

Superconvergent Interpolatory HDG methods for reaction diffusion equations II: HHO-inspired methods

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Received: date / Accepted: date

Abstract In [J. Sci. Comput., 81:2188–2212, 2019], we considered a superconvergent hybridizable discontinuous Galerkin (HDG) method, defined on simplicial meshes, for scalar reaction-diffusion equations and showed how to define an interpolatory version which maintained its convergence properties. The interpolatory approach uses a locally postprocessed approximate solution to evaluate the nonlinear term, and assembles all HDG matrices once before the time integration leading to a reduction in computational cost. The resulting method displays a superconvergent rate for the solution for polynomial degree $k \geq 1$. In this work, we take advantage of the link found between the HDG and the hybrid high-order (HHO) methods, in [ESAIM Math. Model. Numer. Anal., 50(3):635–650, 2016] and extend this idea to the new, HHO-inspired HDG methods, defined on meshes made of general polyhedral elements, uncovered therein. We prove that the resulting interpolatory HDG methods converge at the same rate as for the linear elliptic problems. Hence, we obtain superconvergent methods for $k \geq 0$ by some methods. We present numerical results to illustrate the convergence theory.

Keywords Hybrid high order methods · hybridizable discontinuous Galerkin methods · interpolatory method · superconvergence

1 Introduction

This is the third in a series of papers devoted to the devising of interpolatory HDG methods for the scalar reaction-diffusion model problem

$$\begin{aligned}\partial_t u - \Delta u + F(u) &= f && \text{in } \Omega \times (0, T], \\ u &= 0 && \text{on } \partial\Omega \times (0, T], \\ u(\cdot, 0) &= u_0 && \text{in } \Omega,\end{aligned}\tag{1}$$

where Ω is a Lipschitz polyhedral domain in $\mathbb{R}^d (d = 2, 3)$, with boundary $\Gamma = \partial\Omega$. The interpolatory approach has two main advantageous features. First, it avoids the use of a numerical quadrature typically

G. Chen was supported by supported by National Natural Science Foundation of China (NSFC) grant 11801063, and the Fundamental Research Funds for the Central Universities grant YJ202030. B. Cockburn was partially supported by National Science Foundation grant DMS-1912646. J. Singler and Y. Zhang were supported in part by National Science Foundation grant DMS-1217122.

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Interpolatory HDG	W_h	u_h^*	τ	flux \mathbf{q}	scalar u
(A)	\mathcal{P}^{k+1}	u_h	$1/h$	$k+1$	$k+2$ ($k \geq 0$)
(B)	\mathcal{P}^k	$\mathfrak{p}_h^{k+1}(u_h, \widehat{u}_h)$	$1/h$	$k+1$	$k+2$ ($k \geq 0$)
(C)	\mathcal{P}^{k-1}	$\mathfrak{p}_h^{k+1}(u_h, \widehat{u}_h)$	$1/h$	$k+1$	$k+2$ ($k \geq 2$)

Table 1 Convergence rates for the Interpolatory HDG (ABC) methods. The elementwise postprocessing $\mathfrak{p}_h^{k+1}(u_h, \widehat{u}_h)$ is taken from [14]. The last column gives the orders of convergence of u_h^* to u .

required for the assembly of the global matrix at each iteration in each time step, which is a computationally costly component of standard HDG methods for nonlinear equations. Second, the interpolated nonlinear term and its Jacobian are simple to formulate and evaluate, which yields a straightforward implementation of the method.

In the first paper of this series, [18], we applied this idea to an HDG method defined on simplicial meshes. It is called the HDG_k method since it uses polynomials of degree k to approximate all variables, that is, the flux $\mathbf{q} = -\nabla u$, the solution u , and its numerical trace on the faces of the elements. The interpolatory method was obtained by simply replacing the nonlinearity $F(u_h)$ by a suitably defined linear interpolate $\mathcal{I}_h F(u_h)$. Unfortunately, the resulting method lost the superconvergence property it had in the linear case. In the second paper, [8], we showed that, if instead of $\mathcal{I}_h F(u_h)$, we use $\mathcal{I}_h F(u_h^*)$, where u_h^* was a elementwise postprocessing of the approximate solution, we recovered the superconvergence previously lost. In this paper, we extend this idea to the new, HHO-inspired HDG methods uncovered in [14]. These methods are defined on meshes made of general polyhedral elements, use polynomials of degree k to approximate the flux variable \mathbf{q} and numerical trace, and use different polynomial degrees for the scalar variable u . We refer to them as the HDG (ABC) methods. To deal with non-simplicial elements, the stabilization function incorporates the postprocessed approximation u_h^* , which is the distinctive feature of the HHO methods, see [3, 14]. We prove that the interpolatory technique maintains the convergence rates of the HDG (ABC) methods while retaining all the advantages of the interpolatory approach.

We note that the HDG (A) method is also known as the Lehrenfeld-Schöberl HDG method or the HDG+ method. This HDG method has been investigated in a number of works; see, e.g., [3, 39, 40] and the recent papers [9, 24, 54] and the references therein.

We summarize the convergence rates of the Interpolatory HDG (ABC) methods in Table 1. We see that, in terms of the approximation for u , the Interpolatory HDG (A) method converges optimally for all $k \geq 0$, the Interpolatory HDG (B) superconverges for all $k \geq 0$, and the Interpolatory HDG (C) superconverges for $k \geq 2$. We must emphasize that the superconvergence of the HDG (B) methods for $k = 0$ is fundamentally different from the superconvergence of the HDG_k methods considered in [8], where $k \geq 1$ is required for superconvergence. This reflects the essential different nature of the HDG (ABC) and the HDG_k methods. It is worthwhile to mention that the convergence rate of HDG (C) method stated in [14] has an error when $k = 1$ since, in the linear case, its superconvergence property is only valid for $k \geq 2$. This property is similar to that of the BDM mixed methods [2] which use the same local spaces for \mathbf{q} and u .

Interpolatory finite element approaches for nonlinear partial differential equations have been investigated for many decades because of their computational advantages. There are many different names for these methods, including finite element methods with interpolated coefficients, product approximation, and the group finite element method. For more information, see [7, 10, 11, 22, 23, 25, 26, 30–32, 42, 47–53] and the references therein. Our interest in applying the interpolatory approach to the HDG methods is that, after its introduction [15] in the framework of linear steady-state diffusion problems, they have been extended in the last decade to a wide variety of partial differential equations including nonlinear equations like those of convection-diffusion [17], of the p -Laplacian [17], of the incompressible Navier-Stokes flow [5, 34, 37], of the compressible Navier-Stokes flow [41, 43], of fluid dynamics [38], of continuum solid mechanics [35], of scalar hyperbolic conservation laws [4, 28, 33], of large deformation elasticity [29, 36, 46]; see also the reviews [12, 13]. The popularity of these methods stems from the fact that they are discontinuous Galerkin (DG) methods amenable to static condensation. Therefore, the number of globally-coupled degrees of freedom for HDG methods is significantly lower than for standard DG methods. The applica-

tion of the interpolatory approach to HDG methods for nonlinear problems renders the resulting methods even more efficient to implement.

The paper is organized as follows. We discuss the Interpolatory HDG (ABC) formulations in Section 2. We then analyze the semidiscrete Interpolatory HDG (ABC) methods in Section 3. Finally, we illustrate the performance of the Interpolatory HDG (ABC) methods in Section 4 with numerical experiments. We end with some concluding remarks.

2 Main results

In this section, we introduce the notation, define the interpolatory HDG (ABC) methods, and state and briefly discuss their a priori error estimates.

2.1 Notation

To describe the Interpolatory HDG (ABC) methods, we first introduce the notation used in [15].

2.1.1 Meshes and inner products

Let \mathcal{T}_h be a collection of disjoint elements K that partition Ω , and K is star-shaped with respect to some ball; see [1, Definition 4.2.2]. Set $\partial\mathcal{T}_h$ to be $\{\partial K : K \in \mathcal{T}_h\}$. For an element K in \mathcal{T}_h , let $e = \partial K \cap \Gamma$ denote the boundary face of K if the $d-1$ Lebesgue measure of e is non-zero. For two elements K^+ and K^- of the collection \mathcal{T}_h , let $e = \partial K^+ \cap \partial K^-$ denote the interior face between K^+ and K^- if the $d-1$ Lebesgue measure of e is non-zero. We assume the edges are flat in 2D and are planar in 3D. Let \mathcal{E}_h^o and \mathcal{E}_h^∂ denote the sets of interior and boundary faces, respectively, and let \mathcal{E}_h denote the union of \mathcal{E}_h^o and \mathcal{E}_h^∂ .

For $D \subset \mathbb{R}^d$, let $(\cdot, \cdot)_D$ denote the $L^2(D)$ inner product and, when Γ is the union of subsets of \mathbb{R}^{d-1} , let $\langle \cdot, \cdot \rangle_\Gamma$ denote the $L^2(\Gamma)$ inner product. We finally set

$$(w, v)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (w, v)_K, \quad \langle \zeta, \rho \rangle_{\partial\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K}.$$

2.1.2 Spaces

Set

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v} \in [L^2(\Omega)]^d : \mathbf{v}|_K \in [\mathcal{P}^k(K)]^d, \forall K \in \mathcal{T}_h\}, \\ W_h &:= \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^\ell(K), \forall K \in \mathcal{T}_h\}, \\ Z_h &:= \{z \in L^2(\Omega) : z|_K \in \mathcal{P}^{k+1}(K), \forall K \in \mathcal{T}_h\}, \\ M_h &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_e \in \mathcal{P}^k(e), \forall e \in \mathcal{E}_h, \mu|_{\mathcal{E}_h^\partial} = 0\}, \end{aligned}$$

where $\mathcal{P}^k(D)$ denotes the set of polynomials of degree at most k on a domain D . In what follows, we take $\ell = k+1$, k , and $k-1$ to define the Interpolatory HDG (A), (B), and (C) methods, respectively. Note that the Interpolatory HDG (C) method is only defined for $k \geq 1$.

2.1.3 Interpolators and projections

As in [8], we denote by \mathcal{I}_h the element-wise Lagrange interpolation operator with respect to the finite element nodes for the space Z_h . Thus, $\mathcal{I}_h g \in Z_h$ for any function g that is continuous on each element.

We denote by Π_ℓ^o ($\ell \geq 0$) and Π_k^∂ ($k \geq 0$) the particular L^2 -orthogonal projections $\Pi_\ell^o : L^2(K) \rightarrow \mathcal{P}^\ell(K)$ and $\Pi_k^\partial : L^2(e) \rightarrow \mathcal{P}^k(e)$, respectively, that is,

$$(\Pi_\ell^o u, v_h)_K = (u, v_h)_K, \quad \forall v_h \in \mathcal{P}^\ell(K), \quad (2a)$$

$$\langle \Pi_k^\partial u, \widehat{v}_h \rangle_e = \langle u, \widehat{v}_h \rangle_e, \quad \forall \widehat{v}_h \in \mathcal{P}^k(e). \quad (2b)$$

The following error estimates for the L^2 -projections and the elementwise interpolation operator \mathcal{I}_h from Section 2 are standard and can be found in [1].

