Classical-Nonclassical Polarity of Gaussian States

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(Received 17 October 2023; accepted 14 May 2024; published 13 June 2024)

Gaussian states with nonclassical properties such as squeezing and entanglement serve as crucial resources for quantum information processing. Accurately quantifying these properties within multimode Gaussian states has posed some challenges. To address this, we introduce a unified quantification: the "classical-nonclassical polarity," represented by $\mathcal{P}$. For a single mode, a positive value of $\mathcal{P}$ captures the reduced minimum quadrature uncertainty below the vacuum noise, while a negative value represents an enlarged uncertainty due to classical mixtures. For multimode systems, a positive $\mathcal{P}$ indicates bipartite quantum entanglement. We show that the sum of the total classical-nonclassical polarity is conserved under arbitrary linear optical transformations for any two-mode and three-mode Gaussian states. For any pure multimode Gaussian state, the total classical-nonclassical polarity equals the sum of the mean photon number from single-mode squeezing and two-mode squeezing. Our results provide a new perspective on the quantitative relation between single-mode nonclassicality and entanglement, which may find applications in a unified resource theory of nonclassical features.

DOI: 10.1103/PhysRevLett.132.240201

Gaussian states are continuous-variable (CV) quantum systems that are not only straightforward to describe from a theoretical standpoint, but also convenient to produce and manipulate experimentally [1,2]. Nonclassical Gaussian states such as single-mode squeezed vacuum states and the Einstein-Podolski-Rosen state are essential resources for quantum-enhanced applications [3–7], including quantum teleportation [8], quantum dense coding [9], quantum computing [10,11], and quantum sensing [12–15]. The nonclassical characteristics of multimode Gaussian states may originate from their reduced quadrature variance falling below the vacuum noise level and/or the presence of quantum entanglement among two or more modes. In particular, a single-mode squeezed vacuum state can be distributed in a multimode linear optical network of beam splitters (BSs) and phase shifters to create multipartite entangled states [8,16,17]. In general, qualitative aspects of nonclassicality conversion have been explored [18,19]. However, a quantitative understanding of these nonclassical properties in multimode Gaussian states is crucial for evaluating the enhancement in quantum information applications.

Various quantifications have been proposed to evaluate the degree of single-mode nonclassicality [20–24] and bipartite entanglement [5,25–29] individually. For a single-mode state, the Lee nonclassical depth quantifies the minimum number of thermal photons necessary to destroy whatever nonclassical effects exist in the quantum state [20]. Resource theories of single-mode quantum states [30–33] have been explored to determine their usefulness as a resource, e.g., in metrology [33]. For two-mode Gaussian states, the entanglement of formation gives the amount of the entropy of the state minimized from all possible state decomposition [34]. Yet, it is difficult to calculate this quantity in general [35]. An easy entanglement measure to compute is the logarithmic negativity, which quantifies how much the state fails to satisfy the positive partial transpose (PPT) condition [36,37]. The logarithmic negativity can be written as an analytical function of the minimum symplectic eigenvalue of the partially transposed state and it can quantify the degree of bipartite entanglement of a $1 \times (n-1)$ modes Gaussian state [38]. Yet, it remains an elusive task to define a unified quantification for both single-mode and multimode nonclassicalities. Furthermore, exploring the conversion between these nonclassicalities adds another layer of complexity.

Several previous works [39–43] have discussed the quantitative conversion of nonclassicality and entanglement during BS operations. For example, Ge *et al.* have explored a conservation relation of the two quantities during BS transformation for certain two-mode Gaussian states [39]. Moreover, Arkhipov *et al.* have found an invariant for nonclassical two-mode Gaussian states which comprises the terms describing both local nonclassicality of the reduced states and the entanglement of the whole system related to the symplectic eigenvalues [41], and extended the results to pure three-mode Gaussian states. However, these results only hold for a subset of two-mode or three-mode Gaussian states.

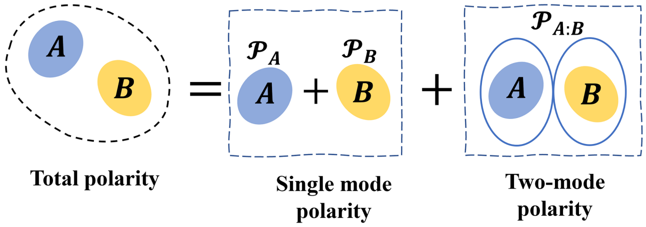

FIG. 1. Total classical-nonclassical polarity of a two-mode Gaussian state.

