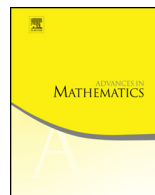




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Tropical and non-Archimedean Monge–Ampère equations for a class of Calabi–Yau hypersurfaces

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ABSTRACT

For a large class of maximally degenerate families of Calabi–Yau hypersurfaces of complex projective space, we study non-Archimedean and tropical Monge–Ampère equations, taking place on the associated Berkovich space, and the essential skeleton therein, respectively. For a symmetric measure on the skeleton, we prove that the tropical equation admits a unique solution, up to an additive constant. Moreover, the solution to the non-Archimedean equation can be derived from the tropical solution, and is the restriction of a continuous semipositive toric metric on projective space. Together with the work of Yang Li, this implies the weak metric SYZ conjecture on the existence of special Lagrangian fibrations in our setting.

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Introduction

Let $f(z) \in \mathbb{C}[z_0, \dots, z_{d+1}]$ be a generic homogeneous polynomial of degree $d+2$, where $d \geq 1$. Then

$$X := \{z_0 z_1 \dots z_{d+1} + t f(z) = 0\} \subset \mathbb{P}^{d+1} \times \mathbb{C}^* \quad (\star)$$

defines a maximally degenerate 1-parameter family of complex Calabi–Yau manifolds, polarized by $L := \mathcal{O}(d+2)|_X$. By Yau’s theorem [63], we can equip each X_t with a Ricci flat metric in the Chern class of $L|_{X_t}$. The structure of X_t as $t \rightarrow 0$ is described by two fundamental conjectures, namely the *SYZ conjecture* [60] and the *Kontsevich–Soibelman conjecture* [39]. These two conjectures have recently been related to a conjecture about solutions to the *non-Archimedean Monge–Ampère equation* [40]. In this paper we address the latter conjecture and prove a weak version of the SYZ conjecture in the setting above.

To explain all this, first note that X defines a smooth projective variety over the non-Archimedean field $K := \mathbb{C}((t))$ of complex Laurent series. Its Berkovich analytification X^{an} has a canonical closed subset $\text{Sk}(X) \subset X^{\text{an}}$, the *essential skeleton*, [39,50], which in this case can be identified with the boundary of a $(d+1)$ -dimensional simplex. The skeleton has a canonical piecewise integral affine structure, and in particular a canonical Lebesgue measure.

The Kontsevich–Soibelman conjecture states that, as $t \rightarrow 0$, X_t converges (after rescaling) in the Gromov–Hausdorff sense to a metric space whose underlying topological space is $\text{Sk}(X)$, and whose metric is determined by the solution to—roughly speaking—a real Monge–Ampère equation on the skeleton, with right hand side given by the Lebesgue measure on $\text{Sk}(X)$. Making sense of this equation is not obvious, but something that we address satisfactorily in Theorem B below in our setting.

As an alternative, one can look at the non-Archimedean Monge–Ampère equation. To any continuous semipositive metric $\|\cdot\|$ on L^{an} is associated a *Chambert–Loir measure* $c_1(L, \|\cdot\|)^d$, a positive Radon measure on X^{an} of mass $(d+2)^{d+1}$ [14,34]. By the main

results in [7,64], any positive Radon measure ν on $\text{Sk}(X)$ of this mass is the Chambert–Loir measure of a continuous semipositive metric, unique up to scaling.

When ν equals Lebesgue measure on the skeleton, it is expected that the solution to the non-Archimedean Monge–Ampère equation can be used to define the metric in the Kontsevich–Soibelman conjecture, as explored by Yang Li in his groundbreaking work [40] (see also [42]). Unfortunately, the proof in [7] is variational in nature, and does not give any information beyond continuity.

Our first main result gives a much more precise description of the solution in terms of convex functions or, put differently, toric metrics.

Theorem A. *If ν is a symmetric positive measure on $\text{Sk}(X)$ of mass $(d+2)^{d+1}$, then any solution to $c_1(L, \|\cdot\|)^d = \nu$ is the restriction of a symmetric toric metric on $\mathcal{O}_{\mathbb{P}^{d+1}}(d+2)^{\text{an}}$.*

Let us be a bit more precise. In Theorem A we assume that the polynomial $f(z)$ used to define X is *admissible* in the following sense: for any intersection Z of coordinate hyperplanes $z_j = 0$ in \mathbb{P}^{d+1} , f does not vanish identically on Z and $V(f|_Z)$ is smooth, see §7. A general polynomial is admissible.

The symmetric group S_{d+2} acts on projective space and its analytification by permuting the coordinates z_i . This action preserves $\text{Sk}(X)$, but not necessarily X^{an} . We say that a measure ν on $\text{Sk}(X)$ is *symmetric* if it is invariant under the action. For example, Lebesgue measure is symmetric.

A particular example of an admissible polynomial is the Fermat polynomial $f(z) = \sum_{i=0}^{d+1} z_i^{d+2}$. The resulting *Fermat family* is the central object in [41]. For this family, Theorem A was obtained independently by Pille-Schneider [57] in the special case when ν is the Lebesgue measure, by using the results from [41].

To prove Theorem A we study the real Monge–Ampère equation on the skeleton $\text{Sk}(X)$, as alluded to above. In doing so we exploit the structure of $X \subset \mathbb{P}^{d+1}$, as in [41]. Namely, we view \mathbb{P}^{d+1} as a toric variety with character lattice M and co-character lattice N . Let $\Delta \subset M_{\mathbb{R}}$ be the polytope for the anticanonical bundle $\mathcal{O}(d+2)$ on \mathbb{P}^{d+1} . There is a bijection between continuous semipositive toric metrics on $\mathcal{O}_{\mathbb{P}^{d+1}}(d+2)^{\text{an}}$ and convex functions $\psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$ whose Legendre transforms are continuous convex functions on Δ .

Both Δ and its polar $\Delta^{\vee} \subset N_{\mathbb{R}}$ are $(d+1)$ -dimensional simplices. It turns out that the boundary $B := \partial\Delta^{\vee}$ can be identified with the essential skeleton of X ; we therefore work on B rather than $\text{Sk}(X)$. Let $\mathcal{Q} \subset C^0(B)$ be the set of restrictions $\psi|_B$, with ψ as above, and $\mathcal{Q}_{\text{sym}} \subset \mathcal{Q}$ the subset of S_{d+2} -invariant functions.

Each d -dimensional face τ_i of B comes with an integral affine structure, and the restriction of any $\psi \in \mathcal{Q}$ to τ_i is a convex function. This allows us to define the real Monge–Ampère measure $\text{MA}_{\mathbb{R}}(\psi|_{\tau_i^{\circ}})$ on the interior τ_i° of τ_i . We show that this Monge–Ampère operator extends naturally to all of B , at least for symmetric functions. Let \mathcal{M}_{sym} denote the space of positive, symmetric measures on B of mass $(d+2)^{d+1}/d!$.

Theorem B. *There exists a unique continuous map $\mathcal{Q}_{\text{sym}} \ni \psi \mapsto \nu_\psi \in \mathcal{M}_{\text{sym}}$ such that*

$$\nu_\psi|_{\tau_i^\circ} = \text{MA}_{\mathbb{R}}(\psi|_{\tau_i^\circ}) \quad (\dagger)$$

for all $\psi \in \mathcal{Q}_{\text{sym}}$ and all i . Moreover, this map induces a homeomorphism $\mathcal{Q}_{\text{sym}}/\mathbb{R} \rightarrow \mathcal{M}_{\text{sym}}$.

The space B is an integral tropical manifold in the sense of [28], and Theorem B can be seen as solving a *tropical Monge–Ampère equation*; slightly more precisely we can define a natural integral affine structure on a subset $B_0 \subset B$, with $B \setminus B_0$ of codimension 2. Any $\psi \in \mathcal{Q}_{\text{sym}}$ can then be viewed as a convex metric on a certain affine \mathbb{R} -bundle over B_0 , in the sense of [36], with real Monge–Ampère measure $\nu_\psi|_{B_0}$; see §3.4 and §4.14 for details. While the real Monge–Ampère measure of this convex metric is only defined on B_0 , Theorem B gives a way of extending this operator over the singular set $B \setminus B_0$.

After the first draft of this paper appeared, it was pointed out to us by Rolf Andreasson that the main result of [13] directly gives a regularity result for solutions ψ to $\nu_\psi = \mu$ which implies they define smooth Hessian metrics over B_0 when μ is the Lebesgue measure on A . See [2, Theorem 3, Lemma 16 and Lemma 17] for details and an extension to other symmetric polytopes.

Combining Theorem B and its proof with the work of Li [40] we obtain a weak version of the SYZ conjecture in our setting. The SYZ conjecture predicts that X_t admits a special Lagrangian fibration for small t .¹

Corollary C. *Given $\delta > 0$, for all sufficiently small t there exists a special Lagrangian torus fibration on an open subset of X_t of normalized Calabi–Yau volume at least $1 - \delta$;*

This is stronger than the main result of [41], in which the analysis is restricted to the Fermat family, where $f(z) = \sum_{j=0}^{d+1} z_j^{d+2}$, and to subsequences X_{t_n} , with $t_n \rightarrow 0$.

In [40], Li gave an argument reducing Corollary C (as well as the corresponding statement for more general families) to a certain conjectural *comparison property* of the solution to the non-Archimedean Monge–Ampère equation. In fact, our proof of Corollary C follows [40], using a weaker version of the comparison property that we derive from Theorem A and its proof.

Li also proved a weak version of the Kontsevich–Soibelman conjecture for the Fermat family in [41]: any subsequential Gromov–Hausdorff limit of X_t as $t \rightarrow 0$ contains a dense subset locally isometric to the regular part of a Monge–Ampère metric on B_0 . The injectivity in Theorem B implies that the dense set in these subsequential Gromov–Hausdorff limits is uniquely determined up to local isometry, as is also obtained in [57].

The solution $\psi \in \mathcal{Q}_{\text{sym}}$ to the equation $\nu_\psi = \nu$, where ν is Lebesgue measure on B , can be used to state a precise version of the Kontsevich–Soibelman conjecture in this setting.

¹ Ruddat and Siebert proved that X_0 itself admits a special Lagrangian fibration, see [58].

Namely, if we knew that (the metric associated to) ψ is smooth and strictly convex on B_0 , then its Hessian would give B_0 the structure of a metric space, whose completion should be homeomorphic to B , and equal to the Gromov–Hausdorff limit of X_t as $t \rightarrow 0$. It seems plausible that having a well-posed global (tropical) Monge–Ampère equation may allow us to improve the local regularity results [23,47–49], which themselves are not sufficient, at least in dimension $d \geq 3$.

See also [17,26,27,31,32,30,46,55,56,59] for related, but slightly different, approaches to the SYZ and Kontsevich–Soibelman conjectures. In particular, a version of the Kontsevich–Soibelman conjecture is known in dimension 2 [30,56].

Strategy

We now describe the main ideas behind Theorem B. While there are satisfactory results for the Monge–Ampère equation on Hessian manifolds [16,22,36,35], extending these to general integral tropical manifolds seems challenging. Instead, our approach heavily uses the large symmetry group of B ; this allows us to adapt the variational approach in [3,5,7] for solving real, complex, and non-Archimedean Monge–Ampère equations, respectively.

More precisely, if $A := \partial\Delta \subset M_{\mathbb{R}}$, then the canonical pairing of $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ induces a *cost function* on $A \times B$, in the sense of optimal transport. From this, one defines the c -transform (generalizing the usual Legendre transform), which can be used to recover Q as the class of c -convex functions, and to define a notion of c -subgradients.

While the c -transform and c -subgradient express some pathological behavior in general, for symmetric functions, they reduce to the usual Legendre transform and subgradient when viewed in coordinate charts for the integral affine structure. For any $\psi \in \mathcal{Q}_{\text{sym}}$, we may then define ν_{ψ} as the pushforward of Lebesgue measure on A under the c -subgradient map of ψ^c , the c -transform of ψ .

Solving $\nu_{\psi} = \nu$, for a given $\nu \in \mathcal{M}_{\text{sym}}$, can now be reformulated as minimizing a certain functional $F = F_{\nu}$ on \mathcal{Q}_{sym} ; as in [3,5,7] the crucial fact that the minimizer is a solution amounts to a differentiability property for F , which we can prove in the symmetric case (and, surprisingly, fails in the non-symmetric case, see Example 4.20).

We now outline how to deduce Theorem A from Theorem B. For this, we need to explain the relation between $\text{Sk}(X)$ and B .

The variety X admits a natural model \mathcal{X} over the valuation ring $\mathbb{C}[[t]]$, given by the same equation as above in (\star) . Its special fiber \mathcal{X}_0 is the union of the coordinate hyperplanes in $\mathbb{P}_{\mathbb{C}}^{d+1}$, and the associated dual complex can be identified with $\text{Sk}(X)$. There are $d+2$ closed points $\xi_i \in \mathcal{X}_0$ where $d+1$ distinct hyperplanes meet, and the preimage of ξ_i under the specialization map $X^{\text{an}} \rightarrow \mathcal{X}_0$ is an open subset $U_i \subset X^{\text{an}}$, whose intersection with $\text{Sk}(X)$ is the relative interior $\tilde{\tau}_i^{\circ}$ of a d -dimensional simplex $\tilde{\tau}_i$; in fact, we have $\text{Sk}(X) = \bigcup_i \tilde{\tau}_i$. We have a natural retraction $U_i \rightarrow \tilde{\tau}_i^{\circ}$, and this retraction is an affinoid torus fibration.

Let $T \subset \mathbb{P}^{d+1}$ be the torus. There is a canonical tropicalization map $\text{trop}: T^{\text{an}} \rightarrow N_{\mathbb{R}}$. One can show that $\text{Sk}(X) \subset T^{\text{an}}$, and that the tropicalization map restricts to a homeomorphism of $\text{Sk}(X)$ onto B , sending $\tilde{\tau}_i$ onto τ_i for each i . On $U_i \subset X^{\text{an}}$, the tropicalization map is also invariant under the retraction to $\tilde{\tau}_i^{\circ}$, and the restriction $\text{trop}: U_i \rightarrow \tau_i^{\circ}$ is an affinoid torus fibration.

Now consider the case of a symmetric measure ν on $\text{Sk}(X) \simeq B$ that is sufficiently smooth, say equivalent to Lebesgue measure; the general case in Theorem A can be treated by approximation. Pick $\psi \in \mathcal{Q}_{\text{sym}}$ with $\nu_{\psi} = \nu$. We can extend ψ to a convex function on $N_{\mathbb{R}}$ whose Legendre transform is a symmetric continuous convex function on Δ . As already mentioned, this induces a symmetric continuous semipositive toric metric on $\mathcal{O}(d+2)^{\text{an}}$, over $\mathbb{P}^{d+1, \text{an}}$, and by restriction a continuous semipositive metric $\|\cdot\|$ on L^{an} .

By construction, the restriction of $\|\cdot\|$ to U_i can be viewed as the pullback of the convex function ψ on τ_i° . Combining (\dagger) with a theorem of Vilsmeier in [61], it follows that the Chambert-Loir measure $c_1(L, \|\cdot\|)^d$ agrees with the measure ν on an open subset of $\text{Sk}(X) \simeq B$, and hence everywhere, as this open set carries all the mass of ν .

Corollary C relies on Theorem A and the ideas of [40]. Namely, while the model \mathcal{X} above is not semistable snc, Theorem A implies that we still have the comparison property for the non-Archimedean and real Monge–Ampère operators in the sense of [40, Definition 3.11]. The arguments in [40] then go through essentially unchanged; see §9 for details.

