



Uniform Upper Bounds on Courant Sharp Neumann Eigenvalues of Chain Domains

Thomas Beck¹ · Yaiza Canzani² · Jeremy L. Marzuola² 

Received: 9 October 2023 / Accepted: 28 May 2024
© Mathematica Josephina, Inc. 2024

Abstract

We obtain upper bounds on the number of nodal domains of Laplace eigenfunctions on chain domains with Neumann boundary conditions. The chain domains consist of a family of planar domains, with piecewise smooth boundary, that are joined by thin necks. Our work does not assume a lower bound on the width of the necks in the chain domain. As a consequence, we prove an upper bound on the eigenvalue of Courant sharp eigenfunctions that is independent of the widths of the necks.

Keywords Pleijel bounds · Courant Sharp Eigenvalues · Neumann boundary conditions

Mathematics Subject Classification 35P05 · 35P15 · 58J50

1 Introduction

A long-studied question is to what extent the spectrum of the Laplacian interacts with the geometry of the domain on which it is defined [17]. In this article, we study *chain domains* which consist of a family of disjoint bounded planar domains, with piecewise smooth boundaries, that are connected by thin necks (see Fig. 2). For these chain domains, we prove bounds on the number of nodal domains of the associated Laplace eigenfunctions when Neumann boundary conditions are imposed.

✉ Jeremy L. Marzuola
marzuola@math.unc.edu

Thomas Beck
tbeck7@fordham.edu

Yaiza Canzani
canzani@email.unc.edu

¹ Department of Mathematics, Fordham University, 407 John Mulcahy Hall, 441 E. Fordham Road, Bronx, NY 10458-5165, USA

² Department of Mathematics, University of North Carolina at Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599, USA

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary and write $0 = \mu_1 < \mu_2 \leq \dots$ for the Laplace eigenvalues of the Neumann problem

$$\begin{cases} \Delta u_m = -\mu_m u_m, & \text{in } \Omega, \\ \partial_n u_m = 0, & \text{on } \partial\Omega. \end{cases} \quad (1)$$

The Courant Nodal Domain Theorem [6] asserts that for all $m = 1, 2, \dots$,

$$v(u_m) \leq m, \quad (2)$$

where $v(u_m)$ is the number of nodal domains of u_m . That is, $v(u_m)$ is the number of connected components of $\{x \in \Omega : u_m(x) \neq 0\}$. If $v(u_m) = m$, then u_m is said to be a *Courant sharp eigenfunction* and μ_m a *Courant sharp eigenvalue*.

In this article, we study the case in which Ω is a chain domain and prove an asymptotic upper bound on $v(u_m)$ that, in particular, provides an upper bound on the values of μ_m that can be Courant sharp. The bound we prove is independent of a lower bound of the widths of the necks in the chain domain, hence notably does not depend upon the cut-distance of $\partial\Omega$.

A chain domain consists of a family of disjoint bounded planar domains, $\{\mathcal{D}_\ell\}_{\ell=1}^M$, and a family of thin necks joining the domains. The planar domains have smooth boundaries except for a finite number of vertices.

Since we are interested in understanding how our estimates respond to the width of the necks shrinking to zero, we actually think of a chain domain as a structure built around a *skeleton* comprised of the domains $\{\mathcal{D}_\ell\}_{\ell=1}^M$ joined by a family of curves $\{\Gamma_i\}_{i=1}^N$. Then, the skeleton is “thickened” into what we call a *base domain*, Ω , by turning each curve Γ_i into a neck that joins two domains. Formally, associated to each curve Γ_i there is a smooth homotopy $\mathcal{G}_i : [0, L_i] \times [-1, 1] \rightarrow \mathbb{R}^2$ such that $\Gamma_i = \mathcal{G}_i([0, L_i] \times \{0\})$ and each of the arcs $\mathcal{G}_i(\{0\} \times [-1, 1])$ and $\mathcal{G}_i(\{L_i\} \times [-1, 1])$ belong to the boundaries of the two domains being joined by Γ_i . The curve Γ_i is then thickened to the neck $\mathcal{N}_i := \mathcal{G}_i([0, L_i] \times (-1, 1))$. The base domain is the set $\Omega = \bigcup_{\ell=1}^M \mathcal{D}_\ell \cup \bigcup_{i=1}^N \mathcal{N}_i$. See Definition 2.2 for a detailed description.

Given a base domain Ω , we allow for the width of each neck to vary by working with a family of neck widths $\mathbf{w} := \{I_i\}_{i=1}^N$, where the interval $I_i \subset [-1, 1]$ for each i . Indeed, we define each neck as $\mathcal{N}_i(\mathbf{w}) := \mathcal{G}_i([0, L_i] \times I_i)$. The *chain domain* is then

$$\Omega(\mathbf{w}) = \bigcup_{\ell=1}^M \mathcal{D}_\ell \cup \bigcup_{i=1}^N \mathcal{N}_i(\mathbf{w}).$$

Note that this definition of a chain domain includes polygonal figures with smooth edges. A toy model with a single neck is depicted in Fig. 1. See Definition 2.3 for a precise definition of a chain domain and Fig. 2 for an illustration of such.

Our main result is the following.

Fig. 1 A simple chain domain, $\Omega(\mathbf{w}) = \mathcal{D}_1 \cup \mathcal{N} \cup \mathcal{D}_2$, in which two squares, $\mathcal{D}_1, \mathcal{D}_2$ of side length 2 are joined by the neck $\mathcal{N} := [0, 2] \times \mathbf{w}$, with $\mathbf{w} = (-w/2, w/2)$

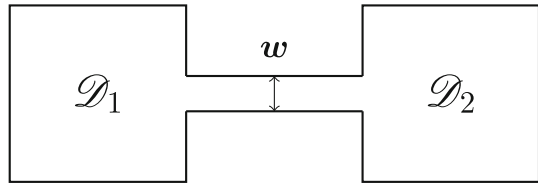
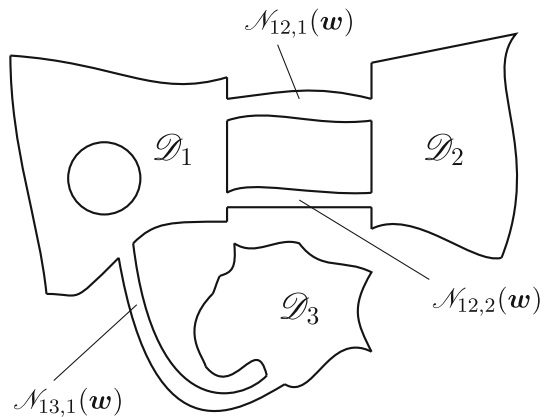


Fig. 2 An example of a domain $\Omega(\mathbf{w})$ from Definition 2.3



Theorem 1.1 Let Ω be a base domain. There exists $C > 0$ such that for every family of neck widths, $\mathbf{w}$, the following holds. If $\{u_m(\mathbf{w})\}_m$ and $\{\mu_m(\mathbf{w})\}_m$ are an orthonormal basis of Neumann eigenfunctions and eigenvalues for the chain domain $\Omega(\mathbf{w})$, then

$$\limsup_{m \rightarrow \infty} \frac{v(u_m(\mathbf{w}))}{m} \leq \frac{4}{\lambda_1(\mathbb{D})}. \quad (3)$$

Moreover, for every Courant sharp eigenvalue $\mu_m(\mathbf{w})$,

$$\text{Area}(\Omega(\mathbf{w}))\mu_m(\mathbf{w}) < C. \quad (4)$$

Here, $\lambda_1(\mathbb{D})$ is the first Dirichlet eigenvalue of the unit disc, $\mathbb{D} \subset \mathbb{R}^2$. It is known that $\lambda_1(\mathbb{D}) = j_{0,1}^2$, where $j_{0,1}$ is the first positive zero of the Bessel function J_0 . Since $4/\lambda_1(\mathbb{D}) \approx 0.692 < 1$, the upper bound in (3) implies that there are only finitely many Courant sharp eigenfunctions. The estimate in (4) then gives an upper bound on the eigenvalue of such an eigenfunction, independent of the choice of neck-widths $\mathbf{w}$.

Remark 1.1 In Section 2.1 we introduce five geometric constants $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$ associated to the geometry of the base domain Ω . The constant C in (4) depends only on $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$. The same constant C can be used for all chain domains whose associated base domains, $\tilde{\Omega}$, have geometric constants controlled by $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$ as explained in Remark 2.1. This implies that the number and structure of the planar domains and necks comprising Ω only play a role in controlling the Courant sharp eigenvalues through how they impact these five geometric constants.

There are several articles that study eigenfunction nodal domain count under various boundary conditions (see Section 1.1 for an account of these). Directly related to our main result is the work of Gittins and Léna [9], who also give a bound on $\text{Area}(\Omega)\mu_m$ when μ_m is Courant sharp. However, applying their result would require imposing a lower bound on the widths of the connecting necks in the chain. A strength of our work is that we prove bounds on the nodal domain counting function that are independent of the width of the necks, and hence obtain that the oscillatory behavior of the Courant sharp eigenfunctions is driven by the geometry of the planar domains that make up the chain, as stated in (4). The bound in (3) was proved by Polterovich [21] when $\Omega \subset \mathbb{R}^2$ has piecewise real analytic boundary and by Léna [18] when the boundary is $C^{1,1}$. Theorem 1.1 extends their work to more general domains, including polygonal regions with smooth edges.

When working with a chain domain on which *Dirichlet boundary conditions* are imposed, the spectrum of the Laplacian approaches the spectrum of the disjoint union of the individual planar domains as the widths of the necks connecting the domains decrease to 0. The corresponding eigenfunctions do not ‘see’ the thin necks and become localized to the connecting planar domains, for example, see [7]. In contrast, when Neumann boundary conditions are imposed, the behavior of the low-energy eigenfunctions is influenced by the presence of the thin necks. Indeed, in this case, the eigenfunctions for the chain behave like the restricted eigenfunctions on the individual smooth domains, but are ‘connected’ along each neck by a function that, to leading order, is an eigenfunction of a one-dimensional Schrödinger operator [1, 14–16].

