrm{d}v\mathrm{d}w \\
&=: C_{s,d} \int\int J \mathrm{d}v\mathrm{d}w.
\end{aligned}$$

We proceed by separating the integration domain into $|v-w| \le C_0$ and $|v-w| > C_0$ for some constant $C_0 > 1$ to be determined and estimate the two integrals individually. Over the domain $|v-w| > C_0$, we have

$$\begin{aligned}
&\left|\int\int_{|v-w|>C_0} J \mathrm{d}v\mathrm{d}w\right| \\
&\le \int\int_{|v-w|>C_0} \frac{|(1+\delta\langle v\rangle)^{b} - (1+\delta\langle w\rangle)^{b}|}{|v-w|^{d+2s}} \frac{1}{(1+\delta\langle v\rangle)^{b}} \frac{|u(v)|}{(1+\delta\langle v\rangle)^{b}} \frac{|u(w)|}{(1+\delta\langle w\rangle)^{b}} \frac{1}{\mathcal{M}(v)} \mathrm{d}w\mathrm{d}v \\
&= \int\int_{|v-w|>C_0, |w|<\frac{1}{2}|v|} + \int\int_{|v-w|>C_0, |w|>2|v|} + \int\int_{|v-w|>C_0, \frac{1}{2}|v|\le|w|\le 2|v|} \\
&=: T_{23} + T_{24} + T_{25}.
\end{aligned}$$

If $|v-w| > C_0$ and $|w| < \frac{1}{2}|v|$, then $|v| > \frac{2}{3}C_0$ and

$$\frac{|(1+\delta\langle v\rangle)^{b} - (1+\delta\langle w\rangle)^{b}|}{|v-w|^{d+2s}} \frac{1}{(1+\delta\langle v\rangle)^{b}} \frac{1}{\mathcal{M}(v)} \le C \frac{(1+\delta\langle v\rangle)^{b}}{|v|^{d+2s}} \frac{1}{(1+\delta\langle v\rangle)^{b}} \frac{1}{\mathcal{M}(v)} \le C.$$

Thus,

$$\begin{aligned}
T_{23} &\le C \int\int_{|v|>\frac{2}{3}C_0} \frac{|u(w)|}{(1+\delta\langle w\rangle)^{b}} \frac{|u(v)|}{(1+\delta\langle v\rangle)^{b}} \mathrm{d}w\mathrm{d}v \\
&\le C \int \frac{|u(w)|}{(1+\delta\langle w\rangle)^{b}} \frac{1}{\sqrt{\mathcal{M}(w)}} \sqrt{\mathcal{M}(w)} \mathrm{d}w \int_{|v|>\frac{2}{3}C_0} \frac{|u(v)|}{(1+\delta\langle w\rangle)^{b}} \frac{1}{\sqrt{\mathcal{M}(v)}} \sqrt{\mathcal{M}(v)} \mathrm{d}v \\
&\le C \|(1+\delta\langle v\rangle)^{-b} u\|^2_{\mathcal{M}^{-1}} \left(\int_{|v|>\frac{2}{3}C_0} \mathcal{M}(v) \mathrm{d}v\right)^{\frac{1}{2}} \\
&\le C C_0^{-s} \|(1+\delta\langle v\rangle)^{-b} u\|^2_{\mathcal{M}^{-1}}.
\end{aligned} \tag{3.13}$$

If $|v-w| > C_0$ and $|w| > 2|v|$, then $|w| > \frac{2}{3}C_0$ and

$$\begin{aligned}
&\frac{|(1+\delta\langle v\rangle)^{b} - (1+\delta\langle w\rangle)^{b}|}{|v-w|^{d+2s}} \frac{1}{(1+\delta\langle v\rangle)^{b}} \frac{1}{\mathcal{M}(v)} \\
&\le C \frac{(1+\delta\langle w\rangle)^{b}}{|w|^{d+2s}} \frac{1}{(1+\delta\langle v\rangle)^{b}} \frac{1}{\mathcal{M}(v)} \le C \frac{(1+\delta\langle w\rangle)^{b} \langle w\rangle^{-d-2s}}{(1+\delta\langle v\rangle)^{b} \langle v\rangle^{-d-2s}},
\end{aligned} \tag{3.14}$$

where in the last inequality we have used $C_1 \langle v\rangle^{-d-2s} \le \mathcal{M}(v) \le C_2 \langle v\rangle^{-d-2s}$ for $0 < C_1 \le C_2$. If we choose $b \le d+2s$, then $(1+\delta\langle w\rangle)^b \langle w\rangle^{-d-2s}$ is a decreasing function in $|w|$ and (3.14) implies

$$\frac{|(1+\delta\langle v\rangle)^b - (1+\delta\langle w\rangle)^b|}{|v-w|^{d+2s}} \frac{1}{(1+\delta\langle v\rangle)^b} \frac{1}{\mathcal{M}(v)} \le C.$$

The rest of the estimate is the same as that of T_{23} and we have

$$T_{24} \le CC_0^{-s} \|(1+\delta\langle v\rangle)^{-b} u\|^2_{\mathcal{M}^{-1}}. \tag{3.15}$$

If $|v-w| > C_0$ and $\frac{1}{2}|v| < |w| < 2|v|$, then

$$\frac{|(1+\delta\langle v\rangle)^b - (1+\delta\langle w\rangle)^b|}{(1+\delta\langle v\rangle)^b} \le C.$$

Hence,

$$\begin{aligned}
T_{25} &\le C \int\int_{|v-w|>C_0, \frac{1}{2}|v|<|w|<2|v|} \frac{1}{|v-w|^{2s+d}} \frac{|u(w)|}{(1+\delta\langle w\rangle)^b} \frac{|u(v)|}{(1+\delta\langle v\rangle)^b} \frac{1}{\mathcal{M}(v)} \mathrm{d}w\mathrm{d}v \\
&\le C \int\int_{|v-w|>C_0, \frac{1}{2}|v|<|w|<2|v|} \frac{1}{|v-w|^{2s+d}} \frac{|h(w)|}{(1+\delta\langle w\rangle)^b} \frac{|u(v)|}{(1+\delta\langle v\rangle)^b} \frac{1}{\sqrt{\mathcal{M}(v)}} \frac{1}{\sqrt{\mathcal{M}(w)}} \mathrm{d}w\mathrm{d}v \\
&\le CC_0^{-2s} \|(1+\delta\langle v\rangle)^{-b} u\|^2_{\mathcal{M}^{-1}}.
\end{aligned} \tag{3.16}$$

Combining (3.13), (3.15) and (3.16), we have

$$\left| \int\int_{|v-w|>C_0} J \mathrm{d}v\mathrm{d}w \right| \le CC_0^{-s} \|(1+\delta\langle v\rangle)^{-b} u\|^2_{\mathcal{M}^{-1}}. \tag{3.17}$$

For the domain $|v-w| \le C_0$, we symmetrize the integral as

$$\begin{aligned}
&\int\int_{|v-w|\le C_0} J \mathrm{d}v\mathrm{d}w \\
&= \frac{1}{2} \int\int_{|v-w|\le C_0} \frac{(1+\delta\langle v\rangle)^b - (1+\delta\langle w\rangle)^b}{|v-w|^{d+2s}} \frac{h(w)}{(1+\delta\langle w\rangle)^b} \frac{h(v)}{(1+\delta\langle v\rangle)^b} \times \\
&\qquad\qquad \times \left[\frac{1}{\mathcal{M}(v)(1+\delta\langle v\rangle)^b} - \frac{1}{\mathcal{M}(w)(1+\delta\langle w\rangle)^b} \right] \mathrm{d}v\mathrm{d}w.
\end{aligned}$$

By the mean value theorem, for $\langle \eta_1\rangle, \langle \eta_2\rangle$ between $\langle v\rangle$ and $\langle w\rangle$, we have

$$\begin{aligned}
&\left| \int\int_{|v-w|\le C_0} J \mathrm{d}v\mathrm{d}w \right| \\
&\le b\delta \int\int \frac{(1+\delta\langle \eta_1\rangle)^{b-1} \frac{|\eta_1|}{\langle \eta_1\rangle}}{|v-w|^{d+2s-2}} \frac{|h(w)|}{(1+\delta\langle w\rangle)^b} \frac{|h(v)|}{(1+\delta\langle v\rangle)^b} \times \\
&\qquad\qquad \times \frac{1}{\mathcal{M}(\eta_2)(1+\delta\langle \eta_2\rangle)^b} \left| \frac{\nabla\mathcal{M}(\eta_2)}{\mathcal{M}(\eta_2)} + \frac{b\delta \frac{\eta_2}{\langle \eta_2\rangle}}{(1+\delta\langle \eta_2\rangle)} \right| \mathrm{d}v\mathrm{d}w \\
&\le b\delta \int\int \frac{1}{|v-w|^{d+2s-2}} \frac{|h(w)|}{(1+\delta\langle w\rangle)^b} \frac{|h(v)|}{(1+\delta\langle v\rangle)^b} \times \\
&\qquad\qquad \times \frac{(1+\delta\langle \eta_1\rangle)^{b-1}}{\mathcal{M}(\eta_2)(1+\delta\langle \eta_2\rangle)^b} \left| \frac{\nabla\mathcal{M}(\eta_2)}{\mathcal{M}(\eta_2)} + \frac{b\delta \frac{\eta_2}{\langle \eta_2\rangle}}{1+\delta\langle \eta_2\rangle} \right| \mathrm{d}v\mathrm{d}w \\
&\le b\delta(1+C_0\delta)^{\max\{b-1,0\}} \int\int \frac{1}{|v-w|^{d+2s-2}} \frac{|h(w)|}{(1+\delta\langle w\rangle)^b} \frac{|h(v)|}{(1+\delta\langle v\rangle)^b} \frac{1}{\mathcal{M}(\eta_2)} \mathrm{d}v\mathrm{d}w,
\end{aligned} \tag{3.18}$$

where we have used $\left|\frac{\nabla\mathcal{M}(\eta_2)}{\mathcal{M}(\eta_2)}\right| \leq C$ thanks for Proposition 1. In addition, if $\frac{1}{2}|v| < |w| \leq 2|v|$, then

$$\langle \eta \rangle \ \leq C \langle v \rangle \quad \text{and} \quad \frac{(1+\delta \langle \eta \rangle)^{b-1}}{(1+\delta\langle v\rangle)^b} \leq C.$$

If $|w| \geq 2|v|$, then we have $2|v| \leq |w| \leq 2C_0$ and

$$(1+\delta\langle v\rangle)^{b-1} \leq \begin{cases} C(1+\delta C_0)^{b-1} & b>1, \\ C & b \leq 1. \end{cases}$$

Therefore, we always have

$$\frac{(1+\delta\langle\eta_1\rangle)^{b-1}}{(1+\delta\langle\eta_2\rangle)^b} \leq C(1+C_0\delta)^{\max\{b-1,0\}}.$$

Since $|v-w| \leq C_0$, if $|w| > 2|v|$, then we have $|v| \leq C_0$ and $|w| \leq 2C_0$. If $|w| < \frac{1}{2}|v|$, then $|v| \leq 2C_0$ and $|w| \leq C_0$. In both cases,

$$\frac{1}{\mathcal{M}(\eta_2)} \leq C \max\left\{\frac{1}{\mathcal{M}(w)}, \frac{1}{\mathcal{M}(v)}\right\} \leq C \frac{1}{\sqrt{\mathcal{M}(w)}} \frac{1}{\sqrt{\mathcal{M}(v)}} C_0^{\frac{d}{2}+s}. \tag{3.19}$$

When $\frac{1}{2}|v| \leq |w| \leq 2|v|$, since $\langle \eta_2\rangle$ is between $\langle v\rangle$, and $\langle w\rangle$, we have

$$\frac{1}{\mathcal{M}(\eta_2)} \leq C \frac{1}{\sqrt{\mathcal{M}(w)}} \frac{1}{\sqrt{\mathcal{M}(v)}}. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18), we have

$$\begin{aligned}
&\left|\int\int_{|v-w|\leq C_0} J \mathrm{d}v\mathrm{d}w\right| \\
&\leq b\delta(1+C_0\delta)^{\max\{b-1,0\}} C_0^{\frac{d}{2}+s} \iint_{|v-w|<C_0} \frac{1}{|v-w|^{d+2s-2}} \frac{|u(w)|}{(1+\delta\langle w\rangle)^b} \times \\
&\qquad\qquad \times \frac{|u(v)|}{(1+\delta\langle v\rangle)^b} \frac{1}{\sqrt{\mathcal{M}(w)}} \frac{1}{\sqrt{\mathcal{M}(v)}} \mathrm{d}v\mathrm{d}w \\
&\leq Cb\delta(1+\delta C_0)^{\max\{b-1,0\}} C_0^{\frac{d}{2}+s} \left(\iint_{|v-w|<C_0} \frac{1}{|v-w|^{2s+d-2}} \frac{|u(v)|^2}{(1+\delta\langle v\rangle)^{2b}} \frac{1}{\sqrt{\mathcal{M}(v)}} \mathrm{d}v\mathrm{d}w\right)^{\frac{1}{2}} \times \\
&\qquad \times \left(\iint_{|v-w|<C_0} \frac{1}{|v-w|^{2s+d-2}} \frac{|u(w)|^2}{(1+\delta\langle w\rangle)^{2b}} \frac{1}{\sqrt{\mathcal{M}(w)}} \mathrm{d}v\mathrm{d}w\right)^{\frac{1}{2}} \\
&\leq Cb\delta(1+\delta C_0)^{\max\{b-1,0\}} C_0^{\frac{d}{2}+s} C_0^{2-2s} \|(1+\delta\langle v\rangle)^{-b} u\|^2_{\mathcal{M}^{-1}}.
\end{aligned} \tag{3.21}$$

