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Frame dimension functions and phase retrievability

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Abstract

The frame dimension function of a frame $\mathcal{F} = \{f_j\}_{j=1}^n$ for an n-dimensional Hilbert space H is the function $d_{\mathcal{F}}(x) = \dim \operatorname{span}\{\langle x, f_j\rangle f_j : j = 1, \ldots, N\}$, $0 \neq x \in H$. It is known that $\mathcal{F}$ does phase retrieval for an n-dimensional real Hilbert space H if and only if $\operatorname{range}(d_{\mathcal{F}}) = \{n\}$. This indicates that the range of the dimension function is one of the good candidates to measure the phase retrievability for an arbitrary frame. In this paper we investigate some structural properties for the range of the dimension function, and examine the connections among different exactness of a frame with respect to its PR-redundance, dimension function and range of the dimension function. A subset Ω of $\{1, \ldots, n\}$ containing n is attainable if $\operatorname{range}(d_{\mathcal{F}}) = \Omega$ for some frame $\mathcal{F}$. With the help of linearly connected frames, we show that, while not every Ω is attainable, every (integer) interval containing n is always attainable by an n-linearly independent frame. Consequently, $\operatorname{range}(d_{\mathcal{F}})$ is an interval for every generic frame for $\mathbb{R}^n$. Additionally, we also discuss and post some questions related to the connections among ranges of the dimension functions, linearly connected frames and maximal phase retrievable subspaces.

Keywords Phase retrievability · Basis and frame · Linearly connected frame · Exact phase retrievable frame · n-Independent frame

Mathematics Subject Classification 15A63 · 42C15 · 46C05

Dedicated to Professor Chi-Kwong Li's 65th Birthday.

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1 Introduction

The phase-retrieval problem asks to determine and/or reconstruct an object after loss of phases. Mathematically, it asks to recover a vector (signal) x in a vector space, up to a unimodular scalar, from some phaseless measurements $\{m_j(x)\}_{j\in\mathcal{J}}$ such that $m_j(\lambda x) = m_j(x)$ for all unimodular scalars λ. The frame-based phase retrieval requires to perform phaseless measurements by using a frame or redundant basis. Recall that a *frame* for a Hilbert space H is a sequence $\mathcal{F} = \{f_j\}_{j\in\mathcal{J}}$ such that there exist $C_1, C_2 > 0$ with the property

$$C_1\|x\|^2 \leq \sum_{j\in\mathcal{J}} |\langle x, f_j\rangle|^2 \leq C_2\|x\|^2, \quad x \in H.$$

In the finite-dimensional space case, frames are exactly the spanning sets, i.e., $\text{span}\{f_j\} = H$. Clearly, frames are generalizations of (linearly independent) bases. A frame $\{f_j\}_{j=1}^N$ is called *phase retrievable* for a finite dimensional Hilbert space H if $|\langle x, f_i\rangle| = |\langle y, f_i\rangle|$ for all j implies that $y = \lambda x$ for some unimodular scalar λ. We refer to [1–14, 18–21] and the references therein for some backgrounds and recent developments on the frame-based phase retrieval.

For a finite-dimensional real Hilbert space, it is known [13] that a frame $\{f_j\}_{j=1}^N$ for H is phase-retrievable if and only if

$$\text{span}\{\langle x, f_j\rangle f_j : j = 1, \ldots, N\} = H$$

holds for every nonzero vector $x \in H$. For the complex Hilbert space case, this condition is still necessary, but not sufficient. A frame $\{f_j\}_{j=1}^n$ for H has the *complement property* if either $\text{span}\{f_j : j \in \Lambda\} = H$ or $\text{span}\{f_j : j \in \Lambda^c\} = H$ for any subset Λ of $\{1, \ldots, N\}$. The following is well known (c.f. [7, 13] and some generalizations [16, 18]):

Proposition 1.1 *Let $\{f_j\}_{j=1}^N$ be a frame for H. Then the following are equivalent*:

(i) $\text{span}\{\langle x, f_j\rangle f_j : j = 1, \ldots, N\} = H$ *for every nonzero vector* $x \in H$.
(ii) $\text{rank}(\sum_{j=1}^N |\langle x, f_j\rangle|^2 f_j \otimes f_j) = \dim H$ *for every nonzero vector* $x \in H$.
(iii) $\{f_j\}_{j=1}^N$ *has the complement property.*

In real world applications, our preferred frames may not be phase retrievable due to some application constraints or other restrictions. Nevertheless, phase-retrieval can still be performed on some subsets of the signal space. This naturally leads to the problem of examining the phase retrievability for arbitrary frames. There are different possible approaches to measure phase retrievability for a frame. Possible candidates include the "measurements" on

(I) The subspace M with largest dimension such that $\mathcal{F}$ is phase retrievable when restricted to M;
(II) The algebraic variety of $\mathcal{S}_2 \cap \ker L_{\mathcal{F}}$, where $\mathcal{S}_2 = \{A \in B(H) : A^* = A,\ \text{rank}(A) \leq 2\}$ and $L_{\mathcal{F}}(A) = \{\langle Af_j, f_j\rangle\}_{j=1}^N$;

(III) $\text{span}\{\langle x, f_j\rangle f_j : j = 1, \dots, N\}$ for every $x \in H$.

In a recent work [17], we investigated the exact phase retrievability in terms of measurement (II), and phase retrievability of (usually not phase retrievable) frames in terms of measurement (I). In this paper, we examine this issue from the frame dimension function point of view. Listed below are few notations and terminologies that are needed in this paper.

- H—a finite real or complex dimensional Hilbert space. We will explicitly state it if a statement is only valid for a real Hilbert space, and otherwise it is valid for both real and complex cases.
- $B(H)$—the space of bounded linear operators on H. We also use $M_d(\mathbb{C})$ or $M_d(\mathbb{R})$ for $B(H)$ in the case that $H = \mathbb{C}^d$ or $\mathbb{R}^d$.
- $S^1(H)$—the set of all unit vectors in H.
- Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for H and $x \in H$. The set $\text{supp}(x) := \{j : \langle x, e_j\rangle \neq 0\}$ is the support of x with respect to the basis, and $|\text{supp}(x)|$ is the cardinality of the support.
- $\langle S, T\rangle = Tr(ST^*)$ is the Hilbert–Schmidt product on $B(H)$.
- For $x, y \in H$, $x \otimes y \in B(H)$ is the rank-one operator defined by $z \to \langle z, y\rangle x$.
- $[N] = \{1, 2, \dots, N\}$, $[K : N] = \{j \in \mathbb{N} : K \leq j \leq N\}$, where $K, N \in \mathbb{N}$.
- For a frame $\mathcal{F} = \{f_j\}_{j=1}^N$ and a subset $\Lambda \subseteq [N]$, we use $\mathcal{F}_\Lambda$ to denote the subframe $\{f_j\}_{j\in\Lambda}$ for $H_\Lambda := \text{span}\mathcal{F}_\Lambda$.

Definition 1.1 Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for H. Then the *frame function* is defined to be the subspace-valued function $\mathcal{F}_x = \text{span}\{\langle x, f_j\rangle f_j : j = 1, \dots, N\}$ on $S^1(H)$, and the integer-valued function $d_{\mathcal{F}}(x) := \dim \mathcal{F}_x$ is called the *frame dimension function* or simply *dimension function* of $\mathcal{F}$.

Proposition 1.2 *Let $\mathcal{F}$ be a frame for H. Then $d \in \text{range}(d_{\mathcal{F}})$ if and only if there exists $\Lambda \subset \{1, \dots, N\}$ such that $d = \dim H_\Lambda$ and $f_i \notin H_{\Lambda^c}$ for every $i \in \Lambda$.*

Proof $\Rightarrow$: Let $d \in \text{range}(d_{\mathcal{F}})$. Then there exists $x \in H$ such that $d = d_{\mathcal{F}}(x)$. Set $\Lambda = \{i : \langle x, f_i\rangle \neq 0\}$. Then clearly $\mathcal{F}_x = H_\Lambda$. Moreover, since $\langle x, f\rangle = 0$ for any $f \in H_{\Lambda^c}$, we have that $f_i \notin H_{\Lambda^c}$ for every $i \in \Lambda$.

