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RESEARCH ARTICLE

PL-Genus of surfaces in homology balls

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Abstract
We consider manifold-knot pairs (Y, K), where Y is a homology 3-sphere that bounds a homology 4-ball. We show that the minimum genus of a PL surface Σ in a homology ball X, such that $\partial(X, \Sigma) = (Y, K)$ can be arbitrarily large. Equivalently, the minimum genus of a surface cobordism in a homology cobordism from (Y, K) to any knot in S^3 can be arbitrarily large. The proof relies on Heegaard Floer homology.

Contents

1. Introduction

Every knot K in S^3 bounds a piecewise-linear (PL) disk in the 4-ball, namely, by taking the cone on the pair (S^3, K) (this disk in not locally flat, and throughout, we will not impose any local-flatness conditions on our PL surfaces). Resolving a conjecture of Zeeman [12], Akbulut [1] gave an example of a contractible 4-manifold X and a knot $K \subset \partial X$, such that K does not bound a PL disk in X. However, Akbulut's K does bound a PL disk in a different contractible 4-manifold X' with $\partial X' = \partial X$. Levine [5] proved the stronger result that there exist manifold-knot pairs (Y, K), such that Y bounds a smooth, contractible 4-manifold X and that K does not bound a PL disk in X nor in any other integer homology ball X' with $\partial X' = Y$. In light of Levine's [5] result, a natural question to ask is: Given a knot K in an integer homology 3-sphere Y, such that Y bounds an integer homology 4-ball, what's the minimum genus of a PL surface Σ in an integer homology ball X, such that $\partial(X, \Sigma) = (Y, K)$? We observe that such a surface Σ always exists, since K is null-homologous and thus bounds a surface in Y, which may be pushed slightly into any bounding 4-manifold.

Our main result is that this notion of PL genus can be arbitrarily large. Throughout, let $(Y_n, K_n) = \left(S^3_{-1}(T_{2n,2n+1}) \# -S^3_{-1}(T_{2n,2n+1}), \mu_{2n-1,-1} \# U\right)$, where $\mu_{2n-1,-1}$ denotes the $(2n-1, -1)$-cable of the meridian in $S^3_{-1}(T_{2n,2n+1})$ and U denotes the unknot in $-S^3_{-1}(T_{2n,2n+1})$.

Theorem 1.1. *Any PL surface Σ in any integer homology ball X, such that $\partial(X, \Sigma) = (Y_n, K_n)$ must have genus at least $n-1$, for $n \in \mathbb{Z}_{>0}$.*

We prove Theorem 1.1 by reinterpreting PL surfaces in terms of cobordisms inside of homology cobordisms. Recall that a *homology cobordism* from Y_0 to Y_1 is a smooth, compact 4-manifold W, such that $\partial W = Y_0 \sqcup Y_1$ and that the map $i_*\colon H_*(Y_j;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ induced by inclusion is an isomorphism for $j = 0, 1$. Let Σ be a genus g PL surface in an integer homology ball X, such that $\partial(X, \Sigma) = (Y_n, K_n)$. Up to isotopy, we may assume that Σ is smooth, except at finitely many singular points, each of which is modeled on the cone of a smooth knot J_i in S^3 (see, for example, [4, Theorem A.1]). By deleting neighborhoods of arcs in Σ connecting the cone points, we obtain a genus g cobordism from the knot $J = J_1 \# \dots \# J_m$ to K in a homology cobordism from S^3 to Y.

Let K be a knot in a homology null-bordant homology sphere Y. We consider cobordisms of pairs

$$(W, S)\colon (S^3, J) \to (Y, K),$$

such that W is a homology cobordism from S^3 to Y. The *cobordism distance* between (Y, K) and (S^3, J) is the minimal genus of S in any such pair (W, S). By the preceding discussion, Theorem 1.1 is an immediate consequence of the following result.

Theorem 1.2. *The cobordism distance between (Y_n, K_n) and any knot in S^3 is at least $n-1$.*

We prove Theorem 1.2 using Heegaard Floer homology [8], specifically Zemke's cobordism maps [13]. Our obstruction relies on two key properties:

1. Consider a cobordism of pairs

   $$(W, S)\colon (S^3, J) \to (Y_n, K_n),$$

   where W is a homology cobordism and S has genus g. For any $(c_1, c_2) \in (2\mathbb{Z})^2$, such that $c_1 + c_2 = -2g$ and $c_1, c_2 \le 0$, there exists a local map

   $$f_{W,S}\colon \mathrm{CFK}(S^3, J) \to \mathrm{CFK}(Y_n, K_n)$$

   with bigrading (c_1, c_2). Similarly, we may consider a cobordism in the opposite direction, from (Y_n, K_n) to (S^3, J) (see [13, Theorems 1.4 and 1.7]).
2. The Heegaard Floer homology of S^3 is especially simple; namely, $\mathrm{HF}^-(S^3) = \mathbb{F}[U]$. In particular, U acts nontrivially on any nontrivial element of $\mathrm{HF}^-(S^3)$, or equivalently, $\mathrm{HF}^-(S^3)$ contains no U-torsion.

The proof of Theorem 1.2 relies on showing that $\mathrm{CFK}(Y_n, K_n)$ is sufficiently complicated so as to not admit local maps to and from $\mathrm{CFK}(S^3, J)$ of certain bigradings (see Section 2 for more details).

For constructing our examples (Y_n, K_n), we rely on recent work of the last author [14], which combines work of Hedden-Levine [3] and Truong [11] to give a description of the knot Floer complex for $(p, 1)$-cables of the meridian in the image of surgery along a knot in S^3. Preliminaries on this filtered mapping cone are given in Section 3 and the computation is carried out in Section 4.

2. Cobordism obstruction

In this section, we introduce a cobordism obstruction for manifold-knot pairs and prove Theorem 1.2 by applying the obstruction to the pairs (Y_n, K_n), calling upon the computational results in the later part of the paper. In addition, we compute the values of the concordance homomorphisms $\varphi_{i,j}$ of [2] on the family (Y_n, K_n), which may be of independent interest.

We start with some preliminaries on knot Floer homology. Knot Floer homology was defined by Ozsváth-Szabó [7] and Rasmussen [10]. We associate, following the conventions of Zemke [13], to a manifold-knot pair (Y, K) a chain complex $\mathrm{CFK}_{\mathbb{F}[\mathcal{U},\mathcal{V}]}(Y, K) = \mathrm{CFK}(Y, K)$ over the polynomial ring

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$\mathbb{F}[\mathcal{U},\mathcal{V}]$, where $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$, called the *knot Floer complex*. This chain complex is a free module generated by intersecting points of two Lagrangians in a symmetric product of a Heegaard surface, equipped with differentials by counting holomorphic disks, weighted over the intersection numbers with the two basepoints. The topological invariance of $\mathrm{CFK}(Y,K)$ up to chain homotopy equivalence over $\mathbb{F}[\mathcal{U},\mathcal{V}]$ is due to Ozsváth-Szabó and Rasmussen.

The knot Floer complex $\mathrm{CFK}(Y,K)$ comes with a bigrading, namely $(\mathrm{gr}_{\mathcal{U}},\mathrm{gr}_{\mathcal{V}})$, where $\mathcal{U},\mathcal{V}$, and ∂ each have bigrading $(-2,0)$, $(0,-2)$ and $(-1,-1)$, respectively. The Alexander grading of a homogeneous element $x\in\mathrm{CFK}(Y,K)$ is defined by $A(x)=\frac{1}{2}(\mathrm{gr}_{\mathcal{U}}(x)-\mathrm{gr}_{\mathcal{V}}(x))$.

A chain map between two complexes is called a *local map* if it induces an isomorphism on the $(\mathcal{U},\mathcal{V})$-localized homology. Following from a special case of [13, Theorem 1.4], the next theorem provides the main technical input for the obstruction.

Theorem 2.1 (Theorem 1.4 in [13]). *Suppose that* $(W,S)\colon (Y_1,K_1)\to(Y_2,K_2)$ *is a cobordism between the manifold-knot pairs* (Y_1,K_1) *and* (Y_2,K_2)*, such that* W *is a homology cobordism and* S *is of genus* g*. Then, for any given* $(c_1,c_2)\in(2\mathbb{Z})^2$*, such that* $c_1+c_2=-2g$ *and* $c_1,c_2\le 0$*, there exists a local map*

$$f_{W,S}\colon \mathrm{CFK}(Y_1,K_1)\to\mathrm{CFK}(Y_2,K_2)$$

with bigrading (c_1,c_2).

In particular, when $g(S)=0$, namely, when (Y_1,K_1) and (Y_2,K_2) are homology concordant, then the cobordism map $f_{W,S}$ is a local map that preserves the bigrading.