In this Letter, we aim to establish a unified quantification for both single-mode nonclassicality and multimode bipartite entanglement that is invariant under passive linear transformations. We introduce classical-nonclassical polarity (CNP) for quantifying single-mode nonclassicality and $1 \times (n-1)$-mode bipartite entanglement. Contrary to the measure quantification of quantum resource theories [44,45], our definitions can be positive or negative. This dual nature is necessitated by the conservation relation to hold for both classical and nonclassical Gaussian states. By taking into account the CNPs of the reduced single modes and all possible bipartite modes, we find the total CNP of an arbitrary two-mode or three-mode Gaussian state is a linear function of the symplectic invariants [38] and the extreme quadrature variances of the reduced single-mode states. Moreover, we show that the total CNP is a conserved quantity for arbitrary two-mode and three-mode Gaussian states under an optical linear network (Figs. 1 and 2). Our results provide a new perspective on the complex structure of nonclassical features in multi-mode Gaussian states, which may find applications in a unified resource theory of nonclassical states in different physical settings.

Gaussian state preliminary.—An n-mode Gaussian state, whose density matrix is denoted as ρ, is fully characterized, up to local displacements, by its covariance matrix (CM) $\boldsymbol{\gamma}$ of elements $\gamma_{jk} = \frac{1}{2}\langle \Delta\hat{\boldsymbol{r}}_j \Delta\hat{\boldsymbol{r}}_k^\dagger + \Delta\hat{\boldsymbol{r}}_k^\dagger \Delta\hat{\boldsymbol{r}}_j \rangle$, where $\langle \hat{X} \rangle = \mathrm{Tr}(\rho\hat{X})$, $\Delta\hat{\boldsymbol{r}}_j$ denotes $\hat{\boldsymbol{r}}_j - \langle \hat{\boldsymbol{r}} \rangle_j$, and $\hat{\boldsymbol{r}} = [\hat{a}_1^\dagger, \hat{a}_1, \dots \hat{a}_n^\dagger, \hat{a}_n]^\dagger$ is the vector of the bosonic field operators [28,46]. For a two-mode Gaussian state ρ_{AB}, the CM $\boldsymbol{\gamma_{AB}}$ is given by a 4×4 matrix

$$\gamma_{\boldsymbol{AB}} = \begin{bmatrix} \gamma_{\boldsymbol{A}} & \boldsymbol{x} \\ \boldsymbol{x}^\dagger & \gamma_{\boldsymbol{B}} \end{bmatrix}. \quad (1)$$

Note that $\boldsymbol{\gamma_A}$ ($\boldsymbol{\gamma_B}$) is a 2×2 matrix, representing the reduced single-mode Gaussian state ρ_A (ρ_B), and $\boldsymbol{x}$ describes the correlation of the two modes.

Symplectic invariants in Gaussian states.—Gaussian operations refer to unitary transformations that map a Gaussian state onto another Gaussian state. The exponent of these unitaries consists of terms up to quadratic in the bosonic field operators [6]. When a Gaussian state ρ undergoes a Gaussian unitary transformation $\hat{U}$, it induces a

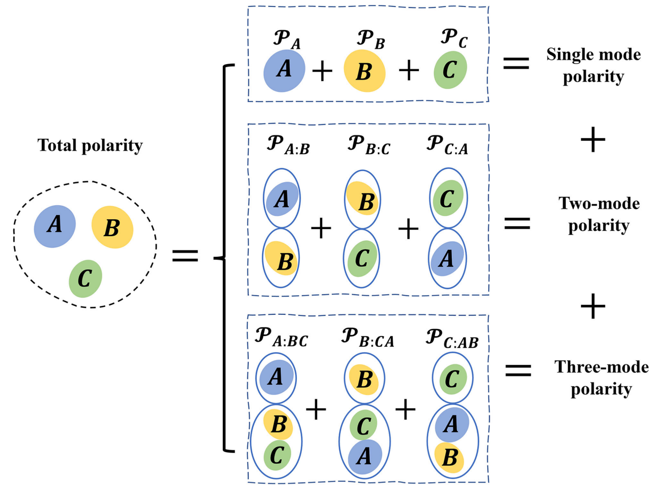

FIG. 2. Total classical-nonclassical polarity of a three-mode Gaussian state.