Lemma 1 Suppose $k, \ell \geq 0$. There exists a constant C independent of $K \in \mathcal{T}_h$ such that

$$\begin{aligned} \|w - \mathcal{I}_h w\|_K &\leq Ch^{k+2}|w|_{k+2,K} & \forall w \in C(\bar{K}) \cap H^{k+2}(K), \\ \|w - \Pi_\ell^\circ w\|_K &\leq Ch^{\ell+1}|w|_{\ell+1,K} & \forall w \in H^{\ell+1}(K), \\ \|w - \Pi_k^\partial w\|_{\partial K} &\leq Ch^{k+1/2}|w|_{k+1,K} & \forall w \in H^{k+1}(K). \end{aligned}$$

We now define an auxiliary projection related to the postprocessings originally developed in [16, 27, 44, 45] but more intimately linked with that of the HHO methods [19–21], see also [14]. On an element $K \in \mathcal{T}_h$, we define the auxiliary projection Π_{k+1}^* as

$$\Pi_{k+1}^* u = \mathbf{p}_h^{k+1}(\Pi_\ell^\circ u, \Pi_k^\partial u), \quad (3)$$

where $\mathbf{p}_h^{k+1}(u_h, \hat{u}_h)$ is the element of $\mathcal{P}^{k+1}(K)$ satisfying

$$(\nabla \mathbf{p}_h^{k+1}(u_h, \hat{u}_h), \nabla z_h)_K = -(u_h, \Delta z_h)_K + \langle \hat{u}_h, \mathbf{n} \cdot \nabla z_h \rangle_{\partial K} \quad \forall z_h \in [\mathcal{P}_\ell^{k+1}(K)]^\perp, \quad (4a)$$

$$(\mathbf{p}_h^{k+1}(u_h, \hat{u}_h), w_h)_K = (u_h, w_h)_K \quad \forall w_h \in \mathcal{P}^\ell(K), \quad (4b)$$

where $[\mathcal{P}_\ell^{k+1}(K)]^\perp := \{v_h \in \mathcal{P}^{k+1}(K) : (v_h, w_h)_K = 0, \forall w_h \in \mathcal{P}^\ell(K)\}$.

2.2 The Interpolatory HDG (ABC) methods

We can now define the Interpolatory HDG (ABC) methods as follows: for all $(\mathbf{r}_h, v_h, \hat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$, find $(\mathbf{q}_h, u_h, \hat{u}_h) \in \mathbf{V}_h \times W_h \times M_h$ such that

$$(\mathbf{q}_h, \mathbf{r}_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \hat{u}_h, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0, \quad (5a)$$

$$(\partial_t u_h, v_h)_{\mathcal{T}_h} - (\mathbf{q}_h, \nabla v_h)_{\mathcal{T}_h} + \langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, v_h \rangle_{\partial \mathcal{T}_h} + (\mathcal{I}_h F(u_h^*), v_h)_{\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}, \quad (5b)$$

$$\langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, \hat{v}_h \rangle_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^\partial} = 0, \quad (5c)$$

$$u_h(0) = \bar{u}_h(0), \quad (5d)$$

where $u_h^* := \mathbf{p}_h^{k+1}(u_h, \hat{u}_h)$. To complete the definition of the methods, we need to define the numerical trace for the flux, $\hat{\mathbf{q}}_h$, and the initial condition $\bar{u}_h(0)$. For any element $K \in \mathcal{T}_h$, we define $\hat{\mathbf{q}}_h \cdot \mathbf{n}$ on ∂K by

$$\hat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + r_{\partial K}^{k*}[h_K^{-1} r_{\partial K}^k(u_h - \hat{u}_h)], \quad (6a)$$

where $r_{\partial K}^{k*}$ is the adjoint of $r_{\partial K}^k$, and $r_{\partial K}^k$ is defined, see [14], by

$$r_{\partial K}^k(u_h - \hat{u}_h) = \Pi_k^\partial u_h^* - \hat{u}_h. \quad (6b)$$

Finally, we define the initial condition $\bar{u}_h(0)$ as one of the components of the HDG (ABC) elliptic approximation. For any $t \in [0, T]$, we define the HDG (ABC) elliptic approximation of $(-\nabla u(t)|_{\mathcal{T}_h}, u(t)|_{\mathcal{T}_h}, u(t)|_{\mathcal{E}_h})$ to be the unique element $(\bar{\mathbf{q}}_h, \bar{u}_h, \bar{\hat{u}}_h)$ of $\mathbf{V}_h \times W_h \times M_h$ which solves

$$(\bar{\mathbf{q}}_h, \mathbf{r}_h)_{\mathcal{T}_h} - (\bar{u}_h, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \bar{\hat{u}}_h, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} = 0, \quad (7a)$$

$$(\nabla \cdot \bar{\mathbf{q}}_h, v_h)_{\mathcal{T}_h} - \langle \bar{\mathbf{q}}_h \cdot \mathbf{n}, \hat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle h_K^{-1}(\Pi_k^\partial \bar{u}_h^* - \bar{\hat{u}}_h), \Pi_k^\partial v_h^* - \hat{v}_h \rangle_{\partial \mathcal{T}_h} = (-\Delta u(t), v_h)_{\mathcal{T}_h} \quad (7b)$$

for all $(\mathbf{r}_h, v_h, \hat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$, where $\bar{u}_h^* = \mathbf{p}_h^{k+1}(\bar{u}_h, \bar{\hat{u}}_h)$ and $v_h^* = \mathbf{p}_h^{k+1}(v_h, \hat{v}_h)$.

2.3 Main result

We assume that the nonlinearity F satisfies a Lipschitz condition:

$$|F(u) - F(v)| \leq L |u - v| \quad \forall u, v \in D. \quad (8)$$

As done in [8], we assume that, when F is globally Lipschitz in a suitably chosen domain D , the solutions of the model problem (1), and those of the semidiscrete Interpolatory HDG (ABC) equations (5), exist and are unique for $t \in [0, T]$.

We also assume the elliptic regularity inequality

$$\|\Phi\|_1 + \|\Psi\|_2 \leq C \|g\|_0, \quad (9a)$$

where (Φ, Ψ) solves the dual problem:

$$\Phi + \nabla \Psi = 0, \quad \nabla \cdot \Phi = g \quad \text{in } \Omega, \quad \Psi = 0 \quad \text{on } \partial\Omega. \quad (9b)$$

We can now state our main result for the Interpolatory HDG (ABC) methods.

Theorem 1 *Assume that the nonlinearity F is globally Lipschitz, that is, it satisfies condition (8) with $D := \mathbb{R}$. Assume that $u \in C^1[0, T; H^{k+2}(\Omega)] \cap C^0[0, T; W^{1,\infty}(\Omega)]$. Finally, assume that the elliptic regularity inequality (9a) holds. Then, for all $0 \leq t \leq T$, the solution $(\mathbf{q}_h, u_h, u_h^*)$ of (5) satisfies*

$$\begin{aligned} \|\mathbf{q}(t) - \mathbf{q}_h(t)\|_{\mathcal{T}_h} &\leq C h^{k+1}, \\ \|u(t) - u_h(t)\|_{\mathcal{T}_h} &\leq C h^{\ell+1}, \\ \|u(t) - u_h^*(t)\|_{\mathcal{T}_h} &\leq C \begin{cases} h^2 & \text{if } (k, l) = (1, 0), \\ h^{k+2} & \text{otherwise.} \end{cases} \end{aligned}$$

The constant C is independent of h , but depends on T and on norms of u and u_t . Moreover, if the nonlinearity F satisfies the Lipschitz condition (8) with $D := [-M, M]$, where

$$M = \max\{|u(t, x)| : x \in \overline{\Omega}, t \in [0, T]\} + \delta, \quad \text{for a fixed } \delta > 0,$$

and the mesh is quasi-uniform and $h \leq \frac{1}{C} \delta^{\frac{1}{k+2-d/2}}$ is small enough, then the same convergence rates hold.

This result states that, provided the solution is smooth enough, we recover the optimal orders of convergence. For HDG (A) with $k \geq 0$, the optimal order of convergence of $k+2$ holds for u_h^* , as it coincides with u_h . Superconvergence of order $k+2$ for u_h^* holds for HDG (B) with $k \geq 0$, and for HDG (C) with $k \geq 2$. When $k=1$, the order of convergence of u_h^* for HDG (C) is only $k+1=2$.

The result can be extended to other initial conditions, as confirmed by our numerical experiments. The one we chose makes the proof simpler.

3 Proof of the error estimates

This section is devoted to proving our main result, the a priori error estimates of Theorem 1. To do that, we essentially follow the approach carried out in [8]. However, we need to use different auxiliary projections to capture the special structure of the stabilization functions of the HDG (ABC) methods.

3.1 Reformulating the HDG (ABC) methods

We begin by rewriting the definition of the Interpolatory HDG (ABC) methods to render it more suitable to our error analysis. Unlike the approach used in [8], here we eliminate the numerical trace of the flux from the equations.

Proposition 1 (Reformulation of the methods) For all $(\mathbf{r}_h, v_h, \widehat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$, the Interpolatory HDG (ABC) formulations can be rewritten as follows: find $(\mathbf{q}_h, u_h, \widehat{u}_h) \in \mathbf{V}_h \times W_h \times M_h$ satisfying

$$\begin{aligned} (\mathbf{q}_h, \mathbf{r}_h)_{\mathcal{T}_h} - (u_h, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \widehat{u}_h, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\ (\partial_t u_h, v_h)_{\mathcal{T}_h} + (\mathcal{I}_h F(u_h^*), v_h)_{\mathcal{T}_h} + (\nabla \cdot \mathbf{q}_h, v_h)_{\mathcal{T}_h} - \langle \mathbf{q}_h \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\ + \langle h_K^{-1} (\Pi_K^\partial u_h^* - \widehat{u}_h), \Pi_K^\partial v_h^* - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, \\ u_h(0) &= \bar{u}_h(0), \end{aligned}$$

where $u_h^* = \mathbf{p}_h^{k+1}(u_h, \widehat{u}_h)$ and $v_h^* = \mathbf{p}_h^{k+1}(v_h, \widehat{v}_h)$.

Proof Inserting the definition of the numerical trace of the flux (6a) into the first two equations defining the HDG (ABC) method (5), we obtain

$$\begin{aligned} (\partial_t u_h, v_h)_{\mathcal{T}_h} - (\mathbf{q}_h, \nabla v_h)_{\mathcal{T}_h} + (\mathcal{I}_h F(u_h^*), v_h)_{\mathcal{T}_h} \\ + \langle \mathbf{q}_h \cdot \mathbf{n} + r_{\partial K}^{k*} [h_K^{-1} (\Pi_K^\partial u_h^* - \widehat{u}_h)], v_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, \\ \langle \mathbf{q}_h \cdot \mathbf{n} + r_{\partial K}^{k*} [h_K^{-1} (\Pi_K^\partial u_h^* - \widehat{u}_h)], \widehat{v}_h \rangle_{\partial \mathcal{T}_h} &= 0. \end{aligned}$$

Subtracting the second equation from the first, and integrating by parts, we get

$$\begin{aligned} (\partial_t u_h, v_h)_{\mathcal{T}_h} + (\mathcal{I}_h F(u_h^*), v_h)_{\mathcal{T}_h} + (\nabla \cdot \mathbf{q}_h, v_h)_{\mathcal{T}_h} - \langle \mathbf{q}_h \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\ + \langle r_{\partial K}^{k*} [h_K^{-1} (\Pi_K^\partial u_h^* - \widehat{u}_h)], v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}. \end{aligned}$$

Since $r_{\partial K}^{k*}$ is the adjoint of $r_{\partial K}^k$, the result follows after using the definition of $r_{\partial K}^k$ in (6b), $r_{\partial K}^k(v_h - \widehat{v}_h) = \Pi_K^\partial \mathbf{p}_h^{k+1}(v_h, \widehat{v}_h) - \widehat{v}_h$, and after recalling that $v_h^* = \mathbf{p}_h^{k+1}(v_h, \widehat{v}_h)$.

