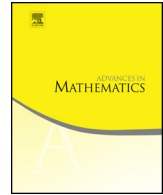Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Hilbert's third problem and a conjecture of Goncharov

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ARTICLE INFO

ABSTRACT

In this paper we reduce the generalized Hilbert's third problem about Dehn invariants and scissors congruence classes to the injectivity of certain Cheeger–Chern–Simons invariants. We also establish a version of a conjecture of Goncharov relating scissors congruence groups of polytopes and the algebraic K-theory of $\mathbf{C}$.

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Introduction

Hilbert's third problem asks the following question: given two polyhedra P and Q, when is it possible to decompose P into finitely many polyhedra and form Q out of the pieces? More formally, is it possible to write $P = \bigcup_{i=1}^{n} P_i$ and $Q = \bigcup_{i=1}^{n} Q_i$ such that $P_i \cong Q_i$ for all i, and such that $\mathrm{meas}(P_i \cap P_j) = \mathrm{meas}(Q_i \cap Q_j) = 0$ for all $i \neq j$? (Two polyhedra for which this is true are called *scissors congruent*.) The generalized version of Hilbert's third problem is the observation that this can be asked in any dimension and any geometry. The question then becomes: describe a complete set of invariants of scissors congruence classes of polytopes in a given dimension and geometry.

Let us briefly consider the classical version. If two polyhedra are scissors congruent then their volumes are equal. The reverse implication is not true; a second invariant, called the *Dehn invariant*, exists. For three-dimensional polyhedra (in Euclidean, spherical or hyperbolic space) this invariant is defined as follows:

$$D(P) = \sum_{e \text{ edge of } P} \mathrm{len}(e) \otimes \theta(e) \in \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}.$$

Here, $\theta(e)$ is the dihedral angle at e; in other words, it is the arc length of the intersection with P of a small circle around e. In dimension n it is possible to define other Dehn invariants, by picking a dimension ℓ and writing a similar sum over all faces of P of dimension ℓ; the measure of the angle will then be a portion of the sphere in dimension $n - \ell - 1$. By the Dehn–Sydler theorem [27,19] in Euclidean space in dimensions 3 and 4, two polytopes are scissors congruent if and only if their volumes and Dehn invariants are equal. Work of Dupont and Sah [11] extended this technique to 3-dimensional spherical and hyperbolic space, classifying the kernel of the Dehn invariant as a group homology group. We thus have the following question:

Generalized Hilbert's Third Problem *([11, Question 1]). In Euclidean, spherical, and hyperbolic geometries, do the volume and generalized Dehn invariant separate the scissors congruence classes of polytopes?*

Remark. Spherical polytopes are often used to measure angles. When a polytope is decomposed into smaller polytopes it produces extra angles, at all of the faces along the cuts. These newly produced angles always add up either to an entire sphere (if they are contained in the interior of the original polytope) or to some type of "flat" angle. Such flatness can be quantified by observing that such angles always arise from an angle in a lower-dimensional sphere; in spherical scissors congruence classes we thus declare all such angles to have "scissors congruence measure 0." The Dehn invariant, and thus the generalized version of Hilbert's third problem, can also be defined for polytopes up to such "measure-0" polytopes. See Definition 1.13.

An algebraic approach to Hilbert's Third Problem defines *scissors congruence groups associated to a geometry X*, $\mathcal{P}(X)$, which are free abelian groups generated by polytopes (in whichever geometry is under consideration) modulo "cutting" and translation by isometries. In addition, when X is a spherical geometry, it is also necessary to quotient by "suspension." (For a more formal definition, see Definition 1.12 and 1.13.) The Euclidean and hyperbolic volume and the generalized Dehn invariant can then be defined as homomorphisms of groups, and we see that the generalized Hilbert's Third Problem has a positive answer exactly when volume is injective when restricted to the kernel of the Dehn invariant.

Motivated by the theory of mixed Tate motives, in [17, Conjecture 1.7] Goncharov proposed the following method for solving generalized Hilbert's third problem. Let $D : \mathcal{P}(S^{2n-1}) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{P}(S^{2i-1}) \otimes \mathcal{P}(S^{2(n-i)-1})$ be the Dehn invariant on the spherical scissors congruence groups. The generalized Hilbert's third problem can then be rephrased to say that volume is injective when restricted to $\ker D$. Conjectures of Ramakrishnan [24, Conjectures 7.1.2, 7.1.8] imply that the Borel regulator produces an injective homomorphism

$$(\mathrm{gr}_n^{\gamma} K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \otimes (\mathbb{Q}^{\sigma})^{\otimes n})^{+} \longrightarrow \mathbb{R}/(2\pi)^n \mathbb{Q}.$$

Here, $\mathbb{Q}^{\sigma}$ is $\mathbb{Q}$ with an action of $\mathbb{Z}/2$ via the sign, $\mathrm{gr}_n^{\gamma} K_{2n-1}(\mathbb{C})$ is the n-th graded piece of the weight filtration on the algebraic K-theory of $\mathbb{C}$ (with action by $\mathbb{Z}/2$ induced by complex conjugation), and $\cdot^{+}$ denotes taking the fixed points of the action. (For a more detailed explanation of this, see Section 5; an in-depth understanding of the terms is not needed for the current discussion.) If it were possible to construct an injective map

$$\ker D \otimes \mathbb{Q} \longhookrightarrow (\mathrm{gr}_n^{\gamma} K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \otimes (\mathbb{Q}^{\sigma})^{\otimes n})^{+} \qquad (0.1)$$

such that the composition with the Borel regulator was equal to the volume, this would imply generalized Hilbert's third problem for spherical scissors congruence groups (at least modulo torsion, and assuming Ramakrishnan's conjecture). Goncharov also made an analogous form of this conjecture for the hyperbolic groups; here the Borel regulator takes values in $\mathbb{R}$. In his paper, Goncharov was able to construct a map of the form (0.1) once polytopes were restricted to polytopes with algebraic vertices and $\mathbb{C}$ was replaced by $\overline{\mathbb{Q}}$; however, he did not show that it is injective.

There are a couple of indications this initial version may not be the most useful form of the conjecture. Restricting to polytopes of a particular dimension restricts us to considering group homology for matrices of a set dimension; this is directly related to the *rank* filtration, rather than the γ-filtration. As the graded pieces of the rank and γ-filtrations are expected to be isomorphic (see, for example, [21, Conjecture 2.6.1] for an in-depth discussion) this is not a major change to the conjecture. Since $K_*(\mathbb{C})_{\mathbb{Q}}$ is isomorphic to the primitive elements in the Hopf algebra $H_*(BGL(\mathbb{C}); \mathbb{Q})$ (where $GL(\mathbb{C})$ is considered as a discrete group), the desired map (0.1) can be described as a

map into a quotient of certain group homology groups. The second is the observation that scissors congruence groups are constructed out of the group homology of orthogonal groups, rather than general linear groups, so it is more likely that the kernel of the Dehn invariant will be related to the group homology of orthogonal groups, rather than general linear groups. It turns out that the correct analog of the quotient in the orthogonal case is simply the groups $H_*(O(n; \mathbb{R}); \mathbb{Z}[\frac{1}{2}]^\sigma)$, where $\cdot^\sigma$ indicates that the group is acting via multiplication by the determinant. These also have a regulator, usually referred to as the Cheeger–Chern–Simons class, which agrees with the Beilinson (and thus Borel) regulator [9] and which is also expected to be injective.

With these two changes we can prove a reverse of Goncharov's conjecture:

Theorem A *(Theorem 5.3). Let D be the Dehn invariant for hyperbolic scissors congruence. There is a homomorphism*

$$H_d(O^+(1, d; \mathbb{R}); \mathbb{Z}[\tfrac{1}{2}]^\sigma) \longrightarrow \ker D$$

which, after composition with volume is equal to the Cheeger–Chern–Simons class. Thus, if this map is surjective and the Cheeger–Chern–Simons class is injective, volume and the Dehn invariant separate scissors congruence classes in hyperbolic geometry in all dimensions.

Here, $O^+(1, d)$ is the subgroup of $GL(1 + d)$ which preserves the hyperbolic quadratic form and the sign of the first coordinate. An analogous statement is true for the spherical case, although it is somewhat more complicated as volume is not well-defined on the spherical scissors congruence groups. For more details, see Theorem 5.4.

Remark. For unreduced spherical scissors congruence groups, generalized Hilbert's Third Problem is resolved modulo torsion (see [26, Proposition 6.3.22]). However, unreduced spherical scissors congruence groups do not appear to have nearly as many interesting applications as the reduced version, and in particular do not appear to relate to the homology of the group $O(d)$, and in this paper we therefore focus on the reduced case.

Goncharov's intuition about scissors congruence classes did not stop at the kernel of the Dehn invariant. He noticed that Dehn invariants can be iterated to produce a chain complex, denoted $\mathcal{P}_*(S^d)$. (See Section 3 for more details.) In [17],[1] he conjectures [17, Conjecture 1.8] that for odd d there exists a homomorphism

$$H_m(\mathcal{P}_*(S^d) \otimes \mathbb{Q}) \longrightarrow \left(\mathrm{gr}^\gamma_{\lfloor \frac{d+1}{2} \rfloor} K_{m + \lfloor \frac{d+1}{2} \rfloor}(\mathbb{C})_\mathbb{Q} \otimes (\mathbb{Q}^\sigma)^{\lfloor \frac{d+1}{2} \rfloor} \right)^+$$

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[1] Goncharov's indexing is somewhat different from ours, as he considers $\mathcal{P}_*(X)$ to be a cohomologically-graded complex starting in degree 1.

for all m. The techniques for proving Theorem A extend to proving a form of this conjecture, as well:

Theorem B *(Theorem 4.6). Let $X = S^d$ or $\mathcal{H}^d$, and let $I(X)$ be the isometry group of X. For all m there are homomorphims*

$$H_{m+\lfloor\frac{d+1}{2}\rfloor}(I(X), \mathbb{Z}[\tfrac{1}{2}]^\sigma) \longrightarrow H_m(\mathcal{P}_*(X)).$$

(Since $I(\mathcal{H}^d) \cong O^+(1, d)$ this is a generalization of Theorem A to the case when $m = \lfloor\frac{d-1}{2}\rfloor$.) In fact, Theorem 4.6 is shown for any field of characteristic 0, not just $\mathbb{R}$; for the definition of scissors congruence groups over a general field see Definition 1.22. It is also important to note that as the right-hand side will have nonzero homology only for $0 \le m \le \lfloor\frac{d-1}{2}\rfloor$, these maps can only be nontrivial in these cases.

The main tool allowing us to prove these theorems is the "geometrization" of the Dehn invariant: a topological model which is both *rigid* and *equivariant with respect to the isometry group of our geometry*. It is rigid in the sense that the structural properties that we desire of the Dehn invariant (described at the beginning of Section 2) already hold for the topological spaces, without having to work "up to homotopy" or "inside homology groups." It is equivariant in the sense that the Dehn invariant is a map of $I(X)$-spaces, rather than simply topological spaces.

The advantage of this construction is that the presence of higher homological information in the coinvariant computations leads to major cancellations. All of the complexity of $\mathcal{P}_*(X)$ is contracted into $\mathbb{Q}^\sigma$. Here, the key observation is that in a topological context homotopy coinvariants and the "total complex" that Goncharov uses to define $\mathcal{P}_*$ commute past one another; thus the rigid and equivariant construction of the Dehn invariant above can be used to explicitly determine the homotopy type of a space modeling this complex. We produce a spectral sequence whose lowest nonzero row is the complex $\mathcal{P}_*(S^d)$ (resp. $\mathcal{P}_*(\mathcal{H}^d)$); the cancellations allowing us to identify the homotopy type of the "total complex" allows us to directly relate this to the homology of the isometry group.

At the end of our analysis we illustrate the connection between our reformulation and Goncharov's original conjectures (see Proposition 5.10).

Outline of the Proof of Theorem B

The key ingredient in this proof is the repeated use of the notions of homotopy cofiber and homotopy coinvariants.

The Dehn complex is defined as the total complex of a cubical diagram in **AbGp** (Definition 3.6), each vertex of which is obtained by taking coinvariants (i.e. H_0) of an action on a Steinberg module. The total complex of a cube is the same as the total homotopy cofiber taken in the category of chain complexes (Example 3.4). Thus, to construct the Dehn complex, one takes homology of a group, and then takes a homotopy

colimit. This order feels unnatural from the point of view of homotopy theory, as one should first construct the space and then analyze its homology.

We begin by replacing each Steinberg module with a space (Definition 1.8), and taking homotopy coinvariants of the group action. The key step is the construction of an equivariant Dehn invariant (Definition 2.12), so that we can analyze the total homotopy cofiber of the original cubical diagram, prior to taking coinvariants. We can then commute the homotopy cofiber past the group action, and analyze them independently. We denote this space $(\mathcal{Y}^X)_{hI(X)}$ (it is defined in Section 4). Sections 1 and 2 are devoted to the construction of this cube.

In the homotopical analysis of $(\mathcal{Y}^X)_{hI(X)}$ a minor miracle occurs: the space is weakly equivalent to the homotopy coinvariants of a sphere with $I(X)$ acting on it (almost) trivially. We offer two proofs of this fact in Section 6. This allows for significant simplification of the spectral sequences that compute its homology groups. The spectral sequences that we use are known as the homotopy orbit spectral sequence (Proposition B.7) and the spectral sequence for the total homotopy cofiber of a cube (Section B.3). We can also use the homotopy orbit spectral sequence to compute $H_*((\mathcal{Y}^X)_{hI(X)})$ to obtain a shift of $H_*(I(X); \mathbb{Q}^\sigma)$; this implies Theorem B. An analysis of the edge homomorphism of the spectral sequence constructs the desired homomorphism in Theorem 4.6; an explicit description of the homomorphism in the case $* = n-1$ in Lemma 4.7 implies Theorem A.

Remark. In this paper we mostly focus on spherical and hyperbolic geometries, as well as work over $\mathbb{R}$ and $\mathbb{C}$, as these were our main examples of interest. However, most of our techniques do not rely on either these choices of geometry or the choice of field. In future work we hope to work out further implications of these approaches in other fields, geometries, and isometry groups. (The Euclidean case is an obvious candidate.)

Remark. What is especially striking about our approach is that most of the topological spaces we work with turn out to be homotopy-equivalent to bouquets of spheres. This means, in essence, that they are *combinatorial* objects, rather than topological. Despite this, the topological approach appears to produce significantly simpler proofs, and stronger results, than has previously appeared in the literature.

Organization

In Section 1 we introduce the basic objects of interest. Although many of the objects and definitions are standard, several key definitions (esp. RT-buildings) differ subtly from standard. We have attempted to highlight these differences in the exposition. Section 2 introduces derived Dehn invariants and states that they agree with the classical definitions; although the comparison between our objects and the classical objects is interesting (and we believe a good introduction to simplicial techniques) we postpone the direct comparison to Appendix A, as it is technical and completely disjoint from the main thrust of the paper. Section 3 recalls Goncharov's definition of the Dehn complex

and shows how to construct a "geometrized" model. Section 4 is the main meat of the topological story: it introduces the key theorem (Theorem 4.1) which allows us to directly compare the homology of the Dehn complex to the group homology of orthogonal groups. Section 5 proves Theorems A and B and explains the connection between Goncharov's original conjectures and the form in which they arise in this paper. Section 6 proves Theorem 4.1 and uses the proof to provide the computation for the claim about volume in Theorem A. This section is largely independent of much of the rest of the paper, dealing mostly with the structure of RT-buildings.

Notation and conventions

All groups considered in this paper are discrete.

We work in the category of **pointed** topological spaces and simplicial sets. Thus homology is reduced, and all constructions on spaces are pointed. In particular, homotopy G-coinvariants—denoted $\bullet_{hG}$—are taken in a correctly-pointed manner, so that $*_{hG} \simeq *$ and $(S^0)_{hG} \simeq (BG)_+$. Here, $\bullet_+$ denotes adding a disjoint basepoint, so that we can think of S^0 as $*_+$. Note the difference with the unpointed constructions: in the unpointed case $*_{hG} \simeq BG$. In general, in order to translate from the unpointed case to the pointed case one adds a disjoint basepoint and then works relative to that point. All groups in this paper are considered **discrete** unless explicitly stated otherwise, so that BG is always the Eilenberg–Mac Lane space $K(G,1)$.

Two spaces (resp. simplicial sets) are weakly equivalent if there exists a map $f : X \longrightarrow Y$ which is a bijection on connected components and such that the induced maps $f_* : \pi_n(X,x) \longrightarrow \pi_n(Y,f(x))$ are isomorphisms for all choices of basepoint x and all $n \geq 1$. When two spaces (resp. simplicial sets) X and Y are weakly equivalent, we denote this by $X \simeq Y$. Two simply-connected spaces are "weakly equivalent after inverting 2" if there exists a map $f : X \longrightarrow Y$ such that the induced maps $f_* : \pi_n(X,x) \otimes \mathbb{Z}[\frac{1}{2}] \longrightarrow \pi_n(Y,f(x)) \otimes \mathbb{Z}[\frac{1}{2}]$ are isomorphisms for all $n \geq 2$. We denote this by $X \simeq_{[2]} Y$.

The notation X/Y will refer to the quotient of spaces (resp. simplicial sets): the topological space (resp. simplicial set) given by collapsing all points to Y to a point. The only exception to this notation will be the group quotient $\mathbb{R}/\mathbb{Z}$ (and scalings thereof) and the group $\mathbb{Z}/2$.

We denote by X^d either d-dimensional hyperbolic ($\mathcal{H}^d$) or spherical (S^d) space. In each case, we think of X^d as sitting inside $\mathbb{R}^{d+1}$, with subspaces being cut out by subspaces of $\mathbb{R}^d$ through the origin (which intersect, respectively, the plane where $x_{d+1} = 1$, the hyperboloid $-x_0^1 + x_1^2 + \cdots + x_d^2$, and the sphere in a nonempty set). When the dimensions is clear from context we write X instead of X^d.

For any abelian group A, we write $A_{\mathbb{Q}} \overset{\text{def}}{=} A \otimes \mathbb{Q}$.

The field k is always assumed to have characteristic 0. (Although many of the techniques should work over finite characteristic, there are several key geometric points where rational coefficients are necessary.)

1. RT-buildings and scissors congruence groups

Our main goal in this section is to establish the basic definitions of the objects we will be using, as many of these definitions are not (quite) standard. Many small variations on these definitions exist in the literature (see, for example, [13, Chapter 2], [5]), leading to a combinatorial explosion of choices. In our experience only the current choices lead to a consistent rigid derived theory. We work over any base field k of characteristic 0.

1.1. RT-buildings

Definition 1.1 (*Based on [5, Definition 1.0.3]*). A *geometry over k*, X, is a vector space equipped with quadratic form (E, q) over k, where q is totally nondegenerate, together with its isometry group $I(X)$. The *dimension* of X is $\dim E - 1$. The isometry group must be a subgroup of $I(E)$, the subgroup of $GL(E; k)$ which preserves the quadratic form.

When we wish to emphasize that a geometry X has dimension d, we write it as X^d.

Definition 1.2. The *neat geometries* are the *spherical* geometry S^d, given by the quadratic form $x_0^2 + \cdots + x_d^2$, and the *hyperbolic* geometry $\mathcal{H}^d$, given by the quadratic form $-x_0^2 + x_1^2 + \cdots + x_d^2$. The isometry group of the spherical geometry of dimension d is $O(d+1)$. The isometry group of the hyperbolic geometry is $O(1, d)$ if k is unordered, and $O^+(1, d)$ if k is ordered; here, $O^+(1, d)$ is the subgroup of $O(1, d)$ of those transformations that preserve the sign of the first coordinate.

When it is not clear from context, we write S_k^d or $\mathcal{H}_k^d$ to emphasize that the geometries are over k.

In later sections, we will often be considering maps of the form $X^d \longrightarrow X^a \widetilde{\star} S^b$ (where $\widetilde{\star}$ is defined in Definition 2.7). A map of this sort states that we fix a type of geometry

(spherical or hyperbolic), and both X's are of this same type, of dimensions d and a, respectively.

Definition 1.3. Let $\mathbb{P}(E)$ be the projectivization of E. For a geometry $X = (E, q)$, where q has signature (n_-, n_+), a *subspace* U of X is a subset of $\mathbb{P}(E)$ corresponding to a linear subspace V of E such that the restriction of q to V is totally nondegenerate and such that the signature (m_-, m_+) of $q|_V$ has $m_- = n_-$. The isometry group $I(U)$ is the subgroup of $I(X)$ that preserves U. Note that a subspace of a neat geometry is another neat geometry of the same type.

An *angular-subspace* U of X is a linear subspace V of E such that the restriction of q to V is totally nondegenerate and such that the signature (m_-, m_+) of $q|_V$ has $m_- = 0$.

