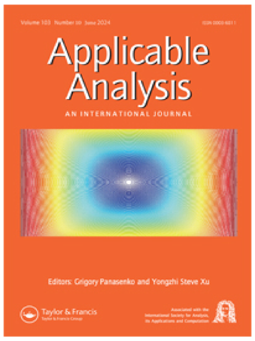

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Asymptotic optimality of a class of controlled non-Markov processes

Francois Dufour[a], Ky Tran[b], Le Yi Wang[c] and George Yin[d]

[a]Institut Polytechnique de Bordeaux, INRIA Bordeaux Sud Ouest, Team: ASTRAL, IMB, Institut de Mathématiques de Bordeaux, Université de Bordeau, France ; [b]Department of Applied Mathematics and Statistics, State University of New York, Incheon, Republic of Korea; [c]Department of Electrical and Computer Engineering, Wayne State University, Detroit, MI, USA; [d]Department of Mathematics, University of Connecticut, Storrs, CT, USA

ABSTRACT

Motivated by applications in power systems and problems arising in simulation of large scale complex system optimizations, this work is concerned with controlled stochastic switching systems. The system of interest displays a multi-time scale structure. In contrast to the so-called singularly perturbed diffusions and multi-scale Markov decision processes, controlled non-Markov processes (also known as non-Markov decision processes) are treated. The novelty of our work is the treatment of the non-Markov controlled processes and the time-scale used. The fast and slow processes are coupled through a stochastic differential equation. Using averaging, it is first shown that the non-Markov switching process has a weak limit that is a Markov decision process. Then asymptotic optimal control of the non-Markov process is obtained by using the limit process.

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1. Introduction

This work is largely motivated by applications in optimization and control of complex systems such as those arising in power systems. Modern power systems (MPS) employ advanced control methodologies in achieving stability, performance, reliability, robustness, and safety. MPS are complex network systems that consist of diversified electrical generators and loads, and include large numbers of buses, renewable energy sources, users, and controllable loads. When dynamic system models for subsystems are all included, a common microgrid can have thousands of state variables. Such large-scale complex systems encounter substantial complexity issues in their state estimation, control design, monitoring, and optimization. These complexity issues lead to extremely high computational complexity, high data flow rates, large memory space, complicated control design, among many other practical constraints and system costs. As a result, complexity reduction becomes a critical task in MPS management.

MPS control and management tasks are naturally time-scale separated. Some tasks are in the millisecond scale such as contingency detection and surging protection, some in the scale of seconds such as primary frequency regulations, followed by slower secondary frequency regulations and regional power dispatch. Such time-scale divisions have provided an important opportunity in treating systems in a hierarchical time structure so that fast subsystems can be grouped and treated together and whose averaging behavior becomes the relevant variables in slower dynamic systems

CONTACT George Yin gyin@uconn.edu

with much reduced model orders. This modeling idea stems from the approach of singular perturbations. In a typical deterministic framework, the idea of singular perturbations has been applied to power systems; see [1–10]. Nevertheless, stochastic properties that can be used to facilitate complexity reduction have not been used in power systems. This paper aims to introduce such a fundamental framework with key properties for potential usage in MPS. However, specific applications in MPS will be considered elsewhere.

Similar to many complex systems, large and complex power networks normally include a number of subsystems, and subsystems consist of numerous components. The complexity makes the analysis of such systems difficult. There is a long history of research on reduction of computational complexity by means of multi-scale formulations. We refer the reader to [1] and colleagues for the formulation using time-scale separation and singular perturbation methods. We also cite the related references on Kokotovic and colleagues as mentioned in the above paragraphs. While these references are interesting, their main focus has been placed on deterministic systems. On the other hand, from the literature of stochastic systems, there has been continuing effort of modeling, analysis, and computation for treating large and complex systems with multi-scales. Not only have deterministic systems been dealt with, but also stochastic systems have been analyzed for various models such as diffusions, Markov chains, jump systems, and various combinations of these.

Two-time-scale or multi-time-scale systems also appear in Monte-Carlo simulations of large and complex systems. In such problems, one uses time-scale separation to facilitate complex optimization problems. With the motivation mentioned above, this paper investigates a related optimization problem of a complex system from a mathematical perspective. In optimization, Markov decision processes have been studied extensively and applied to numerous scenarios. Here we propose to analyze a system that is non-Markovian in continuous time. The non-Markovian setting makes the analysis of such systems difficult. Directly using the techniques of Markov decision processes is not possible. Processes involving random perturbations were pioneered in the work of Khasminskii [11]. Such an effort was much extended to involve wideband noise processes by Kushner [12, 13]. Effort was also devoted to further characterize the associated transition probabilities through the corresponding forward and backward Kolmogorov equations [14–17] in the late 1900s and early 2000s. The structure of fast and slow diffusions, and diffusions with a Markov chain was considered in [18]. Along another line, diffusions modulated by continuous-time switching processes were considered in [19, 20]; the former concentrated on Markovian switching diffusions, whereas the latter treated the switching processes depending on the diffusion processes.

In this paper, we consider a class of controlled non-Markov processes (also called non-Markov decision processes) that are similar to Markov decision processes (or controlled Markov processes) in which the switching processes depend on control actions; see [21] for related works on piecewise deterministic Markov processes and references therein. However, the underlying randomly switching process appears in a stochastic differential equation as a slowly varying component. There is a fast varying diffusion. So the main features are the non-Markovian switching and the presence of the fast and slow processes. Our objective is to minimize an objective function for the control problem.

Our effort here is to show that in lieu of working with the process directly, the control-dependent switching process, in fact, has a limit in an appropriate sense. To begin, the fast-varying diffusion does not blowup, but rather has a stationary or invariant distribution. Then we can show that the diffusion-dependent switching process converges weakly (in the sense of convergence in probability measure) to a process. Interestingly, the limit process is a Markov decision process, or a controlled Markov chain whose generator depends on the control. We then show that the original objective function converges to that of a Markov decision process. Assuming the limit Markov decision process has an optimal control, we proceed to show that if we use the optimal control of the limit process in the original process, we get an optimality of the original process in an appropriate sense.