3.2 Main error estimate

Our analysis is based on estimating the following quantities:

$$e_h^{\mathbf{q}} = \mathbf{q}_h - \bar{\mathbf{q}}_h, \quad e_h^u = u_h - \bar{u}_h, \quad e_h^{\widehat{u}} = \widehat{u}_h - \widehat{\bar{u}}_h, \quad e_h^{u^*} = u_h^* - \bar{u}_h^*.$$

Here, we obtain the main estimates for these functions.

We begin by obtaining the error equations.

Lemma 2 (Error equations) We have

$$\begin{aligned} (e_h^{\mathbf{q}}, \mathbf{r}_h)_{\mathcal{T}_h} - (e_h^u, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle e_h^{\widehat{u}}, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\ (\partial_t e_h^u, v_h)_{\mathcal{T}_h} + (\nabla \cdot e_h^{\mathbf{q}}, v_h)_{\mathcal{T}_h} - \langle e_h^{\mathbf{q}} \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} \\ + \langle h_K^{-1} (\Pi_K^\partial e_h^{u^*} - e_h^{\widehat{u}}), \Pi_K^\partial v_h^* - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + (\mathcal{I}_h F(u_h^*) - F(u), v_h)_{\mathcal{T}_h} &= (\partial_t (\Pi_\ell^o u - \bar{u}_h), v_h)_{\mathcal{T}_h}. \end{aligned}$$

This result can be easily proven by subtracting the equations (7) from those in Proposition 1, and noting that $e_h^{u^*} = \mathbf{p}_h^{k+1}(e_h^u, e_h^{\widehat{u}})$.

Lemma 3 (Error estimates at $t = 0$) We have $e_h^u(0) = 0$ and

$$\|e_h^{\mathbf{q}}(0)\|_{\mathcal{T}_h}^2 + \|h^{-1/2} (\Pi_K^\partial e_h^{u^*}(0) - e_h^{\widehat{u}}(0))\|_{\partial \mathcal{T}_h}^2 = 0.$$

Proof Take $(\mathbf{r}_h, v_h, \widehat{v}_h) := (e_h^{\mathbf{q}}(0), e_h^u(0), e_h^{\widehat{u}}(0))$ in the error equations of Lemma 2, evaluate at $t = 0$ and add the resulting equations. Since $e_h^u(0) = u_h(0) - \bar{u}_h(0) = 0$, we get the result.

Next, we display the main error estimates.

Lemma 4 (Main error estimates) For $t \in [0, T]$, we have

$$\begin{aligned} \|e_h^u(t)\|_{\mathcal{T}_h}^2 + \int_0^t (\|e_h^{\mathbf{q}}\|_{\mathcal{T}_h}^2 + \|h^{-1/2} (\Pi_K^\partial e_h^{u^*} - e_h^{\widehat{u}})\|_{\partial \mathcal{T}_h}^2) &\leq C t \Theta(t), \\ \|e_h^{\mathbf{q}}\|_{\mathcal{T}_h}^2 + \|h^{-1/2} \Pi_K^\partial e_h^{u^*} - e_h^{\widehat{u}}\|_{\partial \mathcal{T}_h}^2 + \int_0^t \|\partial_t e_h^u\|_{\mathcal{T}_h}^2 &\leq C \Theta(t), \end{aligned}$$

where $\Theta(t) := \int_0^t \|\partial_t (\Pi_\ell^o u - \bar{u}_h)\|_{\mathcal{T}_h}^2 + \int_0^t \|F(u) - \mathcal{I}_h F(u_h^*)\|_{\mathcal{T}_h}^2$.

Proof We first take $(\mathbf{r}_h, v_h, \hat{r}_h) := (e_h^{\mathbf{q}}, e_h^u, e_h^{\hat{u}})$ in the error equations of Lemma 2, and add the resulting equations to get

$$(\partial_t e_h^u, e_h^u)_{\mathcal{T}_h} + \|e_h^{\mathbf{q}}\|_{\mathcal{T}_h}^2 + \|h^{-1/2}(\Pi_k^\partial e_h^{u^*} - e_h^{\hat{u}})\|_{\partial\mathcal{T}_h}^2 = (\partial_t(\Pi_\ell^o u - \bar{u}_h), e_h^u)_{\mathcal{T}_h} + (F(u) - \mathcal{I}_h F(u_h^*), e_h^u)_{\mathcal{T}_h}.$$

We now apply the Cauchy-Schwarz inequality to both terms of the right-hand side, and take

$$\begin{aligned} \zeta(t) &= \|e_h^u(t)\|_{\mathcal{T}_h}^2, Z(s) = 2(\|e_h^{\mathbf{q}}\|_{\mathcal{T}_h}^2 + \|h^{-1/2}(\Pi_k^\partial e_h^{u^*} - e_h^{\hat{u}})\|_{\partial\mathcal{T}_h}^2), \\ A(t) &= 0, B(s) = 2(\|\partial_t(\Pi_\ell^o u - \bar{u}_h)\|_{\mathcal{T}_h} + \|F(u) - \mathcal{I}_h F(u_h^*)\|_{\mathcal{T}_h}) \end{aligned}$$

in a Gronwall-like inequality [6, Proposition 3.1] and the fact that $e_h^u(0) = 0$ to obtain

$$\begin{aligned} \|e_h^u(t)\|_{\mathcal{T}_h}^2 + 2 \int_0^t (\|e_h^{\mathbf{q}}\|_{\mathcal{T}_h}^2 + \|h^{-1/2}(\Pi_k^\partial e_h^{u^*} - e_h^{\hat{u}})\|_{\partial\mathcal{T}_h}^2) \leq & \left(\int_0^t \|\partial_t(\Pi_\ell^o u - \bar{u}_h)\|_{\mathcal{T}_h} \right. \\ & \left. + \int_0^t \|F(u) - \mathcal{I}_h F(u_h^*)\|_{\mathcal{T}_h} \right)^2. \end{aligned}$$

The first inequality is obtained after simple manipulations.

Next, we take the partial derivative with respect to t in the first error equation of Lemma 2 and keep the second equation unchanged. We obtain

$$\begin{aligned} & (\partial_t e_h^{\mathbf{q}}, \mathbf{r}_h)_{\mathcal{T}_h} - (\partial_t e_h^u, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \partial_t e_h^{\hat{u}}, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} = 0, \\ & (\partial_t e_h^u, v_h)_{\mathcal{T}_h} + (\nabla \cdot e_h^{\mathbf{q}}, v_h)_{\mathcal{T}_h} - \langle e_h^{\mathbf{q}} \cdot \mathbf{n}, \hat{v}_h \rangle_{\partial\mathcal{T}_h} \\ & + \langle h_K^{-1}(\Pi_k^\partial e_h^{u^*} - e_h^{\hat{u}}), \Pi_k^\partial v_h^* - \hat{v}_h \rangle_{\partial\mathcal{T}_h} + (\mathcal{I}_h F(u_h^*) - F(u), v_h)_{\mathcal{T}_h} = (\partial_t(\Pi_\ell^o u - \bar{u}_h), v_h)_{\mathcal{T}_h}. \end{aligned}$$

Taking $(\mathbf{r}_h, v_h, \hat{r}_h) := (e_h^{\mathbf{q}}, \partial_t e_h^u, \partial_t e_h^{\hat{u}})$ in these equations and adding them, we get

$$\begin{aligned} & (\partial_t e_h^{\mathbf{q}}, e_h^{\mathbf{q}})_{\mathcal{T}_h} + \langle h_K^{-1}(\Pi_k^\partial e_h^{u^*} - e_h^{\hat{u}}), \partial_t \Pi_k^\partial e_h^{u^*} - \partial_t e_h^{\hat{u}} \rangle_{\partial\mathcal{T}_h} + \|\partial_t e_h^u\|_{\mathcal{T}_h}^2 \\ & = (\partial_t(\Pi_\ell^o u - \bar{u}_h), \partial_t e_h^u)_{\mathcal{T}_h} + (F(u) - \mathcal{I}_h F(u_h^*), \partial_t e_h^u)_{\mathcal{T}_h}. \end{aligned}$$

We now apply the Cauchy-Schwarz inequality to each of the two terms of the right-hand side, use Young's inequality and the estimates of the errors at $t = 0$ of Lemma 3 to get the second estimate.

3.3 The Lipschitz conditions on the nonlinearity

Here, we end our error analysis. We bound the term $\|F(u) - \mathcal{I}_h F(u_h^*)\|_{\mathcal{T}_h}$ under different assumptions on the nonlinearity $F(u)$ and conclude. To do that, we need the following auxiliary result. Its proof is given in Appendix A.

Lemma 5 *We have*

$$\|\Pi_{k+1}^* u - u\|_{0,\infty,K} \leq Ch \|\nabla u\|_{0,\infty,K}, \quad (10a)$$

$$\|\Pi_{k+1}^* u - u_h^*\|_{\mathcal{T}_h} \leq C(\|u_h - \Pi_\ell^o u\|_{\mathcal{T}_h} + h\|\mathbf{q}_h - \Pi_k^o \mathbf{q}\|_{\mathcal{T}_h} + h\|\mathbf{q} - \Pi_k^o \mathbf{q}\|_{\mathcal{T}_h}). \quad (10b)$$

Theorem 2 *For any $t \in [0, T]$, we have the following error estimates*

$$\begin{aligned} \|\Pi_k^o \mathbf{q} - \bar{\mathbf{q}}_h\|_{\mathcal{T}_h} & \leq Ch \|\Pi_\ell^o(-\Delta u) + \Delta u\|_{\mathcal{T}_h} \\ & \quad + C(h^{1/2} \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\partial\mathcal{T}_h} + \|h^{-1/2}(\Pi_{k+1}^* u - u)\|_{\partial\mathcal{T}_h}), \\ \|\Pi_\ell^o u - \bar{u}_h\|_{\mathcal{T}_h} & \leq Ch^{1+\min\{1,\ell\}} \|\Pi_\ell^o(-\Delta u) + \Delta u\|_{\mathcal{T}_h} \\ & \quad + C(h^{1/2} \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\partial\mathcal{T}_h} + \|h^{-1/2}(\Pi_{k+1}^* u - u)\|_{\partial\mathcal{T}_h}), \\ \|\partial_t \Pi_\ell^o u - \partial_t \bar{u}_h\|_{\mathcal{T}_h} & \leq Ch^{1+\min\{1,\ell\}} \|\Pi_\ell^o(-\Delta u_t) + \Delta u_t\|_{\mathcal{T}_h} \\ & \quad + C(h^{1/2} \|\Pi_k^o \mathbf{q}_t - \mathbf{q}_t\|_{\partial\mathcal{T}_h} + \|h^{-1/2}(\Pi_{k+1}^* u_t - u_t)\|_{\partial\mathcal{T}_h}). \end{aligned}$$

3.3.1 Error estimates for a global Lipschitz condition

Here, we assume the nonlinearity is globally Lipschitz.

