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Reduction by Symmetries of Simple Hybrid Mechanical Systems ⋆

María Emma Eyrea Irazú * **Leonardo Colombo** ** **Anthony Bloch** ***

* *Departamento de Matemática, Centro de Matemática (CMaLP), Facultad de Ciencias Exactas, Universidad Nacional de La Plata, and CONICET, Argentina. e-mail: maemma@mate.unlp.edu.ar)*
** *Instituto de Ciencias Matemáticas (CSIC-UAM-UCM-UC3M), Calle Nicolás Cabrera 13-15, Campus Cantoblanco, 28049, Madrid, Spain.(e-mail: leo.colombo@icmat.es)*
*** *Department of Mathematics, University of Michigan, 530 Church St. Ann Arbor, 48109, Michigan, USA.(e-mail: abloch@umich.edu)*

Abstract: This paper investigates reduction by symmetries in simple hybrid mechanical systems, in particular, symplectic and Poisson reduction for simple hybrid Hamiltonian and Lagrangian systems. We give general conditions for whether it is possible to perform a symplectic reduction for simple hybrid Lagrangian system under a Lie group of symmetries and we also provide sufficient conditions for performing the reduction by symmetries of hybrid Poisson manifolds for simple hybrid Hamiltonian systems.

Keywords: Hybrid Systems, Lagrangian systems, Hamiltonian systems, Poisson manifolds, Reduction by Symmetries.

1. INTRODUCTION

One of the key symmetry reduction techniques for mechanical systems can be found in the work of Marsden and Weinstein [20] on symplectic reduction for Hamiltonian systems. In a nutshell, given a Hamiltonian action of a Lie group G on a symplectic manifold, one considers a level set of a momentum map modulo the action of a suitable subgroup to form a new symplectic manifold. Since then, a number of papers have been devoted to the geometrization and generalization of this reduction technique. One remarkable approach is the Poisson reduction technique [19], that allows the construction of new Poisson structures out of a given one by combination of a restriction to some submanifolds that satisfy certain compatibility assumptions, and the passage to a quotient space where degeneracies can be eliminated.

Hybrid systems are dynamical systems with continuous-time and discrete-time components of its dynamics. These dynamical systems are capable of modeling several physical systems, such as, multiple UAV systems [18], [24], bipedal robots [22], [23] and embedded computer systems [25], [11], among others. Simple hybrid systems are a class of hybrid system introduced in [16], denoted as such because of their simple structure. A simple hybrid system is characterized by a tuple $\mathbf{H} = (D, X, \mathcal{S}, R)$ where D is a smooth manifold, X is a smooth vector field on D, $\mathcal{S}$ is an embedded submanifold of D with co-dimension 1, and $R : \mathcal{S} \to D$ is a smooth embedding. This type of hybrid system has been mainly employed for the understanding of locomotion gaits in bipeds and insects [3], [13], [26]. In the situation where the vector field X is associated with a mechanical system (Lagrangian or Hamiltonian), alternative approaches for mechanical systems with unilateral constraints have been considered in [9], [10], [14] and [15], and hybrid port-hamiltonian systems in [21], but to the best of our knowledge, the hybrid analogue for symmetry reduction has not been widely discussed in the literature.

A hybrid scheme for Routh reduction for hybrid Lagrangian systems with cyclic variables is found in [2] and [8], inspired to gain a better understanding of bipedal walking models (see also [3] and references therein). Symplectic reduction for hybrid Hamiltonian systems has been introduced in [1] and extended to time-dependent systems in [7]. This paper attempts to go one step further and to consider symmetry reduction of simple hybrid Hamiltonian systems with continuous-time dynamics described in terms of a Poisson bracket and symplectic reduction for simple hybrid Lagrangian systems. Thus reduction by symmetries can be seen as the hybrid version of the Poisson reduction theorem of [19] and the Lagrangian picture of symplectic reduction is obtained from the Hamiltonian one by adapting the scheme developed in [17] to the hybrid setting.

The paper is organized as follows. Sec. II presents the necessary background on the geometry of mechanical systems. Sec. III introduces the class of hybrid Lagrangian and Hamiltonian systems under consideration and the corresponding relation between both formalisms. The existence of symmetries and their associated conserved quantities are derived in Section IV to present the symplectic reduction forsimple hybrid Lagrangian systems, Finally, Sec. V derive sufficient conditions for the

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reduction of hybrid Poisson manifolds for simple hybrid Hamiltonian systems.

2. BACKGROUND AND NOTATION

Let Q be the configuration space of a mechanical system, an n-dimensional differentiable manifold. If $(q^1, \dots, q^n)$ are local coordinates on Q, the corresponding standard coordinates on the tangent bundle TQ and the cotangent bundle T^*Q are, respectively, $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$ and $(q^1, \dots, q^n, p_1, \dots, p_n)$, where $v_q = \dot{q}^i \frac{\partial}{\partial q^i}$ and $p_q = p_i dq^i$ for any $v_q \in T_qQ$ and $p_q \in T_q^*Q$ (the cotangent space at $q \in Q$, the dual of T_qQ). Throughout the paper, we use these induced coordinates.

Definition 2.1. Consider a differentiable function $f : Q \to M$ where Q and M are smooth manifolds. The *tangent lift* (or tangent map) of f at $q \in Q$ is the map $T_qf : T_qQ \to T_{f(q)}M$ and the *cotangent lift* (or cotangent map) of f is the map $T_s^*f : T_s^*M \to T_q^*Q$ defined by

$$\langle T_s^* f(\alpha_s), v_q\rangle = \langle \alpha_s, T_q f(v_q)\rangle \tag{1}$$

where $\alpha_s \in T_s^*M,\ v_q \in T_qQ, s = f(q)$ and $\langle\cdot,\cdot\rangle$ denotes how tangent covectors act on tangent vectors.

