Heavy Traffic Joint Queue Length Distribution without Resource Pooling

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ABSTRACT

This paper studies the Heavy Traffic (HT) joint distribution of queue lengths in an Input-queued switch (IQ switch) operating under the MaxWeight scheduling policy. IQ switchserve as representative of SPNs that do not satisfy the so-called Complete Resource Pooling (CRP) condition, and consequently exhibit a multidimensional State Space Collapse (SSC). Except in special cases, only mean queue lengths of such non-CRP systems is known in the literature. In this paper, we develop the Transform method to study the joint distribution of queue lengths in non-CRP systems. The key challenge is in solving an implicit functional equation involving the Laplace transform of the HT limiting distribution. For the general $n \times n$ IQ switch that has n^2 queues, under a conjecture on uniqueness of the solution of the functional equation, we obtain an exact joint distribution of the HT limiting queue-lengths in terms of a non-linear combination of $2n$ iid exponentials.

1. INTRODUCTION

Stochastic Processing Networks (SPNs) are ubiquitous in engineering with applications in manufacturing, telecommunications, transportation, computer systems, etc. A general SPN consists of jobs or packets that compete for limited resources, and can be modeled using a set of interacting queues. A key performance metric of interest in such systems is queue length. In general, it is not possible to exactly characterize the steady-state queue length behavior in such SPNs. Therefore, SPNs are studied in various asymptotic regimes. In this paper, we consider the HT regime where the system is loaded close to its capacity. The queue length in this case, usually blows up to infinity, at a rate of $1/\epsilon$, where ϵ is the HT parameter that denotes the gap between the arrival rate and the system capacity. Therefore, the objective of interest is typically the asymptotic behavior of the queue length, scaled by ϵ.

Using HT analysis, it was shown that the scaled queue length distribution of a single server queue converges to that of an exponential random variable. Since then, a variety of SPNs has been studied in HT. A key phenomenon in the HT regime is that the multi-dimensional queue-length vector typically collapses to a lower-dimensional subspace. This is called *State Space Collapse* (SSC), and it simplifies the analysis of an SPN. When the so-called Complete Resource Pooling (CRP) condition is satisfied, various SPNs exhibit an SSC to a one-dimension subspace, i.e., a line. In this case, the SPN behaves like a single server queue in HT, and the limiting distribution of scaled queue lengths converges to an exponential random variable.

However, several SPNs that arise in practice do not satisfy the CRP condition, and the SSC occurs to a multi-dimensional subspace. Despite special efforts, except in special cases, the classical diffusion limit approach failed to characterize the HT steady state queue length behaviour. Recent work [3, 1] developed the drift method and used it to characterize the mean of the (weighted) sum of the queue lengths in such systems under great generality. However, it was shown in [1] that the drift method is insufficient to even obtain the individual mean queue lengths. Going beyond the mean queue lengths, the key question we focus on in this paper is: *What is the HT* ***joint distribution*** *of queue lengths in an SPN when the CRP condition is not satisfied?*

In this work, we consider a well-studied stochastic processing networks (SPN), viz., an IQ switch policy. We characterize the HT joint distribution by establishing an implicit functional equation, and also provide the solution to the functional equation under certain condition.

2. MODEL

An Input-Queued switch (IQ switch) is a device that exchanges data from one channel to another in a data center. A switch of size n consists of n input ports and n output ports. The message packets flow from input ports to output ports in a time-slotted manner. For time slot t, we denote the arrival $a_{i+n(j-1)}(t)$ to be the number of packets that come input i to be sent to output port j. As there are n^2 such input-output pairs, the arrival in any time slot can be represented by an n^2 vector $\mathbf{a}(t)$. The architecture of the device doesn't allow all the packets to be transferred in one go, which leads to a queue build up on the inputs. We use $q_{i+n(j-1)}(t)$ (or $\mathbf{q}(t)$ in vector notation) to denote the backlog of packets that needs to be transferred to the output j from input i. We assume that the arrivals are i.i.d. with respect to t and the distribution of the arrivals have a bounded support (i.e. for (i,j) and t, $a_{i+n(j-1)}(t) \leq a_{\max}$). The mean arrival rate vector is given by $\mathbb{E}[\mathbf{a}(t)] = \boldsymbol{\lambda}$ and let $\boldsymbol{\sigma}^2$ be the co-variance matrix of the arrival vector $\mathbf{a}(t)$.