Definition 2.2. Two bigraded chain complexes C_1 and C_2 over $\mathbb{F}[\mathcal{U},\mathcal{V}]$ are *locally equivalent* if there exist bigrading-preserving local maps

$$f\colon C_1\to C_2 \qquad \text{and} \qquad g\colon C_2\to C_1.$$

It is straightforward to verify that local equivalence is an equivalence relation. By turning the cobordism around, we thus obtain that homology concordance induces local equivalence of the knot Floer complexes. Since the cobordism distance is invariant over the homology concordance class, we study the local equivalence class of the knot Floer complexes of the interested manifold-knot pairs.

Due to computational reasons, it is somewhat easier to first consider

$$-(Y_n,K_n)=-\big(S^3_{-1}(T_{2n,2n+1})\#-S^3_{-1}(T_{2n,2n+1}),\mu_{2n-1,-1}\#U\big),$$

that is, the orientation reversal of the manifold-knot pairs that appear in the Section 1. Observe that $-(S^3_{-1}(T_{2n,2n+1}),\mu_{2n-1,-1})$ is equivalent to $(S^3_1(-T_{2n,2n+1}),\mu_{2n-1,1})$. According to Lemma 4.3, over the ring $\mathbb{F}[U,U^{-1}]$, the complex $X^\infty_{2n-1}(-T_{2n,2n+1})\langle 2n-1\rangle$ represents the local equivalence class of $\mathrm{CFK}^\infty(S^3_1(-T_{2n,2n+1}),\mu_{2n-1,1})$ for all $n\ge 3$ (see the beginning of Section 3 for more about the knot Floer complex $\mathrm{CFK}^\infty(Y,K)$ defined over the ring $\mathbb{F}[U,U^{-1}]$).

Recall that the knot Floer complex enjoys a Künneth principle by [7, Theorem 7.1]. Since $-(Y_n,K_n)=\big(S^3_1(-T_{2n,2n+1}),\mu_{2n-1,1})\#\big(S^3_{-1}(T_{2n,2n+1}),U\big)$, the knot Floer complex of the pair $-(Y_n,K_n)$ is locally equivalent to $\mathrm{CFK}^\infty(S^3_1(-T_{2n,2n+1}),\mu_{2n-1,1})$ tensored with a trivial complex, with the Maslov grading adjusted such that the tensored complex has d–invariant equal to 0. Translate this into the ring $\mathbb{F}[\mathcal{U},\mathcal{V}]$; for $n\ge 1$, let C_n denote the complex corresponding to $X^\infty_{2n-1}(-T_{2n,2n+1})\langle 2n-1\rangle$, with a $(d(S^3_{-1}(T_{2n,2n+1})),d(S^3_{-1}(T_{2n,2n+1})))$ bigrading shift. Then C_n represents the local equivalence class of the complex $\mathrm{CFK}_{\mathbb{F}[\mathcal{U},\mathcal{V}]}(-(Y_n,K_n))$ (see Figure 3 for an example when $n=3$).

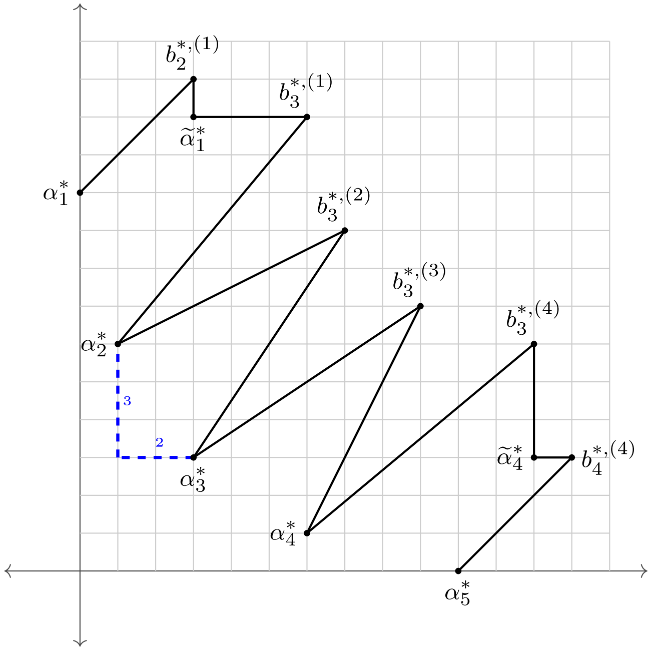

Figure 1. *The complex C_3^*, defined to be the dual complex of C_3. The axes indicate the $\mathcal{U}$ and $\mathcal{V}$ actions. The solid dots are generators, marked abstractly, missing actual $\mathcal{U}, \mathcal{V}$ decorations, and the edges represent the differentials.*

Proposition 2.3. *For $n \geq 3$, the complex C_n is characterized by*

$$\partial \alpha_s = \begin{cases} \mathcal{U}^{\frac{n(n-1)}{2}} \mathcal{V}^{\frac{n(n-1)}{2}} b_{n-1}^{(1)}, & s = 1 \\ \mathcal{U}^{\frac{n(n+1)}{2}-s+1} \mathcal{V}^{\frac{n(n+1)}{2}} b_n^{(s-1)} + \mathcal{U}^{\frac{n(n-1)}{2}} \mathcal{V}^{\frac{n(n-1)}{2}} b_{n-1}^{(s)}, & 2 \leq s \leq n-2 \\ \mathcal{U}^{\frac{n(n-1)}{2}+n-s+1} \mathcal{V}^{\frac{n(n+1)}{2}} b_n^{(s-1)} + \mathcal{U}^{\frac{n(n+1)}{2}} \mathcal{V}^{\frac{n(n-1)}{2}-n+s+1} b_n^{(s)} & n-1 \leq s \leq n+1 \\ \mathcal{U}^{\frac{n(n-1)}{2}} \mathcal{V}^{\frac{n(n-1)}{2}} b_{n+1}^{(s-1)} + \mathcal{U}^{\frac{n(n+1)}{2}} \mathcal{V}^{\frac{n(n-1)}{2}-n+s+1} b_n^{(s)}, & n+2 \leq s \leq 2n-2 \\ \mathcal{U}^{\frac{n(n-1)}{2}} \mathcal{V}^{\frac{n(n-1)}{2}} b_{n+1}^{(2n-2)}, & s = 2n-1 \end{cases} \tag{1}$$

$$\partial \widetilde{\alpha}_s = \begin{cases} \mathcal{U}^n b_n^{(s)} + \mathcal{V}^{n-s-1} b_{n-1}^{(s)}, & 1 \leq s \leq n-2 \\ \mathcal{U}^{s-n} b_{n+1}^{(s)} + \mathcal{V}^n b_n^{(s)}, & n+1 \leq s \leq 2n-2. \end{cases} \tag{2}$$

Proof. This is a direct translation from Lemma 4.4. □

This allows us to compute the values of the family of concordance homomorphisms $\varphi_{i,j}$ defined in [2, Definition 8.1], as follows.

Proposition 2.4. *For each $n \geq 3$, we have*

$$\varphi_{i,0}(C_n) = \begin{cases} -1, & 1 \leq i \leq n-2 \\ -n+2, & i = n. \end{cases} \tag{3}$$

$$\varphi_{\frac{n(n-1)}{2}, \frac{n(n-1)}{2}}(C_n) = -n+2, \tag{4}$$

$$\varphi_{\frac{n(n+1)}{2}, j}(C_n) = -1, \qquad \frac{n(n-1)}{2} \leq j \leq \frac{n(n+1)}{2} - 1, \tag{5}$$

and $\varphi_{i,j}(C_n) = 0$ for all other i and j.

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Proof. The complexes over $\mathbb{F}[\mathcal{U},\mathcal{V}]$ can be translated to complexes over the ring $\mathbb{X}$ defined in [2] using the maps

$$\mathcal{U} \longmapsto U_B + W_{T,0}$$
$$\mathcal{V} \longmapsto V_T + W_{B,0}.$$

Due to its simple form, it is not hard to formulate a change of a basis under which C_n becomes a standard complex (see [2, Section 5.1]). In particular, the invariants a_i of C_n with i odd (see [2, Definition 6.1]) are given by the sequence

$$\Bigg(\underbrace{-\Big(\frac{n(n-1)}{2},\frac{n(n-1)}{2}\Big),-(n,0),\cdots}_{\text{repeats } n-2 \text{ times}},-\Big(\frac{n(n+1)}{2},\frac{n(n-1)}{2}\Big),-\Big(\frac{n(n+1)}{2},\frac{n(n-1)}{2}+1\Big),$$
$$\underbrace{-\Big(\frac{n(n+1)}{2},\frac{n(n-1)}{2}+2\Big),-(1,0),\cdots,-\Big(\frac{n(n+1)}{2},\frac{n(n-1)}{2}+s+1\Big),-(s,0),\cdots}_{\text{for } 1\le s\le n-2}\Bigg).$$

The computations for the values of $\varphi_{i,j}(C_n)$ immediately follow. □

Similarly, for the case $n = 2$, Lemma 4.5 yields the following.

Lemma 2.5. *We have*

$$\varphi_{3,1}(C_2) = \varphi_{3,2}(C_2) = -1,$$

and $\varphi_{i,j}(C_2) = 0$ *for all other* i *and* j*.*

As a consequence, we can compute the τ invariant of the manifold-knot pair (Y_n, K_n).