Definition 2.2. Let P be a smooth manifold. A *Poisson bracket* on P is a bilinear, skew-symmetric operator $\{\cdot,\cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ satisfying the Jacobi identity and Leibniz rule. The pair $(P, \{\cdot,\cdot\})$ is called *Poisson manifold*. For any Hamiltonian function $H : P \to \mathbb{R}$ the *Hamiltonian vector field*, X_H, describing the equations of motion, is uniquely determined by $\dot{F} = \{F, H\}$ for all smooth functions $F : P \to \mathbb{R}$.

Consider the cotangent bundle of a manifold Q. For $F, G \in C^\infty(T^*Q)$ the pair $(T^*Q, \{\cdot,\cdot\})$ is a Poisson manifold with Poisson bracket $\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\cdot\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\cdot\frac{\partial G}{\partial q_i}\right)$. This bracket is called the *canonical Poisson bracket* [4].

3. SIMPLE HYBRID MECHANICAL SYSTEMS

Next, we recall some definitions about simple hybrid Lagrangian and Hamiltonian systems [2] and [1]. We also show how trajectories of both types of hybrid system are related. For more details on the class of hybrid system considered in this work we refer to [16], [26].

3.1 Simple hybrid systems

Definition 3.1. A *simple hybrid system* is characterized by the 4-tuple $\mathscr{H} = (D, \mathcal{S}, R, X)$, where D is a smooth manifold (*the domain*), $\mathcal{S}$ is an embedded submanifold of D with co-dimension one (the *switching surface*), $R : \mathcal{S} \to D$ is a smooth embedding (the *impact map*), and X is a smooth vector field on D. The tuple $D^{\mathscr{H}} := (D, \mathcal{S}, R)$ is called *hybrid manifold*.

The dynamics associated with a simple hybrid system is described by an autonomous system with impulse effect as in [26]. We denote by $\Sigma_{\mathscr{H}}$ the *simple hybrid dynamical system* generated by $\mathscr{H}$, that is,

$$\Sigma_{\mathscr{H}} = \begin{cases} \dot{x}(t) = X(x(t)), & x^-(t) \notin \mathcal{S} \\ x^+(t) = R(x^-(t)), & x^-(t) \in \mathcal{S} \end{cases} \tag{2}$$

where $x : I \subset \mathbb{R} \to D$, and $x^-(t) := \lim_{\tau\to t^-} x(\tau)$, $x^+(t) := \lim_{\tau\to t^+} x(\tau)$ are the left and right limits of the state trajectory $x(t)$, respectively. These limits describe the states immediately before and after the times when integral curves of X intersects $\mathcal{S}$ (i.e., pre and post impact of the solution $x(t)$ with $\mathcal{S}$).

Definition 3.2. A *solution* for the simple hybrid system $\Sigma_{\mathscr{H}}$ is a curve $x : [t_0, t_f) \to D$, with $t_f \in \mathbb{R} \cup \{\infty\}$, $t_f > t_0$ unique from a given initial condition, depending continuously on it, and satisfying:

i) x is right continuous on $[t_0, t_f)$,
ii) left and right limits, denoted by $x^-(t) := \lim_{\tau\longrightarrow t^-} x(\tau)$ and $x^+(t) := \lim_{\tau\longrightarrow t^+} x(\tau)$, exist at each point $t \in [t_0, t_f)$,
iii) there exists a closed discrete subset $\mathcal{T} \subset [t_0, t_f)$, *the impact times*, closed and discrete, such that
   a) $\forall t \notin \mathcal{T}$, $x(t)$ is differentiable, $\frac{dx(t)}{dt} = X(x(t))$ and $x(t) \notin \mathcal{S}$,
   b) $\forall t \in \mathcal{T}$, $x^-(t) \in \mathcal{S}$ y $x^+(t) = R(x^-(t))$.

Note that to exclude Zeno behavior, as in [26], we require that $\mathcal{S} \cap \overline{R}(\mathcal{S}) = \emptyset$, where $\overline{R}(\mathcal{S})$ denotes the closure as a set of $R(\mathcal{S})$ and the set of impact times is closed and discrete.

Definition 3.3. Let $L : TQ \to \mathbb{R}$ be a Lagrangian function. A simple hybrid system $\mathscr{H}$ with $D = TQ$ and $X = X_L$, where $X_L : TQ \to T(TQ)$ is the Lagrangian vector field associated with L is called *simple hybrid Lagrangian system*, and it will be denoted by $\mathscr{H}_L = (TQ, S_{\mathbf{L}}, R_{\mathbf{L}}, X_L)$.

Definition 3.4. Let $H : T^*Q \to \mathbb{R}$ be a Hamiltonian function. A simple hybrid system $\mathscr{H}$ with $D = T^*Q$ and $X = X_H$, where $X_H : T^*Q \to T(T^*Q)$ is the Hamiltonian vector field associated with H is called *simple hybrid Hamiltonian system*, and it will be denoted by $\mathscr{H}_H = (T^*Q, S_{\mathbf{H}}, R_{\mathbf{H}}, X_H)$.

3.2 Relation between simple hybrid Hamiltonian and Lagrangian systems

In the following we introduce hybrid flows and we provide a Legendre transformation between the classes of simple hybrid Lagrangian systems and simple hybrid Hamiltonian systems.

Definition 3.5. A *hybrid flow* for $\mathscr{H}_L$ is a tuple $\chi^{\mathscr{H}_L} = (\Lambda, \mathcal{J}, \mathscr{C})$, where

i) $\Lambda = \{0, 1, 2, ...\} \subseteq \mathbb{N}$ is a finite (or infinite) indexing set,
ii) $\mathcal{J} = \{I_i\}_{i\in\Lambda}$ a set of intervals, called hybrid intervals, where $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, N finite, with $\tau_i, \tau_{i+1}, \tau_N \in \mathbb{R}$ and $\tau_i \leq \tau_{i+1}$,
iii) $\mathscr{C} = \{c_i\}_{i\in\Lambda}$ is a collection of solutions for the vector field X_L specifying the continuous-time Lagrangian dynamics, i.e., $\dot{c}_i = X_L(c_i(t))$ for all $i \in \Lambda$, and such that for each $i, i+1 \in \Lambda$,
   (a) $c_i(\tau_{i+1}) \in S_{\mathbf{L}}$,
   (b) $R_{\mathbf{L}}(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1})$.

