Real factorization of positive semidefinite matrix polynomials ☆

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A R T I C L E I N F O

A B S T R A C T

Suppose $Q(x)$ is a real $n \times n$ regular symmetric positive semidefinite matrix polynomial. Then it can be factored as

$$Q(x) = G(x)^T G(x),$$

where $G(x)$ is a real $n \times n$ matrix polynomial with degree half that of $Q(x)$ if and only if $\det(Q(x))$ is the square of a nonzero real polynomial. We provide a constructive proof of this fact, rooted in finding a skew-symmetric solution to a modified algebraic Riccati equation

$$XSX - XR + R^T X + P = 0,$$

where P, R, S are real $n \times n$ matrices with P and S real symmetric. In addition, we provide a detailed algorithm for computing the factorization.

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1. Introduction

The Fejér-Riesz factorization was first shown for matrix polynomials by Rosenblatt [24] and Helson [19]. Its version on the real line is the following: given a matrix polynomial $Q(x) = \sum_{i=0}^{2m} Q_i x^i$ with Q_i Hermitian and $Q(x)$ positive semidefinite for all $x \in \mathbb{R}$, we can factorize it as

$$Q(x) = G(x)^* G(x),$$

where $G(x) = \sum_{i=0}^{m} G_i x^i$. In 1964, Gohberg generalized this factorization to certain operator-valued polynomials [13]. Later it was further generalized to operator-valued polynomials in general form [25]. The multivariable case has also been studied, e.g. in [22]. For an overview of the work done with the operator-valued Fejér-Riesz theorem, see [8]. Fejér-Riesz factorization has applications in H^∞-control [10], in the construction of compactly supported wavelets [5, Chapter 6], filter design [12], determinantal representations [16], and prediction theory [6, Chapter XII], [11,27]. In some cases, one may want to insist on having a real factorization. Indeed, our motivation for finding a real factorization came from an interest in constructing real symmetric solutions to A. Horn's problem, where the eigenvalues of two real symmetric matrices are prescribed, as well as the eigenvalues of their sum. Adjusting the techniques in [2] to the real case required finding a real Fejér-Riesz factorization.

In this paper, we provide a constructive proof of the real analog of the Fejér-Riesz factorization of matrix-valued polynomials. In particular, given a matrix polynomial $Q(x) = \sum_{i=0}^{2m} Q_i x^i$ with Q_i $n \times n$ real symmetric, $Q(x)$ positive semidefinite for all $x \in \mathbb{R}$, and $\det(Q(x))$ equal to the square of a nonzero real polynomial, we show that $Q(x)$ admits the factorization

$$Q(x) = G(x)^T G(x),$$

where $G(x) = \sum_{i=0}^{m} G_i x^i$ with G_i real $n \times n$ matrices. This result was first shown by Hanselka and Sinn [18] using methods from projective algebraic geometry and number theory. We provide an alternative, linear algebraic proof, inspired by the proof of the Fejér-Riesz factorization presented in Section 2.7 of [1]. That proof, in turn, was taken from [9,17]. For earlier work on factorizations of real symmetric matrix polynomials (not necessarily positive semidefinite), see e.g. [23].

A key part of the Fejér-Riesz factorization proof we follow requires finding a Hermitian solution to an algebraic Riccati equation

$$XDX + XA + A^*X - C = 0, \qquad (*)$$

where D and C are Hermitian. Reducing a factorization problem to solving a Riccati equation is a technique that has been used in many other papers as well (see, e.g., [3],

[10], [20, Chapter 19] and references therein). This technique is useful because Riccati equations have been studied extensively. Early work was done by Willems [26] and Coppel [4] in analyzing properties of solutions of continuous algebraic Riccati equations. Another key paper was [7] where Riccati equations were used to solve H_∞-control problems. For an in depth analysis of algebraic Riccati equations, please see the book by Lancaster and Rodman [20].

For the current real factorization problem, we end up needing to find a real skew-symmetric solution to an equation of the form

$$XSX - XR + R^T X + P = 0, \qquad (**)$$

where P and S are real symmetric. This is not quite an algebraic Riccati equation of the form $(*)$ and thus we call it a modified algebraic Riccati equation. In general, to find a skew-Hermitian solution X, one often considers instead iX, which is Hermitian; however, we want real solutions and thus this method is not applicable here. Thus we instead follow the same steps presented in [20] for finding a real *symmetric* solution to the real algebraic Riccati equation $(*)$ and amend them to our current situation. This is the topic of Section 2, which culminates in giving sufficient conditions for a skew-symmetric solution of our modified Riccati equation $(**)$. In Section 3 we provide additional background on matrix polynomials necessary for our main result, the real factorization of a symmetric positive semidefinite matrix polynomial, presented in Section 4. The major advantage of our proof compared to that by Hanselka and Sinn [18] is that ours is constructive. Thus we provide an explicit algorithm for finding the factorization, along with examples illustrating the construction.

2. A modified algebraic Riccati equation

The goal of this section is to provide necessary and sufficient conditions for the existence of a real skew-symmetric solution X to the modified algebraic Riccati equation

$$XSX - XR + R^T X + P = 0, \qquad (1)$$

where P, R, S are real $n \times n$ matrices with P and S real symmetric. In Chapter 8 of the book *Algebraic Riccati Equations* [20], Lancaster and Rodman show conditions for which there is a real symmetric solution X to the continuous time algebraic Riccati equation

$$XDX + XA + A^T X - C = 0,$$

where A, C, D are real $n \times n$ matrices with C and D real symmetric. We amend these results to the present situation. Define the $2n \times 2n$ real matrices

$$M_r = \begin{bmatrix} R & -S \\ P & R^T \end{bmatrix}, \qquad \hat{H}_r = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad H_r = \begin{bmatrix} P & R^T \\ R & -S \end{bmatrix}. \qquad (2)$$

Then both $\hat{H}_r$ and H_r are real symmetric. Also

$$\hat{H}_r M_r = M_r^T \hat{H}_r \qquad \text{and} \qquad H_r M_r = M_r^T H_r.$$

Using terminology from [20, Section 2.6], we say M_r is both H_r-symmetric and $\hat{H}_r$-symmetric. Next, we define the graph subspace of a real $n \times n$ matrix X by

$$G(X) := \operatorname{Im} \begin{bmatrix} I_n \\ X \end{bmatrix} = \left\{ \begin{bmatrix} I_n \\ X \end{bmatrix} x : x \in \mathbb{R}^n \right\}.$$

We can now give a condition for a real solution of eq. (1) to exist.

Proposition 2.1. *X is a real solution of eq. (1) if and only if the graph subspace $G(X)$ is M_r-invariant, where M_r is defined as in eq. (2).*

Proof. If $G(X)$ is M_r-invariant, then

$$\begin{bmatrix} R & -S \\ P & R^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} Z \tag{3}$$

for some $n \times n$ matrix Z. The first block row gives $Z = R - SX$ and the second gives $P + R^T X = XZ$. Combining the two gives

$$P + R^T X = X(R - SX).$$

Thus X solves eq. (1). Conversely, if X solves eq. (1), then eq. (3) holds for $Z = R - SX$ and thus $G(X)$ is M_r-invariant. $\quad\square$

More than just a real solution, though, we want a skew-symmetric solution. Thus we next strive to give a condition for such a solution. For this we first need a few definitions (see [20, Section 2.6]). For $x, y \in \mathbb{R}^n$, we use the notation $\langle x, y \rangle$ to mean the usual inner product $y^T x$.

Definition 2.2. Let H be an $n \times n$ invertible real symmetric matrix. A subspace $\mathcal{M}$ of $\mathbb{R}^n$ is called

1. *H-nonnegative if $\langle Hx, x \rangle \geq 0$ for all $x \in \mathcal{M}$.*
2. *H-nonpositive if $\langle Hx, x \rangle \leq 0$ for all $x \in \mathcal{M}$.*
3. *H-neutral if $\langle Hx, x \rangle = 0$ for all $x \in \mathcal{M}$.*

Proposition 2.3.