$\Leftarrow$: Let $\Lambda \subset \{1, \dots, N\}$ be a subset such that $d = \dim H_\Lambda$ and $f_i \notin H_{\Lambda^c}$ for every $i \in \Lambda$. Then for each $i \in \Lambda$, the restriction of f_i to $H_{\Lambda^c}^\perp$ is a nonzero linear functional, and so $\{x \in H_{\Lambda^c}^\perp : \langle x, f_i\rangle = 0\}$ has measure zero in $H_{\Lambda^c}^\perp$. Thus there exists $x \in H_{\Lambda^c}^\perp$ (actually for almost all x) such that $\langle x, f_i\rangle \neq 0$ for all $i \in \Lambda$. This implies that $\mathcal{F}_x = H_\Lambda$ and hence $d \in \text{range}(d_{\mathcal{F}})$. □

A frame $\mathcal{F}$ and its proper subframe $\mathcal{F}_\Lambda$ could share the same phase retrievability measured by their PR-redundancy (see Sect. 2 for the definition), the dimension functions and range of the dimension functions. Therefore, to have a meaningful classification of frames with respect to their phase retrievability, it is natural to restrict to only those ones that have the "exact" phase retrievability. Section 2 is devoted to establishing the connections among PR-redundancy exact, dimension function exact and dimension range exact frames. In the real Hilbert space case, we show that PR-redundancy exactness and dimension function exactness are equivalent. However, they do not necessarily imply the range exactness. Section 3 will be focused on examining the structure of the ranges of the dimension functions. Since every frame spans the Hilbert space H, we know that $n = \dim H \in \text{range}(d_{\mathcal{F}})$ for every frame $\mathcal{F}$. We say that a subset Ω of $\{1, \ldots, n\}$ containing n is attainable if $\text{range}(d_{\mathcal{F}}) = \Omega$ for some frame $\mathcal{F}$. We first prove that $\text{range}(d_{\mathcal{F}}) = \{k, k+1, \ldots, n\}$ if $\mathcal{F} = \{f_j\}_{j=1}^N$ is an n-independent (i.e. full spark) frame, where $k = \min\{N-n+1, n\}$. This clearly implies that every (integer) interval containing n in $\{1, \ldots, n\}$ is attainable and $\text{range}(d_{\mathcal{F}})$ is an interval for every generic frame. For the general case, $\text{range}(d_{\mathcal{F}})$ is totally determined by the ranges of dimension functions for linearly connected frames. This allows us to obtain variety of examples where either $\text{range}(d_{\mathcal{F}})$ is not an interval or a subset Ω is not attainable. For example, we will show that $\{k, n-k, n\}$ is attainable for every $1 \le k < n$, and Ω is not attainable if $1 \in \Omega$ and $n-1 \notin \Omega$ when $n \ge 4$. In Sect. 4, we post some questions related to the connections among ranges of the dimension functions, linearly connected frames and maximal phase retrievable subspaces. In particular, we conjecture that $\text{range}(d_{\mathcal{F}})$ is an interval if $\mathcal{F}$ is linearly connected.

2 Exactness

Recall that an exact frame is a frame that $\mathcal{F}_\Lambda$ fails to be a frame for H for any proper subset Λ of $\mathcal{J}$. In the finite-dimensional case, exact frames are precisely the bases of H. While high redundancy of a frame makes is really important for many applications (for example, it wouldn't be possible to perform phase retrieval without enough redundancy of a frame), "exactness" is also an important feature to require in some applications. For this purpose, we introduced the concept of exact phase retrievable frames [17]: An *exact phase-retrievable frame* is a phase retrievable frame $\mathcal{F} = \{f_j\}_{j=1}^N$ such that $\mathcal{F}_\Lambda$ fails to be a phase retrievable frame for H whenever Λ is proper subset of $[N]$. While any generic frame of length $N > 2n-1$ is a non-exact phase retrievable frame for $\mathbb{R}^n$, exact phase retrievable frames for $\mathbb{R}^n$ do exist for every $N : 2n-1 \le N \le \frac{n(n+1)}{2}$. Clearly, exact phase retrievable frames are the ones that have the exact redundancy with respect to phase retrieval property. Based on the well-known fact (c.f. [13]) that a frame $\mathcal{F}$ is phase retrievable if and only if $\mathcal{S}_2 \cap \ker L_{\mathcal{F}} = \{0\}$, it is reasonable to measure the exactness of an arbitrary (not necessarily phase retrievable) frame with respect to its phase retrievability with the following definition:

Definition 2.1 Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for a Hilbert space H. We say that $\mathcal{F}$ has the *exact PR-redundancy* if

$$\mathcal{S}_2 \cap \ker L_{\mathcal{F}} \neq \mathcal{S}_2 \cap \ker L_{\mathcal{F}_\Lambda}$$

for any proper subset Λ of $\{1, \ldots, N\}$.

Since frame dimension functions are also proper candidates for measuring the phase retrievability of a frame, naturally we introduce the following concept of exactness with respect to the dimension function.

Definition 2.2 Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for a Hilbert space H. We say that $\mathcal{F}$ is

(i) *dimension function exact* if $d_{\mathcal{F}} \neq d_{\mathcal{F}_\Lambda}$ for any proper subset Λ of $[N]$.
(ii) *dimension range exact* or simply *range exact* if range$(d_{\mathcal{F}}) \neq$ range$(d_{\mathcal{F}_\Lambda})$ for any proper subset Λ of $[N]$.

Remark 2.1 Let $G_{\mathcal{F}}(x) = \sum_{j=1}^{N} |\langle x, f_j\rangle|^2 f_j \otimes f_j$ (see Theorem.2.1. Property 5 in [7]). Then $d_{\mathcal{F}}(x) = \text{rank}(G_{\mathcal{F}}(x)$. Thus dimension function and dimension range exactness can also be interpreted in terms of the matrix-valued map $G_{\mathcal{F}}(x)$.

Next we examine the connections among the exactness with respect to PR-redundancy, dimension function and range of the dimension function in the case that H is a real Hilbert space. Our first main result tells us that, in the real Hilbert space case, the exactness of PR-redundancy and dimension function are the same.

Theorem 2.1 *Let $\mathcal{F} = \{f_j\}_{j=1}^{N}$ be a frame for a real Hilbert space H. Then $\mathcal{F}$ has the exact PR-redundancy if and only if $\mathcal{F}$ is dimension function exact. The sufficient part is also true for complex Hilbert spaces.*

Proof ($\Rightarrow$): Assume that $\mathcal{F}$ is not dimensional function exact. Then there exists a proper subset Λ of $\{1, \ldots, N\}$ such that $d_{\mathcal{F}}(x) = d_{\mathcal{F}_\Lambda}(x)$ for every $x \in S(H)$. This implies that $\mathcal{F}_x = \mathcal{G}_x$ for every $x \in H$, where $\mathcal{G} = \mathcal{F}_\Lambda$. We show that $\mathcal{S}_2 \cap \ker L_{\mathcal{G}} = \mathcal{S}_2 \cap \ker L_{\mathcal{F}}$, which will lead to a contradiction since $\mathcal{F}$ is assumed to have the exact PR-redundancy.

We only need to show that $\mathcal{S}_2 \cap \ker L_{\mathcal{G}} \subseteq \mathcal{S}_2 \cap \ker L_{\mathcal{F}}$. Let $R \in \mathcal{S}_2 \cap \ker L_{\mathcal{G}}$. Without losing the generality, we can assume that $R = u \otimes u + \lambda v \otimes v$ with $\lambda = \pm 1$. Then we get

$$|\langle u, f_j\rangle|^2 + \lambda |\langle v, f_j\rangle|^2 = 0$$

for every $j \in \Lambda$.

If $\lambda = 1$, then we obtain that $|\langle u, f_j\rangle|^2 = |\langle v, f_j\rangle|^2 = 0$ for every $j \in \Lambda$, and hence $u, v \in (\mathcal{G}_x)^\perp$ for every $x \in H$. Since $\mathcal{F}_x = \mathcal{G}_x$ for every $x \in H$, we have $u, v \in (\mathcal{F}_x)^\perp$ for every $x \in H$. In particular, we get $u \in (\mathcal{F}_u)^\perp$ and $v \in (\mathcal{F}_v)^\perp$. This implies that $|\langle u, f_j\rangle|^2 = |\langle v, f_j\rangle|^2 = 0$ for every $1 \leq j \leq N$, and thus $R \in \mathcal{S}_2 \cap \ker L_{\mathcal{F}}$.

If $\lambda = -1$, then $|\langle u, f_j\rangle|^2 - |\langle v, f_j\rangle|^2 = 0$ for every $j \in \Lambda$. Let $x = u - v$ and $y = u + v$. Since

$$|\langle u, f_j\rangle|^2 - |\langle v, f_j\rangle|^2 = -4\langle x,\ \langle y, f_j\rangle f_j\rangle,$$

we obtain that $x \perp \mathcal{G}_y$ and hence $x \perp \mathcal{F}_y$. This implies that

$$0 = -4\langle x,\ \langle y, f_j\rangle f_j\rangle = |\langle u, f_j\rangle|^2 - |\langle v, f_j\rangle|^2$$

for every $1 \leq j \leq N$. Thus $R \in \mathcal{S}_2 \cap \ker L_{\mathcal{F}}$.