The novelties of the paper are in the following aspects. First, the controlled switching process is non-Markovian. Second, although optimization of two-time scale systems has been treated in the literature, the current effort is different from the existing work. In the treatment of singularly perturbed diffusions, there are usually two equations, one for the fast-varying process and the other for a slowly-changing process [13]. In the work on multi-scale Markov decision processes, both fast and slow controlled processes are modeled using operators. In this paper, the fast process is a diffusion that is coupled with the slow process, whereas the slow process is given by using an operator. In addition, the fast-changing process is both dealt with as a state and also a fast noise process. The main analysis is based on weak convergence methods. After showing that the slow process has a limit process that is a Markov decision process, we use the optimal control of the limit process in the original system. We then show that such controls lead to asymptotic optimality.

The rest of the paper is arranged as follows. Section 2 begins with the formulation of the problem. Then Section 3 establishes the weak convergence of the switching process. Section 4 analyzes the asymptotic optimality of the control process.

This paper is written on the occasion of the celebration of Professor Wei Lin's 90's Birthday. More than 40 years ago, when Professor Lin visited Professor Robert Gilbert at the University of Delaware (UD), George Yin was an undergraduate student at UD. Both of them rented the same house on Elkton Road at that time. George benefited the many discussions with Professor Lin.

While his mathematical work started in the early 1960s, Professor Lin's main line of work has been focused on partial differential equations and applications in mathematical physics and elasticity. His book (English version) coauthored with the renowned mathematician Professor Loo Keng Hua and also Professor Ci-Quian Wu, was published in Pitman's Research Notes in Mathematics in 1985, which has become an important reference for partial differential equations from the particular angle of using the function theory approach. It should also be mentioned that in his early career in the 1970s, Professor Lin worked on control theory and applications. Our current paper is related to Professor Lin's earlier work in that our paper is devoted to a class of stochastic control problems. We join our colleagues to celebrate Professor Lin's achievement in mathematics, wishing him many happy returns.

2. Formulation

We begin with the following formulation. Let $x \in \mathbb{R}^d$ and $\mathcal{M} = \{\alpha_1, \ldots, \alpha_m\}$ be a finite set. Suppose that $\varepsilon > 0$ is a small parameter and that $\alpha^\varepsilon(t)$ is a continuous-time stochastic process taking values in the finite set $\mathcal{M}$ such that $\alpha^\varepsilon(t)$ depends on the small parameter. The dependence of $\alpha^\varepsilon(\cdot)$ is to be specified shortly. Consider a pair of stochastic processes $(X^\varepsilon(t), \alpha^\varepsilon(t))$, which is jointly Markov so that for $H : \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}^d$, a suitable smooth function, we have

$$\mathrm{d}X^\varepsilon(t) = \frac{1}{\varepsilon} H(X^\varepsilon(t), \alpha^\varepsilon(t))\,\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\sqrt{2\rho}\,\mathrm{d}W(t), \quad X^\varepsilon(0) = x, \tag{1}$$

where $\rho > 0$ is a constant, and $W(\cdot)$ is a standard d-dimensional Brownian motion. Assume throughout the work that both $\alpha^\varepsilon(\cdot)$ and $W(\cdot)$ are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, $\alpha^\varepsilon(\cdot)$ and $W(\cdot)$ are independent. The model is originated from the consideration of stochastic simulation in chemical reactions, in which the Brownian perturbation is added to speed up the process. We have adopted this formulation. In fact, there is no problem for treating the diffusion part given by $\frac{1}{\sqrt{\varepsilon}}\widetilde{\sigma}(X^\varepsilon(t), \alpha^\varepsilon(t))\,\mathrm{d}W(t)$. All needed is $\widetilde{\sigma}(x,\alpha)\widetilde{\sigma}'(x,\alpha)$ to be positive definite (i.e. the diffusion is non-degenerate).

Suppose that $u^\varepsilon(t)$ is a control taking values in U a compact subset of $\mathbb{R}^l$, and that the transition probabilities of $\alpha^\varepsilon(\cdot)$ satisfy

$$\begin{aligned}\mathbb{P}(\alpha^\varepsilon(t+\Delta) = \alpha_j | \alpha^\varepsilon(t) = \alpha_i, X^\varepsilon(s), \alpha^\varepsilon(s), u^\varepsilon(s) : s \le t)\\ = q_{ij}(X^\varepsilon(t), u^\varepsilon(t))\Delta + o(\Delta),\end{aligned} \tag{2}$$

for $\Delta \downarrow 0$ with $Q(x,u) = (q_{ij}(x,u)) \in \mathbb{R}^{m\times m}$ matrix-valued function satisfying $q_{ij}(x,u) \ge 0$ for $i \ne j$ and $\sum_{j=1}^m q_{ij}(x,u) = 0$ for each i. Note that the transition probabilities depend on x through (2). We index the process u by ε just for indicating that the control is associated with $\alpha^\varepsilon(\cdot)$ that depends on ε. System (1) has a two-time scale structure; $X^\varepsilon(\cdot)$ is fast varying and $\alpha^\varepsilon(\cdot)$ is slowly changing. We remark that $\alpha^\varepsilon(\cdot)$ alone is not Markov due to the dependence of $Q(x, u)$ on the state x; only the pair $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$ is Markov. If the x dependence in $Q(x, u)$ were missing, then $Q(x, u) = Q(u)$ would be a generator of a continuous-time Markov decision process. In this paper, we aim to solve a problem that is similar to a Markov decision process. Nevertheless, the problem is a non-Markovian optimization problem due to the x dependence. Hence the problem is more complex. Our objective is to minimize a cost function

$$J^\varepsilon(\alpha_\iota, u^\varepsilon(\cdot)) = \mathbb{E}\int_0^T L(\alpha^\varepsilon(t), u^\varepsilon(t))\,\mathrm{d}t, \tag{3}$$

where $L(\alpha_\iota, u)$ is a suitable running cost rate function, $\alpha^\varepsilon(0) = \alpha_\iota$, T is a positive constant. To emphasize the dependence of the initial data $\alpha^\varepsilon(0) = \alpha_\iota$, one often uses $\mathbb{E}_\iota$ in lieu of $\mathbb{E}$. In this paper, to simplify the discussion, we will not use this notation. Recall that the dynamics of $\alpha^\varepsilon(\cdot)$ depend on $X^\varepsilon(\cdot)$. Note also $X^\varepsilon(\cdot)$ is fast varying, and can be viewed as a noise process. Directly solving the problem seems virtually impossible. Nevertheless, the fast changing diffusion given by (1) does not blowup, but has an invariant measure leading to certain averages with respect to the invariant measure taking place, which helps us to solve the underlying problem.