In this work, instead of studying low-energy eigenfunctions, we ask whether the intermediate to high energy eigenfunctions have oscillatory behavior that is more closely related to the geometry of the family of planar domains in the chain than to the structure of the thin necks connecting them. Theorem 1.1 shows that the behavior of Courant sharp eigenfunctions is driven by that of the planar domains and ignores the structure of the connecting necks. Interestingly, the oscillatory behavior of eigenfunctions has been known to be unstable to domain perturbations for some time, see [24, 25]. Indeed, in previous work, the authors explored how boundary perturbations can lower the number of nodal domains of low energy eigenfunctions [2]. It is thus natural to think that, in general, it should be more challenging to have many internal oscillations in chain domains than in the piecewise smooth domains themselves, but we do not address that question here.

The problem we study has a natural geometric interpretation of a community detection problem in data analysis. Many data sets have the structure of stochastic block models, which consist of clusters of sub-networks that are strongly connected with weak connections between them. It is a natural question in learning and community detection algorithms to determine the number of communities that one should actually look for in a data set. This study is a first step in understanding how the spectrum of the Laplacian with Neumann boundary conditions may be used to explore such a question in a continuum setting, which can in many ways be seen as a model for large data sets with manifold structure, see [22, 23]. Indeed, we conjecture that a chain domain of k planar domains connected in a line will have its first k eigenfunctions saturating Courant’s nodal domain bound by oscillating as much as possible, while the eigenfunctions with sufficiently large eigenvalues will not saturate the bound. This article

works towards this conjecture, by giving bounds on the nodal domain count that are independent of the thin connections between domains.

1.1 Prior Results

While we will focus on planar domains with Neumann boundary conditions, the Courant Nodal Domain Theorem (see equation (2)) also holds for eigenfunctions with Dirichlet boundary conditions. In [20], Pleijel proved that for planar domains with Dirichlet boundary conditions, there are only a finite number of Courant sharp eigenfunctions. This was extended by Peetre in [19] to domains on Riemannian surfaces, and by Bérard and Meyer [4] to n -dimensional Riemannian manifolds that are compact or have smooth boundary. In each case, the result is achieved by obtaining an asymptotic upper bound on the ratio between the number of nodal domains of the m -th Dirichlet eigenfunction, v_m , and its index m . In two dimensions, this upper bound is

$$\limsup_{m \rightarrow \infty} \frac{v(v_m)}{m} \leq \frac{4}{\lambda_1(\mathbb{D})}, \quad (5)$$

the same as in Theorem 1.1. Since $\lambda_1(\mathbb{D}) = j_{0,1}^2$ and $4/j_{0,1}^2 \approx 0.692 < 1$, the upper bound in (5) immediately implies that there are only finitely many Courant sharp Dirichlet eigenfunctions.

A key step in the proof of (5) is to obtain a lower bound on the area of a nodal domain using the Faber-Krahn Theorem. This theorem can be used in the Dirichlet case because the restriction of a Dirichlet eigenfunction to a nodal domain is the first Dirichlet eigenfunction of that domain. When Neumann boundary conditions are imposed, this is no longer true for nodal domains that touch the boundary. However, by using a different method to count nodal domains near the boundary, Polterovich [21] in the two-dimensional case with piecewise real analytic boundaries, and Léna [18] in n -dimensions with $C^{1,1}$ -boundaries, showed that (5) continues to hold for Neumann eigenfunctions, and Robin eigenfunctions with non-negative Robin parameter. This was extended to any sign of the Robin parameter, for domains with $C^{1,1}$ -boundaries, in the recent work [11].

While the above results guarantee that there are only finitely many eigenvalues with a Courant sharp eigenfunction, they do not provide an upper bound on the corresponding eigenvalue in terms of geometric quantities of the underlying domain. For a planar bounded domain Ω , a bound on $\text{Area}(\Omega)\lambda$, when λ is a Courant sharp Dirichlet eigenvalue, was established by Bérard and Helffer [3]. Such a bound is given in terms of dilation invariant quantities involving $\text{Area}(\Omega)$ and $\sup_{\delta > 0} \frac{1}{\delta} M_\Omega(\delta)$, where

$$M_\Omega(\delta) := \text{Area}(\{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}). \quad (6)$$

For an open set $\Omega \subset \mathbb{R}^n$ of finite Lebesgue measure, van den Berg and Gittins [26] obtain an upper bound on a Courant sharp Dirichlet eigenvalue in terms of $\text{Area}(\Omega)$ and $M_\Omega(\delta)$. For a planar domain with C^2 -boundary, Gittins and Léna prove an upper bound on $\text{Area}(\Omega)\mu$, when μ is a Courant sharp Neumann eigenvalue [9]. This bound

is given in terms of an upper bound on $\text{Area}(\Omega)$, the isoperimetric ratio, the curvature of $\partial\Omega$, and a lower bound on the cut-distance

$$\inf_{t \in [0, L]} \sup_{\delta > 0} \left\{ \delta : \text{dist} \left(\gamma(t) + sn(t), \partial\Omega \right) = s \text{ for all } s \in [0, \delta] \right\}.$$

Here, $\gamma : [0, L] \rightarrow \mathbb{R}^2$ is a parameterization of $\partial\Omega$ and $n(t)$ is the unit inward normal at $\gamma(t)$.

Note that as the width w shrinks to 0, the chain domain $\Omega(w)$ from Fig. 1 does not fit into the results of Gittins and Léna in [9]. This is because the cut-distance of the neck, $[0, 2] \times (-w/2, w/2)$, shrinks to zero when w approaches 0. Moreover, the sharp corners in $\Omega(w)$ mean that the curvature of the boundary is not bounded from above. Therefore, a major part of our study of the Courant sharp Neumann eigenfunctions of such a chain domain will be focused on the behavior of Neumann eigenfunctions in thin necks and near the vertices of the domain.

Recently, De Ponti, Farinelli, and Violo [8] proved the Pleijel limit for Dirichlet and Neumann eigenfunctions of $RCD(K, N)$ spaces, which are metric measure spaces with a synthetic notion of Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in \mathbb{N}$. As a corollary, they showed that the Pleijel limit holds for the Neumann eigenfunctions of N -dimensional uniform domains, which is a class of subsets of $\mathbb{R}^N$ which includes Lipschitz domains. They proved these results using different techniques, by directly showing that even nodal domains touching the boundary satisfy a Faber-Krahn inequality and avoiding using a reflection argument. For fixed widths w , the chain domains $\Omega(w)$ that we study are uniform domains. However, the constant C in the definition of a uniform domain cannot be chosen uniformly as the widths w tend to 0 (see Definition 3.1 in [8]), and so the results in [8] do not give uniform control as the neck widths shrink.

2 Chain Domains

As described in the introduction, we study the Neumann eigenvalues and eigenfunctions of a class of planar domains consisting of a number of piecewise smooth domains joined by thin necks. We now give the precise definition of the chain domains under consideration. See Fig. 2 for an example of such a domain.

We define the family of chain domains in three steps. First, we introduce the *skeleton* consisting of a finite number of piecewise smooth connected regions, joined by smooth curves. We then define a *base domain*, by using homotopies to replace the smooth curves by fixed necks. Finally, we restrict the domains of these homotopies to allow for a full family of necks, of arbitrarily small width.

Definition 2.1 (*Skeleton*) Let $M \in \mathbb{N}$, $\{K_{i,j}\}_{i,j=1}^M \subset \mathbb{N}$. Then, the skeleton domain $\mathcal{S}$ is given by the disjoint union

$$\mathcal{S} = \left(\bigcup_{\ell=1}^M \mathcal{D}_\ell \right) \cup \left(\bigcup_{1 \leq i \leq j \leq M} \bigcup_{k=1}^{K_{ij}} \Gamma_{ij,k} \right).$$

- For $1 \leq \ell \leq M$, each $\mathcal{D}_\ell$ is a bounded planar domain with smooth boundary except for a finite number of vertices. The domains $\mathcal{D}_\ell$ are pairwise disjoint.
- For $1 \leq i \leq j \leq M$ and $1 \leq k \leq K_{ij}$, $\Gamma_{ij,k}$ is a smooth *neck curve* joining $\mathcal{D}_i$ and $\mathcal{D}_j$, parameterized by arc-length,

$$\Gamma_{ij,k} : [0, L_{ij,k}] \rightarrow \mathbb{R}^2,$$

with $\Gamma_{ij,k}(0) \in \partial\mathcal{D}_i$ and $\Gamma_{ij,k}(L_{ij,k}) \in \partial\mathcal{D}_j$. The linking curves are also pairwise disjoint, and transverse of $\partial\mathcal{D}_i$, $\partial\mathcal{D}_j$ at their endpoints.

Next, we endow a skeleton $\mathcal{S}$ with a family of homotopies that encode how the curves $\{\Gamma_{ij,k}\}$ are thickened into necks.