1. *For $\hat{H}_r$ as defined in eq. (2), $X \in \mathbb{R}^{n \times n}$ is skew-symmetric if and only if the graph subspace $G(X)$ is $\hat{H}_r$-neutral.*

2. Let X be a real solution of eq. (1). Then for H_r as defined in eq. (2), $G(X)$ is H_r-nonpositive if and only if $(X^T + X)(R - SX)$ is negative semi-definite.

Proof. 1. $G(X)$ is $\hat{H}_r$-neutral if and only if for all $z \in \mathbb{R}^n$,

$$\left\langle \hat{H}_r \begin{bmatrix} I \\ X \end{bmatrix} z, \begin{bmatrix} I \\ X \end{bmatrix} z \right\rangle = 0.$$

Rewriting the lefthand side, we have

$$\left\langle \hat{H}_r \begin{bmatrix} I \\ X \end{bmatrix} z, \begin{bmatrix} I \\ X \end{bmatrix} z \right\rangle = z^T \begin{bmatrix} I & X^T \end{bmatrix} \hat{H}_r \begin{bmatrix} I \\ X \end{bmatrix} z = z^T (X + X^T) z.$$

Thus $G(X)$ is $\hat{H}_r$-neutral if and only if $X + X^T = 0$, i.e. X is skew-symmetric.

2. $G(X)$ is H_r-nonpositive if and only if for all $z \in \mathbb{R}^n$,

$$\left\langle H_r \begin{bmatrix} I \\ X \end{bmatrix} z, \begin{bmatrix} I \\ X \end{bmatrix} z \right\rangle \le 0.$$

Rewriting the lefthand side, we have

$$\begin{aligned} \left\langle H_r \begin{bmatrix} I \\ X \end{bmatrix} z, \begin{bmatrix} I \\ X \end{bmatrix} z \right\rangle &= z^T \begin{bmatrix} I & X^T \end{bmatrix} \begin{bmatrix} P & R^T \\ R & -S \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} z \\ &= z^T [P + R^T X + X^T R - X^T S X] z \\ &= z^T [XR - XSX + X^T R - X^T S X] z \qquad \text{by eq. (1)} \\ &= z^T (X^T + X)(R - SX) z. \end{aligned}$$

Thus $G(X)$ is H_r-nonpositive if and only if $(X^T + X)(R - SX)$ is negative semidefinite. $\square$

Proposition 2.3 shows that in order to get a real skew-symmetric solution X to the eq. (1), we need an $\hat{H}_r$-neutral subspace $G(X)$ of dimension n. We consider now conditions for such a subspace to exist. For this we first state a few known results (see [20, Section 2.6]).

Definition 2.4. Let A be a square matrix and λ_i be an eigenvalue of A. We call the sizes of the Jordan blocks of λ_i the *partial multiplicities* of λ_i.

Theorem 2.5. *[20, Part of Theorem 2.6.3] Let A be a real $n \times n$ H-symmetric matrix (meaning H is symmetric and $HA = A^T H$). Suppose the partial multiplicities of the real eigenvalues of A are all even. Then there exists an A-invariant H-neutral subspace of dimension $k - p$ where k is the number of positive eigenvalues of H (counting algebraic multiplicities) and p is the number of distinct pairs of non-real complex conjugate eigenvalues of A with odd algebraic multiplicity.*

Lemma 2.6. *[20, p. 56] Let H be a real symmetric matrix. A real subspace $\mathcal{M}$ is H-neutral if and only if $\langle Hx, y \rangle = 0$ for all $x, y \in \mathcal{M}$.*

Proof. This follows from the relation

$$\langle Hx, y \rangle = \frac{1}{2} \left(\langle H(x+y), x+y \rangle - \langle Hx, x \rangle - \langle Hy, y \rangle \right). \quad \square$$

Now we are ready for the following new result.

Lemma 2.7. *Let eq. (2) hold with P and S real symmetric. Among the following statements, the implications $(iii) \implies (ii) \implies (i)$ hold.*

(i)   *There exists an n-dimensional M_r-invariant H_r-neutral subspace.*
(ii) *There exists an n-dimensional M_r-invariant $\hat{H}_r$-neutral subspace.*
(iii) *All real eigenvalues of M_r have even partial multiplicities and all non-real eigenvalues of M_r have even algebraic multiplicity.*

If in addition M_r is invertible, then we also have $(i) \implies (ii)$.

Proof. Assume (iii). By Theorem 2.5, there is an n-dimensional M_r-invariant $\hat{H}_r$-neutral subspace (since $p = 0$ and $\hat{H}_r$ has $k = n$ positive eigenvalues). Thus $(iii) \implies (ii)$. Next, since $H_r = \hat{H}_r M_r$, it follows by Lemma 2.6 that $(ii) \implies (i)$. Finally, if M_r is invertible, $\hat{H}_r = H_r M_r^{-1}$ and thus again by Lemma 2.6, $(i) \implies (ii)$. $\square$

We need one more result before the main theorem of this section. For this result, we first recall a definition (see, e.g., [20, Section 4.1]).

Definition 2.8. Let A be a real $n \times n$ matrix and B be a real $n \times m$ matrix. The pair (A, B) is said to be *controllable* if

$$\text{rank} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n.$$

For an $n \times n$ real symmetric matrix S, we use the notation $S > 0$ to mean S is positive definite and the notation $S \geq 0$ to mean S is positive semidefinite.

Lemma 2.9. *Let eq. (2) hold with P and S real symmetric. Assume that $S \geq 0$ and the pair (R, S) is controllable. Let $\mathcal{L}$ be an n-dimensional M_r-invariant H_r-nonnegative subspace of $\mathbb{R}^{2n}$. Then $\mathcal{L}$ is a graph subspace, i.e.*

$$\mathcal{L} = \text{Im} \begin{bmatrix} I \\ X \end{bmatrix}$$

for some real $n \times n$ matrix X.

Proof. For $\mathcal{L}$ as defined in the statement, write

$$\mathcal{L} = \mathrm{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

for some real $n \times n$ matrices X_1 and X_2. We shall show that X_1 is invertible. First, since $\mathcal{L}$ is M_r-invariant,

$$\begin{bmatrix} R & -S \\ P & R^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T$$

for some $n \times n$ matrix T. Thus,

$$RX_1 - SX_2 = X_1 T, \tag{4}$$

$$PX_1 + R^T X_2 = X_2 T. \tag{5}$$

Next, since $\mathcal{L}$ is H_r-nonnegative, we know

$$\begin{bmatrix} X_1^T & X_2^T \end{bmatrix} \begin{bmatrix} P & R^T \\ R & -S \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1^T P X_1 + X_1^T R^T X_2 + X_2^T R X_1 - X_2^T S X_2 \geq 0. \tag{6}$$

Let $\mathcal{K} = \ker X_1$. By eq. (6), for every $x \in \mathcal{K}$,

$$0 \leq x^T X_1^T P X_1 x + x^T X_1^T R^T X_2 x + x^T X_2^T R X_1 x - x^T X_2^T S X_2 x = -x^T X_2^T S X_2 x.$$

Since $S \geq 0$, $X_2 x \in \ker S$, so

$$X_2 \mathcal{K} \subset \ker S.$$

Then, eq. (4) implies

$$T\mathcal{K} \subset \mathcal{K}.$$

Consequently, eq. (5) gives

$$R^T X_2 \mathcal{K} \subset X_2 \mathcal{K}.$$

All together, we have

$$R^T X_2 \mathcal{K} \subset \ker S.$$

By induction, we get

$$(R^T)^r X_2 \mathcal{K} \subset \ker S, \qquad r = 0, 1, 2, \ldots$$

Now for every $x \in \mathcal{K}$,

$$
\begin{bmatrix}
S \\
SR^T \\
\vdots \\
S(R^T)^{n-1}
\end{bmatrix} (X_2 x) = 0.
$$

Since (R, S) is controllable, we must have $X_2 x = 0$. The only n-dimensional vector x for which $X_1 x = X_2 x = 0$ is the zero vector (otherwise $\dim \mathcal{L} < n$). Thus $\mathcal{K} = \{0\}$ and X_1 is invertible. Hence

$$
\mathcal{L} = \operatorname{Im} \begin{bmatrix} I \\ X \end{bmatrix},
$$

where $X = X_2 X_1^{-1}$. Thus $\mathcal{L}$ is a graph subspace. □

Now we put everything together to get necessary and sufficient conditions for the existence of a real skew-symmetric solution of eq. (1).