The variational principle we developed in Theorem B has been applied in some more general contexts after the first draft of this paper appeared. In particular, in [43] it has been used to prove the SYZ for families of hypersurfaces in some toric Fano manifolds; this partially extends our approach to the non-symmetric setting, imposing however a condition on the vertices of Δ and Δ^{\vee} , which seems unfortunately rather restrictive. In [2], using to a larger extent the connections to optimal transport, Andreasson and Hultgren provide a necessary and sufficient condition for the solvability of the tropical Monge–Ampère equation on a reflexive polytope, which implies the SYZ conjecture for the corresponding family of Calabi–Yau hypersurfaces.

Structure

The paper is organized as follows: after a discussion of the toric setup and the structure of B as a tropical manifold, we introduce in §3 the class of c-convex functions, and show their basic properties. In §4, we define the Monge–Ampère operator on the subclass of symmetric c-convex functions, and in §5 we solve the tropical Monge–Ampère equation, proving Theorem B. The relation between c-convex functions and toric metrics on the Berkovich analytification on $\mathcal{O}(d+2)$ is explored in §6, whereas the restriction of the tropicalization map to X^{an} is studied in §7. After that, combining all the ingredients, we prove Theorem A in §8 and Corollary C in §9.

Notation

Given a variety X over a non-Archimedean field K , we denote by X^{an} the Berkovich analytification of X , and by $X^{\text{val}} \subset X^{\text{an}}$ the subset of valuations on the function field of X extending the valuation on K . Given an abelian group Γ , we set $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$. If g is a convex function on \mathbb{R}^n , then the subgradient $\partial g(x)$ at $x \in \mathbb{R}^n$ is the set of linear functions $\ell \in (\mathbb{R}^n)^*$ such that the function $g - \ell$ attains its minimum at x . The (real) Monge–Ampère measure $\text{MA}_{\mathbb{R}}(g)$ of g is taken in the sense of Alexandrov, i.e. as the Lebesgue measure of the subgradient image, see e.g. [23, §2.1].

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1. Toric setup

Fix an integer $d \geq 1$. Our toric terminology largely follows [25].

1.1. Lattices and tori

Consider the lattice $M' := \mathbb{Z}^{d+2}$ with basis

$$e_0 = (1, 0, \dots, 0), \dots, e_{d+1} = (0, \dots, 0, 1).$$

Let $T' := \text{Spec } K[M'] \simeq \mathbb{G}_{\text{m}}^{d+2}$ be the corresponding (split) torus. Each $m \in M'$ defines a character on T' . If we denote by z_i the character associated to the basis element e_i , then the character associated to a general element $m = (y_0, \dots, y_{d+1}) \in M'$ is given by

$$z^m := z_0^{y_0} \cdots z_{d+1}^{y_{d+1}}$$

Define a sublattice $M \subset M'$ by $M = \{y \in \mathbb{Z}^{d+2} \mid \sum_0^{d+1} y_i = 0\}$. For any $i \in \{0, \dots, d+1\}$ the set $\{e_j - e_i\}_{j \neq i}$ forms a basis for M . Let $T := \text{Spec } K[M] \simeq \mathbb{G}_{\text{m}}^{d+1}$ be the associated torus. The inclusion $M \subset M'$ induces a morphism $T' \rightarrow T$, allowing us to view T as a quotient of T' . The characters z_i on T' can be viewed as homogeneous coordinates on T .

Set $N' := \text{Hom}(M', \mathbb{Z})$ and $N := \text{Hom}(M, \mathbb{Z})$. Then $N' \simeq \mathbb{Z}^{d+2}$ and

$$N \simeq \mathbb{Z}^{d+2} / \mathbb{Z}(1, \dots, 1).$$

1.2. Tropicalization

We use ‘additive’ conventions for valuations and semivaluations. Thus T^{an} is the set of semivaluations $v: K[M] \rightarrow \mathbb{R} \cup \{+\infty\}$ restricting to the given valuation on K , and equipped with the topology of pointwise convergence. We have a tropicalization map

$$\text{trop}: T^{\text{an}} \rightarrow N_{\mathbb{R}} = \text{Hom}(M, \mathbb{R})$$

characterized by

$$\langle m, \text{trop}(v) \rangle = -v(z^m)$$

for all $m \in M$. This map is continuous and surjective. It admits a natural continuous one-sided inverse, which to $n \in N_{\mathbb{R}}$ associates the valuation $v_n \in T^{\text{val}} \subset T^{\text{an}}$ defined by

$$v_n \left(\sum_{m \in M} a_m z^m \right) = \min_m \{ -v(a_m) - \langle m, n \rangle \};$$

this is the minimal element in the fiber $\text{trop}^{-1}(n)$, with respect to the natural partial ordering on T^{an} .

1.3. Simplices and projective space

Let $\Delta \subset M_{\mathbb{R}}$ be the convex hull of the elements

$$m_i := (d+1)e_i - \sum_{j \neq i} e_j \in M, \quad i = 0, \dots, d+1.$$

Then Δ is a simplex, whose polar polytope²

$$\Delta^{\vee} := \{n \in N_{\mathbb{R}} \mid \sup_{m \in \Delta} \langle m, n \rangle = \max_{0 \leq i \leq d+1} \langle m_i, n \rangle \leq 1\},$$

is also a simplex, with vertices given by

$$n_0 = (-1, 0, \dots, 0), \dots, n_{d+1} = (0, \dots, 0, -1).$$

² We use a different sign convention from [41].

The fan in $N_{\mathbb{R}}$ dual to Δ has rays generated by n_i , $0 \leq i \leq d+1$, and defines a toric variety that we identify with \mathbb{P}^{d+1} . In fact, Δ is the moment polytope for the anticanonical bundle $\mathcal{O}(d+2)$ on \mathbb{P}^{d+1} , and the unique effective torus invariant anticanonical divisor on \mathbb{P}^{d+1} is given by $-K_{\mathbb{P}^{d+1}} = \sum_{i=0}^{d+1} D_i$, where D_i is the prime divisor on \mathbb{P}^{d+1} corresponding to n_i .

For later reference, we note that

$$\langle m_i, n_j \rangle = \begin{cases} -(d+1) & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} \quad (1.1)$$

We can view z_0, \dots, z_{d+1} as homogeneous coordinates on \mathbb{P}^{d+1} . For any $m \in M$, z^m is a rational function on \mathbb{P}^{d+1} . If $m \in \Delta \cap M$, then z^m can be viewed as a global section of $\mathcal{O}(d+2) = \mathcal{O}(-K_{\mathbb{P}^{d+1}})$, in the sense that $\operatorname{div}(z^m) - K_{\mathbb{P}^{d+1}} \geq 0$. More generally, for any $r \geq 1$, the set

$$\{z^m \mid m \in r\Delta \cap M\}$$

is a basis for $H^0(\mathbb{P}^{d+1}, \mathcal{O}(r(d+2)))$.

There is an alternative description in which a global section of $\mathcal{O}(r(d+2))$ is given as a homogeneous polynomial in the z_i of degree $r(d+2)$. Given $m \in r\Delta \cap M$, define a monomial

$$\chi^{r,m} := z^m \prod_{i=0}^{d+1} z_i^r.$$

Then $(\chi^{r,m})_{m \in r\Delta \cap M}$ is a basis of the space of homogeneous polynomials of degree $r(d+2)$ in the z_i , and hence a basis for $H^0(\mathbb{P}^{d+1}, \mathcal{O}(r(d+2)))$. Note that the sections $\chi^{r,rm_i} = z_i^{r(d+2)}$, $0 \leq i \leq d+1$, have no common zeros.

2. Tropical manifolds

Above we defined simplices $\Delta \subset M_{\mathbb{R}}$ and $\Delta^{\vee} \subset N_{\mathbb{R}}$. Their boundaries

$$A := \partial\Delta \quad \text{and} \quad B := \partial\Delta^{\vee}$$

will be key players in what follows. As we will see, they are integral tropical manifolds in the sense of [29]. The exposition below more or less follows [41].

The spaces A and B are naturally equipped with piecewise integral affine structures, and hence a canonical volume form that we refer to as Lebesgue measure. The total mass of A and B is $|A| = (d+2)^{d+1}/d!$ and $|B| = (d+2)/d!$, respectively. It will occasionally be convenient to parametrize A and B as follows:

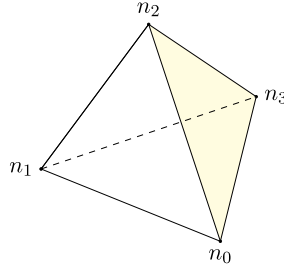


Fig. 1. Subset τ_1 for $d = 2$.

$$A = \left\{ \sum_j \alpha_j m_j \mid \alpha_j \in \mathbb{R}, \min_j \alpha_j = 0, \sum_j \alpha_j = 1 \right\} \quad (2.1)$$

$$B = \left\{ \sum_j \beta_j n_j \mid \beta_j \in \mathbb{R}, \min_j \beta_j = 0, \sum_j \beta_j = 1 \right\}. \quad (2.2)$$

2.1. Singular integral affine structure

Following [29,41], we now upgrade the piecewise integral structures on A and B to singular integral affine structures. This means that we have open dense subsets $A_0 \subset A$ and $B_0 \subset B$, of real codimension 2, such that A_0 and B_0 each admit a sheaf of integral affine functions.

In general, there is a great deal of flexibility in the choice of A_0 and B_0 , see e.g. [51]. We will, however, be interested in symmetric data on A and B , i.e. data invariant under the action of the permutation group $G = S_{d+2}$ on A and B . This gives a canonical choice of our singular set, namely, the barycentric complexes of the $(d-1)$ -dimensional faces of A and B .

Let us now be more precise. First consider the d -dimensional faces of A and B . These are of the form

$$\sigma_i := \{ \max_j n_j = n_i = 1 \} \subset M_{\mathbb{R}} \quad \text{and} \quad \tau_i := \{ \max_j m_j = m_i = 1 \} \subset N_{\mathbb{R}}$$

for $0 \leq i \leq d+1$, and we write $\sigma_i^\circ, \tau_i^\circ$ for the relative interiors (Fig. 1). The integral affine functions on σ_i° (resp. τ_i°) are the restrictions of the integral affine functions on $M_{\mathbb{R}}$ (resp. $N_{\mathbb{R}}$).

Second, we can define the integral affine structure near vertices of A and B , respectively. Let $\text{Star}(m_i) = \bigcup_{j \neq i} \sigma_j$ be the closed star of m_i , and $\text{Star}^\circ(m_i) = A \setminus \sigma_i$ the open star. The stars $\text{Star}(n_i)$ and $\text{Star}^\circ(n_i)$ are defined analogously.

As follows from (1.1), given $i \neq j$, the integral linear map $M_{\mathbb{R}} \rightarrow \mathbb{R}^d$ given by

$$m \mapsto (\langle m, n_j - n_k \rangle)_{k \neq i, j} \quad (2.3)$$

restricts to a piecewise integral affine isomorphism $\text{Star}(m_i) \xrightarrow{\sim} \tilde{S}$, where $\tilde{S} \subset \mathbb{R}^d$ is the simplex with vertices given by

$$(d+2, 0, \dots, 0), \dots, (0, \dots, d+2), \text{ and } -(d+2), \dots, -(d+2).$$

It will be notationally convenient to denote the map in (2.3) by

$$p_{i,j}^{-1}: \text{Star}(m_i) \xrightarrow{\sim} \tilde{S}.$$

In this way, the inverse $p_{i,j}: \tilde{S} \rightarrow \text{Star}(m_i)$ is an integral piecewise affine isomorphism, whose restriction to any simplex spanned by the origin and d of the vertices of \tilde{S} above is an integral affine isomorphism onto a simplex σ_k , $k \neq i$, when σ_k is endowed with the integral affine structure above. We view $p_{i,j}^{-1}$ as coordinates on $\text{Star}(m_i)$.

By using Proposition 2.2 below, one can easily check that for any $j, k, \ell \neq i$, the function $(n_k - n_\ell) \circ p_{i,j}: \tilde{S} \rightarrow \mathbb{R}$ is the restriction of an integral linear function on \mathbb{R}^d . From this, it follows that $p_{i,k}^{-1} \circ p_{i,j}: \tilde{S} \rightarrow \tilde{S}$ is the restriction of an integral linear isomorphism of \mathbb{R}^d .

Similarly, we define coordinates on $\text{Star}(n_i)$ by:

$$q_{i,j}^{-1}(n) = \left(\left\langle \frac{m_k - m_j}{d+2}, n \right\rangle \right)_{k \neq i,j} = (\langle e_k - e_j, n \rangle)_{k \neq i,j} \in \mathbb{R}^d. \quad (2.4)$$

Note the sign change, which makes the duality pairing in the charts compatible with the global pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, see Proposition 2.2. We get a piecewise integral affine isomorphism

$$q_{i,j}: \tilde{T} \xrightarrow{\sim} \text{Star}(n_i),$$

where $\tilde{T} \subset \mathbb{R}^d$ is the simplex spanned by

$$(-1, 0, \dots, 0), \dots, (0, \dots, -1), \text{ and } (1, \dots, 1).$$

If $j, k \neq i$, then $q_{i,k}^{-1} \circ q_{i,j}: \tilde{T} \rightarrow \tilde{T}$ is the restriction of an integral linear isomorphism of \mathbb{R}^d , and for $j, k, l \neq i$, $(m_k - m_l) \circ q_{i,j}: \tilde{T} \rightarrow \mathbb{R}$ is the restriction of an integral linear function on \mathbb{R}^d .

As $p_{i,j}$ and $q_{i,j}$ are integral piecewise integral isomorphisms, they map Lebesgue measure on \mathbb{R}^d to Lebesgue measure on A and B , respectively.

It is tempting to define integral affine structures on $\text{Star}^\circ(m_i)$ and $\text{Star}^\circ(n_i)$ by pulling back the sheaf on integral affine functions on \tilde{S}° and \tilde{T}° , respectively. However, these sheaves don't agree on the overlaps; we need to define branch cuts in the above charts in order to work globally on A and B . This corresponds to choosing the singular part of the singular affine structure, which again we will canonically choose to be the barycentric complex of the $(d-1)$ -dimensional faces.

To describe this explicitly, define subsets $S_i \subset \text{Star}(m_i)$ and $T_i \subset \text{Star}(n_i)$ by

$$S_i := \{n_i = \min_j n_j\} \quad \text{and} \quad T_i := \{m_i = \min_j m_j\}.$$

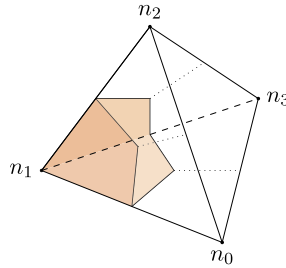


Fig. 2. Subset T_1 for $d = 2$.

Their relative interiors are given by $S_i^\circ = \{n_i < \min_{j \neq i} n_j\}$ and $T_i^\circ = \{m_i < \min_{j \neq i} m_j\}$, respectively, and are open neighborhoods of m_i and n_i in A and B , respectively (Fig. 2). Note that $S_i^\circ \cap S_j^\circ = \emptyset$ and $T_i^\circ \cap T_j^\circ = \emptyset$ if $i \neq j$. We can easily describe these sets in terms of the parametrizations (2.1) and (2.2); for example,

$$S_i = \left\{ \sum_j \alpha_j m_j \mid \alpha_i \geq \max_{j \neq i} \alpha_j \geq \min_{j \neq i} \alpha_j = 0, \sum_j \alpha_j = 1 \right\}.$$

We now define the integral affine structure on S_i° and T_i° as the pullback of the integral affine structures on \mathbb{R}^d under the maps $p_{i,j}^{-1}$ and $q_{i,j}^{-1}$, respectively. This is compatible with the integral affine structure on the open simplices σ_i° and τ_i° as above. Moreover, the integral affine structures on S_i° and S_j° (resp. T_i° and T_j°) are trivially compatible for $i \neq j$, since $S_i^\circ \cap S_j^\circ = \emptyset$ (resp. $T_i^\circ \cap T_j^\circ = \emptyset$). We therefore obtain integral affine structures on

$$A_0 := \bigcup_i \sigma_i^\circ \cup \bigcup_i S_i^\circ \quad \text{and} \quad B_0 := \bigcup_i \tau_i^\circ \cup \bigcup_i T_i^\circ,$$

and $A \setminus A_0, B \setminus B_0$ have codimension two.

