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Toughness of Recursively Partitionable Graphs

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Abstract

A simple graph $G = (V, E)$ on n vertices is said to be *recursively partitionable* (RP) if $G \simeq K_1$, or if G is connected and satisfies the following recursive property: for every integer partition $a_1, a_2, \ldots, a_k$ of n, there is a partition $\{A_1, A_2, \ldots, A_k\}$ of V such that each $|A_i| = a_i$, and each induced subgraph $G[A_i]$ is RP ($1 \leq i \leq k$). We show that if S is a vertex cut of an RP graph G with $|S| \geq 2$, then $G - S$ has at most $3|S| - 1$ components. Moreover, this bound is sharp for $|S| = 3$. We present two methods for constructing new RP graphs from old. We use these methods to show that for all positive integers s, there exist infinitely many RP graphs with an s-vertex cut whose removal leaves $2s + 1$ components. Additionally, we prove a simple necessary condition for a graph to have an RP spanning tree, and we characterise a class of minimal 2-connected RP graphs.

1 Introduction

Let n be a positive integer. An **integer partition** of n is a list $a_1, \ldots, a_k$ of positive integers such that $a_1 \leq a_2 \leq \cdots \leq a_k$ and $a_1 + \cdots + a_k = n$. Let $G = (V, E)$ be a graph of order n. An **$(a_1, \ldots, a_k)$-partition** of G is a partition $\{A_1, \ldots, A_k\}$ of V such that $|A_i| = a_i$ for all i. We say the partition has **connected parts** if, for all $i \in \{1, \ldots, k\}$, the induced subgraphs $G[A_i]$ are connected.

In 1976, Györi and Lovász considered the problem of determining when a graph has an $(a_1, \ldots, a_k)$-partition with connected parts and independently proved the following theorem.

Theorem 1 (Györi-Lovász [15, 21])**.** *Let G be a graph of order n and $a_1, \ldots, a_k$ an integer partition of n. If G is k-connected, then it has an $(a_1, \ldots, a_k)$-partition with connected parts.*

We say G is **arbitrarily partitionable** (or just **AP**) if, for every integer partition $a_1, \ldots, a_k$ of n, there exists an $(a_1, \ldots, a_k)$-partition of V with connected parts. AP graphs were introduced in [1], and a polynomial time algorithm for determining whether a subdivision of $K_{1,3}$ is AP was provided.

The graph G is **recursively partitionable** (**RP**) if $G \simeq K_1$, or G is connected and satisfies the following recursive property: for every integer partition $a_1, \ldots, a_k$ of n, there is an $(a_1, \ldots, a_k)$-partition $\{A_1, \ldots, A_k\}$ of V such that each $G[A_i]$ is RP. RP graphs were introduced in [6, 7].

In [7], RP trees were characterised (among other results), and in [6], a class of RP unicyclic graphs was characterised. In both papers, the authors made heavy use of the following characterisation of RP graphs.

Proposition 2. [6] *An n-vertex graph $G = (V, E)$ is RP if and only if it is connected, and:*

- *$G \simeq K_1$, or*
- *for every partition a, b of n, there is an (a, b)-partition $\{A, B\}$ of V such that both $G[A]$ and $G[B]$ are RP.*

RP graphs were independently introduced (as "partition wonderful graphs") as a result of investigations into rainbow-cycle-free edge colorings (such as in [16]), by Peter Johnson, with the help of Paul Horn, at the MASAMU 2020 workshop.

These graphs arise naturally when considering rainbow-cycle-free edge colorings, which are of recent interest in their own right (see [13, 17, 20]). A **JL-coloring** of an n-vertex graph is an edge coloring using exactly $n-1$ colors that does not contain any rainbow cycles. These colorings are studied for K_n in [11] and [14], $K_{n,m}$ in [19] and complete multipartite graphs in [18].

In [16], the authors introduced the following **standard construction** for creating a JL-coloring of a connected graph G:

1. If $n > 1$, find a partition $V = \{A, B\}$ with connected parts,

2. color edges between A and B with a single color that will not be used again,

3. iterate (1) and (2) on $G[A]$ and $G[B]$.

This leads to the main result of [16]:

Theorem 3. [16] *Every JL-coloring is obtainable by an instance of the standard construction.*

Corollary 4. [16] *Every JL-coloring of a connected graph $G = (V, E)$ is the restriction of a JL-coloring of the complete graph with vertex set V.*

Combining Proposition 2, Theorem 3 and Corollary 4 yields the following observation of Johnson:

Observation 5. *A connected graph $G = (V, E)$ of order n is RP if and only if every JL-coloring φ of K_n can be restricted to a JL-coloring $\varphi|_E$ of a copy of G.*

The rest of this paper is organised as follows. In Section 2, we define useful graph-theoretical tools and constructions that will be used throughout the paper. In Section 3, we list basic observations about the properties of AP and RP graphs. In Section 4, we introduce recursive constructions of RP graphs, which we later use to find infinite classes of RP graphs with a given toughness. It is easy to see that if a graph has an AP (RP) spanning tree, then it is AP (RP). In Section 5, we take a more detailed look at spanning subgraphs of RP graphs and provide a necessary condition for an RP graph to have a spanning tree homeomorphic to $K_{1,k}$. Denote by $\boldsymbol{c(G)}$ the number of components of G. We show that if an RP graph has an RP spanning tree, then for every $S \subseteq V$ we have $c(G-S) \leq |S|+2$. In Section 6, we find lower bounds for the maximum possible values of $c(G-S)$ for $S \subseteq V$ in an RP graph G. In particular, we show that, for any s, there exists an infinite family of RP graphs, each with an s-vertex cut whose removal leaves $2s+1$ components. In Section 7, we show that there exists a finite set of minimal RP graphs for any given possible cut size $|S|$ and $c(G-S)$. In Section 8, we bound $c(G-S)$ from above, by showing that in an RP graph G, for any $S \subseteq V$, we have $c(G-S) \leq 3|S|-1$, which shows that every RP graph is $\frac{1}{3}$-tough. Finally, in Section 9, we list a set of open questions.

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2 Additional definitions

2.1 Properties and parameters

For a positive integer k, let $\boldsymbol{E_k}$ denote the **empty graph** with k vertices and no edges. If G is a graph, then $\boldsymbol{n(G)}$ or $|\boldsymbol{G}|$ is its **order** (number of vertices), $\boldsymbol{m(G)}$ its number of edges. Let $\boldsymbol{\alpha(G)}$ denote the **independence number** of G (the order k of a maximum induced E_k subgraph). A **vertex cut** of a graph G is a set of vertices whose removal disconnects G, and the **vertex connectivity**, or simply connectivity, of G is the minimum cardinality of a vertex cut. Let

$$\boldsymbol{\sigma(G)} = \min\{d(u) + d(v) : u \text{ and } v \text{ are non-adjacent vertices of } G\}.$$

A graph is **traceable** if it has a spanning path (i.e., a Hamiltonian path) and **Hamiltonian** if it has a spanning cycle (i.e., a Hamiltonian cycle).

A **perfect matching** of a graph is a set M of edges that are pairwise disjoint, such that every vertex is incident with an edge in M. A **near-perfect matching** is a set M of edges that are pairwise disjoint, such that every vertex except for one is incident with an edge in M. A graph is **(near) matchable** if it has a (near) perfect matching.

Let $G_1 = (V_1, E_1), \ldots, G_n = (V_n, E_n)$ be graphs. The **sequential join** $G_1 + \cdots + G_n$ is the graph formed by taking the graph union $(V_1 \cup \cdots \cup V_n, E_1 \cup \cdots \cup E_n)$ and adding to it all edges of the form uv, where $u \in V_i$ and $v \in V_{i+1}$ $(1 \leq i < n)$.

Let $G = (V, E)$ be a graph. In particular, for a connected graph G, if $S \subseteq V$, then $c(G - S) \geq 2$ if and only if S is a cut. The **toughness** $\boldsymbol{\tau(G)}$ of G is

$$\tau(G) = \min\left\{\frac{|S|}{c(G-S)} : S \subseteq V, c(G-S) \geq 2\right\}.$$

For a positive real number r, we say G is r**-tough** if $\tau(G) \geq r$.