Definition 2.2 (*Base domain*) Let $\mathcal{S}$ be a skeleton domain as in Definition 2.1. The base domain Ω associated to $\mathcal{S}$ is the disjoint union

$$\Omega = \left(\bigcup_{\ell=1}^M \mathcal{D}_\ell \right) \cup \left(\bigcup_{1 \leq i \leq j \leq M} \bigcup_{k=1}^{K_{ij}} \mathcal{G}_{ij,k}([0, L_{ij,k}] \times (-1, 1)) \right),$$

where for each i, j, k ,

$$\mathcal{G}_{ij,k} : [0, L_{ij,k}] \times [-1, 1] \rightarrow \mathbb{R}^2$$

is a smooth homotopy with the following properties:

- $\mathcal{G}_{ij,k}(s, 0) = \Gamma_{ij,k}(s)$ for all $s \in [0, L_{ij,k}]$;
- $\mathcal{G}_{ij,k}(0, t) \in \partial\mathcal{D}_i$ and $\mathcal{G}_{ij,k}(L_{ij,k}, t) \in \partial\mathcal{D}_j$, for all $t \in [-1, 1]$, and not coinciding with any vertex of $\mathcal{D}_i$ or $\mathcal{D}_j$;
- $\det[\partial_t \mathcal{G}_{ij,k}(s, t), \partial_s \mathcal{G}_{ij,k}(s, t)] \neq 0$ for all (s, t) , i.e. the Jacobian for this coordinate system is non-degenerate;
- the images of the $\mathcal{G}_{ij,k}$ are disjoint, and disjoint from the interiors of all the domains $\mathcal{D}_\ell$.

Note that since $\det[\partial_t \mathcal{G}_{ij,k}(s, t), \partial_s \mathcal{G}_{ij,k}(s, t)] \neq 0$ for all (s, t) , each curve $\mathcal{G}_{ij,k}(\cdot, t)$ meets $\partial\mathcal{D}_i$ and $\partial\mathcal{D}_j$ transversally. We are now ready to introduce the chain domains, in which we allow the widths of the necks to become arbitrarily small.

Definition 2.3 (*Chain domain*) Let Ω be a base domain as in Definition 2.2. We define a collection of widths $\mathbf{w}$ by

$$\mathbf{w} = \{I_{ij,k} : 1 \leq i, j, \leq M, 1 \leq k \leq K_{ij}\},$$

where each $I_{ij,k}$ is an open interval in $(-1, 1)$ containing 0. The chain domain $\Omega(\mathbf{w})$ is then given by the disjoint union

$$\Omega(\mathbf{w}) = \left(\bigcup_{\ell=1}^M \mathcal{D}_\ell \right) \cup \left(\bigcup_{1 \leq i \leq j \leq M} \bigcup_{k=1}^{K_{ij}} \mathcal{N}_{ij,k}(\mathbf{w}) \right).$$

Here $\mathcal{N}_{ij,k}(\mathbf{w})$ is a *neck* joining $\mathcal{D}_i$ and $\mathcal{D}_j$ given by

$$\mathcal{N}_{ij,k}(\mathbf{w}) = \mathcal{G}_{ij,k}([0, L_{ij,k}] \times I_{ij,k}).$$

A measure of the minimum width of the neck $\mathcal{N}_{ij,k}(\mathbf{w})$ is defined by

$$w_{ij,k} := \min_{s \in [0, L_{ij,k}]} |\mathcal{G}_{ij,k}(s, t_2) - \mathcal{G}_{ij,k}(s, t_1)| \quad (7)$$

for $I_{ij,k} = (t_1, t_2)$.

The above definition includes the case $M = 1$, $K_{11} = 0$, where $\Omega(\mathbf{w}) = \mathcal{D}_1$ is a bounded domain, with piecewise smooth boundary except for a finite number of vertices, and with no $\mathbf{w}$ dependence. We define the interval $I_{ij,k}$ to contain 0 to ensure that the chain domain $\Omega(\mathbf{w})$ always contains the skeleton $\mathcal{S}$.

2.1 Geometric Constants

Let Ω be a base domain as in Definition 2.2. We define

$$A^* := \sum_{\ell} \text{Area}(\mathcal{D}_{\ell})$$

and

$$L^* := \sum_{\ell} \text{Length}(\partial \mathcal{D}_{\ell}) + 2 \sum_{i,j,k} \max_{t \in [-1, 1]} \text{Length}(\mathcal{G}_{ij,k}([0, L_{ij,k}] \times \{t\})).$$

We will bound the Courant sharp Neumann eigenvalues of Ω in terms of the five geometric constants, ρ^* , κ^* , δ^* , τ^* , w^* defined as follows.

- *Isoperimetric ratio constant*, ρ^* : We define $\rho^* > 0$ to be the isoperimetric ratio,

$$\rho^* = \frac{(L^*)^2}{A^*}.$$

- *Normalized curvature constant*, κ^* : We define $\kappa^* > 0$ so that the curvature of each smooth segment of $\partial \mathcal{D}_{\ell}$ and the slices $\mathcal{G}_{ij,k}([0, L_{ij,k}] \times \{t\})$ are bounded above by κ^*/L^* for all $t \in [-1, 1]$ and for all choices of ℓ, i, j, k .
- *Vertex control constant*, δ^* : The constant $\delta^* > 0$ is defined so that the following holds. For all ℓ and each vertex $p \in \partial \mathcal{D}_{\ell}$, we have that the connected component of $B(p, L^*\delta^*) \cap \partial \Omega$ containing p consists of two smooth curves joined at p , and after possibly rotating, we may assume that their tangent lines at p agree with the lines $\theta = \frac{\pi}{2} \pm \frac{\theta_0}{2}$ for some $0 < \theta_0 < \pi$. These are graphs with respect to the x_1 -axis, contained within the lines $\theta = \frac{\pi}{2} \pm \frac{\theta_0}{4}$ and $\theta = \frac{\pi}{2} \pm \frac{3\theta_0}{4}$, and with slope bounded by $1/\delta^*$. Moreover, the same properties hold at the points where the sides of $\partial \mathcal{D}_i$, $\partial \mathcal{D}_j$ meet a curve $\mathcal{G}_{ij,k}(\cdot, t)$, for any $t \in [-1, 1]$, and all choices of i, j, k .

- *Normalized cut-distance constant, τ^** : The constant τ^* is defined so that, for $\eta \leq L^* \delta^*$ and all ℓ , the cut-distance, given by

$$\inf_{u \in K_{\eta, \ell}} \sup_{\delta > 0} \left\{ \delta : \text{dist} \left(\gamma(u) + sn(u), \partial \mathcal{D}_\ell \right) = s \text{ for all } s \in [0, \delta] \right\},$$

is bounded from below by $\tau^* \eta$. Here, $\gamma : K_{\eta, \ell} \rightarrow \mathbb{R}^2$ is a parameterization of $\partial \mathcal{D}_\ell$ with the parts of $\partial \mathcal{D}_\ell$ in the discs of radius η centered at each vertex of $\partial \mathcal{D}_\ell$ excluded, for a union of intervals $K_{\eta, \ell}$, and $n(u)$ is the inward unit normal at $\gamma(u) \in \partial \mathcal{D}_\ell$.

- *Neck-width constant, w^** : The constant w^* is defined to provide control on the widths and regularity of the necks so that for all choices of i, j, k ,

$$\begin{aligned} \min_s \frac{\min_{t \in [-1, 1]} |\partial_t \mathcal{G}_{ij, k}(s, t)|}{\max_{t \in [-1, 1]} |\partial_t \mathcal{G}_{ij, k}(s, t)|} &\geq w^*, \quad w^* \leq |\partial_s \mathcal{G}_{ij, k}(s, t)| \leq 1/w^*, \\ \min_{(s, t)} \frac{|\det[\partial_t \mathcal{G}_{ij, k}(s, t), \partial_s \mathcal{G}_{ij, k}(s, t)]|}{|\partial_t \mathcal{G}_{ij, k}(s, t)| |\partial_s \mathcal{G}_{ij, k}(s, t)|} &\geq w^*. \end{aligned} \quad (8)$$

Our main result, Theorem 1.1, produces a bound for a Courant sharp eigenvalue $\mu_m(\mathbf{w})$ of the form $|\Omega(\mathbf{w})| \mu_m(\mathbf{w}) < C$, where C only depends on $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$. Since these control quantities are invariant under dilations of the base domain Ω , the constant C in the theorem can also be chosen uniformly over dilations of the domain. More generally, we have the following remark.

Remark 2.1 (Uniform control over base domains Ω) The constant C in (4) in Theorem 1.1 can be taken to be the same for every chain domain $\tilde{\Omega}(\mathbf{w})$ whose associated base domain $\tilde{\Omega}$ has geometric constants $\tilde{\rho}^*, \tilde{\kappa}^*, \tilde{\delta}^*, \tilde{\tau}^*, \tilde{w}^*$ satisfying

$$\tilde{\rho}^* \leq \rho^*, \quad \tilde{\kappa}^* \leq \kappa^*, \quad \tilde{\delta}^* \geq \delta^*, \quad \tilde{\tau}^* \geq \tau^*, \quad \tilde{w}^* \geq w^*,$$

where $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$ are the geometric constants associated to the base domain Ω .

2.1.1 Uniformity Over $\Omega(\mathbf{w})$ as $\mathbf{w}$ Changes

Next, we explain how the geometric constants for a base domain Ω are uniform as one varies the width of the necks for the associated chain domains $\Omega(\mathbf{w})$.

Let Ω be a base domain with geometric constants $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$. Then, for any collection of widths $\mathbf{w} = \{I_{ij, k}\}$, we have $|\Omega(\mathbf{w})| > A^*$ and $L(\mathbf{w}) < L^*$, where

$$|\Omega(\mathbf{w})| := \text{Area}(\Omega(\mathbf{w})) \quad \text{and} \quad L(\mathbf{w}) := \text{Length}(\partial \Omega(\mathbf{w})).$$

Therefore, we observe the following control on the geometry of $\Omega(\mathbf{w})$:

- (1) *Isoperimetric ratio.* The isoperimetric ratio $L(\mathbf{w})^2/|\Omega(\mathbf{w})|$ of $\Omega(\mathbf{w})$ is bounded above by ρ^* .