2.2. Pairing and symmetries

The pairing $M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ restricts to a pairing

$$A \times B \rightarrow \mathbb{R}.$$

Given $m \in A$ and $n \in B$, write $m = \sum_{j=0}^{d+1} \alpha_j m_j$ and $n = \sum_{j=0}^{d+1} \beta_j n_j$, where $\min_j \alpha_j = \min_j \beta_j = 0$ and $\sum_j \alpha_j = \sum_j \beta_j = 1$. Using (1.1) we then have

$$\langle m, n \rangle = 1 - (d+2) \sum_j \alpha_j \beta_j. \quad (2.5)$$

In §4 it will be important to understand how the pairing interacts with the action of the permutation group $G = S_{d+2}$ on $M' = \mathbb{Z}^{d+2}$, and its various induced actions. Note

that G acts on the sets of simplices σ_i , τ_i and stars $\text{Star}(m_i)$, $\text{Star}(n_i)$, S_i , T_i , mapping relative interiors to relative interiors. We also have $\langle g(m), g(n) \rangle = \langle m, n \rangle$ for $m \in A$, $n \in B$, but not always $\langle m, g(n) \rangle = \langle m, n \rangle$.

Lemma 2.1. *Pick any $m \in A$, $n \in B$, and let $G(m, n) \subset G$ be the set of $g \in G$ such that $\langle m, g(n) \rangle$ is maximal. Then, for any $i \in \{0, 1, \dots, d+1\}$ we have:*

- (i) if $m \in \sigma_i$ (resp. $m \in S_i$), then $g(n) \in T_i$ (resp. $g(n) \in \tau_i$) for some $g \in G(m, n)$;
- (i') if $m \in \sigma_i^\circ$ (resp. $m \in S_i^\circ$), then $g(n) \in T_i$ (resp. $g(n) \in \tau_i$) for all $g \in G(m, n)$;
- (ii) if $n \in \tau_i$ (resp. $n \in T_i$), then $g(m) \in S_i$ (resp. $g(m) \in \sigma_i$) for some $g \in G(m, n)$;
- (ii') if $n \in \tau_i^\circ$ (resp. $n \in T_i^\circ$), then $g(m) \in S_i$ (resp. $g(m) \in \sigma_i$) for all $g \in G(m, n)$.

Proof. It suffices to prove (i) and (i'); the proofs of (ii) and (ii') are analogous. Write $m = \sum_j \alpha_j m_j$ and $n = \sum_j \beta_j n_j$, with $\min_j \alpha_j = \min_j \beta_j = 0$ and $\sum_j \alpha_j = \sum_j \beta_j = 1$.

To prove (i), suppose $m \in \sigma_i$ (resp. $m \in S_i$), so that $\alpha_i = 0$ (resp. $\alpha_i = \max_l \alpha_l$). Pick any $g' \in G(m, n)$, and choose j such that $g'(n) \in T_j$ (resp. $g'(n) \in \tau_j$), that is, $\beta_{g'^{-1}(j)} = \max_l \beta_l$ (resp. $\beta_{g'^{-1}(j)} = 0$). Set $g = h \circ g'$, where $h \in G$ is the transposition of $\{0, 1, \dots, d+1\}$ exchanging i and j . Then $g(n) \in T_i$ (resp. $g(n) \in \tau_i$), and we claim that $g \in G(m, n)$. But (2.5) implies

$$\langle m, g(n) - g'(n) \rangle = (d+2)(\alpha_j - \alpha_i)(\beta_{g'^{-1}(j)} - \beta_{g'^{-1}(i)}) \geq 0.$$

The proof of (i') is similar. Assume $m \in \sigma_i^\circ$ (resp. $m \in S_i^\circ$), so that $\min_{j \neq i} \alpha_j > \alpha_i = 0$ (resp. $\alpha_i > \max_{j \neq i} \alpha_j$). It suffices to prove that if $n \notin T_i$ (resp. $n \notin \tau_i$), then there exists $g \in G$ such that $\langle m, g(n) - n \rangle > 0$. But $n \notin T_i$ (resp. $n \notin \tau_i$) means that $\beta_j > \beta_i$ for some j (resp. $\beta_i > 0$). Let $g \in G$ be transposition exchanging i and j . Then

$$\langle m, g(n) - n \rangle = (d+2)(\alpha_j - \alpha_i)(\beta_j - \beta_i) > 0,$$

completing the proof. \square

2.3. Pairing in coordinate charts

Lemma 2.1 suggests that the pairing between A and B is most natural between σ_i and $\text{Star}(n_i)$, or between $\text{Star}(m_i)$ and τ_i . We now calculate the pairing between elements in compatible coordinate charts defined on these regions.

Proposition 2.2. *Fix indices $i \neq j$. For $x \in p_{j,i}^{-1}(\sigma_i)$ and $y \in \tilde{T} = q_{i,j}^{-1}(\text{Star}(n_i))$, we have:*

$$\langle x, y \rangle = \langle p_{j,i}(x) - m_j, q_{i,j}(y) \rangle. \quad (2.6)$$

Similarly, for all $x \in \tilde{S} = p_{i,j}^{-1}(\text{Star}(m_i))$ and $y \in q_{j,i}^{-1}(\tau_i)$, we have

$$\langle x, y \rangle = \langle p_{i,j}(x), q_{j,i}(y) - n_j \rangle. \quad (2.7)$$

Note that the pairing on the right-hand sides of (2.6), (2.7) is between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, while the pairing on the left-hand side is the scalar product on \mathbb{R}^d .

Proof. Pick $m \in \sigma_i \subset \text{Star}(m_j)$ and $n \in \text{Star}(n_i)$. Write $m = \sum_{k \neq i} \alpha_k m_k$ and $n = \sum_k \beta_k n_k$, where $\alpha_k, \beta_k \geq 0$ and $\sum_k \alpha_k = \sum_k \beta_k = 1$. Then $m - m_j = \sum_{k \neq i, j} \alpha_k (m_k - m_j)$, so that

$$\langle m - m_j, n \rangle = (d+2) \sum_{k \neq i, j} \alpha_k (\beta_j - \beta_k)$$

in view of (1.1). On the other hand, (2.3) and (2.4) give

$$p_{j,i}^{-1}(m) = (d+2)(\alpha_k)_{k \neq i, j} \quad \text{and} \quad q_{i,j}^{-1}(n) = (\beta_j - \beta_k)_{k \neq i, j},$$

which implies $\langle m - m_j, n \rangle = \langle p_{j,i}^{-1}(m), q_{i,j}^{-1}(n) \rangle$. We now obtain (2.6) by inverting the coordinate maps, and (2.7) is proved in the same way. \square

3. The c-transform and the class of c-convex functions

Denote by $L^\infty(A)$ and $L^\infty(B)$ the space of bounded real-valued functions on A and B , respectively.

3.1. General definitions and properties

We start by defining the *c-transforms*

$$L^\infty(A) \rightarrow L^\infty(B) \quad \text{and} \quad L^\infty(B) \rightarrow L^\infty(A)$$

as follows. Given $\phi \in L^\infty(A)$, we define a new function $\phi^c \in L^\infty(B)$ by

$$\phi^c(n) := \sup_{m \in A} \langle m, n \rangle - \phi(m). \quad (3.1)$$

Note that ϕ^c is bounded since $-(d+1) \leq \langle m, n \rangle \leq 1$. Similarly, given $\psi \in L^\infty(B)$, we define $\psi^c \in L^\infty(A)$ by

$$\psi^c(m) := \sup_{n \in B} \langle m, n \rangle - \psi(n). \quad (3.2)$$

Remark 3.1. The c-transform in this setting is inspired by the usual one in optimal transport [1], and can be defined much more generally, e.g. when $X = A$ and $Y = B$ are replaced by arbitrary sets, and $\langle m, n \rangle$ by an arbitrary ‘cost’ function $c: X \times Y \rightarrow \mathbb{R}$. In that generality, ϕ^c and ψ^c may take infinite values, but our cost function is uniformly bounded, so we can restrict to bounded functions.

It is a general fact that $(\phi + a)^c = \phi^c - a$ and $(\psi + a)^c = \psi^c - a$ for any bounded functions ϕ, ψ and any constant a . Moreover, if $\phi_1 \leq \phi_2$, then $\phi_1^c \geq \phi_2^c$, and similarly for the c -transform in the other direction. This formally implies that the c -transforms are contractive: $\|\phi_1^c - \phi_2^c\| \leq \|\phi_1 - \phi_2\|$ and $\|\psi_1^c - \psi_2^c\| \leq \|\psi_1 - \psi_2\|$ for $\phi_i \in L^\infty(A)$ and $\psi_i \in L^\infty(B)$, where $\|\cdot\|$ denotes the sup norm.

In our case, we also have $0^c = 1$, as follows from $\max_{m \in A} \langle m, n \rangle = 1$ for all $n \in B$ and $\max_{n \in B} \langle m, n \rangle = 1$ for all $m \in A$.

Lemma 3.2. *For any bounded functions $\phi: A \rightarrow \mathbb{R}$ and $\psi: B \rightarrow \mathbb{R}$, we have $\phi^{cc} \leq \phi$, $\psi^{cc} \leq \psi$, $\phi^{ccc} = \phi^c$, and $\psi^{ccc} = \psi^c$.*

Proof. This is formal, see [1, p. 8]. \square

Definition 3.3. We define $\mathcal{P} \subset L^\infty(A)$ and $\mathcal{Q} \subset L^\infty(B)$ as the images of the c -transform,

$$\mathcal{P} := \{\phi = v^c \mid v \in L^\infty(B)\} \quad \text{and} \quad \mathcal{Q} := \{\psi = u^c \mid u \in L^\infty(A)\},$$

and equip \mathcal{P} and \mathcal{Q} with the supremum norm.

The functions in \mathcal{P} and \mathcal{Q} are called *c-convex*. It follows from the remarks above that the spaces \mathcal{P} and \mathcal{Q} of c -convex functions are invariant under the addition of a real constant, and they consist of bounded functions. They also contain all constant functions.

Lemma 3.4. *The c -transform defines isometric bijections $\mathcal{P} \rightarrow \mathcal{Q}$ and $\mathcal{Q} \rightarrow \mathcal{P}$ that are inverse to each other.*

Proof. By Lemma 3.2, the two maps are bijective, and inverse to one another. As they are both contractive, they must be isometries. \square

Lemma 3.5. *The functions in \mathcal{P} and \mathcal{Q} are uniformly Lipschitz continuous.*

Proof. Suppose ψ is a bounded function on B . By definition, $\psi^c(m) = \sup_{n \in B} (\langle m, n \rangle - \psi(n))$; this defines a locally bounded function on $M_{\mathbb{R}}$. Each of the $\langle m, n \rangle - \psi(n)$ is linear, with uniform Lipschitz constant, since B is compact. It follows that $\psi^c(m)$ is also Lipschitz on $M_{\mathbb{R}}$, with the same constant. The same argument obviously works for \mathcal{Q} . \square

Corollary 3.6. *The spaces \mathcal{P} and \mathcal{Q} are closed subspaces of $C^0(A)$ and $C^0(B)$, respectively. Moreover, \mathcal{P}/\mathbb{R} and \mathcal{Q}/\mathbb{R} are compact.*

Proof. Lemma 3.5 shows that $\mathcal{P} \subset C^0(A)$. To prove that \mathcal{P} is a closed subspace, consider a sequence $(\phi_k)_1^\infty$ in \mathcal{P} converging uniformly to $\phi \in C^0(A)$. Then $\phi_k^{cc} = \phi_k$ for all k , so

since the double c -transform is a continuous (even contractive) map from $C^0(A) \rightarrow \mathcal{P}$, we must have $\phi^{cc} = \phi$, so that $\phi \in \mathcal{P}$. Thus \mathcal{P} is closed.

To prove that \mathcal{P}/\mathbb{R} is compact, it suffices to show that the closed subspace $\mathcal{P}_0 := \{\phi \in \mathcal{P} \mid \max \phi = 0\}$ is compact. But Lemma 3.5 shows that the functions in \mathcal{P}_0 are uniformly bounded, and equicontinuous, so we conclude using the Arzelà–Ascoli theorem.

The same argument shows that $\mathcal{Q} \subset C^0(B)$ is closed and that \mathcal{Q}/\mathbb{R} is compact. \square

Remark 3.7. For any subset $A' \subset A$ and any bounded function $\phi: A' \rightarrow \mathbb{R}$, the function $\psi: B \rightarrow \mathbb{R}$ defined by $\psi = \sup_{m \in A'} (m - \phi(m))$ is c -convex. Indeed, ψ is the c -transform of the extension of ϕ to A defined by $\phi|_{A \setminus A'} \equiv \sup_{A'} \phi + d + 2$.

Lastly, we have the following definition, also standard in the optimal transport literature:

Definition 3.8. Given $\psi \in \mathcal{Q}$, the c -subgradient of ψ is the multi-valued map $\partial^c \psi: B \rightarrow A$ given by

$$(\partial^c \psi)(n) := \{m \in A \mid \psi(n) + \psi^c(m) = \langle m, n \rangle\}$$

for any $n \in B$.

By continuity, the c -subgradient is nonempty. When it is a singleton, we call it a c -gradient. We make similar definitions for $\phi \in \mathcal{P}$. It is evident that for $\psi \in \mathcal{Q}$, $m \in A$ and $n \in B$, we have $m \in (\partial^c \psi)(n)$ iff $n \in (\partial^c \psi^c)(m)$, so that $\partial^c \psi$ and $\partial^c \psi^c$ are inverses, in the sense of multi-valued maps.

Example 3.9. Let $\psi = \max_i m_i \equiv 1 \in \mathcal{Q}_{\text{sym}}$ where the max is taken over the vertices of A . Then

$$\partial^c \psi^c(n) = \{m \in A : \langle m, n \rangle = 1\}$$

is the face in A dual to the smallest face in B containing n .

3.2. Extension property

In [41], Li studies the class of functions on A and B which satisfy what he calls the extension property, motivated in part by extension theorems for (quasi-)plurisubharmonic functions: see e.g. [19,18,62,54,20]. Here, similarly to [41, Proposition 3.19], we show that these extendable functions are exactly those in \mathcal{P} and \mathcal{Q} , and discuss their canonical extensions to $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$.

We set some notation. As in [3], let \mathcal{P}_+ be the set of convex functions $\phi: M_{\mathbb{R}} \rightarrow \mathbb{R}$ such that $\phi = \max_j n_j + O(1)$, and \mathcal{Q}_+ the set of convex functions $\psi: N_{\mathbb{R}} \rightarrow \mathbb{R}$ with $\psi = \max_j m_j + O(1)$. Using (3.1) and (3.2), we can view the c -transforms as maps

$$L^\infty(A) \rightarrow \mathcal{Q}_+ \quad \text{and} \quad L^\infty(B) \rightarrow \mathcal{P}_+,$$

so that all functions in \mathcal{P} and \mathcal{Q} come from restrictions of functions in \mathcal{P}_+ and \mathcal{Q}_+ . The following proposition shows the converse:

Proposition 3.10. *Suppose that $\psi \in \mathcal{Q}_+$. Then $\psi^{cc} \geq \psi$ on $N_{\mathbb{R}} \setminus (\Delta^\vee)^\circ$. It follows that $\psi^{cc} = \psi$ on $B = \partial\Delta^\vee$. The corresponding statements hold for $\phi \in \mathcal{P}_+$.*

Proof. It suffices to prove $\psi^{cc} \geq \psi$ on $N_{\mathbb{R}} \setminus (\Delta^\vee)^\circ$. Indeed, the inequality $\psi^{cc} \leq \psi$ on B is formal, see Lemma 3.2.