symplectic transformation $\boldsymbol{S}$ on its associated covariance matrix $\boldsymbol{\gamma}$. This correspondence can be succinctly expressed as $\rho' = \hat{U}^\dagger \rho \hat{U} \rightleftharpoons \boldsymbol{\gamma}' = \boldsymbol{S\gamma S}^\dagger$ [49]. According to Williamson's theorem [50], the CM $\boldsymbol{\gamma}$ can always be symplectic diagonalized, namely, $\boldsymbol{S\gamma S}^\dagger = \mathrm{diag}(\nu_1, \nu_1, \nu_2, \nu_2, \dots, \nu_n, \nu_n)$. Here ν_j are the symplectic eigenvalues. Certain quantities remain invariant under symplectic transformations, which are defined as symplectic invariants. As discussed in Refs. [38,51], a natural choice of symplectic invariants for n-mode Gaussian states is given by [38]

$$\mathcal{I}_k^{(n)} = \sum \mathcal{M}_{n-k}(\boldsymbol{\gamma}), \quad (2)$$

where $\mathcal{M}_{n-k}(\boldsymbol{\gamma})$ represents the minors of order $(n-k)$, which are obtained by calculating the determinants of submatrices that result from removing k rows and k columns from the block matrix. Given that there are various ways to select which rows and columns to delete, multiple minors of the same order can be derived. The summation encompasses all possible minors of the $(n-k)$ th order [46]. In terms of the symplectic diagonalized form of the CM, we have [52]

$$\mathcal{I}_k^{(n)} = \sum_{\mathcal{S}_k^n} \prod_{j \in \mathcal{S}_k^n} \nu_j^2, \quad (3)$$

where $\mathcal{S}_k^n$ represents a subset of $n-k$ integers chosen from integers $1, 2, \dots, n$ and the summation goes over all possible subsets [46].

For a general scenario, consider a two-mode Gaussian state whose CM is given by Eq. (1). Two symplectic invariants can be identified: $\mathcal{I}_0^{(2)} = |\boldsymbol{\gamma_{AB}}|$ and $\mathcal{I}_1^{(2)} = |\boldsymbol{\gamma_A}| + |\boldsymbol{\gamma_B}| + 2|\boldsymbol{x}|$ [5,28,53]. Note that $|\cdot|$ signifies the matrix determinant.

For a three-mode Gaussian state, represented as ρ_{ABC}, the CM is described by

$$\boldsymbol{\gamma}_{\boldsymbol{ABC}} = \begin{bmatrix} \boldsymbol{\gamma}_{\boldsymbol{A}} & \boldsymbol{x} & \boldsymbol{z} \\ \boldsymbol{x}^\dagger & \boldsymbol{\gamma}_{\boldsymbol{B}} & \boldsymbol{y} \\ \boldsymbol{z}^\dagger & \boldsymbol{y}^\dagger & \boldsymbol{\gamma}_{\boldsymbol{C}} \end{bmatrix}, \tag{4}$$

where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ denote the interaction of modes A, B, modes B, C, and modes A, C, respectively. The symplectic invariants of the three-mode Gaussian state are $\mathcal{I}_0^{(3)} = |\boldsymbol{\gamma}_{\boldsymbol{ABC}}|$, $\mathcal{I}_1^{(3)} = |\boldsymbol{\gamma}_{\boldsymbol{AB}}| + |\boldsymbol{\gamma}_{\boldsymbol{BC}}| + |\boldsymbol{\gamma}_{\boldsymbol{AC}}| + 2|\boldsymbol{D}_{\boldsymbol{x}}| + 2|\boldsymbol{D}_{\boldsymbol{y}}| + 2|\boldsymbol{D}_{\boldsymbol{z}}|$, and $\mathcal{I}_2^{(3)} = |\boldsymbol{\gamma}_{\boldsymbol{A}}| + |\boldsymbol{\gamma}_{\boldsymbol{B}}| + |\boldsymbol{\gamma}_{\boldsymbol{C}}| + 2|\boldsymbol{x}| + 2|\boldsymbol{y}| + 2|\boldsymbol{z}|$, where $\boldsymbol{D}_{\boldsymbol{x}}, \boldsymbol{D}_{\boldsymbol{y}}, \boldsymbol{D}_{\boldsymbol{z}}$ are 4×4 matrices given by: $\boldsymbol{D}_{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{z} \\ \boldsymbol{\gamma}_{\boldsymbol{B}} & \boldsymbol{y} \end{bmatrix}$, $\boldsymbol{D}_{\boldsymbol{y}} = \begin{bmatrix} \boldsymbol{y} & \boldsymbol{x}^\dagger \\ \boldsymbol{\gamma}_{\boldsymbol{C}} & \boldsymbol{z}^\dagger \end{bmatrix}$, $\boldsymbol{D}_{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{z} & \boldsymbol{\gamma}_{\boldsymbol{A}} \\ \boldsymbol{y} & \boldsymbol{x}^\dagger \end{bmatrix}$. We explain the patterns of the minors for a three-mode system $\boldsymbol{\gamma}_{\boldsymbol{ABC}}$ in [46].