Proposition 2.6. *For all* $n \geq 1$*,*

$$\tau(Y_n, K_n) = 2n^2 - 3n + 1.$$

Proof. The τ invariant can be computed from $\varphi_{i,j}$ by [2, Proposition 1.4]. For $n \geq 3$,

$$\begin{aligned}\tau(C_n) &= n(-n+2) - \sum_{i=1}^{n-2} i - \sum_{i=1}^{n} i \\ &= -2n^2 + 3n - 1.\end{aligned}$$

When $n = 2$,

$$\tau(C_2) = -1 - 2 = -3.$$

The complex C_1 is locally trivial, so $\tau(C_1) = 0$. The result now follows from the fact that τ is additive in the concordance group. □

According to [7, Proposition 3.8], the knot Floer complex of the mirror knot is the dual complex to the original knot. Therefore, the local equivalence class of $\mathrm{CFK}(Y_n, K_n)$ is given by the dual complex of C_n; denote it by C_n^*. Denote by α_s^* and $\widetilde{\alpha}_s^*$ the dual of $\alpha_s, \widetilde{\alpha}_s$, respectively, and similarly denote by $b_i^{*,(s)}$ the dual of $b_i^{(s)}$.

Proposition 2.7. *For* $n \geq 3$*, the complex* C_n^* *is characterized by the following*

$$\partial b_{n-1}^{*,(s)} = \mathcal{U}^{\frac{n(n-1)}{2}}\mathcal{V}^{\frac{n(n-1)}{2}}\alpha_s^* + \mathcal{V}^{n-s-1}\widetilde{\alpha}_s^*, \qquad 1 \leq s \leq n-2 \tag{6}$$

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$$\partial b_n^{*,(s)} = \begin{cases} \mathcal{U}^{\frac{n(n+1)}{2}-s}\mathcal{V}^{\frac{n(n+1)}{2}}\alpha_{s+1}^* + \mathcal{U}^n\widetilde{\alpha}_s^*, & 1 \le s \le n-2 \\ \mathcal{U}^{\frac{n(n+1)}{2}}\mathcal{V}^{\frac{n(n-1)}{2}-n+s+1}\alpha_s^* + \mathcal{U}^{\frac{n(n+1)}{2}-s}\mathcal{V}^{\frac{n(n+1)}{2}}\alpha_{s+1}^*, & n-1 \le s \le n \\ \mathcal{U}^{\frac{n(n+1)}{2}}\mathcal{V}^{\frac{n(n-1)}{2}-n+s+1}\alpha_s^* + \mathcal{V}^n\widetilde{\alpha}_s^*, & n+1 \le s \le 2n-2 \end{cases} \tag{7}$$

$$\partial b_{n+1}^{*,(s)} = \mathcal{U}^{\frac{n(n-1)}{2}}\mathcal{V}^{\frac{n(n-1)}{2}}\alpha_{s+1}^* + \mathcal{U}^{s-n}\widetilde{\alpha}_s^*, \qquad n+1 \le s \le 2n-2. \tag{8}$$

Proof. This follows from Proposition 2.3 and the fact that C_n^* is the dual complex of C_n. □

We record a few salient features of the complex C_n^* for $n \ge 3$ from Proposition 2.7:

Lemma 2.8. *We have the inequalities*

$$\operatorname{gr}_{\mathcal{V}} \alpha_s^*, \operatorname{gr}_{\mathcal{V}} \widetilde{\alpha}_s^* \le \operatorname{gr}_{\mathcal{V}} \alpha_n^* - 2n, \quad \text{for } s \le n-1.$$

Similarly,

$$\operatorname{gr}_{\mathcal{U}} \alpha_s^*, \operatorname{gr}_{\mathcal{U}} \widetilde{\alpha}_s^* \le \operatorname{gr}_{\mathcal{U}} \alpha_n^* - 2n, \quad \text{for } s \ge n+1.$$

Proof. We have

$$\operatorname{gr}_{\mathcal{V}} \alpha_{n-1}^* = \operatorname{gr}_{\mathcal{V}} \alpha_n^* - 2n.$$

Note also the equalities:

$$\operatorname{gr}_{\mathcal{V}} \widetilde{\alpha}_s^* = \operatorname{gr}_{\mathcal{V}} \alpha_{s+1}^* - n(n+1) \qquad \text{for } 1 \le s \le n-2 \tag{9}$$

$$\operatorname{gr}_{\mathcal{V}} \widetilde{\alpha}_s^* = \operatorname{gr}_{\mathcal{V}} \alpha_s^* - n(n-1) + 2(n-s-1) \qquad \text{for } 1 \le s \le n-2. \tag{10}$$

In particular,

$$\operatorname{gr}_{\mathcal{V}} \alpha_s^* \le \operatorname{gr}_{\mathcal{V}} \alpha_{s+1}^* - 2n - 2$$

for $1 \le s \le n-2$.

From here, the claim of the lemma follows for α_s^* for all $1 \le s \le n-1$. The statement for $\widetilde{\alpha}_s^*$ follows from (9).

The case of $\operatorname{gr}_{\mathcal{U}}$ follows similarly, where we use

$$\operatorname{gr}_{\mathcal{U}} \alpha_{n+1}^* = \operatorname{gr}_{\mathcal{U}} \alpha_n^* - 2n,$$

and also calculate:

$$\operatorname{gr}_{\mathcal{U}} \widetilde{\alpha}_s^* = \operatorname{gr}_{\mathcal{U}} \alpha_s^* - n(n+1) \qquad \text{for } n+1 \le s \le 2n-2 \tag{11}$$

$$\operatorname{gr}_{\mathcal{U}} \widetilde{\alpha}_s^* = \operatorname{gr}_{\mathcal{U}} \alpha_{s+1}^* - n(n-1) + 2(s-n) \qquad \text{for } n+1 \le s \le 2n-2. \tag{12}$$

In particular,

$$\operatorname{gr}_{\mathcal{U}} \alpha_{s+1}^* \le \operatorname{gr}_{\mathcal{U}} \alpha_s^* - 2n - 2$$

for $n+1 \le s \le 2n-2$. From here, the claim of the lemma follows for α_s^* for all $n+1 \le s \le 2n-1$. The statement for $\widetilde{\alpha}_s^*$ follows from (11). □

Lemma 2.9. *For $n \ge 3$, let $\phi\colon C_n^* \to C_n^*$ be a chain map of bigrading (c_1, c_2), where $c_1 > -2n$ and $c_2 > -2n$. Then, $\phi(\alpha_n^*)$ is either an $\mathbb{F}[\mathcal{U}, \mathcal{V}]$-multiple of α_n^* or 0.*

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Proof. By Lemma 2.8, all of the other generators of C_n^*, which are cycles, have either $\mathcal{U}$-grading or $\mathcal{V}$-grading less than that of $\phi(\alpha_n^*)$. No linear combination of the b^*-type terms is a cycle, and so $\phi(\alpha_n^*)$ is supported only by $\langle\alpha_n^*\rangle$. □

Lemma 2.10. *For $n \geq 3$, let $\phi\colon C_n^* \to C_n^*$ be a homogeneous chain map with degree as in Lemma 2.9 and so that $\phi(\alpha_n^*)$ is a boundary in $C_n^* \otimes \mathbb{F}[\mathcal{U},\mathcal{V}=1]/(\mathcal{U}^{n-1})$. Then, $\phi(\alpha_n^*)$ must be divisible by $\mathcal{U}^{n-1}$.*

Proof. From Lemma 2.9, $\phi(\alpha_n^*) = c\alpha_n^*$ for some $c \in \mathbb{F}[\mathcal{U},\mathcal{V}]$. Considering the differential of C_n^* mod $\mathcal{V}=1, \mathcal{U}^{n-1}=0$, we obtain that $c\alpha_n^*$ is a boundary over this quotient ring if and only if $\mathcal{U}^{n-1} \mid c$. □

We say that a chain complex D over $\mathbb{F}[\mathcal{U},\mathcal{V}]$ is S^3-*knotlike* if $H_*(D \otimes \mathbb{F}[\mathcal{U},\mathcal{V}=1]) = \mathbb{F}[\mathcal{U}]$. Recall that for a knot $K \subset Y$, by setting $\mathcal{V}=1$ in $\mathrm{CFK}(Y,K)$ and taking the homology, one recovers the Heegaard Floer homology $\mathrm{HF}^-(Y)$. In particular, if D is the knot Floer complex of a knot in S^3, then D is S^3-knotlike.