Theorem 2.10. *Let eq. (2) hold with P and S real symmetric. Assume that $S \geq 0$ and the pair (R, S) is controllable. Then the following are equivalent.*

(i) Equation (1) has a real skew-symmetric solution.
(ii) There exists an n-dimensional M_r-invariant $\hat{H}_r$-neutral subspace.

Proof. By Proposition 2.1 and Proposition 2.3, we know that if X is a real skew-symmetric solution to eq. (1), then $G(X)$ is an n-dimensional M_r-invariant $\hat{H}_r$-neutral subspace. Thus $(i) \implies (ii)$. Finally, assume there exists an n-dimensional M_r-invariant $\hat{H}_r$-neutral subspace, say $\mathcal{L}$. Since $H_r = \hat{H}_r M_r$, $\mathcal{L}$ is also an n-dimensional M_r-invariant H_r-neutral subspace. Clearly $\mathcal{L}$ is an H_r-nonnegative subspace, so by Lemma 2.9, $\mathcal{L}$ is a graph subspace. Since $\mathcal{L} = G(X)$ is M_r-invariant, X is a real solution of eq. (1) by Proposition 2.1. Since $\mathcal{L} = G(X)$ is also $\hat{H}_r$-neutral, by Proposition 2.3, X is skew-symmetric. Thus $(ii) \implies (i)$. □

3. Matrix polynomials

Building toward our goal of factorizing a real symmetric positive semidefinite matrix polynomial, we next state a few relevant results on matrix polynomials (see [15]).

Definition 3.1. For $n \times n$ matrices P_i, we define an $n \times n$ *matrix polynomial $P(x)$ of degree m* by

$$
P(x) = \sum_{i=0}^{m} P_i x^i.
$$

- The matrix polynomial is called *real* if all P_i are real matrices.
- The matrix polynomial is called *monic* if $P_m = I_n$.
- The matrix polynomial is called *self-adjoint* if $P_i = P_i^*$, its conjugate transpose, for all i.
- The matrix polynomial is called *symmetric* if $P_i = P_i^T$ for all i.
- The matrix polynomial is called *positive semidefinite (also nonnegative)* if for all $x \in \mathbb{R}$, $P(x)$ is positive semidefinite.
- The matrix polynomial is called *regular* if $\det(P(x))$ is not identically zero.

Following [21], where the spectrum and Jordan canonical form of a quadratic matrix polynomial were defined, we define these concepts for any degree matrix polynomial in the following ways.[1]

Definition 3.2. Let $P(x)$ be a regular matrix polynomial. Then the set of *eigenvalues* of P, i.e. the *spectrum*, is

$$\sigma(P) := \{x \in \mathbb{C} : \det(P(x)) = 0\}.$$

Definition 3.3. Let $P(x) = \sum_{i=0}^{m} P_i x^i$ be a degree m matrix polynomial with P_m nonsingular. The *Jordan canonical form* for $P(x)$ is defined to be that of the companion matrix

$$C_P := \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n \\ -P_m^{-1}P_0 & -P_m^{-1}P_1 & \cdots & -P_m^{-1}P_{m-2} & -P_m^{-1}P_{m-1} \end{bmatrix}.$$

Theorem 3.4. *[15, part of Theorem 12.8] For a monic self-adjoint matrix polynomial $P(x)$, the following statements are equivalent.*

(i) $P(x)$ is nonnegative.

(ii) The partial multiplicities of $P(x)$ for real points of the spectrum are all even.

In the previous section, we found that if all real eigenvalues of M_r have even partial multiplicities and all non-real eigenvalues of M_r have even algebraic multiplicity, then our eq. (1) has the desired real skew-symmetric solution. Next, using the notion of linearization, we will associate the matrix M_r with a monic non-negative matrix polynomial. For this we begin with a few definitions from [15, Section 1.1].

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[1] Note that the partial multiplicities of an eigenvalue of a matrix polynomial are often defined as powers of the elementary divisors; however, these partial multiplicities are the same as the sizes of the Jordan blocks of our companion matrix. See the Appendix of [15] for a more in depth understanding of matrix polynomial equivalences, linearizations, partial multiplicities and elementary divisors.

Definition 3.5. Two matrix polynomials $M_1(x)$ and $M_2(x)$ of size $n \times n$ are called *equivalent* (notated $M_1(x) \sim M_2(x)$) if

$$M_1(x) = E(x)M_2(x)F(x)$$

for some $n \times n$ matrix polynomials $E(x)$ and $F(x)$ with constant nonzero determinants.

Definition 3.6. Let $P(x)$ be an $n \times n$ monic matrix polynomial of degree m. A linear matrix polynomial $xI_{nm} - A$ is called a *linearization* of $P(x)$ if

$$xI_{nm} - A \sim \begin{bmatrix} P(x) & 0 \\ 0 & I_{n(m-1)} \end{bmatrix}.$$

Note that $xI - C_P$ is a linearization of $P(x)$. For any linearization $xI - A$, the partial multiplicities in every eigenvalue of A and $P(x)$ are the same [15, Section 1.1].

Lemma 3.7. *Let* $Q(x) = \sum_{j=0}^{2m} Q_j x^j$ *be an* $n \times n$ *real symmetric matrix polynomial of degree* $2m$ *with* $Q_0 = I_n$. *If* $m = 1$, *set*

$$M_r := \begin{bmatrix} -\frac{1}{2}Q_1 & -I_n \\ Q_2 - \frac{1}{4}Q_1^2 & -\frac{1}{2}Q_1 \end{bmatrix}.$$

Otherwise, define the $2nm \times 2nm$ *matrix as*

$$M_r := \begin{bmatrix}
-\frac{1}{2}Q_1 & & & & & & -I_n & & & & \\
I_n & 0 & & & & & & 0 & & & \\
& I_n & 0 & & & & & & 0 & & \\
& & \ddots & \ddots & & & & & & \ddots & \\
& & & I_n & 0 & & & & & & 0 \\
\hline
Q_2 - \frac{1}{4}Q_1^2 & \frac{1}{2}Q_3 & & & & & -\frac{1}{2}Q_1 & I_n & & & \\
\frac{1}{2}Q_3 & Q_4 & \frac{1}{2}Q_5 & & & & & 0 & I_n & & \\
& \frac{1}{2}Q_5 & Q_6 & \ddots & & & & & 0 & \ddots & \\
& & \ddots & \ddots & \frac{1}{2}Q_{2m-1} & & & & & \ddots & I_n \\
& & & \frac{1}{2}Q_{2m-1} & Q_{2m} & & & & & & 0
\end{bmatrix}.$$

Then for

$$\operatorname{rev} Q(x) := \sum_{i=0}^{2m} Q_{2m-i} x^i = x^{2m} I_n + \sum_{i=0}^{2m-1} Q_{2m-i} x^i,$$

we have

$$xI_{2nm} - M_r \sim \begin{bmatrix} \operatorname{rev} Q(x) & 0 \\ 0 & I_{2nm-n} \end{bmatrix}.$$