For any subspace or angular-subspace U of X we say that U is *represented* by V. The *dimension* of U is $\dim V - 1$; if $\dim U = 0$ we refer to U as a *point* of X.

When k contains $\sqrt{-1}$, the condition on the signature is vacuous and subspaces and angular-subspaces are equivalent.

Remark 1.4. The condition on the signature may appear artificial, but it is necessary in order to model the types of subspaces in question. A geometry X of dimension d can be considered to be sitting inside k^{d+1} as a submanifold. In the case when k is not algebraically closed, a plane of dimension m may not intersect this submanifold in a subspace of dimension $m - 1$, as desired. The condition on the signature ensures that this will happen in the cases of hyperbolic and spherical geometry.

Remark 1.5. Many of the definitions and results in this paper will also work for the Euclidean geometry, as well as for geometries with signatures other than $(0, d+1)$ and $(1, d)$. However, there are enough subtleties and differences between these cases that in this paper we focus exclusively on the spherical and hyperbolic cases.

The key structure necessary for the program is the presence of an orthogonal complement for any subspace and the notion of a projection onto the orthogonal complement.

Definition 1.6. Let U be an i-dimensional subspace of X, represented by a linear subspace V of E. We define the orthogonal complement $U^\perp$ of U to be the angular-subspace represented by $V^\perp$.

If U is a subspace and U' is an angular-subspace of X, represented by V and V', then we write $U \perp U'$ if $V \perp V'$. We write $U \oplus U'$ for the subspace represented by $V \oplus V'$. If $V \perp V'$ we write $U \perp U'$ instead of $U \oplus U'$ to emphasize this fact.

For subspaces $U \subseteq U'$ of X, we write

$$\mathrm{pr}_{U^\perp} U' \overset{\mathrm{def}}{=} U' \cap U^\perp.$$

The isometry group of $\mathrm{pr}_{U^\perp} U'$ is taken to be the subgroup of the isometry group of U' that fixes U.

We will be using the following three properties of subspaces:

Lemma 1.7. *Let X^d be a neat geometry and let U^i be a subspace of X. Then $\dim U^\perp = d - i - 1$. For a subspace U' containing U, U' is uniquely determined by U and $U^\perp \cap U'$. In addition, the induced quadratic form on $\mathrm{pr}_{U^\perp} U'$ has positive signature.*

The key object of study in this paper is the *RT-building* associated to a geometry X.[2]

Definition 1.8. Let X be a geometry of dimension d over k.

Let $T^m_\bullet(X)$ be the simplicial set whose i-simplices are sequences $U_0 \subseteq \cdots \subseteq U_i$ of nonempty subspaces of X of dimension at most m. The j-th face map deletes U_j; the j-th degeneracy repeats U_j. The isometry group $I(X)$ acts on $T^m_\bullet(X)$.

We define the *RT-building* of X to be the pointed simplicial set given by

$$F^X_\bullet \overset{\mathrm{def}}{=} T^d_\bullet(X)/T^{d-1}_\bullet(X),$$

with the inherited $I(X)$-action. More explicitly, the non-basepoint i-simplices of $F^X_\bullet$ are sequences $U_0 \subseteq \cdots \subseteq U_i$, where each U_j is a nonempty subspace of X and $U_i = X$. The face maps and degeneracies work as before, with the caveat that if $U_{i-1} \neq X$ then d_i sends the simplex $U_0 \subseteq \cdots \subseteq U_i$ to the basepoint.

It turns out that the group $\widetilde{H}_d(F^{S^d}_\bullet)$ contains vital information about scissors congruence. In fact, this is the only nonzero homology group of this space:

Proposition 1.9. *For $i \neq \dim X$, $\widetilde{H}_i(F^X_\bullet) \cong 0$.*

The fact that all homology groups above $\dim X$ are 0 is evident from the fact that all nondegenerate simplices have length at most $d + 1$. The fact that all (reduced) homology groups below degree d are also 0 is more complicated; one can refer to the Solomon–Tits Theorem [23, Section 2], or use the theory developed in Appendix A. As the proof is technical and not illuminating, we defer it to the appendix.

1.2. Classical scissors congruence

We turn our attention to defining the scissors congruence groups. For scissors congruence to be defined we need a notion of a geometry to work within, as well as a notion of "inside" and "outside" for polytopes; thus we will need to be working inside an ordered field. For now we fix $k = \mathbb{R}$, although most of the machinery developed should work equally well over other ordered fields.

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[2] The term "RT-building" is named after Rognes and Tits, as our objects are "halfway" between Tits' original objects—which must start at a nonempty subspace and end at a proper subspace—and Rognes' spaces $D^1(V)$, which have simplices which start at the trivial subspace and end at the full space.

The basic building block of a polytope (and thus of a scissors congruence group) is a simplex, which can be defined as a convex hull.

Definition 1.10. Suppose X is a neat geometry of dimension n.

A *convex hull* of a tuple $(a_0, \ldots, a_m)$ of points in X is any subset of X represented by a cone over b_i

$$\left\{ \sum_{i=0}^{m} c_i b_i \in \mathbb{R}^{d+1} \,\middle|\, c_i \geq 0 \ \forall i \right\}$$

for any choice of $0 \neq b_i \in a_i$ for all i.

An *m-simplex* in X is the convex hull of a tuple $(a_0, \ldots, a_m)$ which is not contained in an $m-1$-dimensional subspace of X. An *m-polytope* in X is a finite union of m-simplices; we make no assumptions of convexity or connectedness. When $m = d$ we omit it from the terminology and refer simply to "simplices in X" or "polytopes in X."

Remark 1.11. When X is hyperbolic, a simplex is uniquely determined by its vertices in the following sense. The hyperboloid $-x_0^2 + x_1^2 + \cdots + x_d^2$ has two connected components, and we think of X as one of these components and a point of X as the intersection of the representing line with this component. A tuple of points $(a_0, \ldots, a_d)$ in X thus defines a tuple of vectors in $\mathbb{R}^{d+1}$, and thus the positive cone above is well-defined.

When X is spherical, there are 2^{d+1} possible choices of "sign" of the representatives b_i. Thus a simplex is no longer uniquely defined by its vertices.

We can now define the scissors congruence group of X:

Definition 1.12. Let X be a neat geometry over $\mathbb{R}$, and let G be a subgroup of $I(X)$. Then the *scissors congruence group of X relative to G*, denoted $\widehat{\mathcal{P}}(X, G)$, is the free abelian group generated by polytopes in X modulo the relations

- $[P \cup Q] = [P] + [Q]$ if $P \cap Q$ is contained in a finite union of $m-1$-dimensional subspaces.
- $[P] = [g \cdot P]$ for any $g \in G$. Here g acts on P pointwise; as it is in $I(X)$ it takes convex hulls to convex hulls.

When $G = I(X)$ we omit it from the notation.

The case $X = S^d$ is more complicated, as there are now 2^{d+1} choices for any convex hull, depending on how the points b_i are chosen. The following definition will make all of the choices in Definition 1.10 equivalent.

Definition 1.13. For any G-module M, the *coinvariants* of G acting on M are defined to be the group

$$M/(m - g \cdot m \,|\, g \in G, \ m \in M).$$

This is isomorphic to the zeroth group homology $H_0(G, M)$.

Let

$$\Sigma: \bigoplus_{\substack{V \subseteq \mathbb{R}^{d+1} \\ \dim V = n}} \widehat{\mathcal{P}}(V \cap S^d, 1) \longrightarrow \widehat{\mathcal{P}}(S^d, 1) \tag{1.14}$$

be the "suspension" map taking a simplex in $V \cap S^n$ to the union of the two simplices defined by the choice of representatives in $V^{\perp}$. Denote by $\mathcal{P}(S^d, G)$ the cokernel of the induced map

$$\Sigma: H_0\!\left(G, \bigoplus_{\substack{V \subseteq \mathbb{R}^{n+1} \\ \dim V = d}} \widehat{\mathcal{P}}(V \cap S^d, 1)\right) \longrightarrow H_0(G, \widehat{\mathcal{P}}(S^d, 1)).$$

When $X \neq S^d$ we define $\mathcal{P}(X, G) \overset{\text{def}}{=} \widehat{\mathcal{P}}(X, G)$.

The cokernel of Σ turns out to be the more "correct" notion of scissors congruence of the sphere, as it is most often used to measure angles. A subdivision of a polytope adds many angle measures that add up to the entire sphere; thus, in order to make our definitions treat subdivisions correctly, the entire sphere should be considered to be zero. When we discuss the Dehn invariant in Section 2 this will become clearer, as Dehn invariants are only well-defined inside these reduced scissors congruence groups.

Remark 1.15. The notation we are using is somewhat nonstandard. The group $\widehat{\mathcal{P}}(X, G)$ is usually denoted $\mathcal{P}(X, G)$, and the group $\mathcal{P}(S^d, G)$ is generally denoted $\widetilde{\mathcal{P}}(S^d, G)$. In this paper, however, the group of interest is $\P(S^d, G)$ for $X = S^d$, and we would like to unify the notation so that this group is the default one.

As motivation for considering scissors congruence as homotopy coinvariants we observe that

$$\mathcal{P}(X, G) \cong H_0(G, \mathcal{P}(X, 1)) \tag{1.16}$$

and that this works for the groups $\widehat{\mathcal{P}}(S^d, G)$ as well.

To end this subsection we mention a classic result which underlies several key steps in the paper:

Lemma 1.17 ([13, Corollary 2.5]). *When $k = \mathbb{R}$, the groups $\mathcal{P}(S^d)$ are 2-divisible.*

1.3. Geometrizing a twist

In the classical literature on scissors congruence, flags often take the place of polytopes (thus motivating our study of RT-buildings). However, a complication arises: in the algebraic story, the action of the isometry on flags is twisted by the determinant map, so that $g \cdot [x] = (\det g)[g \cdot x]$. In order to construct a topological model of such a twist, we need a topological model for "tensoring with a copy of $\mathbb{Z}$ with the sign action."

Definition 1.18. Let S^1 be the pointed simplicial set $\Delta^1/\partial\Delta^1$.

Let S^σ be the pointed simplicial set $\Delta^1 \cup_{\partial\Delta^1} \Delta^1$, with one of the vertices in $\partial\Delta^1$ taken to be the basepoint. This is a model of a circle with two 0-simplices and two 1-simplices. There is an action of $\mathbb{Z}/2$ on S^σ given by swapping the two 1-simplices.

Remark 1.19. The notation S^σ is chosen to be compatible with the standard notation M^σ for a G-module which is twisted by the action of a "sign" map $G \longrightarrow \mathbb{Z}/2$.

The group $I(X)$ acts on S^σ via the map $\det : I(X) \longrightarrow \mathbb{Z}/2$. Note that $H_d(F_\bullet^X) \cong H_{d+1}(S^\sigma \wedge F_\bullet^X)$ as groups. As $I(X)$-modules, these differ only by the action on S^σ, which adds a twist $\bullet^\sigma$ by the determinant. In particular, this means that

$$H_0(G, H_d(F_\bullet^X)^\sigma) \cong H_0(G, H_{d+1}(S^\sigma \wedge F_\bullet^X)).$$

In other words, the $I(X)$-coinvariants of $H_{d+1}(S^\sigma \wedge F_\bullet^X)$ are exactly the "$I(X)$-semi-coinvariants" in $H_d(F_\bullet^X)$. From Proposition 1.9, the homotopy orbit spectral sequence (see Proposition B.7) collapses, and we have

$$H_0(G, H_{d+1}(S^\sigma \wedge F_\bullet^X)) \cong H_{d+1}((S^\sigma \wedge F_\bullet^X)_{hG}).$$

For a detailed discussion of homotopy orbits, see Appendix B.2.

Remark 1.20. It may seem that the approach of "geometrizing" $\bullet^\sigma$ by smashing with S^σ produces a spurious increase of dimension. However, this increase is present in the algebraic story (discussed in [26, Section 6.1], [17], and others) as well: the scissors congruence groups for $S^{n-1} \subseteq \mathbb{R}^n$ must be graded by the ambient dimension n, rather than $n-1$, in order to make the Dehn invariant a graded homomorphism. In addition, Section 5 shows that these dimensions allow the maps from K-theory to have the correct grading. It is thus unsurprising that something of this sort should appear in the topological viewpoint.

1.4. The geometrization of scissors congruence groups

We now state the connection between scissors congruence and RT-buildings:

Theorem 1.21. *Suppose $k = \mathbb{R}$. Let X have dimension n and let G be a subgroup of the isometry group of X. For a neat geometry X,*

$$\mathcal{P}(X, G) \cong H_{d+1}((S^\sigma \wedge F_\bullet^X)_{hG}) \cong H_0(G, H_{d+1}(S^\sigma \wedge F_\bullet^X)).$$

This map is induced by the G-equivariant map $\mathcal{P}(X, 1) \longrightarrow H_{d+1}(S^\sigma \wedge F_\bullet^X)$ which takes a simplex with vertices $\{x_0, \ldots, x_d\}$ to the sum

$$\sum_{\sigma \in \Sigma_{d+1}} \mathrm{sgn}(\sigma)[\mathrm{span}(x_{\sigma(0)}) \subseteq \mathrm{span}(x_{\sigma(0)}, x_{\sigma(1)}) \subseteq \cdots \subseteq \mathrm{span}(x_{\sigma(0)}, \ldots, x_{\sigma(d)})].$$

As the proof of this theorem is technical and not illuminating, we postpone it until Appendix A. A similarly-simple model for $\widehat{\mathcal{P}}(S^d, G)$ is not known.

The theorem above implies that scissors congruence information is contained inside $H_{d+1}(S^\sigma \wedge F_\bullet^X) \cong H_d(F_\bullet^X)^\sigma$. The value added by the topology is that when $G \neq 1$ the space $(S^\sigma \wedge F_\bullet^X)_{hG}$ contains nontrivial higher homological information (see for example Lemma 1.25), and thus remembers more about the algebra of G than the left-hand side.

Inspired by Theorem 1.21 we can now define generalized scissors congruence groups:

Definition 1.22. Let X be a geometry over k, and let $G \leq I(X)$. The *scissors congruence group of X*, written $\mathcal{P}(X, G)$, is

$$\mathcal{P}(X, G) \overset{\mathrm{def}}{=} H_{d+1}((S^\sigma \wedge F_\bullet^X)_{hG}).$$

When $G = I(X)$ we omit it from the notation.

Remark 1.23. At this point it may be tempting to think that since all spaces under consideration are simply-connected, the current topological model can contain no information that is not contained in $\mathcal{P}(X, G)$. However, this misses the important point that we are keeping track not only of the group, but also of the G-action. Taking orbits on the level of homology is the "underived" model, which cannot keep track of higher homotopical information. Taking homotopy coinvariants will remember this higher information, analogously to the way that taking homotopy coinvariants of G acting on a point produces BG, which has many interesting homology groups (even though a point does not).

A large part of the value in these models is that we can define a *rigid* topological model of the Dehn invariant. This allows us to use simple topological techniques to prove Theorem 4.1, none of which work in the case where we are working up to homotopy.

As a first application of the topological viewpoint, we show that when $X = S^{2n}$ there is no interesting scissors congruence information:

Proposition 1.24. *When $n \geq 0$, after inverting 2,*

$$(S^\sigma \wedge F_\bullet^{S^{2n}})_{hO(2n+1)} \simeq_{[2]} *.$$

In other words, the left-hand side is connected and for $i \geq 1$,

$$\pi_i((S^\sigma \wedge F_\bullet^{S^{2n}})_{hO(2n+1)}) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong H_i\left((S^\sigma \wedge F_\bullet^{S^{2n}})_{hO(2n+1)}; \mathbb{Z}[\tfrac{1}{2}]\right) \cong 0.$$

In particular, for $n > 0$, $\mathcal{P}(S^{2n}) = 0$.

Proof. First consider the case when $n > 0$. The matrix $-I \in O(2n+1)$ acts on all homology groups of $S^\sigma \wedge F_\bullet^{S^{2n}}$ by -1. Thus by the "Center Kills Lemma" [13, Lemma 5.4], 2 annihilates $H_i(O(2n+1), \widetilde{H}_{2n}(S^\sigma \wedge F_\bullet^{S^{2n}}; \mathbb{Z}[\tfrac{1}{2}]))$ for all i; since we have inverted 2, these must all be 0. By the homotopy orbit spectral sequence (Proposition B.7),

$$\widetilde{H}_*\left((S^\sigma \wedge F_\bullet^{S^{2n}})_{hO(2n+1)}; \mathbb{Z}[\tfrac{1}{2}]\right) = 0$$

for all i. Since the simplicial set $(S^\sigma \wedge F_\bullet^{S^{2n}})_{hO(2n+1)}$ is simply-connected (as suspensions and homotopy coinvariants commute), it must be contractible after inverting 2.

When $n = 0$ the situation is somewhat more complicated. In this case, $F_\bullet^{S^0} = S^0$, with one non-basepoint 0-simplex represented by the subspace S^0 and no other nondegenerate simplices. Then $S^\sigma \wedge S^0 = S^\sigma$, with $O(1) = \mathbb{Z}/2$ acting on it via the sign representation. By definition, $(S^\sigma)_{h\mathbb{Z}/2} \cong B\mathbb{Z}/2$, which is a nilpotent space. Thus after inverting 2 its homotopy groups are trivial, as desired.

The last part of the proposition follows because by [13, Corollary 2.5], $\mathcal{P}(S_\mathbb{R}^{2n})$ is 2-divisible, so inverting 2 does not affect the lowest homology group. $\square$

In general, the homology of $(S^\sigma \wedge F_\bullet^X)_{hG}$ can be described in terms of the homology of G and the homology of $F_\bullet^X$:

Lemma 1.25. *Let X be a neat geometry of dimension d. For all i,*

$$\widetilde{H}_i\big((S^\sigma \wedge F_\bullet^X)_{hG}\big) \cong H_{i-(d+1)}\big(G, \widetilde{H}_d(F_\bullet^X)^\sigma\big),$$

where $\cdot^\sigma$ denotes that the action of G on the homology is twisted by multiplication by the determinant. In particular, if $i - (d+1)$ is negative the left-hand side is 0.

Proof. This follows directly from the homotopy orbit spectral sequence (see Proposition B.7). Since the reduced homology of $S^\sigma \wedge F_\bullet^X$ is concentrated in degree $d+1$, the homotopy orbit spectral sequence is contained in the d-th column, and thus collapses. This implies that

$$\widetilde{H}_i((S^\sigma \wedge F_\bullet^X)_{hG}) \cong H_{i-(d+1)}(G, \widetilde{H}_{d+1}(S^\sigma \wedge F_\bullet^X)).$$

Using that $\widetilde{H}_{d+1}(S^\sigma \wedge F_\bullet^X) \cong \widetilde{H}_d(F_\bullet^X)^\sigma$ as a G-module completes the proof. $\square$

2. Rigid derived Dehn invariants

The statement (rephrased in modern terminology) of Hilbert's third problem is extremely simple:

Do there exist two polyhedra with the same volume which are not scissors congruent?

The answer, given in 1901 by Dehn is "yes": the cube and regular tetrahedron are not scissors congruent, even if they have the same volume. Dehn proved this statement by constructing a second invariant of polyhedra (these days called the "Dehn invariant") which is zero on a cube and nonzero on any regular tetrahedron. This invariant takes values in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$—a difficult group to work in, but even more startling given that tensor products were only originally defined in 1938. In 1965, Sydler proved that the volume and the Dehn invariant uniquely determine scissors congruence classes; phrased in a more modern fashion (after [19]), this is equivalent to stating that the volume map is injective when restricted to the kernel of the Dehn invariant.

2.1. The classical story: constructing an equivariant Dehn invariant

In this section we give a definition of the classical Dehn invariant (extended to arbitrary dimensions in the form proposed by Sah in [26]) and construct a derived model (a "geometrization"). The homological inspiration for our construction is Cathelineau's approach in [4–6].

Definition 2.1. Let X^d be a neat geometry over $\mathbb{R}$, and consider $\mathcal{P}(X)$. For any integer $0 < i < n$, we define the *i-th classical Dehn invariant* in the following manner. Since $\mathcal{P}(X)$ is generated by simplices, it suffices to define it on simplices. For a simplex σ in X with vertices $\{x_0, \ldots, x_d\}$, we define

$$\widehat{D}_i(\sigma) = \sum_{\substack{J \sqcup J' = \{0,\ldots,d\} \\ |J|=i+1 \\ U = \text{span } x_J}} [x_J] \otimes [\text{pr}_{U^\perp}(x_{J'})] \in \mathcal{P}(X^i, I(X^i)) \otimes \mathcal{P}(S^{d-i-1}, I(S^{d-i-1})).$$

Here, x_J is the set $\{x_j \mid j \in J\}$, $[x_J]$ is the class of the simplex with vertices x_J in an isometric copy of X^i sitting inside X^d, and $[\text{pr}_{U^\perp} x_{J'}]$ is the class in $\mathcal{P}(S^{d-i-1}, I(S^{d-i-1}))$ of the simplex spanned by the projections of the $x_{J'}$. For a more detailed discussion of this, see [26, Section 6.3].