Lemma 2.11. *For $n \geq 3$, let f be a map from C_n^* to an S^3-knotlike complex D, and let g be a map from D to C_n^*. Then, $gf(\alpha_n^*)$ is a boundary in $C_n^* \otimes \mathbb{F}[\mathcal{U},\mathcal{V}=1]/(\mathcal{U}^{n-1})$.*

Proof. We have, by considering $b_n^{*,(n-1)}$, that

$$\mathcal{U}^{n(n-1)/2+1}\mathcal{V}^{n(n+1)/2}\alpha_n^* + \mathcal{U}^{n(n+1)/2}\mathcal{V}^{n(n-1)/2}\alpha_{n-1}^*$$

is a boundary. Setting $\mathcal{V}=1$,

$$\mathcal{U}^{n(n-1)/2+1}(f(\alpha_n^*) + \mathcal{U}^{n(n+1)/2-n(n-1)/2-1}f(\alpha_{n-1}^*)) \text{ is a boundary in } C_n/(\mathcal{V}=1).$$

Since any cycle in an S^3-knotlike complex that is $\mathcal{U}$-torsion in $(\mathcal{V}=1)$ homology is actually zero in homology, we have that

$$f(\alpha_n^*) + \mathcal{U}^{n-1}f(\alpha_{n-1}^*)$$

is a boundary in $D/(\mathcal{V}=1)$. So $f(\alpha_n^*)$ is a boundary in $D/(\mathcal{V}=1,\mathcal{U}^{n-1}=0)$. Since g is a chain map, the same holds for $gf(\alpha_n^*)$. □

Lemma 2.12. *For $n \geq 3$, let f be a local map from C_n^* to a knotlike complex D. There does not exist a local map $g\colon D \to C_n^*$, so that $g \circ f$ is of bigrading (c_1,c_2) with $c_1 > -2n+2$ and $c_2 > -2n$.*

Proof. Suppose such a g exists. Since f and g are local and α_n^* generates the $(\mathcal{U},\mathcal{V})$-localized homology, it follows that $gf(\alpha_n^*) \neq 0$. Hence, by Lemma 2.9, we obtain that

$$gf(\alpha_n^*) = \mathcal{U}^{-c_1/2}\mathcal{V}^{-c_2/2}\alpha_n^*.$$

By Lemma 2.11, $gf(\alpha_n^*)$ is a boundary mod $\mathcal{U}^{n-1}=0, \mathcal{V}=1$, and so by Lemma 2.10, we have $n-1 \leq -c_1/2$. That is, $-2n+2 \geq c_1 > -2n+2$, a contradiction. □

Proof of Theorem 1.2. When $n=2$, for any knot $J \subset S^3$, by [2, Theorem 10.1], we have $\varphi_{i,j}(S^3,J)=0$ for any $j \neq 0$, so Lemma 2.5 obstructs the existence of a homology concordance between (Y_2,K_2) and (S^3,J).

Now suppose $n \geq 3$. Say that there is a pair (W,S) as in the discussion preceding Theorem 1.2, with S of genus $g \leq n-2$. Then, for any choice of $(c_1,c_2),(d_1,d_2) \in (2\mathbb{Z})^2$ so that $c_i,d_i \leq 0$ and $c_1+c_2=-2g=d_1+d_2$, there exist local maps $f\colon C_n \to \mathrm{CFK}(J)$ and $g\colon \mathrm{CFK}(J) \to C_n$ of bigrading $(c_1,c_2),(d_1,d_2)$, respectively. Let f be of bigrading $(0,-2g)$ and g be of bigrading $(-2g,0)$. By hypothesis, $-2g \geq -2n+4$, and so Lemma 2.12 applies to show that such f,g do not exist, a contradiction. □

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3. Preliminaries on the filtered mapping cone formula

We start by reviewing the original definition of the knot Floer complex over the ring $\mathbb{F}[U, U^{-1}]$ by Ozsváth and Szabó, as this is the setting where the filtered mapping cone formula can be most conveniently defined.

In the original definition, the knot Floer complex is freely generated by the intersecting points of the two Lagrangians over the ring $\mathbb{F}[U, U^{-1}]$, where the differentials similarly count the holomorphic disks but are weighted over the intersection number with only one of the basepoints. The datum of the other basepoint is encoded in the Alexander grading. This version of the knot Floer complex is denoted by $\mathrm{CFK}^\infty(Y, K)$, and commonly depicted in an (i, j)–plane, where the j–coordinate is given by the Alexander grading, and the i–coordinate is the normalized filtration level naturally induced by the U–action. We will often think of $\mathrm{CFK}^\infty(Y, K)$ as a chain complex with an extra filtration given by the Alexander grading. By collapsing the Alexander filtration, one recovers a chain complex associated to the underlying three-manifold, $\mathrm{CF}^\infty(Y)$.

There is a Maslov grading on $\mathrm{CFK}^\infty(Y, K)$, corresponding to $\mathrm{gr}_{\mathcal{U}}$; multiplication by U on $\mathrm{CFK}^\infty(Y, K)$ is equivalent to multiplication by $\mathcal{U}\mathcal{V}$ on $\mathrm{CFK}_{\mathbb{F}[\mathcal{U},\mathcal{V}]}(Y, K)$. Although the setting is slightly different, $\mathrm{CFK}^\infty(Y, K)$ contains the same information as $\mathrm{CFK}_{\mathbb{F}[\mathcal{U},\mathcal{V}]}(Y, K)$ does. In the setting of $\mathrm{CFK}^\infty(Y, K)$, the local equivalence reads as follows.

Definition 3.1. Two filtered chain complex C_1 and C_2 over $\mathbb{F}[U, U^{-1}]$ are *locally equivalent* if there exist Maslov grading-preserving filtered local maps

$$f : C_1 \to C_2 \qquad \text{and} \qquad g : C_2 \to C_1.$$

For the rest of the paper, we will always use the knot Floer complex $\mathrm{CFK}^\infty(Y, K)$. Next, we recall the filtered mapping cone formula from [14] for the reader; this is our main computational tool.

Let $K \subset S^3$ be a knot with genus equal to g. For a given positive integer p, let $\mu_{p,1}$ denote the $(p, 1)$-cable of the meridian of K in the $+1$-surgery on K. According to [14, Theorem 1.9], the knot Floer complex $\mathrm{CFK}^\infty(S^3_1(K), \mu_{p,1})$ is a filtered chain homotopy equivalent to the doubly filtered chain complex $X^\infty_p(K)$, defined to be the mapping cone of

$$\bigoplus_{s=-g+1}^{g+p-1} A_s \xrightarrow{v_s + h_s} \bigoplus_{s=-g+2}^{g+p-1} B_s, \tag{13}$$

where each A_s and B_s are isomorphic to $\mathrm{CFK}^\infty(S^3, K)$, coming with the (i, j) coordinate. The map $v_s : A_s \to B_s$ is the identity, and the map $h_s : A_s \to B_{s+1}$ is the reflection along $i = j$ precomposed with U^s. Note that there are corresponding versions of the filtered mapping cone formula for the hat, minus, and infinity flavors of knot Floer homology. In the following computation, we will consistently use the infinity version of the A_s and B_s complexes and v_s and h_s maps, so we repress the superindices.

Let $\mathcal{I}$ and $\mathcal{J}$ be the double filtrations, and let gr_M be the absolute Maslov grading on the filtered mapping cone complex $X^\infty_p(K)$. We will reserve letters $\mathcal{I}$ and $\mathcal{J}$ solely for this purpose throughout the paper. We have

for $[\mathbf{x}, i, j] \in A_s$,

$$\mathcal{I}([\mathbf{x}, i, j]) = \max\{i, j - s\} \tag{14}$$

$$\mathcal{J}([\mathbf{x}, i, j]) = \max\{i - p, j - s\} + ps - \frac{p(p-1)}{2} \tag{15}$$

$$\mathrm{gr}_M([\mathbf{x}, i, j]) = \widetilde{\mathrm{gr}}([\mathbf{x}, i, j]) + s(s-1) \tag{16}$$

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and for $[\mathbf{x}, i, j] \in B_s$,

$$\mathcal{I}([\mathbf{x}, i, j]) = i \tag{17}$$

$$\mathcal{J}([\mathbf{x}, i, j]) = i - p + ps - \frac{p(p-1)}{2} \tag{18}$$

$$\mathrm{gr}_M([\mathbf{x}, i, j]) = \widetilde{\mathrm{gr}}([\mathbf{x}, i, j]) + s(s-1) - 1. \tag{19}$$

Here, $\widetilde{\mathrm{gr}}$ denotes the absolute Maslov grading on the original chain complex $\mathrm{CFK}^\infty(S^3, K)$. It is straightforward to check that for $s < -g + 1$, the map h_s induces an isomorphism on the homology; for $s > g + p - 1$, the map $v_s(K)$ induces an isomorphism on the homology, which justifies the truncation of the mapping cone.

The general strategy for computation involves finding a *reduced* basis for $X_p^\infty(K)$, where every term in the differential strictly lowers at least one of the filtrations. This can be achieved through a cancellation process (see, for example [6, Proposition 11.57]) as follows: suppose $\partial x_i = y_i$ + lower filtration terms, where the double filtration of y_i is the same as x_i, then the subcomplex of $X_p^\infty(K)$ generated by all such $\{x_i, \partial x_i\}$ is acyclic, and the $X_p^\infty(K)$ quotient by this complex is reduced. Alternatively, one can view the above process as a change of basis, that splits off acyclic summands which individually lie entirely in one double-filtration level.