Summation of (3.17) and (3.21) leads to

$$\iint (1+\delta\langle v\rangle)^{-b} u T_2 u \frac{1}{\mathcal{M}} \mathrm{d}x\mathrm{d}v \leq C\left[b\delta(1+\delta C_0)^{\max\{b-1,0\}} C_0^{\frac{d}{2}+2-s} + C_0^{-s}\right] \|(1+\delta\langle v\rangle)^{-b}u\|^2_{\mathcal{M}^{-1}}.$$

Choose $C_0 = \delta^{-\frac{2}{d+4}}$. Then the inequality above becomes

$$\iint (1+\delta\langle v\rangle)^{-b} u T_2 u \frac{1}{\mathcal{M}} \mathrm{d}x\mathrm{d}v \leq C\delta^{\frac{2s}{d+4}} \|(1+\delta\langle v\rangle)^{-b}u\|^2_{\mathcal{M}^{-1}}.$$

Together with (3.12), this implies

$$\left|\iint (1+\delta\langle v\rangle)^{-b} u[(1+\delta\langle v\rangle)^{-b}, \mathcal{L}^s] u \frac{1}{\mathcal{M}} \mathrm{d}x\mathrm{d}v\right| \leq C\delta^{\frac{2s}{d+4}} \|(1+\delta\langle v\rangle)^{-b}u\|^2_{\mathcal{M}^{-1}} + C\varepsilon^{-4s}\delta^{\min\{1,2s\}}.$$

We thereby finish the proof of the commutator estimate. □

4 Error estimate

In this section we prove the uniform accuracy in time for the semi-discrete scheme (1.14). When ε is large compared with Δt, it corresponds to the kinetic regime, in which we show that the numerical error can be controlled by some positive power of Δt. This shows the consistency of the method. When ε is small compared with Δt, we will leverage the asymptotic preserving property of our scheme to control the numerical error by a certain positive power of ε. The specific regimes will be made clear later.

Given the initial data $f_{in}(x,v)$, we have the initial data for ρ and g as

$$\rho_{in} = \langle f_{in} \rangle_v \,, \quad g_{in}(x,v) = f_{in}(x,v) - \rho_{in}(x+\varepsilon v)\mathcal{M}(v).$$

Assume the initial data are chosen such that the following quantities are bounded:

$$\|g_{in}\|_{\mathcal{M}^{-1}}, \ \|g_{in}\|_{L^\infty_{\mathcal{M}^{-1}}}, , \ \|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}}, \ \|\partial_{v_i v_j} g_{in}\|_{\mathcal{M}^{-1}}, \ \|\rho_{in}\|_{H^2_x}, \ \|\rho_{in}\|_{W^{1,\infty}} \ < +\infty. \tag{4.1}$$

Add the last two equations in (1.14) to get

$$\begin{cases} \dfrac{1}{\Delta t}(\eta^{n+1} - \eta^n) = -(-\Delta_x)^s \eta^{n+1}, & (4.2a) \\ \dfrac{\varepsilon^{2s}}{\Delta t}(g^{n+1} - g^n) + \varepsilon v \partial_x g^{n+1} = \mathcal{L}^s(g^{n+\frac{1}{2}}) - \gamma g^{n+\frac{1}{2}} + \gamma g^{n+1} - I(\eta^{n+1}, \mathcal{M}). & (4.2b) \end{cases}$$

Let e_1^n and e_2^n be the local truncation errors defined by

$$\begin{aligned} e_1^n &= \frac{1}{\Delta t^2}(g(t^{n+1}) - g(t^n) - \partial_t g(t^{n+1})), \\ e_2^n &= \frac{1}{\Delta t^2}(\eta(t^{n+1}) - \eta(t^n) - \partial_t \eta(t^{n+1})). \end{aligned}$$

Then (1.12) becomes

$$\begin{cases} \dfrac{1}{\Delta t}\left[\eta(t^{n+1}) - \eta(t^n)\right] = -(-\Delta_x)^s \eta(t^{n+1}) + \Delta t e_2^n, & (4.3a) \\ \dfrac{\varepsilon^{2s}}{\Delta t}\left[g(t^{n+1}) - g(t^n)\right] + \varepsilon v \partial_x g(t^{n+1}) = \mathcal{L}^s g(t^{n+1}) - I(\eta(t^{n+1}), \mathcal{M}) + \varepsilon^{2s} \Delta t e_1^n, & (4.3b) \end{cases}$$

Denote

$$\tilde{\eta}^n(x,v) = \eta(t^n, x, v) - \eta^n(x,v), \qquad \tilde{g}^n(x,v) = g(t^n, x, v) - g^n(x,v)$$

as the numerical error. By subtracting (4.2) from (4.3), we get

$$\begin{cases} \dfrac{1}{\Delta t}(\tilde{\eta}^{n+1} - \tilde{\eta}^n) = -(-\Delta_x)^s \tilde{\eta}^{n+1} + \Delta t e_2^n, & (4.4a) \\ \dfrac{\varepsilon^{2s}}{\Delta t}(\tilde{g}^{n+1} - \tilde{g}^n) + \varepsilon v \partial_x \tilde{g}^{n+1} = \mathcal{L}^s \tilde{g}^{n+1} - I(\tilde{\eta}^{n+1}, \mathcal{M}) + (\mathcal{L}^s - \gamma \mathcal{I})(g^{n+1} - g^{n+\frac{1}{2}}) + \Delta t \varepsilon^{2s} e_1^n. & (4.4b) \end{cases}$$

Denote the error from the operator splitting as

$$e_3^n = (\mathcal{L}^s - \gamma \mathcal{I})(g^{n+1} - g^{n+\frac{1}{2}})$$

and denote

$$\alpha = \frac{\varepsilon^{2s}}{\Delta t}, \qquad \tilde{f}^n = \tilde{\eta}^n \mathcal{M} + \tilde{g}^n. \tag{4.5}$$

Multiply (4.4a) by $\varepsilon^{2s}\mathcal{M}$ and add to (4.4b). Then the error equation (4.4) has the form

$$\alpha(\tilde{f}^{n+1}-\tilde{f}^{n})+\varepsilon v\partial_x\tilde{f}^{n+1}=\mathcal{L}^s\tilde{f}^{n+1}+\Delta te_1^n+e_3^n+\varepsilon^{2s}\Delta te_2^n\mathcal{M}. \tag{4.6}$$

In view of (4.2a), η^n evolves independently from g^n. We summarize the properties of η^n as follows.

Lemma 5. *Denote*

$$\eta^n(x,v)=h^n(x+\varepsilon v).$$

Then $h^n(y)$ solves

$$\frac{1}{\Delta t}(h^{n+1}-h^n)=-(-\Delta_y)^s h^{n+1},\quad h^0(y)=\rho_{in}(y) \tag{4.7}$$

and we have

$$\|h^n\|_{H_x^1}\le\left\|h^0\right\|_{H_x^1}=\|\rho_{in}\|_{H_x^1},\qquad \|h^n\|_{W_x^{1,\infty}}\le\left\|h^0\right\|_{W_x^{1,\infty}}=\|\rho_{in}\|_{W_x^{1,\infty}}.$$

Proof. Equation (4.7) directly follows from (4.2a) and (1.13) and the inequality is the result of the dissipative property of $-(-\Delta_x)^s$. □

We have the following estimates for errors e_1^n and e_2^n.

Lemma 6. *If (η,g) satisfies (4.1), then e_1^n, e_2^n defined in (4.8) and (4.9) satisfy*

$$\|e_1^n\|_{\mathcal{M}^{-1}}\le C\varepsilon^{-4s}\sum_{i,j=1}^{2}\||v|^2(\partial_{v_iv_j}+\partial_{x_ix_j})f_{in}\|_{\mathcal{M}^{-1}}+\frac{1}{2}\left\|(-\Delta_x)^{2s}\rho_{in}\right\|_{L_x^2},$$
$$\|e_2^n\mathcal{M}\|_{\mathcal{M}^{-1}}\le\frac{1}{2}\left\|(-\Delta_x)^{2s}\rho_{in}\right\|_{L_x^2}.$$

Proof. Rewrite e_1^n and e_2^n in their integral forms:

$$e_1^n=\frac{1}{\Delta t^2}\int_{t^n}^{t^{n+1}}(\partial_s g(s)-\partial_s g(t^{n+1}))\mathrm{d}s=\frac{1}{\Delta t^2}\int_{t^n}^{t^{n+1}}\int_s^{t^{n+1}}\partial_{\tau\tau}g(\tau,x,v)\mathrm{d}\tau\mathrm{d}s. \tag{4.8}$$

Similarly,

$$e_2^n=\frac{1}{\Delta t^2}\int_{t^n}^{t^{n+1}}\int_s^{t^{n+1}}\partial_{\tau\tau}\eta(\tau,x,v)\mathrm{d}\tau\mathrm{d}s. \tag{4.9}$$

Recall that $\eta(t,x,v)=h(t,x+\varepsilon v)$ where $h(t,y)$ solves

$$\partial_t h=-(-\Delta_y)^s h,\qquad h(0,y)=\rho_{in}(y).$$

Then $\partial_{tt}h(t,y)$ satisfies

$$\partial_t(\partial_{tt}h)=-(-\Delta_y)^s(\partial_{tt}h),\qquad \partial_{tt}h(0,y)=(-\Delta_y)^{2s}\rho_{in}(y).$$

Note that $\partial_{tt}\eta(t,x,v)=\partial_{tt}h(t,x+\varepsilon v)$. Thus we have

$$\|e_2\mathcal{M}\|_{\mathcal{M}^{-1}}\le\frac{1}{\Delta t^2}\int_{t^n}^{t^{n+1}}\int_s^{t^{n+1}}\|\partial_{\tau\tau}\eta(\tau,\cdot,\cdot)\|_{\mathcal{M}^{-1}}\mathrm{d}\tau\mathrm{d}s\le C\left\|(-\Delta_x)^{2s}\rho_{in}\right\|_{L_x^2}. \tag{4.10}$$

To bound e_1^n, we make use of equation (1.3) for f. Note that $\partial_{tt} f$ solves the same equation as f with initial condition

$$\begin{aligned}\partial_{tt} f(0,x,v) = \varepsilon^{2-4s}(v\cdot\nabla_x)(v\cdot\nabla_x) f_{in}(x,v) - \varepsilon^{1-4s} v\cdot\nabla_x(\mathcal{L}^s f_{in})\\ - \varepsilon^{1-4s}\mathcal{L}^s(v\cdot\nabla_x f_{in}) + \varepsilon^{-4s}\mathcal{L}^s\mathcal{L}^s f_{in}.\end{aligned}$$

Due to the dissipative mechanism of $\mathcal{L}^s$, we have

$$\|\partial_{tt} f\|_{\mathcal{M}^{-1}} \leq \|\partial_{tt} f_{in}\|_{\mathcal{M}^{-1}} \leq C\varepsilon^{-4s}\sum_{i,j=1}^{2} \||v|^2(\partial_{v_iv_j} + \partial_{x_ix_j}) f_{in}\|_{\mathcal{M}^{-1}}. \tag{4.11}$$

Then

$$\begin{aligned}\|e_1^n\|_{\mathcal{M}^{-1}} &= \frac{1}{\Delta t^2}\|\int_{t^n}^{t^{n+1}}\int_s^{t^{n+1}} \partial_{\tau\tau} g(\tau,x,v)\mathrm{d}\tau\mathrm{d}s\|_{\mathcal{M}^{-1}}\\ &\leq \frac{1}{\Delta t^2}\int_{t^n}^{t^{n+1}}\int_s^{t^{n+1}} \|\partial_{\tau\tau}(f-\eta\mathcal{M})(\tau,\cdot,\cdot)\|_{\mathcal{M}^{-1}}\mathrm{d}\tau\mathrm{d}s\\ &\leq \frac{1}{\Delta t^2}\int_{t^n}^{t^{n+1}}\int_s^{t^{n+1}} \|\partial_{\tau\tau} f\|_{\mathcal{M}^{-1}} + \|\partial_{\tau\tau}\eta\mathcal{M}\|_{\mathcal{M}^{-1}}\mathrm{d}\tau\mathrm{d}s.\end{aligned}$$

The desired bound for e_1^n follows from (4.10) and (4.11). □

Next, we bound $g^{n+1} - g^{n+\frac{1}{2}}$ and g^{n+1}. These are for estimating the splitting and asymptotic errors in the fractional diffusion regime. In both cases, we start with the estimate for $I(\eta^{n+1},\mathcal{M})$.

Lemma 7 (Estimate of $I(\eta,\mathcal{M})$)**.** *Denote $\eta(x,v) = h(x+\varepsilon v)$. Then for $s\in(0,1)$, we have*

1) $\|I(\eta,\mathcal{M})\|_{L^2_{x,v}} \leq C\|h\|_{H^1_x}\varepsilon^s$;

2) $\|D^k I(\eta,\mathcal{M})\|_{\mathcal{M}^{-1}} \leq C\|h\|_{H_x^{k+1}}\varepsilon^s$, $0\leq k\leq 2$;

3) $|I(\eta,\mathcal{M})| \leq C\,\|h\|_{W^{1,\infty}}\,\mathcal{M}$.

Here $\|\cdot\|_{\mathcal{M}^{-1}}$ is defined in (1.20), and $D^1 f = \partial_{v_i} f$, $D^2 f = \partial_{v_iv_j} f$.