- (2) *Normalized curvature.* The curvature of each smooth segment of $\partial\Omega(\mathbf{w})$ is bounded above by $\kappa^*/L(\mathbf{w})$.
- (3) *Vertex control.* The vertex control constant δ^* provides the same control on the boundary of $\Omega(\mathbf{w})$ as for $\partial\Omega$, with L^* replaced by $L(\mathbf{w})$ and $[-1, 1]$ replaced by $I_{ij,k}$. In particular, there is a lower bound on the interior and exterior angle of each vertex of $\partial\Omega(\mathbf{w})$ in terms of δ^* , and the number of vertices of $\partial\Omega(\mathbf{w})$ is bounded by $1/\delta^*$.
- (4) *Normalized cut-distance.* For $\eta \leq L(\mathbf{w})\delta^*$ and all ℓ , the cut-distance, given by

$$\inf_{u \in K_{\eta,\ell}} \sup_{\delta > 0} \left\{ \delta : \text{dist} \left(\gamma(u) + sn(u), \partial\mathcal{D}_\ell \right) = s \text{ for all } s \in [0, \delta] \right\},$$

is bounded from below by $\tau^*\eta$. As above, $\gamma : K_{\eta,\ell} \rightarrow \mathbb{R}^2$ is a parameterization of $\partial\mathcal{D}_\ell$ with the parts of $\partial\mathcal{D}_\ell$ in the discs of radius η centered at each vertex of $\partial\mathcal{D}_\ell$ excluded, for a union of intervals $K_{\eta,\ell}$, and $n(u)$ is the inward unit normal at $\gamma(u) \in \partial\mathcal{D}_\ell$.

- (5) *Neck-width.* The constant w^* provides control on the ratio between the maximum and minimum width of the neck $\mathcal{N}_{ij,k}(\mathbf{w})$ for all choices of i, j, k , in the sense that

$$\begin{aligned} \min_s \frac{\min_{t \in I_{ij,k}} |\partial_t \mathcal{G}_{ij,k}(s, t)|}{\max_{t \in I_{ij,k}} |\partial_t \mathcal{G}_{ij,k}(s, t)|} &\geq w^*, & w^* &\leq |\partial_s \mathcal{G}_{ij,k}(s, t)| \leq 1/w^*, \\ \min_{(s,t)} \frac{|\det[\partial_t \mathcal{G}_{ij,k}(s, t), \partial_s \mathcal{G}_{ij,k}(s, t)]|}{|\partial_t \mathcal{G}_{ij,k}(s, t)| |\partial_s \mathcal{G}_{ij,k}(s, t)|} &\geq w^*. \end{aligned} \quad (9)$$

Remark 2.2 (*Uniform control in $\mathbf{w}$*) Our main result, Theorem 1.1, produces a bound for a Courant sharp eigenvalue $\mu_m(\mathbf{w})$ of the form $|\Omega(\mathbf{w})|\mu_m(\mathbf{w}) < C$, where C only depends on $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$. The reason for this being possible is that the constant C depends only on the geometric features of $\Omega(\mathbf{w})$ described in points (1)–(5) above.

For the domains $\Omega(\mathbf{w})$ from Fig. 1, with $0 < w < 1$, we can take

$$\rho^* = 50, \quad \kappa^* = 0, \quad \delta^* = \frac{1}{40}, \quad \tau^* = 1, \quad w^* = 1.$$

Remark 2.3 (*Notation*) From now on, all constants C^*, C_1^* , etc., appearing may depend on $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$, but will be independent of the collection of widths $\mathbf{w} = \{I_{ij,k}\}$ and any other geometric quantities involving $\Omega(\mathbf{w})$. Constants C, C_1 without asterisks will be absolute constants, independent of $\Omega(\mathbf{w})$.

3 Strategy and Ingredients for the Proof of Theorem 1.1

In Sect. 3.1 below, we describe an outline for the proof of Theorem 1.1. The proof relies on introducing a partition of the chain domain into several regions and on classifying the nodal domains of an eigenfunction in terms of which regions they touch. The

partition is introduced in Sect. 3.2 and the classification of the domains is presented in Sect. 3.3

3.1 Outline of the Proof of Theorem 1.1

Let $\Omega(\mathbf{w})$ be a chain domain as in Definition 2.3, and let $\{u_k(\mathbf{w})\}_k$ and $\{\mu_k(\mathbf{w})\}_k$ be the Neumann eigenfunctions and eigenvalues for $\Omega(\mathbf{w})$ introduced in (1). In order to prove Theorem 1.1, we will use the same underlying strategy of proof as the various Pleijel-type results described in the introduction: we will establish upper and lower bounds on the Neumann counting function

$$\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu) := \#\{k : \mu_k(\mathbf{w}) < \mu\}, \quad (10)$$

when $\mu = \mu_m(\mathbf{w})$ is a Courant sharp eigenvalue; that is, $v(u_m(\mathbf{w})) = m$, where we continue to write $v(u_m(\mathbf{w}))$ for the number of nodal domains of $u_m(\mathbf{w})$.

In this Courant sharp case, $\mu_k(\mathbf{w}) \neq \mu_m(\mathbf{w})$ for $k < m$, and so $\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu_m(\mathbf{w})) = m - 1$. We therefore have

$$\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu_m(\mathbf{w})) + 1 = m = v(u_m(\mathbf{w})). \quad (11)$$

We find an upper bound on $v(u_m(\mathbf{w}))$ of the form $C_1^* \mu_m(\mathbf{w}) |\Omega(\mathbf{w})| + O((\mu_m(\mathbf{w}) |\Omega(\mathbf{w})|)^{3/4})$ from deriving lower bounds on the area of a nodal domain of $u_m(\mathbf{w})$ (See Sect. 3.1.1). At the same time, a Weyl law with an explicit bound on the remainder then gives a lower bound on $\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu_m(\mathbf{w}))$ of the form $C_2^* \mu_m(\mathbf{w}) |\Omega(\mathbf{w})| + O((\mu_m(\mathbf{w}) |\Omega(\mathbf{w})|)^{3/4})$ (See Sect. 3.1.2). The constants C_1^*, C_2^* are explicit enough that we can argue $C_2^* > C_1^*$ and hence derive from (11) that

$$\text{Area}(\Omega(\mathbf{w})) \mu_m(\mathbf{w}) < C^*$$

as claimed. The detailed proof of Theorem 1.1 is done in Sect. 7. The upper bound on $v(u_m(\mathbf{w}))$ will hold for any (not necessarily Courant sharp) Neumann eigenfunction, and will imply the limit in (3).

3.1.1 Strategy for Obtaining the Asymptotic Upper Bound on $v(u_m(\mathbf{w}))$

To obtain lower bounds on the area of a nodal domain, we will adapt the strategy from [9] and [18] by splitting each nodal domain D of u_m into four different categories, depending on where the $L^2(D)$ -mass of u_m is concentrated. Roughly speaking, we will estimate the number of nodal domains of u_m in each of the following four cases:

- i) the majority of the mass is concentrated away from the necks $\mathcal{N}_{ij,k}(\mathbf{w})$ and the boundaries of the domains $\mathcal{D}_\ell$;
- ii) some of the mass is concentrated away from the necks $\mathcal{N}_{ij,k}(\mathbf{w})$ and near the smooth parts of the boundary of the domains $\mathcal{D}_\ell$;
- iii) some of the mass is concentrated near a vertex of $\mathcal{D}_\ell$ or the ends of the necks $\mathcal{N}_{ij,k}(\mathbf{w})$;

iv) some of the mass is concentrated in the necks $\mathcal{N}_{ij,k}(\mathbf{w})$.

Using the same techniques as in [9] and [18], cases i) and ii) can be handled using the Faber-Krahn Theorem together with a reflection argument across the smooth part of the boundary of $\Omega(\mathbf{w})$. We will recall this argument and define the above partitioning of the nodal domains in Sects. 4 and 3.2 respectively.

The novelty of our work lies in the remaining cases. For case iii), in Sect. 5 we will exploit properties of a Neumann eigenfunction near a corner in order to bound the number of such nodal domains near each vertex in $\Omega(\mathbf{w})$. Finally for case iv), in Sect. 6 we will prove a non-sharp version of the Faber-Krahn Theorem for thin cylinders in order to obtain a lower bound on the area of nodal domains contained in the neck. In particular, for sufficiently large eigenvalues $\mu_m(\mathbf{w})$, the number of nodal domains in cases ii), iii), and iv) will be small compared to case i).

3.1.2 Strategy for Obtaining the Lower Bound on $\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu_m(\mathbf{w}))$

For the lower bound on (11), we will use a Weyl law with an explicit bound on the remainder. This comes from a Weyl remainder estimate given in [26] involving $M_{\Omega(\mathbf{w})}(\delta)$, as defined in (6). We then bound $M_{\Omega(\mathbf{w})}(\delta)$ in terms of δ and the five geometric constants $\rho^*, \kappa^*, \delta^*, \tau^*, w^*$. This estimate will be given in Sect. 7.1.

3.2 Partition of the Chain Domains

Let $\Omega(\mathbf{w})$ be a chain domain as in Definition 2.3. As outlined in Sect. 3.1.1, we estimate the number of nodal domains by splitting our study into a series of cases. This is achieved by partitioning the domain. In this section, we define the partition, establish its required properties, and use it to define bulk, boundary, corner, and neck nodal domains.

Given $\delta > 0$, we partition $\Omega(\mathbf{w})$ into $\bigcup_{j=0}^4 \Omega_j^\delta(\mathbf{w})$. Roughly speaking,

- $\Omega_0^\delta(\mathbf{w})$ is the part of $\Omega(\mathbf{w})$ a distance δ away from the boundary;
- $\Omega_1^\delta(\mathbf{w})$ is a δ -neighborhood of the smooth part of $\partial\Omega(\mathbf{w})$;
- $\Omega_2^\delta(\mathbf{w})$ is a δ -neighborhood of the vertices of $\mathcal{D}_\ell$.
- $\Omega_3^\delta(\mathbf{w})$ is the part of each neck domain that is distance δ from the ends of the necks (when $w_{ij,k}$ is sufficiently small relative to δ).
- $\Omega_4^\delta(\mathbf{w})$ contains a δ -neighborhood of the ends of each neck domain (when $w_{ij,k}$ is sufficiently small relative to δ).