We aim to show that as $\varepsilon \to 0$, $\alpha^\varepsilon(\cdot)$ converges weakly to $\alpha(\cdot)$, a continuous-time Markov chain whose generator $\overline{Q}(u)$ is an average with respect to the stationary distribution of the fast-varying diffusion. The more precise notion will follow.

2.1. Some background materials

As alluded to in the introduction, we are dealing with a class of stochastic control problems. Normally, stochastic control problems lead to the treatment of the so-called Hamilton–Jacobi–Bellman (HJB) partial differential equations. However, because the problem we are considering is not Markovian. The usual approach does not work. As an alternative, we will use a probability approach. This approach is deeply rooted to the methods of weak convergence of probability measures and martingale averaging. Before proceeding further, we briefly recall some of the basic notions.

The concept of weak convergence is a substantial generalization of convergence in distribution in probability theory. Here we gather a number of definitions and results regarding weak convergence including tightness, martingale problem, Skorohod representation, and Prohorov's theorem, etc. Nevertheless, all the notions are mentioned in a rather intuitive way. Additional references are given for convenience.

First let us recall the definition of weak convergence. Let $\mathbb{P}$ and $\mathbb{P}_n$, $n = 1, 2, \ldots$, denote probability measures defined on a metric space. The sequence $\{\mathbb{P}_n\}$ converges weakly to $\mathbb{P}$ if

$$\int f\,\mathrm{d}\mathbb{P}_n \to \int f\,\mathrm{d}\mathbb{P}$$

for every bounded and continuous function f on the probability space. Let $\{X_n\}$ and X be random variables associated with $\mathbb{P}_n$ and $\mathbb{P}$, respectively. The sequence X_n converges to X weakly if for any bounded and continuous function f,

$$\mathbb{E}f(X_n) \to \mathbb{E}f(X) \quad \text{as} \quad n \to \infty.$$

Use $D([0, T]; \mathbb{R}^r)$ to denote the space of $\mathbb{R}^r$-valued functions on $[0, T]$ that are right-continuous and that have left-hand limits. Then we can define what is called Skorohod topology (see [12] for detail). We say a family of probability measures defined on a metric space is tight if for each $\eta > 0$, there exists a compact set K_η such that $\mathbb{P}(K_\eta) \geq 1 - \eta$. The notion of tightness is closely related to compactness. The well-known Prohorov's theorem indicates that tightness is equivalent to sequential compactness or relative compactness.

Weak convergence techniques usually allow the use of much weaker conditions and results in more general setup. For purely analytic reasons, however, it is often more convenient to work with probability one convergence. A device, known as Skorohod representation, provides us with such opportunities. Let $\mathbb{P}_n$ and $\mathbb{P}$ denote probability measures on $D([0,T];\mathbb{R}^r)$ such that $\mathbb{P}_n$ converges weakly to $\mathbb{P}$. Then corresponding to the original probability space $(\Omega,\mathcal{F},\mathbb{P})$, there exists a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ on which are defined $D([0,T];\mathbb{R}^r)$-valued random variables $\widetilde{X}_n$, $n=1,2,\ldots$, and $\widetilde{X}$ such that for any Borel set B and all $n<\infty$,

$$\widetilde{\mathbb{P}}(\widetilde{X}_n\in B)=\mathbb{P}_n(B) \quad \text{and} \quad \widetilde{\mathbb{P}}(\widetilde{X}\in B)=\mathbb{P}(B) \text{ such that}$$
$$\lim_{n\to\infty}\widetilde{X}_n=\widetilde{X} \text{ w.p.1.}$$

Finally, we mention that a right-continuous process $X(t)$, $t\geq 0$ and $s>0$, is a solution of the martingale problem for the operator $\mathcal{A}$ if and only if for any bounded and continuous function $h(\cdot)$, any sufficiently smooth (the smoothness requirement depends on the problem) function $f(\cdot)$ with compact support, any positive integers κ and ℓ, and any $t_\ell\leq t$,

$$\mathbb{E}h(X_{t_\ell}:\ell\leq\kappa)\left[f(X(t+s))-f(X(t))-\int_t^{t+s}\mathcal{A}f(X(\zeta))\,\mathrm{d}\zeta\right]=0.$$

We only provided a very brief introduction. Further details on the related consequents can be found in, for example, [12, 18] and many references therein.

2.2. Further preparation

The process $(X^\varepsilon(\cdot),\alpha^\varepsilon(\cdot))$ is Markov [20], whose generator (for a fixed u) is given by

$$\mathcal{L}f(x,\alpha_i)=\frac{\rho}{\varepsilon}\mathrm{tr}(\nabla^2 f(x,\alpha_i))+\frac{1}{\varepsilon}(\nabla f(x,\alpha_i))'H(x,\alpha_i)$$
$$+\sum_{j\in\mathcal{M}}q_{ij}(x,u)f(x,\alpha_j),\quad \alpha_i\in\mathcal{M}, \tag{4}$$

where z' denotes the transpose of z, ∇^2 denotes the Hessian matrix (with differentiation with respect to x), $\mathrm{tr}(\Gamma)$ is the trace of Γ, and $f(\cdot)$ is a suitable smooth function. Sometimes, to emphasize on the dependence of u, one may write $\mathcal{L}$ as $\mathcal{L}^u$. Here and henceforth, for simplicity, we simply suppress the dependence of u.

To facilitate the analysis, we use the relaxed control representation; see for example, [13, 22–24]. Although the relaxed controls cannot be used in the actual control systems, they are rather convenient to be used in getting the desired limit problems [23, 24].

Suppose that $\mathcal{B}(U\times[0,T])$ is the σ-algebra of Borel subsets of $U\times[0,T]$. A deterministic relaxed control ß is a measure on $\mathcal{B}(U\times[0,T])$ such that the marginal of ß on $[0,T]$ coincides with Lebesgue measure $\mathrm{d}t$. This space will be denoted by $\mathfrak{R}(U\times[0,T])$. Then every measure ß in $\mathfrak{R}(U\times[0,T])$ can be disintegrated as $ß(\mathrm{d}c\,\mathrm{d}t)=ß_t(\mathrm{d}c)\,\mathrm{d}t$, where $ß_t$ is the derivative of ß (more details will follow). The space $\mathfrak{R}(U\times[0,T])$ can be metrized with the Prohorov metric (see [22, p. 263]). Under such a setup, the weak convergence of a sequence of deterministic relaxed controls $ß^\varepsilon(\cdot)\to ß(\cdot)$ is equivalent to

$$\int\phi(c,s)ß^\varepsilon(\mathrm{d}c\,\mathrm{d}s)\to\int\phi(c,s)ß(\mathrm{d}c\,\mathrm{d}s)$$

for any continuous function $\phi(\cdot)$ on $U\times[0,T]$ having compact support. With the Prohorov metric, $\mathfrak{R}(U\times[0,T]))$ is a compact space.