Single-mode classical-nonclassical polarity.—The origin of all nonclassical effects is that the P function of a quantum state ρ, which is defined through $\rho = \int P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha| d^2\alpha$, are singular and not positive definite [20,46,54]. The Lee nonclassical depth provides a measure for this single-mode nonclassicality. In order to quantify the nonclassicality of each single mode (tracing out the rest modes [39,41]) in an n-mode Gaussian state, we introduce the concept of single-mode classical-nonclassical polarity, denoted as $\mathcal{P}^{(1)}$. For a single-mode Gaussian state with the CM $\boldsymbol{\gamma}^{(1)} = \begin{bmatrix} a & b \\ b^* & a \end{bmatrix}$, the degree of polarity is defined as [46]

$$\mathcal{P}^{(1)} = -\left(\lambda - \frac{1}{2}\right)\left(\Lambda - \frac{1}{2}\right), \tag{5}$$

where $\lambda = a - |b|$, $\Lambda = a + |b|$ are the minimum and the maximum eigenvalues of the CM matrix $\boldsymbol{\gamma}^{(1)}$, describing the minimum and the maximum quadrature variances of the quantum state. In the context of single-mode Gaussian states, $|\boldsymbol{\gamma}| = \lambda\Lambda \geqslant 1/4$ [5]. Thus, $\Lambda \geqslant 1/2$ always holds. For $\lambda < 1/2$, the state has the minimum quadrature variance below that of coherent states [46], therefore it is nonclassical and the CNP $\mathcal{P}^{(1)} > 0$. For $\lambda > 1/2$, the state has the minimum quadrature variance larger than that of coherent states, meaning classical and $\mathcal{P}^{(1)} < 0$ [55]. When $\lambda = 1/2$, $\mathcal{P}^{(1)} = 0$ is the classical-nonclassical boundary, where the states are squeezed thermal coherent states with the minimum quadrature variance being $1/2$. For a pure state, it can be calculated that $\mathcal{P}^{(1)} = \langle \hat{a}^\dagger \hat{a} \rangle - |\langle \hat{a} \rangle|^2$, which quantifies the amount of averaged photon number from the nonclassical process, i.e., single-mode squeezing.

Our definition of single-mode CNP takes the purity $(4|\boldsymbol{\gamma}^{(1)}|)^{-1}$ [5] of the state into account. The smaller the purity, the greater the absolute value of $\mathcal{P}^{(1)}$. An arbitrary single-mode Gaussian state can be described as a squeezed thermal coherent state [6], where $\lambda = (1/2 + \langle n_{\text{th}} \rangle) e^{-2r}$ and $\Lambda = (1/2 + \langle n_{\text{th}} \rangle) e^{2r}$ [46]. Here r is the squeezing parameter and $\langle n_{\text{th}} \rangle$ is the averaged number of thermal photons. For nonclassical states with the same value of λ, smaller purity is represented by a larger value of Λ. Thus, the positive CNP for nonclassical states characterizes the degree of squeezing for both pure and mixed states.