This generalization of the Dehn invariant allows for the following question, the "generalized Hilbert's third problem":

Question 2.2 *(Generalized Hilbert's third problem).* *In a neat geometry X^d, is volume injective when restricted to the kernel of $\bigoplus_{i=1}^{d} \widehat{D}_i$?*

Remark 2.3. There is an important subtlety which is often overlooked in the definition of scissors congruence groups. Although volume is well-defined on $\mathcal{P}(\mathcal{H}^d)$ and $\widehat{\mathcal{P}}(S^d)$,[3] it is *not* well-defined on $\mathcal{P}(S^d)$: inside $\mathcal{P}(S^d)$ we quotient out by *lunes*, which are polytopes which are "suspensions" of lower-dimensional polytopes. Since lunes of any volume can be constructed, the volume map on $\widehat{\mathcal{P}}(S^d)$ does not descend to a well-defined map out of $\mathcal{P}(S^d)$ for $d > 1$. (When $d = 1$ all lunes are semicircles, and thus length is well-defined mod π.)

The Dehn invariant is not well-defined if $\mathcal{P}(X)$ is replaced by $\mathcal{P}(X, 1)$, as there is no natural way to consider $[x_I]$ as sitting inside $\mathcal{P}(X^i, 1)$. Since the goal is to postpone taking the coinvariants of $I(X)$ for as long as possible in order to model D_i on an RT-building, it is necessary to construct the Dehn invariant as the functor of $I(X)$-coinvariants applied to an equivariant homomorphism of $I(X)$-modules; this motivates the later geometric construction. To define a map

$$\widehat{D}_U : \mathcal{P}(X, 1) \longrightarrow \mathcal{P}(U, 1) \otimes \mathcal{P}(U^\perp, 1)$$

consider a simplex σ with vertices $\{x_0, \ldots, x_n\}$ in X, and suppose that $U = \mathrm{span}(x_0, \ldots, x_i)$. Let τ be the projection of $\{x_{i+1}, \ldots, x_n\}$ to $U^\perp$. Define

$$\widehat{D}_U([x_0, \ldots, x_n]) \overset{\mathrm{def}}{=} [x_0, \ldots, x_i] \otimes [\tau].$$

For any simplex $\{x_0, \ldots, x_d\}$ such that there do not exist $0 \le j_0 < \cdots < j_i \le d$ with $U = \mathrm{span}(x_{j_0}, \ldots, x_{j_i})$, define

$$\widehat{D}_U([x_0, \ldots, x_d]) \overset{\mathrm{def}}{=} 0.$$

Lemma 2.4. *With this definition,*

$$\mathcal{P}(X, 1) \xrightarrow{\ \oplus \widehat{D}_U\ } \bigoplus_{\substack{U \subseteq X \\ \dim U = i}} \mathcal{P}(U, 1) \otimes \mathcal{P}(U^\perp, 1)$$

is well-defined and $I(X)$-equivariant. After taking $I(X)$-coinvariants this map becomes $\widehat{D}_i$.

Proof. To prove well-definedness it suffices to show that for any simplex σ all but finitely many of the $\widehat{D}_U$ are 0. This is true because there are only finitely many subspaces U which are the span of a subset of $\{x_0, \ldots, x_d\}$.

---

[3] and also $\mathcal{P}(E^d)$, although E^d is not a neat geometry.

The action of $I(X)$ on the left is simply an action on tuples. The action on the right is a bit more complicated: it acts on the indices of the sum, and acts within each group, as well. However, simplices in U can be thought of as i-simplices in X that happen to be contained in U, on each individual simplex the action is via acting on each vertex of the simplex; in this way the right-hand side is considered to be sitting inside $\bigoplus_{\substack{U \subseteq X \\ \dim U = i}} \mathcal{P}(X, 1) \otimes \mathcal{P}(X, 1)$.

It remains to check that taking $I(X)$-coinvariants makes this map into $\widehat{D}_i$. The left-hand side becomes $\mathcal{P}(X)$. Moreover, $I(X)$ identifies all of the summands on the right-hand side, and the stabilizer of any fixed U is $I(U) \times I(U^\perp)$; thus the right-hand side is $H_0(I(U) \times I(U^\perp), \mathcal{P}(U, 1) \otimes \mathcal{P}(U^\perp, 1))$. We have an isomorphism [3, Section V.2]

$$H_0(I(U), \mathcal{P}(U, 1)) \otimes H_0(I(U^\perp), \mathcal{P}(U^\perp, 1)) \longrightarrow H_0(I(U) \times I(U^\perp), \mathcal{P}(U, 1) \otimes \mathcal{P}(U^\perp, 1)),$$

induced by the cross-product in homology, as the group $\mathcal{P}(U, 1)$ is free (by [13, Theorem 3.5]). Thus the right hand side is $\mathcal{P}(U) \otimes \mathcal{P}(U^\perp)$, as desired.

To see that the map is exactly $\widehat{D}_i$, note that taking the $I(X)$-coinvariants adds up the images of all nonzero $\widehat{D}_U$ on a given simplex σ; this is exactly the definition of $\widehat{D}_i$. □

The classical Dehn invariant can be iterated in the following sense. Suppose that $i < j$; then the following square commutes:

$$
\begin{array}{ccc}
\mathcal{P}(X^d) & \xrightarrow{\widehat{D}_i} & \mathcal{P}(X^i) \otimes \mathcal{P}(S^{d-i-1}) \\
\downarrow{\widehat{D}_j} & & \downarrow{\mathrm{id} \otimes \widehat{D}_{j-i}} \\
\mathcal{P}(X^j) \otimes \mathcal{P}(S^{d-j-1}) & \xrightarrow{\widehat{D}_i \otimes \mathrm{id}} & \mathcal{P}(X^i) \otimes \mathcal{P}(S^{j-i-1}) \otimes \mathcal{P}(S^{d-j-1}).
\end{array}
\tag{2.5}
$$

Goncharov uses this observation to construct a chain complex of Dehn invariants which he conjectures is related to algebraic K-theory. For a discussion of this, see Section 3.

2.2. The derived construction

The goal is to construct a notion of the Dehn invariant on $F_\bullet^X$ which will produce the classical Dehn invariant when $k = \mathbb{R}$, but only *after* taking coinvariants and homology. The idea that the Dehn invariant should be constructed in this manner is key to making the analysis in Section 4 possible. It is also the only perspective from which it appears to be possible to construct the derived Dehn invariant; the authors attempted many non-equivariant constructions before settling on this approach. That this makes the Dehn invariant concise and clean and exposes its combinatorial nature is a minor miracle.

The key idea here is to replace the tensor product of abelian groups with the *reduced join* of simplicial sets.

Remark 2.6. Classically, the tensor product would be replaced by the smash product, and choosing instead the reduced join may appear to be a perverse choice: the reduced join of simplicial sets is not symmetric in the category of simplicial sets, and using it to model a symmetric structure like the tensor product feels unnatural. And, as above, the authors spent considerable time on attempts to rework this material using a smash product. Unfortunately (or perhaps incredibly interestingly), it does not seem possible to construct a topological model of the Dehn invariant using a smash product of spaces.[4] (See also Remark 1.20.)

An interesting corollary of this is that the constructions in this section are fundamentally *unstable*. Smash products of spaces lift naturally to smash products of spectra, and therefore give some hope that analogous constructions could be lifted to stable models of scissors congruence (such as those arising from [8,29]). Unfortunately, this does not appear to be the case, and the natural questions arise: how stable is the Dehn invariant? What parts of it can be seen stably? And which portions are irredeemably unstable?

Definition 2.7. For pointed simplicial sets X and Y, the *reduced join* $X \mathbin{\widetilde{\star}} Y$ is defined by

$$(X \mathbin{\widetilde{\star}} Y)_m = \bigvee_{i+j=m-1} X_i \wedge Y_j.$$

For a simplex $(x, y) \in X_i \wedge Y_j$, the face maps d_ℓ are defined to be $d_\ell \times 1 : X_i \wedge Y_j \longrightarrow X_{i-1} \wedge Y_j$ when $\ell \leq i$, and $1 \times d_{\ell-i-1} : X_i \wedge Y_j \longrightarrow X_i \wedge Y_{j-1}$ otherwise. If $i = \ell = 0$ or $j = m - 1 - \ell = 0$ then the face map takes the simplex to the basepoint. Degeneracies are defined analogously, with the first $i + 1$ acting on the x-coordinate, and the last $m - i - 1$ acting on the y-coordinate. Note that this structure makes the reduced join *asymmetric*.

For those unfamiliar with reduced joins, an introduction and proofs of the most relevant properties of the reduced join are in Section B.1. The most important feature of reduced joins is their relationship to smash products; the proof is given in Section B.1:

Lemma 2.8 *(Lemma B.4). Let X and Y be pointed simplicial sets. There exists a simplicial weak equivalence*

$$S^1 \wedge X \wedge Y \longrightarrow X \mathbin{\widetilde{\star}} Y.$$

To define Dehn invariants on $F_\bullet^X$ (Definition 1.8) the first step is, as above, to define a Dehn invariant indexed by a single subspace.

---

[4] Indeed, it seems that this may not be possible with any symmetric notion of product.

Definition 2.9. Let U be a proper nonempty subspace of X. Define the *rigidly derived Dehn invariant relative to* U, $D_U : F_\bullet^X \longrightarrow F_\bullet^U \widetilde{\divideontimes} F_\bullet^{U^\perp}$ by

$$D_U(U_0 \subseteq \cdots \subseteq U_n) = \begin{cases} * & \text{if } \nexists\, j \text{ s.t. } U_j = U \\ (U_0 \subseteq \cdots \subseteq U_j) \wedge (\mathrm{pr}_{U^\perp}\, U_{j+1} \subseteq \cdots \subseteq \mathrm{pr}_{U^\perp}\, U_n) \\ \quad \text{if } j = \max\{i \,|\, U_i = U\}. \end{cases}$$

We call a j as above the *U-pivot* of $U_0 \subseteq \cdots \subseteq U_n$.

That D_U is compatible with the simplicial structure is direct from the definitions. Observe that if a U-pivot exists then it is strictly less than n, since $U \neq X$.

Lemma 2.10. *Let U be a proper nonempty subspace of X, and write $I(X, U)$ for the subgroup of $I(X)$ of those elements fixing U. The group $I(X, U)$ acts on $U^\perp$ and D_U is $I(X, U)$-equivariant.*

Proof. The first claim follows from the definition of orthogonal complement. To check equivariants it suffices to check that for any $g \in I(X, U)$, the map D_U commutes with the action of g. If a U-pivot exists then the action of g passes to $F_\bullet^{U^\perp}$, and thus commutes with D_U. If no U-pivot exists then this is also true after applying g; since g fixes the basepoint the action of g commutes with D_U. $\square$

This derived Dehn invariant can also be iterated on the nose, analogously to (2.5):

Lemma 2.11. *Let $U \subsetneq V$ be proper nonempty subspaces of X. Then the following diagram commutes:*

$$\begin{array}{ccc} F_\bullet^X & \xrightarrow{\quad D_U \quad} & F_\bullet^U \widetilde{\divideontimes} F_\bullet^{U^\perp} \\ {\scriptstyle D_V}\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id} \,\widetilde{\divideontimes}\, D_{U^\perp \cap V}} \\ F_\bullet^V \widetilde{\divideontimes} F_\bullet^{V^\perp} & \xrightarrow{\quad D_U \,\widetilde{\divideontimes}\, \mathrm{id} \quad} & F_\bullet^U \widetilde{\divideontimes} F_\bullet^{U^\perp \cap V} \widetilde{\divideontimes} F_\bullet^{V^\perp} \end{array}$$

Proof. Fix any m-simplex $U_0 \subseteq \cdots \subseteq U_m$ with U-pivot i and V-pivot j. For any subspace W of X, if $\mathrm{pr}_{U^\perp}(W) \subseteq V \cap U^\perp$ then we must have $W \subseteq V$. In particular,

$$((1 \,\widetilde{\divideontimes}\, D_{U^\perp \cap V}) \circ D_U)(U_0 \subseteq \cdots \subseteq U_m)$$
$$= (1 \,\widetilde{\divideontimes}\, D_{U^\perp \cap V})((U_0 \subseteq \cdots \subseteq U_i) \wedge (\mathrm{pr}_{U^\perp}(U_{i+1}) \subseteq \cdots \subseteq \mathrm{pr}_{U^\perp}(U_m)))$$
$$= (U_0 \subseteq \cdots \subseteq U_i) \wedge (\mathrm{pr}_{U^\perp}(U_{i+1}) \subseteq \cdots \subseteq \mathrm{pr}_{U^\perp}(U_j))$$
$$\qquad \wedge (\mathrm{pr}_{V^\perp}\, \mathrm{pr}_{U^\perp}(U_{j+1}) \subseteq \cdots \subseteq \mathrm{pr}_{V^\perp}\, \mathrm{pr}_{U^\perp}(U_m))$$
$$= (U_0 \subseteq \cdots \subseteq U_i) \wedge (\mathrm{pr}_{U^\perp}(U_{i+1}) \subseteq \cdots \subseteq \mathrm{pr}_{U^\perp}(U_j))$$

$$\wedge \left(\mathrm{pr}_{V^{\perp}}(U_{j+1}) \subseteq \cdots \subseteq \mathrm{pr}_{V^{\perp}}(U_m) \right)$$

where the last step follows because $V^{\perp} \subseteq U^{\perp}$. This is equal to the composition around the bottom, as desired. $\quad\square$

Up to this point, the definitions and results can be constructed for the smash product, instead of the reduced join. However, the authors could not find anything analogous to the definition below when $\widetilde{\star}$ is replaced by $\wedge$:

Definition 2.12. Let $0 < i < d$. Define the *dimension-i derived Dehn invariant* D_i to be the lift in the following diagram:

$$
\begin{array}{ccc}
 & & \bigvee\limits_{\substack{U \subseteq X \\ \dim \overline{U} = i}} F_\bullet^U \widetilde{\star} F_\bullet^{U^{\perp}} \\
 & \nearrow{\scriptstyle D_i} & \Big\downarrow \\
F_\bullet^X & \xrightarrow{\ \Pi\, D_U\ } & \prod\limits_{\substack{U \subseteq X \\ \dim \overline{U} = i}} F_\bullet^U \widetilde{\star} F_\bullet^{U^{\perp}}
\end{array}
$$

This is well-defined: every simplex contains at most one space of dimension i, and thus only a single dimension-i component will be nontrivial on it.

Lemma 2.13. *D_i is well-defined and $I(X)$-equivariant.*

This produces a Dehn invariant for a fixed dimension. Moreover, this Dehn invariant can be put into a square similar to (2.5). When $i < j$ the diagram

$$
\begin{array}{ccc}
F_\bullet^X & \xrightarrow{\ \ D_i\ \ } & \bigvee\limits_{\substack{U \subseteq X \\ \dim \overline{U} = i}} F_\bullet^U \widetilde{\star} F_\bullet^{U^{\perp}} \\
{\scriptstyle D_j}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}\,\widetilde{\star}\,D_{j-i}} \\
\bigvee\limits_{\substack{V \subseteq X \\ \dim \overline{V} = j}} F_\bullet^V \widetilde{\star} F_\bullet^{V^{\perp}} & \xrightarrow{\ D_i\,\widetilde{\star}\,\mathrm{id}\ } & \bigvee\limits_{\substack{U \subseteq V \\ \dim \overline{U} = i \\ \dim \overline{V} = j}} F_\bullet^U \widetilde{\star} F_\bullet^{U^{\perp} \cap V} \widetilde{\star} F_\bullet^{V^{\perp}}
\end{array}
\qquad (2.14)
$$

commutes and is $I(X)$-equivariant.

Analogously to Theorem 1.21, this construction is compatible with the classical story; as before the proof is postponed to Appendix A.

Theorem 2.15. *Suppose* $\dim X = d$ *and* $k = \mathbb{R}$. *Then* $H_0(I(X); H_{d+1}(S^\sigma \wedge D_i))$ *is the classical Dehn invariant.*

3. A geometrization of the Dehn complex

Let X be a neat geometry (in the sense of Definition 1.2).

In [17] Goncharov considers a complex $\mathcal{P}_*(X)$ constructed out of iterations of the Dehn invariant, and gives several conjectures relating these to algebraic K-theory. These conjectures will be discussed in Section 5; here we focus on the construction of this complex and its geometrization.

We begin with an informal outline in the case $k = \mathbb{R}$ and $\dim X = 2n - 1$. Using the square (2.5) the classical Dehn invariant $\widehat{D}_i$ can be iterated by varying over all possible values of $0 < i < 2n - 1$; this produces a commutative cube of dimension $2n - 2$ whose vertices contain tensor products of scissors congruence groups. When j is even, $\mathcal{P}(S^j) = 0$ (Proposition 1.24); removing the coordinates where these appear leaves an $(n-1)$-dimensional cube. Goncharov considers the total complex of this cube in [17]; we refer to this complex as the *Dehn complex* and denote it by $\mathcal{P}_*(X)$. One advantage is that it allows for the following rephrasing of the generalized Hilbert's third problem for reduced spherical and hyperbolic scissors congruence groups:

Question 3.1. *Is volume injective on* $H_{n-1}\mathcal{P}_*(X^d)$?

The goal of this section is to develop a tool for analyzing this complex using total homotopy cofibers of cubical diagrams; as an additional benefit, a definition of the Dehn complex for arbitrary fields k naturally emerges.

More formally:

Definition 3.2. Let I be the category $0 \longrightarrow 1$. An *n-cube* in $\mathcal{C}$ is a functor $I^n \longrightarrow \mathcal{C}$. Suppose that $\mathcal{C}$ is a model category.[5] Write $\widetilde{I}^n$ for the full subcategory of I^n which does not contain the object $(1, \ldots, 1)$.

Let $F : I^n \longrightarrow \mathcal{C}$ be any functor. The *total homotopy cofiber* $\mathrm{cofib}^{th} F$ is the homotopy cofiber of the map

$$\operatorname*{colim}^{h} F|_{\widetilde{I}^n} \longrightarrow F(1, \ldots, 1).$$

For a more in-depth discussion of the total homotopy cofiber, see [22, Section 5.9].

The important examples are the following:

---

[5] Model categories are just one of a wide variety of situations (often called "homotopical categories") in which it is possible to define homotopy cofibers (which is all we need for the current application). For an overview of this, see for example [25].

Example 3.3. In the case $n = 1$ the cube F becomes a morphism $M \longrightarrow M'$ of R-modules, which can be thought of as a morphism of chain complexes concentrated in degree 0. Taking the homotopy cofiber produces

$$\text{cofib}^h(M[0] \longrightarrow M'[0]) = (0 \longrightarrow M \longrightarrow M' \longrightarrow 0),$$

with M' in degree 0 and M in degree 1. Tautologically, this is the total complex of the 1-complex given by the original 1-cube.

Example 3.4. Now consider the general case. Let $F : I^n \longrightarrow \text{Mod}_R$ be a functor; this is an n-complex which has length 2 in each direction. One can check that the total complex of F is quasi-isomorphic to $\text{cofib}^{th} F[0]$.

To construct the Dehn complex it is more convenient to work with a slightly different coordinatization of a cube.

Definition 3.5. Denote by $\mathcal{I}_d$ the category whose objects are sequences $\vec{A} = (b, a_1, \ldots, a_i)$ of positive integers such that $b + a_1 + \cdots + a_i = d$ and in which all a_j are even. There exists a morphism $(b, a_1, \ldots, a_i) \longrightarrow (b', a_1', \ldots, a_\ell')$ if there exist indices $0 \leq i_0 < \cdots < i_\ell = i$ such that $b = b' + a_1' + \cdots + a_{i_0}'$ and $a_j = a_{i_{j-1}+1}' + \cdots + a_{i_j}'$.

Note that $\mathcal{I}_d$ is an $\lfloor \frac{d-1}{2} \rfloor$-cube via the map $(b, a_1, \ldots, a_i) \longrightarrow (\delta_1, \ldots, \delta_{\lfloor \frac{d-1}{2} \rfloor})$, where $\delta_j = 1$ if there exists an index ℓ with $b + a_1 + \cdots + a_\ell = 2j$ and 0 otherwise.

Definition 3.6. Let X be a neat geometry of dimension d.

Define the *Dehn complex* $\mathcal{P}_*(X)$ to be the total complex (equivalently, total homotopy cofiber) of the cube $\mathbf{D} : \mathcal{I}_d \longrightarrow \mathbf{AbGp}$ sending $(b, a_1, \ldots, a_i)$ to

$$\mathbf{D}(b, a_1, \ldots, a_i) = \mathbb{Z}[\tfrac{1}{2}] \otimes \mathcal{P}(X^b) \otimes \bigotimes_{j=1}^{i} \mathcal{P}(S^{a_j - 1})$$

and the map $(b, a_1, \ldots, a_j + a_{j+1}, \ldots, a_i) \longrightarrow (b, a_1, \ldots, a_i)$ to $1 \otimes \cdots \otimes D_{a_j} \otimes \cdots \otimes 1$.