Relaxed controls for controlled stochastic processes (in particular, controlled diffusion and/or controlled jump diffusions) were defined in the aforementioned references. Coming back to the problem that we are considering, with the given probability space, we say that for $\varepsilon > 0$, $\text{ß}^\varepsilon(\cdot)$ is an admissible relaxed (stochastic) control for $\alpha^\varepsilon(\cdot)$ or $(\text{ß}^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$ is admissible, if $\text{ß}^\varepsilon(\cdot)$ is an $\Re(U \times [0,T]))$-values random variable such that $\text{ß}^\varepsilon(A \times [0,t])$ is $\mathcal{F}_t^\varepsilon$-adapted for all $A \in \mathcal{B}(U)$, where $\mathcal{F}_t^\varepsilon$ is the σ-algebra generated by $\{X^\varepsilon(s), \alpha^\varepsilon(s) : s \le t\}$. The set of admissible relaxed stochastic controls for $\alpha^\varepsilon(\cdot)$ will be denoted by $\mathcal{R}^\varepsilon$.

In what follows, working with a limit problem, we will consider the set of admissible relaxed stochastic control for $\alpha(\cdot)$ (the weak-limit of $\alpha^\varepsilon(\cdot)$) as the set of $\Re(U \times [0,T]))$-values random variables ß such that $\text{ß}(A \times [0,t])$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U)$, with $\mathcal{F}_t$ is the σ-algebra generated by $\{\alpha(s) : s \le t\}$. The set of admissible relaxed stochastic controls for $\alpha(\cdot)$ will be denoted by $\mathcal{R}$.

The technical details will be seen in the section on weak convergence of $\alpha^\varepsilon(\cdot)$ to α. Using the given processes $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$ with $X^\varepsilon(0) = x$ and $\alpha^\varepsilon(0) = \alpha_\iota$, we proceed to study the near optimal control with the cost function given by

$$\begin{aligned} J^\varepsilon(\alpha_\iota, u^\varepsilon(\cdot)) &= \mathbb{E}\int_0^T L(\alpha^\varepsilon(t), u^\varepsilon(t))\,\mathrm{d}t \\ &= \mathbb{E}\int_0^T \int L(\alpha^\varepsilon(t), c)\text{ß}_t^\varepsilon(\mathrm{d}c)\,\mathrm{d}t, \end{aligned} \tag{5}$$

where L is a suitable cost rate function, u^ε is the control, $\text{ß}^\varepsilon \in \mathcal{R}^\varepsilon$ is a relaxed control representation of u^ε, and ß_t^ε is the derivative of ß. Note that in fact $\alpha^\varepsilon(\cdot)$ depends on $X^\varepsilon(\cdot)$ as given in (1).

2.2.1. Standing conditions

We shall assume the following conditions throughout the paper.

(A1) The partial derivative of $H(\cdot, \alpha_i)$ with respect to the first variable is continuous for each α_i; $L(\cdot,\cdot)$ is continuous in both variables; $Q(x,u)$ is a bounded matrix-valued function that is continuous with respect to both variables.

Remark 2.1: Choose

$$f(x, \alpha_i) = f(\alpha_i) = 1_{\{\alpha = \alpha_i\}} \quad \text{for a fixed } \alpha_i \in \mathcal{M},$$

where 1_A denotes the indicator function of the set A. Because this $f(\cdot)$ is independent of x, we have

$$\mathcal{L}f(\alpha_i) = Q(x,u)f(\cdot)(\alpha_i),$$

where

$$Q(x,u)f(\cdot)(\alpha_i) = \sum_{j=1}^m q_{ij}(x,u)f(\alpha_j).$$

There are many examples of the function $H(\cdot)$. Here, we only need the stochastic differential equation (1) to have a weak solution. For example, in certain optimization problems, $H(x,\alpha) = -\nabla_x \widehat{H}(x,\alpha)$, where $\widehat{H}(\cdot)$ is a potential function. In this case, since $\rho > 0$ in (1), the diffusion is non-degenerate, by virtue of [20, Theorem 4.3], there is a stationary distribution $\nu^\varepsilon(x,\alpha)$ for the process $(X^\varepsilon(\cdot), \alpha^\varepsilon(\cdot))$. In what follows, our assumptions are more general including many other cases.

We should also like to add that our main interest is on treating the 'marginal' $X^\varepsilon(\cdot)$ as a fast noise process and $\alpha^\varepsilon(\cdot)$ as a slow process so as to obtain certain desired limit. Moreover, we are interested in the associated optimization problem.

To proceed, using a scaling technique, we change both independent and dependent variables by defining

$$Z(t) = X^{\varepsilon}(\varepsilon t), \quad \gamma(t) = \alpha^{\varepsilon}(\varepsilon t), \quad \text{and} \quad \widetilde{W}(t) = W(\varepsilon t)/\sqrt{\varepsilon}.$$

Then (1) can be rewritten as

$$dZ(t) = H(Z(t), \gamma(t))\, dt + \sqrt{2\rho}\, d\widetilde{W}(t). \tag{6}$$

Similar to [13, Chapter 4], we view the process $\gamma(\cdot)$ of being varying very slowly and almost a constant. In fact, given $\gamma(t) = \alpha$, $Z(\cdot) = Z(\cdot|\alpha)$ is given by the solution of

$$dZ(t) = H(Z(t), \alpha)\, dt + \sqrt{2\rho}\, d\widetilde{W}(t). \tag{7}$$

That is, we fix or frozen the $\gamma(\cdot)$ at a constant level (α in (7) is treated as a parameter). To proceed, we need another condition.

(A2) For each α, the stochastic differential equation (7) has a unique weak solution. Moreover, for each α, $Z(\cdot|\alpha)$ has a unique invariant measure $\mu_\alpha(\cdot)$.