$1 \times (n-1)$ modes bipartite classical-nonclassical polarity.—For a single-mode Gaussian state, the CNP characterizes the squeezing property, while in a multimode scenario, it is the entanglement between different subsystems that introduces another degree of nonclassicality. The positive partial transpose (PPT) criterion [5,25,36] is a both necessary and sufficient condition for $1 \times (n-1)$ mode Gaussian entanglement. Therefore, violation of the criterion can be used to construct bipartite entanglement measures. For a partial-transposed state ρ^{T_A} (without loss of generality, transpose the first mode A here), the corresponding CM is $\boldsymbol{\gamma}^{\boldsymbol{T_A}} = \boldsymbol{T}\boldsymbol{\gamma}\boldsymbol{T}$ with $\boldsymbol{T} = \boldsymbol{t} \oplus \mathbb{1}_{2n-2}$, where $\boldsymbol{t} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbb{1}_{2n-2}$ is the $(2n-2)$-dimension identity matrix [36]. For convenience, denote $\boldsymbol{\gamma}^{\boldsymbol{T_A}}$ as $\tilde{\boldsymbol{\gamma}}$. The symplectic invariants $\tilde{\mathcal{I}}_j^{(n)}$ of the PPT state ρ^{T_A} are the summations of minors of $\tilde{\boldsymbol{\gamma}}$, which can be expressed by the symplectic eigenvalues $\tilde{\nu}_j$ of $\tilde{\boldsymbol{\gamma}}$ in the same form as Eq. (3). According to the PPT criterion, if $\tilde{\nu}_j \geqslant 1/2$, ρ is a PPT state, implying mode A is separable from other subsystems. Reference [38] demonstrates that at most one symplectic eigenvalue can be smaller than $1/2$. Therefore, computable entanglement measures have been defined by comparing the minimum partially transposed symplectic eigenvalue $\tilde{\nu}_{\min}$ with $1/2$, such as the negativity and the logarithmic negativity, both of which quantify how much the PPT condition is violated. However, these quantifications do not differentiate states with the same minimum symplectic eigenvalue but with other different properties, such as the purity of the system.

In this context, we introduce the CNP for bipartite $1 \times (n-1)$-mode Gaussian states based on the symplectic invariants of PPT state. Regarding the PPT state, Eq. (3) suggests an analogy with Vieta's formulas if we treat $\tilde{v}_i^2$ as the roots of a polynomial. With this insight, we define the polynomial functions $g^{(n)}(x) \equiv 2\sum_{j=0}^{n} (-1)^{j+1} x^j \tilde{\mathcal{I}}_j^{(n)}$ of degree n $(n \geqslant 2)$, where $\tilde{\mathcal{I}}_n^{(n)}$ is set to be 1. Through Vieta's formulas, the $g(x)$ function can be expressed as $g^{(n)}(x) = -2\prod_{j=1}^{n} (\tilde{v}_j^2 - x)$. Therefore, the separability condition $\tilde{\nu}_j \geq 1/2$ is equivalent to $g^{(n)}(1/4) \leq 0$. It reveals that $g^{(2)}(1/4) > 0$ indicates two-mode Gaussian entanglement exists, while $g^{(n)}(1/4) < 0$ leads to separable states. Hence we define the $1 \times (n-1)$ modes bipartite CNP as [46]

$$\mathcal{P}^{(n)} = g^{(n)}\left(\frac{1}{4}\right) = -2\prod_{j=1}^{n} \left(\tilde{v}_j^2 - \frac{1}{4}\right). \tag{6}$$

Taking $n = 2$, for example, the sign of $\mathcal{P}^{(2)}$ determines whether the two-mode bipartite system is entangled or separable, while the absolute value quantifies the distance of the state to the separability-entanglement boundary. Higher order of the multimode CNPs quantifies any additional separability or entanglement contribution to the $1 \times (n-1)$ modes bipartite system. In terms of the symplectic invariants, the two-mode CNP is given by

$$\mathcal{P}^{(2)}_{A:B} = -\frac{1}{8} + \frac{1}{2}\tilde{\mathcal{I}}^{(2)}_1 - 2\tilde{\mathcal{I}}^{(2)}_0. \tag{7}$$

In [46], we show that $\tilde{\mathcal{I}}^{(2)}_1 = \mathcal{I}^{(2)}_1 - 4|\boldsymbol{x}| = |\boldsymbol{\gamma_A}| + |\boldsymbol{\gamma_B}| - 2|\boldsymbol{x}|$ and $\tilde{\mathcal{I}}^{(2)}_0 = \mathcal{I}^{(2)}_0 = |\boldsymbol{\gamma_{AB}}|$. Similar to the single-mode CNP, the two-mode CNP exhibits linearity in relation to (sub-)matrix determinants and is influenced by the system's purity. For entangled states, the value of $\mathcal{P}^{(2)}$ takes into account the potential extra effort to prepare a more mixed state with the same $\tilde{v}_{\min}$. For separable states, a smaller purity means more classical.