Proof. If $m = 1$,

$$\begin{bmatrix} I_n & -xI_n - \frac{1}{2}Q_1 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} (xI_{2n} - M_r) = \begin{bmatrix} -Q_2 - xQ_1 - x^2 I_n & 0 \\ xI_n + \frac{1}{2}Q_1 & I_n \end{bmatrix}.$$

It is clear then

$$xI_{2n} - M_r \sim \begin{bmatrix} \operatorname{rev} Q(x) & 0 \\ 0 & I_n \end{bmatrix}.$$

Now assume $m \geq 2$. We begin by defining $M_1(x)$ as a permutation of the rows and columns of $xI - M_r$.

$$
M_1(x) := \begin{bmatrix}
\begin{array}{cccc|cccc}
 & & & 0 & 0 & \cdots & 0 & I_n \\
 & & 0 & I_n & \vdots & & 0 & 0 \\
 & \iddots & \iddots & & & \iddots & & \vdots \\
0 & I_n & & & 0 & & \cdots & 0 \\
\hline
I_n & 0 & \cdots & 0 & 0 & & & \\
0 & 0 & & \vdots & I_n & 0 & & \\
\vdots & & \ddots & 0 & & \ddots & \ddots & \\
0 & \cdots & & 0 & & & I_n & 0
\end{array}
\end{bmatrix}
(xI - M_r)
\begin{bmatrix}
\begin{array}{cccc|cccc}
 & & & I_n & & & & \\
 & & \iddots & & & & \mathbf{0} & \\
 & I_n & & & & & & \\
\hline
 & & & & & I_n & & \\
 & \mathbf{0} & & & & & \ddots & \\
 & & & & & & & I_n
\end{array}
\end{bmatrix}
$$

$$
= \left[
\begin{array}{ccccc|ccccc}
-Q_{2m} & -\frac{1}{2}Q_{2m-1} & & 0 & & & & & & xI_n \\
xI_n & -I_n & & & & & & & 0 & \\
 & \ddots & & \ddots & & & & \iddots & & \\
 & & \ddots & & \ddots & & & & & \\
 & & & xI_n & -I_n & & 0 & & & \\
\hline
 & & & xI_n + \frac{1}{2}Q_1 & & I_n & & & & \\
 & & -\frac{1}{2}Q_3 & \frac{1}{4}Q_1^2 - Q_2 & & xI_n + \frac{1}{2}Q_1 & -I_n & & & \\
 & \iddots & -Q_4 & -\frac{1}{2}Q_3 & & & xI_n & \ddots & & \\
 \iddots & \iddots & \iddots & & & & & \ddots & \ddots & \\
-\frac{1}{2}Q_{2m-1} & -Q_{2m-2} & -\frac{1}{2}Q_{2m-3} & & & & & & xI_n & -I_n
\end{array}
\right].
$$

Clearly,

$$xI - M_r \sim M_1(x).$$

Next, define $W, V_1, \ldots, V_{m-2}$ by

$$W = -\frac{1}{2}x^{m-2}Q_3 - x^{m-1}Q_2 - x^m Q_1 - x^{m+1}I_n$$

$$V_1 = -\frac{1}{2}x^{m-3}Q_5 - x^{m-2}Q_4 - \frac{1}{2}x^{m-1}Q_3$$

$$V_2 = -\frac{1}{2}x^{m-4}Q_7 - x^{m-3}Q_6 - \frac{1}{2}x^{m-2}Q_5$$

$$\vdots$$

$$V_{m-3} = -\frac{1}{2}xQ_{2m-3} - x^2 Q_{2m-4} - \frac{1}{2}x^3 Q_{2m-5}$$

$$V_{m-2} = -\frac{1}{2}Q_{2m-1} - xQ_{2m-2} - \frac{1}{2}x^2 Q_{2m-3}.$$

Then set

$$V(x) := \left[\begin{array}{c|cccc|cccc} I_n & x^{m-2}W + \sum_{i=1}^{m-2} x^{m-2-i}V_i & x^{m-3}W + \sum_{i=1}^{m-3} x^{m-3-i}V_i & \cdots & xW + V_1 \quad W & -x^{m-1}(xI_n + \frac{1}{2}Q_1) & x^{m-1}I_n & \cdots & x^2 I_n \quad xI_n \\ \hline \mathbf{0} & \multicolumn{4}{c|}{\mathbf{I}_{n(m-1)}} & \multicolumn{4}{c}{\mathbf{0}} \\ \hline \mathbf{0} & \multicolumn{4}{c|}{\mathbf{0}} & \multicolumn{4}{c}{\mathbf{I}_{nm}} \end{array}\right].$$

Noting that

$$-Q_{2m} - \frac{1}{2}xQ_{2m-1} + x\left(x^{m-2}W + \sum_{i=1}^{m-2} x^{m-2-i}V_i\right) = -\operatorname{rev}Q(x),$$

we get

$$V(x)M_1(x) = \left[\begin{array}{ccc|ccc} -\operatorname{rev}Q(x) & & & & & \\ & -I_n & \mathbf{0} & & \mathbf{0} & \\ * & & \ddots & & & \\ & & -I_n & & & \\ \hline & & & I_n & & \\ & * & & -I_n & \mathbf{0} & \\ & & & * & \ddots & \\ & & & & & -I_n \end{array}\right].$$

Since $V(x)$ has constant nonzero determinant, $xI - M_r \sim V(x)M_1(x)$. It is evident now that

$$xI_{2nm} - M_r \sim \begin{bmatrix} \operatorname{rev}Q(x) & 0 \\ 0 & I_{2nm-n} \end{bmatrix}. \quad \square$$

4. Real factorization of non-negative matrix polynomial

We are now ready for the main result. While the following theorem was previously proven by Hanselka and Sinn [18], we provide a new constructive proof following that

of the complex analogue presented in the monograph by Bakonyi and Woerdeman [1, Section 2.7].

Theorem 4.1. *Let $Q(x) = \sum_{j=0}^{2m} Q_j x^j$ be an $n \times n$ real symmetric positive semidefinite matrix polynomial of degree $2m$ with $Q_0 > 0$. Then the roots of $\det(Q(x))$ all have even multiplicity if and only if there exists an $n \times n$ real matrix polynomial $G(x) = \sum_{j=0}^{m} G_j x^j$ of degree m such that*

$$Q(x) = G(x)^T G(x).$$

Proof. First assume $Q(x) = G(x)^T G(x)$. Then $\det(Q(x)) = \det(G(x))^2$, so clearly all roots have even multiplicity. On the other hand, assume the roots of $\det(Q(x))$ all have even multiplicity. Without loss of generality, assume $Q_0 = I_n$ (otherwise, take $\tilde{Q}(x) := Q_0^{-1/2} Q(x) Q_0^{-1/2}$). Consider the $(m+1)n \times (m+1)n$ real symmetric matrix

$$F_0 = \begin{bmatrix} I_n & \frac{1}{2}Q_1 & & & \\ \frac{1}{2}Q_1 & Q_2 & \ddots & & \\ & \ddots & \ddots & \frac{1}{2}Q_{2m-1} \\ & & \frac{1}{2}Q_{2m-1} & Q_{2m} \end{bmatrix}.$$

Given an $nm \times nm$ real skew-symmetric matrix X, let

$$F_X = F_0 + \begin{bmatrix} 0_{nm \times n} & X \\ 0_n & 0_{n \times nm} \end{bmatrix} - \begin{bmatrix} 0_{n \times nm} & 0_n \\ X & 0_{nm \times n} \end{bmatrix}.$$

It should be noted that in the above line, the matrix decompositions are different; e.g. the X block and the $-X$ block overlap in general. We want to solve

$$X_{\text{opt}} = \arg\min \quad \text{rank}(F_X) \quad \text{such that } F_X \geq 0.$$