(A3) For the sequence $\{ß^{\varepsilon}(\cdot)\}$, there is a nonnegative function $g(\cdot)$ satisfying $0 \le g(x) \to \infty$ as $|x| \to \infty$, a $0 < K_1 < \infty$, and $\Delta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that $\varepsilon/\Delta_\varepsilon \to 0$ and

$$\sup_{t \le T} \frac{1}{\Delta_\varepsilon} \int_t^{t+\Delta_\varepsilon} \mathbb{E} g(X^{\varepsilon}(s))\, ds \le K_1 < \infty.$$

Remark 2.2: Note that the conditions (A2) and (A3) require that the solution of (7) is regular; i.e. the solution is global [25] (see also [20]) or no finite explosion time. The well-known theory of diffusions (see [25]) can provides us with sufficient conditions that ensure the existence of the invariance distribution that associated with the solution of (7), there is an invariant distribution $\mu_\alpha(z)$ such that the transition probability of $Z(\cdot)$ converges to the invariant distribution exponentially fast as $t \to \infty$. However, in this paper, rather than giving sufficient conditions for the existence of invariance distribution, we assume the existence of the invariant distribution. This is for simplicity and also our main interest is in the averaging and optimization based on the existence of the invariant distribution. Note that $X^{\varepsilon}(\cdot)$ can be viewed as a noise process. Although it does not blow up and has an invariant measure, it does not have a limit in the sense of convergence of probability measure. Rather, only the $\alpha^{\varepsilon}(\cdot)$ has a limit in weak sense.

As commented in [13, p. 67], (A3) is satisfied if $\{X^{\varepsilon}(t) : t \le T, \varepsilon > 0\}$ is tight. Under this assumption, for the sequence $\{ß^{\varepsilon}(\cdot)\}$ of relaxed control, for each $\Delta > 0$, define the occupation measure by

$$\widehat{P}_t^{\varepsilon,\Delta}(F) = \frac{\varepsilon}{\Delta} \int_{t/\varepsilon}^{(t+\Delta)/\varepsilon} 1_F(Z(s))\, ds.$$

Then for any bounded sequence $\{t_\varepsilon\}$, the sequence of measure-valued variables $\{\widehat{P}_{t_\varepsilon}^{\varepsilon,\Delta}, \varepsilon > 0\}$ is tight (see [13, p. 68] for a proof).

With the preparation above, we proceed to derive the weak convergence of the process $\alpha^{\varepsilon}(\cdot)$. Recall that $X^{\varepsilon}(\cdot)$ is treated as a fast-varying noise. To obtain the desired convergence, we first show that the sequence is tight. Then we characterize the limit by martingale problem formulation.

3. Limit process

3.1. Tightness of the switching process

Lemma 3.1: *Consider the process $\{\alpha^{\varepsilon}(\cdot)\}$ and let $u^{\varepsilon}(\cdot)$ be the corresponding control with $ß^{\varepsilon}(\cdot)$ being its relaxed control representation. Then $(\alpha^{\varepsilon}(\cdot), ß^{\varepsilon}(\cdot))$ is tight in $D([0, T] : \mathcal{M}) \times \mathfrak{R}(U \times [0, T])$, where*

$D([0,T]:\mathcal{M})$ is the space of functions that are defined on $[0,T]$ taking values in $\mathcal{M}$, that are right continuous, and that have left limits endowed with the Skorohod topology, where $\Re(U\times[0,T])$ is the collection of all relaxed controls.

Proof: We first note that since $\Re(U\times[0,T])$ is compact, it is weakly compact and hence $ß^\varepsilon(\cdot)$ is tight. Thus we need only consider the tightness of $\{\alpha^\varepsilon(\cdot)\}$. Define

$$\chi^\varepsilon(t)=(1_{\{\alpha^\varepsilon(t)=\alpha_1\}},\ldots,1_{\{\alpha^\varepsilon(t)=\alpha_m\}})\in\mathbb{R}^{1\times m}, \tag{8}$$

where 1_G denotes the usual indicator function of the set G. To prove the desired tightness of $\alpha^\varepsilon(\cdot)$, we use the process $\chi^\varepsilon(\cdot)$. Since the process $(X^\varepsilon(\cdot),\alpha^\varepsilon(\cdot))$ is Markov,

$$\chi^\varepsilon(t)-\chi^\varepsilon(0)-\int_0^t\int\chi^\varepsilon(v)Q(X^\varepsilon(v),c)ß^\varepsilon_v(\mathrm{d}c)\,\mathrm{d}v \text{ is a martingale.}$$

As a result, for any $\delta>0$, and $t,s\in[0,T]$ with $s\le\delta$, we have

$$\mathbb{E}\left(\chi^\varepsilon(t+s)-\chi^\varepsilon(t)-\int_t^{t+s}\int\chi^\varepsilon(v)Q(X^\varepsilon(v),c)ß^\varepsilon_v(\mathrm{d}c)\,\mathrm{d}v|\mathcal{F}^\varepsilon_t\right)=0,$$

Recall that $\mathcal{F}^\varepsilon_t$ denotes the σ-algebra generated by $\{X^\varepsilon(v),\alpha^\varepsilon(v):v\le t\}$. Since $Q(x,u)$ is bounded and continuous, it is readily seen that

$$\sup_{0\le s\le\delta}\left|\int_t^{t+s}\int\chi^\varepsilon(v)Q(X^\varepsilon(v),c)ß^\varepsilon(\mathrm{d}c)\,\mathrm{d}v\right|=\sup_{0\le s\le\delta}O(s)=O(\delta).$$

Thus, working with $\chi^\varepsilon(\cdot)$, we obtain

$$\sup_{0\le s\le\delta}\mathbb{E}(|\chi^\varepsilon(t+s)-\chi^\varepsilon(t)|\mathcal{F}^\varepsilon_t)=O(\delta)\quad\text{and}$$
$$\sup_{0\le s\le\delta}\mathbb{E}(|\chi^\varepsilon(t+s)-\chi^\varepsilon(t)|^2|\mathcal{F}^\varepsilon_t)=O(\delta).$$

Taking expectation, then $\limsup_{\varepsilon\to0}$ followed by $\lim_{\delta\to0}$, we obtain

$$\lim_{\delta\to0}\limsup_{\varepsilon\to0}\mathbb{E}\left[\sup_{0\le s\le\delta}E(|\chi^\varepsilon(t+s)-\chi^\varepsilon(t)|^2|\mathcal{F}^\varepsilon_t)\right]=0.$$

The desired tightness of $\{\chi^\varepsilon(\cdot)\}$ thus follows. Note

$$\alpha^\varepsilon(t)=\sum_{i=1}^m\alpha_i1_{\{\alpha^\varepsilon(t)=\alpha_i\}}=\chi^\varepsilon(t)(\alpha_1,\ldots,\alpha_m)',$$

the tightness of $\{\chi^\varepsilon(\cdot)\}$ then implies that of $\{\alpha^\varepsilon(\cdot)\}$. The proof is complete. ■

3.2. Weak convergence

The main result of this section is the following theorem.