Analogously, one can define the notion of hybrid flow $\chi^{\mathscr{H}_H}$ for a simple hybrid Hamiltonian system, $\mathscr{H}_H$.

We will assume in this paper that the Lagrangian is hyper-regular, i.e. that the Legendre transformation $\mathbb{F}L$ is a diffeomorphism between TQ and T^*Q (this is always the case for mechanical Lagrangians). One can then work out the velocities $\dot{q}$ in terms of (q, p) using the inverse of $\mathbb{F}L$ and define the Hamiltonian function $H : T^*Q \to \mathbb{R}$ as

$$H(q, p) = \langle p, \dot{q}(q, p)\rangle - L(q, \dot{q}(q, p)).$$

Definition 3.6. Given $\mathscr{H}_L = (TQ, S_{\mathbf{L}}, R_{\mathbf{L}}, X_L)$ and $\mathscr{H}_H = (T^*Q, S_{\mathbf{H}}, R_{\mathbf{H}}, X_H)$ we define the *Legendre transformation between simple hybrid system* $\mathbb{F}L_{\mathscr{H}} : \mathscr{H}_L \to \mathscr{H}_H$ as the application between simple hybrid system satisfying the following conditions:

i) $\mathbb{F}L(D_{\mathbf{L}}) = D_{\mathbf{H}}$ where $\mathbb{F}L : TQ \to T^*Q$ is the usual Legendre transformation, i.e., $\mathbb{F}L : TQ \to T^*Q$ is define by $\langle \mathbb{F}L(v_q), w_q\rangle = \frac{d}{dt}\Big|_{t=0} L(v_q + tw_q)$, with $v_q, w_q \in T_qQ$.
ii) $\mathbb{F}L(S_{\mathbf{L}}) = S_{\mathbf{H}}$.
iii) $\mathbb{F}L \circ R_{\mathbf{L}} = R_{\mathbf{H}} \circ \mathbb{F}L|_{S_{\mathbf{L}}}$.
iv) $(\mathbb{F}L)_* X_L = X_H$,

where $(\mathbb{F}L)_* X_L$ denotes the push forward of the Lagrangian vector field by $\mathbb{F}L$ (see [4], Ch. 2).

Next, we show the relation between the hybrid flows for simple hybrid Lagrangian and Hamiltonian systems.

Proposition 1. Given $\chi^{\mathscr{H}_L} = (\Lambda, \mathfrak{J}, \mathfrak{C})$ a hybrid flow for $\mathscr{H}_L$ with initial condition $(q_0, \dot{q}_0)$, then $\chi^{\mathscr{H}_H} = (\Lambda, \mathfrak{J}, (\mathbb{F}L)(\mathscr{C}))$ with $(\mathbb{F}L)(\mathscr{C}) = \{(\mathbb{F}L)(c_i)\}_{i\in\Lambda}$ is a hybrid flow for $\mathscr{H}_H$ with initial condition $(q_0, p_0 = \mathbb{F}L(q_0))$.

Proof: As X_L and X_H are $\mathbb{F}L$-related by condition (iv), if $c_i(t)$ is an integral curve for X_L, $\tilde{c}_i(t) = (\mathbb{F}L \circ c_i)(t)$ is an integral curve for X_H. By considering $c_0(t)$ a solution with initial condition $c_0 = (q_0, \dot{q}_0)$ defined on $[\tau_0, \tau_1]$, then $\tilde{c}_0(t)$ is a solution with initial condition $\tilde{c}_0 = (q_0, p_0)$ defined on $[\tau_0, \tau_1]$. In the same way, by taking $c_1(t)$ to be a solution with initial condition $c_1 = (q_1, \dot{q}_1)$ defined on $[\tau_1, \tau_2]$, then $\tilde{c}_1(t)$ is a solution with initial condition $\tilde{c}_1 = (q_1, p_1)$ defined on the same hybrid interval.

Proceeding inductively, one finds $c_i(t) = (q_i, \dot{q}_i)$ defined on $[\tau_i, \tau_{i+1}]$. We can observe that $\tilde{c}_i(t)$ satisfies that $\tilde{c}_i(\tau_{i+1}) \in S_{\mathbf{H}}$ and $R_{\mathbf{H}}(\tilde{c}_i(\tau_{i+1})) = \tilde{c}_{i+1}(\tau_{i+1})$. For the property of $\mathbb{F}L$, we have that

(i) $\tilde{c}_i(\tau_{i+1}) = (\mathbb{F}L \circ c_i)(\tau_{i+1}) = \mathbb{F}L(c_i(\tau_{i+1}))$ and given that $c_i(\tau_{i+1}) \in S_{\mathbf{L}}$ then $\tilde{c}_i(\tau_{i+1}) \in S_{\mathbf{H}}$, because of condition (ii) in Definition 3.6.
(ii) $R_{\mathbf{H}}(\tilde{c}_i(\tau_{i+1})) = R_{\mathbf{H}} \circ \mathbb{F}L \circ c_i(\tau_{i+1}) = \mathbb{F}L \circ R_{\mathbf{L}} \circ c_i(\tau_{i+1}) = \mathbb{F}L \circ c_{i+1}(\tau_{i+1}) = \tilde{c}_{i+1}(\tau_{i+1})$. □

Note that Proposition 1 implies, in particular, that the hybrid flow $\mathscr{H}_L$ is mapped onto the hybrid flow $\mathscr{H}_H$ .The converse is also true by using that $\mathbb{F}L$ is a diffeomorphism.

4. SYMPLECTIC REDUCTION OF SIMPLE HYBRID SYSTEMS

In the following we provide some definitions and results about hybrid actions [1] and we describe the symplectic reduction for simple hybrid systems in the Hamiltonian and Lagrangian pictures by employing Proposition 1.