The *equivariant Dehn cube* is the cube $\mathbf{D}^{\text{eqvt}} : \mathcal{I}_d \longrightarrow \text{Mod}_{I(X)}$ given by

$$\mathbf{D}^{\text{eqvt}}(b, a_0, \ldots, a_i) = \mathbb{Z}[\tfrac{1}{2}] \otimes \bigoplus_{\substack{W \oplus^\perp V_1 \oplus^\perp \cdots \oplus^\perp V_i = X \\ \dim W = b \\ \dim V_j = a_j - 1}} \mathcal{P}(W, 1) \otimes \bigotimes_{j=1}^{i} \mathcal{P}(V_j, 1).$$

By the same reasoning as in the proof of Lemma 2.4, we see that $H_0(I(X), -) \circ \mathbf{D}^{\text{eqvt}} = \mathbf{D}$.[6]

---

[6] The only change that tensoring with $\mathbb{Z}[\tfrac{1}{2}]$ imposes is that when $n = 0$, $\mathcal{P}(X^n) \cong \mathbb{Z}[\tfrac{1}{2}]$, instead of $\mathbb{Z}$ (as it would usually be: it is a count of the number of points). All other classical scissors congruence groups are 2-divisible.

Thus the Dehn complex is obtained by constructing a cube of coinvariants of homology groups and taking its total homotopy cofiber. The goal of this section is to construct, $I(X)$-equivariantly, a "geometrization": a cube of spaces that produces **D** after taking homology and then the $I(X)$-coinvariants.

Remark 3.7. As mentioned in (1.16) and Theorem 1.21, taking coinvariants in the homology can be replaced by taking the homotopy coinvariants of an action on a space. Since homotopy coinvariants and the total homotopy cofiber commute past one another, in future sections these are applied in the opposite order to relate the homology of the Dehn complex to algebraic K-theory.

We proceed as in previous sections: by replacing $\mathcal{P}(X)$ with $F_\bullet^X$. It cannot be done over $\mathbb{Z}$, but instead over $\mathbb{Z}[\frac{1}{2}]$; the difficulties are highlighted in the differences between $F_\bullet^{\vec{A}}$ and $J_\bullet^{\vec{A}}$:

Definition 3.8. Let $\vec{A}$ be any tuple of integers $\vec{A} = (b, a_1, \ldots, a_i)$. Define

$$F_\bullet^{\vec{A}} \stackrel{\text{def}}{=} \bigvee_{\substack{W \oplus^\perp \bigoplus^\perp V_j = X \\ \dim V_j = a_j - 1 \\ \dim W = b}} F_\bullet^W \widetilde{\star} \, \underset{j=1}{\overset{i}{\bigstar}} \, F_\bullet^{V_j}$$

(with $\widetilde{\star}$-factors ordered from left to right) and

$$J_\bullet^{\vec{A}} \stackrel{\text{def}}{=} \bigvee_{\substack{W \oplus^\perp \bigoplus^\perp V_j = X \\ \dim V_j = a_j - 1 \\ \dim W = b}} F_\bullet^W \wedge \bigwedge_{j=1}^{i} (S^\sigma \wedge F_\bullet^{V_j}).$$

The construction of the Dehn complex in spaces can be duplicated in the current context:

Definition 3.9. Define the functor $\mathbf{Y} : \mathcal{I}_d \longrightarrow \mathbf{Top}$ by

$$\vec{A} \longmapsto S^\sigma \wedge F_\bullet^{\vec{A}},$$

with morphisms given by the appropriate D_i. Define the *Dehn space* $\mathcal{Y}^X$ by

$$\mathcal{Y}^X = \operatorname{cofib}^{th} \mathbf{Y}.$$

Theorem 3.10. *Let X be a neat geometry of dimension d. The Dehn complex is quasi-isomorphic to the total complex of the $\lfloor \frac{d-1}{2} \rfloor$-cube given by*

$$H_{d+1}\left(\mathbf{Y}(-)_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right) : \vec{A} \longmapsto H_{d+1}\left((S^\sigma \wedge F_\bullet^{\vec{A}})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right).$$

The rest of this section is dedicated to the proof of this theorem; as everything from this point on will be done with $\mathbb{Z}[\frac{1}{2}]$ coefficients, the coefficients are omitted from the notation. From the homotopy orbit spectral sequence (Proposition B.7) applied to the right-hand side of the given formula, it suffices to construct a natural isomorphism $\mathbf{D} \implies H_0(I(X), H_{d+1}(\mathbf{Y}))$. Because $\mathbf{D} \cong H_0(I(X), \mathbf{D}^{\mathrm{eqvt}})$ it suffices to produce an $I(X)$-equivariant natural isomorphism $\alpha : \mathbf{D}^{\mathrm{eqvt}} \implies H_{d+1} \circ \mathbf{Y}$.

To produce the value $\alpha_{\vec{A}}$ of α on $\vec{A}$, first observe that

$$
\mathbf{D}^{\mathrm{eqvt}}(\vec{A}) = \mathbb{Z}[\tfrac{1}{2}] \otimes \bigoplus_{\substack{W \oplus^\perp \bigoplus^\perp V_j = X \\ \dim V_j = a_j - 1 \\ \dim W = b}} \mathcal{P}(W, 1) \otimes \bigotimes_{j=1}^{i} \mathcal{P}(V_j, 1)
$$

$$
= \bigoplus_{\substack{W \oplus^\perp \bigoplus^\perp V_j = X \\ \dim V_j = a_j - 1 \\ \dim W = b}} H_{b+1}(S^\sigma \wedge F_\bullet^W) \otimes \bigotimes_{j=1}^{i} H_{a_j}(S^\sigma \wedge F_\bullet^{V_j})
$$

$$
\cong H_{d+1}\left(\bigvee_{\substack{W \oplus^\perp \bigoplus^\perp V_j = X \\ \dim V_j = a_j - 1 \\ \dim W = b}} S^\sigma \wedge F_\bullet^W \wedge \bigwedge_{j=1}^{i} (S^\sigma \wedge F_\bullet^{V_j}) \right) = H_{d+1}(S^\sigma \wedge J_\bullet^{\vec{A}}).
$$

Therefore $\alpha_{\vec{A}}$ could be produced by giving maps of simplicial sets $S^\sigma \wedge J_\bullet^{\vec{A}} \longrightarrow S^\sigma \wedge F_\bullet^{\vec{A}}$ which give isomorphisms on H_{d+1}, compatible with the images of arrows in $\mathcal{I}_n$. These arrows are given by Dehn invariants; unfortunately, while Definition 2.12 gives a geometrization of the Dehn invariant on $F_\bullet^{\vec{A}}$, we do not have an analogous geometrization of the Dehn invariant on $J_\bullet^{\vec{A}}$. Therefore the compatibility conditions between the $\alpha_{\vec{A}}$ cannot be stated using only maps of simplicial sets. Instead, *ad-hoc* mappings are constructed between these spaces, which with a scaling correction behave correctly on homology. These maps are homotopy equivalences after tensoring with $\mathbb{Z}[\frac{1}{2}]$, although not *integral* homotopy equivalences.

Remark 3.11. This seems to imply that the original definition of the Dehn invariant had an extra factor of 2 somehow incorporated into the definition. It would be interesting to see a geometric explanation of this phenomenon.

To construct this explicit descriptions of S^1 and S^σ are required. The structure is summarized in the following table; note that in the case of S^1, $\epsilon = 1$ always; in the case of S^σ, $\epsilon = \pm 1$.

		S^1	S^σ
n-simplices		$\{*, 1, \ldots, n\}$	$\{*, \circledast, \pm 1, \ldots, \pm n\}$
$d_j(\epsilon i)$	$j = 0, i = 1$	$*$	$\circledast$
	$j = i = n$	$*$	$*$
	$j < i$	$i - 1$	$\epsilon(i - 1)$
	otherwise	i	ϵi
$s_j(\epsilon i)$	$j < i$	$i + 1$	$\epsilon(i + 1)$
	$j \geq i$	i	ϵi
$\mathbb{Z}/2$-action		none	$\epsilon i \mapsto -\epsilon i$

$$(3.12)$$

All simplices above dimension 1 are degenerate. All face and degeneracy maps on $*$ (resp. $\circledast$) map it to $*$ (resp. $\circledast$) in the appropriate dimension.

In this notation, the simplicial weak equivalence mentioned in Lemma 2.8 and Lemma B.4 is described via

$$(i, x, y) \in (S^1 \wedge X \wedge Y)_n \longmapsto (d_i^{n-i+1} x, d_0^{i+1} y) \in (X \widetilde{\star} Y)_n. \qquad (3.13)$$

The key construction for the desired equivalence is the $\mathbb{Z}/2 \times \mathbb{Z}/2$-equivariant map defined by

$$\gamma : S^\sigma \wedge S^\sigma \longrightarrow S^\sigma \wedge S^1$$
$$(a, b) \longmapsto ((\operatorname{sgn} b)a, |b|)$$

and $\gamma(\star) = *$. Here, $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts on the left coordinatewise, and on the right via the addition mapping $\mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2$. This is a *two-fold* cover of S^2 by S^2. More visually, consider the following illustration:
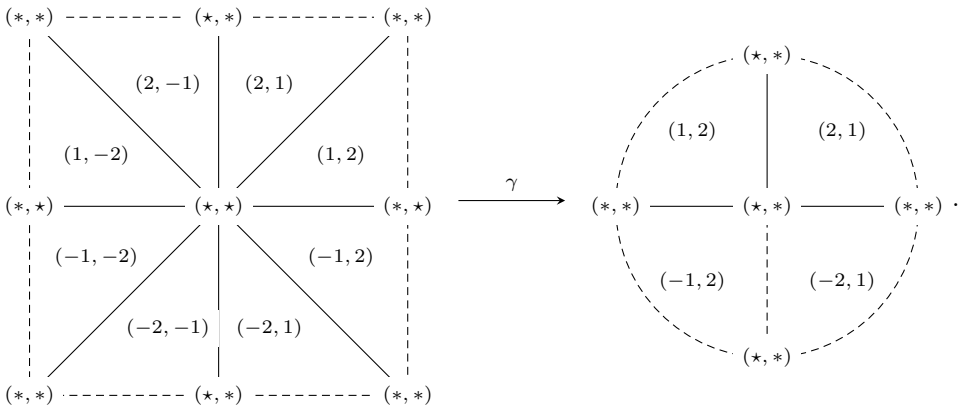
In this diagram all nondegenerate simplices present in $S^\rho \times S^\rho$ and $S^\rho \times S^1$ are drawn; everything drawn dashed is collapsed to a single point in the smash product. Edges are not labeled but 2-simplices are, using the explicit description of the simplicial structures

in (3.12). In effect, the map γ is the endomorphism of the unit disk which multiplies the angle (in polar coordinates) by 2.

It is now possible to define $f_{\vec{A}} : S^\sigma \wedge J_\bullet^{\vec{A}} \longrightarrow S^\sigma \wedge F_\bullet^{\vec{A}}$. For any simplicial sets K and L, let $f : S^1 \wedge K \wedge L \longrightarrow K \widetilde{\star} L$ take (a, x, y) to $(d_a^{n-a+1} x, d_0^{a+1} y)$; by Lemma B.4 it is a weak equivalence. Define $f_{\vec{A}}$ inductively, as an i-fold composition of maps of the following form:

$$ S^\sigma \wedge K \wedge S^\sigma \wedge L \overset{\tau}{\longrightarrow} S^\sigma \wedge S^\sigma \wedge K \wedge L \overset{\gamma}{\longrightarrow} S^\sigma \wedge S^1 \wedge K \wedge L \overset{f}{\longrightarrow} S^\sigma \wedge K \widetilde{\star} L. $$

Lemma 3.14. *After inverting 2, $f_{\vec{A}}$ becomes an $I(X)$-equivariant weak equivalence.*

With $\alpha_{\vec{A}}$ constructed, we can describe the homology groups of $\mathbf{Y}(\vec{A})_{hI(X)}$ explicitly in terms of group homology and scissors congruence groups:

Lemma 3.15. *Let $\vec{A} = (b, a_1, \ldots, a_i)$ be an object of $\mathcal{I}_d$. Then*

$$ \widetilde{H}_q \left(\mathbf{Y}(\vec{A})_{hI(X)}; \mathbb{Q} \right) \cong \bigoplus_{\ell_0 + \cdots + \ell_i = q} H_{\ell_0 - b - 1}(I(X^b), \mathcal{P}(X^b, 1)_{\mathbb{Q}}) $$

$$ \otimes \bigotimes_{j=1}^{i} H_{\ell_j - a_j - 1}(O(a_j + 1), \mathcal{P}(S^{a_j}, 1)_{\mathbb{Q}}). $$

Moreover, when $q = d + 1$ this works with only 2 inverted:

$$ H_{d+1} \left(\mathbf{Y}(\vec{A})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}] \right) \cong \mathcal{P}(X^b) \otimes \bigotimes_{j=1}^{i} \mathcal{P}(S^{a_j}). $$

Proof. Since $\alpha_{\vec{A}}$ is an equivariant weak equivalence, we know that

$$ \widetilde{H}_q \left(\mathbf{Y}(\vec{A})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}] \right) \cong \widetilde{H}_q \left((S^\sigma \wedge J_\bullet^{\vec{A}})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}] \right). $$

We thus focus on the computation of the right-hand side. We can model $(S^\sigma \wedge J_\bullet^{\vec{A}})_{hI(X)}$ as the bisimplicial set $K_{\bullet, \bullet}$ where

$$ K_{p,q} = I(X)^q \wedge (S^\sigma \wedge J_\bullet^{\vec{A}})_p. $$

Here, the horizontal face and degeneracies act only on the $(S^\sigma \wedge J_\bullet^{\vec{A}})$-coordinate. The j-th vertical degeneracies add an identity into the j-th slot in the tuple $I(X)^q$; the vertical face maps are defined as follows:

$$ d_i((g_1, \ldots, g_q), y) = \begin{cases} ((g_1, \ldots, g_{i+1} g_i, \ldots, g_q), y) & \text{if } 0 < i < q \\ ((g_1, \ldots, g_{q-1}), y) & \text{if } i = q \\ ((g_2, \ldots, g_q), g_1 \cdot y) & \text{if } i = 0. \end{cases} $$

In particular, every simplicial set $K_{p,\bullet}$ is the nerve of a category whose objects are $(S^\sigma \wedge J_\bullet^{\vec{A}})_p$ and in which $\mathrm{Hom}(y, y') = \{g \in I(X) \,|\, g \cdot y = y'\}$ (plus a disjoint basepoint).

Fix a single decomposition $X = W \oplus^\perp \bigoplus^\perp V_j$, and write $Y' = (S^\sigma \wedge F_\bullet^W \wedge \bigwedge_{j=1}^i (S^\sigma \wedge F_\bullet^{V_j})$. Consider the bisimplicial subset $K'_{\bullet,\bullet}$ of $K_{\bullet,\bullet}$ containing those simplices $((g_1, \ldots, g_q), y)$ with $y \in Y'$; for each p, $K'_{p,\bullet}$ is also the nerve of a category, which is the full subcategory of $K_{p,\bullet}$ on the simplices in Y'. Note that the inclusion of this subcategory is essentially surjective, and thus induces a weak equivalence on nerves. Thus the inclusion $K'_{p,\bullet} \longrightarrow K_{p,\bullet}$ is a weak equivalence for all p, and thus the inclusion $K'_{\bullet,\bullet} \hookrightarrow K_{\bullet,\bullet}$ is a weak equivalence on geometric realization. Consequently,

$$\widetilde{H}_q\left(\mathbf{Y}(\vec{A})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right) \cong \widetilde{H}_q(K'_{\bullet,\bullet}).$$

Any group element appearing in $K'_{\bullet,\bullet}$ must preserve each of $W, V_1, \ldots, V_i$. Thus (by reversing the construction of $K_{\bullet,\bullet}$ above) we see that

$$K'_{\bullet,\bullet} \cong Y'_{h(I(W) \times \prod_{j=1}^i I(V_j))} \simeq (S^\sigma \wedge F_\bullet^W)_{hI(W)} \wedge \bigwedge_{j=1}^i (S^\sigma \wedge F_\bullet^{V_i})_{hI(V_i)}.$$

After tensoring with $\mathbb{Q}$, the homology of the right-hand side gives the desired formula. Moreover, as the $d+1$-st homology on the right is the lowest nonzero homology group, the second part of the lemma follows by the Künneth theorem. $\square$

It follows that the objects in the desired cube are isomorphic to the objects in Goncharov's cube. In an ideal world, it would be possible to define $\alpha_{\vec{A}} = H_{d+1}(f_{\vec{A}})$. Unfortunately, it is not that simple, as this is not compatible with Dehn invariants. Consider a small example. When $\vec{A} = (d)$, $F_\bullet^{\vec{A}} = J_\bullet^{\vec{A}} = F_\bullet^X$. Fix b, and consider the Dehn invariant D_b corresponding to the morphism $(d) \longrightarrow (b, a)$. This produces the following **noncommutative** diagram:

$$
\begin{array}{ccc}
 & & H_{d+1}(S^\sigma \wedge J_\bullet^{(b,a)}) \\
 & \overset{\widehat{D}_b}{\nearrow} \quad \not\circlearrowleft & \big\downarrow {\scriptstyle H_{d+1}(f_{(b,a)})} \\
H_{d+1}(S^\sigma \wedge F_\bullet^X) & \underset{H_{d+1}(D_b)}{\longrightarrow} & H_{d+1}(S^\sigma \wedge F_\bullet^{(b,a)})
\end{array}
$$

To make the diagram commute it is necessary to multiply the vertical map by $\frac{1}{2}$ (due to γ having degree 2). This is true in general; writing $|\vec{A}|$ for the length of $\vec{A}$, if $|\vec{A}| > 1$ the construction of $f_{\vec{A}}$ contains $|\vec{A}| - 1$ compositions with γ, and thus multiplies by $2^{|\vec{A}|-1}$ in homology. As 2 is inverted, this can be remedied:

Lemma 3.16. $\alpha_{\vec{A}} \overset{\mathrm{def}}{=} 2^{1-|\vec{A}|} H_{d+1}(f_{\vec{A}})$ gives a natural isomorphism

$$\mathbf{D}^{\mathrm{eqvt}} \Longrightarrow H_{d+1}(\mathbf{Y}; \mathbb{Z}[\tfrac{1}{2}]).$$

The proof is now complete. $\quad\square$

4. Large cubes and the Dehn complex

To construct the Dehn complex, Goncharov essentially starts with the groups $\mathcal{P}(X, 1)$, takes their coinvariants with respect to a group action (or, in other words, a homology group), constructs a new differential to make these groups into a chain complex, and then studies the homology of this new chain complex. This produces an object which is difficult to analyze and does not seem to fit into any of the standard methods for taking the homology of homologies. In light of Example 3.4, this chain complex can be thought of as the total homotopy cofiber of a cube of groups. The goal of this section is to compare it with the total homotopy cofiber $(\mathcal{Y}^X)_{hI(X)}$ of derived Dehn invariants defined above. It turns out that the homology of $(\mathcal{Y}^X)_{hI(X)}$ can be described in two ways: one by directly analyzing its homotopy type, and one via a spectral sequence of Munson and Volic [22, Proposition 9.6.14]. The comparison between these computations is incredibly fruitful.

For the rest of this section, the neat geometry X over k has dimension $d = 2n$ or $2n - 1$ (so that $n = \lfloor \frac{d+1}{2} \rfloor$).

Theorem 4.1. *After inverting* 2 *there is an equivalence*

$$(\mathcal{Y}^X)_{hI(X)} \simeq (S^\sigma \wedge S^{n-1})_{hI(X)}.$$

Here $I(X)$ acts by det *on the S^σ-coordinate and trivially on the S^{n-1}-coordinate.*

The proof of this theorem is deferred to Section 6. The key to this theorem is the observation that homotopy coinvariants are both homotopy colimits and therefore *commute*; thus

$$(\mathcal{Y}^X)_{hI(X)} = (\text{cofib}^{th}\, \mathbf{Y})_{hI(X)} \simeq \text{cofib}^{th}(\mathbf{Y}_{hI(X)}).$$

This means that the simple combinatorial nature of $\mathbf{Y}$ can be played against the benefits of taking homotopy coinvariants. This is also where the benefits of constructing an *equivariant* Dehn invariant comes into play: if an equivariant model for the Dehn invariant did not exist it would be impossible to move the homotopy coinvariants outside of the total cofiber.