In a three-mode system, denoted as ρ_{ABC}, bipartite CNP can manifest in two ways: as two-mode polarity and as three-mode polarity, as illustrated in Fig. 2. For the two-mode CNP, by tracing out one mode from ρ_{ABC}, it encompasses three components: $\mathcal{P}^{(2)}_{A:B}$, $\mathcal{P}^{(2)}_{B:C}$, and $\mathcal{P}^{(2)}_{C:A}$. On the other hand, the three-mode CNP is divided into the following three terms: $\mathcal{P}^{(3)}_{A:BC}$, $\mathcal{P}^{(3)}_{B:CA}$, and $\mathcal{P}^{(3)}_{C:AB}$. These terms can indicate the bipartite entanglement or separability of the system. As an example, according to Eq. (6), the CNP for $A{:}BC$ is given by

$$\mathcal{P}^{(3)}_{A:BC} = \frac{1}{32} - \frac{1}{8}\tilde{\mathcal{I}}^{(3)}_2 + \frac{1}{2}\tilde{\mathcal{I}}^{(3)}_1 - 2\tilde{\mathcal{I}}^{(3)}_0, \tag{8}$$

where $\tilde{\mathcal{I}}^{(3)}_1 = \mathcal{I}^{(3)}_1 - 4|\boldsymbol{D_x}| - 4|\boldsymbol{D_y}|$, and $\tilde{\mathcal{I}}^{(3)}_2 = \mathcal{I}^{(3)}_2 - 4|\boldsymbol{x}| - 4|\boldsymbol{z}|$ [46].

In particular, for pure state, the following theorem holds.

Theorem 1.—For any pure multimode Gaussian state, the order of n ($n \geq 3$) CNP bipartite $\mathcal{P}^{(n)}$ equals zero.

As the two-mode polarities $\mathcal{P}^{(2)}_{A:B}$, $\mathcal{P}^{(2)}_{A:C}$ already quantify some amount of entanglement or separability of the three-mode system when one mode is traced out, the three-mode polarity $\mathcal{P}^{(3)}_{A:BC}$ can be zero even if the bipartite $A{:}BC$ is entangled. In this case, the three-mode polarity avoids over-counting the entanglement or separability of the system. For example, for a one-mode biseparable state $\rho_{AB} \otimes \rho_C$ with ρ_{AB} entangled, $\mathcal{P}^{(3)}_{A:BC} = 0$ when ρ_C is pure and $\mathcal{P}^{(3)}_{A:BC} > 0$ when ρ_C is a mixed state. Yet, we show numerically that by adding a separable state ρ_C to the two-mode ρ_{AB}, the total bipartite polarities in general do not increase.

Conservation relation for two-mode Gaussian states before and after a beam splitter.—Beam splitters and phase shifters are linear optical devices that do not generate additional nonclassicality [33,56]. Entanglement can be generated from single-mode nonclassical states using beam splitters [16]. Previous work [39] attempted to find a conservation relation of the total nonclassicality from both single-mode reduced systems and the two-mode system. However, only a subset of two-mode Gaussian states has been shown to satisfy the conservation of nonclassicality under some quantifications. Here we show that the sum of single-mode CNPs and two-mode bipartite CNP is conserved before and after a beam splitter for arbitrary input states ρ_{AB} [Fig. 3(a)]. We obtain that the total CNP [46]