($\Leftarrow$): In this part we assume that H is either a real or complex Hilbert space. Suppose that $\mathcal{F}$ does not the exact PR-redundancy. Then there exists a proper subset Λ of $\{1, \ldots, N\}$ such that $\mathcal{S}_2 \cap \ker L_{\mathcal{G}} = \mathcal{S}_2 \cap \ker L_{\mathcal{F}}$, where $\mathcal{G} = \mathcal{F}_\Lambda$. We show that

$d_{\mathcal{F}}(x) = d_{\mathcal{G}}(x)$ for every $x \in H$, which will imply that F is not dimension function exact.

Assume to the contrary that there exists a vector $x \in H$ such that $d_{\mathcal{F}}(x) \neq d_{\mathcal{G}}(x)$. Then $\mathcal{G}_x \subsetneq \mathcal{F}_x$. Thus there exists $y \in H$ such that $y \perp \mathcal{G}_x$ but $y \notin \mathcal{F}_x^{\perp}$. This implies that $\langle x, f_j\rangle\langle f_j, y\rangle = 0$ for every $j \in \Lambda$ and there exists some i such that $\langle x, f_i\rangle\langle f_i, y\rangle \neq 0$. Let $u = x - y$ and $v = x + y$, and let $R = u \otimes u - v \otimes v$. Then, from

$$\langle R, f_j \otimes f_j\rangle = -4\langle x, f_j\rangle\langle f_j, y\rangle,$$

we get that $R \in \mathcal{S}_2 \cap \ker L_{\mathcal{G}}$ but $R \notin \mathcal{S}_2 \cap \ker L_{\mathcal{F}}$, which contradicts with the equality $\mathcal{S}_2 \cap \ker L_{\mathcal{G}} = \mathcal{S}_2 \cap \ker L_{\mathcal{F}}$. □

The next simple example shows that the necessary part in the above theorem is not true for the complex Hilbert space case.

Example 2.1 Let $H = \mathbb{C}^2$ and

$$f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad f_4 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Then it can be verified that $\mathcal{F} = \{f_j\}_{j=1}^4$ is an exact phase-retrievable frame for $\mathbb{C}^2$. However, $\mathcal{F}$ is not dimension function exact since $d_{\mathcal{F}}(x) = d_{\mathcal{F}_\Lambda}(x) = 2$ for every nonzero vector $x \in \mathbb{C}^2$, where $\Lambda = \{1, 2, 3\}$.

Corollary 2.2 *If a real Hilbert space frame $\mathcal{F}$ has the exact PR-redundancy, then so does every subframe of $\mathcal{F}$.*

Proof Suppose that $\mathcal{F}_\Lambda$ is a subframe which does not have the PR-redundancy. Then there is a proper subset Ω of Λ such that

$$\operatorname{span}\{\langle x, f_j\rangle f_j : j \in \Omega\} = \operatorname{span}\{\langle x, f_j\rangle f_j : j \in \Lambda\}$$

for every $x \in H$. This implies that

$$\operatorname{span}\{\langle x, f_j\rangle f_j : j \in \Omega \cup \Lambda^c\} = \operatorname{span}\{\langle x, f_j\rangle f_j : j \in \Lambda \cup \Lambda^c\}$$

holds for every $x \in H$. Let $I = \Omega \cup \Lambda^c$. Then I is a proper subset of $\{1, \ldots, N\}$ with the property that $(\mathcal{F}_I)_x = \mathcal{F}_x$ for every $x \in H$. Thus, by Theorem 2.1, we have that $\mathcal{F}$ does not have the exact PR-redundancy. □

By the proof of the Theorem 2.1 we also have:

Corollary 2.3 *Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for a real Hilbert space H and $\mathcal{F}_\Lambda$ be a subframe of $\mathcal{F}$. Then $\mathcal{S}_2 \cap \ker L_{\mathcal{F}} = \mathcal{S}_2 \cap \ker L_{\mathcal{F}_\Lambda}$ if and only if $d_{\mathcal{F}} = d_{\mathcal{F}_\Lambda}$.*

The next result shows that range exactness is stronger than dimension function exactness.

Proposition 2.4 *Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for a Hilbert space H. If $\mathcal{F}$ is range exact, then it is dimension function exact. The converse is not true in general. However, the converse is true if $\mathcal{F}$ is a phase-retrievable frame.*

Proof Assume that $\mathcal{F}$ is not a dimension function exact. Then there exists a proper subset Λ of $\{1, \ldots, N\}$ such that $d_{\mathcal{F}}(x) = d_{\mathcal{F}_\Lambda}(x)$ for every $x \in H$. This clearly implies that range$(d_{\mathcal{F}}) =$ range$(d_{\mathcal{F}_\Lambda})$. Thus $\mathcal{F}$ can not be range exact.

For a counterexample of the converse, let $H = \mathbb{R}^3$ and $\mathcal{F} = \{f_j\}_{j=1}^5 = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_2 + e_3\}$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis for H. Let $\mathcal{G}_i = \mathcal{F}\backslash\{f_i\}$, $x_1 = e_1, x_2 = e_2, x_3 = e_3, x_4 = e_1, x_5 = e_1 + e_2$. Then it can be directly verified that $\mathcal{F}_{x_i} \neq (\mathcal{G}_i)_{x_i}$ for each i. Thus for every proper subset Λ of $\{1, \ldots, 5\}$ we have that $d_{\mathcal{F}} \neq d_{\mathcal{F}_\Lambda}$ and so $\mathcal{F}$ is dimension function exact. However, since range$(d_{\mathcal{F}}) =$ range$(d_{\mathcal{G}_4}) = \{2, 3\}$, we know that $\mathcal{F}$ is not range exact.

Now assume $\mathcal{F}$ is a phase retrievable frame and dimension function exact. By Theorem 2.1, it has the exact PR-retrievability and hence it is an exact PR-frame. We show that $\mathcal{F}$ is the dimension range exact. Since $\mathcal{F}$ is phase-retrievable, we know that range$(d_{\mathcal{F}}) = \{n\}$, where $n = \dim H$. Let Λ be a proper subset of $\{1, \ldots, N\}$. Then $\mathcal{F}_\Lambda$ is not phase-retrievable. Thus there exists a nonzero vector $x \in H$ such that $(\mathcal{F}_\Lambda)_x \neq H$, and hence $\dim(\mathcal{F}_\Lambda)_x \neq d$. If $\dim(\mathcal{F}_\Lambda)_x \neq 0$, then we already have range$(d_{\mathcal{F}}) \neq$ range$(d_{\mathcal{F}_\Lambda})$. If $\dim(\mathcal{F}_\Lambda)_x = 0$, then we get that $\langle x, f_j\rangle f_j = 0$ for every $j \in \Lambda$. Since $\mathcal{F}$ is dimension function exact, we know that $f_j \neq 0$. Thus $x \perp f_j$ for $j \in \Lambda$. This implies that $(\mathcal{F}_\Lambda)_z \subseteq \text{span}\mathcal{F}_\Lambda \neq H$ for every $z \in H$. Thus $d \notin$ range$(d_{\mathcal{F}_\Lambda})$ and therefore again we have range$(d_{\mathcal{F}}) \neq$ range$(d_{\mathcal{F}_\Lambda})$. Thus $\mathcal{F}$ is range exact. □

Follows from Theorem 2.1 and Proposition 2.4 we get:

Corollary 2.5 *Assume that $\mathcal{F}$ is a phase-retrievable frame for a real Hilbert space H. Then the following are equivalent:*

(i) *$\mathcal{F}$ has the exact PR-redundancy, i.e. $\mathcal{F}$ is an exact phase phase-retrievable frame.*
(ii) *$\mathcal{F}$ is dimension function exact.*
(iii) *$\mathcal{F}$ is range exact.*

3 Ranges of frame dimension functions

In this section, we assume that $H = \mathbb{R}^n$ or $\mathbb{C}^n$ and $\{e_j\}_{j=1}^n$ is the standard orthonormal basis for H. We are interested in digging out more information on $R(\mathcal{F}) :=$ range$(d_{\mathcal{F}})$. There are two extreme cases for which we know that $R(\mathcal{F})$ is an integer interval: $R(\mathcal{F}) = \{n\} = [n : n]$ if $\mathcal{F}$ is phase-retrievable, and $R(\mathcal{F}) = \{1, \ldots, n\} = [1 : n]$ if $\mathcal{F}$ is a basis for H. This leads to the question of investigating the structure of $R(\mathcal{F})$ with the aim at establishing some kind of possible classifications for frames with respect to their phase-retrievability. We first point out that $R(\mathcal{F})$ could be complicated, in particular, it does not have to be an integer interval.