Proof. 1) & 2): Since part 1) is a direct consequence of part 2) via Cauchy-Schwartz, we only show the proof of part 2). Note that

$$\begin{aligned}\|I(\eta,\mathcal{M})\|_{\mathcal{M}^{-1}} &\leq C\left\|\int_{\mathbb{R}^d} \frac{\|(h(x+\varepsilon v) - h(x+\varepsilon w))\|_{L^2_x}\,|\mathcal{M}(v)-\mathcal{M}(w)|}{|v-w|^{d+2s}\sqrt{\mathcal{M}(v)}}\mathrm{d}w\right\|_{L^2_v}\\ &= C\left\|\int_{|v-w|>\frac{1}{\varepsilon}}\cdot\mathrm{d}w\right\|_{L^2_v} + C\left\|\int_{|v-w|<\frac{1}{\varepsilon}}\cdot\mathrm{d}w\right\|_{L^2_v} =: C\,\|I_1\|_{L^2_v} + C\,\|I_2\|_{L^2_v}.\end{aligned}$$

For I_1, we have

$$\|(h(x+\varepsilon v) - h(x+\varepsilon w))\|_{L^2_x} \leq \|h(x+\varepsilon v)\|_{L^2_x} + \|h(x+\varepsilon w)\|_{L^2_x} \leq 2\,\|h\|_{L^2_x}.$$

Thus,

$$\|I_1\|_{L^2_v} \leq C\,\|h\|_{L^2_x}\left\|\int_{|v-w|>\frac{1}{\varepsilon}} \frac{|\mathcal{M}(v)-\mathcal{M}(w)|}{|v-w|^{d+2s}}\frac{1}{\sqrt{\mathcal{M}(v)}}\mathrm{d}w\right\|_{L^2_v}.$$

If $|v-w|>\frac{1}{\varepsilon}$ and $|w|>\frac{1}{2}|v|$, then $\mathcal{M}(w)\le C\mathcal{M}(v)$. In this case we have

$$\begin{aligned}
&\left\|\int_{|v-w|>\frac{1}{\varepsilon},|w|>\frac{1}{2}|v|}\frac{|\mathcal{M}(v)-\mathcal{M}(w)|}{|v-w|^{d+2s}}\frac{1}{\sqrt{\mathcal{M}(v)}}\mathrm{d}w\right\|_{L^2_v}\\
&\le C\left\|\int_{|v-w|>\frac{1}{\varepsilon},|w|>\frac{1}{2}|v|}\frac{\sqrt{\mathcal{M}(v)}}{|v-w|^{d+2s}}\mathrm{d}w\right\|_{L^2_v}\\
&\le C\int_{|v-w|>\frac{1}{\varepsilon}}\frac{1}{|v-w|^{d+2s}}\mathrm{d}w\,\|\mathcal{M}\|^{\frac{1}{2}}_{L^1_v}=C\varepsilon^{2s}. \qquad (4.12)
\end{aligned}$$

If $|v-w|>\frac{1}{\varepsilon}$ and $|w|\le\frac{1}{2}|v|$, then $\mathcal{M}(v)\le C\mathcal{M}(w)$, $|v|>\frac{2}{3\varepsilon}$ and $|v-w|\ge\frac{1}{2}|v|$, and we have

$$\begin{aligned}
&\left\|\int_{|v-w|>\frac{1}{\varepsilon},|w|\le\frac{1}{2}|v|}\frac{|\mathcal{M}(v)-\mathcal{M}(w)|}{|v-w|^{d+2s}}\frac{1}{\sqrt{\mathcal{M}(v)}}\mathrm{d}w\right\|_{L^2_v}\\
&\le C\left\|\int_{|v-w|>\frac{1}{\varepsilon},|w|\le\frac{1}{2}|v|}\frac{\mathcal{M}(w)}{|v-w|^{d+2s}}\frac{1}{\sqrt{\mathcal{M}(v)}}\mathrm{d}w\right\|_{L^2_v}\\
&\le C\left\|\int_{\mathbb{R}^d}\frac{\mathcal{M}(w)}{|v|^{d+2s}}\frac{1}{\sqrt{\mathcal{M}(v)}}\mathbf{1}_{|v|>\frac{2}{3\varepsilon}}\mathrm{d}w\right\|_{L^2_v}\\
&\le C\Big(\int_{|v|>\frac{2}{3\varepsilon}}\mathcal{M}(v)\mathrm{d}v\Big)^{\frac{1}{2}}\int\mathcal{M}(w)\mathrm{d}w=C\varepsilon^{s}. \qquad (4.13)
\end{aligned}$$

Combining (4.12) and (4.13) gives

$$\|I_1\|_{L^2_v}\le C\|h\|_{L^2_x}\varepsilon^s.$$

For I_2, we rewrite it as

$$\begin{aligned}
\|I_2\|_{L^2_v}&=\left\|\int_{|v-w|\le 1}\frac{\|(h(x+\varepsilon v)-h(x+\varepsilon w))\|_{L^2_x}|\mathcal{M}(v)-\mathcal{M}(w)|}{|v-w|^{d+2s}\sqrt{\mathcal{M}(v)}}\mathrm{d}w\right\|_{L^2_v}\\
&\quad+\left\|\int_{1\le|v-w|\le\frac{1}{\varepsilon}}\frac{\|(h(x+\varepsilon v)-h(x+\varepsilon w))\|_{L^2_x}|\mathcal{M}(v)-\mathcal{M}(w)|}{|v-w|^{d+2s}\sqrt{\mathcal{M}(v)}}\mathrm{d}w\right\|_{L^2_v}\\
&=:\|I_{21}\|_{L^2_v}+\|I_{22}\|_{L^2_v}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\|h(x+\varepsilon v)-h(x+\varepsilon w)\|_{L^2_x}\le\varepsilon\|\nabla h\|_{L^2_x}|v-w|,\\
&\mathcal{M}(v)-\mathcal{M}(w)=\nabla\mathcal{M}(\xi)\cdot(v-w),\quad\text{where}\quad\xi=(1-t)v+tw,\ t\in(0,1).
\end{aligned}$$

Then for $|v-w|\le 1$, we have

$$|\nabla\mathcal{M}(\xi)|\le C|\nabla\mathcal{M}(v)|\le C\mathcal{M}(v), \qquad (4.14)$$

where for the first inequality we have used the fact that $|\xi-v|\le 1$ and the second comes as a result of Proposition 1. Therefore I_{21} satisfies

$$\|I_{21}\|_{L^2_v}\le\varepsilon\|\nabla h\|_{L^2_x}\left\|\int_{|v-w|<1}\frac{1}{|v-w|^{d+2s-2}}\sqrt{\mathcal{M}(v)}\mathrm{d}w\right\|_{L^2_v}\le C\varepsilon\|\nabla h\|_{L^2_x}. \qquad (4.15)$$

Similarly, for I_{22}, we have

$$\begin{aligned}\|I_{22}\|_{L^2_v} &\le C\varepsilon \|\nabla h\|_{L^2_x} \left\| \int_{1\le|v-w|\le\frac{1}{\varepsilon},|w|\le\frac{1}{2}|v|} \frac{1}{|v-w|^{d+2s-1}} \frac{|\mathcal{M}(v)-\mathcal{M}(w)|}{\sqrt{\mathcal{M}(v)}} \mathrm{d}w \right\|_{L^2_v} \\ &\quad + C\varepsilon \|\nabla h\|_{L^2_x} \left\| \int_{1\le|v-w|\le\frac{1}{\varepsilon},|w|>\frac{1}{2}|v|} \frac{1}{|v-w|^{d+2s-1}} \frac{|\mathcal{M}(v)-\mathcal{M}(w)|}{\sqrt{\mathcal{M}(v)}} \mathrm{d}w \right\|_{L^2_v} \\ &=: C\varepsilon \|\nabla h\|_{L^2_x} (I_{22,1} + I_{22,2}). \end{aligned} \tag{4.16}$$

For $I_{22,1}$, using $|v-w| > \frac{1}{2}|v|$, we get

$$I_{22,1} \le C \int \mathcal{M}(w)\mathrm{d}w \left[\int_{\frac{2}{3}<|v|<\frac{2}{\varepsilon}} \left(\frac{1}{|v|^{d+2s-1}} \frac{1}{\sqrt{\mathcal{M}(v)}} \right)^2 \mathrm{d}v \right]^{\frac{1}{2}} \le C\varepsilon^{s-1}. \tag{4.17}$$

For $I_{22,2}$, we have $\mathcal{M}(w) \le C\mathcal{M}(v)$. Thus,

$$I_{22,2} \le C \|\mathcal{M}\|_{L^1_v} \int_{1\le|v-w|\le\frac{1}{\varepsilon}} \frac{1}{|v-w|^{d+2s-1}} \mathrm{d}w \le C\varepsilon^{2s-1}. \tag{4.18}$$

Applying (4.19) and (4.18) to (4.16), we get

$$\|I_{22}\|_{L^2_v} \le C \|\nabla h\|_{L^2_x} \varepsilon^s. \tag{4.19}$$

The combination of (4.15) and (4.19) leads to

$$\|I(\eta,\mathcal{M})\|_{\mathcal{M}^{-1}} \le C \|\nabla h\|_{L^2_x} \varepsilon^s.$$

The first-order derivatives of $\mathcal{M}$ are

$$\begin{aligned}\partial_{v_i} I(\eta,\mathcal{M}) &= \partial_{v_i} \int_{\mathbb{R}^d} \frac{(h(x+\varepsilon v)-h(x+\varepsilon w))(\mathcal{M}(v)-\mathcal{M}(w))}{|v-w|^{1+2s}} \mathrm{d}w \\ &= \partial_{v_i} \int_{\mathbb{R}^d} \frac{(h(x+\varepsilon v)-h(x+\varepsilon v-\varepsilon w))(\mathcal{M}(v)-\mathcal{M}(v-w))}{|w|^{1+2s}} \mathrm{d}w \\ &= \varepsilon I(\partial_{x_i} h,\mathcal{M}) + I(\eta,\partial_{v_i}\mathcal{M}).\end{aligned}$$

These can be estimated similarly as $I(\eta,\mathcal{M})$. In particular, we have $\varepsilon I(\partial_x h,\mathcal{M}) \le C \|h\|_{H^2_x} \varepsilon^{s+1}$ directly following the first estimate in part 2). For $I(h,\partial_{v_i}\mathcal{M})$, the same estimate holds since

$$|\partial_v^k \mathcal{M}| \le C_k \mathcal{M},$$

thanks to Proposition 1. Likewise,

$$\begin{aligned}\partial_{v_iv_j} I(\eta,\mathcal{M}) &= \partial_{v_j}(\varepsilon I(\partial_{x_i}\eta,\mathcal{M}) + I(\eta,\partial_{v_i}\mathcal{M})) \\ &= \varepsilon^2 I(\partial_{x_ix_j}\eta,\mathcal{M}) + \varepsilon I(\partial_{x_j}\eta,\partial_{v_i}\mathcal{M}) + \varepsilon I(\partial_{x_i}\eta,\partial_{v_j}\mathcal{M}) + I(h,\partial_{v_iv_j}\mathcal{M}),\end{aligned}$$

and the rest of the estimate proceeds exactly the same as before.

3) We divide the integral in $I(h,\mathcal{M})$ into several subdomains and estimate them individually:

$$\begin{aligned}|I(\eta,\mathcal{M})| &\le \int \frac{|h(x+\varepsilon v)-h(x+\varepsilon w)||\mathcal{M}(v)-\mathcal{M}(w)|}{|v-w|^{d+2s}} \mathrm{d}w \\ &= \int_{|v-w|>1} \cdot\, \mathrm{d}w + \int_{|v-w|\le 1} \cdot\, \mathrm{d}w =: I_3 + I_4.\end{aligned}$$

For I_3, we separate the two cases where $|w| \geq \frac{1}{2}|v|$ and $|w| < \frac{1}{2}|v|$. When $|w| \geq \frac{1}{2}|v|$, we have $\mathcal{M}(w) \leq C\mathcal{M}(v)$. Thus,

$$\begin{aligned}
&\int_{|v-w|>1, |w|\geq \frac{1}{2}|v|} \frac{|h(x+\varepsilon v) - h(x+\varepsilon w)||\mathcal{M}(v) - \mathcal{M}(w)|}{|v-w|^{d+2s}} \mathrm{d}w \\
&\leq C \|h\|_\infty \mathcal{M}(v) \int_{|v-w|>1} \frac{1}{|v-w|^{d+2s}} \mathrm{d}w = C \|h\|_\infty \mathcal{M}(v).
\end{aligned}$$

When $|w| < \frac{1}{2}|v|$, we have $|v| > \frac{2}{3}$, $|v-w| > \frac{1}{2}|v|$ and $\mathcal{M}(v) \leq C\mathcal{M}(w)$. Therefore,

$$\begin{aligned}
&\int_{|v-w|>1, |w|< \frac{1}{2}|v|} \frac{|h(x+\varepsilon v) - h(x+\varepsilon w)||\mathcal{M}(v) - \mathcal{M}(w)|}{|v-w|^{d+2s}} \mathrm{d}w \\
&\leq C \|h\|_\infty \int_{|v|>\frac{2}{3}} \frac{\mathcal{M}(w)}{|v|^{d+2s}} \mathrm{d}w \leq C \|h\|_\infty \left(\frac{1}{|v|^{d+2s}} \mathbf{1}_{|v|>\frac{2}{3}} \right) \int \mathcal{M}(w) \mathrm{d}w \\
&\leq C \|h\|_\infty \mathcal{M}(v).
\end{aligned}$$

For I_4, we have

$$I_4 \leq C\varepsilon \|\nabla h\|_\infty \int_{|v-w|<1} \frac{|\nabla \mathcal{M}(\xi)|}{|v-w|^{d+2s-2}} \mathrm{d}w, \quad \xi = (1-t)v + tw, \ t \in (0,1),$$

where $|\nabla \mathcal{M}(\xi)|$ has the same estimate as in (4.14). Therefore,

$$I_4 \leq C\varepsilon \|\nabla h\|_\infty \mathcal{M}(v).$$

Combining the estimates above gives the pointwise bound of I. $\square$

To proceed with the error estimate, we will consider two separate regimes sketched in Fig. 2.