Lemma 6 *We have*

$$\|F(u) - \mathcal{I}_h F(u_h^*)\|_{\mathcal{T}_h} \leq \|F(u) - \mathcal{I}_h F(u)\|_{\mathcal{T}_h} + C(\|u - \mathcal{I}_h u\|_{\mathcal{T}_h} + \|u - \Pi_{k+1}^* u\|_{\mathcal{T}_h} + \|\Pi_{k+1}^* u - u_h^*\|_{\mathcal{T}_h}).$$

Proof To bound the error in the nonlinear term, we write $F(u) - \mathcal{I}_h F(u_h^*) = R_1 + R_2 + R_3$, where

$$R_1 := F(u) - \mathcal{I}_h F(u), \quad R_2 := \mathcal{I}_h F(u) - \mathcal{I}_h F(\Pi_{k+1}^* u), \quad R_3 := \mathcal{I}_h F(\Pi_{k+1}^* u) - \mathcal{I}_h F(u_h^*).$$

The result follows since

$$\|R_2\|_{\mathcal{T}_h} \leq C(\|u - \mathcal{I}_h u\|_{\mathcal{T}_h} + \|u - \Pi_{k+1}^* u\|_{\mathcal{T}_h}) \quad \text{and} \quad \|R_3\|_{\mathcal{T}_h} \leq C\|\Pi_{k+1}^* u - u_h^*\|_{\mathcal{T}_h}.$$

as shown in [18, Lemma 6]. This completes the proof.

Remark 1 Although the proof of [18, Lemma 6] only holds for simplex mesh, it can be extend to general polygonal mesh by using a finite number of reference elements. The proof is trivial and we skip it.

Lemma 7 *For $t \in [0, T]$, we have that*

$$\Theta(t) \leq \Theta_{HDG}(t) + \Theta_{APP}(t) + C \int_0^t (\|e_h^u\|_{\mathcal{T}_h}^2 + h^2 \|e_h^q\|_{\mathcal{T}_h}^2),$$

where

$$\begin{aligned} \Theta_{HDG}(t) &:= \int_0^t \|\partial_t(\Pi_\ell^o u - \bar{u}_h)\|_{\mathcal{T}_h}^2 + C \int_0^t (\|\bar{u}_h - \Pi_\ell^o u\|_{\mathcal{T}_h}^2 + h^2 \|\bar{q}_h - \Pi_k^o q\|_{\mathcal{T}_h}^2), \\ \Theta_{APP}(t) &:= C \int_0^t (\|F(u) - \mathcal{I}_h F(u)\|_{\mathcal{T}_h}^2 + \|u - \mathcal{I}_h u\|_{\mathcal{T}_h}^2 + \|u - \Pi_{k+1}^* u\|_{\mathcal{T}_h}^2 + h^2 \|q - \Pi_k^o q\|_{\mathcal{T}_h}^2). \end{aligned}$$

We note that Θ_{HDG} involves the HDG elliptic approximation, while Θ_{APP} involves only approximations of the exact solution of the PDE and related quantities.

Proof We have, by Lemma 6,

$$\begin{aligned} \Theta_h &:= \|F(u) - \mathcal{I}_h F(u_h^*)\|_{\mathcal{T}_h} \\ &\leq \|F(u) - \mathcal{I}_h F(u)\|_{\mathcal{T}_h} + C(\|u - \mathcal{I}_h u\|_{\mathcal{T}_h} + \|u - \Pi_{k+1}^* u\|_{\mathcal{T}_h} + \|\Pi_{k+1}^* u - u_h^*\|_{\mathcal{T}_h}) \\ &\leq \|F(u) - \mathcal{I}_h F(u)\|_{\mathcal{T}_h} + C(\|u - \mathcal{I}_h u\|_{\mathcal{T}_h} + \|u - \Pi_{k+1}^* u\|_{\mathcal{T}_h}) \\ &\quad + C(\|u_h - \Pi_\ell^o u\|_{\mathcal{T}_h} + h\|q_h - \Pi_k^o q\|_{\mathcal{T}_h} + h\|q - \Pi_k^o q\|_{\mathcal{T}_h}), \end{aligned}$$

by (10b). Using the definition of e_h^u and e_h^q , and the triangle inequality, we get

$$\begin{aligned} \|F(u) - \mathcal{I}_h F(u_h^*)\|_{\mathcal{T}_h} &\leq \|F(u) - \mathcal{I}_h F(u)\|_{\mathcal{T}_h} + C(\|u - \mathcal{I}_h u\|_{\mathcal{T}_h} + \|u - \Pi_{k+1}^* u\|_{\mathcal{T}_h}) \\ &\quad + C(\|\bar{u}_h - \Pi_\ell^o u\|_{\mathcal{T}_h} + h\|\bar{q}_h - \Pi_k^o q\|_{\mathcal{T}_h} + h\|q - \Pi_k^o q\|_{\mathcal{T}_h}) \\ &\quad + C(\|e_h^u\|_{\mathcal{T}_h} + h\|e_h^q\|_{\mathcal{T}_h}). \end{aligned}$$

Inserting this bound in the definition of $\Theta(t)$, we obtain the desired result. This completes the proof.

Lemma 8 *For $t \in [0, T]$, we have*

$$\begin{aligned} \|e_h^u(t)\|_{\mathcal{T}_h}^2 + \int_0^t (\|e_h^q\|_{\mathcal{T}_h}^2 + \|h^{-1/2}(\Pi_k^\partial e_h^{u^*} - e_h^{\hat{u}})\|_{\partial\mathcal{T}_h}^2) &\leq 2t\Phi(T), \\ \|e_h^q\|_{\mathcal{T}_h}^2 + \|h^{-1/2}\Pi_k^\partial e_h^{u^*} - e_h^{\hat{u}}\|_{\partial\mathcal{T}_h}^2 + \int_0^t \|\partial_t e_h^u\|_{\mathcal{T}_h}^2 &\leq 2\Phi(T), \end{aligned}$$

where $\Phi(T) := \Theta_{HDG}(T) + \Theta_{APP}(T)$.

Proof By the previous lemma, we have, for all $t \in [0, T]$,

$$\begin{aligned} \Theta(t) &\leq \Theta_{HDG}(T) + \Theta_{APP}(T) + C \int_0^t (\|e_h^u\|_{\mathcal{T}_h}^2 + h^2 \|e_h^q\|_{\mathcal{T}_h}^2) \\ &\leq \Theta_{HDG}(T) + \Theta_{APP}(T) + C \int_0^t (s + h^2) \Theta(s) ds, \end{aligned}$$

by Lemma 4. By applying the Gronwall inequality, we get that $\Theta(t) \leq C(T)\Phi(T)$. The result now follows by using the main estimates of Lemma 4.

3.3.2 Error estimates for a local Lipschitz condition

In this section we assume that the nonlinearity F is only locally Lipschitz, as is the case in many applications. To deal with this case, we assume that the mesh \mathcal{T}_h is quasi-uniform.

Lemma 9 *Assume the mesh \mathcal{T}_h is quasi-uniform. Then for h small enough and $t \in (0, T]$, the error estimates of Lemma 8 hold.*

Proof By (10a), there is an h_0 such that for all $h \in (0, h_0]$ and for all $t \in [0, T]$, there holds

$$\|u - \Pi_{k+1}^* u\|_{0,\infty,\mathcal{T}_h} \leq \frac{\delta}{2}.$$

Therefore, $\Pi_{k+1}^* u \in [-(M - \delta/2), (M - \delta/2)]$, and this implies that

$$\|F(u) - F(\Pi_{k+1}^* u)\|_{\mathcal{T}_h} \leq L\|u - \Pi_{k+1}^* u\|_{\mathcal{T}_h}.$$

By an inverse inequality and the assumption of quasiuniformity of the mesh, we get

$$\begin{aligned} \|\Pi_{k+1}^* u(0) - u_h^*(0)\|_{0,\infty,\mathcal{T}_h} &\leq h^{-d/2} \|\Pi_{k+1}^* u(0) - u_h^*(0)\|_{\mathcal{T}_h} \\ &\leq C(\|u_h(0) - \Pi_\ell^o u(0)\|_{\mathcal{T}_h} + h\|\mathbf{q}_h(0) - \Pi_k^o \mathbf{q}(0)\|_{\mathcal{T}_h} + h\|\mathbf{q}(0) - \Pi_k^o \mathbf{q}(0)\|_{\mathcal{T}_h}) \\ &\leq C(\|\bar{u}_h(0) - \Pi_\ell^o u(0)\|_{\mathcal{T}_h} + h\|\mathbf{q}(0) - \Pi_k^o \mathbf{q}(0)\|_{\mathcal{T}_h}), \end{aligned}$$

where we used (10b) (5d) and Lemma 3. Then by Theorem 2 and Lemma 1 we have

$$\|\Pi_{k+1}^* u(0) - u_h^*(0)\|_{0,\infty,\mathcal{T}_h} \leq Ch^{-d/2}(h^{\ell+2+\min\{\ell,1\}} + h^{k+2}).$$

By the restrictions on d , the upper bound of this error at time zero can be made strictly smaller than $\delta/2$ by taking h sufficiently small, say, for all $h \in (0, h_0^*]$, where $h_0^* \leq h_0$.

Then, for each $h \in (0, h_0^*]$ let $t_h \in (0, T]$ be the largest value such that for all $t \in [0, t_h]$ there holds

$$\|\Pi_{k+1}^* u - u_h^*\|_{0,\infty,\mathcal{T}_h} \leq \frac{\delta}{2}. \quad (11)$$

Therefore, $u_h^* \in [-M, M]$, and again we have

$$\|F(\Pi_{k+1}^* u) - F(u_h^*)\|_{\mathcal{T}_h} \leq L\|\Pi_{k+1}^* u - u_h^*\|_{\mathcal{T}_h}.$$

Now the error estimate of Lemma 9 can be proved in exactly the same way as in Lemma 8. However, the estimate now holds only for all $h \in (0, h_0^*]$ and for all $t \in [0, t_h]$.

By (10b) and the error estimate, for all $t \in [0, t_h]$ we have

$$\|\Pi_{k+1}^* u(t) - u_h(t)\|_{\mathcal{T}_h} = \|e_h^{u^*}(t)\|_{\mathcal{T}_h} \leq Ch^{\ell+2+\min\{1,\ell\}} + Ch^{k+2}.$$

By an inverse inequality, for all $t \in [0, t_h]$ we have

$$\|\Pi_{k+1}^* u(t) - u_h(t)\|_{0,\infty,\mathcal{T}_h} \leq C(h^{\ell+2+\min\{1,\ell\}} + Ch^{k+2})h^{-\frac{d}{2}}.$$

As before, there exists $h_1 \in (0, h_0^*]$ such that for all $h \in (0, h_1]$ and for all $t \in [0, t_h]$ there holds

$$\|\Pi_{k+1}^* u(t) - u_h(t)\|_{0,\infty,\mathcal{T}_h} < \frac{\delta}{2}.$$

Since for each $h \in (0, h_1]$ we have that $t_h \in (0, T]$ is the largest value such that (11) holds for all $t \in [0, t_h]$, therefore $t_h = T$ for all h small enough. This completes the proof.

3.4 Conclusion

We are now ready to conclude the proof of our main result. Indeed, if the nonlinearity is globally Lipschitz, since $\mathbf{q} - \mathbf{q}_h = \mathbf{q} - \Pi_k^o \mathbf{q} + \Pi_k^o \mathbf{q} - \bar{\mathbf{q}}_h + \bar{\mathbf{q}}_h - \mathbf{q}_h$ and $u - u_h = u - \Pi_k^o u + \Pi_k^o u - \bar{u}_h + \bar{u}_h - u_h$, the convergence estimates for $\mathbf{q} - \mathbf{q}_h$ and $u - u_h$ in the main result follow from the triangle inequality, the estimates in Lemma 1, Theorem 2, and Lemma 8. The superconvergence estimate for $u - u_h^*$ in the main result follows from the triangle inequality, Proposition 1, (10b), and the estimates in Theorem 2 and Lemma 8.