2.2 Graph constructions

In this section, we define graph constructions that we will use throughout the paper. See Figures 1 and 2 for examples.

Let $t_1, t_2, \ldots t_k$ be positive integers. The k**-pode graph** $T_k(t_1, t_2, \ldots, t_k)$ is the tree that has one degree k vertex, v, the removal of which leaves k paths having $t_1, t_2, \ldots t_k$ vertices. When $k = 3$, we say this is a **tripode** graph and, when it causes no confusion to the reader, denote it by $T(t_1, t_2, t_3)$.

Let k be a positive integer, and b_i, $1 \leq i \leq k$ be non-negative integers. A **generalized theta graph** graph $\Theta(b_1, b_2, \ldots, b_k)$ consists of a pair of endvertices joined by k internally disjoint paths of lengths $b_1, b_2, \ldots, b_k$ [10]. Note that the authors in [3] refer to these graphs as balloons.

Let k be a positive integer, and let $b_0, b_1, \ldots, b_k$ be non-negative integers. The **semistar** $K_{b_0}(b_1, b_2, \ldots, b_k)$ is the graph formed from the disjoint union of (possibly null) cliques K_{b_0}, K_{b_1}, $\ldots$, K_{b_k} by adding every possible edge between a vertex of K_{b_0} and a vertex not belonging to K_{b_0}. Symbolically,

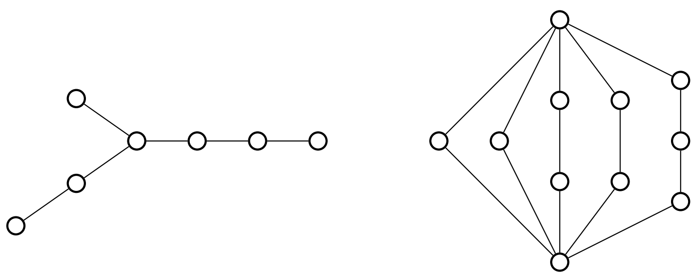
$$K_{b_0}(b_1, \ldots, b_k) = K_{b_0} + \left(\bigcup_{i=1}^{k} K_{b_i} \right).$$

Note that $K_b(0, \ldots, 0) \simeq K_0(0, \ldots, b, \ldots, 0) \simeq K_b$, and that $K_{b_0}(b_1, \ldots, b_k, 0) \simeq K_{b_0}(b_1, \ldots, b_k)$.

Throughout the paper, we will use the notation $H \leq G$ to mean H is a subgraph G or $H < G$ to mean H is a proper subgraph of G. Observe that for semistars $H = K_{a_0}(a_1, \ldots, a_k)$ with $a_i \leq a_{i+1}$ for $1 \leq i \leq k-1$ and $G = K_{b_0}(b_1, \ldots, b_k)$ with $b_i \leq b_{i+1}$ for $1 \leq i \leq k-1$, $H < G$ if and only if $a_i \leq b_i$ for $0 \leq i \leq k$, with at least one of these inequalities being strict.

For a semistar $K_{b_0}(b_1, \ldots, b_k)$ and a set of graphs $\{G_i\}_{i=0}^{k}$ such that $n(G_i) = b_i$, the graph H is a **replacement graph** for $K_{b_0}(b_1, \ldots, b_k)$ with respect to $\{G_i\}_{i=0}^{k}$ if

$$H = G_0 + \left(\bigcup_{i=1}^{k} G_i \right).$$



$T(1,2,3)$ $\quad\quad$ $\Theta(2,2,3,3,4)$

Figure 1: The tripode $T(1,2,3)$ and the generalized theta graph $\Theta(2,2,3,3,4)$.

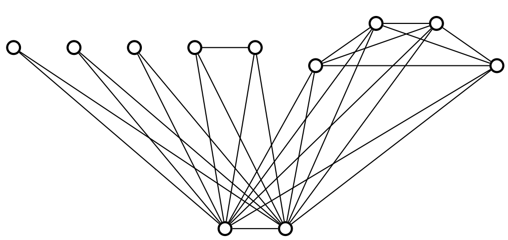


Figure 2: The semistar $K_2(1,1,1,2,4)$.

3 Elementary and known results

In this section, we list a number of useful literature results on AP and RP graphs. We make frequent use of these results and observations, particularly Lemma 6, Theorem 13

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and Observations 15 and 16. We also present a characterisation of AP and RP complete multipartite graphs.

Lemma 6. [7] *If a graph G has an RP (AP) spanning subgraph, then G is itself RP (AP).*

Observation 7. [7] *Let G be a graph. The following implications for properties of G hold, and none of their converses hold:*

$$\textit{traceable} \implies \textit{RP} \implies \textit{AP} \implies \textit{(near) matchable.}$$

The following lemma, by Bondy and Chvatal [9] is a somewhat well-known variation of Ore's Hamiltonicity Theorem [23].

Lemma 8. [9] *Let G be a graph of order n. If $\sigma(G) \geq n-1$, then G is traceable.*

Theorem 9 (Ore's Theorem [23])**.** *Let G be a graph of order n. If $\sigma(G) \geq n$, then G is Hamiltonian.*

With Lemma 8 we easily prove the following.

Proposition 10. *Let G be a graph with $\sigma(G) \geq 2k$ and order n. If $n \leq 2k+1$, then G is RP (and therefore AP), and this bound is sharp.*

Proof. The graph G is RP since it is traceable (Observation 7 and Lemma 8).

To prove the bound is sharp, consider the complete bipartite graph $K_{k,k+2}$. This graph has $\sigma = 2k$ and order $2k+2$. However $K_{k,k+2}$ does not have a perfect matching, and thus by Observation 7, it is not RP. □

In [22], Marczyk showed that the above result can be improved for AP graphs with the extra condition $\alpha(G) \leq \lceil \frac{n(G)}{2} \rceil$.

Theorem 11. [22] *Let G be a connected graph of order n. If $\alpha(G) \leq \lceil \frac{n}{2} \rceil$ and $\sigma(G) \geq n-3$, then G is AP.*

For G to have a (near) perfect matching, it is clearly necessary that $\alpha(G) \leq \lceil \frac{n(G)}{2} \rceil$. For a large class of graphs, including complete multipartite graphs, this condition is also sufficient. We summarize these equivalences in Proposition 12.

Note that there is no possible forbidden subgraph characterisation of AP (RP) graphs. Given any graph G of order n, the graph $K_n + G$ is Hamiltonian, and thus AP (RP).

Proposition 12. *Suppose G is a graph of order n such that $K_{a,b} \leq G \leq K_a + E_b$ for $a \leq b$ positive integers, or that G is a complete multipartite graph. The following are equivalent:*

(i) $\alpha(G) \leq \lceil \frac{n}{2} \rceil$,

(ii) G has a (near) perfect matching,

(iii) G is traceable,

(iv) G is AP,

(v) G is RP.

Proof. It is easy to verify that *(iii)* implies *(ii)* and that *(ii)* implies *(i)*. We now argue that *(i)* implies *(iii)*. If G is complete multipartite or $K_{a,b} \leq G \leq K_a + E_b$, and $\alpha(G) \leq \lceil \frac{n}{2} \rceil$, then the minimum degree satisfies $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$. Consider the join $G + \{v\}$ and note that $\delta(G + \{v\}) \geq \frac{n+1}{2}$. By Ore's Theorem (Theorem 9), $G + \{v\}$ has a spanning cycle C, so $C - v$ is a spanning path of G. Observation 7 completes the proof. □

In [7], Baudon, Gilbert and Woźniak characterised RP trees. In [3], Baudon, Bensmail, Foucaud and Pilsniak described some properties of RP generalized theta graphs.