In what follows, we continue to write $w_{ij,k}$ for the minimum neck widths introduced in (7).

When δ is small compared to the minimum width $w_{ij,k}$ of the neck $\mathcal{N}_{ij,k}(\mathbf{w})$, then $\Omega_2^\delta(\mathbf{w})$ also contains a δ -neighborhood of the vertices where the neck $\mathcal{N}_{ij,k}(\mathbf{w})$ is joined to the domains $\mathcal{D}_i, \mathcal{D}_j$. However, when δ is large compared to the minimum width $w_{ij,k}$ of the neck $\mathcal{N}_{ij,k}(\mathbf{w})$, then $\Omega_4^\delta(\mathbf{w})$ contains a δ -neighborhood of these vertices, and $\Omega_3^\delta(\mathbf{w})$ then contains the rest of the neck $\mathcal{N}_{ij,k}(\mathbf{w})$.

Throughout, we will work with $\delta > 0$ satisfying

$$\delta \leq \min \left\{ \frac{1}{20} L(\mathbf{w}) \delta^*, \frac{L(\mathbf{w})}{\kappa^* \tau^*} \right\}, \quad (12)$$

with $\delta^*, \kappa^*, \tau^*$ the vertex control, normalized curvature, and normalized cut-distance constants of $\Omega(\mathbf{w})$. Recall that $L(\mathbf{w})\delta^*$ gives a lower bound on the distance between vertices of $\cup_\ell D_\ell$, and this bound on δ guarantees that we can cleanly separate a δ neighborhood of each vertex p in $\partial\Omega(\mathbf{w})$ from the rest of its connected component in $B(p, L^*\delta^*) \cap \partial\Omega(\mathbf{w})$. The upper bound $L(\mathbf{w})/(\kappa^*\tau^*)$ on δ will also ensure that, after excluding a disc of radius δ centered at each vertex of $\mathcal{D}_\ell$, the cut-distance of the remaining part of $\partial\mathcal{D}_\ell$ is bounded from below by $\tau^*\delta$. This will be important because it will allow us to apply a diffeomorphism to straighten this resulting part of the boundary.

We proceed to give the precise definition of the partition.

Definition 3.1 (δ -partition of a chain domain) Let $\Omega(\mathbf{w})$ be a chain domain as in Definition 2.3 and $\delta > 0$ satisfy (12). Then, the δ -partition for $\Omega(\mathbf{w})$ is defined as

$$\Omega(\mathbf{w}) = \bigcup_{j=0}^4 \Omega_j^\delta(\mathbf{w}),$$

where the following holds:

- 1) $\Omega_2^\delta(\mathbf{w})$ contains the intersection of $\Omega(\mathbf{w})$ with a disc of radius δ around each vertex of $\mathcal{D}_\ell$ for all $1 \leq \ell \leq M$.
- 2) For each $1 \leq i \leq j \leq M$, $1 \leq k \leq K_{ij}$ one of the following holds.
 - i) If $w_{ij,k} > 4\delta$, then
 - $\Omega_2^\delta(\mathbf{w})$ contains the intersection of $\Omega(\mathbf{w})$ with a disc of radius δ around each vertex formed by the neck $\mathcal{N}_{ij,k}(\mathbf{w})$ and the domains $\mathcal{D}_i, \mathcal{D}_j$;
 - $\Omega_1^\delta(\mathbf{w})$ contains the part of $\mathcal{N}_{ij,k}(\mathbf{w})$ near its boundary, given by

$$\left\{ x \in \mathcal{N}_{ij,k}(\mathbf{w}) : \text{dist}(x, \partial\mathcal{N}_{ij,k}(\mathbf{w})) < \frac{3}{4}\tau^*\delta \right\} \setminus \Omega_2^\delta(\mathbf{w}).$$
 - ii) If $w_{ij,k} \leq 4\delta$, then
 - $\Omega_4^\delta(\mathbf{w})$ contains the δ -neighborhood of the ends of the neck $\mathcal{G}_{ij,k}(\{0\} \times I_{ij,k})$ and $\mathcal{G}_{ij,k}(\{L_{ij,k}\} \times I_{ij,k})$;
 - $\Omega_3^\delta(\mathbf{w})$ then contains the rest of the neck, $\mathcal{N}_{ij,k}(\mathbf{w}) \setminus \Omega_4^\delta(\mathbf{w})$.
- 3) $\Omega_1^\delta(\mathbf{w})$ includes

$$\left\{ x \in \bigcup_{\ell=1}^M \mathcal{D}_\ell : \text{dist}(x, \partial\Omega(\mathbf{w})) < \frac{3}{4}\tau^*\delta \right\} \setminus (\Omega_2^\delta(\mathbf{w}) \cup \Omega_4^\delta(\mathbf{w})).$$

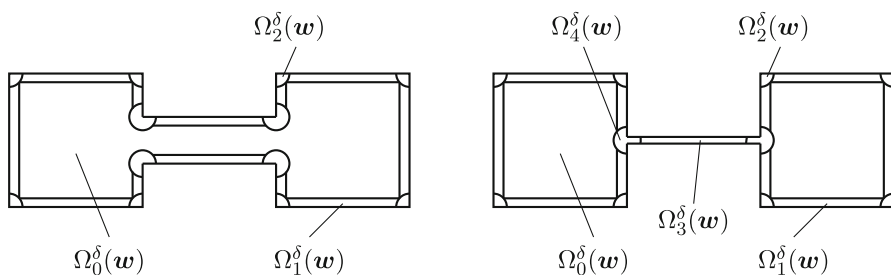


Fig. 3 The two δ -partitions of the chain domain $\Omega(w)$ introduced in Fig. 1 with $w_{ij,k} > 4\delta$ (left) and $w_{ij,k} \leq 4\delta$ (right)

4) $\Omega_0^\delta(w)$ is defined by

$$\Omega_0^\delta(w) = \Omega(w) \setminus (\Omega_1^\delta(w) \cup \Omega_2^\delta(w) \cup \Omega_3^\delta(w) \cup \Omega_4^\delta(w)).$$

See Fig. 3 for an example of the possible δ -partitions for the model domain $\Omega(w)$ introduced in Fig. 1, depending on the relative size of δ , and the neck width w .

We will later work with a partition of unity adapted to the δ -partition of a chain domain. Before introducing it, we present the reader with a lemma that allows us to straighten the boundary of the domain. This lemma will allow us to define the partition of unity and will also be used in Sect. 4.

Lemma 3.1 *Let $\Omega(w)$ be a chain domain. Fix $\eta > 0$ with $\eta \leq L(w) \min \left\{ \delta^*, \frac{1}{\kappa^* \tau^*} \right\}$. Given a side of $\mathcal{D}_\ell$, denote by b_η the part of this side a distance at least η from the vertices of $\mathcal{D}_\ell$. Letting $\{\gamma(s) : s \in I\}$ be a parameterization of b_η , and $n(s)$ be the unit inward normal to $\partial \mathcal{D}_\ell$ at $\gamma(s)$, a neighborhood of b_η in $\mathcal{D}_\ell$ can be straightened in the following sense.*

The function

$$F : I \times [0, \frac{3}{4}\tau^*\eta] \rightarrow \mathcal{D}_\ell, \quad (s, t) \mapsto (x, y) = \gamma(s) + tn(s),$$

is a diffeomorphism onto its image. Moreover, the Jacobian of this change of variables is bounded from above and below by

$$1 + \frac{3}{4}L(w)^{-1}\tau^*\kappa^*\eta \leq \frac{7}{4}, \quad 1 - \frac{3}{4}L(w)^{-1}\tau^*\kappa^*\eta \geq \frac{1}{4}$$

respectively.

Proof By the definition of the normalized cut-distance constant τ^* , since $\eta \leq L(w)\delta^*$, the cut-distance of b_η is bounded from below by $\tau^*\eta$. The proof of this lemma then follows in an identical way to the argument in Section 3 of [9], and so we omit the details. $\square$

Remark 3.1 For each neck $\mathcal{N}_{ij,k}(w)$, we define the diffeomorphism $F_{ij,k}$ by

$$F_{ij,k} : [0, L_{ij,k}] \times (-w_{ij,k}, w_{ij,k}) \rightarrow \mathcal{N}_{ij,k}(\mathbf{w})$$

$$F_{ij,k}(s, t) = \mathcal{G}_{ij,k}(s, t_1 + (t + w_{ij,k})|I_{ij,k}|/(2w_{ij,k})),$$

where we recall $t_1 < 0$ defines an endpoint of the interval $I_{ij,k}$. By the definition of $w_{ij,k}$ in (7) and the neck-width constant w^* , $|\partial_t \mathcal{G}_{ij,k}(s, t)|$ is bounded from above and below by $w_{ij,k}/|I_{ij,k}|$ multiplied by constants depending only on w^* . Therefore, the Jacobian of $F_{ij,k}$ is bounded in terms of the neck-width constant w^* , and so in particular this can be used to straighten the top and bottom boundaries of $\mathcal{N}_{ij,k}(\mathbf{w})$.

We proceed to introduce the partition of unity associated to the δ -partition of a chain domain.