4.1 Hybrid actions

Let $\mathscr{H}_L = (TQ, S_{\mathbf{L}}, R_{\mathbf{L}}, X_L)$ be a simple hybrid Lagrangian system. The starting point for symmetry reduction is a *Lie group action* $\psi\colon G \times Q \to Q$ of some Lie group G on the manifold Q. We will assume that all the actions satisfy some regularity conditions so as to be able to carry out reduction (for instance, one can consider free and proper actions).

There is a natural lift Ψ^{T^*Q} of the action ψ to T^*Q, the *cotangent lift action*, defined by $\Psi_g^{T^*Q} = T^*\psi_{g^{-1}}$. It enjoys the following properties:

- Ψ^{T^*Q} is a *symplectic action*, meaning that $(\Psi_g^{T^*Q})^*\Omega = \Omega$, being Ω the canonical symplectic 2-form on T^*Q.
- It admits an Ad^*-equivariant *momentum map* $J\colon T^*Q \to \mathfrak{g}^*$ given by
$$\langle J(q,p), \xi\rangle = \langle p, \xi_Q\rangle, \quad \forall \xi \in \mathfrak{g},$$
where $\mathfrak{g}$ denotes the Lie algebra of G and $\xi_Q(q) = d(\psi_{\exp(t\xi)}q)/dt$ is the infinitesimal generator of $\xi \in \mathfrak{g}$.

Likewise, Ψ^{TQ} denotes the *tangent lift action* on TQ, defined by $\Psi_g^{TQ} = T\psi_g(q, \dot{q})$ (see [12] Ch. 6 for instance).

To perform hybrid reduction one needs to impose some compatibility conditions between the action and the hybrid system (see e.g. [2]). By an *hybrid action* on the simple hybrid Lagrangian system $\mathscr{H}_L$ we mean a Lie group action $\psi\colon G \times Q \to Q$ such that

- L is invariant under Ψ^{TQ}, i.e. $L \circ \Psi^{TQ} = L$.
- Ψ^{TQ} restricts to an action G on $S_{\mathbf{L}}$.
- $R_{\mathbf{L}}$ is equivariant with respect to the previous action, namely $R_{\mathbf{L}} \circ \Psi_g^{TQ}|_{S_{\mathbf{L}}} = \Psi_g^{TQ} \circ R_{\mathbf{L}}$.

Note that Ψ^{TQ} admits an Ad^*-equivariant momentum map $J_L : TQ \to \mathfrak{g}^*$ given by $J_L = J \circ \mathbb{F}L$. This follows directly from the invariance of L, since it implies that $\mathbb{F}L$ is an equivariant diffeomorphism, i.e. $\mathbb{F}L \circ \Psi_g^{TQ} = \Psi_g^{T^*Q} \circ \mathbb{F}L$.

The hybrid equivalent of momentum map is the notion of hybrid momentum map introduced in [1]. Denoting by $i\colon S_{\mathbf{H}} \hookrightarrow T^*Q$ the canonical inclusion, J is an *hybrid momentum map* if the following diagram commutes

$$\begin{tikzcd}
& \mathfrak{g}^* & \\
T^*Q \arrow[ur, "J"] & S_{\mathbf{H}} \arrow[l, hook', "i"'] \arrow[u, "J|_{S_{\mathbf{H}}}"] \arrow[r, "R_{\mathbf{H}}"] & T^*Q \arrow[ul, "J"']
\end{tikzcd} \tag{3}$$

4.2 Symplectic reduction of simple hybrid Lagrangian systems

For the Lagrangian side, one needs a further regularity condition, sometimes referred to as G-regularity, which is satisfied by mechanical Lagrangians. Precisely, one has the following definition (see [17]):

Definition 4.1. Let L be an invariant Lagrangian on TQ and denote by ξ_Q the infinitesimal generator for the associated action. We say that L is G-regular if, for each $v_q \in TQ$, the map
$$\mathfrak{J}_L^{v_q} : \mathfrak{g} \to \mathfrak{g}^*,$$
$$\xi \mapsto J_L\left(v_q + \xi_Q(q)\right), \; v_q \in T_qQ,$$
is a diffeomorphism.

In a nutshell, G-regularity amounts to regularity "with respect to the group variables". From now on we will assume that the Lagrangian is G-regular.

Consider a simple hybrid Lagrangian system $\mathscr{H}_L$ equipped with an hybrid action ψ. We begin by analyzing the reduction of the associated hybrid Hamiltonian system $\mathscr{H}_H$ given by Proposition 1.

Consider a *hybrid regular value* $\mu \in \mathfrak{g}^*$ of $J\colon T^*Q \to \mathfrak{g}^*$, which means that μ is a regular value of both J and $J\mid_{S_{\mathbf{H}}}\colon S_{\mathbf{H}} \to \mathfrak{g}^*$. When we combine this definition with the commutative diagram (3), we obtain that the following diagram

$$\begin{tikzcd} J^{-1}(\mu) \arrow[d, hook] & J\mid_{S_{\mathbf{H}}}^{-1}(\mu) \arrow[l, hook', "i"'] \arrow[r, "R_{\mathbf{H}}|_{S_{\mathbf{H}}}"] \arrow[d, hook] & J^{-1}(\mu) \arrow[d, hook] \\ T^*Q & S_{\mathbf{H}} \arrow[l, hook', "i"'] \arrow[r, "R_{\mathbf{H}}"] & T^*Q \end{tikzcd}$$

commutes, where $J^{-1}(\mu)$ and $J\mid_{S_{\mathbf{H}}}^{-1}(\mu)$ are embedded submanifolds of T^*Q and $S_{\mathbf{H}}$, respectively.