The bottlenecks in the system don't allow the transfer of all the packets in the queue simultaneously. Every port can send or receive at most one packet in any time slot. Also,

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the packet transfer can happen only among the connected input-output pairs in that time slot. A *schedule* denoted by $\mathbf{s}(t) \in \{0,1\}^{n^2}$ gives the set of input-output pairs that are connected in time slot t. The element $s_{i+n(j-1)}(t) = 1$ if and only if the pair (i,j) is connected in time slot t. It follows that the set of possible schedules $\mathcal{X}$ is given by the set of vectors $\mathbf{s} \in \{0,1\}^{n^2}$ such that $\sum_{i=1}^n s_{i+n(j-1)} = 1\ \forall j$ and $\sum_{j=1}^n s_{i+n(j-1)} = 1\ \forall i$.

A scheduling algorithm is then the policy that chooses the schedule in each time slot. The queue length evolution process is given by

$$\mathbf{q}(t+1) = [\mathbf{q}(t) + \mathbf{a}(t) - \mathbf{s}(t)]^+ = \mathbf{q}(t) + \mathbf{a}(t) - \mathbf{s}(t) + \mathbf{u}(t),$$

where operation $[\cdot]^+$ in the above equation is used because the queue length can't be negative. The terms $\mathbf{u}(t)$ is the unused service, which arises because it might happen that there is a connection between a input-output pair but there is no packet available to be transferred.

As is well-known, the capacity region of the switch system is given by the set of $\boldsymbol{\lambda} \in \mathbb{R}_+^{n^2}$ such that $\sum_{i=1}^n \lambda_{i+n(j-1)} < 1\ \forall j$ and $\sum_{j=1}^n \lambda_{i+n(j-1)} < 1\ \forall i$. Let $\mathcal{F}$ denote the part of boundary of the capacity region given by the convex hull of $\mathcal{X}$. A switch system is in HT when the arrival rate vector $\boldsymbol{\lambda}$ approaches the boundary $\mathcal{F}$. There exists a vector $\boldsymbol{\nu} \in \mathcal{F}$ and the HT parameter $\epsilon \in (0,1)$ such that $\boldsymbol{\lambda} = (1-\epsilon)\boldsymbol{\nu}$.

A well known scheduling algorithm is *MaxWeight* scheduling which chooses the schedule with maximum weight, where *weight* of the schedule is the sum of the queue lengths that are being served in the given time slot. It has been proved in prior literature that MaxWeight scheduling is *throughput optimal*, i.e., the corresponding Markov chain is stable for any arrival rate vector in $\mathcal{C}$. From here onwards, we consider an IQ switch operating under MaxWeight scheduling.

3. RESULTS

Let $\mathbf{B} \in \{0,1\}^{n^2 \times 2n}$ is such that for any $1 \le i, j \le n$,

$$B_{i+n(j-1),i} = B_{i+n(j-1),n+j} = 1,$$

and all other elements are zero. Consider the subspace $\mathcal{S} \subseteq \mathbb{R}^{n^2}$ to be the space spanned by columns of $\mathbf{B}$. For any vector $\mathbf{x} \in \mathbb{R}^{n^2}$, let $\mathbf{x}_{\|\mathcal{S}}$ denote its projection to the subspace $\mathcal{S}$ and $\mathbf{x}_{\perp\mathcal{S}} = \mathbf{x} - \mathbf{x}_{\|\mathcal{S}}$.

Proposition 1. *There exist a C_r independent of ϵ such that* $\mathbb{E}\Big[\|\mathbf{q}_{\perp\mathcal{S}}\|^r\Big] \le C_r, \forall r \ge 1$.

According to Proposition 1, for MaxWeight scheduling algorithm, the moments of $\mathbf{q}_{\perp\mathcal{S}}$ are bounded irrespective of the HT parameter ϵ. We know from [3, Proposition 1], that in HT queue length scales at least at the rate of $\Omega(1/\epsilon)$. This shows that, in HT, $\mathbf{q}_{\perp\mathcal{S}}$ is insignificant compared to $\mathbf{q}$ and so, in HT, $\epsilon\mathbf{q} \approx \epsilon\mathbf{q}_{\|\mathcal{S}}$. This is known as SSC.