Lemma 3.2 *Let $\Omega(\mathbf{w})$ be a chain domain. There exists a constant $C^* > 0$ such that the following holds. For each $\delta > 0$ satisfying (12) there exist smooth functions $\{\chi_j^\delta\}_{j=0}^4$ associated to the δ -partition $\{\Omega_j^\delta(\mathbf{w})\}_{j=0}^4$ of $\Omega(\mathbf{w})$ such that*

$$\sum_{j=0}^4 (\chi_j^\delta)^2 \equiv 1 \quad \text{on } \Omega(\mathbf{w}),$$

1. $\chi_j^\delta \equiv 1$ on $\Omega_j^\delta(\mathbf{w})$ for $j = 0, 3$,
2. $\chi_1^\delta \equiv 1$ on $\Omega_1^{\delta/2}(\mathbf{w})$,
3. $\chi_j^\delta \equiv 1$ on $\Omega_j^{\delta/4}(\mathbf{w})$ for $j = 2, 4$,
4. $\chi_j^\delta \in H^1(\Omega(\mathbf{w}))$ with $|\nabla \chi_j^\delta| \leq C^* \delta^{-1}$ a.e. on $\Omega(\mathbf{w})$ for $j = 0, \dots, 4$.

Proof This result follows from the definition of the δ -partition, and the straightening of the smooth parts of the boundary given in Lemma 3.1 above:

First, the cut-off function χ_2^δ is straightforward to define as a sum of radial functions, centered at the points used to define $\Omega_2^\delta(\mathbf{w})$, and on a length scale comparable to δ .

To define χ_1^δ , we use the upper bound on δ from (12). This allows us to apply Lemma 3.1 above with $\eta = \delta$. Using the diffeomorphism F from this lemma straightens each part of the side of $\mathcal{D}_\ell$ in $\Omega_1^\delta(\mathbf{w})$, via a change of variables with a bounded Jacobian. We can also use the diffeomorphism from Remark 3.1 to straighten the top and bottom boundaries of $\mathcal{N}_{ij,k}(\mathbf{w})$ in the case where $w_{ij,k} > 4\delta$. It is then straightforward to define the cut-off function χ_1^δ with the desired properties.

This straightening of the sides of $\mathcal{D}_\ell$ and the necks $\mathcal{N}_{ij,k}(\mathbf{w})$ also allows for the function χ_4^δ to be defined on a length scale comparable to δ , with the required properties. Note that since the neck and sides are straightened using two different diffeomorphisms, we can use them to define a continuous cut-off function χ_4^δ in $H^1(\Omega(\mathbf{w}))$, with a possibly discontinuous derivative at the intersection of their supports. This is the reason for the almost everywhere nature of the pointwise bound stated in (4) above.

The function χ_3^δ is then defined with support in the necks $\mathcal{N}_{ij,k}(\mathbf{w})$ with $w_{ij,k} \leq 4\delta$, and so that

$$(\chi_3^\delta)^2 + (\chi_4^\delta)^2 \equiv 1 \quad \text{on } \mathcal{N}_{ij,k}(\mathbf{w}).$$

Finally, χ_0^δ is defined so that $\sum_{j=0}^4 (\chi_j^\delta)^2 \equiv 1$ on $\Omega(\mathbf{w})$ as required. $\square$

3.3 Classification of the Nodal Domains

Let u be a Neumann eigenfunction of $\Omega(\mathbf{w})$, not necessarily Courant sharp, with eigenvalue μ , and let D be one of its nodal domains. Fix $\varepsilon, \delta > 0$. We will use ε to measure the extent to which the $L^2(D)$ -mass of u is concentrated away from the boundary. As in Definition 3.1, δ is used to partition the neighborhoods of different parts of the boundary of $\Omega(\mathbf{w})$. We will eventually choose δ in terms of the area of $\Omega(\mathbf{w})$ and the eigenvalue μ . We follow a similar framework to that in [18] and [9], but with more regions.

Let $\{\chi_j^\delta\}_{j=0}^4$ be the associated partition of unity to the δ -partition $\{\Omega_j^\delta(\mathbf{w})\}_{j=0}^4$ of $\Omega(\mathbf{w})$ (see Lemma 3.2). We decompose

$$u^2 = \sum_{j=0}^4 u_j^2, \quad u_j := \chi_j^\delta u.$$

So, u_0 is localized in the interior, u_1 is localized near the smooth part of the boundary, u_2 is localized near corners of $\mathcal{D}_\ell$, and when $w_{ij,k} \leq 4\delta$, u_3 is localized near necks $\mathcal{N}_{ij,k}(\mathbf{w})$, and u_4 is localized near where the necks are joined to the domains $\mathcal{D}_\ell$.

For $\varepsilon > 0$ and $\delta > 0$, the collection of bulk, boundary, corner, and neck nodal domains will be denoted by

$$\mathcal{V}_0^\delta(\varepsilon; u), \mathcal{V}_1^\delta(\varepsilon; u), \mathcal{V}_2^\delta(\varepsilon; u), \mathcal{V}_3^\delta(\varepsilon; u), \quad (13)$$

respectively, and defined in the following way:

- *bulk nodal domains*: $D \in \mathcal{V}_0^\delta(\varepsilon; u)$ if

$$\|u_0\|_{L^2(D)}^2 \geq (1 - \varepsilon) \|u\|_{L^2(D)}^2. \quad (14)$$

- *boundary nodal domains*: $D \in \mathcal{V}_1^\delta(\varepsilon; u)$ if

$$\|u_1\|_{L^2(D)}^2 \geq \frac{1}{4} \varepsilon \|u\|_{L^2(D)}^2. \quad (15)$$

- *corner nodal domains*: $D \in \mathcal{V}_2^\delta(\varepsilon; u)$ if

$$\|u_2\|_{L^2(D)}^2 \geq \frac{1}{4} \varepsilon \|u\|_{L^2(D)}^2 \quad \text{or} \quad \|u_4\|_{L^2(D \cap \cup_\ell \mathcal{D}_\ell)}^2 \geq \frac{1}{8} \varepsilon \|u\|_{L^2(D)}^2. \quad (16)$$

- *neck nodal domains*: $D \in \mathcal{V}_3^\delta(\varepsilon; u)$ if

$$\|u_3\|_{L^2(D)}^2 \geq \frac{1}{4} \varepsilon \|u\|_{L^2(D)}^2 \quad \text{or} \quad \|u_4\|_{L^2(D \cap \cup_{i,j,k} \mathcal{N}_{ij,k}(\mathbf{w}))}^2 \geq \frac{1}{8} \varepsilon \|u\|_{L^2(D)}^2. \quad (17)$$

For $j = 0, 1, 2, 3$ we write

$$v_j^\delta(\varepsilon; u) := \#\mathcal{V}_j^\delta(\varepsilon; u) \quad (18)$$

for the number of domains in each class.

Note that a given nodal domain may fall into more than one of these categories, but we always have that the total number of nodal domains for u is bounded by

$$v(u) \leq \sum_{j=0}^3 v_j^\delta(\varepsilon; u). \quad (19)$$

3.3.1 Green's Formula for Nodal Domains

A key ingredient in bounding the number of each type of nodal domain will be to obtain an upper bound on the first Dirichlet eigenvalue of a region of area comparable to a given nodal domain, and then apply the Faber-Krahn Theorem. In order to obtain this upper bound, we will need that the following Green's formula holds for each nodal domain.

Lemma 3.3 *Any nodal domain D of a Neumann eigenfunction u of $\Omega(\mathbf{w})$, with eigenvalue μ , satisfies*

$$\int_D |\nabla u|^2 = \mu \int_D u^2.$$

Proof To prove the lemma, we will use the following version of Green's identity, given in [10, Lemma 1.5.3.8]: Let Ω be a bounded, open set in $\mathbb{R}^2$, with boundary $\partial\Omega$ given by a $C^{1,1}$ curvilinear polygon. Then, for $v_1 \in H^2(\Omega)$, $v_2 \in H^1(\Omega)$,

$$\int_{\Omega} (\Delta v_1) v_2 = - \int_{\Omega} \nabla v_1 \cdot \nabla v_2 + \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} v_2. \quad (20)$$

In the case where the nodal domain D does not contain a corner of $\Omega(\mathbf{w})$, then the eigenfunction u is smooth in D . Moreover, since $\Omega(\mathbf{w})$ is planar, the boundary of D is piecewise C^1 and meets at equal angles in the interior and on the boundary. Therefore, we have sufficient regularity to apply (20) with $v_1 = v_2 = u$ and $\Omega = D$. Since $\Delta u = -\mu u$, and u satisfies Neumann boundary conditions on $\partial\Omega$, together with Dirichlet boundary conditions on the rest of ∂D , this gives the equality in the statement of the lemma.

To handle the case where ∂D contains a vertex of $\Omega(\mathbf{w})$, we need to use the regularity of the Neumann eigenfunction u at the vertex. Let θ be the interior angle of a given vertex. Then, by the theorem in Section 1 of [27], u satisfies these estimates in a neighborhood of the vertex:

- i) if $\theta < \pi$, then u is C^1 in a neighborhood of the vertex;
- ii) if $\theta > \pi$, then u is Hölder continuous, with exponent π/θ , in a neighborhood of the vertex, and $\limsup_{r \rightarrow 0} r^{1-\pi/\theta} |\nabla u| < \infty$, where r is the distance to the vertex.

In particular, since $\theta < 2\pi$, this ensures that $\nabla u \in L^4$.

We now define D_k to be the set formed by removing from D with discs of radius $\varepsilon_k > 0$ centered at the corners of $\Omega(\mathbf{w})$. We can choose the sequence ε_k , with $\lim_{k \rightarrow \infty} \varepsilon_k = 0$, so that the domains D_k are $C^{1,1}$ -curvilinear polygons. Moreover, u is smooth in D_k , and so applying (20), with $v_1 = v_2 = u$, we obtain

$$-\int_{D_k} \mu |u|^2 = -\int_{D_k} |\nabla u|^2 + \int_{\partial D_k} \frac{\partial u}{\partial \nu} u.$$

By the boundary conditions satisfied by u , the only contribution to the boundary integral is from arcs of the circles of radius ε_k centered at the corners of $\Omega(\mathbf{w})$. By the regularity of u and ∇u , these contributions go to 0 as ε_k tends to 0. Therefore, taking the limit $\varepsilon_k \rightarrow 0$, we obtain the equality in the statement of the lemma, and this completes the proof. $\square$

Remark 3.2 As we are working in two dimensions, the nodal domain D has a Lipschitz boundary, and meets the smooth part of the boundary of $\Omega(\mathbf{w})$ at non-zero angles. For smooth components of the boundary this is proven in Theorem 2.3 in [12], and at corners it is proven in Theorem 2.6 in [12]. We will also assume that $D \cap \{u_j \neq 0\}$ has these same properties for $0 \leq j \leq 4$, which can be achieved by replacing δ by a sequence $\delta_n \rightarrow \delta$ if necessary.