There is an apparent symmetry on the mapping cone as follows. Let $[\mathbf{x}, i, j] \mapsto [\psi(\mathbf{x}), j, i]$ be a homotopy equivalence that realizes the symmetry on the original chain complex $\mathrm{CFK}^\infty(S^3, K)$. In the following lemma, we use a subindex to mark elements from A_s or B_s.

Proposition 3.2. *Let* $\Psi\colon\ X_p^\infty(K) \longrightarrow X_p^\infty(K)$ *be the map, defined as for* $[\mathbf{x}, i, j]_s \in A_s$,

$$\Psi([\mathbf{x}, i, j]_s) = U^{\frac{(p-1)(p-2s)}{2}} [\psi(\mathbf{x}), j, i]_{p-s} \in A_{p-s} \tag{20}$$

for $[\mathbf{x}, i, j]_s \in B_s$,

$$\Psi([\mathbf{x}, i, j]_s) = U^{\frac{p(p-2s+1)}{2}} [\mathbf{x}, j, i]_{p-s+1} \in B_{p-s+1}. \tag{21}$$

Then, Ψ *is a chain map that realizes a homotopy equivalence on the doubly filtered chain complex* $\mathrm{CFK}^\infty(S^3_1(K), \mu_{p,1})$ *that switches the* $\mathcal{I}$ *and* $\mathcal{J}$ *filtrations.*

Proof. By definition, Ψ is U–equivariant, so it suffices to show Ψ realizes the symmetry for any one $\mathcal{I}$ and $\mathcal{J}$ value.

Over each chain complex A_s, by (14) and (15), we have $\{\mathcal{I} = 0\} = \max\{i, j - s\}$ and $\{\mathcal{J} = ps - \frac{p(p-1)}{2}\} = \max\{i - p, j - s\}$. Compute

$$\begin{aligned}
\Psi(\{\mathcal{I} = 0\}_s) &= U^{\frac{(p-1)(p-2s)}{2}} \max\{i - s, j\}_{p-s} \\
&= U^{\frac{(p-1)(p-2s)}{2} + (p-s)} \max\{i - p, j - (p-s)\}_{p-s} \\
&= U^{\frac{p^2+p-2ps}{2}} \{\mathcal{J} = p(p-s) - \frac{p(p-1)}{2}\}_{p-s} \\
&= \{\mathcal{J} = 0\}_{p-s} \\
\Psi(\{\mathcal{J} = ps - \frac{p(p-1)}{2}\}_s) &= U^{\frac{(p-1)(p-2s)}{2}} \max\{i - s, j - p\}_{p-s} \\
&= U^{\frac{(p-1)(p-2s)}{2} - s} \max\{i, j - (p-s)\}_{p-s} \\
&= \{\mathcal{I} = ps - \frac{p(p-1)}{2}\}_{p-s},
\end{aligned}$$

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while the computations for B_s are similar and left for the reader. Moreover, by definition, we have $\Psi \circ v_s = h_s \circ \Psi$, and $\Psi \circ h_s = v_s \circ \Psi$, therefore, Ψ is a chain map. □

4. Cables of the knot meridian of $-T_{2n,2n+1}$

In this section, we perform the filtered mapping cone computation, which determines the knot Floer complex in Proposition 2.3 and Lemma 2.5.

Given $n \geq 1$, let $T_{2n,2n+1}$ be the $(2n, 2n+1)$-torus knot, with genus equal to $n(2n-1)$. It is a fun exercise to compute its Alexander polynomial as follows

$$\frac{(t^{2n(2n+1)}-1)(t-1)}{(t^{2n}-1)(t^{2n+1}-1)} = \frac{t^{(2n-1)(2n+1)}+t^{(2n-2)(2n+1)}+\cdots+1}{t^{2n-1}+t^{2n-2}+\cdots+1} = 1+\sum_{i=0}^{2n-2}\left(t^{(2n-i)(2n-1)-i}-t^{(2n-i)(2n-1)-2i-1}\right).$$

For example, if we let $S_{2n}(i) = \left(t^{(2n-i)(2n-1)-i} - t^{(2n-i)(2n-1)-2i-1}\right)\left(t^{2n-1}+t^{2n-2}+\cdots+1\right)$ for $i = 0, 1, \cdots, 2n-2$, by induction, we obtain that for $0 \leq \ell \leq 2n-2$

$$\sum_{i=0}^{\ell} S_{2n}(i) = t^{(2n-1)(2n+1)}+\cdots+t^{(2n-\ell-1)(2n+1)}-t^{(2n-\ell)(2n-1)-\ell-1}-\cdots-t^{(2n-\ell)(2n-1)-2\ell-1}.$$

Taking ℓ to be $2n-2$ leads to the answer.

Torus knots are L–space knots. Therefore, according to [9, Theorem 1.2], the knot Floer complex $\mathrm{CFK}^\infty(S^3, T_{2n,2n+1})$ is generated by a_i^* with coordinate $(0, \frac{(2n-i)(2n-i+1)}{2} - \frac{i(i-1)}{2})$ for $i \in \{1, \cdots, 2n\}$ and b_i^* with coordinate $(0, \frac{(2n-i)(2n-i+1)}{2} - \frac{i(i+1)}{2})$ for $i \in \{1, \cdots, 2n-1\}$ (this is, in fact, a set of generators coming from a $\widehat{HFK}$ model), where the differentials are given by

$$\partial b_i^* = U^i a_i^* + a_{i+1}^*.$$

It follows from [7, Proposition 3.8], that the knot Floer complex of the mirror knot is the dual complex to the original knot. Therefore, $\mathrm{CFK}^\infty(S^3, -T_{2n,2n+1})$ is generated by a_i with coordinate $(0, -\frac{(2n-i)(2n-i+1)}{2} + \frac{i(i-1)}{2})$ for $i \in \{1, \cdots, 2n\}$ and b_i with coordinate $(0, -\frac{(2n-i)(2n-i+1)}{2} + \frac{i(i+1)}{2})$ for $i \in \{1, \cdots, 2n-1\}$ (simply by taking a_i to be the dual of a_i^* and b_i to be the dual of b_i^*). As a notational shorthand, we will let $g_a(n,i) := -\frac{(2n-i)(2n-i+1)}{2} + \frac{i(i-1)}{2}$ and $g_b(n,i) := -\frac{(2n-i)(2n-i+1)}{2} + \frac{i(i+1)}{2}$. Note that $g_a(n,i) + i = g_b(n,i)$. The differentials are given by

$$\partial a_i = \begin{cases} Ub_i, & i = 1 \\ b_{i-1}, & i = 2n \\ U^i b_i + b_{i-1}, & \text{otherwise.} \end{cases}$$

Note that the (horizontal) arrow from a_i to b_i is of length i, while the (vertical) arrow from a_i to b_{i-1} is of length $2n-i+1$ (see Figure 2 for an example of $\mathrm{CFK}^\infty(S^3, -T_{2n,2n+1})$ when $n = 3$).

The interesting examples are given by the pair $(S^3_1(-T_{2n,2n+1}), \mu_{2n-1,1})$, where $\mu_{2n-1,1}$ is the $(2n-1, 1)$-cable of the dual knot. To compute the knot Floer complex of said examples, we apply the filtered mapping cone formula for the cables of the dual knot on $-T_{2n,2n+1}$, with the surgery coefficient equal to $+1$. Following the recipe described in Section 3, the filtered chain complex $\mathrm{CFK}^\infty(S^3_1(-T_{2n,2n+1}), \mu_{2n-1,1})$ is filtered homotopy equivalent to the filtered complex $X^\infty_{2n-1}(-T_{2n,2n+1})$

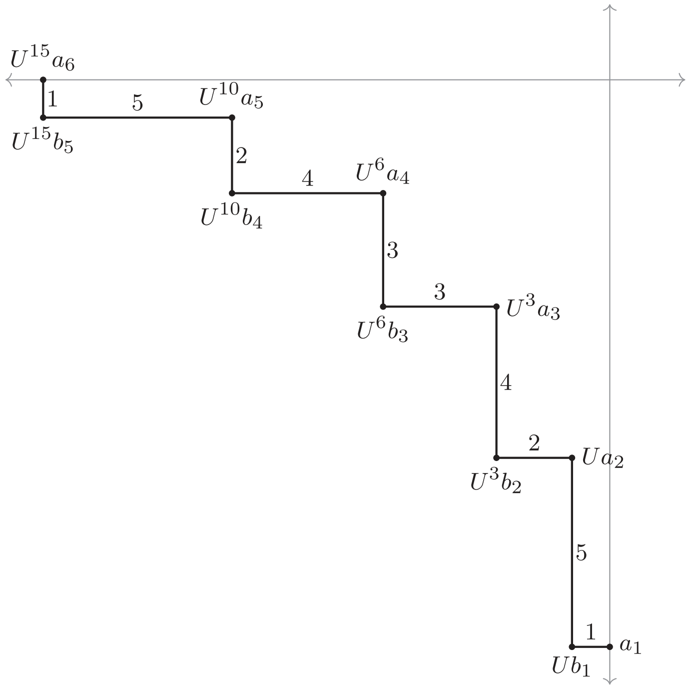

Figure 2. *The knot Floer complex* $\mathrm{CFK}^\infty(S^3, -T_{6,7})$. *The solid dots are generators. The differentials point to lower filtration levels, and the numbers indicate their lengths.*

defined by the mapping cone of

$$\bigoplus_{s=-n(2n-1)+1}^{(n+1)(2n-1)-1} A_s \xrightarrow{v_s+h_s} \bigoplus_{s=-n(2n-1)+2}^{(n+1)(2n-1)-1} B_s.$$