Theorem 4.2. *Write $\mathbb{Z}[\frac{1}{2}]^\sigma$ for a copy of $\mathbb{Z}[\frac{1}{2}]$ with $I(X)$ acting on it via multiplication by the determinant. For all i,*

$$\widetilde{H}_i\left(\mathcal{Y}^X_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right) \cong H_{i-n}\left(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma\right);$$

in particular, when $i < n$ both are zero.

Proof. By Theorem 4.1 there is an isomorphism

$$H_i(\mathcal{Y}^X_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]) \cong H_i((S^\sigma \wedge S^{n-1})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]).$$

By the homotopy orbit spectral sequence (Proposition B.7), and since $\mathbb{Z}[\tfrac{1}{2}]^\sigma \cong \widetilde{H}_1(S^\sigma; \mathbb{Z}[\tfrac{1}{2}])$,
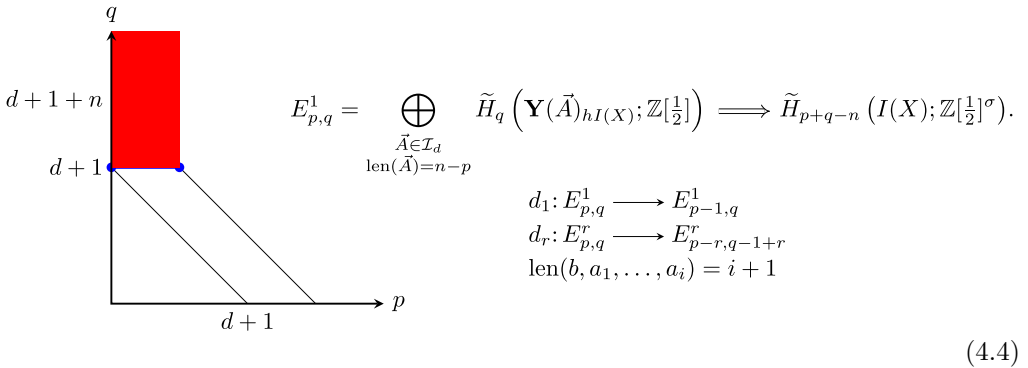
$$\widetilde{H}_i\left((S^\sigma \wedge S^{n-1})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right) \cong H_{i-n}\left(I(X); \widetilde{H}_n\left(S^\sigma \wedge S^{n-1}; \mathbb{Z}[\tfrac{1}{2}]\right)\right)$$

$$\cong H_{i-n}\left(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma\right). \quad \square$$

The spectral sequence for the total homotopy colimit of a cube proved in Proposition B.9 can now be used to connect the homotopy type of $\mathcal{Y}^X_{hI(X)}$ to the Dehn complex. In this case the spectral sequence becomes

$$E^1_{p,q} = \bigoplus_{\vec{A} = (b, a_1, \ldots, a_{n-1-p})} \widetilde{H}_q\left(\mathbf{Y}(\vec{A})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right) \implies \widetilde{H}_{p+q}\left(\mathcal{Y}^X_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right). \qquad (4.3)$$

Since $\mathbf{Y}(\vec{A})$ has no nonzero homology below degree $d+1$ this also holds for $\mathbf{Y}(\vec{A})_{hI(X)}$. Thus all entries in the spectral sequence with $q < d+1$ are 0. When $q = d+1$ the row of the spectral sequence is exactly the Dehn complex.

Below is a picture of the spectral sequence for $\widetilde{H}_*(\mathcal{Y}^X_{I(X)}; \mathbb{Z}[\tfrac{1}{2}])$ in (4.3). The red indicates the non-zero entries in E^1. (For interpretation of the colors in the equation(s), the reader is referred to the web version of this article.) The Dehn complex is the base complex of $\mathbf{Y}$; it is the thick blue line sitting in the row where $q = d+1$.

$$E^1_{p,q} = \bigoplus_{\substack{\vec{A} \in \mathcal{I}_d \\ \mathrm{len}(\vec{A}) = n-p}} \widetilde{H}_q\left(\mathbf{Y}(\vec{A})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}]\right) \implies \widetilde{H}_{p+q-n}\left(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma\right).$$

$$d_1 \colon E^1_{p,q} \longrightarrow E^1_{p-1,q}$$
$$d_r \colon E^r_{p,q} \longrightarrow E^r_{p-r,q-1+r}$$
$$\mathrm{len}(b, a_1, \ldots, a_i) = i+1$$

(4.4)

Definition 4.5. Let $F^*_{p,q}$ be a first-quadrant homologically-graded spectral sequence with lowest nonzero row at $p = m$, converging to the sequence of groups G_{p+q}. The *base complex* of the spectral sequence is the complex $F^1_{*,m}$ with differential d^1. The induced homomorphism $\theta_n : G_n \longrightarrow F^1_{n-m,m}$ is called the *projection to the base*.[7]

---

[7]   This is also sometimes called an "edge homomorphism in the spectral sequence.".

Theorem 4.6. *Projection to the base gives a homomorphism*

$$\theta_m : H_{n+m}(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma) \longrightarrow H_m \mathcal{P}_*(X).$$

This homomorphism is surjective if and only if all differentials $d^r : E^r_{m,d+1} \longrightarrow E^r_{m-r,d+r}$ for $r \geq 2$ are zero. It is injective if and only if $E^\infty_{p,m-p} = 0$ for all $p > d+1$.

Proof. Projection to the base is surjective onto $E^\infty_{m,d+1}$. As the base complex is the lowest nonzero row, the spectral sequence contains no differentials *into* that row; thus, $E^\infty_{p,d+1}$ is a subgroup of $E^2_{p,d+1}$ for all p. This subgroup is exactly the intersection of all of the kernels of the d^r, and is therefore equal to the whole group if and only if all of these differentials are 0. The kernel of projection to the base is exactly the subgroup given by all of the terms in the m-th diagonal above the base complex on the E^∞-page; thus the map is injective if and only if the terms in the diagonal are 0. $\square$

In the case of this spectral sequence, it is possible to give an explicit description of θ_{n-1}:

Lemma 4.7. *Let $k = \mathbb{R}$. In the spectral sequence (4.3), the map*

$$\theta_{n-1} : H_d(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma) \longrightarrow H_{n-1} \mathcal{P}_*(X)$$

is induced by the map taking a chain $(g_1, \ldots, g_d)$ to the scissors congruence class of the d-simplex with vertices

$$\{x_0, g_d x_0, g_d g_{d-1} x_0, \ldots, g_d \cdots g_1 x_0\},$$

(for any chosen point $x_0 \in X$) with the sign given by $\prod_{i=1}^d \det(g_i)$.

The proof of this lemma is technical and not illuminating, so it is postponed to Section 6 (Lemma 6.6); in fact, in that section it is proved over any field k. We state it here for the special case as it makes the result easier to describe and this is the only case of interest in the current paper. This is a generalization to any field of a result claimed by Dupont in [13, Remark 3, p46]; unfortunately, the formula given there is incorrect as stated.

Directly from the spectral sequence it is possible to prove minor generalizations (to all fields of characteristic 0, instead of just the real field, and including all d not just odd ones, and removing rational coefficients) of the following results of Cathelineau:

Theorem 4.8 (*Generalization of [4, Thm. 10.1.1]*). *For $d = 2n$ or $2n-1$ with $n \geq 2$, and any field k of characteristic 0,*

$$H_i(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma) = 0 \qquad if\ i < n$$

and

$$H_n(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma) \cong H_0 \mathcal{P}_*(X^d).$$

Proof. Consider the spectral sequence (4.4). Everything below the $d+1$-st diagonal is 0, which implies the first claim. In the $p+q = d+1$-st diagonal there is exactly one nonzero entry: $H_0(\mathcal{P}_*(X^d))$. $\quad\square$

For the next theorem it is unfortunately necessary to rationalize, rather than simply inverting 2. As the rationalization is used only in computing the group homology of $SO(2)$, in certain cases with good control over the torsion in this group it may be possible to get away with a milder localization.

Theorem 4.9 *(Generalization of [5, Proposition 6.2.2]). Let X be a neat geometry over a field k of characteristic 0. Then*

$$H_1(I(X^1); H_1(F_\bullet^{X^1})^\sigma) = 0.$$

Consequently, for $d = 2n$ or $2n-1$, with $n \geq 2$,

$$H_{n+1}(I(X); \mathbb{Q}^\sigma) \cong H_1(\mathcal{P}_*(X^d)_\mathbb{Q}).$$

Proof. Via the spectral sequence in (4.4), it suffices to check that $E_{0,d+2}^1 = 0$. This term is a direct sum of tensor products, where each tensor product contains exactly one of the groups

$$H_1(I(\mathcal{H}^1), H_1(F_\bullet^{\mathcal{H}^1})^\sigma) \qquad \text{or} \qquad H_1(I(S^1), (F_\bullet^{S^1})^\sigma).$$

When X is spherical, only terms of the second sort will arise. Thus the second statement in the theorem follows directly from the first.

Let $I^+(X^1)$ be the subgroup of $I(X^1)$ of those elements with determinant 1. The Lyndon–Hochschild–Serre spectral sequence for the extension

$$I^+(X) \longrightarrow I(X) \longrightarrow \mathbb{Z}/2,$$

has

$$E_{p,q}^2 = H_p(\mathbb{Z}/2, H_q(I^+(X), H_1(F_\bullet^{X^1})^\sigma)) \implies H_{p+q}(I(X), H_1(F_\bullet^{X^1})^\sigma).$$

In particular, since $H_1(F_\bullet^{X^1})$ is 2-divisible, $E_{p,q}^1 = 0$ whenever $p > 0$. Thus

$$H_n(I(X), H_1(F_\bullet^{X^1})^\sigma) \cong H_0(\mathbb{Z}/2, H_n(I^+(X^1), H_1(F_\bullet^{X^1})^\sigma)). \tag{4.10}$$

We turn our focus to computing $H_n(I^+(X^1), H_1(F_\bullet^{X^1})^\sigma) = H_n(I^+(X^1), H_1(F_\bullet^{X^1}))$, which by the homotopy orbit spectral sequence (Proposition B.7) is isomorphic to $H_{n+1}((F_\bullet^{X^1})_{I^+(X^1)}; \mathbb{Q})$. Consider $(F_\bullet^{X^1})_{hI(X^1)}$ as a bisimplicial set, with the simplices of $F_\bullet^{X^1}$ in the horizontal direction, and the $I(X^1)$-action in the vertical direction. This produces a double complex, whose spectral sequence converges to $H_*((F_\bullet^{X^1})_{I^+(X^1)}; \mathbb{Q})$.

Consider the spectral sequence in which we take first vertical homology, then horizontal homology. As the only nondegenerate simplices in $F_\bullet^{X^1}$ are in dimensions 0 and 1, this spectral sequence will be concentrated in the first two columns. The simplicial set $F_\bullet^{X^1}$ has exactly one non-basepoint 0-simplex $[X^1]$, which has as its stabilizer $I^+(X^1)$; thus

$$E_{0,q}^1 \cong H_q(I^+(X^1), \mathbb{Q}).$$

The simplicial set $F_\bullet^{X^1}$ has as its nondegenerate 1-simplices inclusions $U_0 \subseteq X^1$, which have as their stabilizers the subgroup of $I(X^0) \times I(X^0)$ with determinant 1 (this is the group which fixes U_0 and $U_0^\perp$). This is a 2-group, and we will therefore have

$$E_{1,q}^1 \cong 0 \qquad q > 0.$$

When $q = 0$ this is simply $\mathbb{Q}$, and d_1 will send this $\mathbb{Q}$ isomorphically to $E_{0,0}^1 \cong \mathbb{Q}$. Thus the spectral sequence collapses at E^2, where it is concentrated in the 0-th column, producing

$$H_q(I^+(X^1), H_1(F_\bullet^{X^1})^\sigma) \cong H_{q+1}(I^+(X^1), \mathbb{Q}^\sigma).$$

Combining this with (4.10) produces

$$H_n(I(X^1), H_1(F_\bullet^{X^1})^\sigma) \cong H_0\left(\mathbb{Z}/2, H_{n+1}\left(I^+(X^1), \mathbb{Q}^\sigma\right)\right).$$

In the particular case of interest we have

$$H_1(I(X^1), H_1(F_\bullet^{X^1})^\sigma) \cong H_0\left(\mathbb{Z}/2, H_2\left(I^+(X^1), \mathbb{Q}^\sigma\right)\right).$$

The group $I^+(X^1)$ is abelian, and therefore

$$H_2\left(I^+(X^1), \mathbb{Q}^\sigma\right) \cong \Lambda^2(I^+(X^1) \otimes \mathbb{Q}^\sigma).$$

The group $\mathbb{Z}/2$ acts by -1 on both $\mathbb{Q}$ and on $I^+(X^1)$; thus it will act by -1 on every chain, and thus on the homology group H_2. Thus, since everything is 2-divisible,

$$H_0\left(\mathbb{Z}/2, H_2\left(I^+(X^1), \mathbb{Q}^\sigma\right)\right) \cong 0,$$

as desired. $\quad\square$

Directly from these calculations we can conclude the following:

Corollary 4.11 *(Generalization of [4, Thm 10.2.1]). Projection to the base θ_m is an iso-morphism when $m = 0, 1$ and surjective when $m = 2$.*

5. Goncharov's conjectures

In this section we discuss the connections between Goncharov's original conjectures, Cheeger–Chern–Simons invariants, and the results of the previous sections. All groups in this section are considered as **discrete** groups.

5.1. Projection to the base and the modified conjectures

In [17], Goncharov has a series of three conjectures about possible connections between the Dehn complex and the algebraic K-theory of $\mathbb{C}$. We give a summary of these conjectures here. Our notation does not exactly agree with Goncharov's; in particular, Goncharov's Dehn complex is cohomologically graded and 1-indexed, while ours is homologically graded and 0-indexed. We number the parts of our summary by the number of the conjecture in [17].

All tensor products of $\mathbb{Z}/2$-modules in this section are equipped with a $\mathbb{Z}/2$-action via the diagonal action. Write $\mathbb{Q}^{n\sigma} \overset{\text{def}}{=} (\mathbb{Q}^\sigma)^{\otimes n}$, equipped with the diagonal action of $\mathbb{Z}/2$.

Conjecture 5.1 *([17, Conjectures 1.7-1.9]). Let $\mathcal{P}_*(X^{2n-1})$ be the Dehn complex for the geometry X^{2n-1} over $\mathbb{R}$.*

(1.8) There exist homomorphisms

$$H_i\mathcal{P}_*(S^{2n-1}) \xrightarrow{\phi_i} (\operatorname{gr}_n^\gamma K_{n+i}(\mathbb{C})_\mathbb{Q} \otimes \mathbb{Q}^{n\sigma})^+$$

and

$$H_i\mathcal{P}_*(\mathcal{H}^{2n-1}) \xrightarrow{\phi_i} (\operatorname{gr}_n^\gamma K_{n+i}(\mathbb{C})_\mathbb{Q} \otimes \mathbb{Q}^{n\sigma})^-.$$

(1.7) The homomorphism ϕ_{n-1} is injective, and the diagrams

$$\ker \bigoplus_{i=1}^{n-1} \widehat{D}_i \cong H_{n-1}\mathcal{P}_*(S^{2n-1}) \xrightarrow{\phi_{n-1}} (\operatorname{gr}_n^\gamma K_{2n-1}(\mathbb{C})_\mathbb{Q} \otimes \mathbb{Q}^{n\sigma})^+$$

with maps vol and r to $\mathbb{R}/(2\pi)^n\mathbb{Z}$

and

$$\ker \bigoplus_{i=1}^{n-1} \widehat{D}_i \cong H_{n-1}\mathcal{P}_*(\mathcal{H}^{2n-1}) \xrightarrow{\ \phi_{n-1}\ } (\operatorname{gr}_n^\gamma K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \otimes \mathbb{Q}^{n\sigma})^-$$

with maps vol and r to $\mathbb{R}$.

commute. Here, the right-hand map is the Borel (resp. Beilinson) regulator.
(1.9) *All ϕ_i are isomorphisms.*

Here, $\operatorname{gr}_n^\gamma$ is the n-th graded part of the γ-filtration, and $\mathbb{Q}^{n\sigma}$ is the vector space $\mathbb{Q}$ with $\mathbb{Z}/2$ acting on it via multiplication by $(-1)^n$. The sign in the superscript indicates taking the ± 1 eigenspace with respect to the action by complex conjugation.

For an exposition of the γ-filtration, see for example [18]. For an exposition of the Borel and Beilinson regulators see [1, Chapter 9].

Goncharov proves (1.7) in the case when $\mathbb{C}$ is replaced with $\overline{\mathbb{Q}}$ and simplices in the Dehn complex are restricted to those with algebraic vertices [17, Theorem 1.6]. Note that any polytope which can appear as the fundamental domain of a group action is automatically in the kernel of all Dehn invariants; thus in particular Goncharov's conjectures would imply that all volumes of hyperbolic manifolds must be in the image of the Borel regulator.

Inspired by the conjectures, we propose an alternative method to connect the algebraic K-theory of $\mathbb{C}$ and the scissors congruence groups (see Proposition 5.10). Explaining the γ-filtration and the Borel and Beilinson regulators in the above theorems is extremely nontrivial, while the corresponding notions in our approach are much more elementary. Due to the nature of our methods the morphisms we construct go in the opposite direction to Goncharov's desired morphisms.

As a first observation, Theorem 4.6 constructs exactly the morphism desired by Goncharov's Conjecture 1.8. In the rest of this section we will analyze how well this morphism proves Conjectures 1.7 and 1.9.

We begin with a description of the Cheeger–Chern–Simons class, which plays the same role as the Borel regulator for the case of orthogonal groups (rather than general linear groups).

The construction is originally due to Cheeger and Simons [7, Section 8] (although the authors originally learned it from Dupont [13, Sect. 10]) and works for more general homogeneous spaces. See also [10] and [9, Section 5].

Definition 5.2. The *Cheeger–Chern–Simons construction* is a homomorphism

$$\mathcal{CCS} : H_{2n-1}(O(2n;\mathbb{R});\mathbb{Z}[\tfrac{1}{2}]^{\sigma}) \longrightarrow \widehat{\mathcal{P}}(S^{2n-1})/[S^{2n-1}],$$

defined as follows. Consider the *space* $O(2n;\mathbb{R})/O(2n-1;\mathbb{R})$ with the usual topology. This is homeomorphic to S^{2n-1} with a distinguished point. The group $O(2n;\mathbb{R})$ acts on this on the left, moving the distinguished point. A chain in degree $2n-1$ is represented by a sequence of elements $(g_1,\ldots,g_{2n-1})$; call such a chain *generic* if the points $\{x, g_{2n-1}x, g_{2n-1}g_{2n-2}x, \ldots, g_{2n-1}\cdots g_1 x\}$ are all linearly independent in $\mathbb{R}^{2n}$. For a generic chain, define a geodesic simplex in S^{2n-1} associated to this chain by

$$\Delta_{(g_1,\ldots,g_{2n-1})} \stackrel{\text{def}}{=} (x_0, g_{2n-1}x_0, g_{2n-1}g_{2n-2}x_0, \ldots, g_{2n-1}\cdots g_1 x_0).$$

To check that this morphism is well-defined it suffices to check that given any $2n$-chain $(g_1,\ldots,g_{2n})$ all of whose faces are generic, the sum over the boundary is 0. This holds in $\mathcal{P}(S^{2n-1})/[S^{2n-1}]$, and this construction can be extended to all of $H_{2n-1}(O(2n;\mathbb{R});\mathbb{Z}[\tfrac{1}{2}]^{\sigma})$ as the generic chains are dense ([7, discussion after Theorem 8.14]). Define

$$\mathcal{CCS}: \quad H_{2n-1}(O(2n;\mathbb{R});\mathbb{Z}[\tfrac{1}{2}]^{\sigma}) \quad \longrightarrow \quad \widehat{\mathcal{P}}(S^{2n-1})/[S^{2n-1}]$$
$$[(g_1,\ldots,g_{2n-1})] \quad \longmapsto \quad [\Delta_{(g_1,\ldots,g_{2n-1})}]$$

(See [7, Section 8] and [10] for a more in-depth discussion.)