We can apply a hybrid analog of the symplectic reduction Theorem given in [20] to the hybrid Hamiltonian system $\mathscr{H}_H$. Note that, since L is invariant under Ψ^{TQ}, so is the Hamiltonian H under Ψ^{T^*Q}. The main conclusions are:

(i) The reduced space $J^{-1}(\mu)/G_\mu$ (with G_μ the isotropy group of μ under the coadjoint action) is a symplectic manifold, and the reduced symplectic structure Ω_μ is characterized in terms of the submersion $\pi_\mu\colon J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ and the inclusion $i_\mu\colon J^{-1}(\mu) \hookrightarrow T^*Q$ by means of the relation $\pi_\mu^*\Omega_\mu = i_\mu^*\Omega$.

(ii) If we denote by H_μ the reduction of $H\mid_{J^{-1}(\mu)}$ to $J^{-1}(\mu)/G_\mu$, the evolution vector field X_H projects onto X_{H_μ}.

(iii) $J\mid_{S_{\mathbf{H}}}^{-1}(\mu) \subset S_{\mathbf{H}}$ is G_μ-invariant and hence reduces to a submanifold of the reduced space which we denote $(S_{\mathbf{H}})_\mu \subset J^{-1}(\mu)/G_\mu$.

(iv) Again, using invariance $R_{\mathbf{H}}$ reduces to a map

$$(R_{\mathbf{H}})_\mu\colon (S_{\mathbf{H}})_\mu \to J^{-1}(\mu)/G_\mu.$$

The reduction picture on the Lagrangian side can now be obtained from the Hamiltonian one by adapting the scheme developed in [17] to the symplectic setting. The key idea is that, since L is invariant and hyperregular, the Legendre transformation $\mathbb{F}L$ is a diffeomorphism such that:

- It is equivariant with respect to Ψ^{TQ} and Ψ^{T^*Q},
- Preserves the level sets of the momentum map, that is, $\mathbb{F}L(J_L^{-1}(\mu)) = J^{-1}(\mu)$,
- Relates both symplectic structures, that is, $(\mathbb{F}L)^*\Omega = \Omega_L$, where Ω_L is the Poincaré-Cartan two-form.

It follows that the map $\mathbb{F}L$ reduces to a symplectomorphism $(\mathbb{F}L)_{\text{red}}$ between the reduced spaces. Therefore we get the following commutative diagram of hybrid manifolds:

$$\begin{tikzcd} (TQ, S_{\mathbf{L}}, R_{\mathbf{L}}) \arrow[r, "\mathbb{F}L"] \arrow[d, "\text{Red.}"'] & (T^*Q, S_{\mathbf{H}}, R_{\mathbf{H}}) \arrow[d, "\text{Red.}"] \\ (J_L^{-1}(\mu)/G_\mu, (S_{\mathbf{L}})_\mu, (R_{\mathbf{L}})_\mu) \arrow[r, "(\mathbb{F}L)_{\text{red}}"] & (J^{-1}(\mu)/G_\mu, (S_{\mathbf{H}})_\mu, (R_{\mathbf{H}})_\mu) \end{tikzcd}$$

Special care should be taken when translating the reduction technique to hybrid systems. The reason is that the collisions with the switching surface will, in general, modify the value of the momentum map. Therefore, if $\mathcal{J} = \{I_i\}_{i\in\Lambda}$ is the hybrid interval (see Definition 3.5), the reduced hamiltonian has to be defined in each I_i taking into account the value of the momentum μ_i after the collision at time τ_i. Note that this also has influence on the way the impact map R is reduced.

Let us denote: (1) μ_i the momentum of the system in $I_i = [\tau_i, \tau_{i+1}]$, (2) R_{μ_i} the reduction of $(R_{\mathbf{H}})_{\mu_i}$, and (3) S_{μ_i} the reduction of $J\mid_{S_{\mathbf{H}}^{-1}}(\mu_i)$. There is a sequence of reduced HHS's

$$\begin{tikzcd} {[\tau_0, \tau_1]} \arrow[r, "\text{Red.}"] \arrow[d, "\text{Coll.}"'] & (J^{-1}(\mu_0)/G_{\mu_0}, H_{\mu_0}, S_{\mu_0}, R_{\mu_0}) \arrow[d, "\text{Coll.}"] \\ {[\tau_1, \tau_2]} \arrow[r, "\text{Red.}"] \arrow[d, "\text{Coll.}"'] & (J^{-1}(\mu_1)/G_{\mu_1}, H_{\mu_1}, S_{\mu_1}, R_{\mu_1}) \arrow[d, "\text{Coll.}"] \\ (\dots) \arrow[r, "\text{Red.}"] & (\dots) \end{tikzcd}$$

The same can be stated for Lagrangian hybrid flows by employing Proposition 1. As was remarked in [1], the reconstruction procedure from the reduced hybrid flow to the Hamiltonian hybrid flow involves a recursive integration at each stage in the previous diagram using the solution of the reduced HHS. Roughly speaking, this accounts to imposing the momentum constraint on the reconstructed solution.

5. POISSON REDUCTION OF SIMPLE HYBRID SYSTEMS

Next, we present the reduction by symmetries for simple hybrid Hamiltonian systems as the reduction of hybrid Poisson manifolds.

Definition 5.1. A *hybrid Poisson manifold* is determined by a 4-tuple $\mathscr{H}_{poiss} = (D, \mathcal{S}, R, \{\cdot,\cdot\})$, where D is a smooth manifold, the *domain*, $\mathcal{S}$ is an embedded submanifold of D with co-dimention one, the *switching surface*, $R : \mathcal{S} \to D$ is a smooth embedding, the *impact map* and $\{\cdot,\cdot\}$ is a Poisson bracket on D.