Theorem 2 (Functional equation)**.** *Consider the IQ switch operating under MaxWeight scheduling algorithm. Let* $\boldsymbol{\Theta} = \{\boldsymbol{\theta} \in \mathbb{C}^{n^2} : \boldsymbol{\theta} \in \mathcal{S},\ Re(\mathbf{B}^T\boldsymbol{\theta}) \le \mathbf{0}_{2n}\}$. *Then, for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,*

$$\mathcal{P}(\boldsymbol{\theta}) = \big(2\langle\boldsymbol{\theta}, \mathbf{1}_{n^2}\rangle - n\langle\boldsymbol{\theta}, \boldsymbol{\sigma}^2\boldsymbol{\theta}\rangle\big)\, L(\boldsymbol{\theta}) - 2n\langle\boldsymbol{\theta}, \mathbf{M}(\boldsymbol{\theta})\rangle = 0,$$

where $\forall k \in \{1, 2, \ldots, n^2\}$,

$$L(\boldsymbol{\theta}) = \lim_{\epsilon\to 0} \mathbb{E}_{\pi^\epsilon}[e^{\epsilon\langle\boldsymbol{\theta},\mathbf{q}\rangle}], \quad M_k(\boldsymbol{\theta}) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\mathbb{E}_{\pi^\epsilon}[u_k e^{\epsilon\langle\boldsymbol{\theta},\mathbf{q}\rangle}].$$

The functional equation ($\mathcal{P}(\boldsymbol{\theta}) = 0$ in Theorem 2) is mathematical relationship between the term $L(\boldsymbol{\theta})$, which is the Laplace transform of the limiting HT distribution and the terms $M_k(\boldsymbol{\theta})$, which intuitively denote the Laplace transform under the condition $q_k = 0$ (since $u_k = 1$ implies $q_k = 0$). Further, the set $\boldsymbol{\Theta}$ is appropriately chosen such that the quantities $L(\boldsymbol{\theta})$ and $M_k(\boldsymbol{\theta})$'s are well defined.

The steps to establish the functional equation for IQ switch consists of two steps. The first step is to use the complex exponential as the Lyapunov function and equate its expected drift to zero in steady-state. After that, we use the second-order approximation of the complex exponential in terms of the HT parameter ϵ and eliminate the higher order terms to get the functional equation. Here, SSC plays a key role in the mathematical analysis. To be more specific, due to the SSC, we only have to consider $\mathbf{q}_{\|}$, which leads to a lot of technical simplicity.

Conjecture 3. *There is a unique set of functions $L(\boldsymbol{\theta})$ and $M_k(\boldsymbol{\theta})$'s defined in Theorem 2, that satisfies the functional equation $\mathcal{P}(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.*

A major challenge in solving the implicit functional equation given in Eq. (2) is proving that the functional equation has a unique solution. For simpler systems, where the SSC happens to a two-dimensional subspace, one can prove that the corresponding functional equation has a unique solution using the theory of Carleman boundary value problem. Extending that result to a functional equation with more than two variables, such as for the IQ switch, is open.

Theorem 4. *Assume Conjecture 3 holds. Suppose the variance vector $\boldsymbol{\sigma}^2$ is symmetric, i.e., $\boldsymbol{\sigma}^2 = \sigma^2\mathbf{I}_{n^2}$, where $\mathbf{I}_{n^2}$ is the identity matrix of size n^2. Then,*

$$\epsilon\mathbf{q} \xrightarrow{d} \mathbf{B}(\boldsymbol{\Upsilon} - \tilde{\Upsilon}\mathbf{1}_{2n}),$$

where $\boldsymbol{\Upsilon} = (\Upsilon_1, \ldots, \Upsilon_{2n})$ is a vector of $2n$ independent exponential random variables with mean $\frac{\sigma^2}{2}$ and $\tilde{\Upsilon} = \min_{1\le k\le 2n} \Upsilon_k$.

Assuming Conjecture 3 holds, Theorem 4 completely characterizes the HT distribution of the IQ switch under the symmetric variance condition. The key idea behind the proof is to show that the Laplace transform of the limiting distribution provided in the Theorem 4 is a solution of a functional equation (Eq. (2)) given in Theorem 2 when the variances of the arrival process are symmetric. And under the assumption that the functional equation has a unique solution claimed by Conjecture 3, the solution provided in Theorem 4 is the unique solution for the HT distribution for IQ switch under symmetric variance condition.

For more details on this work, please refer to [2].

4. REFERENCES

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