Example 3.1 Let $\mathcal{F} = \{e_1, e_2, e_3, e_4, e_1 + e_2, e_3 + e_4\}$, where $\{e_1, e_2, e_3, e_4\}$ is the standard orthonormal basis for $\mathbb{R}^4$. Then a simple calculation shows that $R(\mathcal{F}) = \{2, 4\}$.

Naturally, we are interested in answering the following questions: Is it true that for every subset Ω of $\{1, \ldots, n\}$ containing n there exists a frame $\mathcal{F}$ such that $R(\mathcal{F}) = \Omega$? If this is not true, what kind of structure can we say about Ω? In particular, under what conditions do we have that $R(\mathcal{F})$ is an interval? What can we say about two frames $\mathcal{F}$ and $\mathcal{G}$ that have the same length and $R(\mathcal{F}) = R(\mathcal{G})$?

In this section, we will examine the conditions under which $R(\mathcal{F})$ is an integer interval. Since $\dim d_{\mathcal{F}}(x) = n$ for every generic $x \in H$, $R(\mathcal{F})$ has the form $[\ell : n]$ if it is an interval. While the smallest interval $[n : n] = \{n\}$ characterizes all the frames with complement property (and hence phase-retrieval frames for real Hilbert space case), the following tells that the largest interval $[1 : n]$ in principle characterizes all the bases.

Proposition 3.1 *Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for a Hilbert space H. Then $\mathcal{F}$ is a basis if and only if $\mathcal{F}$ is range exact and $R(\mathcal{F}) = [1 : n]$.*

Proof Suppose that $\mathcal{F}$ is a basis. Clearly, we can assume that $\mathcal{F} = \{e_j\}_{j=1}^n$ is an orthonormal basis. For every $j \in [1 : n]$, let $x = e_1 + \cdots + e_j$. Then $\mathcal{F}_x = \operatorname{span}\{e_1, \ldots, e_j\}$ and hence $j \in R(\mathcal{F})$. Thus we have $R(\mathcal{F}) = [1 : n]$. It is also obvious that a basis is always range exact.

Now assume that $R(\mathcal{F}) = [1 : n]$ and $\mathcal{F}$ is range exact. Since $\mathcal{F}$ is a frame, it contains a basis, say $\mathcal{F}_\Lambda$, for H, where Λ is a subset of $[N]$. This implies that $R(\mathcal{F}_\Lambda) = [1 : n]$. Thus we have $R(\mathcal{F}) = R(\mathcal{F}_\Lambda)$. Since $\mathcal{F}$ is range exact, we get that $\Lambda = [N]$ and so $\mathcal{F}$ is a basis. □

Example 3.2 We point out that in the above proposition, the range exactness condition can not be replaced by dimension function exactness: For example, let $\mathcal{F} = \{e_1, e_2, e_3, e_2 + e_3\}$. Clearly $\mathcal{F}$ is not a basis. It is easy to verify the dimension function will be different if we remove any one element from the frame. Thus F is dimension function exact. Moreover, by selecting x to be e_1, $e_2 + e_3$ and $e_1 + e_2$ respectively, we get $R(\mathcal{F}) = [1 : 3]$.

One of our main results of this section is to show that $R(\mathcal{F})$ is an interval if $\mathcal{F}$ is n-independent.

Definition 3.1 Let A be a matrix and $\{f_1, \ldots, f_m\}$ be a (finite) sequence of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$. We say that

(i) A is totally k-nonsingular if every $\ell \times \ell$ submatrix of A is nonsingular for every $1 \leq \ell \leq k$.
(ii) $\{f_1, \ldots, f_m\}$ is totally k-nonsingular if the matrix $A = [f_1, \ldots f_m]$ is totally k-nonsingular.
(iii) $\{f_1, \ldots, f_m\}$ is k-independent if every k-vectors in $\{f_1, \ldots, f_m\}$ are linearly independent. An n-independent frame is also called a full spark frame.

Since every frame $\mathcal{F} = \{f_j\}_{j=1}^N$ is *similar* to a frame of the form $\{e_1, \ldots, e_n, g_1, \ldots, g_k\}$ in the sense that there exists an invertible matrix S such that $Sf_j = e_j$ for $j = 1, \ldots, n$ and $Sf_{n+i} = g_i$ for $i = 1, \ldots, N - n$, and the similarity preserves the frame dimension function and linear independence, it suffices to assume that $\mathcal{F}$ has the form of $\{e_1, \ldots, e_n, f_1, \ldots f_k\}$.

Lemma 3.2 *Let $\mathcal{F} = \{e_1, \dots, e_n, f_1, \dots f_k\}$ be a frame for H with $k < n$. Then F is n-independent if and only if $A = [f_1, \dots, f_k]$ is totally k-nonsingular.*

Proof Write $f_i = \sum_{j=1}^{n} a_{ij} e_j$ for $1 \le i \le k$.

($\Leftarrow$): It is enough to show that $\{e_1, \dots, e_m, f_1, \dots, f_\ell\}$ are linearly independent, where $\ell > 0$ and $m + \ell = n$. Assume that

$$\sum_{j=1}^{m} x_j e_j + \sum_{i=1}^{\ell} y_i f_i = 0.$$

Then we get

$$\sum_{j=1}^{m} (x_j + \sum_{i=1}^{\ell} y_i a_{ij}) e_j + \sum_{j=m+1}^{n} \left(\sum_{i=1}^{\ell} y_i a_{ij} \right) e_j = 0.$$

This implies that $x_j + \sum_{i=1}^{\ell} y_i a_{ij} = 0$ for $1 \le j \le m$ and $\sum_{i=1}^{\ell} y_i a_{ij} = 0$ for $m + 1 \le j \le n$.

Since A is totally k-nonsingular and $\ell \le k$, we get that the submatrix $[a_{ij}]_{1 \le i \le \ell, m+1 \le j \le n}$ of A is invertible (here we use the fact $n - m = \ell$). This implies that $y_1 = \dots = y_\ell = 0$ and hence $x_1 = \dots = x_m = 0$. Therefore $\{e_1, \dots, e_m, f_1, \dots, f_\ell\}$ are linearly independent.

($\Rightarrow$): Suppose that A is not totally k-nonsingular. Then without losing the generality, we can assume that the submatrix $B = [a_{ij}]_{\ell \times \ell}$ is singular for some $1 \le \ell \le k$. Then there exists nonzero vector $y \in \mathbb{R}^\ell$ such that $\sum_{i=1}^{\ell} y_i a_{ij} = 0$ for every $j = 1, \dots, \ell$. Thus we get that

$$\sum_{i=1}^{\ell} y_i f_i = \sum_{i=1}^{\ell} y_i \sum_{j=1}^{n} a_{ij} e_j = \sum_{j=1}^{n} \sum_{i=1}^{\ell} y_i a_{ij} e_j = \sum_{j=\ell+1}^{n} \left(\sum_{i=1}^{\ell} y_i a_{ij} \right) e_j.$$

This implies that $\{e_{\ell+1}, \dots e_n, f_1, \dots, f_\ell\}$ are linearly dependent, which contradicts with the n-linearly independence assumption of $\mathcal{F}$. Thus A is totally k-nonsingular. □

Remark 3.1 Although we proved for the case when $k < n$ in the above lemma, it can be easily generalized (with exactly the same proof) to any k: Let $\mathcal{F} = \{e_1, \dots, e_n, f_1, \dots f_k\}$ be a frame for H and $m = \min\{k, n\}$. Then F is n-independent if and only if $A = [f_1, \dots, f_k]$ is totally m-nonsingular.

Theorem 3.3 *Let $\mathcal{F} = \{e_1, \dots, e_n, f_1, \dots f_k\}$ be a frame for H with $k < n$. Then $R(\mathcal{F}) = [k + 1 : n]$ if and only if $A = [f_1, \dots, f_k]$ is totally k-nonsingular.*

Proof Write $f_i = \sum_{j=1}^{n} a_{ij} e_j$ for $1 \le i \le k$.

"$\Leftarrow$": Assume that A is totally k-nonsingular. We first show that $\dim F_x > k$ for every nonzero vector $x = (x_1, \dots, x_n)$. This is clearly true if $|\mathrm{supp}(x)| > k$ since in this case $e_j \in \mathcal{F}_x$ for every $j \in \mathrm{supp}(x)$. So we can assume that $|\mathrm{supp}(x)| \le k$. Without losing the generality, we can assume that $\mathrm{supp}(x) = \{1, \dots, m\}$ with $m \le k$.

