On the Duration of a Gambler's Ruin Problem

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Abstract

Consider a gambler who on each bet either wins 1 with probability p or loses 1 with probability q = 1   p; with the results of successive bets being independent. The gambler will stop betting when they are either up k or down k: Letting N be the number of bets made, we show that N is a new better than used random variable. Moreover, we show that if k is even then N=2 has an increasing failure rate, and if k is odd then (N + 1)=2 has an increasing failure rate.

Keywords: gambler's ruin, new better than used, increasing failure rate, reverse hazard rate ordering, likelihood ratio ordering

1 Introduction

Consider a gambler who on each bet either wins 1 with probability p or loses 1 with probability $q = 1 - p$; with the results of successive bets being independent. Suppose that the gambler will stop betting when they are either up k or down k. With $Y_i = 1$ or -1, depending on whether the gambler wins or loses bet i, $i \geq 1$, let $S_n = \sum_{i=1}^{n} Y_i$, represent the gambler's winnings after n bets. Further, let

$$N = \min\{n : S_n = k \text{ or } -k\}$$

be the number of bets made before the gambler stops.

In Section 2 we show that N is a new better than used random variable. In Section 3 we show that either $N/2$ or $(N+1)/2$ is an increasing failure rate random variable, with the former being true when k is even, and the latter when k is odd.

2 New Better than Used

The positive integer valued random variable T is said to be a New Better than Used (NBU) random variable, if $T - m | T > m$ is stochastically smaller than T for all $m \geq 0$. That is, it is NBU if

$$P(T - m > r | T > m) \leq P(T > r) \text{ for all } m \geq 0, r \geq 0.$$

To prove that N is NBU, we will make use of the following lemma, whose proof can be found in [2] or [4]

Lemma 1. $|S_n|, n \geq 0$ is a Markov chain with transition probabilities $Q_{i,j}$ given by

$$\begin{aligned} Q_{0,1} &= 1 \\ Q_{i,i-1} &= \frac{p^i q + q^i p}{p^i + q^i}, \; i \geq 1 \\ Q_{i,i+1} &= \frac{p^{i+1} + q^{i+1}}{p^i + q^i}, \; i \geq 1 \end{aligned}$$

Proposition 1. N is NBU.

Proof: First note that $N = \min\{n : |S_n| = k\}$. Let N_i, $i < k$, denote the number of transitions it takes the Markov chain $|S_n|$ to go from state i to state k. Because N, the number of transitions for $|S_n|$ to go from state 0 to state k, is the number of transitions to go from 0 to i plus the number of transitions to go from i to k, it follows that $N_i \le_{st} N$. Consequently,

$$
\begin{aligned}
P(N - m > r \mid N > m) &= \sum_{i=0}^{k-1} P(N - m > r \mid N > m, |S_m| = i) P(|S_m| = i \mid N > m) \\
&= \sum_{i=0}^{k-1} P(N_i > r) P(|S_m| = i \mid N > m) \\
&\le \sum_{i=0}^{k-1} P(N > r) P(|S_m| = i \mid N > m) \\
&= P(N > r)
\end{aligned}
$$

which proves the result. QED

Remark. Proposition 1 does not mean that if the gambler is still playing then, no matter what has so far transpired, the remaining number of bets is always stochastically smaller then when they began. For instance, when $k = 3$ and $p = .9$, if the first two bets both resulted in losses, then the remaining number of bets is not stochastically smaller than at the beginning. (Indeed, it is because of this that it may not be intuitively obvious that N is NBU - though it does become more so when we know that $|S_n|$ is a Markov chain.) What Proposition 1 yields is that if all we know is that the gambler has not yet stopped, and not the results of those bets already made, then the remaining number of bets is stochastically smaller than N.

3 Increasing Failure Rate

The positive integer valued random variable T is said to be an increasing failure rate (IFR) random variable if $T - m \mid T > m$ is stochastically decreasing in m. That is, if $P(T - m > r \mid T > m)$ is a decreasing function of m for all r. (This can be shown to be equivalent to the condition that $P(T = m \mid T \ge m)$ is an increasing function of m.)

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To prove the next result, we will make us of the following result, whose proof can be found in [1] or [3].

Lemma 2. Consider a Markov chain with states $0; 1; :::;$ and let N_i denote the next state from state i: If $P(N_i = r|N_i \ r)$ is an increasing function of i for all r, then the number of transitions to go from state 0 to a state larger than m is IFR for any m:

Remark. The function $P(T = n|T \ n); n \ 0;$ is called the reverse hazard rate function of T: We say that T_1 is reversed hazard rate larger than T_2; written $T_1 \ _{rh} \ T_2$, if $P(T_1 = n|T_1 \ n) \ P(T_2 = n|T_2 \ n)$ for all $n > 0$: Thus, the condition of Lemma 2 is that the reversed hazard rate of N_i increases in i: It is known that if T_1 is likelihood ratio larger than T_2, meaning that $P(T_1 = n)=P(T_2 = n)$ increases in n; then $T_1 \ _{rh} \ T_2$.

Proposition 2. If k is even, then $N=2$ is IFR; if k is odd, then $(N + 1)=2$ is IFR.

Proof: Suppose that k is even. Noting that N would be even in this case, we have that

$$N=2 = \min\{n \ 1 : |S_{2n}| = k\}:$$

Letting $X_n = |S_{2n}|=2$; then $X_n; n \ 0$ is a Markov chain with transition probabilities $P_{i;j} = Q^2_{2i;2j}$ given by

$$\begin{aligned} P_{0;0} &= 2pq \\ P_{0;1} &= p^2 + q^2 \\ P_{i;i \ 1} &= \frac{p^{2i}q^2 + q^{2i}p^2}{p^{2i} + q^{2i}}; \ i \ 1 \\ P_{i;i} &= 2pq \\ P_{i;i+1} &= \frac{p^{2i+2} + q^{2i+2}}{p^{2i} + q^{2i}}; \ i \ 1 \end{aligned}$$

To prove that $N=2$ is IFR, we will show that N_i, the next state from i of the Markov chain X_n, is likelihood ratio increasing in i: To do so, we will need to show

$$\text{(a)} \quad \frac{P_{1;1}}{P_{0;1}} \quad \frac{P_{1;0}}{P_{0;0}}$$

and

$$\text{(b)} \quad \frac{P_{i+1;i+1}}{P_{i;i+1}} \quad \frac{P_{i+1;i}}{P_{i;i}}$$

To prove (a), we need show that

$$4p^2q^2 \quad (p^2 + q^2) \; \frac{p^2q^2 + q^2p^2}{p^2 + q^2}$$

which is immediate. To prove (b), we need show that

$$4p^2q^2\,(p^{2i+2} + q^{2i+2})(p^{2i} + q^{2i}) \quad (p^{2i+2}q^2 + q^{2i+2}p^2)(p^{2i+2} + q^{2i+2})$$

which is equivalent to

$$4(p^{2i+2} + q^{2i+2}) \quad p^{2i+2} + q^{2i+2}$$

Hence, the rst time the Markov chain X_n reaches k=2 is IFR, showing that N=2 is IFR. The case where k is odd follows similarly.

References

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