Pick any $n_0 \in N_{\mathbb{R}} \setminus (\Delta^\vee)^\circ$. To see that $\psi(n_0) \leq \psi^{cc}(n_0)$, it will suffice to find an $m \in A$ such that:

$$\psi(n_0) \leq \langle m, n_0 \rangle - \psi^c(m),$$

since the right-hand side is dominated by $\psi^{cc}(n_0)$. Let m' be a subgradient of ψ at n_0 , i.e.

$$\psi(n) \geq \langle m', n - n_0 \rangle + \psi(n_0),$$

for all $n \in N_{\mathbb{R}}$. Since $\psi \in \mathcal{Q}_+$, the subgradients for ψ satisfy $\partial\psi(N_{\mathbb{R}}) \subseteq \Delta$ (see e.g. [3, Lemma 2.5]). Also, as n_0 is not in the interior of Δ^\vee , we can find a hyperplane, represented by $m_0 \in M_{\mathbb{R}}$, such that $\sup_{n \in B} \langle m_0, n \rangle \leq \langle m_0, n_0 \rangle$.

Now let $\lambda \geq 0$ be such that $m := m' + \lambda m_0 \in A$. Then we have that:

$$\begin{aligned} \psi^c(m) &= \sup_{n \in B} \langle m, n \rangle - \psi(n) \leq -\psi(n_0) + \sup_{n \in B} \langle m' + \lambda m_0, n \rangle + \langle m', n_0 - n \rangle \\ &= \langle m', n_0 \rangle - \psi(n_0) + \lambda \sup_{n \in B} \langle m_0, n \rangle \leq \langle m, n_0 \rangle - \psi(n_0), \end{aligned}$$

and we are done. \square

Corollary 3.11. *The spaces \mathcal{P} and \mathcal{Q} are convex.*

Proof. This is clear since the spaces \mathcal{P}_+ and \mathcal{Q}_+ are convex. \square

Remark 3.12. Unlike the plurisubharmonic case, functions in \mathcal{P} (resp. \mathcal{Q}) admit a canonical extension to $M_{\mathbb{R}}$ (resp. $N_{\mathbb{R}}$), namely the supremum of all such extensions. We omit the proof.

3.3. Convexity in coordinate charts

Following Li [41], we show that the functions in \mathcal{P} and \mathcal{Q} are convex in the coordinate charts defined in §2.1, up to adding a piecewise linear term.

Lemma 3.13. [41, Proposition 3.26] If $\psi \in \mathcal{Q}$, then for any $i \neq j$, the function

$$\psi_{i,j} := (\psi - m_j) \circ q_{i,j}$$

is convex on $q_{i,j}^{-1}(\text{Star}(n_i))$. As a consequence, $\psi \circ q_{i,j}$ is convex on $q_{i,j}^{-1}(\tau_k)$ for any $k \neq i$. Similarly, if $\phi \in \mathcal{P}$, then for any $i \neq j$, the function

$$\phi_{i,j} := (\phi - n_j) \circ p_{i,j}$$

is convex on $p_{i,j}^{-1}(\text{Star}(m_i))$, and $\phi \circ p_{i,j}$ is convex on $p_{i,j}^{-1}(\sigma_k)$ for any $k \neq i$.

In the terminology of [41], the lemma says that the functions in \mathcal{P} and \mathcal{Q} are locally convex.

Proof. We prove the statement about \mathcal{Q} ; functions in \mathcal{P} are handled in the same way. Thus pick $\psi \in \mathcal{Q}$. We shall in fact prove the following: if $n \in \tau_j \subset \text{Star}(n_i)$ and $m \in \partial^c \psi(n)$, then $p_{j,i}^{-1}(m)$ is a subgradient for $\psi_{i,j}$ at $q_{i,j}^{-1}(n)$ (here we are thinking of $p_{j,i}^{-1}$ as a global map from A to \mathbb{R}^d , and make no assumption on where m is inside A). Accepting this, and noting that $\partial^c \psi(n)$ is non-empty for any $n \in B$ by compactness, it follows that $\psi_{i,j}$ is convex on $\text{Star}(n_i)$. The proposition then follows by noting that

$$\psi_{i,k} = (\psi_{i,j} + (m_j - m_k) \circ q_{i,j}) \circ q_{i,j}^{-1} \circ q_{i,k},$$

$(m_j - m_k) \circ q_{i,j}$ is affine on $q_{i,j}^{-1}(\text{Star}(n_i))$ for any $j, k \neq i$, and that the maps $q_{i,j}^{-1} \circ q_{i,k}$ are linear on $q_{i,k}^{-1}(\text{Star}(n_k))$.

First, from the definition of the c -subgradient, we have:

$$\psi(n) = \langle m, n \rangle - \psi^c(m) \leq \langle m, n - n' \rangle + \psi(n')$$

for all $n' \in \text{Star}(n_i)$. With $y := q_{i,j}^{-1}(n)$, $y' := q_{i,j}^{-1}(n')$, we have that

$$\psi_{i,j}(y') - \psi_{i,j}(y) \geq \langle m - m_j, n' - n \rangle,$$

and it remains to estimate the right-hand side in terms of the coordinates.

We can write $m = m' + rm_i$, where $m' \in \sigma_i$ and $r \geq 0$. Indeed, if $m = \sum_k \alpha_k m_k$, then we can pick $m' = \sum_{k \neq i} (\alpha_k + \frac{\alpha_i}{d+1}) m_k$ and $r = 2\alpha_i$. Now set $x := p_{j,i}^{-1}(m)$, $x' := p_{j,i}^{-1}(m')$, and $x_i := p_{j,i}^{-1}(m_i)$. By Proposition 2.2, we have

$$\langle x', y' - y \rangle = \langle m' - m_j, n' - n \rangle.$$

On the other hand, a direct calculation as in the proof of Proposition 2.2 yields

$$\langle x_i, y \rangle = \langle m_i, n \rangle + (d+1)\langle m_j, n \rangle \quad \text{and} \quad \langle x_i, y' \rangle = \langle m_i, n' \rangle + (d+1)\langle m_j, n' \rangle.$$

By linearity, $x = x' + rx_i$, and hence

$$\begin{aligned}\langle x, y' - y \rangle &= \langle m' - m_j, n' - n \rangle + r(\langle m_i, n' - n \rangle + (d+1)\langle m_j, n' - n \rangle) \\ &= \langle m - m_j, n' - n \rangle + r(d+1)\langle m_j, n' - n \rangle \leq \langle m - m_j, n' - n \rangle,\end{aligned}$$

where the inequality holds since $n \in \tau_j$ implies $\langle m_j, n \rangle = 1 \geq \langle m_j, n' \rangle$. Altogether, this yields

$$\psi_{i,j}(y') - \psi_{i,j}(y) \geq \langle x, y' - y \rangle,$$

and completes the proof. \square

3.4. A principal \mathbb{R} -bundle

We can interpret the convexity statement in Lemma 3.13 geometrically as follows. For $0 \leq j \leq d+1$, set $Y_j := N_{\mathbb{R}}$, and define a topological space Λ by $\Lambda := \coprod_j Y_j \times \mathbb{R} / \sim$, where $(n, \lambda) \in Y_j \times \mathbb{R}$ and $(n', \lambda') \in Y_{j'} \times \mathbb{R}$ are equivalent iff $n = n'$ and $\lambda' - \lambda = \langle m_{j'} - m_j, n \rangle$. The evident map $\pi: \Lambda \rightarrow N_{\mathbb{R}}$ gives Λ the structure of a principal \mathbb{R} -bundle.³

Let $Z \subseteq N_{\mathbb{R}}$. A continuous *section* of Λ over Z is a continuous function $s: Z \rightarrow \Lambda$ such that $\pi \circ s = \text{id}$. By construction, Λ is trivial, and comes equipped with isomorphisms $\theta_j: \Lambda \xrightarrow{\sim} Y_j \times \mathbb{R}$. These give rise to a canonical reference section s_{ref} over $N_{\mathbb{R}}$, defined by $\theta_j(s_{\text{ref}}(n)) = (n, \langle m_j, n \rangle)$. For any continuous section s over Z , $s - s_{\text{ref}}$ is a continuous function on Z . We set $s_j := s_{\text{ref}} + m_j$.

A continuous *metric* on Λ over Z can be viewed as a continuous function $\Psi: \pi^{-1}(Z) \rightarrow \mathbb{R}$ which respects the \mathbb{R} -action, i.e. $\Psi(s+r) = \Psi(s) + r$, for $s \in \pi^{-1}(Z)$, $r \in \mathbb{R}$. By checking its representations in coordinate charts, Ψ is naturally a section of the “dual” bundle $-\Lambda$; this is defined in exactly the same way as Λ , except we require $\lambda' - \lambda = \langle m_j - m_{j'}, n \rangle$. It follows that $-s_{\text{ref}}$ is a canonical reference metric on Λ .

The restriction of Λ (and $-\Lambda$) to the integral affine manifold $B_0 \subset N_{\mathbb{R}}$ can be equipped with the structure of an integral affine \mathbb{R} -bundle in the sense of [36]. Namely, we declare that, for any i , a continuous section s of Λ over τ_i° (resp. T_i°) is integral affine iff the function $s - s_j$ on τ_i° (resp. T_i°) is integral affine for some (equivalently, any) $j \neq i$.

Lemma 3.13 now implies that for any $\psi \in \mathcal{Q}$, the metric $\Psi = \psi - s_{\text{ref}}$ on Λ is convex over B_0 , since it is convex in any affine trivializations (equivalently, Ψ is a convex section of $-\Lambda$ [36]).

4. Symmetric c-convex functions and their Monge–Ampère measures

The c-transform is modeled on the Legendre transform between convex functions on a vector space and its dual, and as shown in Lemma 3.13, leads to a seemingly satisfactory

³ One can also view Λ as the skeleton of the analytification of the line bundle $\mathcal{O}(d+2)$ over \mathbb{P}^{d+1} , restricted to $N_{\mathbb{R}} \subset \mathbb{P}^{d+1, \text{an}}$, see [9, §2.1].

notion of “local convexity” on A and B . However, if one attempts to generalize Alexandrov’s definition of the weak Monge–Ampère measure to this setting, some interesting problems manifest.

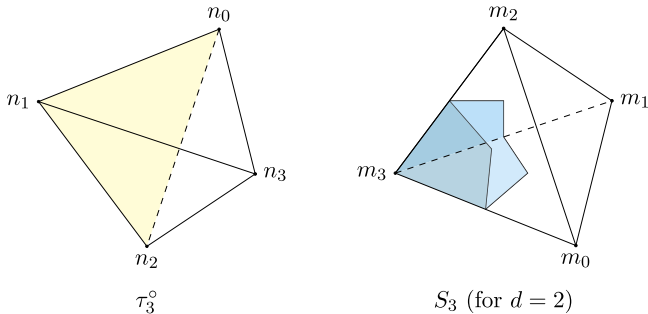
As suggested by Li [41], these issues disappear if we take into account the action of the permutation group $G = S_{d+2}$, and restrict ourselves to symmetric data.

4.1. Controlling the c -gradients

We denote by $\mathcal{P}_{\text{sym}} \subset \mathcal{P}$ and \mathcal{Q}_{sym} the set of symmetric functions, that is, G -invariant functions. These are closed subsets of \mathcal{P} and \mathcal{Q} , respectively, so the quotients $\mathcal{P}_{\text{sym}}/\mathbb{R}$ and $\mathcal{Q}_{\text{sym}}/\mathbb{R}$ are compact by Corollary 3.6. The c -transform is equivariant for the G -action, and restrict to isometric bijections between \mathcal{P}_{sym} and \mathcal{Q}_{sym} .

As we now show, symmetry places a number of strong restrictions on the possible c -subgradients a function could have.

Lemma 4.1. *For any $\psi \in \mathcal{Q}_{\text{sym}}$, we have $\partial^c \psi(T_i^\circ) \subseteq \sigma_i$ and $\partial^c \psi(\tau_i^\circ) \subseteq S_i$. The analogous inclusions hold for $\phi \in \mathcal{P}_{\text{sym}}$.*



Proof. By symmetry of ψ and ψ^c , $m \in (\partial^c \psi)(n)$ implies $\langle m, n \rangle = \max_{g \in G} \langle g^{-1}(m), n \rangle$. The result now follows from Lemma 2.1. \square

Since $\partial^c \psi$ and $\partial^c \psi^c$ are inverses, applying Lemma 4.1 to ψ^c gives:

Corollary 4.2. *For any $\psi \in \mathcal{Q}_{\text{sym}}$, we have $S_i^\circ \subseteq \partial^c \psi(\tau_i)$ and $\sigma_i^\circ \subseteq \partial^c \psi(T_i)$, with analogous results for $\phi \in \mathcal{P}_{\text{sym}}$.*

Next we look at the subgradients in charts. Recall that the function $\psi_{i,j} := (\psi - m_j) \circ q_{i,j}$ is convex on $q_{i,j}^{-1}(\text{Star}(n_i))$, see Lemma 4.1.

Remark 4.3. Lemma 4.1 gives an alternative proof a weaker version of Lemma 3.13, namely, that $\psi_{i,j}$ is convex on $q_{i,j}^{-1}(T_i^\circ)$ for every $\psi \in \mathcal{Q}_{\text{sym}}$. Indeed, the lemma implies that $\psi|_{T_i^\circ}$ is a supremum of functions of the form $\sum_{k \neq i} \theta_k m_k + c$, with $c \in \mathbb{R}$ $\theta_k \geq 0$, and $\sum_k \theta_k = 1$. For each $j, k \neq i$, the function $(m_k + c - m_j) \circ q_{i,j}$ is affine, and this

implies that $\psi_{i,j}$ is convex. In fact, even when $k = i$, the function $(m_k + c - m_j) \circ q_{i,j}$ is convex (although not affine); hence a similar argument can be used to prove the full statement of Lemma 3.13.

Lemma 4.4. *Suppose that $\psi \in \mathcal{Q}_{\text{sym}}$ and $i \neq j$. Then $p_{j,i}^{-1}$ gives a bijection of $(\partial^c \psi)(n)$ onto $\partial \psi_{i,j}(q_{i,j}^{-1}(n))$ for any $n \in T_i^\circ$. The same result holds for any $n \in \tau_j^\circ$. Moreover, the analogous results hold for \mathcal{P}_{sym} .*

Proof. First suppose $n \in T_i^\circ$. By Lemma 4.1, we have $(\partial^c \psi)(n) \subset \sigma_i$. Lemma 2.1 and symmetry of ψ show that, for $m \in \sigma_i$, we have:

$$\psi^c(m) = \sup_{n \in T_i} \langle m, n \rangle - \psi(n) = \sup_{n \in \text{Star}(n_i)} \langle m, n \rangle - \psi(n).$$

Thus, $m \in (\partial^c \psi)(n)$ iff

$$\langle m, n \rangle - \psi(n) \geq \langle m, n' \rangle - \psi(n')$$

for all $n' \in \text{Star}(n_i)$. Writing $m = p_{j,i}(x)$ and $n = q_{i,j}(y)$, Proposition 2.2 implies that the above inequality is equivalent to

$$\langle x, y \rangle - \psi_{i,j}(x) \geq \langle x, y' \rangle - \psi_{i,j}(y')$$

for all $y' \in q_{i,j}^{-1}(\text{Star}(n_i))$, which amounts to $x \in \partial \psi_{i,j}(y)$. The case when $n \in \tau_j^\circ$ is proved in the same way, using Lemma 4.1, and the proof for functions in \mathcal{P}_{sym} is completely analogous. \square

Lemma 4.4 allows us to apply many standard results for convex functions to c -convex functions. For example, we have:

Lemma 4.5. *If $\phi \in \mathcal{P}_{\text{sym}}$, then the following properties hold:*

- (i) *the c -subgradient $(\partial^c \phi)(m)$ is a singleton for almost every $m \in A$ (i.e. ϕ has a c -gradient a.e.);*
- (ii) *the a.e. defined function $(\partial^c \phi): A \rightarrow B$ is measurable, and the set:*

$$\{n \in (\partial^c \phi)(m) \cap (\partial^c \phi)(m') \mid m, m' \in A_0, m \neq m'\}$$

has Lebesgue measure 0.