Theorem 3.2: *Under our standing assumptions, $\alpha^\varepsilon(\cdot)$ converges weakly to $\alpha(\cdot)$, a continuous-time Markov chain with state space $\mathcal{M}$ and generator $\overline{Q}$ given by*

$$\overline{Q}(c)=(\overline{q}_{ij}(c))\text{ with }\overline{q}_{ij}(c)=\int_{\mathbb{R}^d}q_{ij}(x,c)\mu_{\alpha_i}(\mathrm{d}x), \tag{9}$$

where $\mu_{\alpha_i}(x)$ is the stationary distribution of the diffusion process $Z(\cdot|\alpha_i)$.

Proof: Since $\{\alpha^\varepsilon(\cdot), ß^\varepsilon(\cdot)\}$ is tight, we can extract a weakly convergent subsequence by the well-known Prohorov theorem. Extract such a sequence and still index the subsequence by ε for notational simplicity. Denote the limit by $(\alpha(\cdot), ß(\cdot))$. We proceed to characterize its limit. By the Skorohod representation theorem, we may assume (with a slight abuse of notation) that $(\alpha^\varepsilon(\cdot), ß^\varepsilon(\cdot))$ converges to $(\alpha(\cdot), ß(\cdot))$ with probability one (w.p.1) and the convergence is uniform on any bounded interval.

First note that for any real-valued function $f(\cdot)$ defined on $\mathcal{M}$,

$$f(\alpha^\varepsilon(t)) = \sum_{i=1}^m 1_{\{\alpha^\varepsilon(t)=\alpha_i\}} f(\alpha_i) = \chi^\varepsilon(t)(f(\alpha_1), \ldots, f(\alpha_m))'. \tag{10}$$

Then

$$\begin{aligned}
&\int Q(X^\varepsilon(t), c) f(\cdot)(\alpha^\varepsilon(t)) ß_t^\varepsilon(\mathrm{d}c) \\
&\quad = \sum_{i=1}^m 1_{\{\alpha^\varepsilon(t)=\alpha_i\}} \int Q(X^\varepsilon(t), c) f(\cdot)(\alpha_i) ß_t^\varepsilon(\mathrm{d}c) \\
&\quad = \int \chi^\varepsilon(t) Q(X^\varepsilon(t), c)(f(\alpha_1), \ldots, f(\alpha_m))' ß_t^\varepsilon(\mathrm{d}c).
\end{aligned}$$

Using the operator $\mathcal{L}$ defined in (4) and noting $f(\alpha)$ does not depend on x explicitly, define

$$\begin{aligned}
M_f^\varepsilon(t) &= f(\alpha^\varepsilon(t)) - f(\alpha^\varepsilon(0)) - \int_0^t \mathcal{L} f(\alpha^\varepsilon(v))\, \mathrm{d}v \\
&= f(\alpha^\varepsilon(t)) - f(\alpha^\varepsilon(0)) \\
&\quad - \int_0^t \chi^\varepsilon(v) \int Q(X^\varepsilon(v), c)(f(\alpha_1), \ldots, f(\alpha_m))' ß_v^\varepsilon(\mathrm{d}c)\, \mathrm{d}v.
\end{aligned}$$

Then $M_f^\varepsilon(t)$ is a martingale. It follows that for real-valued function $h(\cdot)$, any positive integer κ, κ_1, any $t, s > 0$, any $t_\ell \le t$ with $\ell \le \kappa$, and any continuous functions $\phi_1(\cdot), \ldots, \phi_{\kappa_1}(\cdot)$ on $U \times [0, T]$ having compact support,

$$\begin{aligned}
&\mathbb{E} h(\alpha^\varepsilon(t_\ell), (ß^\varepsilon, \phi_{\ell_1})_{t_\ell} : \ell \le \kappa, \ell_1 \le \kappa_1) \Big[f(\alpha^\varepsilon(t+s)) - f(\alpha^\varepsilon(t)) \\
&\quad - \int_t^{t+s} \int \chi^\varepsilon(v) Q(X^\varepsilon(v), c)(f(\alpha_1), \ldots, f(\alpha_m))' ß_v^\varepsilon(\mathrm{d}c)\, \mathrm{d}v \Big] = 0.
\end{aligned}$$

The weak convergence of $\alpha^\varepsilon(\cdot)$ to $\alpha(\cdot)$, the Skorohod representation, and the fact $\alpha^\varepsilon(t) \in \mathcal{M}$ for any $t > 0$ imply that

$$\begin{aligned}
&\mathbb{E} h(\alpha^\varepsilon(t_\ell), (ß^\varepsilon, \phi_{\ell_1})_{t_\ell} : \ell \le \kappa, \ell_1 \le \kappa_1)[f(\alpha^\varepsilon(t+s)) - f(\alpha^\varepsilon(t))] \\
&\quad \to \mathbb{E} h(\alpha(t_\ell), (ß, \phi_{\ell_1})_{t_\ell} : \ell \le \kappa, \ell_1 \le \kappa_1)[f(\alpha(t+s)) - f(\alpha(t))] \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{11}$$

Next, we aim to show that as $\varepsilon \to 0$,

$$\begin{aligned}
&\mathbb{E} h(\alpha^\varepsilon(t_\ell), (ß^\varepsilon, \phi_{\ell_1})_{t_\ell} : \ell \le \kappa, \ell_1 \le \kappa_1)) \\
&\quad \times \left[\int_t^{t+s} \int \chi^\varepsilon(v) Q(X^\varepsilon(v), c)(f(\alpha_1), \ldots, f(\alpha_m))' ß_v^\varepsilon(\mathrm{d}c)\, \mathrm{d}v \right]
\end{aligned}$$

$$\to \mathbb{E}h(\alpha(t_\ell),(ß,\phi_{\ell_1})_{t_\ell}:\ell\le\kappa,\ell_1\le\kappa_1))$$
$$\times\left[\int_t^{t+s}\int \chi(v)\overline{Q}(c)(f(\alpha_1),\ldots,f(\alpha_m))'ß_v(\mathrm{d}c)\,\mathrm{d}v\right], \tag{12}$$

where $\overline{Q}(\cdot)$ is given in (9). We note that $X^\varepsilon(\cdot)$ varies an order of magnitude faster that $\alpha^\varepsilon(\cdot)$, and $\alpha^\varepsilon(\cdot)$ is almost a constant in a small interval.