Through the isomorphism with $\mathrm{CFK}^\infty(S^3, -T_{2n,2n+1})$, denote the corresponding generators in A_s by $a_i^{(s)}$ and $b_i^{(s)}$, and the generators in B_s by $a_i'^{(s)}$ and $b_i'^{(s)}$, for suitable i and s. Recall that we use $\mathcal{I}$ and $\mathcal{J}$ specifically for the double filtrations on the entire mapping cone complex. Using the formulas given by (14), (15), (17), and (18), the computations for the $\mathcal{I}$ and $\mathcal{J}$ filtrations of the generators described above are quite straightforward. We collect the result in a following lemma, with $g_a(n,i)$ and $g_b(n,i)$ the quantities defined in the previous paragraph. Also define a notational shorthand

$$f(n,s) := -\frac{(n-1)n}{2} + ns. \tag{22}$$

Note that $f(n, s-1) + n = f(n,s)$.

Lemma 4.1. *In the complex* $X^\infty_{2n-1}(-T_{2n,2n+1})$, *we have*

$$\mathcal{J}(a_i^{(s)}) = \begin{cases} f(n,s) + g_a(n,i) - s, & s \le g_a(n,i) + 2n - 1 \\ f(n,s-1), & s > g_a(n,i) + 2n - 1 \end{cases} \tag{23}$$

$$\mathcal{J}(b_i^{(s)}) = \begin{cases} f(n,s) + g_b(n,i) - s, & s \le g_b(n,i) + 2n - 1 \\ f(n,s-1), & s > g_b(n,i) + 2n - 1; \end{cases} \tag{24}$$

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$$\mathcal{I}(a_i^{(s)}) = \mathcal{I}(b_i^{(s)}) = \mathcal{I}(a_i'^{(s)}) = \mathcal{I}(b_i'^{(s)}) = 0; \tag{25}$$

$$\mathcal{J}(a_i'^{(s)}) = \mathcal{J}(b_i'^{(s)}) = f(n, s-1). \tag{26}$$

For the rest of the computation, we assume that $n \geq 3$ (the case when $n = 1, 2$ does not fit into the following model. Instead, the results of those two cases are recorded in Lemma 4.5.).

We first aim to obtain a reduced model of the generators of $X^{\infty}_{2n-1}(-T_{2n,2n+1})$. Up to filtered homotopy equivalence (as a subcomplex of $X^{\infty}_{2n-1}(-T_{2n,2n+1})$), each B_s is one-dimensional. Indeed, quotienting out $\{a_i'^{(s)}, \partial a_i'^{(s)}\}_{2\leq i\leq 2n}$ leaves us with a sole generator $a_1'^{(s)}$ in each B_s.

Each A_s is a subcomplex of the quotient complex $\bigoplus_s A_s$, which inherits the $(\mathcal{I}, \mathcal{J})$ filtration naturally. We would like to obtain a reduced model for each A_s. For the next part, let ∂ temporarily denote the differential restricted to each subquotient-complex A_s, as opposed to the differential on the entire chain complex $X^{\infty}_{2n-1}(-T_{2n,2n+1})$. There are two types of complex A_s, depending on the dimension of the reduced model.

When $s \in [-n(2n-1), -n(2n-1)+2n] \cup \{-n(2n-1)+2jn-1, -n(2n-1)+2jn\}_{1\leq j\leq 2n-2} \cup [n(2n-1)-1, (n+1)(2n-1)-1]$, after quotienting out $\{a_i^{(s)}, \partial a_i^{(s)}\}_{2\leq i\leq 2n-1}$, the reduced model of A_s is three-dimensional.

- When $s \in [-n(2n-1), -n(2n-1)+2n]$, the reduced model is generated by $\{a_{2n}^{(s)}, b_1^{(s)}, a_1^{(s)}\}$ with modified differentials:

$$\partial a_1^{(s)} = U b_1^{(s)}, \qquad \partial a_{2n}^{(s)} = U^{-n(2n-1)+1} b_1^{(s)}.$$

- When $s \in \{-n(2n-1)+2jn-1, -n(2n-1)+2jn\}_{1\leq j\leq 2n-2}$,the reduced model is generated by $\{a_{2n}^{(s)}, b_j^{(s)}, a_1^{(s)}\}$ and modified differentials are

$$\partial a_1^{(s)} = U^{\frac{j(j+1)}{2}} b_j^{(s)}, \qquad \partial a_{2n}^{(s)} = U^{-n(2n-1)+\frac{j(j+1)}{2}} b_j^{(s)}.$$

- When $s \in [n(2n-1)-1, (n+1)(2n-1)-1]$, the reduced model is generated by $\{a_{2n}^{(s)}, b_{2n-1}^{(s)}, a_1^{(s)}\}$ with modified differentials

$$\partial a_1^{(s)} = U^{n(2n-1)} b_{2n-1}^{(s)}, \qquad \partial a_{2n}^{(s)} = b_{2n-1}^{(s)}.$$

When $s \in \bigcup_{1\leq j\leq 2n-2}[-n(2n-1)+2jn+1, -n(2n-1)+2(j+1)n-2]$, the reduced model of A_s is five-dimensional. Indeed, quotienting out $\{a_i^{(s)}, \partial a_i^{(s)}\}_{j\in[2,j]\cup[j+2,2n-1]}$ leaves us with generators $\{a_{2n}^{(s)}, b_{j+1}^{(s)}, a_{j+1}^{(s)}, b_j^{(s)}, a_1^{(s)}\}$. The difference here from the previous case is that both terms in $\partial a_{j+1}^{(s)}$ strictly decrease $\mathcal{I}$ or $\mathcal{J}$ grading, and therefore survive into the reduced complex. The modified differentials are given by

$$\partial a_1^{(s)} = U^{\frac{j(j+1)}{2}} b_j^{(s)}, \qquad \partial a_{j+1}^{(s)} = b_j^{(s)} + U^{j+1} b_{j+1}^{(s)},$$
$$\partial a_{2n}^{(s)} = U^{-n(2n-1)+\frac{(j+1)(j+2)}{2}} b_{j+1}^{(s)}.$$

Finally, consider the entire chain complex $X^{\infty}_{2n-1}(-T_{2n,2n+1})$, using the reduced models for both A_s and B_s. Let ∂ denote the differential on the entire mapping cone complex (including v_s and h_s maps). Observe that $h_s(U^{-s}a_{2n}^{(s)}) = a_1'^{(s+1)} = v_{s+1}(a_1^{(s+1)})$ for $-n(2n-1)+1 \leq s \leq n(2n-1)$, while $\mathcal{I}(U^{-s}a_{2n}^{(s)}) = \mathcal{I}(a_1'^{(s+1)}) = \mathcal{I}(a_1^{(s+1)})$ and $\mathcal{J}(U^{-s}a_{2n}^{(s)}) = \mathcal{J}(a_1'^{(s+1)}) \leq \mathcal{J}(a_1^{(s+1)})$, where the last equality is reached when $s \geq -(n-1)(2n-1)$. Thus, we may quotient out $\{a_{2n}^{(s)}, \partial a_{2n}^{(s)}\}$ for $-n(2n-1)+1 \leq s \leq n(2n-1)$.

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If we let α_s denote the image of $a_1^{(s)}$ in the quotient for $-n(2n-1)+1 \le s \le n(2n-1)+1$, notice that for $-n(2n-1)+2 \le s \le n(2n-1)+1$, this amounts to a change of basis $a_1^{(s)} \mapsto U^{-s+1}a_{2n}^{(s-1)} + a_1^{(s)}$ followed by a homotopy equivalence. Similarly, we may quotient out $\{a_1^{(s)}, \partial a_1^{(s)}\}$ for $n(2n-1)+2 \le s \le (n+1)(2n-1)-1$.

We have obtained a reduced model for $X^{\infty}_{2n-1}(-T_{2n,2n+1})$. Observe that no generator in B_s survives into the reduced basis. Moreover, from the viewpoint of the quotient complex, the induced differential ∂ restricted to A_s is a map $\partial\colon\ A_s \to A_s \oplus A_{s-1}$ for $-n(2n-1)+2 \le s \le n$, viewing α_s as an element of A_s. However, we will generally adopt the viewpoint of a change of basis, and view α_s as an element of $A_s \oplus A_{s-1}$, mainly because this plays well with the symmetry on the mapping cone complex.