If the nonlinearity is locally Lipschitz, the estimates of the main result in this case now follow from the above result in the same way. This concludes the proof of the main result, Theorem 1.

Table 2 History of convergence. Top is HDG (A), middle is HDG (B) and bottom is HDG (C)

Degree	$\frac{h}{\sqrt{2}}$	$\ q - q_h\ _{0,\Omega}$		$\ u - u_h\ _{0,\Omega}$		$\ u - u_h^*\ _{0,\Omega}$	
		Error	Rate	Error	Rate	Error	Rate
$k = 0$	2^{-1}	1.18		2.93E-01		2.93E-01	
	2^{-2}	6.33E-01	0.89	9.53E-02	1.62	9.53E-02	1.62
	2^{-3}	3.23E-01	0.97	2.47E-02	1.95	2.47E-02	1.95
	2^{-4}	1.62E-01	0.99	6.24E-03	1.99	6.24E-03	1.99
	2^{-5}	8.12E-02	0.98	1.56E-03	2.00	1.56E-03	2.00
$k = 1$	2^{-1}	3.39E-02		8.81E-02		8.81E-02	
	2^{-2}	9.15E-03	1.97	1.14E-02	2.95	1.14E-02	2.95
	2^{-3}	2.33E-02	1.99	1.44E-03	3.00	1.44E-03	3.00
	2^{-4}	5.86E-03	1.99	1.80E-04	3.00	1.80E-04	3.00
	2^{-5}	1.47E-03	2.00	2.25E-05	3.00	2.25E-05	3.00

Degree	$\frac{h}{\sqrt{2}}$	$\ q - q_h\ _{0,\Omega}$		$\ u - u_h\ _{0,\Omega}$		$\ u - u_h^*\ _{0,\Omega}$	
		Error	Rate	Error	Rate	Error	Rate
$k = 0$	2^{-1}	1.21		3.23E-01		2.41E-01	
	2^{-2}	6.40E-01	0.92	1.41E-01	1.20	6.47E-01	1.90
	2^{-3}	3.24E-01	0.98	6.68E-01	1.08	1.66E-02	1.97
	2^{-4}	1.62E-01	1.00	3.29E-02	1.02	4.17E-03	2.00
	2^{-5}	8.13E-02	1.00	1.64E-02	1.00	1.04E-03	2.00
$k = 1$	2^{-1}	3.41E-01		9.33E-01		6.31E-02	
	2^{-2}	9.02E-02	1.90	2.12E-02	2.14	9.05E-03	2.80
	2^{-3}	2.28E-02	1.98	5.07E-02	2.07	1.16E-03	2.96
	2^{-4}	5.73E-03	2.00	1.25E-03	2.02	1.46E-04	2.99
	2^{-5}	1.43E-03	2.00	3.11E-04	2.00	1.83E-05	3.00

Degree	$\frac{h}{\sqrt{2}}$	$\ q - q_h\ _{0,\Omega}$		$\ u - u_h\ _{0,\Omega}$		$\ u - u_h^*\ _{0,\Omega}$	
		Error	Rate	Error	Rate	Error	Rate
$k = 1$	2^{-1}	6.28E-01		2.58E-01		1.16E-01	
	2^{-2}	1.78E-01	1.82	1.32E-01	0.97	3.20E-02	1.86
	2^{-3}	4.58E-02	1.96	6.56E-02	1.00	8.24E-02	1.96
	2^{-4}	1.15E-02	1.99	3.28E-02	1.00	2.07E-03	1.99
	2^{-5}	2.89E-03	2.00	1.64E-02	1.00	5.20E-04	2.00
$k = 2$	2^{-1}	1.06E-01		7.39E-02		1.27E-02	
	2^{-2}	1.44E-02	2.88	1.95E-02	1.92	9.39E-04	3.76
	2^{-3}	1.85E-03	2.96	4.95E-03	1.98	6.18E-05	3.92
	2^{-4}	2.33E-04	2.99	1.24E-03	1.99	3.92E-06	3.98
	2^{-5}	2.93E-05	3.00	3.11E-04	2.00	2.47E-07	4.00

4 Numerical Results

We test the Chaffee-Infante equation with an exact solution to illustrate the convergence theory. The domain is the unit square $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$, the nonlinear term is $F(u) := u^3 - u$, and the source term f is chosen so that the exact solution is $u = \sin(t) \sin(\pi x) \sin(\pi y)$. The meshes are uniform and made of triangles. The Crank-Nicolson method is used for the time discretization. The initial condition is the simple L^2 -projection of u_0 into W_h . For Interpolatory HDG (AB), the time step is chosen as $\Delta t = h$ when $k = 0$ and $\Delta t = h^2$ when $k = 1$, where k is the polynomial degree. We choose $\Delta t = h$ when $k = 1$ and $\Delta t = h^2$ when $k = 2$ for Interpolatory HDG (C). We report the errors at the final time $T = 1$ in Table 2. The observed convergence rates match the theory.

5 Conclusion

In [8], we proposed a superconvergent Interpolatory HDG method to approximate the solution of nonlinear reaction diffusion PDEs. The new method uses a postprocessing procedure along with an interpolation operator to evaluate the nonlinear term. This simple change recovers the superconvergence that was lost in our earlier Interpolatory HDG work [18]. Furthermore, this method retains the computational advantages of our Interpolatory HDG method from [18].

We extended the idea developed previously and devised superconvergent Interpolatory HDG methods inspired by hybrid high-order methods [14]. We proved that the interpolatory procedure does not reduce the convergence rate.

The devising of superconvergent HDG methods for equations with the more general nonlinear term $F(\nabla u, u)$ constitutes a subject of ongoing work.

A Approximation estimates of auxiliary projections

A.1 Proof of (10a)

Here we prove the estimate for $\Pi_{k+1}^* u - u$ in (10a).

We are going to use the following auxiliary result.

Lemma 10 *For any $K \in \mathcal{T}_h$, we have*

$$\|\Pi_{k+1}^* u - u\|_K \leq C \left(h \|\nabla u - \nabla \Pi_{k+1}^o u\|_K + \|u - \Pi_{k+1}^o u\|_K \right).$$

Proof By definitions (3) and (4), we obtain

$$\begin{aligned} (\nabla \Pi_{k+1}^* u, \nabla z_h)_K &= -(\Pi_{k+1}^o u, \Delta z_h)_K + \langle \Pi_{k+1}^o u, \mathbf{n} \cdot \nabla z_h \rangle_{\partial K}, \\ (\Pi_{k+1}^* u, w_h)_K &= (\Pi_{k+1}^o u, w_h)_K, \end{aligned}$$

for all $(z_h, w_h) \in [\mathcal{P}_\ell^{k+1}(K)]^\perp \times \mathcal{P}_\ell(K)$. This leads to

$$\begin{aligned} (\nabla \Pi_{k+1}^* u, \nabla z_h)_K &= (\nabla u, \nabla z_h)_K, \\ (\Pi_{k+1}^* u, w_h)_K &= (\Pi_{k+1}^o u, w_h)_K. \end{aligned}$$

The last equation implies that $\Pi_{k+1}^* u - \Pi_{k+1}^o u \in [\mathcal{P}_\ell^{k+1}(K)]^\perp$ and so, we can then take $z_h := \Pi_{k+1}^* u - \Pi_{k+1}^o u$ in the first equation to get

$$\|\nabla \Pi_{k+1}^* u - \nabla \Pi_{k+1}^o u\|_K^2 = (\nabla \Pi_{k+1}^* u - \nabla \Pi_{k+1}^o u, \nabla u - \nabla \Pi_{k+1}^o u)_K,$$

and

$$\|\nabla \Pi_{k+1}^* u - \nabla \Pi_{k+1}^o u\|_K \leq \|\nabla u - \nabla \Pi_{k+1}^o u\|_K.$$

Since $\Pi_{k+1}^* u - \Pi_{k+1}^o u \in [\mathcal{P}_\ell^{k+1}(K)]^\perp$, we have

$$(\Pi_{k+1}^* u - \Pi_{k+1}^o u, 1)_K = 0,$$

and using Poincaré's inequality, we obtain

$$\|\Pi_{k+1}^* u - \Pi_{k+1}^o u\|_K \leq Ch \|\nabla \Pi_{k+1}^* u - \nabla \Pi_{k+1}^o u\|_K \leq Ch \|\nabla u - \nabla \Pi_{k+1}^o u\|_K.$$

Then the estimate follows by applying the triangle inequality. This completes the proof.

We are now ready to prove (10a). Using inverse inequalities, Poincaré's inequality, and the approximation properties for Π_{k+1}^o , one gets

$$\begin{aligned} \|u - \Pi_{k+1}^* u\|_{0,\infty,K} &\leq \|\Pi_{k+1}^* u - \Pi_{k+1}^o u\|_{0,\infty,K} + \|\Pi_{k+1}^o u - u\|_{0,\infty,K} \\ &\leq Ch^{-d/2} \|\Pi_{k+1}^* u - \Pi_{k+1}^o u\|_{0,K} + \|\Pi_{k+1}^o u - u\|_{0,\infty,K} \\ &\leq Ch^{1-d/2} \|u - \Pi_{k+1}^o u\|_{1,K} + \|\Pi_{k+1}^o u - u\|_{0,\infty,K}. \end{aligned} \tag{12}$$

By [1, Lemma 4.3.8], there exists $Q^{k+1}u \in \mathcal{P}^{k+1}(K)$ such that

$$\|Q^{k+1}u - u\|_{1,\infty,K} \leq C \|u\|_{1,\infty,K}, \quad \|Q^{k+1}u - u\|_{0,\infty,K} \leq Ch \|u\|_{1,\infty,K}.$$

Hence, by (12) we have

$$\begin{aligned} \|u - \Pi_{k+1}^* u\|_{0,\infty,K} &\leq Ch^{1-d/2} \left(\|u - Q^{k+1}u\|_{1,K} + \|Q^{k+1}u - \Pi_{k+1}^o u\|_{1,K} \right) + \left(\|u - Q^{k+1}u\|_{0,\infty,K} + \|Q^{k+1}u - \Pi_{k+1}^o u\|_{0,\infty,K} \right) \\ &= Ch^{1-d/2} \left(\|u - Q^{k+1}u\|_{1,K} + \|\Pi_{k+1}^o (Q^{k+1}u - u)\|_{1,K} \right) + C \left(\|u - Q^{k+1}u\|_{0,\infty,K} + h^{-d/2} \|Q^{k+1}u - \Pi_{k+1}^o u\|_K \right) \\ &\leq Ch^{1-d/2} \left(\|u - Q^{k+1}u\|_{1,K} + \|\Pi_{k+1}^o (Q^{k+1}u - u)\|_{1,K} \right) + C \left(\|u - Q^{k+1}u\|_{0,\infty,K} + h^{-d/2} \|\Pi_{k+1}^o (Q^{k+1}u - u)\|_K \right) \\ &\leq Ch^{1-d/2} \left(\|u - Q^{k+1}u\|_{1,K} + \|Q^{k+1}u - u\|_{1,K} \right) + C \left(\|u - Q^{k+1}u\|_{0,\infty,K} + h^{-d/2} \|Q^{k+1}u - u\|_K \right) \\ &\leq Ch^{1-d/2} \|u - Q^{k+1}u\|_{1,K} + Ch^{-d/2} \|u - Q^{k+1}u\|_K + C \|u - Q^{k+1}u\|_{0,\infty,K} \\ &\leq Ch \|u - Q^{k+1}u\|_{1,\infty,K} + C \|u - Q^{k+1}u\|_{0,\infty,K} \\ &\leq Ch \|u\|_{1,\infty,K}. \end{aligned}$$

This completes the proof of (10a).