To complete the proof, we use an argument similar to [13, pp. 67–73] or Remark 2.2. By a change of variable argument, using the tightness of $\{\alpha^\varepsilon(\cdot)\}$, the scaling technique discussed in the preparation subsection, and Remark 2.2, there is $\Delta=\Delta_\varepsilon>0$ satisfying $\Delta_\varepsilon\to0$ as $\varepsilon\to0$ and $\varepsilon/\Delta\to0$, and that there is a sequence $t_\varepsilon\to t$ such that

$$\begin{aligned}&\frac{1}{\Delta_\varepsilon}\int_{t_\varepsilon}^{t_\varepsilon+\Delta_\varepsilon}\chi^\varepsilon(v)Q(X^\varepsilon(v),c)(f(\alpha_1),\ldots,f(\alpha_m))'ß_v^\varepsilon(\mathrm{d}c)\,\mathrm{d}v\\
&=\frac{\varepsilon}{\Delta_\varepsilon}\int_{t_\varepsilon/\varepsilon}^{(t_\varepsilon+\Delta_\varepsilon)/\varepsilon}\chi^\varepsilon(t)Q(Z^\varepsilon(v),c)(f(\alpha_1),\ldots,f(\alpha_m))'ß_v^\varepsilon(\mathrm{d}c)\,\mathrm{d}v+\delta^{\varepsilon,\Delta_\varepsilon}(t),\\
&=\frac{\varepsilon}{\Delta_\varepsilon}\int_{t_\varepsilon/\varepsilon}^{(t_\varepsilon+\Delta_\varepsilon)/\varepsilon}\chi^\varepsilon(t)Q(Z(v),c)(f(\alpha_1),\ldots,f(\alpha_m))'\widehat{P}_{t_\varepsilon}^{\varepsilon,\Delta}(\mathrm{d}z)ß_v^\varepsilon(\mathrm{d}c)+\delta^{\varepsilon,\Delta_\varepsilon}(t),\end{aligned}\tag{13}$$

where as $\varepsilon\to0$, $\delta^{\varepsilon,\Delta_\varepsilon}(t)\to0$ in probability uniformly in $t\le T$. [Note that $\chi^\varepsilon(\cdot)=(1_{\{\alpha^\varepsilon(t)=\alpha_1\}},\ldots,1_{\{\alpha^\varepsilon(t)=\alpha_m\}})$.] Using the definition of the sample occupation measure $\widehat{P}_t^{\varepsilon,\Delta}(\cdot)$ of Remark 2.2, $\{\widehat{P}_t^{\varepsilon,\Delta}(\cdot)\}$ is tight. Select a weakly convergent subsequence (with index (ε,Δ)), it can be argued that the limit sample occupation measure $\widehat{P}_t=\mu_{\alpha(t)}$ (some details can be found in [13, pp. 71–73]).

The weak convergence of $\alpha^\varepsilon(\cdot)$ to $\alpha(\cdot)$, the Skorohod representation, the boundedness and continuity of $Q(\cdot)$, and the ergodicity of the diffusion process $Z(\cdot)$ then yield

$$\begin{aligned}&\frac{\varepsilon}{\Delta_\varepsilon}\int_{t_\varepsilon/\varepsilon}^{(t_\varepsilon+\Delta_\varepsilon)/\varepsilon}\int\chi^\varepsilon(t)Q(Z(v),c))(f(\alpha_1),\ldots,f(\alpha_m))'ß_v^\varepsilon(\mathrm{d}c)\,\mathrm{d}v\\
&\quad\to\int\chi(t)\overline{Q}(c)(f(\alpha_1),\ldots,f(\alpha_m))'ß_v(\mathrm{d}c)\text{ in probability as }\varepsilon\to0.\end{aligned}\tag{14}$$

Thus (14) further leads to (12). Combining (11) and (12), we arrive at that as $\varepsilon\to0$,

$$\begin{aligned}&\mathbb{E}h(\alpha^\varepsilon(t_\ell),(ß^\varepsilon,\phi_{\ell_1})_{t_\ell}:\ell\le\kappa,\ell_1\le\kappa_1))\big[f(\alpha^\varepsilon(t+s))-f(\alpha^\varepsilon(t))\\
&\quad-\int_t^{t+s}\int\chi^\varepsilon(v)Q(X^\varepsilon(v),c)(f(\alpha_1),\ldots,f(\alpha_m))'ß_v(\mathrm{d}c)\,\mathrm{d}v\Big]\\
&\to\mathbb{E}h(\alpha(t_\ell),(ß,\phi_{\ell_1})_{t_\ell}:\ell\le\kappa,\ell_1\le\kappa_1))\big[f(\alpha(t+s))-f(\alpha(t))\\
&\quad-\int_t^{t+s}\int\chi(v)\overline{Q}(c)(f(\alpha_1),\ldots,f(\alpha_m))'ß_v(\mathrm{d}c)\mathrm{d}v\Big]\quad\text{as}\quad\varepsilon\to0.\end{aligned}$$

That is,

$$M_f(t)=f(\alpha(t))-f(\alpha(0))-\int_t^{t+s}\int\chi(v)\overline{Q}(c)(f(\alpha_1),\ldots,f(\alpha_m))'ß_v(\mathrm{d}c)\,\mathrm{d}v$$

is a martingale, so the desired limit is obtained. The proof is concluded. ■

Given that $X^\varepsilon(0)=x$, $\alpha^\varepsilon(0)=\alpha_\imath$, $ß^\varepsilon\in\mathcal{R}^\varepsilon$, and $ß\in\mathcal{R}$, using the relaxed control representation, similar to (5), we write the cost function for the limit as

$$J(\alpha_\imath,ß)=\mathbb{E}\int_0^T\int L(\alpha(t),c)ß_t(\mathrm{d}c)\,\mathrm{d}t. \tag{15}$$

Theorem 3.3: *Under the conditions of Theorem 3.2, $J^\varepsilon(\alpha_\iota, ß^\varepsilon) \to J(\alpha_\iota, ß)$ as $\varepsilon \to 0$.*

Remark 3.4: The assertion essentially follows that of the weak convergence of $(\alpha^\varepsilon(\cdot), ß^\varepsilon(\cdot))$ to $(\alpha(\cdot), ß(\cdot))$, the Skorohod representation, and the continuity of the cost rate $L(\cdot)$. We omit the detailed argument for simplicity.