Now fix a volume form $v \in \Omega^{2n-1}(S^{2n-1})$ (which we normalize to so that $\int_{S^{2n-1}} v_{S^{2n-1}} = (2\pi)^n$). The *Cheeger–Chern–Simons class* is the homomorphism

$$\text{CCS} : H_{2n-1}(O(2n;\mathbb{R});\mathbb{Z}[\tfrac{1}{2}]^{\sigma}) \xrightarrow{\;\mathcal{ccs}\;} \widehat{\mathcal{P}}(S^{2n-1})/[S^{2n-1}] \xrightarrow{\;\text{vol}\;} \mathbb{R}/(2\pi)^n\mathbb{Z}[\tfrac{1}{2}].$$

Here we restricted to the odd-dimensional case because (by the "center-kills" lemma [13, Lemma 5.4]) the homology groups $H_{2n}(O(2n+1;\mathbb{R});\mathbb{Z}[\tfrac{1}{2}]^{\sigma})$ are all 0. By considering $\mathcal{H}^d \cong O^+(1,d;\mathbb{R})/O(d;\mathbb{R})$ one obtains an analogous homomorphism

$$\mathcal{CCS} : H_d(O^+(1,d);\mathbb{Z}[\tfrac{1}{2}]^{\sigma}) \longrightarrow \mathcal{P}(\mathcal{H}^d)$$

and

$$\text{CCS} : H_d(O^+(1,d;\mathbb{R});\mathbb{Z}[\tfrac{1}{2}]^{\sigma}) \longrightarrow \mathbb{R}.$$

The definition of the Cheeger–Chern–Simons construction yields the following analog of Goncharov's Conjecture 1.7 for hyperbolic geometry:

Theorem 5.3. *When $X = \mathcal{H}^d$ and $d = 2n - 1$, projection to the base (Definition 4.5)*

$$H_d(O^+(1,d;\mathbb{R}),\mathbb{Z}[\tfrac{1}{2}]^{\sigma}) \longrightarrow H_{n-1}\mathcal{P}_*(\mathcal{H}^d_{\mathbb{R}})$$

fits into a commutative diagram

$$H_d(O^+(1,d;\mathbb{R}), \mathbb{Z}[\tfrac{1}{2}]^\sigma) \xrightarrow{\theta_{n-1}} H_{n-1}\mathcal{P}_*(\mathcal{H}_\mathbb{R}^d)$$

$$\text{CCS} \searrow \qquad \swarrow \text{vol}$$

$$\mathbb{R}.$$

In particular, vol *is injective if θ_{n-1} is surjective and* CCS *is injective.*

Proof. The key observation to prove the theorem is that the group homomorphism $\mathcal{CCS}$ agrees with the explicit description of θ_{n-1} in Lemma 4.7 after composition with the isomorphism $\mathcal{P}(\mathcal{H}^d) \longrightarrow H_{d+1}((S^\sigma \wedge F_\bullet^{\mathcal{H}^d})_{I(\mathcal{H}^d)}; \mathbb{Z}[\tfrac{1}{2}])$ in Theorem 1.21. □

The main difficulty in relating the spherical Cheeger–Chern–Simons homomorphism to volume is that (as mentioned in Remark 2.3) volume is not well-defined on $\mathcal{P}(S^{2n-1})$. An interesting question is whether it is possible to give a well-defined definition of volume of $H_{n-1}\mathcal{P}_*(S^{2n-1})$—i.e., on the kernel of the Dehn invariant. However, it may be the case that this group is nonzero, and yet still a well-defined lift of the volume is possible, in which case an analogous statement to the hyperbolic one should exist. The precise relationship between the Cheeger–Chern–Simons class and projection to the base in the spherical case is the following; we omit the proof as it is directly analogous to the hyperbolic case.

Let $p : \widehat{P}(S^{2n-1}) \longrightarrow \mathcal{P}(S^{2n-1})$ be the projection. Let

$$L_{2n-1} \overset{\text{def}}{=} p^{-1}(H_{n-1}\mathcal{P}_*(S^{2n-1}));$$

this is the subgroup of spherical polytopes with Dehn invariant equal to 0 after reduction by lunes.

Theorem 5.4. *In the spherical case, the Cheeger–Chern–Simons class factors through the inclusion*

$$L_{2n-1}/[S^{2n-1}] \lhook\joinrel\longrightarrow \widehat{\mathcal{P}}(S^{2n-1})/[S^{2n-1}]$$

and is related to the projection to the base via the following commutative diagram:

$$H_{2n-1}(O(2n;\mathbb{R}); \mathbb{Z}[\tfrac{1}{2}]^\sigma) \xrightarrow{\ \mathcal{ccs}\ } L_{2n-1}/[S^{2n-1}] \xrightarrow{\ \text{vol}\ } \mathbb{R}/(2\pi)^n\mathbb{Z}[\tfrac{1}{2}]$$

with CCS over the top arc, θ_{n-1} down to $H_{n-1}\mathcal{P}_*(S^{2n-1})$, and p on the vertical.

Moreover, the analysis in Theorem 4.8 produces the following version of Goncharov's conjecture 1.8, with special cases of 1.9:

Theorem 5.5. *Let $d = 2n$ or $2n - 1$. For a neat geometry X of dimension d, projection to the base gives a homomorphism*

$$\theta_m : H_{n+m}(I(X), \mathbb{Q}^\sigma) \longrightarrow H_m \mathcal{P}_*(X)_{\mathbb{Q}}.$$

This homomorphism is an isomorphism when $m = 0$ or 1 and is surjective when $m = 2$.

5.2. Relationship to algebraic K-theory

In order to relate the homology of isometry groups to algebraic K-theory, we introduce the rank filtration.

Definition 5.6 *([28, p. 296]).* The *higher rational K-theory of k* is defined to be

$$K_*(k)_{\mathbb{Q}} \overset{\text{def}}{=} \text{primitive elements of the Hopf algebra } H_* GL(k).$$

(Recall that all homology is taken with rational coefficients.) The rank filtration on $K_*(k)_{\mathbb{Q}}$ is defined by

$$F_i K_*(k)_{\mathbb{Q}} \overset{\text{def}}{=} K_*(k)_{\mathbb{Q}} \cap \big(\text{im}\,(H_* GL(i; k) \longrightarrow H_* GL(k))\big).$$

Then

$$\text{gr}_n^{\text{rk}} K_*(k)_{\mathbb{Q}} \overset{\text{def}}{=} F_n K_*(k)_{\mathbb{Q}} / F_{n-1} K_*(k)_{\mathbb{Q}}.$$

We will also need an auxiliary object; define

$$CL_{n,m}(k) \overset{\text{def}}{=} \text{coker}\,\big(H_m GL(n - 1; k)_P \longrightarrow H_m GL(n; k)_P\big),$$

where $H_m GL(i; k)_P$ is the subgroup of $H_m GL(i; k)$ of those elements whose images in $H_* GL(k)$ is primitive. Observe that there is a natural surjection $CL_{n,m}(k) \twoheadrightarrow \text{gr}_n^{\text{rk}} K_m(k)_{\mathbb{Q}}$.

Analogously to $CL_{n,m}$ we define

$$CO_{n,m}(k) \overset{\text{def}}{=} \text{coker}(H_m SO(2n - 2; k)_P \longrightarrow H_m SO(2n; k)_P),$$

where $(H_m SO(i; k))_P$ denotes those elements whose image in $H_* SO(k)$ is primitive. We assume that the stabilization map $SO(2n - 2; k) \longrightarrow SO(2n; k)$ adds coordinates at positions n and $2n$, rather than at $2n - 1$ and $2n$.

There is an action of $\mathbb{Z}/2$ on $H_*SO(n;k)$ given by conjugation by a matrix with determinant -1. From the exact sequence

$$SO(n;k) \longrightarrow O(n;k) \longrightarrow \mathbb{Z}/2$$

the homology of $SO(n;k)$ splits into eigenspaces

$$H_m SO(n;k) \cong H_m O(n;k) \oplus H_m(O(n;k), \mathbb{Q}^\sigma),$$

where the first component is the $+1$-eigenspace and the second is the -1-eigenspace [6, p.489]. Since stabilization is equivariant with respect to this action, it induces an action on $CO_{n,m}(k)$. When n is odd $H_m(O(n;k);\mathbb{Q}^\sigma) = 0$ for all m ([6, Theorem 1.4], or [13, Lemma 5.4]); in particular, this implies that

$$CO_{n,m}(k)^- \cong H_m(O(2n;k), \mathbb{Q}^\sigma)_P. \tag{5.7}$$

We now turn our attention to explaining the connection between rational homology of orthogonal groups and algebraic K-theory. The comparison between the two is very well-studied (see, for example, [2]), and it is known that, after rationalizing, Hermitian K-theory is isomorphic to the homotopy fixed points of algebraic K-theory under the action sending a matrix to its transpose [2, (1-c)] via the hyperbolic map (defined below). This is explored in more detail below, and used to explain the connection between Goncharov's conjectures and the theorems proved above. In particular, we will explain why, in the real case, both spherical and hyperbolic geometries arise, and how Goncharov's curious twisting factors $\mathbb{Q}^{n\sigma}$ arise.

Definition 5.8. The *hyperbolic map*:

$$\mathrm{hyp} : GL(n;k) \longrightarrow SO(n,n;k) : \ M \longmapsto \begin{pmatrix} M & \\ & (M^\sigma)^{-1} \end{pmatrix}.$$

When k contains $i = \sqrt{-1}$ and $\sqrt{2}$, conjugation by the matrix

$$D_n \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ -iI_n & iI_n \end{pmatrix} \qquad \det D_n = i^n$$

induces an isomorphism $SO(n,n;k) \cong SO(2n;k)$. Conjugation by the matrix

$$D_n^{[1]} \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ -\mathrm{diag}(1,i,\ldots,i) & \mathrm{diag}(1,i,\ldots,i) \end{pmatrix} \qquad \det D_n^{[1]} = i^{n-1}$$

induces an isomorphism $SO(n,n;k) \cong SO(1,2n-1;k)$.

This map is not an isomorphism on homology, and it is known that in the limit as $n \longrightarrow \infty$ it has a large kernel.

Definition 5.9. For any field extension L/k, the groups $H_m GL(n; k)$ and $H_m SO(2n; k)$ have an induced action by $\mathrm{Gal}(L/k)$. We call this the *Galois action* and write it on the *left*. We call the action induced by conjugation by a matrix with determinant -1 on $H_* SO(n; k)$ the *conjugation action*, and we write it on the *right*. These two actions commute, and are equivariant with respect to the stabilization maps, so induce actions on $CL_{n,m}$ and $CO_{n,m}$. In the current context only quadratic extensions are considered, and ± 1-eigenspaces are denoted with a $+$ or a $-$, as before. Thus, for example, the space ${}^+CO_{n,m}(k)^-$ is the subspace of those vectors which are -1-eigenvectors for the conjugation action and $+1$-eigenvectors for the Galois action.

The module $\mathbb{Q}^\sigma$ is considered to have "Galois action" by the sign action, and abuse notation to consider the "Galois action" on $CL_{m,n}(L) \otimes \mathbb{Q}^{n\sigma}$ via the diagonal action.

In the case when a field k does not contain $\sqrt{-1}$ it is possible to compare the scissors congruence of k and the algebraic K-theory of $k(i)$. The case when $k = \mathbb{R}$ is the case of Goncharov's original conjectures.

Proposition 5.10. *Let k be a field containing $\sqrt{2}$ and not containing $i = \sqrt{-1}$. There exist natural zigzags*

$$^+(\mathrm{gr}_n^{\mathrm{rk}} K_m(k(i))_\mathbb{Q} \otimes \mathbb{Q}^{n\sigma}) \longleftarrow {}^+(CL_{n,m}(k(i)) \otimes \mathbb{Q}^{n\sigma}) \longrightarrow {}^+H_{m-n}\mathcal{P}_*(S^{2n-1}_{k(i)})$$

$$\longleftarrow H_{m-n}\mathcal{P}_*(S^{2n-1}_k)$$

and

$$^-(\mathrm{gr}_n^{\mathrm{rk}} K_m(k(i))_\mathbb{Q} \otimes \mathbb{Q}^{n\sigma}) \longleftarrow {}^-(CL_{n,m}(k(i)) \otimes \mathbb{Q}^{n\sigma})$$

$$\longrightarrow {}^+H_{m-n}(\mathcal{P}_*(\mathcal{H}^{2n-1}_{k(i)})) \longleftarrow H_{m-n}\mathcal{P}_*(\mathcal{H}^{2n-1}_k).$$

Here, the middle map is induced by the hyperbolic map and projecting to the base (Definition 4.5). All homology is taken with rational coefficients.

To make the above analysis as satisfying as possible, it is desirable to prove the following algebraic conjecture:

Conjecture 5.11. *The map $H_*(SO(2n; k); \mathbb{Q}) \longrightarrow {}^+H_*(SO(2n; k(i)); \mathbb{Q})$ is an isomorphism.*

If this conjecture were true then the zigzags in Corollary 5.10 would be shortened to a length-2 because the rightmost inclusions would be isomorphisms.

6. Proof of Theorem 4.1

In this section we prove Theorem 4.1. First, some notation. Write $\widehat{\mathcal{I}}_d$ for the category whose objects are sequences $\vec{A} = (b, a_1, \ldots, a_i)$ of nonnegative integers such that $b + a_1 + \cdots + a_i = d$ and all a_i are positive, and morphisms defined as in Definition 3.5. Recall the definition of $F_\bullet^{\vec{A}}$ in Definition 3.8. Let $\Phi^{\mathrm{eqvt}} : \widehat{\mathcal{I}}_n \longrightarrow \mathbf{Top}_*$ be defined by

$$\Phi^{\mathrm{eqvt}}(\vec{A}) \overset{\mathrm{def}}{=} S^\sigma \wedge F_\bullet^{\vec{A}}$$

and $\Phi : \widehat{\mathcal{I}}_d \longrightarrow \mathbf{Top}_*$ be defined by

$$\Phi(\vec{A}) = \Phi^{\mathrm{eqvt}}(\vec{A})_{hI(X)}.$$

The functor Φ^{eqvt} is defined the same as the definition of $\mathbf{Y}$ in Definition 3.9, extended from $\mathcal{I}_d$ to $\widehat{\mathcal{I}}_d$. Define

$$Z^{\mathrm{eqvt}} \overset{\mathrm{def}}{=} \mathrm{cofib}^{th} \, \Phi^{\mathrm{eqvt}} \qquad \text{and} \qquad Z \overset{\mathrm{def}}{=} \mathrm{cofib}^{th} \, \Phi.$$

Then (as total homotopy cofibers and homotopy coinvariants commute)

$$(Z^{\mathrm{eqvt}})_{hI(X)} \simeq Z.$$

Surprisingly, it is possible to identify the homotopy type of Z^{eqvt}.

Proposition 6.1. *There is an $I(X)$-equivariant weak equivalence*

$$Z^{\mathrm{eqvt}} \simeq S^\sigma \wedge S^d.$$

Here, the $I(X)$-action is trivial on the S^d-coordinate and acting by the determinant on S^σ.

As the proof of this is technical we postpone it to the end of the section; for now we assume it and complete the proof of Theorem 4.1. Note that this proposition is an integral statement; it is not necessary to invert 2.

We now characterize $Z_{hI(X)}$ from a different perspective. For $\vec{A} = (b, a_1, \ldots, a_i)$, by Lemma 3.14,

$$\left(S^\sigma \wedge F_\bullet^{\vec{A}} \right)_{hI(X)} \simeq_{[2]} (S^\sigma \wedge F_\bullet^W)_{hI(X^b)} \wedge \bigwedge_{j=1}^i (S^\sigma \wedge F_\bullet^{V_j})_{hO(a_j)}.$$

By Proposition 1.24, if any of the a_j are odd then $(S^\sigma \wedge F_\bullet^{V_i})_{hO(a_i)} \simeq_{[2]} *$; thus if $\vec{A}$ has some a_j odd then $\Phi(\vec{A})$ is contractible. For any atomic morphism $\iota :$

$(b, a_1, \ldots, a_i) \longrightarrow (b, a_1, \ldots, a'_\ell, a''_\ell, a_{\ell+1}, \ldots, a_i)$ (where $a_\ell = a'_\ell + a''_\ell$) we say that the morphism is *in direction r* if

$$r = a''_\ell + a_{\ell+1} + \cdots + a_i.$$

If r is odd then $\Phi(b, a_1, \ldots, a'_\ell, a''_\ell, \ldots, a_i)_{hI(X)}$ is contractible; thus all morphisms in $\Phi(\widehat{\mathcal{I}}_d)$ in odd directions have contractible codomain. Note that $\mathcal{I}_d$ is exactly the subcategory of $\widehat{\mathcal{I}}_d$ containing all atomic morphisms in even directions.

Note that there are $\lfloor \frac{d-1}{2} \rfloor$ even directions and $\lfloor \frac{d+1}{2} \rfloor$ odd directions.

It is possible to compute total homotopy cofibers *iteratively*: taking all cofibers in a single direction r, it produces a cube one dimension lower; the total homotopy cofiber of this cube is equivalent to the total homotopy cofiber of the original cube. Take homotopy cofibers in all of the even directions first: this leaves a $\lfloor \frac{d+1}{2} \rfloor$-cube with a single entry $\mathcal{Y}_{hI(X)}^X$ (at the source) and all other entries contractible; since the homotopy cofiber of any map $X \longrightarrow *$ is ΣX,

$$Z \simeq \Sigma^{\lfloor \frac{d+1}{2} \rfloor} (\mathcal{Y}_{hI(X)}^X).$$

By the homotopy orbit spectral sequence (see Proposition B.7) and Proposition 6.1,

$$H_i \left((Z^{\mathrm{eqvt}})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}] \right) \cong H_{i-(d+1)} \left(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma \right).$$

Thus

$$H_i \left(\mathcal{Y}_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}] \right) \cong H_{\lfloor \frac{d+1}{2} \rfloor + i} \left(Z; \mathbb{Z}[\tfrac{1}{2}] \right) \cong H_{\lfloor \frac{d+1}{2} \rfloor + i} \left((Z^{\mathrm{eqvt}})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}] \right)$$

$$\cong H_{i - \lfloor \frac{d-1}{2} \rfloor} \left((S^\sigma \wedge S^{n-1})_{hI(X)}; \mathbb{Z}[\tfrac{1}{2}] \right),$$

completing the proof of the theorem. $\square$

It remains to prove Proposition 6.1.

We begin by computing the homotopy cofiber of a single Dehn invariant. In order to be able to do this for any general map in the cube, it is necessary to generalize the definition of the Dehn invariant.

Let $W, U_1, \ldots, U_i$ be any decomposition of X into orthogonal subspaces. Define

$$d_j = \dim W + \sum_{\ell=1}^{j} \dim U_\ell \qquad j \geq 0.$$

(Thus $d_0 = \dim W$ and $d_i = \dim X$.) Let ℓ be an integer distinct from $d_1, \ldots, d_i$. Let j be the minimal index such that $d_j > \ell$. For convenience, define

$$D_\ell : F_\bullet^W \widetilde{\ast} F_\bullet^{U_1} \widetilde{\ast} \cdots \widetilde{\ast} F_\bullet^{U_i} \longrightarrow \bigvee_{\substack{V_j \subseteq U_j \\ \dim V_j = \ell - d_{j-1}}} F_\bullet^W \widetilde{\ast} F_\bullet^{U_1} \widetilde{\ast} \cdots \widetilde{\ast} F_\bullet^{V_j} \widetilde{\ast} F_\bullet^{V_j^\perp \cap U_j} \widetilde{\ast} \cdots \widetilde{\ast} F_\bullet^{U_i}$$

to be $1 \widetilde{\star} \cdots \widetilde{\star} D_{\ell - d_{j-1}} \widetilde{\star} \cdots \widetilde{\star} 1$.

Definition 6.2. For any subset $I \subseteq \{1, \ldots, d\}$ let $N_I F_\bullet^X$ be the subspace of $F_\bullet^X$ containing no subspace with dimension contained in I.

This definition gives a convenient way to identify the total homotopy cofiber of a Dehn cube.

Lemma 6.3. *Let X be a pointed simplicial set, and let $Y_1, \ldots, Y_n$ be subspaces of X. Write $P(n)$ for the partial order of subsets of $\{1, \ldots, n\}$. Define a functor*

$$F : P(n) \longrightarrow \mathbf{Top}_* \qquad by \qquad I \longmapsto X \Big/ \bigcup_{i \in I} Y_i,$$

with the induced morphisms given by the quotient maps. Then

$$\mathrm{cofib}^{th} F \simeq \Sigma^n \bigcap_{i=1}^{n} Y_i.$$

Proof. We prove this by induction on n. When $n = 0$ the cube is trivial and the statement holds. When $n = 1$ the cube is $X \longrightarrow X/Y$, and the total homotopy cofiber is ΣY, as desired.