Let

$$A = \begin{bmatrix} 0_n & & & \\ I_n & 0_n & & \\ & \ddots & \ddots & \\ & & I_n & 0_n \end{bmatrix} \in \mathbb{R}^{nm \times nm} \quad \text{and} \quad B = \begin{bmatrix} I_n \\ 0_n \\ \vdots \\ 0_n \end{bmatrix} \in \mathbb{R}^{nm \times n}.$$

Then (A, B) is controllable,

$$\begin{bmatrix} 0_{n \times nm} & 0_n \\ X & 0_{nm \times n} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & 0_{n \times nm} \\ XB & XA \end{bmatrix}, \quad \text{and}$$

$$\begin{bmatrix} 0_{nm \times n} & X \\ 0_n & 0_{n \times nm} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & B^T X \\ 0_{nm \times n} & A^T X \end{bmatrix}.$$

Split F_0 into four blocks as follows,

$$F_0 = \left[\begin{array}{c|ccc} I_n & \frac{1}{2}Q_1 & & \\ \hline \frac{1}{2}Q_1 & Q_2 & \ddots & \\ & \ddots & \ddots & \frac{1}{2}Q_{2m-1} \\ & & \frac{1}{2}Q_{2m-1} & Q_{2m} \end{array} \right] =: \begin{bmatrix} I_n & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}.$$

Noting $\Gamma_{12}^T = \Gamma_{21}$ and Γ_{22} is real symmetric, we can recast the condition $F_X \geq 0$ as

$$\begin{bmatrix} I_n & \Gamma_{12} + B^T X \\ \Gamma_{12}^T - XB & \Gamma_{22} + A^T X - XA \end{bmatrix} \geq 0.$$

Consider the Schur complement with respect to I_n,

$$\Gamma_{22} + A^T X - XA - (\Gamma_{12}^T - XB)I_n^{-1}(\Gamma_{12} + B^T X).$$

Setting this equal to zero, we get the modified algebraic Riccati equation

$$P + R^T X - XR + XSX = 0, \tag{7}$$

where

$$P = \Gamma_{22} - \Gamma_{12}^T \Gamma_{12},$$

$$R = A - B\Gamma_{12},$$

$$S = BB^T.$$

Note $P = P^T$ and $S = S^T$ with $S \geq 0$. In this case, the associated M_r matrix is

$$M_r = \begin{bmatrix} A - B\Gamma_{12} & -BB^T \\ \Gamma_{22} - \Gamma_{12}^T \Gamma_{12} & A^T - \Gamma_{12}^T B^T \end{bmatrix}$$

$$
= \left[
\begin{array}{cccc|cccc}
-\frac{1}{2}Q_1 & & & & -I_n & & & \\
I_n & 0 & & & & 0 & & \\
& \ddots & \ddots & & & & \ddots & \\
& & I_n & 0 & & & & 0 \\
\hline
Q_2 - \frac{1}{4}Q_1^2 & \frac{1}{2}Q_3 & & & -\frac{1}{2}Q_1 & I_n & & \\
\frac{1}{2}Q_3 & Q_4 & \ddots & & & 0 & \ddots & \\
& \ddots & \ddots & \frac{1}{2}Q_{2m-1} & & & \ddots & I_n \\
& & \frac{1}{2}Q_{2m-1} & Q_{2m} & & & & 0
\end{array}
\right].
$$

By Lemma 3.7, $xI - M_r$ is a linearization of $\operatorname{rev} Q(x)$. Since $Q(x) \geq 0$ for all $x \in \mathbb{R}$, $\operatorname{rev} Q(x) = x^{2m} Q\left(\frac{1}{x}\right) \geq 0$ for all nonzero $x \in \mathbb{R}$. Thus by continuity, $\operatorname{rev} Q(x) \geq 0$ for all $x \in \mathbb{R}$. Then by Theorem 3.4, the partial multiplicities of every real eigenvalue of $\operatorname{rev} Q(x)$ are all even. Since $xI - M_r$ is a linearization of $\operatorname{rev} Q(x)$, all the partial multiplicities of every eigenvalue of M_r and $\operatorname{rev} Q(x)$ are the same, so the partial multiplicities of every real eigenvalue of M_r are all even. Now since all roots of $\det(Q(x))$ have even multiplicity, all roots of $\det(\operatorname{rev} Q(x))$ have even multiplicity (note the multiplicity of zero as a root is also known to be even since $Q(x)$ has even degree). Then since

$$
\det(xI - M_r) = \det(\operatorname{rev} Q(x)),
$$

all eigenvalues of M_r have even algebraic multiplicity. In particular, all non-real eigenvalues of M_r have even algebraic multiplicity. Hence by Lemma 2.7 and Theorem 2.10, there is a skew-symmetric solution, $\tilde{X}$ of eq. (7). Then since the Schur complement with respect to I_n is zero, we know

$$
F_{\tilde{X}} = \begin{bmatrix} I_n & \Gamma_{12} + B^T \tilde{X} \\ \Gamma_{12}^T - \tilde{X}B & \Gamma_{22} + A^T \tilde{X} - \tilde{X}A \end{bmatrix} \geq 0,
$$

and

$$
\operatorname{rank}(F_{\tilde{X}}) = \operatorname{rank}\left(\begin{bmatrix} I_n & \Gamma_{12} + B^T \tilde{X} \\ \Gamma_{12}^T - \tilde{X}B & \Gamma_{22} + A^T \tilde{X} - \tilde{X}A \end{bmatrix} \right) = \operatorname{rank} I_n = n.
$$

We can factorize

$$
F_{\tilde{X}} = \begin{bmatrix} I_n & \Gamma_{12} + B^T \tilde{X} \\ \Gamma_{12}^T - \tilde{X}B & \Gamma_{22} + A^T \tilde{X} - \tilde{X}A \end{bmatrix} = \begin{bmatrix} G_0^T \\ \vdots \\ G_m^T \end{bmatrix} \begin{bmatrix} G_0 & \cdots & G_m \end{bmatrix},
$$

with $G_0 = I_n$ and G_i real $n \times n$ matrices for $i = 1, 2, \ldots, m$. Then,

$$Q(x) = \begin{bmatrix} I_n & xI_n & \cdots & x^m I_n \end{bmatrix} F_{\tilde{X}} \begin{bmatrix} I_n \\ xI_n \\ \vdots \\ x^m I_n \end{bmatrix}$$

$$= \begin{bmatrix} I_n & xI_n & \cdots & x^m I_n \end{bmatrix} \begin{bmatrix} G_0^T \\ \vdots \\ G_m^T \end{bmatrix} \begin{bmatrix} G_0 & \cdots & G_m \end{bmatrix} \begin{bmatrix} I_n \\ xI_n \\ \vdots \\ x^m I_n \end{bmatrix}.$$

Thus for $G(x) = \sum_{j=0}^{m} G_j x^j$, $Q(x) = G(x)^T G(x)$. $\quad\square$

Theorem 4.1 required $Q_0 > 0$. We can relax this condition as follows.