Assume to the contrary that $\dim F_x = \ell \le k$. Then $m \le \ell \le k$ since $e_1, \ldots, e_m \in \mathcal{F}_x$. We also have $k - (\ell - m) = k - \ell + m \ge m$.

We claim that $f_i \in \mathcal{F}_x$ for exactly $\ell - m$ number of indices i, and $f_i \notin \mathcal{F}_x$ for $\ell - m$ for the rest of indices i. Indeed, note that $e_1, \ldots, e_m \in \mathcal{F}_x$ and $e_{m+1}, \ldots, e_n \in \mathcal{F}_x^{\perp}$. This implies that we need at least $\ell - m$ number of indices i such that $f_i \in \mathcal{F}_x$. Assume that say $f_{i_1}, \ldots f_{i_{\ell-m}}, f_{i_{\ell-m+1}} \in \mathcal{F}_x$. Since $m + (\ell - m) + 1 = \ell + 1 \le n$, we get by Lemma 3.2 that $e_1, \ldots, e_m, f_{i_1}, \ldots f_{i_{\ell-m}}, f_{i_{\ell-m+1}}$ are linearly independent, which implies that $\dim \mathcal{F}_x \ge \ell + 1$, a contradiction. Hence there exists exactly $\ell - m$ number of indices i such that $f_i \in \mathcal{F}_x$.

So without losing the generality, we can assume that $f_i \in \mathcal{F}_x$ for $i = 1, \ldots, \ell - m$ and $f_i \notin \mathcal{F}_x$ for $j = (\ell - m) + 1, \ldots, k$. This implies that $\sum_{i=1}^{m} x_j \bar{a}_{ij} = \langle x, f_i \rangle = 0$ for each $i = (\ell - m) + 1, \ldots, k$. Thus every $m \times m$ submatrix of the matrix $[a_{ij}]_{\ell-m+1 \le i \le k, 1 \le j \le m}$ is singular. This leads to a contradiction since $m \le k - (\ell - m)$ and the submatrix $\{f_{\ell-m+1}, \ldots, f_k]$ of A is totally m-nonsingular. Therefore we have proved that $\dim \mathcal{F}_x > k$.

Now we need to show that for every $\ell \in [k + 1 : n]$, there is $x \in H$ such that $\dim \mathcal{F}_x = \ell$. Given such an ℓ, we write $\ell = m + k$. Then $1 \le m \le n - k$. Since $\{f_1, \ldots, f_k\}$ is totally k-nonsingular, we know that $a_{ij} \ne 0$ for all $1 \le i \le k$ and $1 \le j \le n$. So there exist nonzero scalars $x_1, \ldots, x_m$ such that $\bar{a}_{i1}x_1 + \cdots + \bar{a}_{im}x_m \ne 0$ for every $1 \le i \le m$. Let $x = (x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}^n$. Then $\langle x, e_i \rangle = 0$ for $i = m + 1, \ldots, n$, $\langle x, e_j \rangle = x_i \ne 0$ for $i = 1, \ldots, m$, and

$$\langle x, f_i \rangle = \bar{a}_{i1}x_1 + \cdots + \bar{a}_{im}x_m \ne 0$$

for every $i = 1, \ldots, m$. This implies that $\mathcal{F}(x) = \operatorname{span}\{e_1, \ldots, e_m, f_1, \ldots, f_k\}$. Again, from Lemma 3.2, we obtain that $\{e_1, \ldots, e_m, f_1, \ldots, f_k\}$ are linearly independent. Therefore we get $\dim F_x = m + k = \ell$. Therefore we proved that $R(\mathcal{F}) = [k + 1 : n]$.

"$\Rightarrow$": Assume that $R(\mathcal{F}) = [k+1 : n]$ but A is not totally k-nonsingular. Then there exists a singular $\ell \times \ell$ submatrix, say $B = [a_{ij}]_{1 \le i,j \le \ell}$, of A for some $\ell \in [1 : k]$. Let y be a nonzero vector in $\mathbb{R}^{\ell}$ such that $By = 0$ and let $x = (y_1, \ldots, y_\ell, 0, \ldots, 0) \in \mathbb{R}^n$. Then we get

$$\mathcal{F}_x \subseteq \operatorname{span}\{e_1, \ldots, e_\ell, f_{\ell+1}, \ldots, f_k\},$$

since $\langle x, e_i \rangle = 0$ and $\langle x, f_j \rangle = 0$ for $i = \ell + 1, \ldots, n$ and $j = 1, \ldots, \ell$. This implies that $\dim \mathcal{F}_x \le \ell + (k - \ell) = k$. Thus we have proved that A is totally k-nonsingular if $R(\mathcal{F}) = [k + 1 : n]$. This completes the proof. □

Theorem 3.4 *Let $\mathcal{F} = \{f_1, \ldots, f_N\}$ be a frame for H. If $\mathcal{F}$ is n-independent, then $R(\mathcal{F}) = [m : n]$, where $m = \min\{n, N - n + 1\}$.*

Proof If $m = n$, then $N \ge 2n - 1$. Since $\mathcal{F}$ is n-independent, this implies that $\mathcal{F}$ has the complement property and hence $\dim \mathcal{F}_x = n$ for every nonzero vector x. Therefore $R(\mathcal{F}) = \{n\} = [m : n]$. If $m < n$, then $N < 2n - 1$ and $m = N - n + 1$. Without losing the generality, we can assume that $\mathcal{F} = \{e_1, \ldots, e_n, f_1, \ldots, f_k\}$ with $k < n - 1$.

Since $\mathcal{F}$ is n-independent, then, by Lemma 3.2, we have that $[f_1, \ldots, f_k]$ is totally k-nonsingular. Thus, by Theorem 3.3, $R(\mathcal{F}) = [k+1 : N] = [N-n+1 : n] = [m : n]$. □

Remark 3.2 The n-independent condition is Theorem 3.4 is not necessary for $R(\mathcal{F})$ to be an integer interval even with the additional assumption that $\mathcal{F}$ is dimension range exact. For example, by [17], for $n(n+1)/2 \geq N > 2n-1$, there exists an exact phase-retrievable frame $\mathcal{F} = \{f_j\}_{j=1}^N$ for $\mathbb{R}^n$. So $R(\mathcal{F}) = \{n\}$ is an interval and $\mathcal{F}$ is dimension range exact. However, $\mathcal{F}$ can not be n-independent. Indeed, if $\mathcal{F}$ is n-linearly independent, then $\{f_1, \ldots, f_{N-1}\}$ is also n-independent and $N-1 \geq 2n-1$. This implies that $\{f_1, \ldots, f_{N-1}\}$ has the complement property and hence it is phase-retrievable, which is impossible since $\mathcal{F}$ is an exact phase-retrievable frame.

Corollary 3.5 *Let $\mathcal{F} = \{f_1, \ldots, f_N\}$ be a frame for H. Assume Λ is a subset of $\{1, \ldots, N\}$ with the largest cardinality such that $\mathcal{F}_\Lambda$ is n-independent. Then $R(\mathcal{F}) \subseteq [m : n]$, where $m = \min\{n, |\Lambda| - n + 1\}$.*

Proof By Theorem 3.4, we have that $R(\mathcal{F}_\Lambda) = [m : n]$. Let $\ell \in R(\mathcal{F})$. Then there exists x such that $\dim F_x = \ell$. Since $(F_\Lambda)_x \subseteq \mathcal{F}_x$, we have that $n \geq \ell \geq \dim(F_\Lambda)_x \geq m$. Therefore $R(\mathcal{F}) \subseteq [m : n]$. □

The next proposition allows us to easily construct a rich class of examples whose dimension functions do not have an interval range. Let $\Omega_1, \ldots, \Omega_k$ be subsets of $\mathbb{N}$. We use $\sum_{j=1}^k \Omega_j$ to denote the set $\{d : d = \sum_{j=1}^k d_j,\ d_j \in \Omega_j\}$.

Proposition 3.6 *Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for H, $\{\Lambda_1, \ldots, \Lambda_k\}$ be a partition of $[N]$ and $n_j = \dim \operatorname{span}\mathcal{F}_{\Lambda_j}$. If $n = \sum_{j=1}^k n_j$, then*

$$R(\mathcal{F}) \cup \{0\} = \sum_{j=1}^k (R(\mathcal{F}_{\Lambda_j}) \cup \{0\}).$$

Proof Let $H_j = \operatorname{span}\mathcal{F}_{\Lambda_j}$. Then the condition $n = \sum_{j=1}^k n_j$ implies that $H = \sum_{j=1}^k H_j$ is a direct sum. Since similar frames preserve the dimension function, we can assume that $H = \sum_{j=1}^k H_j$ is an orthogonal direct sum.