4.1 Regime I: kinetic regime with $\varepsilon^{2s} \geq \Delta t^{2s\beta}$

First we show that when ε is large compared to Δt and γ is chosen to satisfy (4.20) and (4.29), the accuracy is controlled by some positive power of Δt. In the following estimates, we do not keep track of the error dependence on γ, but it is expected that the constant increases with larger values of γ. So in practice, we always choose γ to be a constant slightly larger than 2 that satisfies the aforementioned conditions.

Lemma 8 (Estimate of $\|g^n\|$ in Regime I)**.** *Let (η^n, g^n) be the solution to* (4.2a)–(4.2b) *and $h^n(x+\varepsilon v) = \eta^n(x,v)$. Suppose $\varepsilon^{2s} \geq \Delta t^{2s\beta}$ with $\beta < \frac{1}{4s}$. Then for any γ such that*

$$0 < \gamma \leq \sqrt{\frac{\lambda_0 - 1}{\lambda_0}} \alpha = \sqrt{\frac{\lambda_0 - 1}{\lambda_0}} \frac{\varepsilon^{2s}}{\Delta t}, \qquad \textit{for some fixed} \quad \lambda_0 > 1, \tag{4.20}$$

we have the following estimates for g^n and $g^{n+\frac{1}{2}}$:

1) $\|g^n\|_{\mathcal{M}^{-1}} \leq C\varepsilon^{-s} \left(\varepsilon^s \|g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^1_x} \right)$, $\quad \|g^n\|_{L^2_{x,v}} \leq C\varepsilon^{-s} \left(\varepsilon^s \|g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^1_x} \right)$;

2) $\|g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}} \leq C\varepsilon^{-s} \left(\varepsilon^s \|g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^1_x} \right)$, $\quad \left\| g^{n+\frac{1}{2}} \right\|_{L^2_{x,v}} \leq C\varepsilon^{-s} \left(\varepsilon^s \|g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^1_x} \right)$;

3) $\|g^n\|_{L^\infty_{\mathcal{M}^{-1}}} := \left\| \frac{g^n}{\mathcal{M}} \right\|_\infty \leq C\varepsilon^{-2s} (\varepsilon^s \|g_{in}\|_{L^\infty_{\mathcal{M}^{-1}}} + \|h_{in}\|_{W^{1,\infty}})$.

Proof. 1) Without loss of generality, we assume that the final time is $t = 1$ such that

$$N\Delta t = 1, \qquad 0 \le n \le N.$$

By (1.14b) and (1.14c), we have

$$g^{n+\frac{1}{2}} = (\alpha + \gamma - \mathcal{L}^s)^{-1}(\alpha g^n - I^{n+1}), \qquad g^{n+1} = (\alpha - \gamma + \varepsilon v \cdot \nabla_x)^{-1} \alpha g^{n+\frac{1}{2}}. \tag{4.21}$$

Then we have

$$\begin{aligned}
\frac{g^{n+1}}{\sqrt{\mathcal{M}}} &= (\alpha - \gamma + \varepsilon v \cdot \nabla_x)^{-1} \frac{\alpha}{\sqrt{\mathcal{M}}} (\alpha + \gamma - \mathcal{L}^s)^{-1} \alpha \sqrt{\mathcal{M}} \left(\frac{g^n}{\sqrt{\mathcal{M}}} - \frac{1}{\alpha} \frac{I^{n+1}}{\sqrt{\mathcal{M}}} \right) \\
&= \mathcal{H}\left(\frac{g^n}{\sqrt{\mathcal{M}}} \right) - \mathcal{H}\left(\frac{1}{\alpha} \frac{I^{n+1}}{\sqrt{\mathcal{M}}} \right) \\
&= \mathcal{H}^{n+1}\left(\frac{g_{in}}{\sqrt{\mathcal{M}}} \right) - \sum_{p=1}^{n+1} \mathcal{H}^p \left(\frac{1}{\alpha} \frac{I^{n+1-p}}{\sqrt{\mathcal{M}}} \right),
\end{aligned} \tag{4.22}$$

where

$$\mathcal{H} = (\alpha - \gamma + \varepsilon v \partial_x)^{-1} \frac{\alpha}{\sqrt{\mathcal{M}}} (\alpha + \gamma - \mathcal{L}^s)^{-1} \alpha \sqrt{\mathcal{M}} =: \mathcal{H}_1 \mathcal{H}_2,$$

with

$$\mathcal{H}_1 = (\alpha - \gamma + \varepsilon v \cdot \nabla_x)^{-1} \alpha, \qquad \mathcal{H}_2 = \frac{1}{\sqrt{\mathcal{M}}} (\alpha + \gamma - \mathcal{L}^s)^{-1} \alpha \sqrt{\mathcal{M}}. \tag{4.23}$$

It is easy to check that $\mathcal{H}_1$ is bounded operator from $L^2_{x,v}$ to itself with bound

$$\|\mathcal{H}_1\|_{L^2_{x,v} \to L^2_{x,v}} \le \frac{\alpha}{\alpha - \gamma}.$$

For $\mathcal{H}_2$, we conduct the energy estimate. Suppose F and G satisfy $\mathcal{H}_2 F = G$, or equivalently,

$$F = \frac{\alpha + \gamma}{\alpha} G - \frac{1}{\alpha} \frac{1}{\sqrt{\mathcal{M}}} \mathcal{L}^s(\sqrt{\mathcal{M}} G).$$

Multiply the equation by G and integrate with respect to x and v. Using the fact that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{G}{\sqrt{\mathcal{M}}} \mathcal{L}^s(\sqrt{\mathcal{M}} G) \mathrm{d}x \mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{\mathcal{M}} \sqrt{\mathcal{M}} G \mathcal{L}^s(\sqrt{\mathcal{M}} G) \mathrm{d}x \mathrm{d}v \le 0,$$

one has

$$\|G\|_{L^2_{x,v}} \le \frac{\alpha}{\alpha + \gamma} \|F\|_{L^2_{x,v}} \Longrightarrow \|\mathcal{H}_2\|_{L^2_{x,v} \to L^2_{x,v}} \le \frac{\alpha}{\alpha + \gamma}. \tag{4.24}$$

Therefore $\|\mathcal{H}\|_{L^2_{x,v} \to L^2_{x,v}} \le \frac{\alpha^2}{\alpha^2 - \gamma^2}$. Plugging it into (4.22), one has

$$\|g^{n+1}\|_{\mathcal{M}^{-1}} \le \left(\frac{\alpha^2}{\alpha^2 - \gamma^2} \right)^{n+1} \|g_{in}\|_{\mathcal{M}^{-1}} + \frac{1}{\alpha} \sum_{p=1}^{n+1} \left(\frac{\alpha^2}{\alpha^2 - \gamma^2} \right)^p \|I^{n+2-p}\|_{\mathcal{M}^{-1}}. \tag{4.25}$$

Note that since $\varepsilon^{2s} \geq \Delta t^{2s\beta}$ with $\beta < \frac{1}{4s}$, we have

$$\alpha^2 - \gamma^2 \geq \frac{1}{\Delta t^{2-4s\beta}} - \gamma^2 \geq N - \gamma^2 \geq n - \gamma^2.$$

Therefore,

$$\left(\frac{\alpha^2}{\alpha^2 - \gamma^2}\right)^{n+1} \leq \left(1 + \frac{\gamma^2}{n - \gamma^2}\right)^{n+1} \leq Ce^{\gamma^2}, \quad \text{for all } n \geq \gamma^2 + 1. \tag{4.26}$$

In the case when $n < \gamma^2 + 1$, recall that $\alpha^2 = \left(\varepsilon^{2s}/\Delta t\right)^2 \geq \Delta t^{2(\beta-1)}$. Therefore, as long as γ is chosen such that $\gamma^2 \leq \frac{\lambda_0 - 1}{\lambda_0}\alpha^2$ for some fixed $\lambda_0 > 1$, we have

$$\left(\frac{\alpha^2}{\alpha^2 - \gamma^2}\right)^{n+1} \leq \lambda_0^{n+1}. \tag{4.27}$$

In sum, (4.25) continues as

$$\|g^{n+1}\|_{\mathcal{M}^{-1}} \leq C\|g_{in}\|_{\mathcal{M}^{-1}} + \frac{C}{\alpha}\sum_{p=1}^{n+1}\|I^p\|_{\mathcal{M}^{-1}}, \quad C = \max\left\{e^{\gamma}, \lambda_0^{[\gamma^2+1]}\right\}.$$

Using Lemma 7 part 2) and Lemma 5, one has

$$\|g^{n+1}\|_{\mathcal{M}^{-1}} \leq C\varepsilon^{-s}\left(\varepsilon^s\|g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^1_x}\right).$$

The bound for $\left\|g^{n+1}\right\|_{L^2_{x,v}}$ is a direct consequence of the boundedness of $\|g^{n+1}\|_{\mathcal{M}^{-1}}$.

2) By (4.21) we have

$$\begin{aligned}
\frac{g^{n+\frac{1}{2}}}{\sqrt{\mathcal{M}}} &= \frac{1}{\sqrt{\mathcal{M}}}(\alpha + \gamma - \mathcal{L}^s)^{-1}\sqrt{\mathcal{M}}(\alpha\frac{g^n}{\sqrt{\mathcal{M}}} - I^{n+1}) \\
&= \mathcal{H}_2\left(\frac{g^n}{\sqrt{\mathcal{M}}} - \frac{1}{\alpha}I^{n+1}\right) \\
&= \mathcal{H}_2^{n+1}\left(\frac{g_{in}}{\sqrt{\mathcal{M}}}\right) - \sum_{p=1}^{n+1}\mathcal{H}_2^p\left(\frac{1}{\alpha}\frac{I^{n+1-p}}{\sqrt{\mathcal{M}}}\right).
\end{aligned}$$

Using the bound (4.24), we have for any $\gamma > 0$,

$$\|g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}} \leq \|g_{in}\|_{\mathcal{M}^{-1}} + \frac{1}{\alpha}\sum_{p=1}^{n+1}\|I^p\|_{\mathcal{M}^{-1}}.$$

The estimate for $\|g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}}$ again follows from Lemma 7 part 2) and Lemma 5.

3) Denote

$$\mathcal{H}_0 = (\alpha - \gamma + \varepsilon v \cdot \nabla_x)^{-1}\alpha(\alpha + \gamma - \mathcal{L}^s)^{-1}\alpha, \qquad \mathcal{H}_3 = (\alpha + \gamma - \mathcal{L}^s)^{-1}\alpha.$$

Then by using (4.21), one can write g^{n+1} as

$$g^{n+1} = \mathcal{H}_0 g^n - \frac{1}{\alpha}\mathcal{H}_0 I.$$

Then our goal is to find the bounds of $\mathcal{H}_3$ and $\mathcal{H}_1$ defined in (4.23) with respect to the norm $L^\infty_{\mathcal{M}^{-1}}$. To this end, we will use the maximum principle argument. Let q_1 satisfies

$$q_1 = \mathcal{H}_3 g = (\alpha + \gamma - \mathcal{L}^s)^{-1}\alpha g.$$

Then $g = \left(\frac{\alpha+\gamma}{\alpha} - \frac{1}{\alpha}\mathcal{L}^s\right) q_1$. Since $g \le \left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}$, we have

$$g - \left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M} = \left(\frac{\alpha+\gamma}{\alpha} - \frac{1}{\alpha}\mathcal{L}^s\right)\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right) \le 0.$$

Multiply the above inequality by $\mathrm{sgn}^+\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)$, where $\mathrm{sgn}^+(x) = 1$ if $x > 0$ and 0 otherwise. Then it becomes

$$\begin{aligned}\frac{\alpha+\gamma}{\alpha}\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)^+ - \frac{1}{\alpha}\nabla_v \cdot \left[v\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)^+\right] \\ + \frac{1}{\alpha}\left(q_1 - \frac{\alpha}{\alpha+\gamma}\mathcal{M}\right)^+ (-\Delta_v)^s\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right) \le 0. \end{aligned} \tag{4.28}$$

Note that for any $b(v)$, we have

$$\begin{aligned}(\mathrm{sgn}^+ b)(-\Delta_v)^s b &= C_{s,d}(\mathrm{sgn}^+ b)(v)\int \frac{b(v)-b(w)}{|v-w|^{1+2s}}\mathrm{d}w \ge C_{s,d}(\mathrm{sgn}^+ b)(v)\int \frac{b(v)-b^+(w)}{|v-w|^{1+2s}}\mathrm{d}w \\ &= C_{s,d}\int \frac{b^+(v) - (\mathrm{sgn}^+ b)(v) b^+(w)}{|v-w|^{1+2s}}\mathrm{d}w \ge C_{s,d}\int \frac{b^+(v)-b^+(w)}{|v-w|^{1+2s}}\mathrm{d}w = (-\Delta_v)^s b^+.\end{aligned}$$