Theorem 13. [7] *A tree is RP if and only if it is either a path, the tripode $T(2,4,6)$, or a tripode $T(a,b,c)$, where (a,b,c) is one of the triples in Table 1.*

$(1,1,c)$	$c \equiv 0 \pmod 2$		$(1,4,c)$	$c \in \{5,6,8,10,13,18\}$
$(1,2,c)$	$c \equiv 0 \pmod 3$ or $c \equiv 1 \pmod 3$		$(1,5,6)$	
$(1,3,c)$	$c \equiv 0 \pmod 2$		$(1,6,c)$	$c \in \{7,8,10,12,14\}$

Table 1: Table of triples $(1,b,c)$, $1 \leq b \leq c$, for which the graph $T(a,b,c)$ is RP.

Theorem 14. [3] *Let B be the generalized theta graph $\Theta(b_1,\ldots,b_k)$ with $b_1 \leq \cdots \leq b_k$. If B is RP, then $k \leq 5$. Further, if B is RP and $k \in \{4,5\}$, then $b_1 \leq 8$ and $b_2 \leq 40$, but b_k can be arbitrarily large.*

Tripodes and generalized theta graphs are "universal" for RP graphs with connectivity 1 and 2, respectively, as the following observation from [5] shows.

Observation 15. [5] *Let S be a vertex cut of a connected graph G, let $C_1,\ldots,C_k$ denote the components of $G - S$, and let c_i denote $n(C_i)$.*

- *If $|S| = 1$ and G is RP (AP), then the k-pode $T_k(c_1,\ldots,c_k)$ is RP (AP),*
- *If $|S| = 2$ and G is RP (AP), then the generalized theta graph $\Theta(c_1+1,\ldots,c_k+1)$ is RP (AP).*

To discuss AP and RP graphs of arbitrary connectivity, we find it easiest to work with the semistars $K_{b_0}(b_1,\ldots,b_k)$, as they are also "universal".

Observation 16. *Let S be an s-vertex cut of a graph G, let $C_1,\ldots,C_k$ denote the components of $G - S$, and let c_i denote $n(C_i)$. If G is RP (AP), then the semistar $K_s(c_1,\ldots,c_k)$ is RP (AP).*

Proof. Notice that G is a spanning subgraph of $K_s(c_1,\ldots,c_k)$ and apply Lemma 6. □

Per Observations 15 and 16, the triples (a,b,c) in Table 1 for which $T(a,b,c)$ is RP are also the triples for which $K_1(a,b,c)$ is RP.

The 'universality' of trees and balloons (generalised theta graphs) can be used to bound the toughness of RP graphs with small cuts. Using Theorem 13, Theorem 14, and Observation 15, we obtain Proposition 17 below.

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Proposition 17. [3] *Let G be an RP graph, and S a subset of $V(G)$.*

- *If $|S| = 1$, then $c(G - S) \leq 3$.*
- *If $|S| = 2$, then $c(G - S) \leq 5$.*

4 New RP graphs from old

In this section, we present two operations for combining RP graphs to obtain new RP graphs: the well-known sequential join, and a "subgraph replacement" operation. These constructions, in tandem with Lemma 6, allow us to easily prove that many graphs encountered in the rest of the paper are RP. Of particular interest is the use of replacement graphs in Section 6 to construct RP graphs with large vertex cuts that leave many components.

There is a generalisation of the fact that paths are RP. In particular, the sequential join of RP graphs is RP.

Proposition 18. *Let $H_1, \ldots, H_k$ be RP graphs. If G is the sequential join of the graphs H_i, $G = H_1 + \cdots + H_k$, then G is RP.*

Proof. Let n_i be the order of H_i, and $n = n_1 + \cdots + n_k$ the order of G. We proceed by induction on n. The base case $n = 1$ is trivial, as then $G \simeq K_1$, which is RP.

Let $n \geq 2$, assume the proposition is true for all positive integers less than n, and let G be an n-vertex sequential join of RP graphs $H_1, \ldots, H_k$. It suffices to show that for any $a \in [1, n-1]$, there exists a partition of G into two RP graphs $G[A]$ and $G[B]$ such that $G[A]$ has order a. To do this, we will pick the subgraph induced by the 'leftmost' a vertices of G in a manner that breaks apart at most one of the graphs H_i.

Let $m_0 = 0$, and for all $i \in [1, k]$, let $m_i = \sum_{j=1}^{i} n_j$. Denote by s the largest non-negative integer such that $a \geq m_s$. Since the graph H_{s+1} is RP, it can be partitioned into two RP parts $H_{s+1}[X]$ and $H_{s+1}[Y]$, such that $|X| = a - m_s$. We can thus pick $A = V(H_1) \cup \cdots \cup V(H_s) \cup X$ and $B = Y \cup V(H_{s+2}) \cup \cdots \cup V(H_k)$. Note that $G[A] = H_1 + \cdots + H_s + H_{s+1}[X]$ and $G[B] = H_{s+1}[Y] + H_{s+2} + \cdots + H_k$. By the induction hypothesis, both $G[A]$ and $G[B]$ are RP, completing the proof. □

A consequence of Proposition 18 is that the suspension $K_1 + G$ of an RP graph G is RP.

Corollary 19. *Suppose t is a positive integer. If $K_{a_0}(a_1, \ldots, a_k)$ and $K_{b_0}(b_1, \ldots, b_j)$ are RP, then so is the graph*

$$K_{a_0+b_0+t}(a_1, \ldots, a_k, b_1, \ldots, b_j).$$

Proof. The graph $K_{a_0+b_0+t}(a_1, \ldots, a_k, b_1, \ldots, b_j)$ has a spanning subgraph isomorphic to

$$K_{a_0}(a_1, \ldots, a_k) + K_t + K_{b_0}(b_1, \ldots, b_j),$$

which is RP by Proposition 18. □

Let $\{G_i\}_{i=1}^k$, $k \geq 2$ be a set of graphs, $J = G_1 + \cdots + G_k$, and $m = \min\{\tau(G_i) : 1 \leq i \leq k\}$ be the minimum toughness among the graphs G_i. Note that $\tau(J) \geq \frac{1}{2}$ as J contains a path with a vertex in each G_i and paths are $\frac{1}{2}$-tough. If $m < \frac{1}{2}$, then $\tau(J) > m$. Thus, there are limitations to how low the toughness of a sequential join of RP graphs can be. However, replacement graphs can provide RP graphs with high connectivity and low toughness (see Corollary 25).

Theorem 20 (RP Replacement Theorem)**.** *Every replacement graph for an RP semistar with respect to a set of RP graphs is RP. That is, suppose $K_{b_0}(b_1, \ldots, b_k)$ is RP, and $\{G_i\}_{i=0}^k$ is a set of RP graphs such that $n(G_i) = b_i$. Then the graph H is RP, where*

$$H = G_0 + \left(\bigcup_{i=1}^{k} G_i \right).$$

Proof. We use induction on the order of the replacement graph with RP parts. Clearly every replacement graph of order at most 3 is RP. Suppose that every replacement graph of order at most $n-1$ for an RP semistar with respect to a set of RP graphs is RP, and let H be a replacement graph of order n for an RP semistar $K = K_{b_0}(b_1, \ldots, b_k)$ with respect to a set of RP graphs $\{G_i\}_{i=0}^k$. Let λ be any positive integer such that $1 \leq \lambda < n$. It suffices to prove that there is a partition $V(H) = \{X', Y'\}$ such that $|X'| = \lambda$ and $H[X']$, $H[Y']$ are both RP.

Since K is RP, there is a partition $V(K) = \{X, Y\}$ such that $|X| = \lambda$, and the induced subgraphs $K[X] = K_{x_0}(x_1, \ldots, x_k)$ and $K[Y] = K_{y_0}(y_1, \ldots, y_k)$ are RP. Note that $x_i + y_i = b_i$, and that we may have $x_i = 0$ ($y_i = 0$) for some i. Since G_i is RP, it has a partition $V(G_i) = \{X_i, Y_i\}$ with $|X_i| = x_i$ and $|Y_i| = y_i$ such that $G_i[X_i]$ and $G_i[Y_i]$ are RP.