4 Estimates on Bulk and Boundary Nodal Domains

In this section, we bound the number of bulk and boundary nodal domains. See Sect. 3.3 for the classification and (18) for the notation.

4.1 Number of Bulk Nodal Domains

We recall that for δ, ε fixed and u being a Neumann eigenfunction for the chain domain, a bulk nodal domain, $D \in \mathcal{V}_0^\delta(\varepsilon, u)$, is one for which (14) holds. As introduced in (18), we continue to denote the corresponding number of bulk nodal domains by $v_0^\delta(\varepsilon; u)$. Our main result is the following.

Proposition 4.1 (Number of bulk nodal domains) *Let $\Omega(\mathbf{w})$ be a chain domain. There exists a constant $C_0^* > 0$ such that for all $0 < \varepsilon < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$ the following holds. If u is a Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue μ and $|\Omega(\mathbf{w})|\mu \geq (C_0^*)^{1/\beta}$, then*

$$v_0^\delta(\varepsilon; u) \leq \frac{1}{\pi \lambda_1(\mathbb{D})} \left(\frac{1 + \varepsilon}{1 - \varepsilon} (|\Omega(\mathbf{w})|\mu) + \frac{1 + \frac{1}{\varepsilon}}{1 - \varepsilon} C_0^* (|\Omega(\mathbf{w})|\mu)^{2\beta} \right) \quad (21)$$

for $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$.

Proof To bound the number of bulk nodal domains of u , we follow exactly the argument that was used to derive [9, equation (16)] and [18, equation (2.3)], using the properties of the δ -partition from Lemma 3.2, provided δ satisfies (12). This involves applying the Faber-Krahn Theorem to each bulk domain, where we use $u_0 = \chi_0^\delta u$ as a test function to obtain an upper bound on its first Dirichlet eigenvalue. This argument gives the estimate on the area of a bulk nodal domain, $D \in \mathcal{V}_0^\delta(\varepsilon, u)$, of the form

$$\pi \lambda_1(\mathbb{D}) |D|^{-1} \leq \frac{1+\varepsilon}{1-\varepsilon} \frac{\int_D |\nabla u|^2 dx}{\int_D |u|^2 dx} + \frac{(1+\frac{1}{\varepsilon})\tilde{C}_0^*}{(1-\varepsilon)\delta^2}. \quad (22)$$

for a constant $\tilde{C}_0^*$. Using Lemma 3.3, this gives an upper bound on the number of such bulk nodal domains of

$$v_0^\delta(\varepsilon; u) \leq \frac{1}{\pi \lambda_1(\mathbb{D})} \left(\frac{1+\varepsilon}{1-\varepsilon} \mu + \frac{1+\frac{1}{\varepsilon}}{1-\varepsilon} \tilde{C}_0^* \delta^{-2} \right) |\Omega(\mathbf{w})|. \quad (23)$$

The result would then hold once we show that the upper bound on δ from (12) holds. To see this, note that the isoperimetric inequality yields $|\Omega(\mathbf{w})|^{-1/2} L(\mathbf{w}) \geq 2\pi^{1/2}$. Thus, setting

$$c_0^* := 2\pi^{1/2} \min \left\{ \frac{\delta_*}{20}, \frac{1}{\kappa^* \tau^*} \right\},$$

we have

$$\delta |\Omega(\mathbf{w})|^{-1/2} = (|\Omega(\mathbf{w})|\mu)^{-\beta} \leq c_0^* \leq |\Omega(\mathbf{w})|^{-1/2} L(\mathbf{w}) \min \left\{ \frac{\delta_*}{20}, \frac{1}{\kappa^* \tau^*} \right\}, \quad (24)$$

as needed, provided $|\Omega(\mathbf{w})|\mu \geq (c_0^*)^{-1/\beta}$. $\square$

We will later choose $\varepsilon > 0$ small so that $\frac{1+\varepsilon}{1-\varepsilon}$ is sufficiently close to 1, and $\beta = 3/8$ so that the lower bound on $|\Omega(\mathbf{w})|\mu$ can be written as simply C_0^* .

The first term in the estimate in Proposition 4.1 will be the leading order term in the count of the number of nodal domains. It is therefore important that the second term is sub-linear in $|\Omega(\mathbf{w})|\mu$ for $0 < \beta < \frac{1}{2}$.

4.2 Number of Boundary Nodal Domains

We recall that for δ, ε fixed and u being a Neumann eigenfunction for the chain domain, a boundary nodal domain, $D \in \mathcal{V}_1^\delta(\varepsilon, u)$, is one for which (15) holds. As introduced in (18), we write $v_1^\delta(\varepsilon; u)$ for the number of boundary nodal domains of u .

In order to bound $v_1^\delta(\varepsilon; u)$, we use the argument leading to [9, Equation (18)] and [18, Equation (2.13)]. The first step in the proof is the following result.

Lemma 4.1 *Let $\Omega(\mathbf{w})$ be a chain domain. There exist constants c^*, C^* such that the following holds. Let $\delta > 0$ satisfy (12), $\varepsilon > 0$, and u be a Neumann eigenfunction of*

$\Omega(\mathbf{w})$ with eigenvalue μ . For each $D \in \mathcal{V}_1^\delta(\varepsilon, u)$ there are a set V_D , with a Lipschitz boundary, and a function $v \in H_0^1(V_D)$ with the following properties:

$$\text{Area}(V_D) \leq C^* \text{Area}(\{x \in D : \text{dist}(x, \partial\Omega(\mathbf{w})) < \tfrac{3}{4}\tau^*\delta\}), \quad (25)$$

and

$$\int_{V_D} v^2 \geq c^* \varepsilon, \quad \int_{V_D} |\nabla v|^2 \leq C^*(\mu + (\tau^*\delta)^{-2}).$$

Proof Lemma 4.1 follows from the straightening results of Lemma 3.1 (with $\eta = \delta$) and Remark 3.1, using an identical argument to the proof of the estimates (13), (15), and (17) in Section 5 of [9], and so we omit the details of the proof here. In particular, this proof relies upon the Lipschitz properties of $D \cap \{u_1 \neq 0\}$ from Remark 3.2. $\square$

When bounding $v_1^\delta(\varepsilon, u)$, we will use Lemma 4.1 along with the Faber-Krahn Theorem to obtain that $v_1^\delta(\varepsilon, u)$ is bounded by a multiple of $M_{\Omega(\mathbf{w})}(\frac{3}{4}\tau^*\delta)$, where $M_{\Omega(\mathbf{w})}(t)$ is as defined in (6). To deal with this new upper bound, we will use the following lemma.

Lemma 4.2 *Let $\Omega(\mathbf{w})$ be a chain domain. There exists $C^* > 0$ such that*

$$M_{\Omega(\mathbf{w})}(t) \leq C^* L(\mathbf{w})t,$$

for $t \leq \frac{3}{4}L(\mathbf{w}) \min\{\tau^*\delta^*, \frac{1}{\kappa^*}\}$.

Proof To bound $M_{\Omega(\mathbf{w})}(t)$, we will break the function into three parts:

- 1) the contribution from the part of the necks $\{\mathcal{N}_{ij,k}(\mathbf{w})\}$ that are a distance of at least t from the vertices of $\Omega(\mathbf{w})$,
- 2) the contribution from the parts of $\{\partial\mathcal{D}_\ell\}$ with an appropriate neighborhood of each vertex excluded,
- 3) the contribution from neighborhoods of the vertices of $\Omega(\mathbf{w})$.

We will denote the contribution to $M_{\Omega(\mathbf{w})}(t)$ from each of these parts of $\partial\Omega(\mathbf{w})$ by $M_{\Omega(\mathbf{w})}^{(j)}(t)$, so that

$$M_{\Omega(\mathbf{w})}(t) = \sum_{j=1}^3 M_{\Omega(\mathbf{w})}^{(j)}(t).$$

To handle 1), we use the diffeomorphisms $F_{ij,k}$ from Remark 3.1 to straighten the upper and lower boundaries of each neck $\mathcal{N}_{ij,k}(\mathbf{w})$. Then, there exists a constant C^* , depending only on the neck-width constant w^* such that

$$M_{\Omega(\mathbf{w})}^{(1)}(t) \leq C^* \sum_{1 \leq i \leq j \leq M} \sum_{k=1}^K \text{Length}(\partial\mathcal{N}_{ij,k}(\mathbf{w}))t \leq C^* L(\mathbf{w})t. \quad (26)$$

For 2), we exclude an η neighborhood of each vertex of $\mathcal{D}_\ell$, with η given by $\eta = \frac{4}{3}(\tau^*)^{-1}t$. The upper bound on t from the statement of the lemma then ensures that $\eta \leq L(\mathbf{w}) \min \left\{ \delta^*, \frac{1}{\kappa^* \tau^*} \right\}$. Therefore, we can apply Lemma 3.1 with this value of η to obtain a diffeomorphism that straightens this part of the boundary of $\mathcal{D}_\ell$. Since this change of variables has a Jacobian bounded by $\frac{7}{4}$, we obtain

$$M_{\Omega(\mathbf{w})}^{(2)}(t) \leq \frac{7}{4} \sum_{\ell=1}^M \text{Length}(\partial \mathcal{D}_\ell) t \leq \frac{7}{4} L(\mathbf{w}) t. \quad (27)$$

Finally, since, by the definition of the vertex control constant, $\Omega(\mathbf{w})$ has at most $1/\delta^*$ vertices, the remaining contribution to

$$\{x \in \Omega(\mathbf{w}) : \text{dist}(x, \partial \Omega(\mathbf{w})) < t\}$$

is contained within $1/\delta^*$ discs of radius 2η . Therefore,

$$M_{\Omega(\mathbf{w})}^{(3)}(t) \leq 4\pi(\delta^*)^{-1}\eta^2 = \frac{64}{9}\pi(\delta^*)^{-1}(\tau^*)^{-2}t^2. \quad (28)$$

Combining (26), (27), and (28), using the upper bound on t gives the desired estimate on $M_{\Omega(\mathbf{w})}(t)$. $\square$

Our main result in this section, which bounds the number of boundary nodal domains for an eigenfunction u , is the following.