Now consider the general case. The total homotopy cofiber can be computed iteratively [22, Proposition 5.9.3] by first taking cofibers in the direction of "adding n to a set": the morphisms in which each subset $J \in P(n-1)$ is mapped to $J \cup \{n\}$. Taking the homotopy cofiber for each such J produces the cube $G : P(n-1) \longrightarrow \mathbf{Top}_*$ given by

$$J \longmapsto \Sigma Y_n \Big/ \bigcup_{j \in J} \Sigma(Y_j \cap Y_n).$$

This is an $n-1$-cube of the same type as in the proposition; by the induction hypothesis,

$$\mathrm{cofib}^{th} G \simeq \Sigma^{n-1} \bigcap_{i=1}^{n-1} \Sigma(Y_i \cap Y_n).$$

$\Sigma(Y_i \cap Y_n)$ sits inside ΣX as $\Sigma Y_i \cap \Sigma Y_n$; then

$$\Sigma^{n-1} \bigcap_{i=1}^{n-1} \Sigma(Y_i \cap Y_n) = \Sigma^{n-1} \bigcap_{i=1}^{n} (\Sigma Y_i) \cap (\Sigma Y_n) = \Sigma^{n-1} \bigcap_{i=1}^{n} \Sigma Y_i = \Sigma^n \bigcap_{i=1}^{n} Y_i,$$

as desired. $\square$

Proposition 6.4. *Let $I \subseteq \{1, \ldots, d\}$. Consider the sub-$|I|$-cube formed by D_i for $i \in I$ and containing the initial point. This cube has total homotopy cofiber $\Sigma^{|I|} N_I F_\bullet^X$.*

Proof. For conciseness, write D_I for the composition of the D_i for $i \in I$. Since the Dehn cube commutes, the order of composition is irrelevant. Let $I = \{i_0, \ldots, i_{j-1}\}$. We claim that

$$D_I : F_\bullet^X \longrightarrow \bigvee_{\substack{W \oplus^\perp U_1 \oplus^\perp \cdots \oplus^\perp U_j = X \\ \dim W = i_0 \\ \dim U_\ell = i_\ell - i_{\ell-1} - \cdots - i_0 \ \ell < j}} F_\bullet^W \widetilde{\star} F_\bullet^{U_1} \widetilde{\star} \cdots \widetilde{\star} F_\bullet^{U_j}$$

is isomorphic to the map

$$F_\bullet^X \longrightarrow F_\bullet^X / \bigcup_{i \in I} N_{\{i\}} F_\bullet^X$$

via the isomorphism

$$\bigvee_{\substack{W \oplus^\perp U_1 \oplus^\perp \cdots \oplus^\perp U_j = X \\ \dim W = i_0 \\ \dim U_\ell = i_\ell \ \ell < j}} F_\bullet^W \widetilde{\star} F_\bullet^{U_1} \widetilde{\star} \cdots \widetilde{\star} F_\bullet^{U_j} \longrightarrow F_\bullet^X / \bigcup_{i \in I} N_{\{i\}} F_\bullet^X$$

taking an ℓ_j-simplex

$$(U_0 \subseteq \cdots \subseteq U_{\ell_0}, U_{\ell_0+1} \subseteq \cdots \subseteq U_{\ell_1}, \ldots, U_{\ell_{j-1}+1} \subseteq \cdots \subseteq U_{\ell_j})$$

to ℓ_j-simplex corresponding to the flag

$$U_0 \subseteq \cdots \subseteq U_{\ell_0} \subset U_{\ell_0} \oplus U_{\ell_0+1} \subseteq U_{\ell_0} \oplus U_{\ell_0+2} \subseteq \cdots \subseteq U_{\ell_0} \oplus \cdots \oplus U_{\ell_j}.$$

Every flag in the image contains subspaces of all dimensions contained in I, and any face map that removes one of them takes the simplex to the basepoint. This map is bijective on simplices, showing that it is an isomorphism of simplicial sets.

Since

$$N_I F_\bullet^X \cong \bigcap_{i \in I} N_{\{i\}} F_\bullet^X,$$

the Dehn cube is isomorphic to a cube of the form in Lemma 6.3. Applying the lemma we see that the total homotopy cofiber is

$$\Sigma^{|I|} \bigcap_{i \in I} N_{\{i\}} F_\bullet^X = \Sigma^{|I|} N_I F_\bullet^X,$$

as desired. $\square$

This can be used to prove Proposition 6.1:

Proof of Proposition 6.1. By Proposition 6.4, $Z^{\mathrm{eqvt}} \simeq S^\sigma \wedge \Sigma^d N_{\{0,1,\dots,d-1\}} F_\bullet^X$. However, $N_{\{0,1,\dots,d-1\}} F_\bullet^X \cong S^0$, as it has exactly two simplices in each dimension: the basepoint and $X = \cdots = X$. The functor $S^\sigma \wedge \cdot$ commutes with taking homotopy cofibers; thus

$$Z^{\mathrm{eqvt}} \simeq S^\sigma \wedge S^d. \quad \square$$

To finish up this section we use this calculation to prove Lemma 4.7. First, a few definitions. Denote by $\vec{g}$ a tuple $(g_1, \dots, g_j)$ of elements in $I(X)$; this tuple can be of any length j. For $0 \le \ell \le j$ and a coefficient $m \in \mathbb{Z}[\frac{1}{2}]$, define the notation

$$d_\ell(m\vec{g}) \overset{\mathrm{def}}{=} \begin{cases} m(\det g)(g_2, \dots, g_j) & \text{if } \ell = 0 \\ m(g_1, \dots, g_{\ell+1} g_\ell, \dots, g_j) & \text{if } 1 \le \ell < j \\ m(g_1, \dots, g_{j-1}) & \text{if } j = \ell. \end{cases}$$

Write, for $1 \le a \le b \le j$,

$$\Pi_a^b \vec{g} \overset{\mathrm{def}}{=} g_b \cdots g_a \qquad \text{and} \qquad \amalg_a^b \vec{g} \overset{\mathrm{def}}{=} g_a^{-1} \cdots g_b^{-1} = \left(\Pi_a^b \vec{g} \right)^{-1}.$$

The double complex C_{**} is a homologically-graded double complex in which for $j \ge 0$ and $0 \le i < d$ the group C_{ij} is generated by symbols of the form

$$(g_1, \dots, g_j)\{x_1 | \cdots | x_i\},$$

where $g_1, \dots, g_j \in I(X)$ and $x_1, \dots, x_i \in X$. When j is clear from context we sometimes write this $\vec{g}\{x_1 | \cdots | x_i\}$. Define boundary maps $\partial^h : C_{ij} \longrightarrow C_{(i-1)j}$ and $\partial^v : C_{ij} \longrightarrow C_{i(j-1)}$ by

$$\partial^h(\vec{g}\{x_0 | \cdots | x_i\}) \overset{\mathrm{def}}{=} \vec{g} \sum_{\ell=0}^i (-1)^\ell \{x_0 | \cdots | \widehat{x}_\ell | \cdots | x_i\} \qquad \text{and}$$

$$\partial^v(\vec{g}\{x_0 | \cdots | x_i\}) \overset{\mathrm{def}}{=} (d_0 \vec{g})\{g_1 x_0 | \cdots | g_1 x_i\} + \sum_{\ell=1}^{j-1} (-1)^\ell (d_\ell \vec{g})\{x_0 | \cdots | x_i\}.$$

Lemma 6.5. *Let $\sigma \overset{\mathrm{def}}{=} \sum_i m_i \vec{g}^{(i)}$ be a cycle representing an element in $H_d(I(X); \mathbb{Z}[\frac{1}{2}]^\sigma)$ and fix any point $x \in X$. Inside the total complex of C_{**} the cycle $\sum_i m_i \vec{g}^{(i)}\{\}$ is homologous to*

$$(-1)^{d+1} \sum_i m_i (\det \Pi_1^d \vec{g}^{(i)}) \sum_{\ell=0}^d (-1)^\ell \left\{ x \Big| g_d^{(i)} \cdot x \Big| \cdots \Big| \Pi_{d-\ell+2}^d \widehat{\vec{g}^{(i)}} \cdot x \Big| \cdots \Big| \Pi_2^d \vec{g}^{(i)} \cdot x \Big| \Pi_1^d \vec{g}^{(i)} \cdot x \right\},$$

(where the $\ell = 0$ term removes the $g_1^{(i)} x$).

Proof. Fix $x \in S^d$. For any $\vec{g} = (g_1, \ldots, g_j)$, any $1 \leq \lambda \leq j\}$, and any point $y \in S^d$, and any $m \in \mathbb{Z}[\frac{1}{2}]$, define

$$\Delta^\lambda(m\vec{g}, y) \overset{\text{def}}{=} m(g_1, \ldots, g_\lambda)\Big\{y \Big| \amalg_1^j \vec{g} \cdot x \Big| \amalg_1^{j-1} \vec{g} \cdot x \Big| \cdots \Big| \amalg_1^\lambda \vec{g} \cdot x\Big\} \in C_{(j-\lambda+2)\lambda}$$

$$B^\lambda(m(a_1, \ldots, a_j)) \overset{\text{def}}{=} m(g_1, \ldots, g_\lambda)\Big\{\amalg_1^j \vec{g} \cdot x \Big| \amalg_1^{j-1} \vec{g} \cdot x \Big| \cdots \Big| \amalg_1^\lambda \vec{g} \cdot x\Big\} \in C_{(j-\lambda+1)\lambda}.$$

For any $\vec{g} = (g_1, \ldots, g_j)$ we have

$$\partial^v \Delta^\lambda(\vec{g}, y) = \Delta^{\lambda-1}(d_0\vec{g}, g_1 y)$$

$$+ \sum_{\ell=1}^{\lambda-1} (-1)^\ell \Delta^{\lambda-1}(d_\ell \vec{g}, y) + (-1)^\lambda (g_1, \ldots, g_{j-1})\Big\{y \Big| \amalg_1^j \vec{g} \cdot x \Big| \cdots \Big| \amalg_1^\lambda \vec{g} \cdot x\Big\}, \text{ and}$$

$$\partial^h \Delta^{\lambda-1}(\vec{g}, y) = B^{\lambda-1}(\vec{g})$$

$$+ (-1)^{j+1} \sum_{\ell=\lambda}^{j} (-1)^\ell \Delta^{\lambda-1}(d_\ell \vec{g}, y) + (-1)^{j+\lambda}(g_1, \ldots, g_{\lambda-1})\Big\{y \Big| \amalg_1^j \vec{g} \cdot x \Big| \cdots \Big| \amalg_1^\lambda \vec{g} \cdot x\Big\}.$$

Now consider σ, so that in the formulas above $j = d$. Define

$$\alpha^\lambda \overset{\text{def}}{=} \sum_i m_i \Big(\Delta^\lambda(d_0\vec{g}^{(i)}, g_1^{(i)}x) + \sum_{\ell=1}^{d}(-1)^\ell \Delta^\lambda(d_\ell \vec{g}^{(i)}, x)\Big) \in C_{(d+1-\lambda)\lambda}.$$

The above calculations (which will have $j = d - 1$) imply that

$$\partial^v \alpha^\lambda = \partial^h \alpha^{\lambda-1},$$

using the fact that (since σ is a cycle)

$$\sum_i m_i \sum_{\ell=0}^{j}(-1)^\ell B^{\lambda-1}(d_\ell \vec{g}^{(i)}) = 0.$$

Thus, in the total complex,

$$\partial \sum_{\ell=1}^{d}(-1)^\ell \alpha^\lambda = \partial^v \alpha^1 + (-1)^d \partial^h \alpha^d.$$

Plugging in the definitions produces that $\partial^h \alpha^d = \sigma$ and

$$\partial^v \alpha^1 = \sum_i m_i(\det g_1^{(i)}) \sum_{\ell=0}^{d}(-1)^\ell \Big\{g_1^{(i)}x \Big| \amalg_2^d \vec{g}^{(i)} \cdot x \Big| \cdots \Big| \amalg_2^{\widehat{d+1-\ell}} \vec{g}^{(i)} \cdot x \Big| \cdots \Big| \amalg_2^2 \vec{g}^{(i)}x \Big| x\Big\}$$

$$+ \sum_i m_i \sum_{\ell=0}^{d}(-1)^\ell \Delta^0(d_\ell \vec{g}^{(i)}, x).$$

(Here we abuse notation and declare that $\Pi_2^{d+1}\vec{g}^{(i)} = g_1$.) As σ is a cycle, the second sum is 0. To complete the proof observe that for any $d+1$-tuple of points $(y_0, \ldots, y_d)$, the class $\sum_{\ell=0}^d \{y_0| \cdots |\widehat{y_\ell}| \cdots |y_d\}$ is a horizontal cycle, and is therefore homologous to

$$\sum_{\ell=0}^d \{g \cdot y_0| \cdots |\widehat{g \cdot y_\ell}| \cdots |g \cdot y_d\}$$

for any $g \in I(X)$. Thus each term in the sum (over i) for $\partial^v \alpha^1$ is a cycle. Acting on the i-th term by $\prod_2^d \vec{g}^{(i)}$ gives the desired expression up to permuting the first element to the end. As this requires d swaps, it changes the sign by $(-1)^d$. □

We are now ready to prove the general case of Lemma 4.7:

Lemma 6.6. *Let $d = 2n$ or $2n-1$. In the spectral sequence of (4.4) the map*

$$\epsilon_{n-1} : H_d(I(X); \mathbb{Z}[\tfrac{1}{2}]^\sigma) \longrightarrow H_{n-1}\mathcal{P}_*(X)$$

is induced by the map taking a chain $(g_1, \ldots, g_d)$ to the sum

$$\left(\prod_{i=1}^d \det(g_i)\right) \sum_{\sigma \in \mathrm{Aut}\{0,\ldots,d\}} \mathrm{sgn}(\sigma)\, [V_{\sigma,0} \subseteq V_{\sigma,1} \subseteq \cdots \subseteq V_{\sigma,d}].$$

Here we define

$$V_{\sigma,i} = \mathrm{span}(h_{\sigma(0)}x_0, h_{\sigma(1)}x_0, \ldots, h_{\sigma(i)}x_0),$$

with $h_d = 1$, $h_i = g_d \cdots g_{d-i}$, and x_0 any fixed point in X.

When $k = \mathbb{R}$ this is the class of the d-simplex with vertices

$$\{h_0 x_0, h_1 x_0, \ldots, h_d x_0\}.$$

Proof. The map ϵ_{n-1} is induced by the edge homomorphism $\mathrm{cofib}^{th}\, \mathbf{Y}_{hI(X)} \longrightarrow \mathrm{cofib}^{th}\, \widehat{\mathbf{Y}}_{hI(X)}$, where

$$\widehat{\mathbf{Y}}(\vec{A}) \overset{\mathrm{def}}{=} \begin{cases} \mathbf{Y}((d)) & \text{if } \vec{A} = (d) \\ * & \text{otherwise.} \end{cases}$$

(This is the quotient of the $n-1$-st filtration level by the $n-2$-nd in the spectral sequence for the total homotopy cofiber of a cube.) Both of these total homotopy cofibers can be computed simultaneously and $I(X)$-equivariantly via the methods above. As all maps in $\widehat{\mathbf{Y}}$ map to the basepoint, each direction will simply suspend the (d)-case. This implies that ϵ_{n-1} is the map on H_d induced by taking $I(X)$-coinvariants of the map

$$S^\sigma \wedge \Sigma^d S^0 \xrightarrow{\ S^\sigma \wedge \iota_0\ } S^\sigma \wedge \Sigma^d F_\bullet^X$$

where ι_0 is the inclusion of the 0-skeleton into $F_\bullet^X$.

This map can be described in an alternate manner. Model homotopy coinvariants via taking an extra simplicial direction (see, for example, [16, Example IV.1.1]) and take the double chain complex associated to a bisimplicial set. Then the homology of the geometric realization is isomorphic to the homology of this total complex. Due to the suspension coordinates, the bottom $d+1$ rows of this double complex are 0. Above this (assuming that the group-coordinate is vertical and the flag-coordinate is vertical) the map above includes the standard bar construction for $H_*(I(X); \mathbb{Z}[\frac{1}{2}]^\sigma)$ as the leftmost column. In order to show that the given formula for ϵ_{n-1} holds it is therefore sufficient to show the following: given a cycle σ in $C_d(I(X); \mathbb{Z}[\frac{1}{2}]^\sigma)$ (which lies in coordinate $(0, 2d+1)$ in the double complex), it is homologous to the cycle given by the formula in the statement of the lemma (which lies in coordinate $(d, d+1)$).

In the double complex associated to the above bisimplicial construction, the group at coordinate $(m, \ell + d + 1)$ is generated by diagrams of the form

$$
\begin{array}{ccccccc}
X_0 & \hookrightarrow & X_1 & \hookrightarrow & \cdots & \hookrightarrow & X_m \\
\downarrow & & \downarrow & & & & \downarrow \\
g_1 X_0 & \hookrightarrow & g_1 X_1 & \hookrightarrow & \cdots & \hookrightarrow & g_1 X_m \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
g_\ell X_0 & \hookrightarrow & g_\ell X_1 & \hookrightarrow & \cdots & \hookrightarrow & g_\ell X_m
\end{array}
\qquad (6.7)
$$

We denote such a diagram by

$$(g_1, \ldots, g_\ell)[X_0 \subseteq \cdots \subseteq X_m].$$

We define

$$\{x_0 | \cdots | x_i\} = \sum_{\sigma \in \mathrm{Aut}\{0,\ldots,i\}} \mathrm{sgn}(\sigma)[\mathrm{span}(x_{\sigma(0)}) \subseteq \mathrm{span}(x_{\sigma(0)}, x_{\sigma(1)})$$

$$\subseteq \cdots \subseteq \mathrm{span}(x_{\sigma(0)}, \ldots, x_{\sigma(i)}) \subseteq X].$$

This double complex contains as a (vertically shifted by $d+1$) subcomplex the complex C_{**} of Lemma 6.5; the conclusion of the lemma is exactly the desired formula.

The last claim in the lemma follows by Theorem 1.21. $\square$

Appendix A. Comparing RT-buildings to the classical constructions

In this appendix we prove our claims in Theorem 1.21 and Theorem 2.15 that the construction of the scissors congruence groups in our account agrees with the classical constructions. To begin we introduce an object closely related to the configuration space of points in X. The simplices in this space are tuples of points in X; unlike in the configuration space, points are allowed to be repeated, and this can produce nondegenerate simplices. For example, for any two distinct points $a, b \in X$, (a, b, a) is a nondegenerate 2-simplex in $\mathrm{Tuple}_{\bullet}^{m}(X)$.

Definition A.1. $\mathrm{Tuple}_{\bullet}^{m}(X)$ is the simplicial set whose i-simplices are given by the subset of $\prod_{j=0}^{i} X$ of those tuples $(x_0, \ldots, x_i)$ such that any subset of the tuple has a nondegenerate span of dimension at most m. The j-th face map is given by dropping the j-th element of the tuple; the j-th degeneracy is given by repeating the j-th element of the tuple.

The homology of $\mathrm{Tuple}_{\bullet}^{m}(X)$ is directly related to scissors congruence groups, as the following results illustrate:

Theorem A.2. *[13, Theorem 2.10] Let $k = \mathbb{R}$. The map taking a tuple of points to its convex hull defines a $I(\mathcal{H}^n)$-equivariant isomorphism*

$$H_n(\mathrm{Tuple}_{\bullet}^{n}(\mathcal{H}^n)/\mathrm{Tuple}_{\bullet}^{n-1}(\mathcal{H}^n))^{\sigma} \longrightarrow \mathcal{P}(\mathcal{H}^n, 1).$$

Here, $\cdot^{\sigma}$ means that the action is twisted by the determinant: for any $g \in I(\mathcal{H}^n)$, g acts on a homology on the left by $(-1)^{\det g}$ as well as by the usual action on $\mathcal{H}^n$.

The spherical case is more complicated. Recall the map Σ, defined as the suspension of a polytope, from Definition 1.13.

Theorem A.3 *([12, Corollary 5.18]). The map taking a simplex to its convex hull induces a $O(n+1)$-equivariant isomorphism*

$$H_n(\mathrm{Tuple}_{\bullet}^{n}(S^n)/\mathrm{Tuple}_{\bullet}^{n-1}(S^n))^{\sigma} \longrightarrow (\mathrm{coker}\,\Sigma).$$

In particular, since Σ is $O(n+1)$-equivariant, Σ induces an isomorphism on coinvariants

$$H_0(O(n+1), H_n(\mathrm{Tuple}_{\bullet}^{n}(S^n)/\mathrm{Tuple}_{\bullet}^{n-1}(S^n)^{\sigma}) \longrightarrow \mathcal{P}(S^n, O(n+1)).$$

It follows that in order to relate scissors congruence groups and the homology of $S^{\sigma} \wedge F_{\bullet}^{X}$ it suffices to show that $\mathrm{Tuple}_{\bullet}^{\dim X}(X)/\mathrm{Tuple}_{\bullet}^{\dim X-1}(X)$ and $F_{\bullet}^{X}$ are $I(X)$-equivariantly homotopy equivalent.