Thus, let

$$X' = \bigcup_{i=0}^{k} X_i \quad \text{and} \quad Y' = \bigcup_{i=0}^{k} Y_i.$$

Note that $\{X', Y'\}$ is a partition of $V(H)$ such that $|X'| = x_0 + \cdots + x_k = |X| = \lambda$. Further, both $H[X']$ and $H[Y']$ are replacement graphs for RP semistars with respect to sets of RP graphs:

$$H[X'] = G_0[X_0] + \left(\bigcup_{i=1}^{k} G_i[X_i] \right) \quad \text{and} \quad H[Y'] = G_0[Y_0] + \left(\bigcup_{i=1}^{k} G_i[Y_i] \right).$$

By the induction hypothesis, both $H[X']$ and $H[Y']$ are RP, so H is RP. □

5 RP spanning subgraphs

It is clear that every graph with an RP (AP) spanning tree is RP (AP). In [7], it was shown that an AP graph need not have an AP spanning tree. Using the sequential join, it is easy to construct RP graphs that do not have a spanning tripode $T(a, b, c)$ (and thus do not have an RP spanning tree). For example, the graph $T(1, 1, 2) + K_1 + T(1, 1, 2)$ is RP but has no spanning tripode. In this section, we give a necessary condition for a graph to have a spanning tree homeomorphic to $K_{1,k}$ ($k \in \mathbb{N}$).

Theorem 21. *Let G be a graph, $k \geq 2$ a positive integer and S a subset of $V(G)$. If $c(G-S) \geq |S|+k$, then G does not have a spanning subdivision of $K_{1,k}$.*

Proof. Let $c(G-S) = c$ and $|S| = s$. Let $G_1, G_2, \ldots, G_c$ denote the components of $G-S$. Assume contrary to the theorem statement that there is a subdivision T of $K_{1,k}$ spanning G, and that $c - s \geq k$. Denote by v the vertex of T such that $d_T(v) = k$, and let $P_1, P_2, \ldots, P_k$ be the k maximal paths of $T - v$. There are two cases to consider. In both cases, we count the number of components G_i that each path P_j intersects.

Case 1: $v \notin S$.

Assume without loss of generality that $v \in G_1$. Let $\zeta(P_i) = |\{j \geq 2 : V(G_j) \cap V(P_i) \neq \emptyset\}|$ be the number of components (other than G_1) that contain a vertex of P_i. Between any two vertices of P_i that lie in different components of $G-S$, there must be a vertex of S. Therefore, for all i, we have

$$\zeta(P_i) \leq |S \cap V(P_i)|. \tag{1}$$

Each of the components $G_2, G_3, \ldots, G_c$ must intersect at least one path P_i, so

$$c - 1 \leq \sum_{i=1}^{k} \zeta(P_i). \tag{2}$$

Since the paths P_i are disjoint, we have

$$\sum_{i=1}^{k} |S \cap V(P_i)| = |S| = s. \tag{3}$$

Combining Inequalities 1, 2 and 3, we obtain the following inequality

$$c - 1 \leq \sum_{i=1}^{k} \zeta(P_i) \leq \sum_{i=1}^{k} |S \cap V(P_i)| = s.$$

But this contradicts the fact that $c - s \geq k \geq 2$.

Case 2: $v \in S$

Let $\eta(P_i) = |\{j : V(G_j) \cap V(P_i) \neq \emptyset\}|$ be the number of components that contain a vertex of P_i. Between any two vertices of P_i from different components of $G-S$, there must be a vertex of $S \cap V(P_i)$. However, it is possible that the end vertices of P_i are not in S. Thus, for all i, the following inequality holds

$$\eta(P_i) \leq |S \cap V(P_i)| + 1. \tag{4}$$

Since every component G_i intersects at least one path P_i:

$$c \leq \sum_{i=1}^{k} \eta(P_i). \tag{5}$$

Since the paths P_i are disjoint, and none contain the vertex $v \in S$, we have

$$\sum_{i=1}^{k} |S \cap V(P_i)| = |S - \{v\}| = s - 1. \tag{6}$$

Putting Inequalities 4, 5 and 6 together, we obtain

$$c \leq \sum_{i=1}^{k} \eta(P_i) \leq \sum_{i=1}^{k} |S \cap V(P_i)| + k = s - 1 + k.$$

But this contradicts the fact that $c \geq s + k$. □

By Theorem 13, every RP tree on at least 3 vertices is either a path (subdivided $K_{1,2}$) or a subdivided $K_{1,3}$. Further, every RP generalized theta graph is spanned by a subdivided $K_{1,k}$ with $k \leq 5$, per Theorem 14. Thus, we have the following corollary.

Corollary 22. *If $G = (V, E)$ contains an RP spanning tree, then every $S \subset V$ satisfies $c(G - S) \leq |S| + 2$. If G is spanned by an RP generalized theta graph, then every $S \subset V$ satisfies $c(G - S) \leq |S| + 4$.*

6 Bounding $c(G - S)$ from below

Let G be an RP graph and $S \subseteq V(G)$. Per Theorem 13 and Observation 15, if $|S| = 1$, then $c(G - S) \leq 3$, and the infinite family of RP tripodes $\{T(1, 1, 2k)\}_{k\in\mathbb{N}}$ all achieve this bound. Theorem 14 and Observation 15 show that if $|S| = 2$, then $c(G - S) \leq 5$, and the RP generalized theta graphs $\{\Theta(2, 2, 3, 4, 2k + 1)\}_{k\in\mathbb{N}}$ achieve this bound [7]. In this section, we bound the maximum possible value of $c(G - S)$ from below. In particular, we show that for all s, there are infinitely many RP graphs with an s-vertex cut S such that $c(G - S) = 2s + 1$. Further, we prove that there exists an RP graph G with a cut S such that $|S| = 3$ and $c(G - S) = 8$.

Lemma 23. *The following graphs are RP:*

(i) $K_1(a, b, c)$ *for* $(a, b, c) = (2, 4, 6)$ *and all* (a, b, c) *in Table 1,*

(ii) $K_2(a, b, c, d)$ *for* $(a, b, c) = (2, 4, 6)$ *and all* (a, b, c) *in Table 1, and for all* $d \in \mathbb{N}$*,*

(iii) $K_0(0, \ldots, d, \ldots, 0)$ *for all* $d \in \mathbb{N}$*,*

(iv) $K_{b_0}(b_1, \ldots, b_k)$ *whenever* $k \leq b_0 + 1$*,*

(v) $K_2(1, 1, 1, 2, 4)$*, and* $K_2(1, 1, 2, 3, c)$ *for all* $c \equiv 0 \pmod 2$*.*

Proof. Part *(i)* follows from Theorem 13 and Observation 16. Part *(ii)* follows from *(i)*, Proposition 18, and the fact that $K_2(a, b, c, d)$ is spanned by $K_1(a, b, c)+K_1+K_d$. The graph $K_0(0, \ldots, d, \ldots, 0)$ is a complete graph of order d, from which *(iii)* follows. Part *(iv)* follows from an application of Observation 7 to the traceable graph $K_{b_0}(b_1, \ldots, b_k)$ for $k \leq b_0 + 1$. Finally, *(v)* is proven in [7]. □

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We begin by finding a convenient infinite family of RP graphs with toughness $\frac{2}{5}$.

Theorem 24. *For all $k \geq 0$, $k \in \mathbb{Z}$, the graph $K_2(1,1,2,6,k)$ is RP.*

Proof. We first prove that $G_k = K_2(1,1,2,6,k)$ is RP for all $k \in \{1,\ldots,10\}$. Note that $n(G_k) = 12 + k$. Thus, it suffices to prove that for all $\lambda \in \{1,\ldots,\lfloor\frac{12+k}{2}\rfloor\}$, there is a partition $\{A,B\}$ of $V(G_k)$ such that $|A| = \lambda$ and $G_k[A]$, $G_k[B]$ are both RP.