A.2 Proof of 10b

Here, we prove the estimate for $\Pi_{k+1}^* u - u_h^*$ in (10b).

Let $z_h \in [\mathcal{P}_\ell^{k+1}(K)]^\perp$ and take $\mathbf{r}_h = \nabla z_h$ in the first equation of Proposition 1 to get

$$(\mathbf{q}_h, \nabla z_h) - (u_h, \Delta z_h)_{\mathcal{T}_h} + \langle \widehat{u}_h, \nabla z_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0.$$

Combined with (4a) one gets

$$(\nabla u_h^*, \nabla z_h) = -(\mathbf{q}_h, \nabla z_h) \quad \forall z_h \in [\mathcal{P}_\ell^{k+1}(K)]^\perp.$$

By the definition of Π_{k+1}^* , as in the proof of Proposition 1 one gets

$$(\nabla \Pi_{k+1}^* u, \nabla z_h)_K = -(\Pi_\ell^o u, \Delta z_h)_K + \langle \Pi_k^\partial u, \mathbf{n} \cdot \nabla z_h \rangle_{\partial K} = (\nabla u, \nabla z_h)_K.$$

Let $e_h = u_h^* - u_h + \Pi_\ell^o u - \Pi_{k+1}^* u$, and then $e_h \in [\mathcal{P}_\ell^{k+1}(K)]^\perp$. By the two previous equations, $\mathbf{q} = -\nabla u$, and an inverse inequality we have

$$\begin{aligned} \|\nabla e_h\|_K^2 &= (\nabla(u_h^* - u_h), \nabla e_h)_K + (\nabla(\Pi_\ell^o u - \Pi_{k+1}^* u), \nabla e_h)_K \\ &= (-\mathbf{q}_h - \nabla u_h, \nabla e_h)_K + (\nabla(\Pi_\ell^o u - u), \nabla e_h)_K \\ &= ((\mathbf{q} - \Pi_k^o \mathbf{q}) - (\mathbf{q}_h - \Pi_k^o \mathbf{q}) + \nabla(\Pi_\ell^o u - u_h), \nabla e_h)_K \\ &\leq C(h^{-1}\|u_h - \Pi_\ell^o u\|_K + \|\mathbf{q}_h - \Pi_k^o \mathbf{q}\|_K + \|\mathbf{q} - \Pi_k^o \mathbf{q}\|_K) \|\nabla e_h\|_K. \end{aligned}$$

Since $(e_h, 1)_K = 0$, we can now apply the Poincaré inequality to get

$$\|e_h\|_K \leq Ch \|\nabla e_h\|_K \leq C(\|u_h - \Pi_\ell^o u\|_K + h\|\mathbf{q}_h - \Pi_k^o \mathbf{q}\|_K + h\|\mathbf{q} - \Pi_k^o \mathbf{q}\|_K).$$

This means

$$\|e_h\|_{\mathcal{T}_h} \leq C(\|u_h - \Pi_\ell^o u\|_{\mathcal{T}_h} + h\|\mathbf{q}_h - \Pi_k^o \mathbf{q}\|_{\mathcal{T}_h} + h\|\mathbf{q} - \Pi_k^o \mathbf{q}\|_{\mathcal{T}_h}).$$

Hence, we have

$$\begin{aligned} \|\Pi_{k+1}^* u - u_h^*\|_{\mathcal{T}_h} &\leq \|\Pi_{k+1}^* u - \Pi_\ell^o u - u_h^* + u_h\|_{\mathcal{T}_h} + \|\Pi_\ell^o u - u_h\|_{\mathcal{T}_h} \\ &\leq C(\|u_h - \Pi_\ell^o u\|_{\mathcal{T}_h} + h\|\mathbf{q}_h - \Pi_k^o \mathbf{q}\|_{\mathcal{T}_h} + h\|\mathbf{q} - \Pi_k^o \mathbf{q}\|_{\mathcal{T}_h}). \end{aligned}$$

This completes the proof of (10b).

B Proof of Theorem 2

This appendix is devoted to the proof of the approximation estimates of Theorem 2. We only give the proofs of the estimates for $\|\Pi_k^o \mathbf{q} - \bar{\mathbf{q}}_h\|_{\mathcal{T}_h}$ and $\|\Pi_\ell^o u - \bar{u}_h\|_{\mathcal{T}_h}$. The proof of the estimate for $\|\partial_t \Pi_\ell^o u - \partial_t \bar{u}_h\|_{\mathcal{T}_h}$ is very similar and is omitted. We use the notation

$$\varepsilon_h^{\mathbf{q}} = \Pi_k^o \mathbf{q} - \bar{\mathbf{q}}_h, \quad \varepsilon_h^u = \Pi_\ell^o u - \bar{u}_h, \quad \varepsilon_h^{\widehat{u}} = \Pi_k^\partial u - \widehat{\bar{u}}_h, \quad \text{and} \quad \varepsilon_h^{u^*} = \Pi_{k+1}^* u - \bar{u}_h^*,$$

and split the proof into four steps.

Step 1: Equations for the projections of the errors

Lemma 11 For all $(\mathbf{r}_h, v_h, \widehat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$, we have

$$\begin{aligned} (\varepsilon_h^{\mathbf{q}}, \mathbf{r}_h)_{\mathcal{T}_h} - (\varepsilon_h^u, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{u}}, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\ (\nabla \cdot \varepsilon_h^{\mathbf{q}}, v_h)_{\mathcal{T}_h} - \langle \varepsilon_h^{\mathbf{q}} \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle h^{-1}(\Pi_k^\partial \varepsilon_h^{u^*} - \varepsilon_h^{\widehat{u}}), \Pi_k^\partial v_h^* - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} &= RHS_h, \end{aligned}$$

where

$$\begin{aligned} RHS_h &:= ((\mathbb{I} - \Pi_\ell^o)(-\Delta u), (\mathbb{I} - \Pi_\ell^o)v_h^*) + E_h(\mathbf{q}, u; v_h, \widehat{v}_h), \\ E_h(\mathbf{q}, u; v_h, \widehat{v}_h) &:= -\langle (\Pi_k^o \mathbf{q} - \mathbf{q}) \cdot \mathbf{n}, \widehat{v}_h - v_h^* \rangle_{\partial \mathcal{T}_h} + \langle h^{-1}(\Pi_{k+1}^* u - u), \Pi_k^\partial v_h^* - \widehat{v}_h \rangle_{\partial \mathcal{T}_h}, \end{aligned}$$

and \mathbb{I} is the identity operator.

Proof We begin by noting that, by the properties of Π_k^o , Π_ℓ^o , and Π_k^∂ , we have

$$(\Pi_k^o \mathbf{q}, \mathbf{r}_h)_{\mathcal{T}_h} - (\Pi_\ell^o u, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \Pi_k^\partial u, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = (\mathbf{q}, \mathbf{r}_h)_{\mathcal{T}_h} - (u, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle u, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0,$$

since $\mathbf{q} + \nabla u = 0$. Also, since $\langle \mathbf{q} \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} = 0$, we have

$$\begin{aligned} (\nabla \cdot \Pi_k^o \mathbf{q}, v_h)_{\mathcal{T}_h} - \langle \Pi_k^o \mathbf{q} \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} &= ((\nabla \cdot \Pi_k^o \mathbf{q}, v_h^*)_{\mathcal{T}_h} - \langle \Pi_k^o \mathbf{q} \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h}) \\ &= (\nabla \cdot \mathbf{q}, v_h^*)_{\mathcal{T}_h} - \langle (\Pi_k^o \mathbf{q} - \mathbf{q}) \cdot \mathbf{n}, \widehat{v}_h - v_h^* \rangle_{\partial \mathcal{T}_h} \\ &= (-\Delta u, v_h^*)_{\mathcal{T}_h} - \langle (\Pi_k^o \mathbf{q} - \mathbf{q}) \cdot \mathbf{n}, \widehat{v}_h - v_h^* \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

As a consequence,

$$\begin{aligned} (\Pi_k^o \mathbf{q}, \mathbf{r}_h)_{\mathcal{T}_h} - (\Pi_\ell^o u, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \Pi_k^\partial u, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\ (\nabla \cdot \Pi_k^o \mathbf{q}, v_h)_{\mathcal{T}_h} - \langle \Pi_k^o \mathbf{q} \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle h^{-1}(\Pi_k^\partial \Pi_{k+1}^* u - \Pi_k^\partial u), \Pi_k^\partial v_h^* - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} &= (-\Delta u, v_h^*)_{\mathcal{T}_h} \\ &\quad + E_h(\mathbf{q}, u; v_h, \widehat{v}_h). \end{aligned}$$

The wanted equations can be now obtained by subtracting these equations from the equations defining the HDG elliptic approximation (7). This completes the proof.

Step 2: Estimate for ε_h^q by an energy argument

Lemma 12 *We have*

$$\begin{aligned} \|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h} + \|\varepsilon_h^q\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h} \\ \leq C \left(h \|(\Pi_\ell^o - \mathbb{I})(-\Delta u)\|_{\mathcal{T}_h} + h^{1/2} \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\partial \mathcal{T}_h} + \|h^{-1/2}(\Pi_{k+1}^* u - u)\|_{\partial \mathcal{T}_h} \right). \end{aligned}$$

This result implies the estimate for the approximate flux in Theorem 2. To prove this lemma, we need the following auxiliary result.

Lemma 13 *We have*

$$\|\varepsilon_h^q\|_{\mathcal{T}_h} \leq C \left(\|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h} \right), \quad (13a)$$

$$\|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h} \leq \left(\|\varepsilon_h^q\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h} \right). \quad (13b)$$

Proof Using the first equation of Lemma 11, the definition of \mathbf{p}_h^{k+1} in (4), and $\nabla \cdot \mathbf{r}_h \in W_h$, we have

$$(\varepsilon_h^q, \mathbf{r}_h)_{\mathcal{T}_h} - (\varepsilon_h^{u^*}, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \widehat{\varepsilon}_h, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0.$$

Integration by parts gives

$$(\varepsilon_h^q, \mathbf{r}_h)_{\mathcal{T}_h} + (\nabla \varepsilon_h^{u^*}, \mathbf{r}_h)_{\mathcal{T}_h} + \langle \widehat{\varepsilon}_h - \Pi_k^\partial \varepsilon_h^{u^*}, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0.$$

Since $\nabla \varepsilon_h^{u^*} \in \mathbf{V}_h$, by taking first $\mathbf{r}_h := \varepsilon_h^q$ and then $\mathbf{r}_h := \nabla \varepsilon_h^{u^*}$, one gets

$$\begin{aligned} \|\varepsilon_h^q\|_{\mathcal{T}_h} &\leq C \left(\|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h} \right), \\ \|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h} &\leq C \left(\|\varepsilon_h^q\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h} \right), \end{aligned}$$

respectively. This completes the proof.