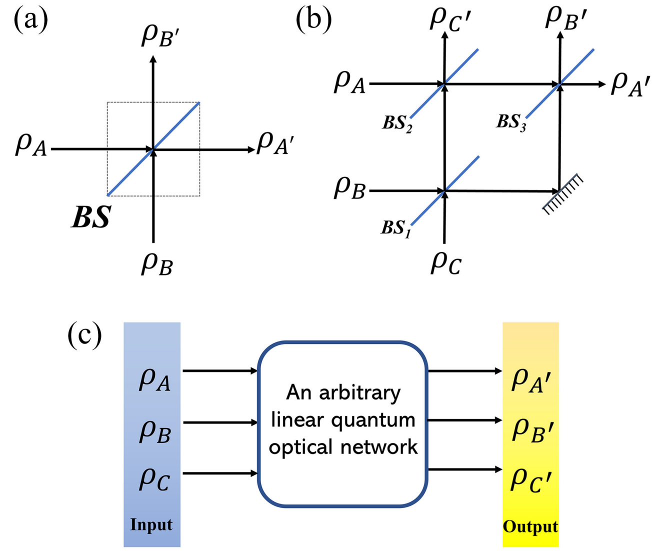

FIG. 3. (a) Two-mode Gaussian state, ρ_{AB}, undergoes mixing by a BS to produce a new two-mode Gaussian state, $\rho_{A'B'}$. (b),(c) Three-mode Gaussian state, ρ_{ABC}, passes through linear optical networks.

$$\begin{aligned}\mathcal{P} &\equiv \mathcal{P}^{(1)}_A + \mathcal{P}^{(1)}_B + \mathcal{P}^{(2)}_{A:B} \\ &= \frac{1}{2}(\lambda_A + \Lambda_A + \lambda_B + \Lambda_B) - \frac{1}{2}\mathcal{I}^{(2)}_1 - \frac{5}{8} - 2\mathcal{I}^{(2)}_0,\end{aligned} \tag{9}$$

where $\lambda_{A(B)}$ and $\Lambda_{A(B)}$ represent the minimum and the maximum eigenvalues of $\boldsymbol{\gamma_{A(B)}}$. In addition to the invariants $\mathcal{I}^{(2)}_1$ and $\mathcal{I}^{(2)}_0$, it can be shown that $\lambda_A + \Lambda_A + \lambda_B + \Lambda_B$ is related to the average number of photons and it is also invariant before and after a BS [46]. Therefore, the total CNP of a two-mode Gaussian state is conserved under linear optical transformations. It is worth noting that although $\mathcal{P}$ is the sum of the total classical-nonclassical polarity, $\mathcal{P} < 0$ does not necessarily mean the system is classical. For example, a two-mode system consisting of a weak single-mode squeezed vacuum state and a large thermal state, which has a non-positive-definite P function. We note that a similar nonclassicality invariant under linear unitary transformations is introduced in Ref. [41]. As an example, we calculate the total CNP for a two-mode squeezed vacuum state [57], which gives $\mathcal{P} = \langle \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \rangle$. We conclude that the total CNP of a single-mode squeezed vacuum equals that of a two-mode squeezed vacuum for the same average number of photons.

Conservation relation for three-mode Gaussian states in linear optical networks.—The conservation relation can be extended to arbitrary three-mode Gaussian states using the concept of the single-mode and multimode bipartite CNPs (Fig. 2) in a linear optical network.

The total polarity $\mathcal{P}$, determined by summing the single-mode, two-mode, and three-mode CNPs as illustrated in Fig. 2, follows the cluster-expansion structure outlined below: [46]

$$\begin{aligned}\mathcal{P} &= \sum_{\alpha=A,B,C} \mathcal{P}_{\alpha}^{(1)} + \frac{1}{2} \sum_{\alpha,\beta=A,B,C} \mathcal{P}_{\alpha:\beta}^{(2)} + \frac{1}{2} \sum_{\alpha,\beta,\kappa=A,B,C} \mathcal{P}_{\alpha:\beta\kappa}^{(3)} \\ &= -\frac{3}{8}\mathcal{I}_2^{(3)} - \frac{1}{2}\mathcal{I}_1^{(3)} - 6\mathcal{I}_0^{(3)} - \frac{33}{32} + \frac{1}{2}\bar{\Lambda}, \end{aligned} \quad (10)$$

where α, β, κ take all the permutations of A, B, C in the summations, the factor $1/2$ in the first line is due to the commutativity of indexes. $\mathcal{I}_0^{(3)}$, $\mathcal{I}_2^{(3)}$, and $\mathcal{I}_3^{(3)}$ are symplectic invariants of the three-mode system, which stay conserved during any Gaussian operation, while $\bar{\Lambda} \equiv \lambda_A + \Lambda_A + \lambda_B + \Lambda_B + \lambda_C + \Lambda_C$ also stay invariant before and after a BS [46]. Hence, $\mathcal{P}$ is a conserved quantity during linear optical transformations. An example of a BS network is given in Fig. 3(b). A general scenario is shown by Fig. 3(c) where a three-mode Gaussian state is input into an arbitrary linear optical network comprising BSs, wave plates, and phase shifters [30,58]. The conserved total CNP describes the conversion between all kinds of classical-nonclasscial features within Gaussian states through a passive Gaussian transformation using our definitions Eqs. (5) and (6).