Let $x \in H$ and $x_i = P_i x$, where P_j is the orthogonal projection of x onto H_j. Thus we have

$$(\mathcal{F}_{\Lambda_j})_x = \operatorname{span}\{\langle x, f_i\rangle f_i : i \in \Lambda_j\} = \operatorname{span}\{\langle x_j, f_i\rangle f_i : i \in \Lambda_j\} = (\mathcal{F}_{\Lambda_j})_{x_j} \subseteq H_j.$$

Since $H = \sum_{j=1}^k H_j$ is an orthogonal direct sum, we get

$$\mathcal{F}_x = \sum_{j=1}^k (\mathcal{F}_{\Lambda_j})_x = \sum_{j=1}^k (\mathcal{F}_{\Lambda_j})_{x_j}$$

is also a direct sum, and so

$$\dim \mathcal{F}_x = \sum_{j=1}^{k} \dim(\mathcal{F}_{\Lambda_j})_{x_j} \in \sum_{j=1}^{k} (R(\mathcal{F}_{\Lambda_j}) \cup \{0\}).$$

Thus $R(\mathcal{F}) \cup \{0\} \subseteq \sum_{j=1}^{k} (R(\mathcal{F}_{\Lambda_j}) \cup \{0\})$.

Conversely, let $d \in \sum_{j=1}^{k} (R(\mathcal{F}_{\Lambda_j}) \cup \{0\})$ be nonzero positive integer. Then there exist (not necessarily nonzero) vectors $x_j \in H_j$ such that

$$d = \sum_{j=1}^{k} \dim(\mathcal{F}_{\Lambda_j})_{x_j}.$$

Let $x = \sum_{j=1}^{k} x_j \in H$. Then $\mathcal{F}_x = \sum_{j=1}^{k} (\mathcal{F}_{\Lambda_j})_{x_j}$. Thus we get $d = d_{\mathcal{F}}(x) \in R(\mathcal{F}) \cup \{0\}$. This completes the proof. □

By selecting $\mathcal{F}_{\Lambda_j}$ such that $R(\mathcal{F}_{\Lambda_j}) = \{n_j\}$ and $H = \sum_{j=1}^{k} H_j$ is a direct sum of H, we get the following:

Corollary 3.7 *If $\{n_1, \ldots, n_k\}$ is a partition of n, i.e., $n = \sum_{j=1}^{k} n_j$, then there exists a frame $\mathcal{F}$ for an n-dimensional space H such that*

$$R(\mathcal{F}) = \{n_{j_1} + \cdots + n_{j_\ell} : 1 \leq \ell \leq k \text{ and } j_1, \ldots, j_\ell \text{ are distinct indices}\}.$$

We say that a subset Ω of $[n]$ containing n is *attainable* if there is a frame $\mathcal{F}$ for an n-dimensional Hilbert space H such that $R(\mathcal{F}) = \Omega$.

Remark 3.3 The above corollary implies that $\{k, n-k, n\}$, in particular $\{n/2, n\}$ when n is even, is attainable for every $1 \leq k < n$.

Clearly, every subset is attainable for $n = 2$. We will show that there exists subset that is not attainable for any $n \geq 3$. In [15], the first named author and co-authors introduced the concept of linearly connected frames. Two vectors f_i and f_j in a frame $\mathcal{F}$ are called $\mathcal{F}$-*connected* if there exist linearly independent vectors $\{f_{k_1}, \ldots, f_{k_\ell}\}$ in $\mathcal{F}$ and nonzero scalars $c, c_1, \ldots, c_m$ such that

$$f_i = cf_j + \sum_{m=1}^{\ell} c_m f_{k_m}.$$

We say that $\mathcal{F}$ is *linearly connected* if every two vectors in $\mathcal{F}$ are $\mathcal{F}$-connected. The following main result was used in [15] to characterize spectrally optimal dual frames for erasures.

Theorem 3.8 [15] *Let $\mathcal{F} = \{f_j\}_{j=1}^{N}$ be a frame for an n-dimensional Hilbert space H with $n \geq 2$. Then the following are equivalent*:

(i) *$\mathcal{F}$ is linearly connected.*
(ii) *$H_\Lambda \cap H_{\Lambda^c} \neq \{0\}$ for any proper subset Λ of $[N]$.*
(iii) *There exists a subset Λ of $[N]$ such that $\mathcal{F}_\Lambda$ is n-independent and $|\Lambda| = n+1$.*

Lemma 3.9 *If $\mathcal{F}$ is a linearly connected frame for an n-dimensional Hilbert space H and $n \geq 2$, then $1 \notin R(\mathcal{F})$. More generally,* $\min R(\mathcal{F}) \geq m := \min\{n, L-n+1\}$, *where L is the largest cardinality such that there is a subset Λ of $[N]$ with the properties that $|\Lambda| = L$ and $\mathcal{F}_\Lambda$ is n-independent.*

Proof Assume that $\mathcal{F}$ is linearly connected. Then, by Theorem 3.8, we know that $L \geq n+1$. So, by Theorem 3.3, $R(\mathcal{F}_\Lambda) = [m : n]$. This implies that

$$\dim \mathcal{F}_x \geq \dim(\mathcal{F}_\Lambda)_x \geq m$$

for any nonzero vector $x \in H$. So $\min R(\mathcal{F}) \geq m$. Since $m \geq 2$ when $n \geq 2$, we particularly get that $1 \notin R(\mathcal{F})$ if $\mathcal{F}$ is linearly connected. □

Now we are ready to prove our last main result of this section.

Theorem 3.10 *Let H be an n-dimensional Hilbert space.*

(i) *$[k : n]$ is attainable for every $1 \leq k \leq n$.*
(ii) *$R(\mathcal{F})$ is an interval for every generic frame for H.*
(iii) *Let Ω be a subset of $[n]$ containing n. If $1 \in \Omega$ and $n-1 \notin \Omega$, then Ω is not attainable. In particular, $\{1, n\}$ is not attainable for $n \geq 3$.*

Proof (i) follows from Theorem 3.3 or Theorem 3.4, and (ii) follows from Theorem 3.4 and the fact that every generic frame is n-independent. So we only need to prove (iii).

Suppose that $R(\mathcal{F}) = \{1, n\}$ for some frame $\mathcal{F}$. By Lemma 3.9, $\mathcal{F}$ is not linearly connected. By Theorem 3.8, there exists a partition $\{\Lambda_1, \ldots, \Lambda_k\}$ of $[n]$ such that $H = \sum_{j=1}^k H_j$ is a direct sum and each $\mathcal{F}_{\Lambda_j}$ is linearly connected, where $H_j = \mathrm{span}\mathcal{F}_{\Lambda_j}$. Since

$$1 \in R(\mathcal{F}) \cup \{0\} = \sum_{j=1}^k (R(\mathcal{F}_{\Lambda_j}) \cup \{0\}),$$

we get that 1 is in one of the $R(\mathcal{F}_{\Lambda_j})$'s, say $1 \in R(\mathcal{F}_{\Lambda_1})$. Thus, again by Lemma 3.9, $\dim H_1 = 1$. Now let $\Lambda = \Lambda_1$. Then $\Lambda^c = \Lambda_2 \cup \cdots \cup \Lambda_k$ and $H_\Lambda \cap H_{\Lambda^c} = \{0\}$. By Proposition 3.6, we have that

$$R(\mathcal{F}) \cup \{0\} = R(\mathcal{F}_\Lambda) \cup \{0\} + R(\mathcal{F}_{\Lambda^c}) \cup \{0\}.$$

Since $\dim H_\Lambda = 1$, we know that $\dim H_{\Lambda^c} = n-1$. Thus $n-1 \in R(\mathcal{F}_{\Lambda^c}) \subseteq R(\mathcal{F})$ which leads to a contradiction. Therefore Ω is not attainable. □

Example 3.3 For $n = 4$, $\{1, 3\}$ and $\{1, 2, 4\}$ are the only sets that are not attainable. Indeed, we already know that every interval containing 4 is attainable, $\{2, 4\}$ and $\{1, 3, 4\}$ are also attainable by Remark 3.3. By Theorem 3.10 (iii), we also have that $\{1, 3\}$ and $\{1, 2, 4\}$ are not attainable.