Similar results hold for $\psi \in \mathcal{Q}_{\text{sym}}$.

Proof. The convex function $\phi_{i,j}$ (defined analogously to $\psi_{i,j}$) is almost everywhere differentiable, so applying Lemma 4.4 to each of the S_i° , say, implies that $(\partial^c \phi)(m)$ is a singleton for a.e. $m \in A$, showing (i).

The second point follows similarly, since the $q_{i,j}$ are measurable and A is covered, up to a set of measure zero, by the sets S_i° . \square

Next we relate the c -transform on symmetric functions to the usual Legendre transform on \mathbb{R}^d . Denote by $L_{\text{sym}}^\infty(A)$ and $L_{\text{sym}}^\infty(B)$ the sets of symmetric bounded functions on A and B , respectively.

Lemma 4.6. *If $\psi \in L_{\text{sym}}^\infty(B)$ and $i \neq j$, then the convex function $(\psi^c - n_j) \circ p_{i,j}$ on $p_{i,j}^{-1}(S_i) \subset \mathbb{R}^d$ is the Legendre transform of the bounded function $\psi \circ q_{j,i}$ on $q_{j,i}^{-1}(\tau_i) \subset \mathbb{R}^d$. Similarly, the convex function $\psi^c \circ p_{i,j}$ on $p_{i,j}^{-1}(\sigma_j)$ is the Legendre transform of the convex function $\psi_{j,i} = (\psi - m_i) \circ q_{j,i}$ on $q_{j,i}^{-1}(T_j)$. The analogous statements hold for $\phi \in L^\infty(A)$.*

Proof. If $m \in S_i$, then Lemma 2.1 implies that $\psi^c(m) = \sup_{n \in \tau_i} (\langle m, n \rangle - \psi(n))$. Writing $m = p_{i,j}(x)$, $n = q_{j,i}(y)$, and using Proposition 2.2, we see that

$$(\psi^c - n_j)(x) = \sup_{y \in q_{j,i}^{-1}(\tau_i)} (\langle x, y \rangle - \psi(p_{j,i}(y))),$$

which proves the first assertion. The remaining statements are proved in the same way. \square

Denote by $C_{\text{sym}}^0(B)$ the set of symmetric continuous functions on B .

Lemma 4.7. *Suppose $\psi \in \mathcal{Q}_{\text{sym}}$ and $v \in C_{\text{sym}}^0(B)$. Then, for almost every $m \in A$, the function $t \mapsto (\psi + tv)^c(m)$ is differentiable at $t = 0$, with derivative $-v((\partial^c \psi^c)(m))$.*

Proof. Working in charts, using Lemma 4.6, this follows from the corresponding result about the Legendre transform on \mathbb{R}^d , as stated in e.g. [3, Lemma 2.7].

A more direct proof goes as follows. Note that $(\psi + tv)^c(m)$ is convex in t . This means its left and right derivatives exist at $t = 0$ and

$$\left. \frac{d(\psi + tv)^c(m)}{dt} \right|_{t=0-} \leq \left. \frac{d(\psi + tv)^c(m)}{dt} \right|_{t=0+}. \quad (4.1)$$

Assume $\partial^c \psi^c(m) = \{n\}$, and for each $t \neq 0$, pick n_t such that

$$(\psi + tv)^c(m) = \langle m, n_t \rangle - \psi(n_t) - tv(n_t), \quad (4.2)$$

which is possible by compactness of B . By compactness of B , $\{n_t\}$ converges up to passing to subsequence to some $n_0 \in B$ when $t \rightarrow 0$. We get, by continuity of the c -transform, that n_0 satisfies $\psi^c(m) + \psi(n_0) = \langle m, n_0 \rangle$; hence $n_0 = n$ and $n_t \rightarrow n$. Using (4.2), this yields

$$\frac{(\psi + tv)^c(m) - \psi^c(m)}{t} = -v(n_t) - \frac{\psi(n_t) - \psi(n) - \langle m, n_t - n \rangle}{t}. \quad (4.3)$$

The numerator on the right hand side is positive since $m \in \partial^c \psi(n)$; hence

$$\left. \frac{d(\psi + tv)^c(m)}{dt} \right|_{t=0+} \leq -v(n) \leq \left. \frac{d(\psi + tv)^c(m)}{dt} \right|_{t=0-}.$$

Combining this with (4.1) proves the lemma. \square

By Dominated Convergence, we obtain the following result, which is the analogue of the differentiability result needed to solve the complex and non-Archimedean Monge–Ampère equations, respectively, see [4, Theorem B] and [7, §7]. In what follows, μ denotes Lebesgue measure on A , of total mass $(d+2)^{d+1}/d!$.

Corollary 4.8. *If $\psi \in \mathcal{Q}_{\text{sym}}$ and $v \in C_{\text{sym}}^0(B)$, then the function*

$$t \mapsto \int_A (\psi + tv)^c d\mu$$

is differentiable at $t = 0$, with derivative $-\int_A v((\partial^c \psi)^c(m)) d\mu$.

4.2. The tropical Monge–Ampère measure

We can now use Corollary 4.8 to assign a symmetric positive measure ν_ψ on B to any *symmetric* function $\psi \in \mathcal{Q}_{\text{sym}}$, in a way which is compatible with the variational approach to the Alexandrov Monge–Ampère operator. Hence, we will think of ν_ψ as the Monge–Ampère measure of ψ .

Definition 4.9. Given $\psi \in \mathcal{Q}_{\text{sym}}$ we define a positive Radon measure ν_ψ on B of mass $|A|$ by declaring

$$\int_B v d\nu_\psi := -\left. \frac{d}{dt} \right|_{t=0} \int_A (\psi + tv)^c d\mu = -\int_A (v \circ \partial^c \psi^c) d\mu$$

for every $v \in C_{\text{sym}}^0(B)$.

Proposition 4.10. *For any $\psi \in \mathcal{Q}_{\text{sym}}$ and Lebesgue measurable $U \subset B$, we have*

$$\nu_\psi(U) = \mu((\partial^c \psi)(U)).$$

Proof. First, by Lemma 4.5, the multivalued map $\partial^c \psi^c$ is μ -a.e. single valued, so by the standard change of variables formula and Corollary 4.8, we have:

$$\nu_\psi = (\partial^c \psi^c)_* \mu.$$

Since $\partial^c \psi^c$ and $\partial^c \psi$ are inverses, the result now follows from the definition of the push-forward measure. \square

That ν_ψ is compatible with the Monge–Ampère measure in charts now follows immediately from Lemma 4.4:

Corollary 4.11. *For any $\psi \in \mathcal{Q}_{\text{sym}}$ and any $i \neq j$, we have:*

$$\nu_\psi|_{T_i^\circ} = (q_{i,j})_* \text{MA}_{\mathbb{R}} \left(\psi_{i,j}|_{q_{i,j}^{-1}(T_i^\circ)} \right) \quad \text{and} \quad \nu_\psi|_{\tau_j^\circ} = (q_{i,j})_* \text{MA}_{\mathbb{R}} \left(\psi_{i,j}|_{q_{i,j}^{-1}(\tau_j^\circ)} \right),$$

and for any j we have

$$\nu_\psi|_{\tau_j^\circ} = \text{MA}_{\mathbb{R}}((\psi \circ q_{i,j})|_{q_{i,j}^{-1}(\tau_j^\circ)}).$$

Corollary 4.11 allows us to now apply the standard theory for the Monge–Ampère operator. For instance, we see that ν_ψ is weakly continuous under uniform convergence of the potentials, using the following result:

Lemma 4.12. [37, Theorem 2.1.22] *Let $\Omega \subset \mathbb{R}^d$ be an open convex subset, $u_j, u: \Omega \rightarrow \mathbb{R}$ convex functions, and assume that $u_j \rightarrow u$ pointwise on Ω . Then there exists a subset $E \subset \Omega$ of full measure such that for every $x \in \Omega$, u_j and u are differentiable at x , and $u'_j(x) \rightarrow u'(x)$.*

Proposition 4.13. *If a sequence $(\psi_k)_{k=1}^\infty$ of functions in \mathcal{Q}_{sym} converges uniformly to $\psi \in \mathcal{Q}_{\text{sym}}$, then $\nu_{\psi_k} \rightarrow \nu_\psi$ weakly as measures on B .*

Proof. By definition, we have $\nu_\psi = (\partial^c \psi^c)_* \mu$, so by Dominated Convergence it suffices to prove that $\partial^c \psi_k^c \rightarrow \partial^c \psi^c$ a.e. Now, the c-transform is 1-Lipschitz, so we have $\psi_k^c \rightarrow \psi^c$ uniformly on A . Further, on the open stars S_i° , which together have full measure, the c-gradient is computed as the gradient of a convex function on an open subset of \mathbb{R}^d , see Lemma 4.4. The result now follows from Lemma 4.12. \square

Remark 4.14. As noted in §3.4, any $\psi \in \mathcal{Q}$ defines a convex metric Ψ on an integral affine \mathbb{R} -bundle Λ on the integral affine manifold B_0 . Such a convex metric has a natural Monge–Ampère measure $\text{MA}_{\mathbb{R}}(\Psi)$, defined as $\text{MA}_{\mathbb{R}}(\Psi \circ s)$ for any local affine section s , and Corollary 4.11 shows that the restriction of ν_ψ to B_0 equals $\text{MA}_{\mathbb{R}}(\Psi)$. Note, however, that ν_ψ may put mass also on $B \setminus B_0$, see Example 4.17.

4.3. Examples

We conclude by giving a few examples. Recall that the total mass of ν_ψ is always $\frac{(d+2)^{d+1}}{d!}$ for $\psi \in \mathcal{Q}_{\text{sym}}$. For instance, when $d = 2$, the total mass is 32.

Example 4.15. Let $\psi = \max_i m_i \equiv 1 \in \mathcal{Q}_{\text{sym}}$ and $m \in A$. The supremum

$$\sup_{n \in B} \langle m, n \rangle - \psi(n) = \sup_{n \in B} \langle m, n \rangle - 1$$

is achieved at one of the vertices $\{n_0, \dots, n_{d+1}\}$. It follows that $\partial^c \psi^c(m)$ contains a vertex for each $m \in B$. Consequently, since $\partial^c \psi^c$ is single valued almost everywhere, ν_ψ is supported at the vertexes and by symmetry $\nu_\psi = \sum_i \frac{(d+2)^d}{d!} \delta_{n_i}$.

Example 4.16. For each i , let $n'_i := \frac{-n_i}{3}$ be the barycenters of the τ_i . Using basic properties for the c -gradient, one can see that $\psi := (\max_i n'_i)^c$ satisfies $\nu_\psi := \sum_i \frac{(d+2)^d}{d!} \delta_{n'_i}$. This can be computed explicitly, for example when $d = 2$,

$$\psi = \max \left\{ \max_i m'_i - \frac{1}{9}, \max_{i \neq j} \frac{m_i + m_j}{2} - \frac{1}{3} \right\},$$

where $m'_i = \frac{-m_i}{3}$.

Example 4.17. ψ can also charge the singular set – indeed, when $d = 2$, one can verify that $\psi = \max \{ \max_i m'_i, \frac{1}{3} \}$ does not charge B_0 at all, and so we have $\nu_\psi = \sum_{x_i \in B \setminus B_0} \frac{16}{3} \delta_{x_i}$, by symmetry. One can also check that, while $\psi_{j,k}$ is actually convex on all of $q_{j,k}^{-1}(\text{Star}(n_j))$, $\text{MA}_{\mathbb{R}}(\psi_{j,k})(x_i) = \frac{80}{9} > \frac{16}{3}$ for each $x_i \in \text{Star}(n_j)^\circ \setminus B_0$, so the equalities in Corollary 4.11 cannot be extended to all of $\text{Star}(n_j)^\circ$.

For $\psi \in \mathcal{Q}_{\text{sym}}$, we have two equivalent definitions for ν_ψ (Definition 4.9 and Proposition 4.10), which agree with the Monge–Ampère measure of $\psi_{i,j}$ in coordinates (Corollary 4.11). For non-symmetric $\psi \in \mathcal{Q}$, none of these are well-defined in general, and when they are, they need not agree with the Monge–Ampère computed in coordinates, as the following examples show.

Example 4.18. Let $d = 2$ and $\psi := m_i$, for some fixed i ; then the Monge–Ampère measure of $\psi_{i,j}$ is $\text{MA}_{\mathbb{R}}(\psi_{i,j}) = 128\delta_0$. Since this gives a total mass larger than 32, we conclude that Corollary 4.11 cannot hold for this ψ .

Example 4.19. Let $d = 2$, and fix $0 \leq i \leq 3$. If $\psi = \max_{j \neq i} m_j$, then one checks that $\mu(\partial^c \psi) = 8\delta_{n_i} + 32\delta_{n'_i}$; hence the right hand side in Proposition 4.10 does not assign the correct total mass for non-symmetric ψ .

Example 4.20. Let $d = 1$, and $\psi(n) = \max_i \langle m_i, n - n'_0 \rangle$, with $n'_j = \frac{-n_j}{2}$ for $j = 0, 1, 2$. Let $v \geq 0$ be a piecewise linear function with $v(n_0) = 1$ and $v(n'_1) = v(n'_2) = 0$. Then $(\psi + tv)^c(m)$ will not be differentiable in $t \in (-\varepsilon, \varepsilon)$ for all $m \in \sigma_0$; hence Definition 4.9 does not make sense for this ψ .

5. The tropical Monge–Ampère equation

We are now ready to study the (symmetric) tropical Monge–Ampère equation. Thus, given a symmetric positive measure ν on B of mass $|A|$, we seek to find $\psi \in \mathcal{Q}_{\text{sym}}$ such that $\nu_\psi = \nu$. In particular, we will prove Theorem B in the introduction.

5.1. Variational formulation

Given a measure ν as above, we define a functional

$$F = F_\nu: \mathcal{Q}_{\text{sym}} \rightarrow \mathbb{R}$$

by

$$F(\psi) := \int_A \psi^c d\mu + \int_B \psi d\nu.$$

Lemma 5.1. *A function $\psi \in \mathcal{Q}_{\text{sym}}$ minimizes the functional F iff $\nu_\psi = \nu$.*

Proof. First suppose $\nu = \nu_\psi$. For any $\psi' \in \mathcal{Q}_{\text{sym}}$ we then have

$$F(\psi') = \int_A (\psi'^c + \psi' \circ (\partial^c \psi^c)) d\mu.$$

For almost every $m \in A$, we have

$$\psi'^c(m) + \psi'((\partial^c \psi^c)(m)) \geq \langle m, (\partial^c \psi^c)(m) \rangle = \psi^c(m) + \psi((\partial^c \psi^c)(m)),$$

and it follows that $F(\psi') \geq F(\psi)$, so that ψ is a minimizer for F .