Therefore, (4.28) leads to

$$\frac{\alpha+\gamma}{\alpha}\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)^+ - \mathcal{L}^s\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)^+ \le 0.$$

Multiply the above equation by $\frac{1}{\mathcal{M}}\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)^+$ and integrate in v. We get

$$\left\|\left(q_1 - \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)^+\right\|_{\mathcal{M}^{-1}} \le 0,$$

which implies $q_1 \le \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}$. Similarly, we can show that $-q_1 \le \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}$. Therefore,

$$\left\|\frac{q_1}{\mathcal{M}}\right\|_\infty \le \frac{\alpha}{\alpha+\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \qquad \text{or} \qquad \|\mathcal{H}_3\|_{L^\infty_{\mathcal{M}^{-1}}} \le \frac{\alpha}{\alpha+\gamma}.$$

Likewise, to bound $\mathcal{H}_1$, let $q_2 = \mathcal{H}_1 g = (\alpha - \gamma + \varepsilon v\cdot\nabla_x)^{-1}\alpha g$. Then

$$\left(\frac{\alpha-\gamma}{\alpha} + \frac{\varepsilon}{\alpha}v\cdot\nabla_x\right) q_2 = g.$$

Since $g - \left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M} \le 0$, we have

$$\left(\frac{\alpha-\gamma}{\alpha} + \frac{\varepsilon}{\alpha}v\cdot\nabla\right)\left(q_2 - \frac{\alpha}{\alpha-\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right) = g - \left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M} \le 0.$$

Multiplying the above inequality by $\left(q_2 - \frac{\alpha}{\alpha-\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}\right)^+$ and conducting an energy estimate leads to $q_2 \le \frac{\alpha}{\alpha-\gamma}\left\|\frac{g}{\mathcal{M}}\right\|_\infty \mathcal{M}$. Consequently, we have

$$\|\mathcal{H}_1\|_{L^\infty_{\mathcal{M}^{-1}}} \le \frac{\alpha}{\alpha-\gamma} \qquad \text{and} \qquad \|\mathcal{H}_0\|_{L^\infty_{\mathcal{M}^{-1}}} \le \frac{\alpha^2}{\alpha^2-\gamma^2}.$$

Then following the same proof as for part 1), we have

$$\|g^{n+1}\|_{L^\infty_{\mathcal{M}^{-1}}} \le \left(\frac{\alpha^2}{\alpha^2-\gamma^2}\right)^{n+1} \|g_{in}\|_{L^\infty_{\mathcal{M}^{-1}}} + \frac{1}{\alpha}\sum_{p=1}^{n+1}\left(\frac{\alpha^2}{\alpha^2-\gamma^2}\right)^p \|I^{n+1-p}\|_{L^\infty_{\mathcal{M}^{-1}}}.$$

This leads to

$$\|g^{n+1}\|_{L^\infty_{\mathcal{M}^{-1}}} \le C\|g_{in}\|_{L^\infty_{\mathcal{M}^{-1}}} + \frac{C}{\alpha}\sum_{p=1}^{n+1}\|I^p\|_{L^\infty_{\mathcal{M}^{-1}}}, \le C\|g_{in}\|_{L^\infty_{\mathcal{M}^{-1}}} + \frac{C}{\varepsilon^{2s}}\|h_{in}\|_{W^{1,\infty}},$$

where $C = \max\{e^\gamma, \lambda_0^{[\gamma^2+1]}\}$ and we have used part 3) of Lemma 7 and Lemma 5. $\square$

Next we bound the derivatives of $g^n p$ and $g^n h$.

Lemma 9 (Estimate of $\|\partial_{v_i} g^{n+1}\|_{\mathcal{M}^{-1}}$ and $\|\partial_{v_i} g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}}$ in Regime I)**.** *Suppose $\varepsilon^{2s} \ge \Delta t^{2s\beta}$ and*

$$2 < \gamma \le \sqrt{\frac{\lambda_2 - 4}{\lambda_2}\frac{\varepsilon^{2s}}{\Delta t}} = \sqrt{\frac{\lambda_2-4}{\lambda_2}}\alpha \quad \textit{for some fixed } \lambda_2 > 4. \tag{4.29}$$

Then the following bounds hold:

1) for any $i = 1, 2, \cdots, d$,

$$\|\partial_{v_i} g^{n+1}\|_{\mathcal{M}^{-1}} \le C(\varepsilon^{-s} + \varepsilon^{1-3s})L_1^0, \quad \|\partial_{v_i} g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}} \le C(\varepsilon^{-s} + \varepsilon^{1-3s})L_1^0,$$

where

$$L_1^0 = \varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^2_x}; \tag{4.30}$$

2) for any $i, j = 1, 2, \cdots, d$,

$$\|\partial_{v_i v_j} g^{n+1}\|_{\mathcal{M}^{-1}} + \|\partial_{v_i v_j} g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}} \le C(\varepsilon^{-s} + \varepsilon^{1-3s} + \varepsilon^{2-5s})L_2^0,$$

where

$$\begin{aligned} L_2^0 = {} & \varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{x_j} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} \\ & + \varepsilon^s\|\partial_{v_i v_j} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^2_x}. \end{aligned} \tag{4.31}$$

Proof. Taking the derivative in v_i of (1.14b) and (1.14c), we have

$$\left\{\begin{aligned} & \alpha(\partial_{v_i} g^{n+\frac{1}{2}} - \partial_{v_i} g^n) = \mathcal{L}^s(\partial_{v_i} g^{n+\frac{1}{2}}) - (\gamma-1)\partial_{v_i} g^{n+\frac{1}{2}} - \partial_{v_i} I(\eta^{n+1}, \mathcal{M}), && \text{(4.32a)} \\ & \alpha(\partial_{v_i} g^{n+1} - \partial_{v_i} g^{n+\frac{1}{2}}) + \varepsilon v\cdot\nabla_x(\partial_{v_i} g^{n+1}) = \gamma\partial_{v_i} g^{n+1} - \varepsilon\partial_{x_i} g^{n+1}, && \text{(4.32b)} \end{aligned}\right.$$

which leads to

$$\begin{aligned} \partial_{v_i} g^{n+1} = {} & (\alpha - \gamma + \varepsilon v\cdot\nabla_x)^{-1}\alpha(\alpha - \mathcal{L}^s + \gamma - 1)^{-1}(\alpha\partial_{v_i} g^n - \partial_{v_i} I^{n+1}) \\ & - (\alpha - \gamma + \varepsilon v\cdot\nabla_x)^{-1}\varepsilon\partial_{x_i} g^{n+1}. \end{aligned}$$

Then similar to the proof of part 1) of Lemma 8, we write

$$\begin{aligned}\frac{\partial_{v_i} g^{n+1}}{\sqrt{\mathcal{M}}} &= \mathcal{H}_1\mathcal{H}_4\left(\frac{\partial_{v_i} g^n}{\sqrt{\mathcal{M}}}\right) - \mathcal{H}_1\mathcal{H}_4\left(\frac{1}{\alpha}\frac{\partial_{v_i} I^{n+1}}{\sqrt{\mathcal{M}}}\right) - \mathcal{H}_1\varepsilon^{1-2s}\Delta t \partial_{x_i} g^{n+1} \\ &= (\mathcal{H}_1\mathcal{H}_4)^{n+1}\left(\frac{\partial_{v_i} g_{in}}{\sqrt{\mathcal{M}}}\right) - \sum_{p=1}^{n+1}(\mathcal{H}_1\mathcal{H}_4)^p\left(\frac{1}{\alpha}\frac{\partial_{v_i} I^{n+2-p}}{\sqrt{\mathcal{M}}}\right) \\ &\quad - \sum_{p=1}^{n+1}(\mathcal{H}_1\mathcal{H}_4)^{p-1}\mathcal{H}_1\varepsilon^{1-2s}\Delta t \partial_{x_i} g^{n+2-p},\end{aligned} \tag{4.33}$$

where $\mathcal{H}_1$ is defined in (4.23) and

$$\mathcal{H}_4 = \frac{1}{\sqrt{\mathcal{M}}}(\alpha - \mathcal{L}^s + \gamma - 1)^{-1}\alpha\sqrt{\mathcal{M}}.$$

Similar to (4.24), we see that

$$\|\mathcal{H}_4\|_{L^2_{x,v}\to L^2_{x,v}} \le \frac{\alpha}{\alpha+\gamma-1}. \tag{4.34}$$

Denote $c = \frac{\alpha^2}{(\alpha-\gamma)(\alpha+\gamma-1)}$. Then via iterations, (4.33) can be estimated as

$$\begin{aligned}\|\partial_{v_i} g^{n+1}\|_{\mathcal{M}^{-1}} \le c^{n+1}\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^{-2s}\Delta t\sum_{p=1}^{n+1} c^p\|\partial_{v_i} I^{n+2-p}\|_{\mathcal{M}^{-1}} \\ + \sum_{p=1}^{n+1} c^{p-1}\frac{\alpha}{\alpha-\gamma}\varepsilon^{1-2s}\Delta t\left\|\partial_{x_i} g^{n+2-p}\right\|_{L^2_{x,v}}.\end{aligned}$$

By Lemma 8 part 1), Lemma 7 part 2) and Lemma 5, we obtain

$$\begin{aligned}\|\partial_{v_i} g^{n+1}\|_{\mathcal{M}^{-1}} \le c^{n+1}\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^{-s}\Delta t\,\|h_{in}\|_{H^2_x}\sum_{p=1}^{n+1} c^p \\ + C\varepsilon^{1-3s}\frac{\alpha}{\alpha-\gamma}\Delta t\left(\varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^2_x}\right)\sum_{p=1}^{n+1} c^{p-1}.\end{aligned}$$

Following the same argument as in (4.26) and (4.27), if we choose γ such that

$$0 < \gamma \le \sqrt{\frac{\lambda_1 - 2}{\lambda_1}\frac{\varepsilon^{2s}}{\Delta t}} = \sqrt{\frac{\lambda_1-2}{\lambda_1}}\alpha$$

for some fixed $\lambda_2 > 2$, then the above inequality reduces to

$$\begin{aligned}\|\partial_{v_i} g^{n+1}\|_{\mathcal{M}^{-1}} &\le C\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + C\varepsilon^{-s}\,\|h_{in}\|_{H^2_x} + C\varepsilon^{1-3s}\left(\varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^2_x}\right) \\ &\le C\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + C(\varepsilon^{-s} + \varepsilon^{1-3s})\left(\varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H^2_x}\right).\end{aligned}$$

In addition, from (4.32a) one has

$$\partial_{v_i} g^{n+\frac{1}{2}} = (\alpha - \mathcal{L}^s + \gamma - 1)^{-1}(\alpha\partial_{v_i} g^n - \partial_{v_i} I^{n+1}).$$

Thus,

$$\frac{\partial_{v_i} g^{n+\frac{1}{2}}}{\sqrt{\mathcal{M}}} = \mathcal{H}_4 \left(\frac{\partial_{v_i} g^n}{\sqrt{\mathcal{M}}} - \frac{1}{\alpha} \frac{1}{\sqrt{\mathcal{M}}} \partial_{v_i} I^{n+1} \right),$$

which immediately leads to

$$\|\partial_{v_i} g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}} \leq C(\|\partial_{v_i} g^n\|_{\mathcal{M}^{-1}} + \frac{1}{\alpha}\|\partial_{v_i} I^{n+1}\|_{\mathcal{M}^{-1}}),$$

thanks to (4.34) with $\gamma > 1$. Combining the bound for $\partial_{v_i} g^n$ above with Lemma 7 part 2) and Lemma 5, we further have

$$\begin{aligned}\|\partial_{v_i} g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}} &\leq C(\varepsilon^{-s} + \varepsilon^{1-3s}) \left(\varepsilon^s \|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H_x^2} \right) + C\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + C\Delta t \varepsilon^{-s} \|h_{in}\|_{H_x^2} \\ &\leq C(\varepsilon^{-s} + \varepsilon^{1-3s}) \left(\varepsilon^s \|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H_x^2} \right) + C\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}}.\end{aligned}$$

For the second-order derivatives, differentiating (4.32a) and (4.32b) in v_j, we have

$$\begin{cases} \alpha(\partial_{v_i v_j} g^{n+\frac{1}{2}} - \partial_{v_i v_j} g^n) = \mathcal{L}^s(\partial_{v_i v_j} g^{n+\frac{1}{2}}) - (\gamma-2)\partial_{v_i v_j} g^{n+\frac{1}{2}} - \partial_{v_i v_j} I(\eta^{n+1}, \mathcal{M}), & (4.35a) \\ \alpha(\partial_{v_i v_j} g^{n+1} - \partial_{v_i v_j} g^{n+\frac{1}{2}}) + \varepsilon v \cdot \nabla_x(\partial_{v_i v_j} g^{n+1}) = \gamma \partial_{v_i v_j} g^{n+1} - \varepsilon \partial_{x_i v_j} g^{n+1} - \varepsilon \partial_{x_j v_i} g^{n+1}, & (4.35b) \end{cases}$$

which implies

$$\begin{aligned}\partial_{v_i v_j} g^{n+1} = &(\alpha - \gamma + \varepsilon v \cdot \nabla_x)^{-1} \alpha (\alpha - \mathcal{L}^s + \gamma - 2)^{-1} (\alpha \partial_{v_i v_j} g^n - \partial_{v_i v_j} I) \\ &- (\alpha - \gamma + \varepsilon v \cdot \nabla_x)^{-1} \varepsilon (\partial_{x_i} \partial_{v_j} g^{n+1} + \partial_{x_j} \partial_{v_i} g^{n+1}). \end{aligned} \tag{4.36}$$