The dynamics associated with a hybrid Poisson manifold is described by an autonomous system with impulse effect. We denote by $\Sigma_{\mathscr{H}_{poiss}}$ the dynamics generated by the hybrid Poisson manifold $\mathscr{H}_{poiss}$, that is,

$$\Sigma_{\mathscr{H}_{poiss}} = \begin{cases} \dot{x}(t) = \{x(t), H\}, & x^-(t) \notin \mathcal{S}, \\ x^+(t) = R(x^-(t)), & x^-(t) \in \mathcal{S} \end{cases}.$$

Definition 5.2. Let $\mathscr{H}_{poiss} = (D, \mathcal{S}, R, \{\cdot,\cdot\})$ be a hybrid Poisson manifold, and $\psi : G \times D \to D$ be an action of a Lie group G on D. ψ is a *hybrid Poisson action* if $\psi|_{\mathcal{S}}$ is a Poisson action of G on $\mathcal{S}$ and for all $g \in G$ and $F, W \in \mathcal{F}(D)$ satisfies

$$\{F, W\} \circ \psi_g = \{F \circ \psi_g, W \circ \psi_g\}$$

together with

$$\{M, N\} \circ \psi_g|_{\mathcal{S}} = \{M \circ \psi_g|_{\mathcal{S}}, N \circ \psi_g|_{\mathcal{S}}\}, \text{ for all } M, N \in \mathcal{F}(\mathcal{S}).$$

The action ψ is called a hybrid Poisson free and proper action, if ψ is a free and proper action and it is also a hybrid Poisson action.

Given a hybrid Poisson manifold $\mathscr{H}_{poiss} = (D, S, R, \{\cdot,\cdot\})$, a Lie group of G and a hybrid Poisson action ψ, we define the *hybrid orbit space* associated to the hybrid Poisson action by $(D/G)^{\mathscr{H}_{poiss}} := (D/G, S/G, \hat{R})$ where D/G and S/G are the orbit spaces obtained through ψ and $\psi|_S$, respectively, and $\hat{R} : S/G \to D/G$ is the impact map induced in the orbit space. That is, if $\pi : D \to D/G$ is given by $\pi(x) = [x] \in D/G$ (the application shifting x in the ψ-orbit of x, i.e., $x \sim \psi_g(x)$ for all $g \in G$, being $\sim$ the equivalence relation defining the orbit space), then $\hat{R}([x]) = [R(x)] = \pi(R(x))$.

Proposition 2. Let G be a Lie group and ψ a hybrid Poisson action which acting on the hybrid Poisson manifold $\mathscr{H}_{poiss}$. Assume that D/G and $\mathcal{S}/G$ are differentiable manifolds and, $\pi : D \to D/G$ and $\pi|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}/G$ are submersions, on D/G and $\mathcal{S}/G$ respectively. Then, there is a unique Poisson bracket $\{\cdot,\cdot\}_{red}$ on D/G called reduced Poisson bracket, such that

$$\{F, K\}_{red} \circ \pi := \{F \circ \pi, K \circ \pi\} \tag{4}$$

and

$$\{M, N\}_{red} \circ \pi|_{\mathcal{S}} := \{M \circ \pi|_{\mathcal{S}}, N \circ \pi|_{\mathcal{S}}\} \tag{5}$$

for all $F, K \in \mathcal{F}(D/G)$ and $M, N \in \mathcal{F}(\mathcal{S}/G)$ so that $\hat{\mathscr{H}}_{poiss} = (D/G, \mathcal{S}/G, \hat{R}, \{\cdot,\cdot\}_{red})$ is a hybrid Poisson manifold.

Proof: For each $F, K \in \mathcal{F}(D/G)$ and $g \in G$, since ψ is a hybrid Poisson action and π is G-invariant, it follows that $\{F\circ\pi, K\circ\pi\}\circ\psi_g = \{F\circ\pi\circ\psi_g, K\circ\pi\circ\psi_g\} = \{F\circ\pi, K\circ\pi\}$, and therefore $\{F \circ \pi, K \circ \pi\}$ is G-invariant.

Observe that for each $M, N \in \mathcal{F}(\mathcal{S}/G)$ and $g \in G$, since ψ is a hybrid Poisson action and $\pi|_{\mathcal{S}}$ is G-invariant, we have that $\{M\circ\pi|_{\mathcal{S}}, N\circ\pi|_{\mathcal{S}}\}\circ\psi_g = \{M\circ\pi|_{\mathcal{S}}\circ\psi_g, N\circ\pi|_{\mathcal{S}}\circ\psi_g\} = \{M\circ \pi|_{\mathcal{S}}, N\circ\pi|_{\mathcal{S}}\}$ and therefore $\{M\circ\pi|_{\mathcal{S}}, N\circ\pi|_{\mathcal{S}}\}$ is G-invariant.

The functions $\{F \circ \pi, K \circ \pi\}$ and $\{M \circ \pi|_{\mathcal{S}}, N \circ \pi|_{\mathcal{S}}\}$ can be expressed as $\beta \circ \pi$ and $\beta|_{\mathcal{S}} \circ \pi|_{\mathcal{S}}$, respectively, for $\beta \in \mathcal{F}(D/G)$ and $\beta|_{\mathcal{S}} \in \mathcal{F}(\mathcal{S}/G)$. We denote β as $\{F, K\}_{red}$, which defines a Poisson bracket $\{\cdot,\cdot\}_{red}$ on D/G, since it satisfies the equations (4) and (5). Note that $\{\cdot,\cdot\}_{red}$ is the unique Poisson bracket on D/G defined by π. Then $(D/G, S/G, \hat{R}, \{\cdot,\cdot\}_{red})$ is a hybrid Poisson manifold. □

Definition 5.3. If $H : D \to \mathbb{R}$ is a G-invariant function on D, then the *reduced function* $H_{red} : D/G \to \mathbb{R}$ on D/G is determined univocally by the relation $H_{red} \circ \pi = H$.

Proposition 3. Consider $\mathscr{H}_{poiss}$ and $\hat{\mathscr{H}}_{poiss}$ as in Proposition 2. If $\chi^{\mathscr{H}_{poiss}}(x_0)$ is a hybrid flow for $\mathscr{H}_{poiss}$ with $x_0 \in D$, then the hybrid flow $\chi^{\hat{\mathscr{H}}_{poiss}}$ associated with a reduced hybrid Poisson manifold $\hat{\mathscr{H}}_{poiss}$ is given by $\chi^{\hat{\mathscr{H}}_{poiss}}(\pi(x_0)) = (\Lambda, \mathcal{J}, \pi(\mathcal{C}))$, where $\pi(\mathcal{C}) := \{\pi(c_i) : c_i \in \mathcal{C}\}$ and $\mathcal{C}$, Λ, $\mathcal{J}$ are given as in Definition 3.5.