Corollary 4.2. *Let $Q(x) = \sum_{j=0}^{2m} Q_j x^j$ be a real $n \times n$ regular symmetric positive semidefinite matrix polynomial of degree $2m$. Then all roots of $\det(Q(x))$ have even multiplicity if and only if there exists an $n \times n$ real matrix polynomial $G(x) = \sum_{j=0}^{m} G_j x^j$ of degree m such that*

$$Q(x) = G(x)^T G(x).$$

Proof. First, assume $Q(x) = G(x)^T G(x)$. Then $\det(Q(x)) = \det(G(x))^2$, so clearly all roots have even multiplicity. On the other hand, assume all roots of $\det(Q(x))$ have even multiplicity. Let $x_0 \in \mathbb{R}$ be such that $\det(Q(x_0)) \neq 0$ (note such x_0 exists since $Q(x)$ is regular, i.e. the determinant is not identically zero). Consider

$$P(x) := Q(x_0 - x).$$

Then $P(x)$ is an $n \times n$ real symmetric matrix polynomial of degree $2m$ such that

$$P(0) = Q(x_0) > 0 \qquad \text{and} \qquad \det(P(x)) = \det(Q(x_0 - x)).$$

Thus the roots of $\det(P(x))$ all have even multiplicity, so by Theorem 4.1, there is an $n \times n$ real matrix polynomial $H(x) = \sum_{j=0}^{m} H_j x^j$ of degree m such that

$$P(x) = H(x)^T H(x).$$

Define

$$G(x) := H(x_0 - x).$$

Then $G(x) = \sum_{j=0}^{m} G_j x^j$ is an $n \times n$ real matrix polynomial such that

$$Q(x) = P(x_0 - x) = H(x_0 - x)^T H(x_0 - x) = G(x)^T G(x). \quad\square$$

5. M_r-invariant $\hat{H}_r$-neutral subspace

The proof of Theorem 4.1 is constructive. It hinges on finding the real skew-symmetric solution X to the modified algebraic Riccati equation. Back in Section 2, we found such a solution by constructing an mn-dimensional M_r-invariant $\hat{H}_r$-neutral subspace. Following the proof of Theorem 2.5, found in [20, Theorem 2.6.3], also to be found in Gohberg, Lancaster, and Rodman's later book *Indefinite Linear Algebra and Applications* [14, Theorem I.3.21], to find such a subspace, we must first convert M_r to its real Jordan form. For this, assume there are k Jordan blocks corresponding to real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (note some eigenvalues may be repeated as they can occur in multiple blocks). The block corresponding to eigenvalue λ_j has size r_j and is denoted by

$$
J_{r_j}(\lambda_j) = \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_j & 1 \\ & & & \lambda_j \end{bmatrix} \in \mathbb{R}^{r_j \times r_j}.
$$

Next, assume there are ℓ real Jordan blocks corresponding to pairs of non-real eigenvalues $\alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2, \ldots, \alpha_\ell \pm i\beta_\ell$ (note again some eigenvalues may be repeated as they can occur in multiple blocks). The block corresponding to the pair of eigenvalues $\alpha_j \pm i\beta_j$ has size $2s_j$ and is denoted by

$$
J_{2s_j}(\alpha_j \pm i\beta_j) = \begin{bmatrix} C(\alpha_j, \beta_j) & I_2 & & \\ & \ddots & \ddots & \\ & & C(\alpha_j, \beta_j) & I_2 \\ & & & C(\alpha_j, \beta_j) \end{bmatrix} \in \mathbb{R}^{2s_j \times 2s_j},
$$

$$
\text{where} \quad C(\alpha_j, \beta_j) = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}.
$$

Then for some real invertible matrix S,

$$
M_r = SJS^{-1},
$$

where

$$
J = J_{r_1}(\lambda_1) \oplus \cdots \oplus J_{r_k}(\lambda_k) \oplus J_{2s_1}(\alpha_1 \pm i\beta_1) \oplus \cdots \oplus J_{2s_\ell}(\alpha_\ell \pm i\beta_\ell). \tag{8}
$$

We will form the desired mn-dimensional M_r-invariant $\hat{H}_r$-neutral subspace by extracting nm columns of S and putting these columns together to form a $2nm \times nm$ matrix Y. The desired M_r-invariant $\hat{H}_r$-neutral subspace is then the column space of this matrix Y. The construction of Y is outlined as follows.

Construction of Y:

1. For each Jordan block $J_{r_j}(\lambda_j)$ of a real eigenvalue, there are r_j corresponding columns in S. Note r_j is known to be even. Take the first $\frac{r_j}{2}$ of those columns.
2. For each real Jordan block $J_{2s_j}(\alpha_j \pm i\beta_j)$ of a complex conjugate pair of eigenvalues, there are $2s_j$ corresponding columns in S. If s_j is even, take the first s_j of those columns.
3. Each remaining real Jordan block $J_{2s_j}(\alpha_j \pm i\beta_j)$ of a complex conjugate pair of eigenvalues has $2s_j$ corresponding columns in S, where s_j is odd. Since the algebraic multiplicity of each eigenvalue is even, we can pair up each of these blocks with another such block of the same eigenvalue, say $J_{2s_j}(\alpha_j \pm i\beta_j)$ pairs with $J_{2s_p}(\alpha_p \pm i\beta_p)$ where $\alpha_j = \alpha_p$ and $\beta_j = \beta_p$. Take the first $s_j - 1$ of the columns of S corresponding to $J_{2s_j}(\alpha_j \pm i\beta_j)$ and the first $s_p - 1$ of the columns of S corresponding to $J_{2s_p}(\alpha_p \pm i\beta_p)$. Lastly, take
   (a) The s_jth column of S corresponding to $J_{2s_j}(\alpha_j \pm i\beta_j)$ plus the $s_p + 1$st column of S corresponding to $J_{2s_p}(\alpha_p \pm i\beta_p)$.
   (b) The $s_j + 1$st column of S corresponding to $J_{2s_j}(\alpha_j \pm i\beta_j)$ minus the s_pth column of S corresponding to $J_{2s_p}(\alpha_p \pm i\beta_p)$.

Putting these columns together to form a $2nm \times nm$ matrix Y, we get that the desired M_r-invariant $\hat{H}_r$-neutral subspace as the column space of this matrix Y. The reason why this works hinges on the following theorem.

Theorem 5.1. *[20, Theorem 2.6.1] Let H be a nonsingular real $n \times n$ symmetric matrix and let A be a real H-symmetric matrix. Then there exists an invertible real matrix S such that $J := S^{-1}AS$ can be written as in eq. (8) and $P := S^*HS$ has the form*

$$P = \epsilon_1 P_{r_1} \oplus \cdots \oplus \epsilon_k P_{r_k} \oplus P_{2s_1} \oplus \cdots \oplus P_{2s_\ell},$$

where $\lambda_1, \ldots, \lambda_k$, $\alpha_1, \ldots, \alpha_\ell$, and $\beta_1, \ldots, \beta_\ell$ are positive numbers; $\epsilon_j = \pm 1, j = 1, 2, \ldots, k$; P_j is the $j \times j$ reversal matrix, i.e.

$$P_j = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{j \times j}.$$

Moreover, the canonical form (J, P) of (A, H) is uniquely determined by (A, H) up to permutation of blocks $(J_{r_j}(\lambda_j), \epsilon_j P_{r_j})$ for $j = 1, \ldots, k$ and $(J_{2s_j}(\alpha_j \pm i\beta_j), P_{2s_j})$ for $j = 1, \ldots, \ell$.