4 Remarks and questions

4.1 Linearly connected frames

Proposition 3.6 tells us that $R(\mathcal{F})$ is constructed by the building blocks $R(\mathcal{F}_{\Lambda_j})$, where each $\mathcal{F}_{\Lambda_j}$ is linearly connected frame for $H_j = \mathrm{span}\mathcal{F}_{\Lambda_j}$. Therefore it is essential to have a better understanding of the dimension functions for linearly connected frames. In particular, we ask:

Question 1 *Is $R(\mathcal{F})$ an interval for every linearly connected frames*?

Remark 4.1 We conjecture that the answer is yes. Clearly, it is true for $n = 1, 2, 3$ since we already know that every attainable set is an interval. However, we still do not know the answer for $n = 4$, By Example 3.3, $\{1, 3, 4\}$ and $\{2, 4\}$ are the only non-interval attainable sets. From Lemma 3.9, $\mathcal{F}$ is not linearly connected if $R(\mathcal{F}) = \{1, 3, 4\}$. So to confirm the conjecture for $n = 4$, it is sufficient to show that $\{2, 4\}$ is not attainable by a connected frame.

Remark 4.2 Every phase-retrievable frame is linearly connected, since if otherwise then it clearly will contradict the complement property. Clearly, the converse is false. Moreover, we have the following:

Example 4.1 For every $N \in [2n-1, n(n+1)/2)$, then there exists a linearly connected frame $\mathcal{G}$ of length N for $\mathbb{R}^n$ such that $\mathcal{G}$ has exact PR-redundancy but it is not phase-retrievable.

Proof Since $2n - 1 < N + 1 < n(n + 1)/2$, by Theorem 2.1 in [17] there exists an exact PR-frame $\mathcal{F}$ of length $N + 1$. Thus F is linearly connected. By Theorem 3.8, there exists a subset Λ with the largest cardinality such that $L = |\Lambda| > n$ and $\mathcal{F}_\Lambda$ is n-independent. Hence by Theorem 3.8 again, $\mathcal{F}_\Lambda$ is linearly connected.

We claim that Λ is a proper subset of $[N + 1]$. Indeed, if otherwise $\Lambda = [N + 1]$, then $\mathcal{F}$ is n-independent. In particular any N number of vectors in $\mathcal{F}$ is n-independent and hence they form a PR- frame, which is impossible since $\mathcal{F}$ is an exact PR-frame. Pick any $i \notin \Lambda$ and let $\mathcal{G} = \{f_j : j \neq i\}$. Then the $\mathcal{G}$ is a frame of length N which is not phase-retrievable. Since $\mathcal{G} \supseteq \mathcal{F}_\Lambda$ and $\mathcal{F}_\Lambda$ is linearly connected, we get that $\mathcal{G}$ is linearly connected. By Corollary 2.2, $\mathcal{G}$ has exact PR-redundancy. This completes the proof. □

While not every subframe of a linearly connected frame is linearly connected, the following results tell us that the ones that share the same PR-redundancy with $\mathcal{F}$ remain to be linearly connected.

Proposition 4.1 *Let $\mathcal{F}$ be a linearly connected frame for H and $\mathcal{F}_\Lambda$ be a subframe of $\mathcal{F}$ such that*

$$\mathcal{S}_2 \cap \ker L_{\mathcal{F}} = \mathcal{S}_2 \cap \ker L_{\mathcal{F}_\Lambda}.$$

Then $\mathcal{F}_\Lambda$ is linearly connected.

Proof Suppose that $\mathcal{F}_\Lambda$ is not linearly connected. Then there exists a proper subset Ω of Λ such that $H_\Omega \cap H_{\Lambda\backslash\Omega} = \{0\}$. Assume that for every $i \notin \Lambda$ we have either $f_i \in H_\Omega$ or $f_i \in H_{\Lambda\backslash\Omega}$. Set $I = \{j : f_j \in H_\Omega\}$. Then $I^c = \{j : f_j \in H_{\Lambda\backslash\Omega}\}$. This implies that

$$H_I \cap H_{I^c} = H_\Omega \cap H_{\Lambda\backslash\Omega} = \{0\},$$

which contradicts with the linearly connectedness of $\mathcal{F}$. Thus there exists $i_0 \notin \Lambda$ such that $f_{i_0} \notin H_\Omega$ and $f_{i_0} \notin H_{\Lambda\backslash\Omega}$.

Since $f_j, f_{i_0} \notin H_\Omega$ for every $j \in \Lambda\backslash\Omega$, there exist $x \in H$ such that $\langle x, f_i\rangle = 0$ for $i \in \Omega$ but $\langle x, f_j\rangle \neq 0$ for every $j \in \Lambda\backslash\Omega$ and $\langle x, f_{i_0}\rangle = 1$, here we use the fact that the restriction of $f_j(j \in \Lambda\backslash\Omega)$ and f_{i_0} to $H_\Lambda^\perp$ are nonzero linear functionals. Similarly, there exists $y \in H$ such that $\langle y, f_i\rangle = 0$ for $i \in \Lambda\backslash\Omega$ but $\langle y, f_j\rangle \neq 0$ for every $j \in \Omega$ and $\langle y, f_{i_0}\rangle = 1$.

Let $u = x - y$ and $v = x + y$. Then we have $|\langle u, f_j\rangle| = |\langle v, f_j\rangle|$ for every $j \in \Lambda$, but $|\langle u, f_{i_{i_0}}\rangle| = 0$ and $|\langle v, f_{i_0}\rangle| = 2$. This implies that $u \otimes u - v \otimes v \in \ker L_{\mathcal{F}_\Lambda}$ but $u \otimes u - v \otimes v \notin \ker L_{\mathcal{F}}$, which leads to a contradiction. Hence $\mathcal{F}_\Lambda$ is linearly connected. □

4.2 Equivalence

Definition 4.1 Two frames $\mathcal{F}$ and $\mathcal{G}$ of same length for a Hilbert space H are called *switching equivalent* (or just equivalent in short) if one of them is obtained by composition of similarity, nonzero rescaling and permutation. We use notation $\mathcal{F} \sim \mathcal{G}$ for two equivalent frames $\mathcal{F}$ and $\mathcal{G}$.

Clearly, equivalence preserve all the exactness and the dimension functions. In particular, if $\mathcal{F} \sim \mathcal{G}$, then $R(\mathcal{F}) = R(\mathcal{G})$. The next example shows that the converse is not true.

Example 4.2 Suppose that $n \geq 3$. Let $2n - 1 < N \leq n(n+1)/2$. By Theorem 2.1 in [17], there exists an exact PR-frame $\mathcal{F} = \{f_j\}_{j=1}^N$. Let $\mathcal{G} = \{g_j\}_{j=1}^N$ be an n-independent frame for $\mathbb{R}^n$. Then $\mathcal{G}$ is phase-retrievable. Thus $R(\mathcal{F}) = R(\mathcal{G})\{n\}$. Let $\Lambda = \{1, \ldots, N-1\}$. Since $\mathcal{G}_\Lambda$ is also n-independent with $|\Lambda| = N - 1 \geq 2n - 1$, we know that $\mathcal{G}_\Lambda$ is phase-retrievable for $\mathbb{R}^n$. Thus $\mathcal{G}$ is not exact. Since equivalence preserves exactness, we get that $\mathcal{F}$ and $\mathcal{G}$ are not equivalent.

Question 2 *Suppose that two frames $\mathcal{F} = \{f_j\}_{j=1}^N$ and $\mathcal{G} = \{g_j\}_{j=1}^N$ are range exact and satisfy the condition $R(\mathcal{F}) = R(\mathcal{G})$. Is it true that $\mathcal{F} \sim \mathcal{G}$? In particular, is it true that $\mathcal{F} \sim \mathcal{G}$ if both $\mathcal{F}$ and $\mathcal{G}$ are exact PR-frames?*

4.3 Maximal PR subspaces

Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a frame for H and M be a subspace of H. Recall from [17] that M is called a *phase retrievable subspace* with respect to $\mathcal{F}$ if $\{P_M f_i\}_{i=1}^N$ is a phase retrievable frame for M, where P_M is the orthogonal projection from H onto M. The largest k such that there is a k-dimensional phase-retrievable subspace will be called the *phase retrievability index* of $\mathcal{F}$:

$$\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) := \max\{\dim M : M \text{ is a phase-retrievable subspace w.r.t. } \mathcal{F}\}.$$

Theorem 4.2 [17] *Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for $\mathbb{R}^n$. Then*

$$\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = \min_{\Lambda \subseteq [N]} \max\{\dim H_\Lambda, \dim H_{\Lambda^c}\},$$

where $H_\Lambda = \mathrm{span}\{f_j : j \in \Lambda\}$.