Conversely, suppose that $\psi \in \mathcal{Q}_{\text{sym}}$ is a minimizer for F , and let us show that $\nu_\psi = \nu$. We must prove that $\int_B v d\nu_\psi = \int_B v d\nu$ for all $v \in C^0(B)$. As ν and ν_ψ are both symmetric, it suffices to establish this for $v \in C_{\text{sym}}^0(B)$. Indeed, the function $\bar{v} = \frac{1}{|G|} \sum_{g \in G} v \circ g$ is symmetric, and we have $\int_B v d\nu_\psi = \int_B \bar{v} d\nu_\psi$, $\int_B v d\nu = \int_B \bar{v} d\nu$.

Thus suppose $v \in C_{\text{sym}}^0(B)$, and consider the function on \mathbb{R} defined by

$$f(t) := \int_A (\psi + tv)^c d\mu + \int_B (\psi + tv) d\nu.$$

It follows from Corollary 4.8 that $f(t)$ is differentiable at $t = 0$, with

$$f'(0) = - \int_B v d\nu_\psi + \int_B v d\nu,$$

so we are done if we can prove that $f(t)$ has a (global) minimum at $t = 0$.

Now $(\psi + tv)^{cc} \in \mathcal{Q}_{\text{sym}}$, and $(\psi + tv)^{ccc} = (\psi + tv)^c$, whereas $(\psi + tv)^{cc} \leq \psi + tv$. Thus

$$f(t) \geq \int_A (\psi + tv)^c d\mu + \int_B (\psi + tv)^{cc} d\nu = F((\psi + tv)^{cc}) \geq F(\psi) = f(0),$$

completing the proof. \square

5.2. Existence and uniqueness

We will prove

Theorem 5.2. *For any symmetric positive measure ν on B of total mass $|A|$, there exists a function $\psi \in \mathcal{Q}_{\text{sym}}$, such that $\nu_\psi = \nu$. Moreover, ψ is unique up to an additive constant, and the map $\psi \mapsto \nu_\psi$ is a homeomorphism from $\mathcal{Q}_{\text{sym}}/\mathbb{R}$ to the space $\mathcal{M}_{\text{sym}}(B)$ of positive, symmetric measures of mass $|A|$.*

Proof. We use Lemma 5.1. To prove existence of a solution, it suffices to show that the functional F_ν admits a minimizer on \mathcal{Q}_{sym} . But as $\phi \mapsto \phi^c$ is Lipschitz continuous, one sees that F is Lipschitz continuous. It is also translation invariant, so the existence of a minimizer follows from compactness of $\mathcal{Q}_{\text{sym}}/\mathbb{R}$, see Corollary 3.6.

We now show uniqueness. It suffices to prove that if $\psi_0, \psi_1 \in \mathcal{Q}_{\text{sym}}$ are two minimizers of F , normalized by $\int \psi_i d\nu = 0$, $i = 0, 1$, then $\psi_0 = \psi_1$. Set $\psi := \frac{1}{2}(\psi_0 + \psi_1)$. Then $\psi \in \mathcal{Q}_{\text{sym}}$, by convexity of \mathcal{Q}_{sym} , and we have $\int_B \psi d\nu = 0$. Now

$$\psi^c = \sup_{n \in B} (n - \psi(n)) \leq \frac{1}{2} \sup_{n \in B} (n - \psi_0(n)) + \frac{1}{2} \sup_{n \in B} (n - \psi_1(n)) = \frac{1}{2}(\psi_0^c + \psi_1^c),$$

pointwise on A . As $\int_B \psi d\nu = 0$, this leads to

$$F_\nu(\psi) = \int_A \psi^c d\mu \leq \frac{1}{2} \int_A \psi_0^c d\mu + \frac{1}{2} \int_A \psi_1^c d\mu = \min_{\mathcal{Q}_{\text{sym}}} F_\nu.$$

Thus equality holds, so since $\psi_0^c, \psi_1^c, \psi^c$ are continuous, we must have $\psi^c = \frac{1}{2}(\psi_0^c + \psi_1^c)$. For a.e. $m \in A$, ψ_0^c, ψ_1^c and ψ^c all admit a c -gradient at m . If we set $n := (\partial^c \psi^c)(m)$, then

$$\begin{aligned} \psi^c(m) &= \langle m, n \rangle - \psi(n) \\ &= \frac{1}{2}(\langle m, n \rangle - \psi_0(n)) + \frac{1}{2}(\langle m, n \rangle - \psi_1(n)) \\ &\leq \frac{1}{2}\psi_0^c(m) + \frac{1}{2}\psi_1^c(m) \\ &= \psi^c(m). \end{aligned}$$

Thus $\psi_i^c(m) = \langle m, n \rangle - \psi_i(n)$ for $i = 1, 2$, so we must have

$$(\partial^c \psi_0^c)(m) = (\partial^c \psi_1^c)(m) = n$$

for a.e. m . We may assume m lies in some open star S_i° . Pick any $j \neq i$. By Lemma 4.4, the convex functions $(\psi_0^c - m_j) \circ p_{j,i}$ and $(\psi_1^c - m_j) \circ p_{j,i}$ on $p_{j,i}^{-1}(S_i^\circ)$ have the same gradient at a.e. point. By Lemma 5.3 below, these two functions differ by an additive constant, so $\psi_0^c - \psi_1^c$ is constant on S_i° . By continuity of the elements of \mathcal{P} and density of $\bigcup_i S_i^\circ$ in A , we get that $\psi_0^c - \psi_1^c$ is constant on A . It follows that $\psi_0 - \psi_1$ is also constant, and hence zero, by our normalization.

It follows that the Monge–Ampère operator $\psi \mapsto \nu_\psi$ defines a bijection between the compact Hausdorff spaces $\mathcal{Q}_{\text{sym}}/\mathbb{R}$ and $\mathcal{M}_{\text{sym}}(B)$. By Proposition 4.13, this bijection is continuous, and hence a homeomorphism. \square

Lemma 5.3. *If $\Omega \subset \mathbb{R}^n$ is open and convex, and u_0, u_1 are convex functions on Ω such that $\nabla u_0 = \nabla u_1$ a.e. on Ω , then $u_0 - u_1$ is constant on Ω .*

Proof. Let $E \subset \Omega$ be the set of points where $\nabla u_0 = \nabla u_1$. Pick any point $x_0 \in \Omega$. After adding a constant to u_1 , we may assume $u_0(x_0) = u_1(x_0)$. Pick $r > 0$ such that $B(x_0, 2r) \subset \Omega$. It suffices to prove that $u_0 = u_1$ on $B(x_0, r)$. Fubini's theorem implies that for almost every point v on the unit sphere in \mathbb{R}^n , we have $x_0 + tv \in E$ for almost every $t \in (-r, r)$. For such v it follows that the convex functions $f_i(t) := u_i(x_0 + tv)$ on $(-r, r)$ satisfy $f_0'(t) = f_1'(t)$ for a.e. t . As $f_0(0) = f_1(0)$, this implies that $f_0 = f_1$, see [24, Theorem 3.35]. Thus $u_0(x_0 + tv) = u_1(x_0 + tv)$ for almost every v and all $t \in (-r, r)$. By continuity, we see that $u_0 = u_1$ on $B(x_0, r)$. \square

Proof of Theorem B. It is clear from Theorem 5.2 and Corollary 4.11 that $\psi \mapsto \nu_\psi$ satisfies all the properties stated in Theorem B. Now let $\psi \mapsto \nu'_\psi$ be a continuous map from \mathcal{Q}_{sym} to \mathcal{M}_{sym} such that $\nu'_\psi|_{\tau_i^\circ} = \text{MA}_{\mathbb{R}}(\psi|_{\tau_i^\circ})$ for every i . Thus $\nu'_\psi = \nu_\psi$ for all $\psi \in \mathcal{Q}_{\text{sym}}$ such that ν_ψ puts full mass on $\bigcup_i \tau_i^\circ$. But it follows from Theorem 5.2 that the set of such functions is dense in \mathcal{Q}_{sym} ; indeed, the set of measures on \mathcal{M}_{sym} putting full mass on $\bigcup_i \tau_i^\circ$ is dense. The result follows. \square

Remark 5.4. It is of interest to the SYZ conjecture to investigate the regularity of the solution ψ when ν is Lebesgue measure on B , using classical and more recent results, see [23, 47–49]. We hope to address this in future work.

6. Induced metrics on the Berkovich projective space

Here we define a procedure that to a symmetric c-convex function on B associates a symmetric toric continuous psh metric on the Berkovich analytification of $\mathcal{O}_{\mathbb{P}^{d+1}}(d+2)$. As before, K is any non-Archimedean field.

6.1. Continuous psh functions and metrics

The study of continuous psh (or semipositive) metrics in non-Archimedean geometry goes back to Zhang [65] and Gubler [33], who defined the notion of a continuous psh metric on the analytification of an ample line bundle on a projective variety defined over a nontrivially valued non-Archimedean field K . This theory is global in nature.

More recently, a local theory was developed by Chambert-Loir and Ducros [15]. Given any non-Archimedean field K , any K -analytic space⁴ Z , can be endowed with a sheaf of continuous psh functions. For example, if f_1, \dots, f_n are invertible analytic functions on Z , $\Omega \subset \mathbb{R}^n$ is an open subset such that $(\log |f_1(z)|, \dots, \log |f_n(z)|) \in \Omega$ for all $z \in Z$, and $\chi: \Omega \rightarrow \mathbb{R}^n$ is a convex function, then the function $z \mapsto \chi(\log |f_1(z)|, \dots, \log |f_n(z)|)$ is a continuous psh function on Z . A general continuous psh function is locally a uniform limit of functions of this type.

If L is a line bundle (in the analytic sense) on Z , then a continuous metric on L in the ‘multiplicative’ sense, is a continuous function $\|\cdot\|$ on the total space of L with values in $\mathbb{R}_{\geq 0}$, and a suitable homogeneity property along the fibers of $L \rightarrow Z$. We say that $\|\cdot\|$ is semipositive if for some (equivalently, any) local analytic section $s: Z \rightarrow L$, the continuous function $-\log \|s\|$ on Z is psh. It will be natural for us to instead use ‘additive’ terminology, and view a continuous metric on L as an \mathbb{R} -valued function $\Psi = -\log \|\cdot\|$ on the total space with the zero section removed. If Ψ is a continuous metric on L and s is a nonvanishing section of L over an open set $U \subset Z^{\text{an}}$, then we can view $\Psi - \log |s|$ as a continuous function on U , and we say that Ψ is psh if $\Psi - \log |s|$ is psh on U ; this is equivalent to $\|\cdot\| = \exp(-\Psi)$ being semipositive.

In fact, the global notion in [33,6] is a priori stronger. Let X be a projective variety, and L an ample line bundle on L . Then a continuous metric Ψ on L^{an} is globally psh if it can be uniformly approximated by *Fubini–Study metrics*, i.e. metrics of the form

$$\Psi = \frac{1}{r} \max_{1 \leq j \leq N} (\log |s_j| + \lambda_j), \quad (6.1)$$

where $r \geq 1$, $s_j \in H^0(X, rL)$ are global nonzero sections with no common zero, and $\lambda_j \in \mathbb{R}$. Such a metric is continuous psh in the sense above, and we shall only consider globally psh metrics.

6.2. Monge–Ampère measures

To any continuous psh function φ on a pure-dimensional K -analytic space Z is associated a ‘non-Archimedean’ Monge–Ampère measure $\text{MA}_{\text{NA}}(\varphi)$, a positive Radon measure on Z . We refer to [15] for the definition, but note that the Monge–Ampère operator is continuous under locally uniform convergence.

⁴ All K -analytic spaces will be assumed good and boundaryless.

If L is an analytic line bundle on Z and Ψ is a continuous psh metric on L , then the Monge–Ampère measure $\text{MA}_{\text{NA}}(\Psi)$ is a well-defined positive Radon measure on Z with the following property: for any nonvanishing section s of L over some open subset $U \subset Z$, we have $\text{MA}_{\text{NA}}(\Psi)|_U = \text{MA}_{\text{NA}}((\Psi - \log |s|)|_U)$. In ‘multiplicative’ notation, this measure is written $c_1(L, \|\cdot\|)^d$, where $d = \dim Z$ and $\|\cdot\| = \exp(-\Psi)$, as first introduced by Chambert–Loir [14].

In this paper, all computations involving the non-Archimedean Monge–Ampère measure will be deduced from the following result, essentially due to Vilsmeier [61].

Lemma 6.1. *Let $T \simeq \mathbb{G}_m^n$ be a split torus, with tropicalization map $\text{trop}: T^{\text{an}} \rightarrow N_{T, \mathbb{R}}$. Let $\Omega \subset N_{T, \mathbb{R}} \simeq \mathbb{R}^n$ be an open subset, and $g: \Omega \rightarrow \mathbb{R}$ a convex function. Then the composition $g \circ \text{trop}: \text{trop}^{-1}(\Omega) \rightarrow \mathbb{R}$ is a continuous psh function, and we have*

$$\text{MA}_{\text{NA}}(g \circ \text{trop}) = n! \text{MA}_{\mathbb{R}}(g)$$

on $\text{trop}^{-1}(\Omega)$, where the left-hand side denotes the non-Archimedean Monge–Ampère measure on $\text{trop}^{-1}(\Omega)$, and the right-hand side denotes the real Monge–Ampère measure on $\Omega \subset \text{trop}^{-1}(\Omega)$.

Proof. We argue as in the proof of [61, Corollary 5.10]. By ground field extension, we may assume K is algebraically closed and non-trivially valued, and in particular has dense value group. We may also assume $T = \text{Spec } K[z_1^{\pm}, \dots, z_n^{\pm}]$ and $N_{T, \mathbb{R}} = \mathbb{R}^n$. The statement is local on $\Omega \subset \mathbb{R}^n$, so pick any point $\underline{t} = (t_1, \dots, t_n) \in \Omega$, and nonzero elements $a, b_1, \dots, b_n \in K$ such that the set

$$\{\underline{s} = (s_1, \dots, s_n) \in \mathbb{R}^n \mid s_j \geq \log |b_j|, \log |a|^{-1} + \sum_j \log |b_j| \geq \sum_j s_j\}$$

is contained in Ω and contains \underline{t} in its interior. After performing the change of coordinates $z_j \mapsto b_j z_j$ we may assume $b_j = 1$ for all j . Now consider the formal scheme

$$\mathfrak{X} = \text{Spf}(K^{\circ}\langle z_0, \dots, z_n \rangle / (z_0 \dots z_n - a)).$$

The generic fiber of \mathfrak{X} is isomorphic to the Laurent domain $|z_j| \leq 1$, $\prod_j |z_j| \geq |a|$ in T^{an} , and the skeleton Δ of \mathfrak{X} is the simplex $\{s_j \geq 0, \sum s_j \leq \log |a|^{-1}\}$. As this simplex contains the point \underline{t} in its interior, the result now follows from Corollary 5.7 in [61]. \square

Remark 6.2. Lemma 6.1 can also be deduced from [11] which systematically studies pluripotential theory for tropical toric varieties, and its relation to complex and non-Archimedean pluripotential theory.

6.3. The Fubini–Study operator

We now return to the setup earlier in the paper. There is a natural *Fubini–Study operator* that associates to any continuous convex function $\phi: \Delta \rightarrow \mathbb{R}$ a continuous psh metric $\text{FS}(\phi)$ on $\mathcal{O}(d+2)^{\text{an}}$, see for example Theorem 4.8.1 in [12].⁵ It is characterized by the following two properties:

- $\phi \rightarrow \text{FS}(\phi)$ is continuous;
- if ϕ is \mathbb{Q} -PL, then for any sufficiently divisible $r \geq 1$, we have

$$\text{FS}(\phi) = \max_{m \in r\Delta \cap M} (r^{-1} \log |\chi^{r,m}| - \phi(r^{-1}m)).$$

Here ϕ is \mathbb{Q} -PL if it is the maximum of finitely many rational affine functions, i.e. functions of the form $n + \lambda$, where $n \in N_{\mathbb{Q}}$ and $\lambda \in \mathbb{Q}$. Any continuous convex function on Δ is a uniform limit of \mathbb{Q} -PL convex functions, so the two conditions above completely determine the operator FS .