This theorem tells us that when we write

$$M_r = SJS^{-1},$$

we also have

$$\hat{H}_r = S^{-*}PS^{-1},$$

where P is as defined in the theorem and $S^{-*} = (S^*)^{-1}$. We illustrate three cases with simple examples. First assume $J = J_{r_1}(\lambda_1)$, where we know r_1 is even. Then by Theorem 5.1, $P = \epsilon_1 P_{r_1}$ where $\epsilon_1 = \pm 1$. Let Y be the matrix formed from the first $\frac{r_1}{2}$ columns of S. We claim that the column space of Y is M_r-invariant and $\hat{H}_r$-neutral. Indeed,

$$M_r Y = SJS^{-1}Y = SJ \begin{bmatrix} I_{r_1/2} \\ 0_{r_1/2} \end{bmatrix} = Y J_{r_1/2}(\lambda_1),$$

and

$$Y^* \hat{H}_r Y = Y^* S^{-*} \epsilon_1 P_{r_1} S^{-1} Y = \epsilon_1 \begin{bmatrix} I_{r_1/2} & 0_{r_1/2} \end{bmatrix} \begin{bmatrix} 0_{r_1/2} & P_{r_1/2} \\ P_{r_1/2} & 0_{r_1/2} \end{bmatrix} \begin{bmatrix} I_{r_1/2} \\ 0_{r_1/2} \end{bmatrix} = 0.$$

Next assume $J = J_{2s_1}(\alpha_1 \pm i\beta_1)$, where s_1 is even. Then by Theorem 5.1, $P = P_{2s_1}$. Let Y be the matrix formed from the first s_1 columns of S. Then the column space of Y is M_r-invariant and $\hat{H}_r$-neutral. Indeed,

$$M_r Y = SJS^{-1}Y = Y J_{s_1}(\alpha_1 \pm i\beta_1),$$

and

$$Y^* \hat{H}_r Y = Y^* S^{-*} P_{2s_1} S^{-1} Y = \begin{bmatrix} I_{s_1} & 0_{s_1} \end{bmatrix} \begin{bmatrix} 0_{s_1} & P_{s_1} \\ P_{s_1} & 0_{s_1} \end{bmatrix} \begin{bmatrix} I_{s_1} \\ 0_{s_1} \end{bmatrix} = 0.$$

Finally, assume

$$J = J_{2s_1}(\alpha_1 \pm i\beta_1) \oplus J_{2s_2}(\alpha_2 \pm i\beta_2)$$

where s_1 and s_2 are odd, $\alpha_1 = \alpha_2$, and $\beta_1 = \beta_2$. Form the $(2s_1 + 2s_2) \times (s_1 + s_2)$ matrix Y using columns $1, 2, \ldots, s_1 - 1$ and $2s_1 + 1, 2s_1 + 2, 2s_1 + s_2 - 1$ from S. Additionally, form the second to last column of Y as column s_1 of S plus column $2s_1 + s_2 + 1$ of S. Finally, form the last column of Y as column $s_1 + 1$ of S minus column $2s_1 + s_2$ of S. Then the column space of Y is M_r-invariant and $\hat{H}_r$-neutral. Indeed, let S_j denote column j of S. Let $\alpha = \alpha_1 = \alpha_2$ and $\beta = \beta_1 = \beta_2$. Then

$$Y = \begin{bmatrix} S_1 & \cdots & S_{s_1-1} & S_{2s_1+1} & \cdots & S_{2s_1+s_2-1} & S_{s_1} + S_{2s_1+s_2+1} & S_{s_1+1} - S_{2s_1+s_2} \end{bmatrix}.$$

Letting 0_j denote the $j \times j$ zero matrix (and generic 0 may represent a zero matrix whose size can be inferred in the context of the other elements of the matrix), we have

$$M_r Y = SJS^{-1}Y = SJ \begin{bmatrix} I_{s_1-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0_{s_1-1} & 0 & 0 & 0 \\ 0 & I_{s_2-1} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0_{s_2-1} & 0 & 0 \end{bmatrix}$$

$$= S \begin{bmatrix} J_{s_1-1}(\alpha \pm i\beta) & 0 & * & * \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \\ 0_{s_1-1} & 0 & 0 & 0 \\ 0 & J_{s_2-1}(\alpha \pm i\beta) & * & * \\ 0 & 0 & \beta & -\alpha \\ 0 & 0 & \alpha & \beta \\ 0 & 0_{s_2-1} & 0 & 0 \end{bmatrix},$$

so Y is M_r-invariant. Next, by Theorem 5.1, we know $\hat{H}_r = S^{-*}PS^{-1}$ where

$$P = P_{2s_1} \oplus P_{2s_2}.$$

Then

$$Y^*\hat{H}_r Y = Y^*S^{-*}(P_{2s_1} \oplus P_{2s_2})S^{-1}Y$$

$$= \begin{bmatrix} I_{s_1-1} & 0 & 0 & 0_{s_1-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{s_2-1} & 0 & 0 & 0_{s_2-1} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} (P_{2s_1} \oplus P_{2s_2})$$

$$\times \begin{bmatrix} I_{s_1-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0_{s_1-1} & 0 & 0 & 0 \\ 0 & I_{s_2-1} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0_{s_2-1} & 0 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} I_{s_1-1} & 0 & 0 & 0_{s_1-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{s_2-1} & 0 & 0 & 0_{s_2-1} \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0_{s_1-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ P_{s_1-1} & 0 & 0 & 0 \\ 0 & 0_{s_2-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & P_{s_2-1} & 0 & 0 \end{bmatrix} = 0.$$

These three cases together show why the general procedure for picking columns of S generates the desired subspace.

6. Algorithm

Now that we have the construction of the invariant subspace, we can put everything together to get an explicit algorithm as outlined below. To the best of our knowledge, no algorithm for finding the real factorization $Q(x) = G(x)^T G(x)$ exists in the literature.

Algorithm 1: Real Factorization of PSD Matrix Polynomial.

Input: A real $n \times n$ regular symmetric positive semidefinite matrix polynomial $Q(x) = \sum_{j=0}^{2m} Q_j x^j$ for which $\det(Q(x))$ has only roots of even multiplicity.

1. Fix $x_0 \in \mathbb{R}$ such that $\det(Q(x_0)) \neq 0$.
2. Set $\hat{P}(x) = Q(x_0 - x) =: \sum_{j=0}^{2m} \hat{P}_j x^j$.
3. Set $P(x) = \hat{P}_0^{-1/2} \hat{P}(x) \hat{P}_0^{-1/2} =: \sum_{j=0}^{2m} P_j x^j$.
4. Set

$$
M_r = \left[
\begin{array}{cccc|cccc}
-\frac{1}{2}P_1 & & & & -I_n & & & \\
I_n & 0 & & & & 0 & & \\
& \ddots & \ddots & & & & \ddots & \\
& & I_n & 0 & & & & 0 \\
\hline
P_2 - \frac{1}{4}P_1^2 & \frac{1}{2}P_3 & & & -\frac{1}{2}P_1 & I_n & & \\
\frac{1}{2}P_3 & P_4 & \ddots & & & 0 & \ddots & \\
& \ddots & \ddots & \frac{1}{2}P_{2m-1} & & & \ddots & I_n \\
& & \frac{1}{2}P_{2m-1} & P_{2m} & & & & 0
\end{array}
\right].
$$

5. Find the real Jordan canonical form $M_r = SJS^{-1}$.
6. Choose columns of the matrix S to form a $2nm \times nm$ matrix Y as outlined under 'Construction of Y' in the previous section.
7. Define the $nm \times nm$ matrices X_1 and X_2 by $Y = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and compute $X = X_2 X_1^{-1}$.
8. Set

$$
F_X = \begin{bmatrix}
I_n & \frac{1}{2}P_1 & & \\
\frac{1}{2}P_1 & P_2 & \ddots & \\
& \ddots & \ddots & \frac{1}{2}P_{2m-1} \\
& & \frac{1}{2}P_{2m-1} & P_{2m}
\end{bmatrix}
+ \begin{bmatrix} 0_{nm \times n} & X \\ 0_n & 0_{n \times nm} \end{bmatrix}
- \begin{bmatrix} 0_{n \times nm} & 0_n \\ X & 0_{nm \times n} \end{bmatrix}.
$$

9. By construction, F_X is positive semidefinite of rank n, so factorize as

$$
F_X = \begin{bmatrix} H_0^T \\ \vdots \\ H_m^T \end{bmatrix} \begin{bmatrix} H_0 & \cdots & H_m \end{bmatrix},
$$

where H_j is real $n \times n$ for all j.