Recall $\mathcal{H}_1$ defined in (4.23) and denote

$$\mathcal{H}_5 = \frac{1}{\sqrt{\mathcal{M}}} (\alpha - \mathcal{L}^s + \gamma - 2)^{-1} \alpha \sqrt{\mathcal{M}}.$$

Then (4.36) can be written as

$$\begin{aligned}\frac{\partial_{v_i v_j} g^{n+1}}{\sqrt{\mathcal{M}}} &= \mathcal{H}_1 \mathcal{H}_5 \left(\frac{\partial_{v_i v_j} g^n}{\sqrt{\mathcal{M}}} \right) - \mathcal{H}_1 \mathcal{H}_5 \left(\frac{1}{\alpha} \frac{\partial_{v_i v_j} I}{\sqrt{\mathcal{M}}} \right) - \mathcal{H}_1 \varepsilon^{1-2s} \Delta t (\partial_{x_i} \partial_{v_j} g^{n+1} + \partial_{x_j} \partial_{v_i} g^{n+1}) \\ &= (\mathcal{H}_1 \mathcal{H}_5)^{n+1} \left(\frac{\partial_{v_i v_j} g_{in}}{\sqrt{\mathcal{M}}} \right) - \sum_{p=1}^{n+1} (\mathcal{H}_1 \mathcal{H}_5)^p \left(\frac{1}{\alpha} \frac{\partial_{v_i v_j} I^{n+1-p}}{\sqrt{\mathcal{M}}} \right) \\ &\quad - \sum_{p=1}^{n+1} (\mathcal{H}_1 \mathcal{H}_5)^{p-1} \mathcal{H}_1 \varepsilon^{1-2s} \Delta t (\partial_{x_i} \partial_{v_j} g^{n+1-p} + \partial_{x_j} \partial_{v_i} g^{n+1-p}). \end{aligned} \tag{4.37}$$

Similar to (4.24) and (4.34), we have

$$\|\mathcal{H}_5\|_{L^2_{x,v} \to L^2_{x,v}} \leq \frac{\alpha}{\alpha + \gamma - 2}.$$

Denote $c_2 = \frac{\alpha^2}{(\alpha-\gamma)(\alpha+\gamma-2)}$. Then via iterations, (4.37) satisfies

$$\begin{aligned}\|\partial_{v_i v_j} g^{n+1}\|_{\mathcal{M}^{-1}} \leq &\, c^{n+1} \|\partial_{v_i v_j} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^{-2s} \Delta t \sum_{p=1}^{n+1} c^p \|\partial_{v_i v_j} I^{n+1-p}\|_{\mathcal{M}^{-1}} \\ &+ \sum_{p=1}^{n+1} c^{p-1} \frac{\alpha}{\alpha - \gamma} \varepsilon^{1-2s} \Delta t \left(\left\| \partial_{x_i} \partial_{v_j} g^{n+1-p} \right\|_{L^2_{x,v}} + \left\| \partial_{x_j} \partial_{v_i} g^{n+1-p} \right\|_{L^2_{x,v}} \right).\end{aligned}$$

By Lemma 7 part 2), Lemma 5 and Lemma 8, we have

$$
\begin{aligned}
\|\partial_{v_iv_j} g^{n+1}\|_{\mathcal{M}^{-1}} \leq c^{n+1}\|\partial_{v_iv_j} g_{in}\|_{\mathcal{M}^{-1}} + C\varepsilon^{-2s}\|h_{in}\|_{H_x^3}\varepsilon^s \Delta t \sum_{p=1}^{n+1} c^p \\
+ C\frac{\alpha}{\alpha-\gamma}\varepsilon^{1-2s}\Delta t(\varepsilon^{-s}+\varepsilon^{1-3s})\left(\sum_{p=1}^{n+1} c^{p-1}\right) \times \\
\times \left(\varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{x_j} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H_x^2}\right).
\end{aligned}
$$

Following the same argument as in (4.26) and (4.27), if we chose γ such that

$$
0 < \gamma \leq \sqrt{\frac{\lambda_2 - 4}{\lambda_2}\frac{\varepsilon^{2s}}{\Delta t}} = \sqrt{\frac{\lambda_2 - 4}{\lambda_2}}\alpha
$$

for some fixed $\lambda_2 > 4$, then the above inequality reduces to

$$
\begin{aligned}
&\|\partial_{v_iv_j} g^{n+1}\|_{\mathcal{M}^{-1}} \\
&\leq C\|\partial_{v_iv_j} g_{in}\|_{\mathcal{M}^{-1}} + C\varepsilon^{-s}\|h_{in}\|_{H_x^3} \\
&\quad + C(\varepsilon^{1-3s}+\varepsilon^{2-5s})\left(\varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{x_j} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H_x^2}\right) \\
&\leq C(\varepsilon^{-s}+\varepsilon^{1-3s}+\varepsilon^{2-5s}) \times \\
&\quad \times \left(\varepsilon^s\|\partial_{x_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{x_j} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{v_i} g_{in}\|_{\mathcal{M}^{-1}} + \varepsilon^s\|\partial_{v_iv_j} g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H_x^2}\right).
\end{aligned}
$$

Additionally, by (4.35a) one has that

$$
\partial_{v_iv_j} g^{n+\frac{1}{2}} = (\alpha - \mathcal{L}^s + \gamma - 2)^{-1}(\alpha\partial_{v_iv_j} g^n - \partial_{v_iv_j} I^{n+1}).
$$

This implies

$$
\frac{\partial_{v_iv_j} g^{n+\frac{1}{2}}}{\sqrt{\mathcal{M}}} = \mathcal{H}_4\left(\frac{\partial_{v_iv_j} g^n}{\sqrt{\mathcal{M}}} - \frac{1}{\alpha}\frac{1}{\sqrt{\mathcal{M}}}\partial_{v_iv_j} I^{n+1}\right).
$$

Therefore, for $\gamma > 2$, we have

$$
\|\partial_{v_i} g^{n+\frac{1}{2}}\|_{\mathcal{M}^{-1}} \leq C(\|\partial_{v_iv_j} g^n\|_{\mathcal{M}^{-1}} + \frac{1}{\alpha}\|\partial_{v_iv_j} I\|_{\mathcal{M}^{-1}})
$$

and the desired bound follows from the estimate for $\|\partial_{v_iv_j} g^n\|_{\mathcal{M}^{-1}}$. □

Now we estimate the difference $g^{n+1} - g^{n+\frac{1}{2}}$.

Lemma 10 (Estimate of $g^{n+1} - g^{n+\frac{1}{2}}$ in Regime I)**.** *For $\varepsilon^{2s} \geq \Delta t^{2s\beta}$ with $\beta < \frac{1}{4s}$, we have the following estimates for $g^{n+1} - g^{n+\frac{1}{2}}$:*

1) $\left\|\langle v\rangle^m (g^{n+1} - g^{n+\frac{1}{2}})\right\|_{L^2_{x,v}} \leq C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{s+\frac{d}{2}-m,1\}}\varepsilon^{-s}L_0^0,$
where $m < s + \frac{d}{2}$ and $L_0^0 = \varepsilon^s\|g_{in}\|_{\mathcal{M}^{-1}} + \|h_{in}\|_{H_x^1}$;

2) $\left\|\langle v\rangle^m \partial_{v_i}(g^{n+1} - g^{n+\frac{1}{2}})\right\|_{L^2_{x,v}} \leq C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{s+\frac{d}{2}-m,1\}}(\varepsilon^{-s}+\varepsilon^{1-3s})L_1^0,$
where $m < s + \frac{d}{2}$, $i = 1, 2, \cdots, d$, and L_1^0 is defined in (4.30);

3) $\left\| \langle v\rangle^m \partial_{v_iv_j}(g^{n+1}-g^{n+\frac{1}{2}})\right\|_{L^2_{x,v}} \le C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{s+\frac{d}{2}-m,1\}}(\varepsilon^{-s}+\varepsilon^{2-5s})L_2^0$*,*
where $m<s+\frac{d}{2}$*,* $i=1,2,\cdots,d$ *and* L_2^0 *is defined in* (4.31)*;*

4) $\left\| \langle v\rangle^p (-\Delta_v)^s(g^{n+1}-g^{n+\frac{1}{2}})\right\|_{L^2_{x,v}} \le C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{s+\frac{d}{2}-p,1\}}(\varepsilon^{-s}+\varepsilon^{2-5s})(L_0^0+L_2^0)$*,*
where $-2s<p<s+\frac{d}{2}$*.*

Proof. 1) From (1.14c), one sees that

$$g^{n+1}-g^{n+\frac{1}{2}}=\frac{1}{\alpha}(\gamma g^{n+1}-\varepsilon v\cdot\nabla_x g^{n+1}),\qquad \alpha=\frac{\varepsilon^{2s}}{\Delta t}. \tag{4.38}$$

Then for a fixed m,

$$\begin{aligned}
&\left\|\langle v\rangle^m (g^{n+1}-g^{n+\frac{1}{2}})\right\|^2_{L^2_{x,v}} = \iint \langle v\rangle^{2m}|g^{n+1}-g^{n+\frac{1}{2}}|^{2\theta}|g^{n+1}-g^{n+\frac{1}{2}}|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&=\frac{C}{\alpha^{2(1-\theta)}}\iint\langle v\rangle^{2m}|g^{n+1}-g^{n+\frac{1}{2}}|^{2\theta}|\gamma g^{n+1}-\varepsilon v\cdot\nabla_x g^{n+1}|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&\le\frac{C}{\alpha^{2(1-\theta)}}\iint\langle v\rangle^{2m}(|g^{n+1}|^{2\theta}+|g^{n+\frac{1}{2}}|^{2\theta})(|\gamma g^{n+1}|^{2(1-\theta)}+\varepsilon^{2(1-\theta)}|v\cdot\nabla_x g^{n+1}|^{2(1-\theta)})\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

Expand the product in the above inequality. The most difficult term is

$$\iint\langle v\rangle^{2m}|g^{n+\frac{1}{2}}|^{2\theta}\varepsilon^{2(1-\theta)}|v\cdot\nabla_x g^{n+1}|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v,$$

which can be estimated as follows:

$$\begin{aligned}
&\iint\langle v\rangle^{2m}|g^{n+\frac{1}{2}}|^{2\theta}\varepsilon^{2(1-\theta)}|v\cdot\nabla_x g^{n+1}|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&=\varepsilon^{2(1-\theta)}\iint\langle v\rangle^{2m+2(1-\theta)}|g^{n+\frac{1}{2}}|^{2\theta}|\nabla_x g^{n+1}|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&\le\varepsilon^{2(1-\theta)}\left(\iint\langle v\rangle^{2m+2(1-\theta)}|g^{n+\frac{1}{2}}|^2\mathrm{d}x\mathrm{d}v\right)^{\theta}\left(\iint\langle v\rangle^{2m+2(1-\theta)}|\nabla_x g^{n+1}|^2\mathrm{d}x\mathrm{d}v\right)^{1-\theta}.
\end{aligned}$$

To use Lemma 8 to bound the right hand side of the inequality above, we need

$$2m+2(1-\theta)\le d+2s.$$

Since $\theta\in(0,1)$, this requires $m<\frac{d}{2}+s$. In this case, we can choose

$$\theta=\max\{1-(d/2+s-m),0\}.$$

and apply Lemma 8. The other terms are bounded similarly.

2) Equation (4.38) implies that

$$\partial_{v_i}(g^{n+1}-g^{n+\frac{1}{2}})=\frac{1}{\alpha}\left\{\gamma\partial_{v_i}g^{n+1}-\varepsilon[v\cdot\nabla_x(\partial_{v_i}g^{n+1})+\partial_{x_i}g^{n+1}]\right\}. \tag{4.39}$$

Then for a fixed m,

$$\begin{aligned}
&\left\|\langle v\rangle^m\partial_{v_i}(g^{n+1}-g^{n+\frac{1}{2}})\right\|^2_{L^2_{x,v}}=\iint\langle v\rangle^{2m}|\partial_{v_i}(g^{n+1}-g^{n+\frac{1}{2}})|^{2\theta}|\partial_{v_i}(g^{n+1}-g^{n+\frac{1}{2}})|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&=\frac{C}{\alpha^{2(1-\theta)}}\iint\langle v\rangle^{2m}|\partial_{v_i}(g^{n+1}-g^{n+\frac{1}{2}})|^{2\theta}|\gamma\partial_{v_i}g^{n+1}-\varepsilon[v\cdot\nabla_x(\partial_{v_i}g^{n+1})+\partial_{x_i}g^{n+1}]|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&\le\frac{C}{\alpha^{2(1-\theta)}}\iint\langle v\rangle^{2m}[|\partial_{v_i}g^{n+1}|^{2\theta}+|\partial_{v_i}g^{n+\frac{1}{2}}|^{2\theta}]\times\\
&\qquad\times[|\gamma\partial_{v_i}g^{n+1}|^{2(1-\theta)}+\varepsilon^{2(1-\theta)}|v\cdot\nabla_x(\partial_{v_i}g^{n+1})|^{2(1-\theta)}+|\partial_{x_i}g^{n+1}|^{2(1-\theta)}]\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

The rest follows similarly as for part 1) by replacing Lemma 8 with Lemma 9 in the proof.