We can now prove Lemma 12.

Proof We take $(\mathbf{r}_h, v_h, \widehat{v}_h) := (\varepsilon_h^q, \varepsilon_h^u, \widehat{\varepsilon}_h)$ in the error equations of Lemma 11, and add them to get

$$\|\varepsilon_h^q\|_{\mathcal{T}_h}^2 + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h}^2 = R_1 + R_2 + R_3,$$

where

$$\begin{aligned} R_1 &:= ((\mathbb{I} - \Pi_\ell^o)(-\Delta u), (\mathbb{I} - \Pi_\ell^o) \varepsilon_h^{u^*})_{\mathcal{T}_h}, \\ R_2 &:= -\langle (\Pi_k^o \mathbf{q} - \mathbf{q}) \cdot \mathbf{n}, \varepsilon_h^u - \varepsilon_h^{u^*} \rangle_{\partial \mathcal{T}_h} \\ R_3 &:= \langle h^{-1}(\Pi_{k+1}^* u - u), \Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

Since

$$\begin{aligned} |R_1| &\leq Ch \|(\mathbb{I} - \Pi_\ell^o)(-\Delta u)\|_{\mathcal{T}_h} \|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h}, \\ |R_2| &\leq Ch^{1/2} \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\partial \mathcal{T}_h} \left(\|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h} \right), \\ |R_3| &\leq \|h^{-1/2}(\Pi_{k+1}^* u - u)\|_{\partial \mathcal{T}_h} \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \widehat{\varepsilon}_h)\|_{\partial \mathcal{T}_h}, \end{aligned}$$

using the last two estimates of Lemma 13 and simple algebraic manipulations, we get the desired result.

Step 3: Estimate for $\varepsilon_h^{u^*}$ by a duality argument

Lemma 14 Assume that the elliptic regularity inequality (9a) holds. Then, we have

$$\begin{aligned} \|\varepsilon_h^{u^*}\|_{\mathcal{T}_h} &\leq Ch^{1+\min\{\ell,1\}} \|(\mathbb{I} - \Pi_\ell^o)(-\Delta u)\|_{\mathcal{T}_h} \\ &\quad + C(h^{3/2} \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\partial\mathcal{T}_h} + h\|h^{-1/2}(\Pi_{k+1}^* u - u)\|_{\partial\mathcal{T}_h}). \end{aligned}$$

Proof Setting $g := \varepsilon_h^{u^*}$ in the dual problem, and proceeding as in the proof of Lemma 11, we get

$$\begin{aligned} (\Pi_k^o \Phi, \mathbf{r}_h)_{\mathcal{T}_h} - (\Pi_\ell^o \Psi, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle \Pi_k^\partial \Psi, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} &= 0, \\ (\nabla \cdot \Pi_k^o \Phi, v_h)_{\mathcal{T}_h} - \langle \Pi_k^o \Phi \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial\mathcal{T}_h} & \end{aligned} \quad (14a)$$

$$+ \langle h^{-1}(\Pi_k^\partial \Pi_{k+1}^* \Psi - \Pi_k^\partial \Psi), \Pi_k^\partial v_h^* - \widehat{v}_h \rangle_{\partial\mathcal{T}_h} = (\varepsilon_h^{u^*}, v_h^*)_{\mathcal{T}_h} + E_h(\Phi, \Psi; v_h, \widehat{v}_h), \quad (14b)$$

where

$$E_h(\Phi, \Psi; v_h, \widehat{v}_h) = -\langle (\Pi_k^o \Phi - \Phi) \cdot \mathbf{n}, \widehat{v}_h - v_h^* \rangle_{\partial\mathcal{T}_h} + \langle h^{-1}(\Pi_{k+1}^* \Psi - \Psi), \Pi_k^\partial v_h^* - \widehat{v}_h \rangle_{\partial\mathcal{T}_h}.$$

Then taking $(v_h, \widehat{v}_h) := (\varepsilon_h^u, \varepsilon_h^{\widehat{u}})$ in (14b), we get

$$\begin{aligned} \|\varepsilon_h^{u^*}\|_{\mathcal{T}_h}^2 &= (\nabla \cdot \Pi_k^o \Phi, \varepsilon_h^u)_{\mathcal{T}_h} - \langle \Pi_k^o \Phi \cdot \mathbf{n}, \varepsilon_h^{\widehat{u}} \rangle_{\partial\mathcal{T}_h} \\ &\quad + \langle h^{-1}(\Pi_k^\partial \Pi_{k+1}^* \Psi - \Pi_k^\partial \Psi), \Pi_k^\partial v_h^* - \widehat{v}_h \rangle_{\partial\mathcal{T}_h} - E_h(\Phi, \Psi; \varepsilon_h^u, \varepsilon_h^{\widehat{u}}) \\ &= (\varepsilon_h^q, \Pi_k^o \Phi)_{\mathcal{T}_h} + \langle h^{-1}(\Pi_k^\partial \Pi_{k+1}^* \Psi - \Pi_k^\partial \Psi), \Pi_k^\partial \varepsilon_h^{u^*} - \varepsilon_h^{\widehat{u}} \rangle_{\partial\mathcal{T}_h} - E_h(\Phi, \Psi; \varepsilon_h^u, \varepsilon_h^{\widehat{u}}), \end{aligned}$$

by the first equation of Lemma 11 with $\mathbf{r}_h := \Pi_k^o \Phi$. By (14a) with $\mathbf{r}_h := \varepsilon_h^q$, we obtain

$$\begin{aligned} \|\varepsilon_h^{u^*}\|_{\mathcal{T}_h}^2 &= (\Pi_\ell^o \Psi, \nabla \cdot \varepsilon_h^q)_{\mathcal{T}_h} - \langle \Pi_k^\partial \Psi, \varepsilon_h^q \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} + \langle h^{-1}(\Pi_k^\partial \Pi_{k+1}^* \Psi - \Pi_k^\partial \Psi), \Pi_k^\partial \varepsilon_h^{u^*} - \varepsilon_h^{\widehat{u}} \rangle_{\partial\mathcal{T}_h} \\ &\quad - E_h(\Phi, \Psi; \varepsilon_h^u, \varepsilon_h^{\widehat{u}}) \\ &= ((\mathbb{I} - \Pi_\ell^o)(-\Delta u), \Pi_{k+1}^* \Psi - \Pi_\ell^o \Psi) + E_h(\mathbf{q}, u; \Pi_\ell^o \Psi, \Pi_k^\partial \Psi) - E_h(\Phi, \Psi; \varepsilon_h^u, \varepsilon_h^{\widehat{u}}), \end{aligned}$$

by the second equation of Lemma 11 with $(v_h, \widehat{v}_h) := (\Pi_\ell^o \Psi, \Pi_k^\partial \Psi)$. Inserting the definitions of the E_h -terms, we finally get

$$\begin{aligned} \|\varepsilon_h^{u^*}\|_{\mathcal{T}_h}^2 &= ((\mathbb{I} - \Pi_\ell^o)(-\Delta u), \Pi_{k+1}^* \Psi - \Pi_\ell^o \Psi) \\ &\quad - \langle (\Pi_k^o \mathbf{q} - \mathbf{q}) \cdot \mathbf{n}, \Pi_k^\partial \Psi - \Pi_{k+1}^* \Psi \rangle_{\partial\mathcal{T}_h} + \langle h^{-1}(\Pi_{k+1}^* u - u), \Pi_k^\partial \Pi_{k+1}^* \Psi - \Pi_k^\partial \Psi \rangle_{\partial\mathcal{T}_h} \\ &\quad + \langle (\Pi_k^o \Phi - \Phi) \cdot \mathbf{n}, \varepsilon_h^{\widehat{u}} - \varepsilon_h^{u^*} \rangle_{\partial\mathcal{T}_h} - \langle h^{-1}(\Pi_{k+1}^* \Psi - \Psi), \Pi_k^\partial \varepsilon_h^{u^*} - \varepsilon_h^{\widehat{u}} \rangle_{\partial\mathcal{T}_h}, \end{aligned}$$

which leads to

$$\begin{aligned} \|\varepsilon_h^{u^*}\|_{\mathcal{T}_h}^2 &\leq Ch^{\min\{\ell,1\}+1} \|(\mathbb{I} - \Pi_\ell^o)(-\Delta u)\|_{\mathcal{T}_h} |\Psi|_{\min\{\ell,1\}+1} \\ &\quad + Ch^{3/2} \|\Pi_k^o \mathbf{q} - \mathbf{q}\|_{\partial\mathcal{T}_h} |\Psi|_2 + Ch\|h^{-1/2}(\Pi_{k+1}^* u - u)\|_{\partial\mathcal{T}_h} |\Psi|_2 \\ &\quad + Ch \left(\|\nabla \varepsilon_h^{u^*}\|_{\mathcal{T}_h} + \|h^{-1/2}(\Pi_k^\partial \varepsilon_h^{u^*} - \varepsilon_h^{\widehat{u}})\|_{\partial\mathcal{T}_h} \right) (|\Phi|_1 + |\Psi|_2). \end{aligned}$$

Using the elliptic regularity inequality (9a) and the first inequality of Lemma 12, we finally obtain the wanted result.

Step 4: Estimate for u_h

Lemma 15 We have that $\|\varepsilon_h^u\|_{\mathcal{T}_h} \leq \|\varepsilon_h^{u^*}\|_{\mathcal{T}_h}$.

Combining this result and the one in the previous step gives the estimate in the approximation for u in Theorem 2. To complete the proof of Theorem 2, it only remains to prove the above lemma.

Proof Since $u_h^* = \mathbf{p}_h^{k+1}(u_h, \widehat{u}_h)$, $\Pi_{k+1}^* u = \mathbf{p}_h^{k+1}(\Pi_\ell^o u, \Pi_k^\partial u)$, and the operator \mathbf{p}_h^{k+1} is linear, we have that $\varepsilon_h^{u^*} = \mathbf{p}_h^{k+1}(\varepsilon_h^u, \varepsilon_h^{\widehat{u}})$. Proceeding as in the proof of Proposition 1, it can be shown that $\varepsilon_h^u \in [\mathcal{P}_\ell^{k+1}(K)]^\perp$. Then, by equation (4b), the wanted inequality follows. This completes the proof.