10. Set $H(x) = \sum_{j=0}^m H_j x^j$.
11. Set $G(x) = \sum_{j=0}^m P_0^{1/2} H_j (x_0 - x)^j =: \sum_{j=0}^m G_j x^j$.

Output: A real $n \times n$ matrix polynomial $G(x) = \sum_{j=0}^m G_j x^j$ such that $Q(x) = G(x)^T G(x)$.

6.1. Examples

Let us illustrate the above ideas in a few examples. Note that the Jordan canonical factorization in the algorithm relies on exact computation, so the examples were computed in Maple. The first example is the real eigenvalue case. The second example is the non-real eigenvalue case with even s_j.

Example 6.1. Take

$$Q(x) = \begin{bmatrix} 2x^2 + 2x + 1 & -4x^2 - 3x \\ -4x^2 - 3x & 8x^2 + 4x + 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} + x^2 \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}.$$

Then $M_r = SJS^{-1}$ for

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad S = \begin{bmatrix} \frac{2}{9} & \frac{37}{54} & \frac{-5}{18} & \frac{17}{54} \\ \frac{1}{9} & \frac{10}{27} & \frac{5}{18} & \frac{-10}{27} \\ \frac{-1}{18} & \frac{-19}{54} & \frac{-5}{36} & \frac{19}{54} \\ \frac{1}{9} & \frac{19}{108} & \frac{-5}{36} & \frac{-19}{108} \end{bmatrix}.$$

We have $J = J_2(0) \oplus J_2(-3)$, so $r_1 = r_2 = 2$. Thus we take the 1st column of S corresponding to the first Jordan block as well as the 1st column of S corresponding to the second Jordan block.

$$\begin{bmatrix} \frac{2}{9} & \frac{-5}{18} \\ \frac{1}{9} & \frac{5}{18} \\ \frac{-1}{18} & \frac{-5}{36} \\ \frac{1}{9} & \frac{-5}{36} \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Our invariant subspace is thus

$$\mathrm{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathrm{Im} \begin{bmatrix} I \\ X_2 X_1^{-1} \end{bmatrix}.$$

Then,

$$X = X_2 X_1^{-1} = \begin{bmatrix} 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Thus

$$F_X = F_0 + \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 1 & -1 & 2 & -4 \\ -2 & 2 & -4 & 8 \end{bmatrix}.$$

We factorize as

$$
F_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \end{bmatrix}.
$$

Thus

$$
G_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}, \qquad G(x) = G_0 + xG_1.
$$

We can verify $Q(x) = G(x)^T G(x)$.

Example 6.2. Take now

$$
Q(x) = \begin{bmatrix} 2x^2 + 2x + 1 & x^2 + 2x \\ x^2 + 2x & 13x^2 + 4x + 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} + x^2 \begin{bmatrix} 2 & 1 \\ 1 & 13 \end{bmatrix}.
$$

Then $M_r = SJS^{-1}$ for

$$
J = \begin{bmatrix} \frac{-3}{2} & \frac{\sqrt{11}}{2} & 1 & 0 \\ \frac{-\sqrt{11}}{2} & \frac{-3}{2} & 0 & 1 \\ 0 & 0 & \frac{-3}{2} & \frac{\sqrt{11}}{2} \\ 0 & 0 & \frac{-\sqrt{11}}{2} & \frac{-3}{2} \end{bmatrix}, \quad
S = \begin{bmatrix} \frac{4}{11} & \frac{-2\sqrt{11}}{11} & \frac{1}{2} & \frac{5\sqrt{11}}{242} \\ \frac{-3}{11} & \frac{-\sqrt{11}}{11} & 0 & \frac{5\sqrt{11}}{121} \\ \frac{-6}{11} & \frac{-2\sqrt{11}}{11} & 0 & \frac{-12\sqrt{11}}{121} \\ \frac{-8}{11} & \frac{4\sqrt{11}}{11} & 0 & \frac{6\sqrt{11}}{121} \end{bmatrix}.
$$

We have $J = J_4\left(\frac{3}{2} \pm i\frac{\sqrt{11}}{2}\right)$, so $s_1 = 2$. Thus we take the first two columns of S corresponding to the first and only Jordan block.

$$
\begin{bmatrix} \frac{4}{11} & \frac{-2\sqrt{11}}{11} \\ \frac{-3}{11} & \frac{-\sqrt{11}}{11} \\ \frac{-6}{11} & \frac{-2\sqrt{11}}{11} \\ \frac{-8}{11} & \frac{4\sqrt{11}}{11} \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.
$$

Our invariant subspace is thus

$$
\mathrm{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathrm{Im} \begin{bmatrix} I \\ X_2 X_1^{-1} \end{bmatrix}.
$$

Then,

$$
X = X_2 X_1^{-1} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.
$$

Thus

$$F_X = F_0 + \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 13 \end{bmatrix}.$$

We factorize as

$$F_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 2 \end{bmatrix}.$$

Thus

$$G_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \qquad G(x) = G_0 + x G_1.$$

We can verify $Q(x) = G(x)^T G(x)$.

6.2. Numerical considerations

Note that the algorithm presented here relies on being able to compute the real Jordan canonical form of a matrix. This is not numerically stable and thus the algorithm assumes exact computation. However, if all eigenvalues have algebraic multiplicity 2 and geometric multiplicity 1 (i.e. in eq. (8), $r_j = 2, j = 1, 2, \ldots, k$ and $s_j = 2, j = 1, 2, \ldots, \ell$), we can still find the required M_r-invariant $\hat{H}_r$-neutral subspace. Indeed, for size 2 blocks of real eigenvalues, our algorithm says to take the first column of S associated with that block. This is simply the eigenvector associated with that eigenvalue. Similarly, for a size 4 block of a pair of non-real eigenvalues, our algorithm says to take the first two columns of S associated with that block. These are the real and imaginary parts of the eigenvector associated with that eigenvalue pair. In short, all we need here are the eigenvectors of the M_r matrix.

We tested a numerical implementation of our algorithm in MATLAB by running 100 trials. In each trial, a random matrix size n and degree m were chosen between 2 and 8. A matrix polynomial $G(x) = \sum_{i=0}^{m} G_i x^i$ of size $n \times n$ and degree m was created by taking $G_0 = I_n$ and randomly generating $G_1, \ldots G_m$. Then the coefficients $Q_0, Q_1, \ldots, Q_{2m}$ of the matrix polynomial $Q(x) = \sum_{i=0}^{2m} Q_i x^i$ were computed by setting $Q(x) = G(x)^T G(x)$. Operating under the assumption that all eigenvalues have algebraic multiplicity 2 and geometric multiplicity 1, Algorithm 1 was implemented, where the matrix Y in step 6 was formed from the eigenvectors of M_r as outlined in the previous paragraph. The algorithm outputted coefficients $\hat{G}_i$ for the matrix polynomial $\hat{G}(x) = \sum_{i=0}^{m} \hat{G}_i x^i$. We computed coefficients $\hat{Q}_i$ for the matrix polynomial $\hat{Q}(x) = \hat{G}(x)^T \hat{G}(x)$. Since in application, only $Q(x)$ and not $G(x)$ would be known before the implementation of the algorithm, the error of the trial was calculated as the maximum absolute entry of

$$\begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{2m} \end{bmatrix} - \begin{bmatrix} \hat{Q}_0 & \hat{Q}_1 & \cdots & \hat{Q}_{2m} \end{bmatrix}.$$

The overall worst error among all 100 trials was on the order of 10^{-6}. It is important to note, though, that for particular examples where the multiplicities of eigenvalues are higher, the algorithm does not work in any reliable way.

Finally, one can also use numerical methods to conclude reliably that if an eigenvalue of M_r has algebraic multiplicity equal to one, then the corresponding matrix polynomial $Q(x)$ will not have a real factorization.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

Acknowledgement

We thank the referee for their careful reading of the manuscript and their detailed advice, which, among other things, lead to the addition of the subsection on numerical considerations.

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