Table 2-11 list all the (subgraphs induced by) partitions needed to show that G_k is RP for $k \leq 10$. All the subgraphs induced by the partitions are RP either by Lemma 23, or by the previous cases. For example, the $|A| = 5$ row of Table 2 shows how to partition $V(G_1) = \{A,B\}$ so that $|A| = 5$ and $G_1[A]$, $G_1[B]$ are both RP (see Figure 3).

To prove G_k is RP for $k \geq 11$, we use induction. Let $k \geq 11$, assume G_k is RP for all $j < k$, and let λ be any integer in $\{1,\ldots,\lfloor\frac{12+k}{2}\rfloor\}$. Then we can partition $V(G_k)$ into two parts $\{A,B\}$ where $|A| = \lambda$ by picking A such that $G_k[A] = K_0(0,0,0,0,\lambda) \simeq K_\lambda$ and $G_k[B] = K_2(1,1,2,6,k-\lambda)$. $G_k[A]$ is RP since it is a complete graph, and $G_k[B]$ is RP by induction, completing the proof. □

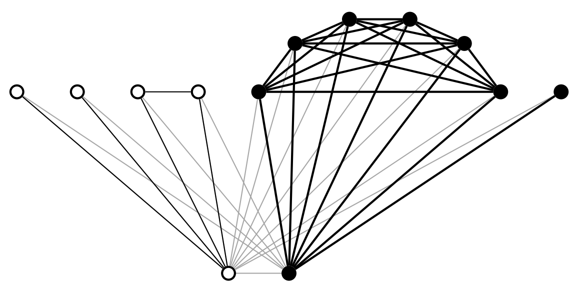

Figure 3: $V(G_1) = \{A,B\}$ where $|A| = 5$, $G_1[A] = K_1(1,1,2,0,0)$ and $G_1[B] = K_1(0,0,0,6,1)$. The subgraph $G_1[B]$ is bolded, and the edges not belonging to either $G_1[A]$ or $G_1[B]$ are light grey.

Table 2: Partitions of G_1 for $\lambda \leq \lfloor\frac{12+1}{2}\rfloor = 6$.

λ	$G_1[A]$	$G_1[B]$	λ	$G_1[A]$	$G_1[B]$
1	$K_0(0,0,0,0,1)$	$K_2(1,1,2,6,0)$	2	$K_0(0,0,2,0,0)$	$K_2(1,1,0,6,1)$
3	$K_1(1,1,0,0,0)$	$K_1(0,0,2,6,1)$	4	$K_1(1,0,2,0,0)$	$K_1(0,1,0,6,1)$
5	$K_1(1,1,2,0,0)$	$K_1(0,0,0,6,1)$	6	$K_0(0,0,0,6,0)$	$K_2(1,1,2,0,1)$

Table 3: Partitions of G_2 for $\lambda \leq \lfloor\frac{12+2}{2}\rfloor = 7$.

λ	$G_2[A]$	$G_2[B]$	λ	$G_2[A]$	$G_2[B]$
≤ 2	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,2-\lambda)$	3	$K_0(0,0,0,3,0)$	$K_2(1,1,2,3,2)$
4	$K_1(1,0,2,0,0)$	$K_1(0,1,0,6,2)$	5	$K_1(1,1,2,0,0)$	$K_1(0,0,0,6,1)$
6	$K_0(0,0,0,6,0)$	$K_2(1,1,2,0,2)$	7	$K_1(1,0,2,3,0)$	$K_1(0,1,0,3,2)$

Table 4: Partitions of G_3 for $\lambda \leq \lfloor \frac{12+3}{2} \rfloor = 7$.

λ	$G_3[A]$	$G_3[B]$
≤ 3	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,3-\lambda)$
5	$K_1(1,1,2,0,0)$	$K_1(0,0,0,6,3)$
7	$K_1(1,0,2,0,3)$	$K_1(0,1,0,6,0)$

λ	$G_3[A]$	$G_3[B]$
4	$K_0(0,0,0,4,0)$	$K_2(1,1,2,2,3)$
6	$K_0(0,0,0,6,0)$	$K_2(1,1,2,0,3)$

Table 5: Partitions of G_4 for $\lambda \leq \lfloor \frac{12+4}{2} \rfloor = 8$.

λ	$G_4[A]$	$G_4[B]$
≤ 4	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,4-\lambda)$
6	$K_0(0,0,0,6,0)$	$K_2(1,1,2,0,4)$
8	$K_1(1,0,2,0,4)$	$K_1(0,1,0,6,0)$

λ	$G_4[A]$	$G_4[B]$
5	$K_1(1,1,2,0,0)$	$K_1(0,0,0,6,4)$
7	$K_1(1,0,2,0,3)$	$K_1(0,1,0,6,1)$

Table 6: Partitions of G_5 for $\lambda \leq \lfloor \frac{12+5}{2} \rfloor = 8$.

λ	$G_5[A]$	$G_5[B]$
≤ 5	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,5-\lambda)$
7	$K_1(1,0,0,0,5)$	$K_1(0,1,2,6,0)$

λ	$G_5[A]$	$G_5[B]$
6	$K_0(0,0,0,6,0)$	$K_2(1,1,2,0,5)$
8	$K_1(0,0,2,0,5)$	$K_1(1,1,0,6,0)$

Table 7: Partitions of G_6 for $\lambda \leq \lfloor \frac{12+6}{2} \rfloor = 9$.

λ	$G_6[A]$	$G_6[B]$
≤ 6	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,6-\lambda)$
8	$K_1(1,0,0,0,6)$	$K_1(0,1,2,6,0)$

λ	$G_6[A]$	$G_6[B]$
7	$K_1(1,0,2,3,0)$	$K_1(0,1,0,3,6)$
9	$K_1(1,1,0,0,6)$	$K_1(0,0,2,6,0)$

Table 8: Partitions of G_7 for $\lambda \leq \lfloor \frac{12+7}{2} \rfloor = 9$.

λ	$G_7[A]$	$G_7[B]$
≤ 7	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,7-\lambda)$
9	$K_1(1,1,0,6,0)$	$K_1(0,0,2,0,7)$

λ	$G_7[A]$	$G_7[B]$
8	$K_1(1,0,0,6,0)$	$K_1(0,1,2,0,7)$

Table 9: Partitions of G_8 for $\lambda \leq \lfloor \frac{12+8}{2} \rfloor = 10$.

λ	$G_8[A]$	$G_8[B]$
≤ 8	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,8-\lambda)$
10	$K_1(1,0,2,6,0)$	$K_1(0,1,0,0,8)$

λ	$G_8[A]$	$G_8[B]$
9	$K_1(1,1,0,6,0)$	$K_1(0,0,2,0,8)$

Table 10: Partitions of G_9 for $\lambda \leq \lfloor \frac{12+9}{2} \rfloor = 10$.

λ	$G_9[A]$	$G_9[B]$
≤ 9	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,9-\lambda)$

λ	$G_9[A]$	$G_9[B]$
10	$K_1(1,0,2,6,0)$	$K_1(0,1,0,0,9)$

Table 11: Partitions of G_{10} for $\lambda \leq \lfloor \frac{12+9}{2} \rfloor = 10$.

λ	$G_{10}[A]$	$G_{10}[B]$
≤ 10	$K_0(0,0,0,0,\lambda)$	$K_2(1,1,2,6,10-\lambda)$

λ	$G_{10}[A]$	$G_{10}[B]$
11	$K_1(1,0,2,0,7)$	$K_1(0,1,0,6,3)$

Using replacement graphs formed from the graphs $K_2(1,1,2,6,j)$, j a positive integer, we can create arbitrarily large RP graphs with arbitrarily large cuts S leaving $2|S|+1$ components.