6.4. From c -convex functions to continuous psh metrics

To any symmetric c -convex function $\psi \in \mathcal{Q}_{\text{sym}}$ we now associate a continuous psh metric on $\mathcal{O}(d+2)^{\text{an}}$. This metric, which we slightly abusively denote by $\text{FS}(\psi)$, is defined by

$$\text{FS}(\psi) := \text{FS}(\psi^c|_{\Delta}),$$

where we view the c -transform ψ^c as a convex function on $M_{\mathbb{R}}$, defined using (3.2). The map $\psi \mapsto \text{FS}(\psi)$ is contractive, and equivariant for addition of constants.

The metric $\text{FS}(\psi)$ is closely related to the canonical extension of ψ to $N_{\mathbb{R}}$ in §3.2. Indeed, if $m \in M \cap \Delta$, and $\chi^m := \chi^{1,m} \in H^0(\mathbb{P}^{d+1}, \mathcal{O}(d+2))$ is the corresponding section, then $\log |\chi^m|$ is a continuous metric on $\mathcal{O}(d+2)^{\text{an}}$ over T^{an} . Then $\text{FS}(\psi) - \log |\chi^m|$ is a continuous psh function on T^{an} , and we have

$$\text{FS}(\psi) - \log |\chi^m| = (\psi - m) \circ \text{trop}, \quad (6.2)$$

on T^{an} , where $\text{trop}: T^{\text{an}} \rightarrow N_{\mathbb{R}}$ is the tropicalization map.

Lemma 6.1 and (6.2) allow us to compute the Monge–Ampère measure of $\text{FS}(\psi)$ on T^{an} , but that is not what we want to do. Instead, we will consider the restriction of $\text{FS}(\psi)$ to a Calabi–Yau hypersurface $X \subset \mathbb{P}^{d+1}$.

⁵ In [12], the authors consider concave rather than convex functions on Δ .

7. Calabi–Yau hypersurfaces

From now on, $K = \mathbb{C}((t))$. Consider a hypersurface $X \subset \mathbb{P}_K^{d+1}$ of the form

$$X = V(z_0 \cdot \dots \cdot z_{d+1} + tf(z)),$$

where $f(z) \in \mathbb{C}[z]$ is a homogeneous polynomial of degree $d+2$ such that for any intersection Z of coordinate hyperplanes $z_j = 0$ in \mathbb{P}^{d+1} , f does not vanish identically on Z and $V(f|_Z)$ is smooth. We call such a polynomial *admissible*. The set of admissible polynomials determines an open in $|\mathcal{O}_{\mathbb{P}^{d+1}}(d+2)|$, which is not empty as, for instance, the Fermat polynomial $f_{d+2}(z) = \sum_{i=0}^{d+1} z_i^{d+2}$ is admissible. Moreover, X is smooth for any admissible polynomial.

7.1. Models and skeletons

Let Y be any smooth and proper variety over K . Given a scheme \mathcal{Y} over $R := \mathbb{C}[[t]]$, we denote by \mathcal{Y}_K , respectively \mathcal{Y}_0 , the base change of \mathcal{Y} to K , respectively to the residue field of R . A model of Y is a flat R -scheme \mathcal{Y} such that $\mathcal{Y}_K \simeq Y$. We say that the model is strict normal crossing (snc), respectively divisorially log terminal (dlt), if the pair $(\mathcal{Y}, \mathcal{Y}_{0,\text{red}})$ is so; see [38, Definitions 1.7, 1.18] for more details.

Given any snc, more generally dlt, model \mathcal{Y} of Y , we denote by $\mathcal{D}(\mathcal{Y}_0)$ the dual complex of \mathcal{Y}_0 (see [38, Definition 3.62]); this admits a canonical embedding $\mathcal{D}(\mathcal{Y}_0) \xrightarrow{\sim} \text{Sk}(\mathcal{Y}) \subset Y^{\text{val}}$ in Y^{val} , whose image is called the skeleton of \mathcal{Y} .

7.2. The essential skeleton of X

Let X^{an} be the analytification of X , and $X^{\text{val}} \subset X^{\text{an}}$ the set of valuations on the function field of X . We have that X^{an} is a closed subset of $\mathbb{P}^{d+1, \text{an}}$. Being the analytification of a Calabi–Yau variety over $\mathbb{C}((t))$, X^{an} admits a canonical subset $\text{Sk}(X) \subset X^{\text{val}}$, the *essential skeleton*, defined in two equivalent ways. On one side, $\text{Sk}(X)$ is the locus where a certain weight function on X^{an} takes its minimal values, [39, 50]; on the other side, $\text{Sk}(X)$ is the skeleton associated with any minimal dlt model of X , [52].

In our case, $\text{Sk}(X)$ can be concretely described as follows. Consider the model

$$\mathcal{X} := V(z_0 \cdot \dots \cdot z_{d+1} + tf(z)) \subset \mathbb{P}_R^{d+1}$$

of X over $R = \mathbb{C}[[t]]$. The special fiber \mathcal{X}_0 is simply given by $V(z_0 \cdot \dots \cdot z_{d+1}) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$, and its dual complex $\mathcal{D}(\mathcal{X}_0)$ is evidently the boundary of a $(d+1)$ -dimensional simplex, hence topologically a sphere. We have an anticontinuous specialization map $\text{sp}_{\mathcal{X}}: X^{\text{an}} \rightarrow \mathcal{X}_0$.

Now \mathcal{X} is smooth away from $\text{Sing}(\mathcal{X}) = \bigcup_{i \neq j} V(z_i, z_j, t, f(z))$. Since f is admissible,

- for any $\xi \in V(z_{i_1}, \dots, z_{i_m}, t, f(z)) \subset \text{Sing}(\mathcal{X})$ for some maximal $m \in \{2, \dots, d\}$, étale locally around ξ we have the isomorphism

$$\mathcal{X} \simeq V(x_1 \cdot \dots \cdot x_m - ty) \subset \mathbb{A}_{x_i, t, y}^{m+2} \times \mathbb{A}^{d-m};$$

- $(\mathcal{X}, \mathcal{X}_0)$ is snc at the generic point of each stratum of the special fiber \mathcal{X}_0 .

It follows that locally around any singular point, \mathcal{X} is a toric subvariety of \mathbb{A}^{d+2} and the special fiber \mathcal{X}_0 consists of toric divisors; by [21, Proposition 11.4.24] the pair $(\mathcal{X}, \mathcal{X}_0)$ is log canonical around the singular points of \mathcal{X} . Finally, one can check that \mathcal{X} is dlt by considering the small resolution of \mathcal{X} obtained as blow-up of all but one irreducible component of the special fiber (see [38, §2.1, 4.2]), and is minimal since $K_{\mathcal{X}} \sim \mathcal{O}_{\mathcal{X}}$. We conclude that $\mathrm{Sk}(X) = \mathrm{Sk}(\mathcal{X}) \simeq \mathcal{D}(\mathcal{X}_0)$.

Let us be a bit more precise, and describe the embeddings of the d -dimensional faces of $\mathcal{D}(\mathcal{X}_0)$ in X^{val} . Such a face is determined by a zero-dimensional stratum of the special fiber, say the point ξ_i , where $z_j = 0$ for $j \neq i$. At ξ_i , \mathcal{X} is smooth, the rational functions

$$w_{i,j} = \frac{z_j}{z_i}, \quad j \neq i,$$

form a coordinate system at ξ_i , and $u_i := -z_i^{d+2}/f(z)$ is a unit at ξ_i . We can write

$$z^{m_i} = \prod_{j \neq i} w_{i,j}^{-1} = t^{-1} u_i \tag{7.1}$$

and, for $j \neq i$,

$$z^{m_j} = w_{i,j}^{d+1} \cdot \prod_{l \neq i,j} w_{i,l}^{-1}. \tag{7.2}$$

Given numbers $\lambda_j \in \mathbb{R}_{>0}$, $j \neq i$ with $\sum \lambda_j = 1$, there exists a unique minimal valuation $v_{i,\lambda}$ on $\mathcal{O}_{\mathcal{X},\xi_i}$ such that $v_{i,\lambda}(w_{i,j}) = \lambda_j$ for all j . Then $v_{i,\lambda}$ defines a point in X^{val} . With a bit of work, one shows that $\lambda \mapsto v_{i,\lambda}$ extends to a homeomorphism of the closed simplex $\{\sum_{j \neq i} \lambda_j = 1\} \subset \mathbb{R}_{\geq 0}^{\{0, \dots, d+1\} \setminus \{i\}}$ onto a compact subset $\tilde{\tau}_i$ of X^{val} . Moreover, $\mathrm{Sk}(X)$ is a (non-disjoint) union of the $\tilde{\tau}_i$.

We equip each $\tilde{\tau}_i$ with an integral affine structure, in which the affine functions are integral linear combinations of $v \mapsto v(w_{i,j})$, $j \neq i$.

Let $U_i := \mathrm{sp}_{\mathcal{X}}^{-1}(\xi_i) \subset X^{\mathrm{an}}$ be the open subset of points specializing to the point ξ_i . Each point $v \in U_i$ lies in T^{an} as $f(z)$ is generic, and defines a semivaluation on $\mathcal{O}_{\mathcal{X},\xi_i}$ such that $v(w_{i,j}) > 0$ for all $j \neq i$, and $\sum v(w_{i,j}) = 1$. There is a canonical retraction $r_i: U_i \rightarrow \tilde{\tau}_i^{\circ}$ defined by $r_i(v) = v_{i,\lambda}$ where $\lambda_j = v(w_{i,j})$.

7.3. Tropicalization

The complement $\mathbb{P}^{d+1, \mathrm{an}} \setminus T^{\mathrm{an}}$ consists of the hyperplanes $z_i = 0$ and do not meet X^{val} , so $X^{\mathrm{val}} \subset T^{\mathrm{an}}$. Note, however, $X^{\mathrm{val}} \cap T^{\mathrm{val}} = \emptyset$.

By definition, the tropicalization of $X \cap T$ (viewed as a subset of T and with respect to the given valuation on the ground field), is the image

$$(X \cap T)^{\text{trop}} := \text{trop}(X^{\text{an}} \cap T^{\text{an}}) \subset N_{\mathbb{R}}.$$

By the Fundamental Theorem of Tropical Geometry (FTTG for short, due to Kapranov in the case of hypersurfaces [45, Theorem 3.1.3]), the tropicalization admits a different description. Namely, the Laurent polynomial $g(z) = 1 + t \frac{f(z)}{z_0 \cdots z_{d+1}} \in K[M]$ can be written as

$$g(z) = 1 + t \sum_{m \in \Delta} a_m z^m,$$

where $a_m \in \mathbb{C}$ and $a_m \neq 0$ whenever m is a vertex of Δ . Then we have $X \cap T = V(g)$. The tropicalization of g is the convex, piecewise affine function on $N_{\mathbb{R}}$ given by

$$g^{\text{trop}}(n) = \max\{0, -1 + \max_{m \in \Delta \cap M, a_m \neq 0} \langle m, n \rangle\} = \max\{0, -1 + \max_{0 \leq i \leq d+1} \langle m_i, n \rangle\},$$

and the FTTG says that $(X \cap T)^{\text{trop}}$ is the locus where the function g^{trop} fails to be locally affine. Its complement in $N_{\mathbb{R}}$ has a unique bounded component, namely the set

$$\{n \in N_{\mathbb{R}} \mid \max_i \langle m_i, n \rangle < 1\},$$

whose closure is exactly the simplex Δ^\vee above. In particular, $B = \partial \Delta^\vee \subset (X \cap T)^{\text{trop}}$.

Lemma 7.1. *The map $\text{trop}: X^{\text{an}} \cap T^{\text{an}} \rightarrow (X \cap T)^{\text{trop}} \subset N_{\mathbb{R}}$*

(1) *induces a homeomorphism of $\tilde{\tau}_i$ onto τ_i , and of $\text{Sk}(X)$ onto B ;*

(2) *fits in the commutative diagram*

$$\begin{array}{ccc} & & \tau_i^\circ \\ & \nearrow^{\text{trop}} & \uparrow \simeq \\ U_i & & \tilde{\tau}_i^\circ \\ & \searrow_{r_i} & \downarrow \text{trop} \end{array}$$

(3) *in particular, satisfies $U_i := \text{sp}_{\mathcal{X}}^{-1}(\xi_i) = \text{trop}^{-1}(\tau_i^\circ)$;*

(4) *induces an isomorphism between the integral affine structures on $\tilde{\tau}_i$ and τ_i .*

Proof. We consider a d -dimensional simplex $\tilde{\tau}_i$ in $\text{Sk}(X)$ as described in §7.2. Given numbers $\lambda_j \in \mathbb{R}_{\geq 0}$, $j \neq i$ with $\sum \lambda_j = 1$, let $v_{i,\lambda}$ be the minimal valuation on $\mathcal{O}_{\mathcal{X}, \xi_i}$ such that $v_{i,\lambda}(w_{i,j}) = \lambda_j$ for all $j \neq i$. By definition, $\text{trop}(v_{i,\lambda}) \in N_{\mathbb{R}}$ satisfies

$$\langle m, \text{trop}(v_{i,\lambda}) \rangle = -v_{i,\lambda}(z^m)$$

for all $m \in M$. From (7.1) and (7.2) we have $\langle m_i, \text{trop}(v_{i,\lambda}) \rangle = -v_{i,\lambda}(t^{-1}u_i) = 1$ and

$$\begin{aligned} \langle m_j, \text{trop}(v_{i,\lambda}) \rangle &= -(d+1)v_{i,\lambda}(w_{i,j}) + \sum_{l \neq i,j} v_{i,\lambda}(w_{i,l}) \\ &= -(d+1)\lambda_j + \sum_{l \neq i,j} \lambda_l = -(d+2)\lambda_j + 1 \leq 1, \end{aligned} \quad (7.3)$$

for $j \neq i$. This means that $\text{trop}(v_{i,\lambda})$ lies in $\tau_i \subset B$, thus $\text{trop}(\tilde{\tau}_i) \subseteq \tau_i$. The composition

$$\begin{array}{ccccc} \{\sum_{j \neq i} \lambda_j = 1\} & \xrightarrow{\cong} & \tilde{\tau}_i & \xrightarrow{\text{trop}} & \tau_i \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}_{\geq 0}^{\{0, \dots, d+1\} \setminus \{i\}} & & \text{Sk}(X) & & B \end{array}$$

is injective by (7.3) and surjective. Indeed, given $n \in \tau_i$, set

$$\lambda_j = \frac{1}{d+2}(1 - \langle n, m_j \rangle)$$

for $j \neq i$. As $-(d+1) \leq \langle n, m_j \rangle \leq 1$ we have $\lambda_j \in \mathbb{R}_{\geq 0}$. Moreover,

$$\sum_{j \neq i} \lambda_j = \frac{1}{d+2}(d+1 - \langle n, \sum_{j \neq i} m_j \rangle) = \frac{1}{d+2}(d+1 + \langle n, m_i \rangle) = 1.$$

It follows that the restriction $\text{trop}: \tau_i \rightarrow \tilde{\tau}_i$ is a homeomorphism, and $\text{trop}: \text{Sk}(X) \rightarrow B$ does too.