Considering the symmetry on $X^{\infty}_{2n-1}(-T_{2n,2n+1})$ (see Proposition 3.2), our strategy would be to focus on the "first half" of the complex, namely, the mapping cone of

$$\bigoplus_{s=-n(2n-1)+1}^{n-1} A_s \xrightarrow{v_s+h_s} \bigoplus_{s=-n(2n-1)+2}^{n} B_s,$$

which under the current basis is simply the chain complex

$$\bigoplus_{s=-n(2n-1)+1}^{n-1} A_s.$$

So, let us summarize the generators and relations of this first half complex in the following lemma (we also include those A_s where s is in the interval $[n, 2n-1]$ for the continuity). Let $\widetilde{\alpha}_s$ denote $a_{j+1}^{(s)}$ when $s \in [-n(2n-1)+2jn+1, -n(2n-1)+2(j+1)n-2]$ for each $1 \le j \le n$.

Lemma 4.2. *Under the reduced basis chosen above, we have*

- *For $s \in [-n(2n-1)+1, -n(2n-1)+2n]$, the complex A_s is generated by α_s and $b_1^{(s)}$, where the differentials are given by*

$$\partial\alpha_s = \begin{cases} Ub_1^{(s)}, & s = -n(2n-1)+1 \\ U^{-s+2}b_1^{(s-1)} + Ub_1^{(s)}, & s > -n(2n-1)+1. \end{cases} \tag{27}$$

- *For $s \in [-n(2n-1)+2jn+1, -n(2n-1)+2(j+1)n-2]$ with some $1 \le j \le n$, the complex A_s is generated by $\alpha_s, \widetilde{\alpha}_s, b_j^{(s)}$ and $b_{j+1}^{(s)}$, where the differentials are given by*

$$\partial\alpha_s = \begin{cases} U^{-s+1+\frac{j(j+1)}{2}}b_j^{(s-1)} + U^{\frac{j(j+1)}{2}}b_j^{(s)}, & s = -n(2n-1)+2jn+1, \\ U^{-s+1+\frac{(j+1)(j+2)}{2}}b_{j+1}^{(s-1)} + U^{\frac{j(j+1)}{2}}b_j^{(s)}, & s > -n(2n-1)+2jn+1, \end{cases} \tag{28}$$

$$\partial\widetilde{\alpha}_s = b_j^{(s)} + U^{j+1}b_{j+1}^{(s)}. \tag{29}$$

- *For $s \in \{-n(2n-1)+2jn-1, -n(2n-1)+2jn\}$ with some $2 \le j \le n$, the complex A_s is generated by α_s and $b_j^{(s)}$, where the differentials are given by*

$$\partial\alpha_s = U^{-s+1+\frac{j(j+1)}{2}}b_j^{(s-1)} + U^{\frac{j(j+1)}{2}}b_j^{(s)}. \tag{30}$$

Proof. This follows from the earlier discussion. □

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We prove in the next lemma that, up to local equivalence, we can further truncate the mapping cone. Define $X^\infty_{2n-1}(-T_{2n,2n+1})\langle\ell\rangle$ for $\ell \in \mathbb{Z}$ to be the filtered mapping cone

$$\bigoplus_{s=-\ell+2n-1}^{\ell} A_s \xrightarrow{v_s+h_s} \bigoplus_{s=-\ell+2n}^{\ell} B_s,$$

which under the reduced basis simplifies to the filtered chain complex

$$\bigoplus_{s=-\ell+2n-1}^{\ell} A_s.$$

Note that under this notation $X^\infty_{2n-1}(-T_{2n,2n+1}) = X^\infty_{2n-1}(-T_{2n,2n+1})\langle(n+1)(2n-1)-1\rangle$.

Lemma 4.3. *Up to a change of basis, the filtered complex $X^\infty_{2n-1}(-T_{2n,2n+1})$ is isomorphic to $X^\infty_{2n-1}(-T_{2n,2n+1})\langle 2n-1\rangle \oplus D$, where $H_*(D) = 0$.*

Proof. It suffices to show for any $2n \le \ell \le (n+1)(2n-1)-1$, the complex $X^\infty_{2n-1}(-T_{2n,2n+1})\langle\ell\rangle$ is isomorphic to $X^\infty_{2n-1}(-T_{2n,2n+1})\langle\ell-1\rangle \oplus D'$ up to a change of basis, where $H_*(D') = 0$. For every such ℓ, we will demonstrate a filtered change of basis, such that the complex $A_{-\ell+2n-1}$ becomes a summand. Following from the symmetry given by Proposition 3.2, there is also a filtered change of basis, such that A_ℓ becomes a summand under the new basis as required. Let $s = -\ell+2n-1$. Recall that we view α_s as an element in $A_s \oplus A_{s-1}$.

- For $s \in [-n(2n-1)+2, -n(2n-1)+2n+1]$, perform the change of basis

$$\alpha_s \longmapsto \alpha_s + U^{-s+1}\alpha_{s-1}.$$

  According to (27) and (28), this splits off an acyclic summand as required. Since $\mathcal{J}(\alpha_s) = \mathcal{J}(a_1^s) > \mathcal{J}(a_1^{s-1}) = \mathcal{J}(\alpha_{s-1})$, by (23), this change of basis is clearly filtered.
- For $s \in [-n(2n-1)+2jn+2, -n(2n-1)+2(j+1)n-1]$ for some $1 \le j \le n-1$, and when $s \le 1$, perform the change of basis

$$\alpha_s \longmapsto \alpha_s + U^{-s+1}\left(\alpha_{s-1} + U^{\frac{j(j+1)}{2}}\widetilde{\alpha}_{s-1}\right).$$

  According to (28), (29), and (30), this splits off an acyclic summand as required. This change of basis is clearly filtered when $s \le 1$ (the equality is reached in the interval associated to $j = n-1$).
- For $s \in \{-n(2n-1)+2jn, -n(2n-1)+2jn+1\}$ with some $2 \le j \le n-1$ (noting that $s < 0$ always holds), perform the change of basis

$$\alpha_s \longmapsto \alpha_s + U^{-s+1}\alpha_{s-1}.$$

  This change of basis is again clearly filtered. □

Therefore, the local equivalence class of $X^\infty_{2n-1}(-T_{2n,2n+1})$ is given by $\bigoplus_{s=0}^{2n-1} A_s$ under the reduced basis. The differentials in this complex are already given by Lemma 4.2, and the filtrations of the generators are given by Lemma 4.1. In the following lemma, we will work out the $\mathcal{J}$–filtration shifts between the generators that are related by a differential.

Suppose $U^c\beta$ is a nontrivial term in $\partial\alpha$, where β is used to represent some $b_i^{(s)}$ and α is used to represent some α_s or $\widetilde{\alpha}_s$. Define

$$\Delta_{\mathcal{I},\mathcal{J}}(\alpha,\beta) = (\mathcal{I},\mathcal{J})(\alpha) - (\mathcal{I},\mathcal{J})(U^c\beta) \tag{31}$$

and, similarly define $\Delta_{\mathcal{I}}$ and $\Delta_{\mathcal{J}}$.

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Table 1. *The filtrations of the generators in the reduced basis of* $\bigoplus_{s=0}^{2n-1} A_s$.

Generators	The $\mathcal{J}$–filtrations	Range
α_s	$f(n, s-1)$	$1 \le s \le 2n-1$
$\widetilde{\alpha}_s$	$f(n, s-1) + n - 1 - s$	$1 \le s \le n-2$
	$f(n, s-1)$	$n+1 \le s \le 2n-2$
$b_n^{(s)}$	$f(n, s) - s$	$1 \le s \le 2n-2$
$b_{n-1}^{(s)}$	$f(n, s-1)$	$1 \le s \le n-2$
$b_{n+1}^{(s)}$	$f(n, s) + 2n + 1 - s$	$n+1 \le s \le 2n-2$