Corollary 25. *For all $s \geq 1$, there exists an infinite family $\mathcal{G}_s$ of RP graphs such that each graph G in $\mathcal{G}_s$ has a vertex cut S with $|S| = s$ and $c(G-S) = 2s+1$.*

Proof. For j a positive integer, let $H_1(j) = T(1,1,2j)$, and let $H_2(j) = K_2(1,1,2,6,j)$. These graphs are all RP by Theorems 13 and 24. Define $H_{s+2}(j)$ inductively by setting

$$H_{s+2}(j) = K_2 + (K_1 \cup K_1 \cup K_2 \cup K_6 \cup H_s(j))$$

By Theorems 20 and 24, the graph $H_{s+2}(j)$ is RP. It's clear that the graph $H_s(j)$ has a vertex cut S with $|S| = s$ and $c(G-S) = 2s+1$ (for example, see Figure 4). To complete the proof, we let $\mathcal{G}_s = \{H_s(j)\}_{j\in\mathbb{N}}$. □

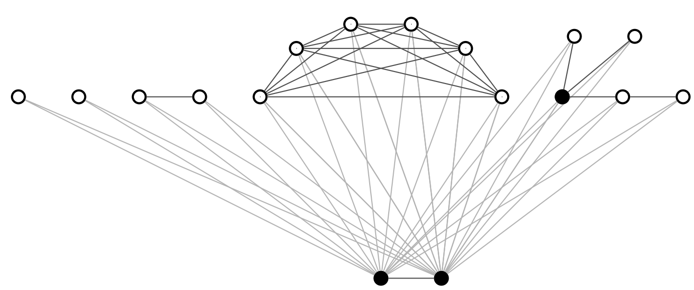

Figure 4: The graph $H_3(1)$. The vertices of a cut set S with $|S| = 3$ and $c(H_3(1) - S) = 7$ are bolded.

Lemma 26. *The graphs $K_2(1,2,3,4,6)$ and $K_2(1,2,2,3,4)$ are RP.*

Proof. The semistar $K_2(1,2,3,4,6)$ has 18 vertices. Table 12 below shows that for all $\lambda \in \{1,\ldots,9\}$, the graph $K_2(1,2,3,4,6)$ has a partition $\{A,B\}$ such that both parts induce RP graphs and $|A| = \lambda$. The parts are RP by Lemma 23 and Theorem 24.

The proof that the 14-vertex graph $K_2(1,2,2,3,4)$ is RP follows similarly by considering Table 13.

□

Table 12: Partitions of $G = K_2(1,2,3,4,6)$ for $\lambda \leq 9$.

λ	$G[A]$	$G[B]$	λ	$G[A]$	$G[B]$
1	$K_0(1,0,0,0,0)$	$K_2(0,2,3,4,6)$	2	$K_0(0,2,0,0,0)$	$K_2(1,0,3,4,6)$
3	$K_0(0,0,3,0,0)$	$K_2(1,2,0,4,6)$	4	$K_0(0,0,0,4,0)$	$K_2(1,2,3,0,6)$
5	$K_1(1,0,3,0,0)$	$K_1(0,2,0,4,6)$	6	$K_0(0,0,0,0,6)$	$K_2(1,2,3,4,0)$
7	$K_1(1,2,3,0,0)$	$K_1(0,0,0,4,6)$	8	$K_1(0,0,3,4,0)$	$K_1(1,2,0,0,6)$
9	$K_1(1,0,3,4,0)$	$K_1(0,2,0,0,6)$			

Table 13: Partitions of $G = K_2(1,2,2,3,4)$ for $\lambda \leq 7$.

λ	$G[A]$	$G[B]$	λ	$G[A]$	$G[B]$
1	$K_0(0,1,0,0,0)$	$K_2(1,1,2,3,4)$	2	$K_0(0,2,0,0,0)$	$K_2(1,0,2,3,4)$
3	$K_0(0,0,0,3,0)$	$K_2(1,2,2,0,4)$	4	$K_0(0,0,0,0,4)$	$K_2(1,2,2,3,0)$
5	$K_1(1,1,2,0,0)$	$K_1(0,1,0,3,4)$	6	$K_1(0,0,2,3,0)$	$K_1(1,2,0,0,4)$
7	$K_1(0,0,2,0,4)$	$K_1(1,2,0,3,0)$			

Theorem 27. *The semistar $K_3(1,1,1,2,2,3,4,6)$ is RP.*

Proof. Let $G = K_3(1,1,1,2,2,3,4,6)$, and note that $n(G) = 23$. Let S denote the cut set of size 3 in G. We show that for all $\lambda \leq 11$, the vertex set V of G has a partition $\{A, B\}$ such that $|A| = \lambda$, and the induced graphs $G[A] = S_\lambda$ and $G[B] = T_\lambda$ are RP. Note that the following cases make use of Lemma 6.

$\boldsymbol{\lambda = 1}$ **:** Let $S_1 = K_1$ be a 1-vertex component of $G - S$, and $T_1 = K_3(1,1,2,2,3,4,6)$. By Theorem 20, Lemma 23 and Lemma 26, we can construct an RP spanning subgraph H of T_1. H is an RP replacement graph made using $K_1(1,6,14)$ and $K_2(1,2,2,3,4)$:

$$T_1 \geq H = K_1 + (K_1 \cup K_6 \cup K_2(1,2,2,3,4)).$$

$\boldsymbol{\lambda = 2}$ **:** Let $S_2 = K_2$ be a 2-vertex component of $G - S$, and $T_2 = K_3(1,1,1,2,3,4,6)$. By Theorem 24, the graph $K_2(1,1,2,6,9)$ is RP. Thus, we can construct an RP spanning subgraph H of T_2 using $K_2(1,1,2,6,9)$ and $K_1(1,3,4)$:

$$T_2 \geq H = K_2 + (K_1 \cup K_1 \cup K_2 \cup K_6 \cup K_1(1,3,4)).$$

$\boldsymbol{\lambda = 3}$ **:** Let $S_3 = K_3$ be the 3-vertex component of $G - S$, and $T_3 = K_3(1,1,1,2,2,4,6)$. Using $K_2(1,1,2,6,8)$ and $K_1(1,2,4)$, we construct an RP replacement graph H that spans T_3:

$$T_3 \geq H = K_2 + (K_1 \cup K_1 \cup K_2 \cup K_6 \cup K_1(1,2,4)).$$

$\boldsymbol{\lambda = 4}$ **:** Let $S_4 = K_4$ be the 4-vertex component of $G - S$, and $T_4 = K_3(1,1,1,2,2,3,6)$. We construct an RP spanning subgraph H of T_4:

$$T_4 \geq H = K_2 + (K_1 \cup K_1 \cup K_2 \cup K_6 \cup K_1(1,2,3)).$$

$\boldsymbol{\lambda = 5}$ **:** Let $S_5 = K_1(1,1,2)$ and $T_5 = K_2(1,2,3,4,6)$.

$\boldsymbol{\lambda = 6}$ **:** Let $S_6 = K_6$ and $T_6 = K_3(1,1,1,2,2,3,4)$. The graph H below is an RP spanning subgraph of T_6, constructed using $K_2(1,1,2,3,8)$ and $T_1(1,2,4)$:

$$T_6 \geq H = K_2 + (K_1 \cup K_1 \cup K_2 \cup K_3 \cup K_1(1,2,4)).$$

$\boldsymbol{\lambda = 7}$ **:** Let $S_7 = K_1(1,2,3)$ and $T_7 = K_2(1,1,2,4,6)$.
$\boldsymbol{\lambda = 8}$ **:** Let $S_8 = K_1(1,2,4)$ and $T_8 = K_2(1,1,2,3,6)$.
$\boldsymbol{\lambda = 9}$ **:** Let $S_9 = K_1(1,3,4)$ and $T_9 = K_2(1,1,2,2,6)$.
$\boldsymbol{\lambda = 10}$ **:** Let $S_{10} = K_1(1,2,6)$ and $T_{10} = K_2(1,1,2,3,4)$.
$\boldsymbol{\lambda = 11}$ **:** Let $S_{11} = K_2(1,1,2,2,3)$ and $T_{11} = K_1(1,4,6)$. □

7 Minimal RP graphs

Let $b_0, \ldots, b_k$ be positive integers. Call $K_{b_0}(b_1, \ldots, b_k)$ a **minimal** $\boldsymbol{(b_0, k)}$ **RP semistar** if there do not exist positive integers $c_1, \ldots, c_k$ such that both the following hold:

- $K_{b_0}(c_1, \ldots, c_k)$ is RP, and
- $K_{b_0}(c_1, \ldots, c_k)$ is a proper subgraph of $K_{b_0}(b_1, \ldots, b_k)$.