Proof: Let $c_i^{red}(t) = \pi|_{\mathcal{S}}(c_i(t))$. We must check that $c_i^{red}(\tau_{i+1}) \in \mathcal{S}/G$ and $\hat{R}(c_i^{red}(\tau_{i+1})) = c_{i+1}^{red}(\tau_{i+1})$:

i) $c_i^{red}(\tau_{i+1}) = \pi|_{\mathcal{S}}(c_i(\tau_{i+1}))$ and given that $c_i(\tau_{i+1}) \in \mathcal{S}$ then $c_i^{red}(\tau_{i+1}) \in \mathcal{S}/G$.
ii) $\hat{R}(c_i^{red}(\tau_{i+1})) = \hat{R}(\pi|_{\mathcal{S}}(c_i(\tau_{i+1}))) = \pi(R(c_i(\tau_{i+1}))) = \pi(c_{i+1}(\tau_{i+1})) = c_{i+1}^{red}(\tau_{i+1})$. □

Remark 1. Let $\mathscr{H}_{poiss} = (D, \mathcal{S}, R, \{\cdot,\cdot\})$ a hybrid Poisson manifold, and $J : D \to \mathfrak{g}^*$ an Ad^*-equivariant hybrid momentum map. Let $\mu \in D$ be a hybrid regular value of J. If D is a symplectic manifold then the reduced Poisson structure on $J^{-1}(\mu)/G_\mu$ is just the reduced symplectic structure Ω_μ defined on Section 4.2.

Remark 2. We are using the notion of hybrid Poisson manifold and not deriving a hybrid Poisson bracket. The non-existence of a hybrid Poisson bracket has been commented in [6]

5.1 *Example: Spherical pendulum hitting a surface*

Consider an inverted spherical pendulum with length R hitting on a surface. The configuration space is $Q = \mathbb{S}^2$ with local coordinates $q = (\theta, \varphi) \in \mathbb{S}^2$. The associated momentum for q is denoted by $p \in T_q^*\mathbb{S}^2$, $p = (p_\theta, p_\varphi)$. The Hamiltonian function $H : T^*\mathbb{S}^2 \to \mathbb{R}$ for the system is given by $H(\theta, \varphi, p_\theta, p_\varphi) = \frac{1}{2mR^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2(\theta)}\right) - mgR\cos(\theta)$.

Consider the function $h : \mathbb{S}^2 \to \mathbb{R}$ given by $h_P(\theta, \varphi) = R\cos(\theta)$ describing the impact of the pendulum with the surface. Then, the associated simple hybrid Hamiltonian system $\mathscr{H}_H$ is given by $\mathscr{H}_H = (D, S_{\mathbf{H}}, R_{\mathbf{H}}, X_H)$ where

i) $D = \{(\theta, \varphi, p_\theta, p_\varphi) \in T^*\mathbb{S}^2 : \cos(\theta) \geq 0\}$ is the domain,
ii) $S_{\mathbf{H}} = \{(\theta, \varphi, p_\theta, p_\varphi) \in T^*\mathbb{S}^2 \mid \cos(\theta) = 0 \text{ and } p_\theta \geq 0\}$ is the switching surface,
iii) $R_{\mathbf{H}}(\theta, \varphi, p_\theta, p_\varphi) = (\theta, \varphi, -ep_\theta, p_\varphi)$ is obtained by using the so-called impact equation [1; 5] where $0 \leq e \leq 1$ is the coefficient of restitution, which is a measure of the energy dissipated through the impact, e.g., for a perfectly elastic impact $e = 1$, and for a perfectly plastic impact $e = 0$.
iv) The Hamiltonian vector field X_H is given by $X_H(q,p) = \left(\frac{p_\theta}{mR^2}, \frac{p_\varphi}{mR^2\sin^2(\theta)}, \frac{p_\varphi^2\cos(\theta)}{mR^2\sin^3(\theta)} - mgR\sin(\theta), 0\right)$.

The Poisson bracket $\{\cdot,\cdot\}_{\mathbb{S}^2} : \mathcal{F}(\mathbb{S}^2) \times \mathcal{F}(\mathbb{S}^2) \to \mathcal{F}(\mathbb{S}^2)$ is

$$\{f, g\}_{\mathbb{S}^2} = \frac{\partial f}{\partial\theta}\frac{\partial g}{\partial p_\theta} + \frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial p_\varphi} - \frac{\partial f}{\partial p_\theta}\frac{\partial g}{\partial\theta} - \frac{\partial f}{\partial p_\varphi}\frac{\partial g}{\partial\varphi}.$$

Therefore $\mathscr{H}_{poiss} = (D, S_{\mathbf{H}}, R_{\mathbf{H}}, \{\cdot,\cdot\}_{\mathbb{S}^2})$ is a hybrid Poisson manifold and by using the Poisson bracket and the fact that $\dot{q} = \{q, H\}$ and $\dot{p} = \{p, H\}$, Hamilton equations are

$$\dot{\theta} = \{\theta, H\} = \frac{p_\theta}{mR^2}, \quad \dot{\varphi} = \{\varphi, H\} = \frac{p_\varphi}{mR^2\sin^2\theta},$$

$$\dot{p}_\theta = \{p_\theta, H\} = \frac{p_\varphi^2\cos\theta}{mR^2\sin^3\theta} - mgR\sin\theta, \; \dot{p}_\varphi = \{p_\varphi, H\} = 0.$$