Lemma 4.4. *Generators in the reduced basis of* $\bigoplus_{s=0}^{2n-1} A_s$ *satisfy the following.*

$$\Delta_{\mathcal{I},\mathcal{J}}(\alpha_s, b_{n-1}^{(s)}) = \Big(\frac{n(n-1)}{2}, \frac{n(n-1)}{2}\Big), \qquad 1 \le s \le n-2 \tag{32}$$

$$\Delta_{\mathcal{I},\mathcal{J}}(\alpha_s, b_{n+1}^{(s-1)}) = \Big(\frac{n(n-1)}{2}, \frac{n(n-1)}{2}\Big), \qquad n+2 \le s \le 2n-1 \tag{33}$$

$$\Delta_{\mathcal{I},\mathcal{J}}(\alpha_s, b_n^{(s-1)}) = \Big(\frac{n(n+1)}{2} - s + 1, \frac{n(n+1)}{2}\Big), \qquad 2 \le s \le n \tag{34}$$

$$\Delta_{\mathcal{I},\mathcal{J}}(\alpha_s, b_n^{(s)}) = \Big(\frac{n(n+1)}{2}, \frac{n(n-1)}{2} + s - n + 1\Big), \qquad n \le s \le 2n-2 \tag{35}$$

$$\Delta_{\mathcal{I},\mathcal{J}}(\alpha_{n-1}, b_n^{(n-1)}) = \Big(\frac{n(n+1)}{2}, \frac{n(n-1)}{2}\Big), \tag{36}$$

$$\Delta_{\mathcal{I},\mathcal{J}}(\alpha_{n+1}, b_n^{(n)}) = \Big(\frac{n(n-1)}{2}, \frac{n(n+1)}{2}\Big), \tag{37}$$

$$\Delta_{\mathcal{I},\mathcal{J}}(\widetilde{\alpha}_s, b_{n-1}^{(s)}) = \big(0, n-1-s\big), \qquad \Delta_{\mathcal{I},\mathcal{J}}(\widetilde{\alpha}_s, b_n^{(s)}) = \big(n, 0\big), \qquad 1 \le s \le n-2 \tag{38}$$

$$\Delta_{\mathcal{I},\mathcal{J}}(\widetilde{\alpha}_s, b_n^{(s)}) = \big(0, n\big), \qquad \Delta_{\mathcal{I},\mathcal{J}}(\widetilde{\alpha}_s, b_{n+1}^{(s)}) = \big(s-n, 0\big), \qquad n+1 \le s \le 2n-2. \tag{39}$$

Proof. We collect in Table 1 the filtrations of the generators in the reduced basis of $\bigoplus_{s=0}^{2n-1} A_s$ from Lemma 4.1. Note that $g_b(n,n) = 0$ and $g_a(n,n) = -n$. The $\mathcal{I}$ filtrations of the generators are all 0 (so this is, in fact, a reduced model of $\widehat{HFK}$.)

To show (32) and (33), first, by (28), we have $\Delta_{\mathcal{I}}(\alpha_s, b_{n-1}^{(s)}) = \frac{n(n-1)}{2}$. Compute

$$\begin{aligned}\Delta_{\mathcal{J}}(\alpha_s, b_{n-1}^{(s)}) &= \mathcal{J}(\alpha_s) - \mathcal{J}(b_{n-1}^{(s)}) + \frac{n(n-1)}{2} \\ &= \frac{n(n-1)}{2},\end{aligned}$$

which proves (32), and (33) follows from the symmetry given by Proposition 3.2.

To show (34) and (35), first, by (28) and (30), we have $\Delta_{\mathcal{I}}(\alpha_s, b_n^{(s-1)}) = \frac{n(n+1)}{2} - s + 1$. Compute

$$\begin{aligned}\Delta_{\mathcal{J}}(\alpha_s, b_n^{(s-1)}) &= \mathcal{J}(\alpha_s) - \mathcal{J}(b_n^{(s-1)}) + \frac{n(n+1)}{2} - s + 1 \\ &= \frac{n(n+1)}{2},\end{aligned}$$

which proves (34), and (35) follows from the symmetry given by Proposition 3.2.

The rest of the results follow from similar computations and are left for the reader. □

When $n = 1$ and 2, the local equivalence class of the complex $X_n^\infty(-T_{2n,2n+1})$ can be decided following a similar vein. We record the result in the next lemma, and the computations are left to the reader as an exercise.

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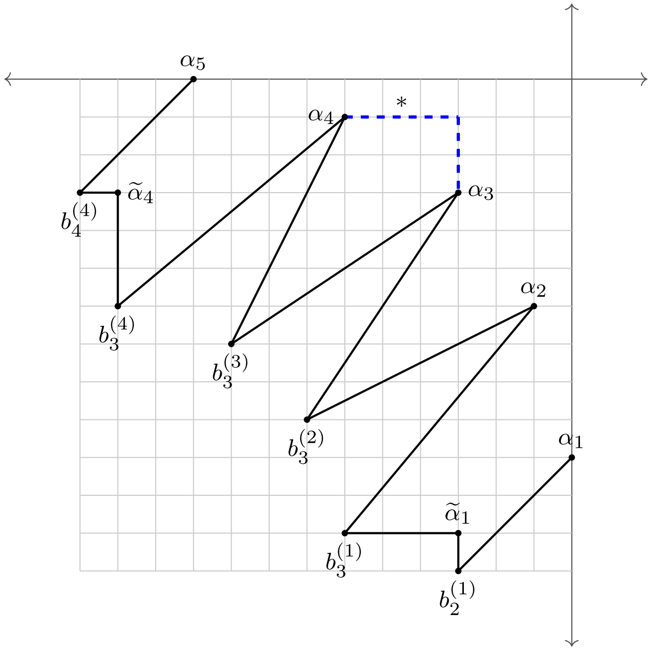

Figure 3. *A reduced basis for the complex $X_5^\infty(-T_{6,7})\langle 5\rangle$, where the coordinates are given by $\mathcal{I}$ and $\mathcal{J}$ filtrations. The generators are marked abstractly, without U powers. The edges represent the differentials; the edge with $*$ depicts an instance of the fact that $\Delta_{\mathcal{I}}(\alpha_n, b_n^{(n)}) = \Delta_{\mathcal{I}}(\alpha_{n+1}, b_n^{(n)}) + n$.*

Lemma 4.5. *When $n = 1$, the complex $X_1^\infty(-T_{2,3})$ is locally trivial.*
When $n = 2$, the complex $X_2^\infty(-T_{4,5})$ has a local complex characterized by the following.

$$\begin{aligned}
\partial\alpha_1 &= U^3 b_2^{(1)},\\
\partial\alpha_2 &= U^2 b_2^{(1)} + U^3 b_2^{(2)},\\
\partial\alpha_3 &= U b_2^{(1)};\\
\Delta_{\mathcal{I},\mathcal{J}}(\alpha_1, b_2^{(1)}) &= (3,1),\\
\Delta_{\mathcal{I},\mathcal{J}}(\alpha_2, b_2^{(1)}) &= (2,3),\\
\Delta_{\mathcal{I},\mathcal{J}}(\alpha_2, b_2^{(2)}) &= (3,2),\\
\Delta_{\mathcal{I},\mathcal{J}}(\alpha_3, b_2^{(2)}) &= (1,3).
\end{aligned}$$

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Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

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References

[1] S. Akbulut, 'A solution to a conjecture of Zeeman', *Topology* **30**(3), (1991), 513–515. doi:10.1016/0040-9383(91)90028-3.

[2] I. Dai, J. Hom, M. Stoffregen and L. Truong, 'Homology concordance and knot Floer homology', Preprint, 2021, arXiv:2110.14803.

[3] M. Hedden and A. S. Levine, 'A surgery formula for knot Floer homology', Preprint, 2019, arXiv:1901.02488.

[4] J. Hom, A. S. Levine and T. Lidman, 'Knot concordance in homology cobordisms', *Duke Math. J.* **171** (2022), 3089–3131. doi:10.1215/00127094-2021-0110.

[5] A. S. Levine, 'Nonsurjective satellite operators and piecewise-linear concordance', *Forum Math. Sigma* **4** (2016), Paper No. e34, 47. doi:10.1017/fms.2016.31.

[6] R. Lipshitz, 'Bordered Heegaard Floer homology', *Mem. Amer. Math. Soc*. **254** (2018), viii+279. doi:10.1090/memo/1216.

[7] P. Ozsváth and Z. Szabó, 'Holomorphic disks and knot invariants', *Adv. Math.* **186**(1) (2004), 58–116. doi:10.1016/j.aim.2003.05.001.

[8] P. Ozsváth and Z. Szabó, 'Holomorphic disks and topological invariants for closed three-manifolds', *Ann. of Math. (2)* **159**(3) (2004), 1027–1158. doi:10.4007/annals.2004.159.1027.

[9] P. Ozsváth and Z. Szabó, 'On knot Floer homology and lens space surgeries', *Topology* **44**(6) (2005), 1281–1300. doi:10.1016/j.top.2005.05.001.

[10] J. Rasmussen, *Floer homology and knot complements*. *Thesis*, 2016, https://arxiv.org/pdf/math/0306378.pdf.

[11] L. Truong, 'A refinement of the Ozsváth-Szabó large integer surgery formula and knot concordance', *Proc. Amer. Math. Soc.* **149**(4) (2021), 1757–1771. doi:10.1090/proc/15212.

[12] E. C. Zeeman, 'On the dunce hat', *Topology* **2**(4) (1964), 341–358. doi:10.1016/0040-9383(63)90014-4.

[13] I. Zemke, 'Link cobordisms and absolute gradings on link Floer homology', *Quantum Topol.* **10**(2) (2019), 207–323. doi:10.4171/QT/124.

[14] H. Zhou, 'A filtered mapping cone formula for cables of the knot meridian', Preprint, 2022, arXiv:2208.11289.