It is easy to see that $K_1(1,1,2)$ is the unique minimal $(1,3)$ RP semistar. Every $(1,3)$ semistar G has $n(G) \geq 4$, and the only such graph with $n(G) = 4$ is $K_{1,3}$. Since it does not have a $(2,2)$-partition, $K_{1,3}$ is not RP. In this section, we show that $K_2(1,1,2,2,3)$ and $K_2(1,1,1,2,4)$ are the only minimal $(2,5)$ RP semistars. Thus, every graph G with a 2-vertex cut S such that $c(G-S) = 5$ has order 11 or more. Further, we show that the RP semistar $K_3(1,1,1,2,2,3,4,6)$ is minimal.

Let $\mathcal{G}(b_0, k) = \{K_{b_0}(b_1, \ldots, b_k) : 1 \leq b_1 \leq \cdots \leq b_k\}$. The poset $\mathcal{G}(b_0, k)$ ordered by subgraph inclusion embeds into $\mathbb{N}^k$ (with the product order) in the obvious way. (See Section 2.2.) Dickson's Lemma states that the product $\mathbb{N}^k$ contains neither infinite anti-chains, nor infinite strictly descending sequences [12].

Remark 28. *For each pair (b_0, k) of positive integers, there are finitely many minimal (b_0, k) RP semistars.*

A well known theorem of Tutte states that a graph G has a perfect matching if and only if for every vertex cut S of G, the graph $G - S$ has at most $|S|$ odd components [24]. The next lemma shows that this necessary condition can be generalised to partitions with connected parts of any size.

If S is a finite set, then let $|S|_k$ denote the number j in $\{0, 1, \ldots, k-1\}$ such that $|S| \equiv j \pmod{k}$. If G is a graph, and $S \subseteq V(G)$, then let

$$w_k(G,S) = \frac{1}{k-1} \cdot \sum \{|V(C)|_k : C \text{ a component of } G - S\}.$$

The following result is given in [5].

Lemma 29. [5] *Let G be a connected graph, S a vertex cut of G with $|S| < c(G-S)$, and $k \geq 2$ a positive integer. If G is AP, then*

$$|S| + 1 \geq w_k(G,S).$$

We give a slight sharpening of this lemma. The proof is similar to the proof in [5], with care taken to track the term $\frac{|V(G)|_k}{k-1}$.

Lemma 30. *Let G be a connected graph, S a subset of $V(G)$, and $m, k > 1$ integers. If G has a partition into connected parts $T_1, T_2, \ldots, T_m$ such that $|T_i| = k$ for all $i \leq m-1$, and $|T_m| \leq k$, then*

$$|S| + \frac{|V(G)|_k}{k-1} \geq w_k(G,S).$$

Proof. Note that either $|T_m| = k$ or $|T_m| = |V(G)|_k$. We begin by considering the following subgraph G' of G:

$$G' = \bigcup \{G[T_i] : V(T_i) \cap S \neq \emptyset\} \cup G[S] \cup G[T_m].$$

Observe that $S \subseteq V(G')$ and $|V(G)|_k = |V(G')|_k$. Further notice that the vertex set of each component of $G - S$ is a union of the vertex sets of components of $G' - S$, and possibly some of the sets T_i, $i < m$. Therefore, we get $w_k(G', S) \geq w_k(G,S)$.

Consider the subgraph $G^* = G' - T_m$, and let $S^* = S \setminus V(T_m)$. Since $|V(G^*)|_k = 0$, and T_m has either k vertices or $|V(G')|_k$ vertices, we obtain

$$\begin{aligned}\left(|S| + \frac{|V(G)|_k}{k-1}\right) - \left(|S^*| + \frac{|V(G^*)|_k}{k-1}\right) &= \left(|S| + \frac{|V(G')|_k}{k-1}\right) - (|S^*| + 0)\\ &\geq \frac{|V(G')|_k}{k-1}\\ &\geq w_k(G', S) - w_k(G^*, S^*)\\ &\geq w_k(G,S) - w_k(G^*, S^*).\end{aligned}$$

To complete the proof, it suffices to show that $|S^*| \geq w_k(G^*, S^*)$. Each component of $G^* - S^*$ is of the form $T_i - S^*$ for some $i < m$, and each such T_i has exactly k vertices. Thus, we have

$$w_k(G^*, S^*) = \frac{|G^* - S^*|}{k-1}.$$

Further, each vertex of G^* is in some T_i, $i < m$. Each T_i has at least one vertex of S^* and at most $k-1$ vertices not in S^*. Therefore, $|G^* - S^*| \leq (k-1)|S^*|$, so

$$|S^*| \geq \frac{|G^* - S^*|}{k-1} = w_k(G^*, S^*),$$

completing the proof. □

Corollary 31. *If G is an AP (RP) graph with $S \subseteq V(G)$, and $k \geq 2$ a positive integer, then*

$$|S| + \frac{|V(G)|_k}{k-1} \geq w_k(G,S).$$

Theorem 32. *The graphs $K_2(1,1,1,2,4)$ and $K_2(1,1,2,2,3)$ are the unique minimal $(2,5)$ RP semistars.*

Proof. Recall from Lemma 23 that $K_2(1,1,1,2,4)$, and $K_2(1,1,2,3,c)$ are RP for all $c \equiv 0 \pmod 2$. Let $G = K_2(b_1, \ldots, b_5)$ be a minimal $(2,5)$ RP semistar with $b_1 \le \cdots \le b_5$. We can remove a single vertex of G and still have an RP graph remaining. The vertex removed cannot be a vertex of the 2-vertex cut, since no $(1,5)$ semistar is RP per Proposition 17. Thus, by minimality of G, we have $b_1 = 1$. Similarly, we can remove two adjacent vertices, so $b_i = 2$ for some i.

By Proposition 17, the RP subgraph induced by removing three vertices cannot be $K_2(1)$ or $K_1(2)$. Thus, there are two possibilities for removing three vertices.

Case 1: The RP subgraph induced by the three removed vertices is $K_1(1,1)$, so $b_2 = 1$, and $G = K_2(1,1,2,s,t)$ for some positive integers $s \le t$. First suppose that $s = 1$, so $G = K_2(1,1,1,2,t)$ for some positive integer t. Let $S = K_2(0,0,0,0,0)$ (so $|S| = 2$) and let $k = 3$, then apply Corollary 31. This yields the inequality

$$|S| + \frac{|V(G)|_3}{2} \ge w_3(G,S), \text{ so}$$
$$2 + \frac{1}{2} \cdot (7 + t \mod 3) \ge \frac{5}{2} + \frac{1}{2} \cdot (t \mod 3)$$

From this inequality, it follows that $t \ne 2$. Again apply Corollary 31 to G, with the same choice $S = K_2(0,0,0,0,0)$, but $k = 2$, and obtain the following inequality:

$$2 + (7 + t \mod 2) \ge 3 + (t \mod 2)$$

From the above inequality, we see that t must be even, so $t \notin \{1,3\}$. Thus, if $s = 1$, the single minimal RP semistar is $K_2(1,1,1,2,4)$.

The semistar $K_2(1,1,2,2,2)$ is not RP — apply Corollary 31 with $S = K_2(0,0,0,0,0)$ and $k = 3$. Thus, if $s = 2$, then $t \ge 3$ and so the only minimal RP semistar is $K_2(1,1,2,2,3)$. When $s \ge 3$, we have $K_2(1,1,2,2,3) < K_2(1,1,2,s,t)$, and if $s \ge 4$, then $K_2(1,1,1,2,4) < K_2(1,1,2,s,t)$, which proves uniqueness in Case 1.