We begin with a basic lemma about the homotopy type of $\mathrm{Tuple}_{\bullet}(X)$ in the absence of dimension restrictions:

Lemma A.4.

$$\text{Tuple}_\bullet^{\dim X}(X) \simeq *.$$

Proof. By [5, Proposition 2.2.1], since k is infinite $\widetilde{H}_*(\text{Tuple}_\bullet^{\dim X}(X)) = 0$. (In fact, Cathelineau proves this only with rational coefficients, but his proof works equally well integrally.) To see that $\text{Tuple}_\bullet^{\dim X}(X)$ is contractible it suffices to check that it is simply-connected. By [5, Proposition 2.2.2] for any pair of points (x, y) spanning a subspace of X, the subset $W_{x,y}$ of those points in X such that (x, y, w) spans a subspace is a Zariski-open subspace of X. Suppose that we are given a loop represented by the sequence of 1-simplices $(x_0, x_1), (x_1, x_2), \ldots, (x_i, x_0)$. Then, since k is infinite, there exists a point w such that (x_j, w) spans a subspace for all j, and the loop is homotopic to a loop of the form $(x_0, w), (w, x_0)$. This is contracted by the 2-simplex (x_0, w, x_0), so $\text{Tuple}_\bullet^{\dim X}(X)$ is contractible. $\square$

For any simplicial set $K_\bullet$, let $\text{Sd}\, K_\bullet$ be the barycentric subdivision of $K_\bullet$ [16, Section III.4]. Define the map $h : \text{Sd}\,\text{Tuple}_\bullet^m(X) \longrightarrow T_\bullet^m(X)$ to be the map induced by taking a tuple of points in X to their span. More explicitly, an i-simplex in $\text{Sd}\,\text{Tuple}_\bullet^m(X)$ is a sequence $\vec{x}_0 \subseteq \vec{x}_1 \subseteq \cdots \subseteq \vec{x}_i$, where $\vec{x}_j$ is a tuple in X and $\vec{x}_{j-1}$ is an (ordered) subset of $\vec{x}_j$ for all j. Taking the spans of each tuple produces an i-simplex in $T_\bullet^m(X)$; as taking spans is G-equivariant, this map is G-equivariant.

Proposition A.5. *The map*

$$h : \text{Sd}\,\text{Tuple}_\bullet^m(X) \longrightarrow T_\bullet^m(X)$$

induced by taking tuples in X to their spans is a G-equivariant weak equivalence.

Proof. We use Theorem A' [15, p.578], which states that a map of simplicial sets is a weak equivalence if the "naive left homotopy fiber" above every simplex in the codomain is contractible. Here, for a given a q-simplex $y \in T_q^m(X)$ represented by $(U_0 \subseteq \cdots \subseteq U_q)$, the naive left homotopy fiber is the simplicial set

$$(h|y)_p = \left\{ (\vec{x}_0 \subseteq \cdots \subseteq \vec{x}_p) \in \text{Sd}\,\text{Tuple}_p^m(X) \,\middle|\, \text{all entries of } \vec{x}_p \text{ are in } U_0 \right\}.$$

(For a precise definition of the naive homotopy fiber see for example [20, Defn. 3.1].) In this case, $(h|y)_p$ is isomorphic to the simplicial set $\text{Sd}\,\text{Tuple}_\bullet^m(U_0)$; as this is isomorphic to $\text{Sd}\,\text{Tuple}_\bullet^{\dim U_0}(U_0)$ it is contractible by Lemma A.4. $\square$

The $m = \dim X$ case of the following theorem shows that $F_\bullet^X$ is G-equivariantly homotopy equivalent to a quotient of tuple spaces.

Theorem A.6. *For all $m \geq 0$*

$$\operatorname{Tuple}_\bullet^m(X)/\operatorname{Tuple}_\bullet^{m-1}(X) \simeq T_\bullet^m(X)/T_\bullet^{m-1}(X)$$

via a zigzag of G-equivariant maps.

Proof. We have the G-equivariant commutative diagram

$$
\begin{array}{ccccc}
\operatorname{Tuple}_\bullet^{m-1}(X) & \xleftarrow{\;\sim\;} & \operatorname{Sd}\operatorname{Tuple}_\bullet^{m-1}(X) & \xrightarrow[h]{\;\sim\;} & T_\bullet^{m-1}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Tuple}_\bullet^m(X) & \xleftarrow{\;\sim\;} & \operatorname{Sd}\operatorname{Tuple}_\bullet^m(X) & \xrightarrow[h]{\;\sim\;} & T_\bullet^m(X)
\end{array}
$$

where the vertical maps are injective on all i-simplices, hence cofibrations. Taking vertical cofibers gives the desired result, as the cofibers of the vertical maps are also the homotopy cofibers. $\square$

Corollary A.7.

$$\widetilde{H}_i(F_\bullet^X) = 0 \qquad \text{for } i \neq \dim X.$$

Proof. By definition $F_\bullet^X = T_\bullet^{\dim X}(X)/T_\bullet^{\dim X-1}(X)$. As all simplices of $T_\bullet^{\dim X}(X)$ above $\dim X$ are degenerate (since they must repeat at least one subspace) it must be the case that $\widetilde{H}_i(F_\bullet^X) = 0$ for $i > \dim X$. By Theorem A.6, $F_\bullet^X \simeq \operatorname{Tuple}_\bullet^{\dim X}(X)/\operatorname{Tuple}_\bullet^{\dim X-1}(X)$. However, all simplices of $\operatorname{Tuple}_\bullet^{\dim X}(X)$ of dimension less than $\dim X$ are contained in $\operatorname{Tuple}_\bullet^{\dim X-1}(X)$, since the span of i points has dimension at most $i-1$. Thus $\widetilde{H}_i(F_\bullet^X) = 0$ for $i < \dim X$. $\square$

Using these results we can finally prove Theorem 1.21:

Proof of Theorem 1.21. Theorems A.2, A.3 and A.6, together with (1.16) demonstrate that scissors congruence groups are group homology with coefficients in $H_n(F_\bullet^X)^\sigma$; this is exactly $H_{n+1}(S^\sigma \wedge F_\bullet^X)$. By the homotopy orbit spectral sequence (Proposition B.7) this is $H_{n+1}((S^\sigma \wedge F_\bullet^X)_{hI(X)})$, as desired. The formula for the class represented by the vertices of a simplex follows from Theorem A.6 and the fact that on homology the inverse

to the map $\operatorname{Sd}\operatorname{Tuple}_\bullet^{m-1}(X) \xrightarrow{\;\sim\;} \operatorname{Tuple}_\bullet^{m-1}(X)$ is given by the formula

$$[x_0,\ldots,x_n] \longmapsto \sum_{\sigma \in \operatorname{Aut}\{1,\ldots,n\}} \operatorname{sgn}(\sigma)\big[(x_{\sigma(0)})' \subseteq (x_{\sigma(0)}, x_{\sigma(1)})' \subseteq \cdots \subseteq (x_{\sigma(0)},\ldots,x_{\sigma(n)})'\big].$$

Here, $(x_{\sigma(0)},\ldots,x_{\sigma(i)})'$ is ordered *not* by the ordering $0,\ldots,i$ but rather by the ordering induced on the x's from the tuple $(x_0,\ldots,x_n)$. $\square$

We wrap up this section by proving our claim in Theorem 2.15, that the derived definition of the Dehn invariant is compatible with the classical Dehn invariant. As we saw in Theorem 1.21, in order to translate between classical scissors congruence groups and RT-buildings we must take "semi-coinvariants," and thus the twist by S^σ (Definition 1.18) appears here as well.

Proof of Theorem 2.15. Rewriting $\widehat{D}_i$ using Lemma 2.4, we see that it suffices to construct an $I(X)$-equivariant diagram relating $\bigoplus_U \widehat{D}_U$ to $H_{n+1}(S^\sigma \wedge D_i)$.

For a geometry W of dimension i, write

$$R_\bullet(W) \overset{\text{def}}{=} \operatorname{Sd} \operatorname{Tuple}_\bullet^i(W) / \operatorname{Sd} \operatorname{Tuple}_\bullet^{i-1}(W)$$

for the quotient of barycentric subdivisions Sd. To define $D_i^R : R_\bullet(X) \longrightarrow \bigvee_U R_\bullet(U) \widetilde{\star} R_\bullet(U^\perp)$ consider a j-simplex of $R_\bullet(W)$: this is represented by a sequence $T_0 \subseteq \cdots \subseteq T_j$ of tuples of points in W such that the span of T_j is W. If there exists a maximal ℓ such that $\dim \operatorname{span} T_\ell = i$, we map this j-simplex to the simplex $(T_0 \subseteq \cdots \subseteq T_\ell) \wedge (\operatorname{pr}_{U^\perp} T_{\ell+1} \subseteq \cdots \subseteq \operatorname{pr}_{U^\perp} T_j)$, indexed by $\operatorname{span} T_\ell$. Otherwise, we map to the basepoint. This is a well-defined simplicial map for the same reason that D_i is.

Consider the following diagram:

$$
\begin{array}{ccc}
H_{n+1}(S^\sigma \wedge F_\bullet^X) & \xrightarrow{\;H_{n+1}(S^\sigma \wedge D_i)\;} & H_{n+1}\left(S^\sigma \wedge \bigvee_{\substack{U \subseteq X \\ \dim U = i}} F_\bullet^U \widetilde{\star} F_\bullet^{U^\perp}\right) \\
\Big\uparrow{\wr}\, {\scriptstyle h} & & \Big\uparrow{\wr}\, {\scriptstyle h} \\
H_{n+1}(S^\sigma \wedge R_\bullet(X)) & \xrightarrow{\;H_{n+1}(S^\sigma \wedge D_i^R)\;} & H_{n+1}\left(S^\sigma \wedge \bigvee_{\substack{U \subseteq X \\ \dim U = i}} R_\bullet(U) \widetilde{\star} R_\bullet(U^\perp)\right) \\
{\scriptstyle p}\,\Big\downarrow{\wr} & & {\scriptstyle p}\,\Big\downarrow{\wr} \\
\mathcal{P}(X,1) & \xrightarrow{\qquad D_\mathbb{Q} \qquad} & \bigoplus_{\substack{U \subseteq X \\ \dim U = i}} \mathcal{P}(U,1) \otimes \mathcal{P}(S^{n-i-1},1)
\end{array}
$$

Here, the vertical maps h are induced by the map h in Theorem A.6 (and are thus isomorphisms). The vertical maps p are defined as in Theorem 1.21 (with G trivial) and are therefore isomorphisms. The map $D_\mathbb{Q}$ is the rationalized Dehn invariant. Since all maps in this diagram are $I(X)$-equivariant, the lemma follows. □

Appendix B. Technical miscellany

B.1. Reduced joins

In this section we restate the definition of a reduced join and prove several important properties.

Definition B.1 *(Definition 2.7)*. For pointed simplicial sets X and Y, the *reduced join* $X \widetilde{\star} Y$ is defined by

$$(X \widetilde{\star} Y)_m = \bigvee_{i+j=m-1} X_i \wedge Y_j.$$

For a simplex $(x, y) \in X_i \wedge Y_j$, the face maps d_ℓ are defined to be $d_\ell \times 1 : X_i \wedge Y_j \longrightarrow X_{i-1} \wedge Y_j$ when $\ell \leq i$, and $1 \times d_{\ell-i-1} : X_i \wedge Y_j \longrightarrow X_i \wedge Y_{j-1}$ otherwise. If $i = \ell = 0$ or $j = m - 1 - \ell = 0$ then the face map takes the simplex to the basepoint. Degeneracies are defined analogously, with the first $i + 1$ acting on the x-coordinate, and the last $m - i - 1$ acting on the y-coordinate. Note that this structure makes the reduced join *asymmetric*.

Lemma B.2. *Reduced joins distribute over wedge products.*

Proof. We have

$$\left(\bigvee_{\alpha \in A} (X_\alpha \widetilde{\star} Y) \right)_n = \bigvee_{\alpha \in A} \bigvee_{i+j=n-1} (X_\alpha)_i \wedge Y_j$$

$$= \bigvee_{i+j=n-1} \left(\bigvee_{\alpha \in A} X_\alpha \right)_i \wedge Y_j = \left(\left(\bigvee_{\alpha \in A} X_\alpha \right) \widetilde{\star} Y \right)_n.$$

Since each step of this expression commutes with simplicial maps, the two are isomorphic as simplicial sets. $\square$

Lemma B.3. *Let $f : X \longrightarrow Y$ be a quotient of simplicial sets. Then the map $f \widetilde{\star} 1 : X \widetilde{\star} Z \longrightarrow Y \widetilde{\star} Z$ is also a quotient of simplicial sets. $(f \widetilde{\star} 1)^{-1}(*) = f^{-1}(*) \widetilde{\star} Z$.*

Proof. It suffices to show that every nonbasepoint simplex in the codomain has a unique preimage in the domain. Consider a non-basepoint n-simplex in $Y \widetilde{\star} Z$; this is a pair of the form (y_i, z_j) with $y_i \in Y_i$, $z_j \in Z_j$ and $i + j = n - 1$. As $y_i \in Y_i$ is non-basepoint, it has a unique preimage $x_i \in X_i$. As the given map takes (x, z) to $(f(x), z)$ the preimage of (y_i, z_j) is exactly $(f^{-1}(y_i), z_j)$, which is unique.

The simplices that map to the basepoint are exactly those that f maps to the basepoint, with anything in the Z-coordinate. $\square$

We end by giving a map relating the smash product and the reduced join.

Lemma B.4. *Let X and Y be pointed simplicial sets. The map $f : S^1 \wedge X \wedge Y \longrightarrow X \widetilde{\star} Y$ given by sending $(i, x, y) \in (S^1 \wedge X \wedge Y)_n$ to $(d_i^{n-i+1} x, d_0^{i+1} y)$ is a simplicial weak equivalence.*

Proof. The fact that f is well-defined is direct from the definition. We define $X *_w Y$ to be the double mapping cylinder of the diagram

$$X \xleftarrow{\mathrm{pr}_X} X \times Y \xrightarrow{\mathrm{pr}_Y} Y.$$

We can thus think of $X *_w Y$ as the quotient of $I \times X \times Y$ given by the mapping cylinder relations $(x, 0, y) \sim (x', 0, y)$ and $(x, 1, y) \sim (x, 1, y')$ for all $x, x' \in X$ and $y, y' \in Y$. Consider the following commutative square:

$$
\begin{array}{ccc}
X *_w Y & \xrightarrow{\;g\;} & S^1 \wedge X \wedge Y \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
X * Y & \xrightarrow{\;g'\;} & X \,\widetilde{\ast}\, Y
\end{array}
$$

The maps g and g' are both weak equivalences because they are quotients by contractible subspaces. The map f' is a weak equivalence by [14, Corollary 3.4]. Thus, by 2-of-3, f is a weak equivalence, as desired. $\quad\square$

B.2. Homotopy coinvariants

All of the results in this section are well-known to experts, although we could not find references for them for the specific cases we were interested in.

Definition B.5 *([16, Example IV.1.10]).* Let X be a (pointed) simplicial set with an action by a discrete group G. The *homotopy coinvariants* (or *homotopy orbits*) of G acting on X, denoted X_{hG}, is the diagonal of the bisimplicial set with (m, n)-simplices given by diagrams

$$x \xrightarrow{\;g_1\;} g_1 x \xrightarrow{\;g_2\;} g_2 g_1 x \longrightarrow \cdots \xrightarrow{\;g_n\;} g_n \cdots g_1 x$$

for $x \in X_m$.

Directly from the definition we see that $*_{hG} \cong *$ and $S^0_{hG} \cong BG_+$.

Remark B.6. This agrees with the more standard definition of homotopy coinvariants, defined as

$$X_{hG} \stackrel{\mathrm{def}}{=} EG_+ \wedge_G X.$$

(In the unpointed context, $\wedge$ is replaced by $\times$.)

There is a spectral sequence for computing the homology of the homotopy orbits from the group homology of G with coefficients in the homology of X:

Proposition B.7. *There is a spectral sequence*

$$H_p(G, \widetilde{H}_q(X)) \Longrightarrow \widetilde{H}_{p+q}(X_{hG}).$$

The proposition holds for all simplicial sets with G-action, which is the case of concern in this paper.

Proof. Consider X as an unpointed simplicial set; write this space $\overline{X}$. The homology of the diagonal simplicial set of a bisimplicial set is the homology of the total complex of the associated simplicial abelian group. The spectral sequence associated to a simplicial abelian group $A_{\bullet\bullet}$ has

$$E_{p,q}^2 = H_q^{\text{vert}} H_p^{\text{horiz}}(A_{\bullet\bullet}) \Longrightarrow H_{p+q}(\text{diag } A_{\bullet\bullet}).$$

Applying this in the current case to both $\overline{X}_{hG}$ and $*_{hG}$ gives us the following pair of spectral sequences:

$$H_p(G, H_q(X)) \Longrightarrow H_{p+q}(\overline{X}_{hG}) \qquad \text{and} \qquad H_p(G, H_q(*)) \Longrightarrow H_{p+q}(BG).$$

The second is a retract of the first; if we take the other summand, we get a spectral sequence

$$H_p(G, \widetilde{H}_q(X)) \Longrightarrow \widetilde{H}_{p+q}(X_{hG}),$$

as desired. $\square$

Lastly we present a technical proposition relating certain kinds of homotopy orbits.

Proposition B.8. *Let G be a group acting on a pointed simplicial set $X_\bullet$. Suppose that $Y_\bullet$ is a subspace of $X_\bullet$ such that the following two conditions hold:*

(1) *If $g \in G$ is such that there exists a (non-basepoint) simplex $y \in Y_\bullet$ such that $g \cdot y \in Y_\bullet$ then for all $y' \in Y_\bullet$, $g \cdot y' \in Y_\bullet$.*
(2) *For all n and for all $x \in X_n$ there exists $g \in G$ such that $g \cdot x \in Y_n$.*

Let H be the subgroup of G that takes $Y_\bullet$ to $Y_\bullet$. Then

$$(X_\bullet)_{hG} \simeq (Y_\bullet)_{hH}.$$

Proof. Let $Z_{\bullet\bullet}$ be the bisimplicial set whose (n,m)-simplex consist of diagrams

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} x_m,$$

where the $x_i \in X_n$ for $i = 0, \ldots, m$ and $g_i \cdot x_{i-1} = x_i$. Then diag $Z_{\bullet\bullet} = (X_\bullet)_{hG}$. In addition, if we let $W_{\bullet\bullet}$ be the sub-bisimplicial set containing those diagrams where the $x_i \in Y_\bullet$ and the $g_i \in H$ then diag $W_{\bullet\bullet} = Y_{hH}$. Thus it suffices to check that the inclusion $W_{\bullet\bullet} \longrightarrow Z_{\bullet\bullet}$ induces an equivalence on diagonals. To prove this, it suffices (by [16, Proposition IV.1.9]) to show that for all n, $W_{n\bullet} \longrightarrow Z_{n\bullet}$ is a weak equivalence of simplicial sets.

$Z_{n\bullet}$ (resp. $W_{n\bullet}$) is the nerve of the category whose objects are X_n (resp. Y_n) and whose morphisms are induced by the action of G (resp. H); call these categories $\mathcal{C}$ and $\mathcal{D}$. $\mathcal{D}$ is clearly a subcategory of $\mathcal{C}$; thus to show that the map induces an equivalence on nerves it suffices to check that the inclusion is full and essentially surjective. That it is full follows from condition (1), since if we are given $y, y' \in Y_n$ then any g such that $g \cdot y = y'$ is in H. That it is essentially surjective follows from condition (2), since every element of X_n is isomorphic via the action of G to an element of Y_n. $\square$

B.3. The spectral sequence for the total homotopy cofiber of a cube

The technical result that we need in order to understand the Dehn cube is the spectral sequence for the total homotopy cofiber of a cube. As the usual spectral sequence is stated only for ordinary, rather than reduced, homology, we state our analog here. We use the notation introduced in Section 3.

Proposition B.9. *Let* $F : \widehat{\mathcal{I}}_n \longrightarrow \mathbf{Top}_*$ *be a functor. There is a spectral sequence*

$$\bigoplus_{\vec{A} = (b, a_1, \ldots, a_{n-p-1})} \widetilde{H}_q(F(\vec{A})) \Longrightarrow \widetilde{H}_{p+q}(\mathrm{cofib}^{th} F).$$

Proof. By [22, Proposition 9.6.14], for a functor $G : \widehat{\mathcal{I}}_n \longrightarrow \mathbf{Top}$ there is a spectral sequence

$$\bigoplus_{\vec{A} = (b, a_1, \ldots, a_{n-p-1})} H_q(G(\vec{A})) \Longrightarrow H_{p+q}(\mathrm{cofib}^{th} G).$$

Each of the spaces we have is pointed, thus the functor $C : \widehat{\mathcal{I}}_n \longrightarrow \mathbf{Top}$ defined by $C(\vec{A}) = *$ is a retract of G. In particular, this means that the spectral sequence given by the kernel of the induced map $G \Longrightarrow C$ is also a spectral sequence, which converges to $\ker(H_{p+q}(\mathrm{cofib}^{th} G) \longrightarrow H_{p+q}(\mathrm{cofib}^{th} C))$. Since $\mathrm{cofib}^{th} C \simeq *$, this reduces to the desired spectral sequence. $\square$

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