3) This part is similar to Part 2). From (4.39), we have

$$\partial_{v_iv_j}(g^{n+1}-g^{n+\frac{1}{2}})=\frac{1}{\alpha}\left\{\gamma\partial_{v_i}g^{n+1}-\varepsilon[v\cdot\nabla_x(\partial_{v_iv_j}g^{n+1})+\partial_{x_i}\partial_{v_j}g^{n+1}+\partial_{x_j}\partial_{v_i}g^{n+1}]\right\}.$$

Then for a fixed m,

$$\begin{aligned}
&\left\|\langle v\rangle^m\partial_{v_iv_j}(g^{n+1}-g^{n+\frac{1}{2}})\right\|^2_{L^2_{x,v}}\\
&=\iint\langle v\rangle^{2m}|\partial_{v_iv_j}(g^{n+1}-g^{n+\frac{1}{2}})|^{2\theta}|\partial_{v_iv_j}(g^{n+1}-g^{n+\frac{1}{2}})|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&=\frac{C}{\alpha^{2(1-\theta)}}\iint\langle v\rangle^{2m}|\partial_{v_iv_j}(g^{n+1}-g^{n+\frac{1}{2}})|^{2\theta}|\gamma\partial_{v_iv_j}g^{n+1}\\
&\qquad\qquad-\varepsilon[v\cdot\nabla_x(\partial_{v_iv_j}g^{n+1})+\partial_{x_i}\partial_{v_j}g^{n+1}+\partial_{x_j}\partial_{v_i}g^{n+1}]|^{2(1-\theta)}\mathrm{d}x\mathrm{d}v\\
&\le\frac{C}{\alpha^{2(1-\theta)}}\iint\langle v\rangle^{2m}[|\partial_{v_iv_j}g^{n+1}|^{2\theta}+|\partial_{v_iv_j}g^{n+\frac{1}{2}}|^{2\theta}][|\gamma\partial_{v_iv_j}g^{n+1}|^{2(1-\theta)}+|\partial_{x_i}\partial_{v_j}g^{n+1}|^{2(1-\theta)}\\
&\qquad\qquad+\varepsilon^{2(1-\theta)}|v\cdot\nabla_x(\partial_{v_iv_j}g^{n+1})|^{2(1-\theta)}+|\partial_{x_j}\partial_{v_i}g^{n+1}|^{2(1-\theta)}]\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

The rest of the estimate follows similarly as that in part 2).

4) For simplicity, we denote $\bar{g}=g^{n+1}-g^{n+\frac{1}{2}}$ and rewrite

$$\langle v\rangle^p(-\Delta_v)^s\bar{g}=(-\Delta_v)^s(\langle v\rangle^p\bar{g})-[\langle v\rangle^p,(-\Delta_v)^s]\bar{g},$$

where $[\cdot,\cdot]$ is the commutator. For the first term, by interpolation we have for any $p<s+\frac{d}{2}$,

$$\begin{aligned}
\|(-\Delta_v)^s(\langle v\rangle^p\bar{g})\|^2_{L^2_{x,v}}&\le C\iint|\langle v\rangle^p\bar{g}|^2\mathrm{d}x\mathrm{d}v+\iint|\nabla_v^2(\langle v\rangle^p\bar{g})|^2\mathrm{d}x\mathrm{d}v\\
&\le C(L_2^0)^2\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{2s+d-2p,2\}}(\varepsilon^{-s}+\varepsilon^{1-3s}+\varepsilon^{2-5s})^2.
\end{aligned}\tag{4.40}$$

For the second term, from Lemma 3, we have, for $-2s<p<\frac{d}{2}+2s$,

$$\|[\langle v\rangle^p,(-\Delta_v)^s]\bar{g}\|_{L^2_{x,v}}\le C\|\langle v\rangle^p\bar{g}\|_{L^2_{x,v}}+\|\langle v\rangle^{p-1}|\nabla_v\bar{g}|\|_{L^2_{x,v}}.$$

Substitute part 1) and 2) into the right-hand side of the inequality. Then it becomes

$$\begin{aligned}
\|[\langle v\rangle^p,(-\Delta_v)^s]\bar{g}\|_{L^2_{x,v}}&\le C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{s+\frac{d}{2}-p,1\}}\varepsilon^{-s}L_0^0\\
&\quad+C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{s+\frac{d}{2}+1-p,1\}}(\varepsilon^{-s}+\varepsilon^{1-3s})L_1^0,\qquad -2s<p<s+\frac{d}{2},
\end{aligned}$$

which together with (4.40) implies

$$\|\langle v\rangle^p(-\Delta_v)^s\bar{g}\|_{L^2_{x,v}}\le C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{s+\frac{d}{2}-p,1\}}(\varepsilon^{-s}+\varepsilon^{1-3s}+\varepsilon^{2-5s})(L_0^0+L_2^0),$$

with $-2s<p<s+\frac{d}{2}$. □

We are now ready to state our main theorem in Regime I. The main idea is to construct a new weighted norm that can compensate the slow decay of the equilibrium at the tail.

Theorem 2 (Error estimate in Regime I)**.** *Assume* (4.1) *and suppose*

$$\varepsilon^{2s} \geq \Delta t^{2s\beta}, \qquad \beta \leq \min\{\beta_0, \frac{1}{4s}\},$$

where β_0 is the biggest constant that satisfies (4.49)–(4.53). *Then the solution $f^n = \eta^n \mathcal{M} + g^n$ obtained from solving* (1.14) *with γ chosen to satisfy* (4.20) *and* (4.29) *has the following estimate*

$$\| \langle v \rangle^{-b} \tilde{f}^n \|_{\mathcal{M}^{-1}} \leq C \Delta t^{\zeta}$$

for $1 < b \leq \min\{d + 2s, \frac{d}{2} + 3s^-\}$ and $\zeta(\beta_0, b) > 0$ explicitly computable. Here $\tilde{f}^n$ is defined in (4.5)*, $n\Delta t = T$, and C is a constant depending on T.*

Proof. Multiply (4.6) by the weight $(1 + \delta \langle v \rangle)^{-b}$, where b and δ are two constants that will be made precise later. We have

$$\begin{aligned}
&\alpha \left[(1 + \delta \langle v \rangle)^{-b} \tilde{f}^{n+1} - (1 + \delta \langle v \rangle)^{-b} \tilde{f}^n \right] + \varepsilon v \partial_x ((1 + \delta \langle v \rangle)^{-b} \tilde{f}^{n+1}) \\
&= \mathcal{L}^s((1 + \delta \langle v \rangle)^{-b} \tilde{f}^{n+1}) + [(1 + \delta \langle v \rangle)^{-b}, \mathcal{L}^s] \tilde{f}^{n+1} \\
&\qquad + (1 + \delta \langle v \rangle)^{-b} (\mathcal{L}^s - \gamma I)(g^{n+1} - g^{n+\frac{1}{2}}) + (1 + \delta \langle v \rangle)^{-b} e^n,
\end{aligned}$$

where

$$e^n = \Delta t (e_1^n + \varepsilon^{2s} e_2^n \mathcal{M}). \tag{4.41}$$

Conduct an energy estimate for the equation by multiplying it by $(1 + \delta \langle v \rangle)^{-b} \frac{\tilde{f}^{n+1}}{\mathcal{M}}$ and integrating with respect to x and v. We obtain

$$\begin{aligned}
&\frac{\alpha}{2} \left(\|(1 + \delta \langle v \rangle)^{-b} \tilde{f}^{n+1}\|^2_{\mathcal{M}^{-1}} - \|(1 + \delta \langle v \rangle)^{-b} \tilde{f}^n\|^2_{\mathcal{M}^{-1}} \right) \\
&\leq \iint (1 + \delta \langle v \rangle)^{-b} \tilde{f}^{n+1} [(1 + \delta \langle v \rangle)^{-b}, \mathcal{L}^s] \tilde{f}^{n+1} \frac{1}{\mathcal{M}} \mathrm{d}v \mathrm{d}x \\
&\quad + \iint (1 + \delta \langle v \rangle)^{-b} \mathcal{L}^s (g^{n+1} - g^{n+\frac{1}{2}}) (1 + \delta \langle v \rangle)^{-b} \tilde{f}^{n+1} \frac{1}{\mathcal{M}} \mathrm{d}v \mathrm{d}x \\
&\quad + \iint (1 + \delta \langle v \rangle)^{-2b} e^n \frac{\tilde{f}^{n+1}}{\mathcal{M}} \mathrm{d}v \mathrm{d}x,
\end{aligned} \tag{4.42}$$

where we have used the fact that $\int \frac{f \mathcal{L}^s f}{\mathcal{M}} \mathrm{d}v \leq 0$. By the definition of $\mathcal{L}^s$ in (1.1),

$$\begin{aligned}
&\frac{1}{\sqrt{\mathcal{M}}} (1 + \delta \langle v \rangle)^{-b} \mathcal{L}^s (g^{n+1} - g^{n+\frac{1}{2}}) \\
&= \frac{1}{\sqrt{\mathcal{M}}} (1 + \delta \langle v \rangle)^{-b} \left(\nabla_v \cdot (v (g^{n+1} - g^{n+\frac{1}{2}})) - (-\Delta_v)^s (g^{n+1} - g^{n+\frac{1}{2}}) \right).
\end{aligned}$$

To proceed, first note that the lower bound in Part 4) of Lemma 10 can be removed. This is because for $p < -2s$, we have

$$\begin{aligned}
\left\| \langle v \rangle^p (-\Delta_v)^s (g^{n+1} - g^{n+\frac{1}{2}}) \right\|_{L^2_{x,v}} &\leq C \left\| \langle v \rangle^{-2s^-} (-\Delta_v)^s (g^{n+1} - g^{n+\frac{1}{2}}) \right\|_{L^2_{x,v}} \\
&\leq C \left(\frac{\Delta t}{\varepsilon^{2s}} \right)^{\min\{\frac{d}{2} + 3s^-, 1\}} (\varepsilon^{-s} + \varepsilon^{2-5s}) (L_0^0 + L_2^0).
\end{aligned}$$

where s^- denotes any number smaller than s. Therefore, if b satisfies

$$-b + \frac{1}{2}(d+2s) + 1 < s + \frac{d}{2}, \quad \text{or equivalently,} \quad b > 1, \tag{4.43}$$

then by Lemma 10, we have the estimate

$$\begin{aligned}
&\|(1+\delta\langle v\rangle)^{-b}\mathcal{L}^s(g^{n+1} - g^{n+\frac{1}{2}})\|_{\mathcal{M}^{-1}} \\
&\leq C\delta^{-b}\left\{\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{b-1,1\}}(\varepsilon^{-s} + \varepsilon^{1-2s})L_1^0 + \left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{\frac{d}{2}+3s^-,1\}}(\varepsilon^{-s} + \varepsilon^{2-5s})(L_0^0 + L_2^0)\right\}.
\end{aligned}$$

If we further assume that

$$b < 3s^- + \frac{d}{2} + 1\,, \tag{4.44}$$

the above inequality becomes

$$\|(1+\delta\langle v\rangle)^{-b}\mathcal{L}^s(g^{n+1} - g^{n+\frac{1}{2}})\|_{\mathcal{M}^{-1}} \leq C\delta^{-b}\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{b-1,1\}}(\varepsilon^{-s} + \varepsilon^{2-5s})L^0, \tag{4.45}$$

where $L^0 = L_0^0 + L_2^0$. This is because in Region I we have $\Delta t/\varepsilon^{2s} \leq 1$.

Next, we control the commutator term $[(1+\delta\langle v\rangle)^{-b}, \mathcal{L}^s]\tilde{f}^{n+1}$. By Lemma 4, if

$$\|\tilde{f}^{n+1}\|_{L^\infty_{\mathcal{M}^{-1}}} \leq C\varepsilon^{-2s}, \tag{4.46}$$

then we have

$$\begin{aligned}
&\left|\iint (1+\delta\langle v\rangle)^{-b}\tilde{f}^{n+1}[(1+\delta\langle v\rangle)^{-b}, \mathcal{L}^s]\tilde{f}^{n+1}\frac{1}{\mathcal{M}}\mathrm{d}x\mathrm{d}v\right| \\
&\leq C\delta^{\frac{2s}{d+4}}\|(1+\delta\langle v\rangle)^{-b}\tilde{f}^{n+1}\|^2_{\mathcal{M}^{-1}} + C\varepsilon^{-4s}\delta^{\min\{1,2s\}}.
\end{aligned} \tag{4.47}$$

Here (4.46) is fulfilled by combining part 2) of Lemma 8, the bound $\|g(t^n)\|_{L^\infty_{\mathcal{M}^{-1}}} \leq C\varepsilon^{-2s}$ and the fact that $\tilde{f}^n = \tilde{\eta}^n\mathcal{M} + \tilde{g}^n$. Indeed, from the argument of maximum principle that similar to the proof of part 2) in Lemma 8, the boundedness of $\|g(t^n)\|_{L^\infty_{\mathcal{M}^{-1}}}$ is guaranteed. Denote

$$F^n = \|(1+\delta\langle v\rangle)^{-b}\tilde{f}^n\|_{\mathcal{M}^{-1}}.$$

Plug (4.45) and (4.47) into (4.42). Then we get

$$(F^{n+1})^2 - (F^n)^2 \leq a\Delta t(F^{n+1})^2 + r, \tag{4.48}$$

where

$$\begin{aligned}
a &= 1 + C\delta^{\frac{2s}{d+4}}\varepsilon^{-2s}, \\
r &= C\frac{\Delta t}{\varepsilon^{6s}}\delta^{\min\{1,2s\}} + C\delta^{-2b}\frac{\Delta t}{\varepsilon^{4s}}\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\min\{2,2b-2\}}(\varepsilon^{-s} + \varepsilon^{2-5s})^2(L^0)^2 + \frac{\Delta t}{\varepsilon^{4s}}\|e^n\|^2_{\mathcal{M}^{-1}}.
\end{aligned}$$