Case 2: The RP subgraph induced by the three removed vertices is a K_3, so $b_i = 3$ for some i. Thus, $G = K_2(1,2,3,s,t)$ for some $1 \le s \le t$. An analysis similar to that in Case 1 shows that $K_2(1,1,2,2,3)$ is the only minimal RP semistar in Case 2. ☐

Corollary 33. *Let G be an RP graph of order n. If G has a cut S with $|S| = 2$ and $c(G - S) = 5$, then $n \ge 11$.*

Proposition 34. *The graph $K_3(1,1,1,2,2,3,4,6)$ is a minimal $(3,8)$ RP semistar.*

Proof. Let $G = K_3(1,1,1,2,2,3,4,6)$. By Theorem 27, this graph is RP. To prove minimality, it suffices to show that G does not have an RP proper subgraph of the form $H = K_3(1,1,1,b_1,b_2,b_3,b_4,b_5)$, where $1 \le b_1 \le \cdots \le b_5$. Assume to the contrary that it does, and let $S = K_3(0,0,0,0,0,0,0,0)$ be the 3-vertex cut of H.

Case 1: $b_1 = 1$. Apply Corollary 31 to H using the 3-vertex cut S and $k = 2$ to get

$$3 + (7 + b_2 + b_3 + b_4 + b_5 \mod 2) \ge 4 + (b_2 \mod 2) + \cdots + (b_5 \mod 2),$$

from which it follows that b_2, b_3, b_4 and b_5 are all even. Thus, $b_2 = b_3 = 2$ and $b_4 \in \{2,4\}$. Again use Corollary 31 with the cut S but $k = 3$ to obtain

$$3 + \frac{1}{2} \cdot (11 + b_4 + b_5 \mod 3) \ge \frac{8}{2} + \frac{1}{2} \cdot (b_4 \mod 3) + \frac{1}{2} \cdot (b_5 \mod 3).$$

Whether $b_4 = 2$ or $b_4 = 4$, this inequality yields a contradiction, completing Case 1.

Case 2: $b_1 = 2$. Since $H < G$, we have $b_2 = 2$ and $b_3 \in \{2, 3\}$. Use Corollary 31 with the 3 vertex cut S and $k = 3$ to get

$$3 + \frac{1}{2} \cdot (10 + b_3 + b_4 + b_5 \mod 3) \geq \frac{7}{2} + \frac{1}{2} \cdot (b_3 \mod 3) + ... + \frac{1}{2} \cdot (b_5 \mod 3).$$

If $b_3 = 2$, then the right-hand side of the inequality is at least $\frac{9}{2}$, which is impossible. Therefore $b_3 = 3$. Applying Corollary 31 to the 3-vertex cut S with $k = 2$, we see that both b_4 and b_5 are even. Thus, $b_4 = 4$. Since $H < G$, we have $b_5 = 4$. Now use Corollary 31 with the cut S and $k = 5$, to get

$$3 + \frac{1}{4} \cdot (21 \mod 5) \geq \frac{18}{4}.$$

However, this is a contradiction, completing Case 2.

In either case, we derive a contradiction, so G does not have such a subgraph H. □

8 Bounding $c(G - S)$ from above

In this section, we show that RP graphs are $\frac{1}{3}$-tough. We have seen that there exist RP graphs with a cut-vertex v such that $c(G - v) = 3$. However, as we show in Theorem 35, for cuts S of greater size in RP graphs, we must have $c(G - S) < 3|S|$, and this bound is sharp when $|S| = 2$ or $|S| = 3$.

We say that an RP graph G of order n is **minimal with respect to S**, if there is no $(\lambda, n - \lambda)$-partition, for any λ, of G into RP graphs G_1 and G_2 such that G_1 is a proper induced subgraph of any of the connected components of $G - S$. Suppose G is minimal with respect to S, and $G - S$ does not have a component of size λ. If we partition G into two RP graphs G_1 and G_2 such that G_1 has λ vertices, then both G_1 and G_2 contain at least one vertex of S. In other words, the partition must split the cut S across its two parts.

Theorem 35. *Let S be a cut of a graph G with $|S| \geq 2$. If $c(G - S) \geq 3|S|$, then G is not RP.*

Proof. By Proposition 17, the result holds when $|S| = s = 2$. It is also useful to note that if $|S| = 1$, then $c(G - S) \leq 3$ (see Proposition 17). We proceed by strong induction, assuming the result holds for all integers i such that $2 \leq i < s$.

Suppose that G is RP, and let S be a cut in G, with $|S| = s \geq 3$. We can assume that G is minimal with respect to S (as defined in the paragraph before Theorem 35) — for if G is not minimal with respect to S, we can repeatedly remove some proper subset T of a component of $G - S$ so that $G - T$ is RP, until it is no longer possible to do so, resulting in an RP subgraph of G that is minimal with respect to S. Let $C_1, C_2, \ldots, C_k$ be the connected components of $G - S$, with $|C_1| \leq |C_2| \leq \cdots \leq |C_k|$. Suppose that $|C_k| = |C_{k-1}| + 1$. Let $\lambda = |C_k| + 1$ and find a $(\lambda, n - \lambda)$-partition of G into RP graphs G_1 and G_2. Since $|C_i| \notin \{\lambda, n - \lambda\}$ for every $1 \leq i \leq k$, we must have that both $S_1 = S \cap G_1$ and $S_2 = S \cap G_2$

are non-empty. Furthermore, since $|S| \geq 3$, we cannot have that $|S_1| = |S_2| = 1$. Therefore, by induction, we have

$$c(G - S) \leq c(G_1 - S_1) + c(G_2 - S_2) \leq 3|S_1| + 3|S_2| - 1 = 3|S| - 1.$$

Now, suppose that $|C_k| \neq |C_{k-1}| + 1$. Let $\lambda = |C_{k-1}| + 1$. Find a $(\lambda, n - \lambda)$-partition of G into graphs G_1 and G_2, and recall that, by minimality, we cannot have $G_1 \leq C_k$. Then, a similar argument holds. □

As Theorems 14, 24, 27 and 32 demonstrate, the bound in Theorem 35 is sharp when $|S| \in \{2, 3\}$.

Corollary 36. *Every RP graph is* $\frac{1}{3}$*-tough.*

9 Further Questions

We mention a few open questions.

1. Consider all pairs (G, S), where G is an RP graph and S is an s-vertex subset of $V(G)$, and let $\zeta(s) = \max\{c(G - S) : (G, S)\}$. When $s > 1$, Theorem 35 and Corollary 25 show that $2s + 1 \leq \zeta(k) \leq 3s - 1$. Can either of these bounds be improved? Is the $3s - 1$ upper bound sharp?

2. Is there some constant c such that every c-tough graph is AP (RP)?

3. If $K_{b_0}(b_1, b_2, \ldots, b_k)$ is RP, is the graph $K_{b_0}(b_1, \ldots b_{i-1}, b_{i+1}, \ldots, b_k)$ also RP for each $i \in \{1, 2, \ldots, k\}$?

4. In light of Remark 28 and Proposition 34, $K_3(1, 1, 1, 2, 2, 3, 4, 6)$ is one of finitely many minimal $(3, 8)$ RP semistars. Are there others? If so, what are they?

5. Both minimal $(2, 5)$ RP semistars are subgraphs of infinitely many $(2, 5)$ RP semistars. For example, $K_2(1, 1, 2, 2, 3)$ is a subgraph of every $K_2(1, 1, 2, 3, k)$ where $k \equiv 0 \pmod 2$ is positive, and $K_2(1, 1, 1, 2, 4)$ is a subgraph of $K_2(1, 1, 2, 6, k)$ where $k \geq 1$. Is $K_3(1, 1, 1, 2, 2, 3, 4, 6)$ a subgraph of infinitely many $(3, 8)$ RP semistars?

6. Do there exist pairs of positive integers (b_0, k) for which there exist a finite, but positive number of (b_0, k) RP semistars?

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