Corollary 4.3 *If $\mathcal{F} = \{f_j\}_{j=1}^N$ is a frame for $\mathbb{R}^n$, then $\mathrm{ind}_{\mathrm{pr}}(F) \geq \lfloor \frac{n+1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x.*

Proof We can assume that $\mathcal{G} = \{f_1, \ldots, f_n\}$ is a basis for $\mathbb{R}^n$. By Theorem 4.2, we know that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{G}) = \lfloor \frac{n+1}{2} \rfloor$. Thus $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) \geq \mathrm{ind}_{\mathrm{pr}}(\mathcal{G}) = \lfloor \frac{n+1}{2} \rfloor$. □

Let $\mathcal{F} = \{e_1, \ldots, e_n, f_1, \ldots, f_k\}$ be an n-independent frame for $\mathbb{R}^n$ with $k < n-1$. Then, by Theorem 3.4, $R(\mathcal{F}) = [k+1 : n]$ and so $\min R(\mathcal{F}) = k+1$. By n-linear independence of the frame, we get that

$$\min_{\Lambda \subseteq [N]} \max\{\dim H_\Lambda, \dim H_{\Lambda^c}\} = \left\lfloor \frac{n+k+1}{2} \right\rfloor.$$

In the case that $k \geq n-1$, we have that $\mathcal{F}$ is phase-retrievable. Thus $R(\mathbb{R}) = \{n\}$ and so $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = n$. Therefore we get:

Example 4.3 If $\mathcal{F}$ is an n-linearly independent frame for $\mathbb{R}^n$, then $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = \lfloor \frac{n+\min R(\mathcal{F})}{2} \rfloor$.

However, this is not true in general if $\mathcal{F}$ is not n-independent. For example, let $f_1 = e_1$ and $f_j \in H_2 := \mathrm{span}\{e_2, \ldots, e_n\}$ such that $\{f_j\}_{j=2}^N$ is a phase retrievable frame for H_2. Then the range of the dimension function of $\mathcal{F} = \{f_j\}_{j=1}^N$ is $\{1, n-1, n\}$ by Proposition 3.6. Clearly, $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = n-1 \neq \lfloor \frac{n+1}{2} \rfloor$. So we ask the following question:

Question 3 *What can we say about $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F})$ from $R(\mathcal{F})$? In particular, is it true that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = \mathrm{ind}_{\mathrm{pr}}(\mathcal{G})$ if $R(\mathcal{F}) = R(\mathcal{G})$? Is it true that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = \lfloor \frac{n+\min R(\mathcal{F})}{2} \rfloor$ if $R(\mathcal{F})$ is an interval?*

Proposition 4.4 *Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for $\mathbb{R}^n$. If $R(F) = [n-1 : n]$, then $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = n-1$.*

Proof Since $\mathcal{F}$ is not phase retrievable, we get that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) < n$. By Theorem 4.2, there is a subset Λ of $[N]$ such that

$$\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = \dim H_\Lambda \geq \dim H_{\Lambda^c}.$$

Pick a nonzero vector $x \in H$ such that $x \in H_{\Lambda^c}^{\perp}$. Then $\mathcal{F}_x = (\mathcal{F}_\Lambda)_x$. This implies that

$$n - 1 \leq \dim \mathcal{F}_x = \dim(\mathcal{F}_\Lambda)_x \leq \dim H_\Lambda = \mathrm{ind}_{\mathrm{pr}}(\mathcal{F}).$$

Therefore $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = n - 1$. □

Remark 4.3 (i) The above proof also shows that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) \geq \min R(\mathcal{F})$ for any frame $\mathcal{F}$. (ii) The following result tells us that we only need to consider the frame dimension function exact frames.

Proposition 4.5 *Let $\mathcal{F} = \{f_j\}_{j=1}^N$ be a frame for $\mathbb{R}^n$ and Λ be a subset of $[N]$. If $\mathcal{F}$ and $\mathcal{F}_\Lambda$ have the same frame dimension functions, then* $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}_\Lambda) = \mathrm{ind}_{\mathrm{pr}}(\mathcal{F})$.

Proof Since every phase retrievable subspace with respect to $\mathcal{F}_\Lambda$ is also phase retrievable with respect to $\mathcal{F}$, we immediately get that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}_\Lambda) \leq \mathrm{ind}_{\mathrm{pr}}(\mathcal{F})$. By Theorem 4.2, there exists a subset Ω of Λ such that

$$\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}_\Lambda) = \max\{\dim H_\Omega, \dim H_{\Lambda\backslash\Omega}\}.$$

Let $A = \Omega \cup \{i \in \Lambda^c : f_i \notin H_{\Lambda\backslash\Omega}\}$ and $B = (\Lambda\backslash\Omega) \cup \{i \in \Lambda^c : f_i \in H_{\Lambda\backslash\Omega}\}$. Clearly $\{A, B\}$ is a partition of $[N]$, and $H_B = H_{\Lambda\backslash\Omega}$. Next we show that $H_A = H_\Omega$. We claim that $f_i \in H_\Omega$ if $f_i \notin H_{\Lambda\backslash\Omega}$. Indeed, if $f_i \notin H_{\Lambda\backslash\Omega}$, then we can pick $x \in (H_{\Lambda\backslash\Omega})^\perp = H_B^\perp$ such that $\langle x, f_i\rangle \neq 0$. This implies that $f_i \in \mathcal{F}_x$ and $(\mathcal{F}_\Lambda)_x \subseteq H_\Lambda$. If $\mathcal{F}$ and $\mathcal{F}_\Lambda$ have the same frame dimension functions and $(F_\Lambda)_x \subseteq F_x$, we get that $\mathcal{F}_x = (F_\Lambda)_x$ and hence

$$f_i \in \mathcal{F}_x = (\mathcal{F}_\Lambda)_x \subseteq H_\Lambda.$$

Therefore $H_A = H_\Lambda$. This together with Theorem 4.2 imply that

$$\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) \leq \max\{\dim H_A, \dim H_B\} = \max\{\dim H_\Omega, \dim H_{\Lambda\backslash\Omega}\} = \mathrm{ind}_{\mathrm{pr}}(\mathcal{F}_\Lambda).$$

Therefore we get $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}_\Lambda) = \mathrm{ind}_{\mathrm{pr}}(\mathcal{F})$. □

Therefore we can restrict to the range exact frames for Question 3.

Example 4.4 If $\mathcal{F} = \{f_j\}_{j=1}^N$ is range exact and $R(\mathcal{F})$ is an interval, then $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = \lfloor\frac{n+\min R(\mathcal{F})}{2}\rfloor$ for $n \leq 4$.

Proof We only need to consider for the cases when $n = 3$ and $n = 4$.

(i) $n = 3$: By Proposition 4.4, the statement is true when $R(\mathcal{F}) = \{2, 3\}$. If $R(\mathcal{F}) = \{1, 2, 3\}$, then, by Proposition 3.1, $\mathcal{F}$ is a basis for $\mathbb{R}^3$, and so $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = 2$ by Example 4.3, and hence $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = \lfloor\frac{3+1}{2}\rfloor$ is true.

(ii) $n = 4$: Similar to (i), the statement is true when $R(\mathcal{F}) = \{1, 2, 3, 4\}$ or $\{3, 4\}$. Suppose that $R(\mathcal{F}) = \{2, 3, 4\}$. We need to show that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = 3$. It is enough to show that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) \neq 2$. Assume to the contrary that $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) = 2$. Then, by Theorem 4.2, there exists a subset Λ of $[N]$ such that

$$\max\{\dim H_\Lambda, \dim H_{\Lambda^c}\} = 2,$$

which implies that $\dim H_\Lambda = \dim H_{\Lambda^c} = 2$. Since $1 \notin R(\mathcal{F})$, we know by Proposition 3.6 that $1 \notin R(\mathcal{F}_\Lambda)$ and $1 \notin R(\mathcal{F}_{\Lambda^c})$. This implies that $R(\mathcal{F}_\Lambda) = R(\mathcal{F}_{\Lambda^c}) = \{2\}$ and, so by Proposition 3.6 again, $R(\mathcal{F}) = \{2, 4\}$, which leads to a contradiction.

$\square$

It is natural to expect that the phase retrievability index is related to the "size" of the algebraic variety $\mathcal{S}_2 \cap \ker L_{\mathcal{F}}$. It certainly makes sense that larger size of $\mathcal{S}_2 \cap \ker L_{\mathcal{F}}$ indicates smaller phase retrievability index.

Question 4 *How is* $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F})$ *related to the dimension of the algebraic variety* $\mathcal{S}_2 \cap \ker L_{\mathcal{F}}$? *In particular, is it true that* $\mathrm{ind}_{\mathrm{pr}}(\mathcal{F}) + \dim(\mathcal{S}_2 \cap \ker L_{\mathcal{F}}) = n$?

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