Additionally, the following theorem holds:

Theorem 2.—For any pure two-mode or three-mode Gaussian state, the total classical-nonclassical polarity equals the sum of the mean photon number from single-mode squeezing and two-mode squeezing.

The proof of Theorem 2, along with a concrete example using CV GHZ/W Gaussian states [52,59] are provided in Supplemental Material [46]. We also note that the treatment of CV GHZ/W using CNP may not be conclusive due to the complex nature of the CV states.

Discussion.—We have established a quantitative relation of classical-nonclassical polarity within Gaussian states up to three modes. Our results on CNP quantification and the conservation relation on Gaussian states suggest a new method for evaluating different classical-nonclassical properties on the same footing, which has multiple implications.

First, the quantitative conversion between various nonclassical properties provides the basis for preparing quantum resources from one to another. This suggests that in order to maximize the resource output of one type, e.g., entanglement, we can design unitary transformations to deplete the input resource of the other kind [60]. Second, our results provide a unified quantification for single-mode, two-mode, and three-mode Gaussian states in terms of the total CNPs. We can compare the degree of CNP for resources from completely different processes, such as single-mode squeezed vacuum states for $SU(2)$ interferometers and two-mode squeezed vacuum states characterized by $SU(1,1)$ interferometers [57]. The unified quantification may also support a more general resource theory of quantum states with linear optical unitaries as free operations [30–33]. Third, our findings offers innovative methods for implementation in experimental settings. For instance, in the case of pure two-mode states, quantum entanglement can be determined through measurements of single-mode squeezing or the total average number of photons using the conservation relation or Theorem 2. This method circumvents the need for two-mode tomography, which, despite its complexity, is often essential for detecting entanglement within quantum light fields or quantum superconducting circuit systems. Fourth, our Letter may inspire the future study of higher-mode Gaussian states. For example, how the nonclassicality of a single-mode squeezed vacuum state is distributed in a multimode linear network for distributed quantum metrology [14,61]. Yet, the structure of the system would be more complex as the dimension grows. For four-mode Gaussian states, the positive-partial-transpose (PPT) criterion we used to indicate $1 \times n$ bipartite entanglement is no longer valid for $n \times m$ Gaussian entanglement ($m, n \geqslant 2$). For $n \geqslant 4$ scenarios, there is a lack of an efficient, well-accepted theorem for determining $n \times m (n, m \geqslant 2)$ Gaussian entanglement in the current research field. However, if we suppose the validity of the conservation relation, the total CNP $\mathcal{P}$ contains only invariant value which is usually related with the symplectic invariants of the CM $\boldsymbol{\gamma_{ABCD}}$. It provides us a way to derive back the formula of 2×2 entanglement witness $\sum_{\alpha,\beta,\kappa,\delta=A,B,C,D} \mathcal{P}_{\alpha\beta:\kappa\delta}^{(4)}$ [46].

Conclusion.—The current research field faces challenges in finding a universally accepted measure of entanglement, not to mention a comprehensive metric for nonclassicality. Within the realm of Gaussian states, we have proposed a unified quantification approach that encompasses both single-mode squeezing and multimode bipartite entanglement through the lens of classical-nonclassical polarity. We demonstrated that the sum of the single-mode and multimode classical-nonclassical polarities is conserved under linear optical transformations for arbitrary two-mode and three-mode Gaussian states. These findings highlight a quantitative conversion relation between different classical-nonclassical features in multimode systems, enriching our understanding of multimode entanglement phenomena and sheds lights on the nonclassicality research of general quantum states.

We thank Kangle Li from HKUST for helpful discussions. This research is supported by the project NPRP 13S-0205-200258 of the Qatar National Research Fund (QNRF). The research of W. G. is supported by NSF Grant No. 2243591.

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