Part (3) follows directly from (2). For (2), we first check that $\text{trop}(v) \in \tau_i^\circ = \{\max_{j \neq i} m_j < m_i = 1\}$ for any $v \in U_i$. Indeed, we have $\langle m_i, \text{trop}(v) \rangle = -v(t^{-1}u_i) = 1$ and $\langle m_j, \text{trop}(v) \rangle = 1 - (d+2)v(w_{i,j}) < 1$ since $z_j = 0$ at ξ_i . To conclude, it is enough to check that $\text{trop}(v) = \text{trop} \circ r_i(v)$ on an integral basis for M :

$$\langle \text{trop}(v), e_j - e_i \rangle = -v(w_{i,j}) = -\lambda_j = -v_{i,\lambda}(w_{i,j}) = \langle \text{trop}(v_{i,\lambda}), e_j - e_i \rangle.$$

For (4), we recall that the affine functions on $\tilde{\tau}_i$ are integral linear combinations of $v \mapsto v(w_{i,j})$, for $j \neq i$. The affine functions on τ_i are the elements of M ; as the set $\{e_j - e_i\}_{j \neq i}$ forms an integral basis for M , the affine functions on τ_i are integral linear combinations of $n \mapsto \langle e_j - e_i, n \rangle$. From $\langle \text{trop}(v_{i,\lambda}), e_j - e_i \rangle = -v_{i,\lambda}(w_{i,j})$ we obtain (4). \square

7.4. Affinoid torus fibration

As explored in [39,53,51], the analytification X^{an} of the Calabi–Yau variety X admits various affinoid torus fibrations. Here we show that the tropicalization map induces affinoid torus fibrations on the open subsets U_i of X^{an} .

We recall that a continuous map $\rho : Y^{\text{an}} \rightarrow S$ to a topological space S is an n -dimensional affinoid torus fibration at a point $s \in S$ if there exists an open neighborhood U of s in S , such that the restriction to $\rho^{-1}(U)$ fits into a commutative diagram:

$$\begin{array}{ccc} \rho^{-1}(U) & \xrightarrow{\cong} & \text{trop}^{-1}(V) \\ \rho \downarrow & & \downarrow \text{trop} \\ U & \xrightarrow{\cong} & V, \end{array}$$

V being an open subset of \mathbb{R}^n , the upper horizontal map an isomorphism of analytic spaces, the lower horizontal map a homeomorphism, and the map trop defined as in §1.2 for a torus of dimension n . In particular, an affinoid torus fibration induces an integral affine structure on the base S ; see [39, §4.1] for more details.

Corollary 7.2. *For any i , the map $\text{trop} : U_i \rightarrow \tau_i^\circ$ is an affinoid torus fibration. Moreover, the induced integral affine structure on τ_i° agrees with the one constructed in §2.1.*

Proof. By Lemma 7.1, $\text{trop}|_{U_i}$ is homeomorphic to the retraction r_i that is an affinoid torus fibration, see for instance the proof of [53, Theorem 6.1]. Again by Lemma 7.1, $\tilde{\tau}_i^\circ$ and τ_i° are isomorphic as integral affine manifolds, hence the integral affine structure induced on τ_i° by the affinoid torus fibration trop is isomorphic to the one constructed in §2.1. \square

8. Solution to the non-Archimedean Monge–Ampère equation

Let $L := \mathcal{O}(d+2)|_X$. We are now ready to prove Theorem A in the introduction. Namely, we show that the preceding method recovers the solution to the non-Archimedean Monge–Ampère equation [7], for a symmetric measure supported on the skeleton.

8.1. Existence and uniqueness

As mentioned above, to any (global) continuous psh metric Ψ on L^{an} is associated a measure $\text{MA}_{\text{NA}}(\Psi)$ on X^{an} . By [64] we have $\text{MA}_{\text{NA}}(\Psi_1) = \text{MA}_{\text{NA}}(\Psi_2)$ iff $\Psi_1 - \Psi_2$ is a constant, whereas the main result of [7] (see also [10]) states that for any measure ν supported on $\text{Sk}(X) \subset X^{\text{an}}$, there exists Ψ such that $\text{MA}_{\text{NA}}(\Psi) = \nu$.

8.2. Comparing Monge–Ampère measures

Given $\psi \in \mathcal{Q}_{\text{sym}}$ we want to compare the tropical Monge–Ampère measure ν_ψ on B with the non-Archimedean Monge–Ampère measure of the continuous psh metric $\text{FS}(\psi)$ on L^{an} , via the embedding $B \xrightarrow{\sim} \text{Sk}(X) \subset X^{\text{an}}$.

Theorem 8.1. *Let $\psi \in \mathcal{Q}_{\text{sym}}$ be any symmetric c -convex function. Then the associated continuous psh metric $\text{FS}(\psi)$ on L^{an} has Monge–Ampère measure $\text{MA}_{\text{NA}}(\text{FS}(\psi)) = d! \nu_\psi$, viewed as a measures on $B \simeq \text{Sk}(X) \subset X^{\text{an}}$.*

To prove the theorem, we want to use Vilsmeier’s result in Lemma 6.1, but our torus T is of the wrong dimension. Instead, we use the fact that X^{an} admits local affinoid torus fibrations with bases that are open subsets of $B \simeq \text{Sk}(X)$, and that these fibrations are compatible with the embedding of X into the toric variety \mathbb{P}^{d+1} , see §7.4.

Proof. We first consider the case when the measure ν_ψ gives full mass to the union of the open d -dimensional simplices τ_i° of B . By Corollary 7.2, the tropicalization map gives a affinoid torus fibration $\text{trop}: U_i \rightarrow \tau_i^\circ$. For any $i \neq j$ we have

$$\text{FS}(\psi) - \log |\chi^{m_j}| = (\psi - m_j) \circ \text{trop}$$

on U_i , by (6.2). Lemma 6.1 and Corollary 4.11 now give that $\text{MA}_{\text{NA}}(\text{FS}(\psi)) = d! \nu_\psi$ on $\tau_i^\circ \subset U_i$. As $\text{MA}_{\text{NA}}(\text{FS}(\psi))$ and ν_ψ have mass $d!|A|$ and $|A|$, respectively, we have accounted for all the mass of $\text{MA}_{\text{NA}}(\text{FS}(\psi))$, so $\text{MA}_{\text{NA}}(\text{FS}(\psi)) = d! \nu_\psi$.

Now consider the general case. We can find a sequence (ν_n) of symmetric measures on B of mass $|A|$ converging weakly to ν_ψ and such that ν_n gives full mass to $\bigcup_i \tau_i^\circ$. By what precedes, there exists a unique $\psi_n \in \mathcal{Q}_{\text{sym}}$ such that $\nu_{\psi_n} = \nu_n$ and $\int_B \psi_n d\nu = 0$. By compactness of $\mathcal{Q}_{\text{sym}}/\mathbb{R}$ we may assume that ψ_n converges uniformly to a function $\psi \in \mathcal{Q}_{\text{sym}}$. By continuity of the Fubini–Study operator, $\text{FS}(\psi_n)$ converges uniformly to $\text{FS}(\psi)$. It follows that the Monge–Ampère measures $\text{MA}_{\text{NA}}(\text{FS}(\psi_n))$ converge weakly to $\text{MA}_{\text{NA}}(\text{FS}(\psi))$. Since $\text{MA}_{\text{NA}}(\text{FS}(\psi_n)) = d! \nu_n$ for all n , we get $\text{MA}_{\text{NA}}(\text{FS}(\psi)) = d! \nu_\psi$, and we are done. \square

8.3. Invariance under retraction

Let ν be a symmetric positive measure of mass $(d+2)^{d+1}$ on $\text{Sk}(X)$. The results above give a rather explicit description of the solution (which is unique, up to a constant) to the non-Archimedean Monge–Ampère equation $\text{MA}_{\text{NA}}(\psi) = \nu$ on the Calabi–Yau variety $X \subset \mathbb{P}^{d+1}$. For one thing, ψ is the restriction of a torus invariant metric on $\mathcal{O}_{\mathbb{P}^{d+1}}(d+2)$. Note that we are *not* assuming that X is invariant under any torus action.

Here we investigate further properties of the solution.

Corollary 8.2. *Let ν be any symmetric positive measure on $\text{Sk}(X)$ of mass $(d+2)^{d+1}$, and let Ψ be a continuous psh metric on L^{an} , whose Monge–Ampère measure equals ν . Then Ψ is invariant under retraction to $\text{Sk}(X)$, in the following sense: for any $j \neq i$, the function $\Psi_{i,j} := (\Psi - \log |\chi^{m_j}|)|_{U_i}$ satisfies $\Psi_{i,j} = \Psi_{i,j} \circ r_i$.*

Proof. By Theorem 5.2 there exists a function $\psi \in \mathcal{Q}_{\text{sym}}$ such that $\nu_\psi = \nu$, and by Theorem 8.1 $\Psi = \text{FS}(\psi)$. By (6.2) we have $\text{FS}(\psi) - \log |\chi^{m_j}| = (\psi - m_j) \circ \text{trop}$ on T^{an} . By Lemma 7.1, we have $\text{trop} = \text{trop} \circ r_i$ on U_i , hence the claim follows. \square

9. Applications to the SYZ conjecture

In this section we prove Corollary C, following the work of Li. As f is admissible, for any $t \in \mathbb{C}^*$,

$$X_t := \{z_0 \dots z_{d+1} + t f_{d+2}(z) = 0\} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$$

is a smooth complex projective variety, which we view as a complex manifold. Set

$$\alpha_t := \frac{1}{(\log |t|^{-1})} c_1(\mathcal{O}_{\mathbb{P}^{d+1}}(d+2)|_{X_t}).$$

We equip X_t with the unique Ricci flat Kähler metric in α_t . Let ν_t be the corresponding smooth positive measure on X_t , and write (X_t, d_t) for associated metric space.

9.1. Proof of Corollary C

We follow [40]. Let us identify

$$\mathcal{X} = \{z_0 \dots z_{d+1} + t f_{d+2}(z) = 0\}$$

with the associated (singular) $d+1$ -dimensional complex analytic subspace of $\mathbb{P}^{d+1} \times \mathbb{C}$. The central fiber \mathcal{X}_0 consists of the $d+2$ coordinate hyperplanes $H_i = \{z_i = 0\}$ in $\mathbb{P}^{d+1} \simeq \mathbb{P}^{d+1} \times \{0\}$. For each i , let $\xi := \bigcap_{j \neq i} H_j \subset \mathcal{X}_0$. Then \mathcal{X} is smooth at ξ_i , and we may find local holomorphic coordinates w_j , $j \neq i$, at ξ_i such that $t = w_0 \dots w_d$. If we pick small (disjoint) neighborhoods W_i of ξ_i , and set $W := \bigcup_i W_i$, then we have a continuous map

$$\text{Log}_{\mathcal{X}}: W \setminus \mathcal{X}_0 \rightarrow \bigcup_i \tau_i^\circ \subset B$$

defined by $\text{Log}_{\mathcal{X}} = \sum_{j \neq i} \frac{\log |w_j|}{\log |t|} n_j$ on W_i .

By [8] (see [40, §3.1]), most of the mass of X_t lies in W for $t \approx 0$. Indeed:

Lemma 9.1. *We have $\lim_{t \rightarrow 0} \nu_t(X_t \cap W) / \nu_t(X_t) = 1$.*

Proof. Viewed as a scheme over $\mathbb{C}[[t]]$, \mathcal{X} is a minimal dlt model, whose skeleton $\text{Sk}(\mathcal{X})$ equals the essential skeleton $\text{Sk}(X)$. If \mathcal{X} were a semistable model, the lemma would follow from [8]. Now, there exists a projective birational morphism $\pi: \mathcal{X}' \rightarrow \mathcal{X}$ such that \mathcal{X}' is an snc model of X , and such that π is an isomorphism over the regular part of

\mathcal{X} . It follows that the only d -dimensional simplices of $\text{Sk}(\mathcal{X}')$ contained in $\text{Sk}(X)$ are associated to $\xi'_i := \pi^{-1}(\xi_i)$, $0 \leq i \leq d$. We can pick corresponding neighborhoods W'_i of ξ'_i such that $\pi(W'_i) \subset W_i$ for all i , and hence $X_t \cap W' \subset X_t$, where $W' = \bigcup_i W'_i$. By [8, Theorem 3.4], we have $\lim_{t \rightarrow 0} \nu_t(X_t \cap W')/\nu_t(X_t) = 1$, and the result follows. \square

Let $\psi \in \mathcal{Q}_{\text{sym}}$ be such that ν_ψ equals Lebesgue measure on $B \simeq \text{Sk}(X)$. In particular, the restriction of ψ to a d -dimensional face τ_i° satisfies the real Monge–Ampère equation, and is therefore smooth outside a small closed subset.

Let $\phi := \psi^c$ be the c-transform of ψ , viewed as a continuous convex function on Δ . This defines a continuous psh metric on $\mathcal{O}_{\mathbb{P}^{d+1}}(d+2)^{\text{an}}$ whose restriction to X^{an} has Monge–Ampère measure equal to ν , by Theorem 8.1.

The continuous convex function on Δ also defines a continuous psh metric on the holomorphic line bundle $\mathcal{O}(d+2)$ on \mathbb{P}^{d+1} . Approximating ψ by a smooth strictly convex function, we can approximate this metric uniformly by a Kähler metric on $\mathcal{O}(d+2)$. As in [40, Lemma 4.1], this leads to the existence, given $\varepsilon > 0$, of a Kähler metric ω_t on (X_t, α_t) , such that, for any i , ω_t has a local potential on $W_i \cap X_t$ that differs from the function $\psi \circ \text{Log}_{\mathcal{X}}$ by at most ε .

As explained in [40, Lemma 4.2], we may, for small t and ε , by shrinking W (so that $\text{Log}_{\mathcal{X}}(W \cap X_t)$ is contained in the smooth locus of ψ) find Lipschitz functions f_t of C^0 -norm on the order of ε , smooth outside a set of measure 0, such that the $(1, 1)$ -currents $\omega_{f,t} = \omega_t + dd^c f_t$ is positive on X_t and approximate $dd^c(\psi \circ \text{Log}_{\mathcal{X}})$ on the smooth locus of f in W and the measure $\omega_{f,t}^d$ is close to the Calabi–Yau volume form ν_t on X_t in total variation. The proof then proceeds to get the C^0 convergence on W of the potentials of the Calabi–Yau metrics on X_t to $\psi \circ \text{Log}_{\mathcal{X}}$ and the resulting C^∞ -convergence and special Lagrangian torus fibration on W in the same way as in §4.3 and §4.5 of [40].

9.2. Gromov–Hausdorff convergence

Consider the Fermat family

$$X_t = \{z_0 z_1 \dots z_{d+1} + t(z_0^{d+2} + \dots + z_{d+1}^{d+2})\} \subset \mathbb{P}_{\mathbb{C}}^{d+1}.$$

As above, we write (X_t, d_t) for the corresponding metric space.

Let $\psi \in \mathcal{Q}_{\text{sym}}$ be a solution (unique, up to a constant) to the tropical Monge–Ampère equation $\nu_\psi = \nu$, where ν is Lebesgue measure on B , see Theorem 5.2, and let Ψ be the corresponding metric on the affine \mathbb{R} -bundle Λ , see Remark 4.14. By the regularity theory for the real Monge–Ampère equation on \mathbb{R}^d , there exists an open subset $\mathcal{R}_\psi \subset B_0$ such that $B \setminus \mathcal{R}_\psi$ has $(d-1)$ -Hausdorff measure zero, and such that Ψ is smooth and strictly convex over \mathcal{R}_ψ . The Hessian of Ψ on \mathcal{R}_ψ then defines a metric on \mathcal{R} , and we let $(\mathcal{R}_\psi, d_\psi)$ be the resulting metric space.

By diameter bounds proved in [44], (X_t, d_t) converges in the sense of Gromov–Hausdorff after passing to subsequence. By [41, Theorem 5.1], any subsequential limit of

(X_t, d_t) contain a dense subset locally isomorphic to the regular part of a Monge–Ampère metric on B_0 . By the injectivity in Theorem B, the latter space is uniquely determined as $(\mathcal{R}_\psi, d_\psi)$.

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