Let $G = \mathbb{S}^1$ which acts by rotations about the vertical axis, i.e., $\psi : \mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$ is the action given by $\psi(\alpha, (\theta, \varphi)) = (\theta, \alpha + \varphi)$. The cotangent lift action on $T^*\mathbb{S}^2$, i.e. $\Psi^{T^*\mathbb{S}^2} : \mathbb{S}^1 \times T^*\mathbb{S}^2 \to T^*\mathbb{S}^2$, is given by $\Psi^{T^*\mathbb{S}^2}(\alpha, (\theta, \varphi, p_\theta, p_\varphi)) = (\theta, \varphi + \alpha, p_\theta, p_\varphi)$. Let's check it is also a hybrid Poisson action, i.e.,

$$\{f, g\}_{\mathbb{S}^2} \circ \Psi^{T^*\mathbb{S}^2} = \{f \circ \Psi^{T^*\mathbb{S}^2}, g \circ \Psi^{T^*\mathbb{S}^2}\}_{\mathbb{S}^2}, \tag{6}$$

for functions $f, g : T^*\mathbb{S}^2 \to \mathbb{R}$. Denote by $F_1(\theta, \varphi, p_\theta, p_\varphi) = \theta$, $F_2(\theta, \varphi, p_\theta, p_\varphi) = \varphi$, $F_3(\theta, \varphi, p_\theta, p_\varphi) = p_\theta$ and $F_4(\theta, \varphi, p_\theta, p_\varphi) = p_\varphi$. Then, for the left side of equation (6),

$\{\cdot,\cdot\}_{\mathbb{S}^2}$	θ	φ	p_θ	p_φ
θ	0	0	1	0
φ	0	0	0	1
p_θ	-1	0	0	0
p_φ	0	-1	0	0

Therefore,

$$\{f, g\}_{\mathbb{S}^2}\circ\Psi^{T^*\mathbb{S}^2} = \begin{cases} 1, \text{ if } f = \theta, \; g = p_\theta \text{ or, } \; f = \varphi, \; g = p_\varphi \\ -1, \text{ if } f = p_\theta, \; g = \theta \text{ or } f = p_\varphi, \; g = \varphi \\ 0, \text{in another case} \end{cases}$$

For the right side of equation (6) we have

$\{\cdot,\cdot\}_{\mathbb{S}^2}$	$\theta \circ \Psi^{T^*\mathbb{S}^2}$	$\varphi \circ \Psi^{T^*\mathbb{S}^2}$	$p_\theta \circ \Psi^{T^*\mathbb{S}^2}$	$p_\varphi \circ \Psi^{T^*\mathbb{S}^2}$
$\theta \circ \Psi^{T^*\mathbb{S}^2}$	0	0	1	0
$\varphi \circ \Psi^{T^*\mathbb{S}^2}$	0	0	0	1
$p_\theta \circ \Psi^{T^*\mathbb{S}^2}$	-1	0	0	0
$p_\varphi \circ \Psi^{T^*\mathbb{S}^2}$	0	-1	0	0

Therefore, $T^*\mathbb{S}^2$ is an hybrid Poisson action.

Next, we describe the reduced hybrid Poisson manifold. Let $D/G = T^*\mathbb{S}^2/\mathbb{S}^1 \approx T^*(\mathbb{S}^2/\mathbb{Z}^1)$, where $\mathbb{S}^2/\mathbb{Z}^1 \simeq (0, 2\pi)$. The reduced Hamilton $H_{red} : T^*\mathbb{S}^2/\mathbb{S}^1 \to \mathbb{R}$ is obtain by

$$H_{red}(\theta, p_\theta, p_\varphi) = H(\theta, \varphi, p_\theta, p_\varphi) \circ \pi(\theta, \varphi, p_\theta, p_\varphi),$$

and it is given by

$$H_{red}(\theta, p_\theta, p_\varphi) = \tfrac{1}{2mR^2}\left(p_\theta^2 + \tfrac{p_\varphi^2}{\sin^2(\theta)}\right) - mgR\cos(\theta),$$

where $\pi : T^*\mathbb{S}^2 \to T^*\mathbb{S}^2/\mathbb{S}^1$ is the projection map given by $\pi(\theta, \varphi, p_\theta, p_\varphi) = (\theta, p_\theta, p_\varphi)$. The reduced Poisson bracket is

$$\{f, g\}_{red} = \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial p_\theta} - \frac{\partial f}{\partial p_\theta}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial p_\varphi}\frac{\partial g}{\partial \varphi}.$$

Using the reduced Poisson bracket, we obtain the reduced Hamilton equations

$$\dot{\theta} = \{\theta, H_{red}\} = \frac{p_\theta}{mR^2}, \quad \dot{p}_\varphi = \{p_\varphi, H_{red}\} = 0. \tag{7}$$

$$\dot{p}_\theta = \{p_\theta, H_{red}\} = \frac{p_\varphi^2 \cos\theta}{mR^2 \sin^3\theta} - mgR\sin\theta. \tag{8}$$

Therefore $\hat{\mathscr{H}}_{poiss} = (D/G, S/G, \hat{R}, \{\cdot,\cdot\}_{red})$ is a reduced hybrid Poisson manifold with

i) $D/G = \{(\theta, 0, p_\theta, p_\varphi) \in T^*\mathbb{S}^2/\mathbb{S}^1 : \cos(\theta) \geq 0\}$,
ii) $S/G = \{(\theta, 0, p_\theta, p_\varphi) \in T^*\mathbb{S}^2/\mathbb{S}^1 : \cos(\theta) = 0,\ p_\theta \geq 0\}$,
iii) $\hat{R}(\theta, 0, p_\theta, p_\varphi) = (\theta, -ep_\theta)$.

Note that, by using equation (7), we have that $p_\varphi = \mu = cte$, and by using this conserved quantity we can recover the same expression for the reduced equations as the ones given in the symplectic reduction of hybrid Hamiltonian systems in [1].

CONCLUSION

In this paper we have described symmetry reduction of simple hybrid Lagrangian and Hamiltonian systems with symmetries. In particular, we provided sufficient conditions for the symplectic reduction of simple hybrid Lagrangian systems and Poisson reduction of simple hybrid Hamiltonian systems.

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