Note that here we have used the fact that for $b > 0$, $\|(1+\delta\langle v\rangle)^{-b}e^n\|_{\mathcal{M}^{-1}} \leq \|e^n\|_{\mathcal{M}^{-1}}$. Optimize in r by choosing

$$\delta = \begin{cases} \left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\frac{\min\{2,2b-2\}}{1+2b}}\varepsilon^{\frac{4-8s}{2b+1}}, & s > \frac{1}{2}, \\ \left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\frac{\min\{1,b-1\}}{b+s}}, & s < \frac{1}{2}. \end{cases}$$

This correspondingly gives

$$a=\begin{cases} 1+C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\frac{\min\{2,2b-2\}}{1+2b}\frac{2s}{d+4}}\varepsilon^{\frac{4-8s}{2b+1}\frac{2s}{d+4}-2s}, & s>\frac{1}{2}\\ 1+C\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\frac{\min\{1,b-1\}}{b+s}\frac{2s}{d+4}}\varepsilon^{-2s}, & s\le\frac{1}{2}\end{cases}$$

and

$$\frac{r}{\Delta t}=\begin{cases} C\left[\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\frac{\min\{2,2b-2\}}{1+2b}}\varepsilon^{\frac{4-8s}{2b+1}-6s}+\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\frac{\min\{2,2b-2\}}{1+2b}}\varepsilon^{4-14s-2b\frac{4-8s}{2b+1}}(L^0)^2+\frac{1}{\varepsilon^{4s}}\|e^n\|^2_{\mathcal{M}^{-1}}\right] & s>\frac{1}{2},\\ C\left[\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{2s\frac{\min\{1,b-1\}}{b+s}}\varepsilon^{-6s}+\left(\frac{\Delta t}{\varepsilon^{2s}}\right)^{\frac{2s\min\{1,b-1\}}{b+s}}\varepsilon^{-6s}(L^0)^2+\frac{1}{\varepsilon^{4s}}\|e^n\|^2_{\mathcal{M}^{-1}}\right] & s\le\frac{1}{2}.\end{cases}$$

Recall the definition of e^n in (4.41) and the estimate of e_1^n, e_2^n in Lemma 6. Then β needs to satisfy

$$s>\frac{1}{2}: \qquad (1-2s\beta)\frac{\min\{2,2b-2\}}{1+2b}\frac{2s}{d+4}-2s\beta\left(\frac{8s-4}{2b+1}+1\right)\ge 0, \tag{4.49}$$

$$(1-2s\beta)\frac{\min\{2,2b-2\}}{1+2b}-2\beta\left(\frac{4s-2}{2b+1}+3s\right)\ge 0, \tag{4.50}$$

$$(1-2s\beta)\frac{\min\{2,2b-2\}}{1+2b}-2\beta\left(2-7s+b\frac{4s-2}{2b+1}\right)\ge 0\,; \tag{4.51}$$

$$s\le\frac{1}{2}: \qquad (1-2s\beta)\frac{\min\{1,b-1\}}{b+s}\frac{2s}{d+4}-2s\beta\ge 0, \tag{4.52}$$

$$(1-2s\beta)\frac{\min\{1,b-1\}}{b+s}-3\beta\ge 0. \tag{4.53}$$

Since $\beta\in(0,1)$, it is clear that there exists $\beta_0\in(0,1)$ such that if $\beta\in(0,\beta_0)$ then the inequalities above are satisfied. Then (4.48) leads to

$$(F^{n+1})^2\le e^{aT}(F^0)^2+C\frac{e^{aT}}{a}\frac{r}{\Delta t}. \tag{4.54}$$

Moreover, since $F^0=0$ and

$$\|(1+\delta\langle v\rangle)^{-b}\tilde f^n\|_{\mathcal{M}^{-1}}\ge\|((1+\delta)\langle v\rangle)^{-b}\tilde f^n\|_{\mathcal{M}^{-1}}\ge 2^{-b}\|\langle v\rangle^{-b}\tilde f^n\|_{\mathcal{M}^{-1}},$$

the bound (4.54) reduces to

$$\|\langle v\rangle^{-b}\tilde f^n\|_{\mathcal{M}^{-1}}\le Ce^{\frac{aT}{2}}\sqrt{\frac{r}{\Delta t}},$$

where C_T is a constant only depends on final time T. □

Remark 1. We can choose $b=1+2s$, then one choice of β_0 is

$$\beta_0=\begin{cases}\frac{1}{2s+(d+4)(12s-2)}, & s>\frac{1}{2},\\ \frac{s}{(1+3s)(d+4)+2s^2}, & s\le\frac{1}{2}.\end{cases}$$

In this case we have $\|\langle v\rangle^{-b}\tilde f^n\|_{\mathcal{M}^{-1}}\sim\Delta t^{\zeta}$, where

$$\zeta=\begin{cases}\frac{4s}{3+4s}-2\beta\frac{13s-2+16s^2}{3+4s}, & s>\frac{1}{2},\\ \frac{2s}{1+3s}-\beta\frac{3+9s+4s^2}{1+3s}. & s\le\frac{1}{2},\end{cases}\qquad \beta\in(0,\beta_0).$$

4.2 Regime II: diffusive and intermediate regime with $\varepsilon^{2s} \leq \Delta t^{2s\beta}$

We show that when ε is small compared with Δt, g^n can be bounded in terms of ε. In this regime, the choice of γ will depend on the relative relationship between Δt and ε specified as follows.

Lemma 11 (Estimate of g^n in Regime II)**.** *For $\varepsilon^{2s} \leq \Delta t^{2s\beta}$, we choose γ according to*

$$\gamma = \begin{cases} \sqrt{3} & \text{if } \varepsilon^{2s} \leq \Delta t \\ \sqrt{3}\Delta t^{\beta-1} & \text{if } \Delta t \leq \varepsilon^{2s} \leq \Delta t^{2s\beta}. \end{cases} \tag{4.55}$$

Then

$$\|g^n\|_2 \leq C\Delta t(\|\rho_{in}\|_{H^1_x} + \|g_{in}\|_{\mathcal{M}^{-1}}),$$

for $n \geq n_0$ outside an initial layer.

Proof. Combining the two equations in (4.21) leads to

$$g^{n+1} = (\alpha - \gamma + \varepsilon v\partial_x)^{-1}\alpha(\alpha + \gamma - \mathcal{L}^s)^{-1}(\alpha g^n - I^{n+1}).$$

Recall the estimate (4.25), then

$$\|g^{n+1}\|_{\mathcal{M}^{-1}} \leq \left|\frac{\alpha^2}{\alpha^2-\gamma^2}\right|^{n+1} \|g_{in}\|_{\mathcal{M}^{-1}} + \frac{\alpha}{\alpha^2-\gamma^2}\left[1 + \cdots + \left|\frac{\alpha^2}{\alpha^2-\gamma^2}\right|^{n}\right] \|I^{n+2-p}\|_{\mathcal{M}^{-1}}. \tag{4.56}$$

When $\varepsilon^{2s} \leq \Delta t$ and $\gamma = \sqrt{3}$, we have $\alpha \leq 1$, $\left|\frac{\alpha^2}{\alpha^2-\gamma^2}\right| < \frac{1}{2}$ and $\left|\frac{\alpha}{\alpha^2-\gamma^2}\right| < \frac{1}{2}$. Furthermore, by Lemma 7 Part 2) and Lemma 5 we have $\|I\|_{\mathcal{M}^{-1}} \leq C\varepsilon^{2s}\|\rho_{in}\|_{H^1_x}$. Therefore,

$$\|g^{n+1}\|_{\mathcal{M}^{-1}} \leq \frac{1}{2^n}\|g_{in}\|_{\mathcal{M}^{-1}} + C\varepsilon^{2s}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n+1}}\right)\|\rho_{in}\|_{H^1_x} \leq C\varepsilon^{2s}\|\rho_{in}\|_{H^1_x} + \frac{1}{2^n}\|g_{in}\|_{\mathcal{M}^{-1}}.$$

Choose $n \geq n_0$ with $n_0 = \frac{\log \Delta t}{\log \frac{1}{2}}$ such that $\frac{1}{2^n} \leq \Delta t$. Then the conclusion follows from the bound $\|g^n\|_{L^2_{x,v}} \leq C\|g^n\|_{\mathcal{M}^{-1}}$.

When $\Delta t < \varepsilon^{2s} \leq \Delta t^{2s\beta}$, $\gamma = \sqrt{3}\Delta t^{2s\beta-1}$, we again have $\left|\frac{\alpha^2}{\alpha^2-\gamma^2}\right| < \frac{1}{2}$. Then (4.56) leads to

$$\|g^{n+1}\|_{\mathcal{M}^{-1}} \leq \frac{1}{2^n}\|g_{in}\|_{\mathcal{M}^{-1}} + C\varepsilon^{2s}\left|\frac{\alpha}{\alpha^2-\gamma^2}\right|\|\rho_{in}\|_{H^1_x} \leq \Delta t\|g_{in}\|_{\mathcal{M}^{-1}} + C\Delta t\|\rho_{in}\|_{H^1_x}.$$ ☐

Theorem 3 (Error estimate in Regime II)**.** *Suppose $\varepsilon^{2s} < \Delta t^{2s\beta}$ and $t \geq t_0$ with t_0 being a fixed constant greater than $C\varepsilon^{2s}$. Then the numerical error $\tilde{f}^n$ satisfies*

$$\left\|\tilde{f}^n\right\|^2_{L^2_{x,v}} \leq C\Delta t^2\|e_2^n\mathcal{M}\|^2_{L^2_{x,v}} + C_{s,t_0}\varepsilon^{2s}\|\rho_{in}\|^2_{L^1} + C\left[\varepsilon\|v\cdot\nabla_x f_{in}\|_{L^1_{x,v}} + C_{t_0}\varepsilon^{2s}\|vf_{in}\|_{L^1_{x,v}}\right]$$
$$+ C\varepsilon^{\frac{d}{2}+2s}\|\rho_{in}\|_\infty + C\varepsilon^{d+4s}\|(-\Delta_x)^s\rho_{in}\|^2_x.$$

Proof. Recall the definition of the numerical error in (4.5). We have

$$\begin{aligned}\left\|\tilde{f}^n\right\|^2_{L^2_{x,v}} &= \|(\eta(t^n,x,v) - \eta^n(x,v))\mathcal{M} + (g(t^n,x,v) - g^n(x,v))\|^2_{L^2_{x,v}} \\ &\leq 2\Delta t^2\|e_2^n\mathcal{M}\|^2_{L^2_{x,v}} + 2\|g(t^n,x,v)\|^2_{L^2_{x,v}} + 2\|g^n(x,v)\|^2_{L^2_{x,v}}.\end{aligned}$$

Note that

$$\begin{aligned}\|g(t^n,x,v)\|_{L^2_{x,v}} &= \|f(t^n,x,v)-\rho(t^n,x)\mathcal{M}(v)+\rho(t^n,x)\mathcal{M}(v)-\eta(t^n,x,v)\mathcal{M}(v)\|_{L^2_{x,v}}\\ &\leq \|f(t^n,x,v)-\rho(t^n,x)\mathcal{M}(v)\|_{L^2_{x,v}}+\|(\rho(t^n,x)-\rho(t^n,x+\varepsilon v))\mathcal{M}(v)\|_{L^2_{x,v}}.\end{aligned}$$

Bound of the first term follows from Theorem 1. For the first term, write

$$\begin{aligned}&\|(\rho(t^n,x)-\rho(t^n,x+\varepsilon v))\mathcal{M}(v)\|^2_{L^2_{x,v}}\\ \leq{}& C\int\int_{|v|\leq\frac{1}{\varepsilon}}\frac{|\rho(x+\varepsilon v)-\rho(x)|^2}{\langle v\rangle^{2(d+2s)}}\mathrm{d}x\mathrm{d}v+C\int\int_{|v|>\frac{1}{\varepsilon}}\frac{|\rho(x+\varepsilon v)-\rho(x)|^2}{\langle v\rangle^{2(d+2s)}}\mathrm{d}x\mathrm{d}v=:I_5+I_6.\end{aligned}$$

For I_6, we have

$$I_6\leq C\left\|\rho_{in}\right\|_\infty\int\int_{|v|>\frac{1}{\varepsilon}}\frac{1}{\langle v\rangle^{2(d+2s)}}\mathrm{d}x\mathrm{d}v\leq C\varepsilon^{\frac{d}{2}+2s}\left\|\rho_{in}\right\|_\infty.$$

For I_5, use the change of variables

$$y=x+\varepsilon v,\quad \text{then } v=\frac{y-x}{\varepsilon},\ \mathrm{d}v=\varepsilon^{-d}\mathrm{d}y.$$

Then

$$I_5\leq C\varepsilon^{d+4s}\int\int_{|x-y|\leq 1}\frac{|\rho(y)-\rho(x)|^2}{|y-x|^{d+4s}}\mathrm{d}x\mathrm{d}y\leq C\varepsilon^{d+4s}\left\|(-\Delta_x)^s\rho\right\|_x^2\leq C\varepsilon^{d+4s}\left\|(-\Delta_x)^s\rho_{in}\right\|_x^2.$$

The rest follows from Theorem 1, Lemma 6 and Lemma 11. □

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