4. Asymptotic optimality

Our objective in this section is to demonstrate the asymptotic optimality. The difficulty of the original optimization problem is due to the non-Markov properties. Here we show how we may resolve the problem by use of limit process. To proceed, denote

$$\begin{aligned} v^\varepsilon(\alpha_\iota) &= \inf_{ß^\varepsilon \in \mathcal{R}^\varepsilon} J^\varepsilon(\alpha_\iota, ß^\varepsilon), \quad \text{and} \\ v(\alpha_\iota) &= \inf_{ß \in \mathcal{R}} J(\alpha_\iota, ß). \end{aligned} \tag{16}$$

Thus they are the value functions of the original problem and the limit problem in the class of stochastic relaxed controls.

Theorem 4.1: *Under the conditions of Theorem 3.3, the following results hold:*

(a) $\lim_{\varepsilon\to 0} v^\varepsilon(\alpha_\iota) = v(\alpha_\iota)$.
(b) *For any $\eta > 0$, there is a finite set $\{a^1, \ldots, a^\eta_{k_\eta}\} = U^\eta \subset U$ and a $\delta > 0$ satisfying that there is a piecewise constant (in t) and locally Lipschitz continuous in α (uniformly in t) control $u^\eta(\cdot)$ such that $u^\eta(t) = u^\eta(\alpha(i\delta), i\delta)$ for $t \in [i\delta, i\delta + \delta]$ is an η-optimal control in that*

$$\lim_{\varepsilon\to 0} |J^\varepsilon(\alpha_\iota, u^\eta) - v^\varepsilon(\alpha_\iota)| = 0.$$

Remark 4.2: In what follows, we mainly concentrate on the proof of (b) above. The rationale is that we can use the chattering lemma to select a Lipschitz continuous feedback control $u^\eta(\alpha, t)$ that is η-optimal for the limit problem. Then we can use this control in the actual control systems. The difference of the cost function uses such a control and the value function for the original problem will be asymptotically diminishing. Such a result is practically useful. It enables us to use a control derived in the limit Markov decision processes in the original non-Markov decision processes. In this way, we obtain asymptotic optimality.

Proof: According to the chattering lemma [26, Theorem 4] (see also [13, 23, 24]) for more details), $u^\eta(\cdot)$ as in the theorem (b) exists such that

$$\inf_{ß \in \mathcal{R}} J(\alpha_\iota, ß) \le J(\alpha_\iota, u^\eta) \le \inf_{ß \in \mathcal{R}} J(\alpha_\iota, ß) + \eta. \tag{17}$$

Observe that

$$J^\varepsilon(\alpha_\iota, u^\eta(\cdot)) = \mathbb{E} \int_0^T L\left(\alpha^\varepsilon(t), u^\eta(t)\right)\, \mathrm{d}t,$$

the weak convergence implies that

$$\begin{aligned} &\alpha^\varepsilon(u^\eta, \cdot) \Rightarrow \alpha(u^\eta, \cdot) \quad \text{and} \\ &J^\varepsilon(\alpha_\iota, u^\eta) \to J(\alpha_\iota, u^\eta) \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{18}$$

Because u^η is an η-optimal control,

$$J(\alpha_\iota, u^\eta) \le v(\alpha_\iota) + \eta.$$

By (18), we also have

$$\begin{aligned} J^\varepsilon(\alpha_\iota, u^\eta) &= J(\alpha_\iota, u^\eta) + o_\varepsilon(1) \\ &\le v(\alpha_\iota) + \eta + o_\varepsilon(1), \end{aligned} \tag{19}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$.

Using the property of the value function, we can choose $ß^\varepsilon$, such that

$$v^\varepsilon(\alpha_\iota) \ge J^\varepsilon(\alpha_\iota, ß^\varepsilon) - \varepsilon.$$

Because $ß^\varepsilon$ is relatively compact, there exists a subsequence, still denoted by $ß^\varepsilon$ for simplicity such that $ß^\varepsilon$ converges to ß. Therefore, by combining Theorem 3.3 and the previous inequality, it follows that

$$\begin{aligned} v(\alpha_\iota) &\le J(\alpha_\iota, ß) \\ &= v^\varepsilon(\alpha_\iota) + o_{\varepsilon,1}(1), \end{aligned} \tag{20}$$

where $o_{\varepsilon,1}(1) \to 0$ as $\varepsilon \to 0$. Using (19), (20), and $v^\varepsilon(\alpha_\iota) \le J^\varepsilon(\alpha_\iota, u^\eta)$, we obtain

$$\begin{aligned} v^\varepsilon(\alpha_\iota) &\le J^\varepsilon(\alpha_\iota, u^\eta) \\ &\le v(\alpha_\iota) + \eta + o_\varepsilon(1) \\ &\le v^\varepsilon(\alpha_\iota) + \eta + o_\varepsilon(1) + o_{\varepsilon,1}(1). \end{aligned} \tag{21}$$

Therefore,

$$\limsup_{\varepsilon\to 0} |v^\varepsilon(\alpha_\iota) - v(\alpha_\iota)| \le \eta.$$

Because η is arbitrary, $\lim_{\varepsilon\to 0} v^\varepsilon(\alpha_\iota) = v(\alpha_\iota)$. Using (21),

$$\begin{aligned} 0 &\le J^\varepsilon(\alpha_\iota, u^\eta) - v^\varepsilon(\alpha_\iota) \\ &\le \eta + o_{\varepsilon,2}(1), \end{aligned} \tag{22}$$

where $o_{\varepsilon,2}(1) \to 0$ as $\varepsilon \to 0$. Thus $\limsup_{\varepsilon\to 0} |J^\varepsilon(\alpha_\iota, u^\eta) - v^\varepsilon(\alpha_\iota)| \le \eta$. The desired result follows. ■

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