Acknowledgements J. Singler and Y. Zhang thank the IMA for funding research visits, during which some of this work was completed.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. Brenner, S.C., Scott, L.R.: The mathematical theory of finite element methods, *Texts in Applied Mathematics*, vol. 15, third edn. Springer, New York (2008). URL <https://doi.org/10.1007/978-0-387-75934-0>
2. Brezzi, F., Douglas, Jr., J., Marini, L.D.: Two families of mixed finite elements for second order elliptic problems. *Numer. Math.* **47**, 217–235 (1985)
3. Burman, E., Ern, A.: An unfitted hybrid high-order method for elliptic interface problems. *SIAM J. Numer. Anal.* **56**(3), 1525–1546 (2018). DOI 10.1137/17M1154266. URL <https://doi.org/10.1137/17M1154266>
4. Casoni, E., Peraire, J., Huerta, A.: One-dimensional shock-capturing for high-order discontinuous Galerkin methods. *Internat. J. Numer. Methods Fluids* **71**(6), 737–755 (2013)
5. Cesmelioglu, A., Cockburn, B., Qiu, W.: Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier-Stokes equations. *Math. Comp.* **86**(306), 1643–1670 (2017). DOI 10.1090/mcom/3195. URL <http://dx.doi.org/10.1090/mcom/3195>
6. Chabaud, B., Cockburn, B.: Uniform-in-time superconvergence of HDG methods for the heat equation. *Math. Comp.* **81**(277), 107–129 (2012). DOI 10.1090/S0025-5718-2011-02525-1
7. Chen, C.M., Larsson, S., Zhang, N.Y.: Error estimates of optimal order for finite element methods with interpolated coefficients for the nonlinear heat equation. *IMA J. Numer. Anal.* **9**(4), 507–524 (1989). DOI 10.1093/imanum/9.4.507
8. Chen, G., Cockburn, B., Singler, J., Zhang, Y.: Superconvergent interpolatory HDG methods for reaction diffusion equations I: An HDG_k method. *J. Sci. Comput.* **81**(3), 2188–2212 (2019). DOI 10.1007/s10915-019-01081-3
9. Chen, G., Cui, J.: On the error estimates of a hybridizable discontinuous Galerkin method for second-order elliptic problem with discontinuous coefficients. *IMA J. Numer. Anal.* **40**(2), 1577–1600 (2020). DOI 10.1093/imanum/drz003
10. Chen, Z., Douglas Jr., J.: Approximation of coefficients in hybrid and mixed methods for nonlinear parabolic problems. *Mat. Apl. Comput.* **10**(2), 137–160 (1991)
11. Christie, I., Griffiths, D.F., Mitchell, A.R., Sanz-Serna, J.M.: Product approximation for nonlinear problems in the finite element method. *IMA J. Numer. Anal.* **1**(3), 253–266 (1981)
12. Cockburn, B.: Static condensation, hybridization, and the devising of the HDG methods. In: G. Barrenechea, F. Brezzi, A. Cagniani, E. Georgoulis (eds.) *Building Bridges: Connections and Challenges in Modern Approaches to Numerical Partial Differential Equations, Lect. Notes Comput. Sci. Engrg.*, vol. 114, pp. 129–177. Springer Verlag, Berlin (2016). LMS Durham Symposia funded by the London Mathematical Society. Durham, U.K., on July 8–16, 2014.
13. Cockburn, B.: Discontinuous Galerkin methods for computational fluid dynamics. In: E. Stein, R. Borst, T. Hughes (eds.) *Encyclopedia of Computational Mechanics Second Edition*, vol. 5, pp. 141–203. J. Wiley & Sons, Ltd., Chichester, U.K. (2018)
14. Cockburn, B., Di Pietro, D.A., Ern, A.: Bridging the hybrid high-order and hybridizable discontinuous Galerkin methods. *ESAIM Math. Model. Numer. Anal.* **50**(3), 635–650 (2016). URL <https://doi.org/10.1051/m2an/2015051>
15. Cockburn, B., Gopalakrishnan, J., Lazarov, R.: Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.* **47**(2), 1319–1365 (2009). DOI 10.1137/070706616. URL <http://dx.doi.org/10.1137/070706616>
16. Cockburn, B., Gopalakrishnan, J., Sayas, F.J.: A projection-based error analysis of HDG methods. *Math. Comp.* **79**, 1351–1367 (2010)
17. Cockburn, B., Shen, J.: A hybridizable discontinuous Galerkin method for the p -Laplacian. *SIAM J. Sci. Comput.* **38**(1), A545–A566 (2016). DOI 10.1137/15M1008014. URL <http://dx.doi.org/10.1137/15M1008014>
18. Cockburn, B., Singler, J.R., Zhang, Y.: Interpolatory HDG Method for Parabolic Semilinear PDEs. *J. Sci. Comput.* **79**, 1777–1800 (2019)
19. Di-Pietro, D., Ern, A.: A hybrid high-order locking-free method for linear elasticity on general meshes. *Comput. Meth. Appl. Mech. Engrg.* **283**, 1–21 (2015)
20. Di-Pietro, D., Ern, A.: Hybrid high-order methods for variable-diffusion problems on general meshes. *C. R. Acad. Sci Paris, Ser. I* **353**, 31–34 (2015)
21. Di-Pietro, D., Ern, A., Lemaire, S.: An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. *Comput. Meth. Appl. Math.* **14**(4), 461–472 (2014)
22. Dickinson, B.T., Singler, J.R.: Nonlinear model reduction using group proper orthogonal decomposition. *Int. J. Numer. Anal. Model.* **7**(2), 356–372 (2010)
23. Douglas Jr., J., Dupont, T.: The effect of interpolating the coefficients in nonlinear parabolic Galerkin procedures. *Math. Comput.* **20**(130), 360–389 (1975)
24. Du, S., Sayas, F.J.: New analytical tools for HDG in elasticity, with applications to elastodynamics. *Math. Comp.* **89**(324), 1745–1782 (2020). DOI 10.1090/mcom/3499
25. Fletcher, C.A.J.: The group finite element formulation. *Comput. Methods Appl. Mech. Engrg.* **37**(2), 225–244 (1983). DOI 10.1016/0045-7825(83)90122-6
26. Fletcher, C.A.J.: Time-splitting and the group finite element formulation. In: *Computational techniques and applications: CTAC-83* (Sydney, 1983), pp. 517–532. North-Holland, Amsterdam (1984)
27. Gastaldi, L., Nochetto, R.: Sharp maximum norm error estimates for general mixed finite element approximations to second order elliptic equations. *RAIRO Modél. Math. Anal. Numér.* **23**, 103–128 (1989)
28. Huerta, A., Casoni, E., Peraire, J.: A simple shock-capturing technique for high-order discontinuous Galerkin methods. *Internat. J. Numer. Methods Fluids* **69**(10), 1614–1632 (2012)
29. Kabaria, H., Lew, A.J., Cockburn, B.: A hybridizable discontinuous Galerkin formulation for non-linear elasticity. *Comput. Methods Appl. Mech. Engrg.* **283**, 303–329 (2015)
30. Kim, D., Park, E.J., Seo, B.: Two-scale product approximation for semilinear parabolic problems in mixed methods. *J. Korean Math. Soc.* **51**(2), 267–288 (2014). DOI 10.4134/JKMS.2014.51.2.267
31. Larsson, S., Thomée, V., Zhang, N.Y.: Interpolation of coefficients and transformation of the dependent variable in finite element methods for the nonlinear heat equation. *Math. Methods Appl. Sci.* **11**(1), 105–124 (1989). DOI 10.1002/mma.1670110108
32. López Marcos, J.C., Sanz-Serna, J.M.: Stability and convergence in numerical analysis. III. Linear investigation of nonlinear stability. *IMA J. Numer. Anal.* **8**(1), 71–84 (1988)
33. Moro, D., Nguyen, N.C., Peraire, J.: A hybridized discontinuous Petrov-Galerkin scheme for scalar conservation laws. *Internat. J. Numer. Methods Engrg.* **91**, 950–970 (2012)

34. Nguyen, N., Peraire, J., Cockburn, B.: An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier-Stokes equations. *J. Comput. Phys.* **230**, 1147–1170 (2011)
35. Nguyen, N.C., Peraire, J.: Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *J. Comput. Phys.* **231**, 5955–5988 (2012)
36. Nguyen, N.C., Peraire, J., Cockburn, B.: An implicit high-order hybridizable discontinuous Galerkin method for nonlinear convection-diffusion equations. *J. Comput. Phys.* **228**(23), 8841–8855 (2009). DOI 10.1016/j.jcp.2009.08.030. URL <https://doi.org/10.1016/j.jcp.2009.08.030>
37. Nguyen, N.C., Peraire, J., Cockburn, B.: A hybridizable discontinuous Galerkin method for the incompressible Navier-Stokes equations (AIAA Paper 2010-362). In: Proceedings of the 48th AIAA Aerospace Sciences Meeting and Exhibit. Orlando, Florida (2010)
38. Nguyen, N.C., Peraire, J., Cockburn, B.: A class of embedded discontinuous Galerkin methods for computational fluid dynamics. *J. Comput. Phys.* **302**, 674–692 (2015)
39. Oikawa, I.: A hybridized discontinuous Galerkin method with reduced stabilization. *J. Sci. Comput.* **65**, 327–340 (2015)
40. Oikawa, I.: Analysis of a Reduced-Order HDG Method for the Stokes Equations. *J. Sci. Comput.* **67**(2), 475–492 (2016)
41. Peraire, J., Nguyen, N.C., Cockburn, B.: A hybridizable discontinuous Galerkin method for the compressible Euler and Navier-Stokes equations (AIAA Paper 2010-363). In: Proceedings of the 48th AIAA Aerospace Sciences Meeting and Exhibit. Orlando, Florida (2010)
42. Sanz-Serna, J.M., Abia, L.: Interpolation of the coefficients in nonlinear elliptic Galerkin procedures. *SIAM J. Numer. Anal.* **21**(1), 77–83 (1984). DOI 10.1137/0721004
43. Schütz, J., May, G.: A hybrid mixed method for the compressible Navier-Stokes equations. *J. Comput. Phys.* **240**, 58–75 (2013)
44. Stenberg, R.: A family of mixed finite elements for the elasticity problem. *Numer. Math.* **53**, 513–538 (1988)
45. Stenberg, R.: Postprocessing schemes for some mixed finite elements. *RAIRO Modél. Math. Anal. Numér.* **25**, 151–167 (1991)
46. Terrana, S., Nguyen, N., Bonet, J., Peraire, J.: A hybridizable discontinuous galerkin method for both thin and 3d nonlinear elastic structures. *Comput. Methods Appl. Mech. Engrg.* **352**, 561–585 (2019)
47. Tourigny, Y.: Product approximation for nonlinear Klein-Gordon equations. *IMA J. Numer. Anal.* **10**(3), 449–462 (1990). DOI 10.1093/imanum/10.3.449
48. Wang, C.: Convergence of the interpolated coefficient finite element method for the two-dimensional elliptic sine-Gordon equations. *Numer. Methods Partial Differential Equations* **27**(2), 387–398 (2011). DOI 10.1002/num.20526
49. Wang, Z.: Nonlinear model reduction based on the finite element method with interpolated coefficients: semilinear parabolic equations. *Numer. Methods Partial Differential Equations* **31**(6), 1713–1741 (2015). DOI 10.1002/num.21961
50. Xie, Z., Chen, C.: The interpolated coefficient FEM and its application in computing the multiple solutions of semilinear elliptic problems. *Int. J. Numer. Anal. Model.* **2**(1), 97–106 (2005)
51. Xiong, Z., Chen, C.: Superconvergence of rectangular finite element with interpolated coefficients for semilinear elliptic problem. *Appl. Math. Comput.* **181**(2), 1577–1584 (2006). DOI 10.1016/j.amc.2006.02.040
52. Xiong, Z., Chen, C.: Superconvergence of triangular quadratic finite element with interpolated coefficients for semilinear parabolic equation. *Appl. Math. Comput.* **184**(2), 901–907 (2007). DOI 10.1016/j.amc.2006.05.192
53. Xiong, Z., Chen, Y., Zhang, Y.: Convergence of FEM with interpolated coefficients for semilinear hyperbolic equation. *J. Comput. Appl. Math.* **214**(1), 313–317 (2008). DOI 10.1016/j.cam.2007.02.023
54. Yu, Y., Chen, G., Pi, L., Zhang, Y.: A new ensemble HDG method for parameterized convection diffusion PDEs. *Numer. Math. Theor. Meth. Appl.* DOI 10.4208/nmtma.OA-2019-0190. To appear