Proposition 4.2 (Number of boundary nodal domains) *Let $\Omega(\mathbf{w})$ be a chain domain. There exists a constant $C_1^* > 0$ such that for all $0 < \varepsilon < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$ the following holds. If u is a Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue μ and $|\Omega(\mathbf{w})|\mu \geq (C_1^*)^{1/\beta}$, then*

$$v_1^\delta(\varepsilon; u) \leq C_1^* \varepsilon^{-1} (|\Omega(\mathbf{w})|\mu)^{1-\beta}$$

for $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$.

Proof Let $D \in \mathcal{V}_1^\delta(\varepsilon, u)$ and let $V_D \subset \mathbb{R}^2$ and $v \in H_0^1(V_D)$ be as in Lemma 4.1. Then,

$$\text{Area}(V_D) \leq C^* \text{Area}(\{x \in D : \text{dist}(x, \partial \Omega(\mathbf{w})) < \frac{3}{4}\tau^*\delta\}). \quad (29)$$

On the other hand, writing $\lambda_1(\mathbb{D}_{V_D})$ for the first Dirichlet eigenvalue of the disc of area equal to that of V_D , the Faber-Krahn Theorem yields

$$\begin{aligned} \pi \lambda_1(\mathbb{D}) |V_D|^{-1} &= \lambda_1(\mathbb{D}_{V_D}) \\ &\leq \lambda_1(V_D) \leq \frac{\int_{V_D} |\nabla v|^2}{\int_{V_D} v^2} \leq C^* \varepsilon^{-1} \left(\mu + (\tau^*\delta)^{-2} \right). \end{aligned} \quad (30)$$

We conclude from (29) that there is C^* such that, for every $D \in \mathcal{V}_1^\delta(\varepsilon, u)$,

$$\varepsilon \left(\mu + (\tau^* \delta)^{-2} \right)^{-1} \leq C^* \text{Area}(\{x \in D : \text{dist}(x, \partial\Omega(\mathbf{w})) < \tfrac{3}{4} \tau^* \delta\}).$$

We then conclude

$$v_1^\delta(\varepsilon; u) \leq C^* \varepsilon^{-1} \left(\mu + (\tau^* \delta)^{-2} \right) M_{\Omega(\mathbf{w})}(\tfrac{3}{4} \tau^* \delta). \quad (31)$$

As explained in (24), $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$ satisfies (12) for $(|\Omega(\mathbf{w})| \mu)^{1/\beta}$ sufficiently large, and so we can apply Lemma 4.2 with $t = \tfrac{3}{4} \tau^* \delta$ to obtain

$$\begin{aligned} M_{\Omega(\mathbf{w})}(\tfrac{3}{4} \tau^* \delta) &\leq \tfrac{3}{4} C^* \tau^* L(\mathbf{w}) \delta = \tfrac{3}{4} C^* \tau^* L(\mathbf{w}) |\Omega(\mathbf{w})|^{-1/2} |\Omega(\mathbf{w})| (|\Omega(\mathbf{w})| \mu)^{-\beta} \\ &\leq \tfrac{3}{4} C^* \tau^* (\rho^*)^{1/2} |\Omega(\mathbf{w})| (|\Omega(\mathbf{w})| \mu)^{-\beta}. \end{aligned}$$

Here ρ^* is the isoperimetric ratio constant. In addition,

$$\mu + (\tau^* \delta)^{-2} = |\Omega(\mathbf{w})|^{-1} \left(|\Omega(\mathbf{w})| \mu + (\tau^*)^{-2} (|\Omega(\mathbf{w})| \mu)^{2\beta} \right),$$

and so the result follows from (31). $\square$

5 Estimates on Corner Nodal Domains

In this section, we will obtain an upper bound on the number of corner nodal domains. Let $\Omega(\mathbf{w})$ be a chain domain, $\delta, \varepsilon > 0$, and u be a Neumann eigenfunction for $\Omega(\mathbf{w})$. We recall that a corner nodal domain, $D \in \mathcal{V}_2^\delta(\varepsilon, u)$, is one for which (16) holds. As introduced in (18), we write $v_2^\delta(\varepsilon; u)$ for the number of corner nodal domains of u .

We write

$$S_{\theta_0}(\delta) = \{(r, \theta) \in \Omega(\mathbf{w}) : r < \delta, |\theta| < \theta_0/2\},$$

where we use polar coordinates. The first step in controlling $v_2^\delta(\varepsilon; u)$ is the following analogue of Lemma 4.1.

Lemma 5.1 *Let $\Omega(\mathbf{w})$ be a chain domain. There exist constants c^*, C^* such that the following holds. Let $\delta > 0$ satisfy (12), $\varepsilon > 0$, and u be a Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue μ . Let $D \in \mathcal{V}_2^\delta(\varepsilon, u)$ such that*

$$\|u_2\|_{L^2(D)}^2 \geq \tfrac{1}{4} \varepsilon \|u\|_{L^2(D)}^2.$$

Then, there are a vertex v_0 of $\Omega(\mathbf{w})$ with interior angle θ_0 , a set $V \subset S_{\theta_0}(\delta)$ with a Lipschitz boundary and a function $v \in H^1(V)$, with $v \equiv 0$ on $\partial V \cap S_{\theta_0}(\delta)$, and the following properties:

$$\text{Area}(V) \leq C^* \text{Area}(\{x \in D : \chi_2^\delta u \neq 0, \text{dist}(x, v_0) < \delta\}), \quad (32)$$

and

$$c^* \varepsilon \leq \int_V v^2 \leq C^*, \quad \int_V |\nabla v|^2 \leq C^* (\mu + \delta^{-2}). \quad (33)$$

Proof By the definition of the vertex control constant δ^* , the number of vertices of $\Omega(\mathbf{w})$ is bounded by $1/\delta^*$. Therefore, since $\|u_2\|_{L^2(D)}^2 \geq \frac{1}{4}\varepsilon\|u\|_{L^2(D)}^2$, there exists a vertex v_0 , opening angle θ_0 , such that

$$\int_{D \cap \{\text{dist}(x, v_0) < \delta\}} u_2^2 \geq \frac{1}{4}\varepsilon\delta^* \int_D u^2.$$

We now apply a transformation to u_2 to translate and straighten the sides of $\Omega(\mathbf{w})$ meeting at v_0 . This will give us a function v defined on $S_{\theta_0}(\delta)$ which is smooth away from the origin, and vanishes on the portion of the boundary of its support in the interior of the sector (i.e., $v \equiv 0$ on $\partial V \cap S_{\theta_0}(\delta)$). The transformation is constructed as follows:

By the upper bound on δ from (12), and using the definition of the vertex control constant δ^* , we can rotate and translate so that, without loss of generality, the vertex v_0 is at the origin, and in a disc of radius 20δ centered at the vertex, this part of the boundary of $\partial\Omega(\mathbf{w})$ can be written as $y = f_{\pm}(x)$ for $\pm x > 0$. Moreover, there exists a constant θ^* , depending only on δ^* , such that the opening angle θ_0 of each vertex satisfies $\theta^* < \theta_0 < 2\pi - \theta^*$. Therefore, the rotation can be chosen to ensure that $\lim_{x \rightarrow 0^{\pm}} f_{\pm}(x)/x = a_{\pm}$, with

$$c^* \leq |a_{\pm}| \leq C^*, \quad c^* \leq |f_{\pm}(x)/x| \leq C^*, \quad |f'_{\pm}(x)| \leq C^*,$$

for constants c^* and C^* only depending on δ^* .

Next, there exists a subset S of a rotation of $S_{\theta_0}(\delta)$ such that the function

$$F : S \rightarrow \Omega(\mathbf{w}), \quad (s, t) \mapsto \left(s, \frac{f_{\pm}(s)}{a_{\pm}s} t \right),$$

is a Lipschitz homeomorphism onto its image (and a diffeomorphism away from $s = 0$), with the image containing the part of $\Omega(\mathbf{w})$ in the disc of radius δ centered at v_0 .

By Lemmas 3.2 and 3.3, the function u_2 satisfies

$$\int_D |\nabla u_2|^2 = \int_D |\nabla (\chi_2^{\delta} u)|^2 \leq C^* (\mu + \delta^{-2}).$$

Therefore, we set $v = u_2 \circ F$, and $V = F^{-1}(D \cap \{\text{dist}(x, v_0) < \delta, u_2 \neq 0\})$, which by Remark 3.2 we may take to be Lipschitz. To complete the proof of the lemma, it is sufficient to show that the Jacobian of this change of variables is bounded from above and below, and that the off-diagonal entries are bounded. This Jacobian is given by the determinant of

$$\begin{pmatrix} 1 & t \frac{s f'_{\pm}(s) - f_{\pm}(s)}{a_{\pm} s^2} \\ 0 & \frac{f_{\pm}(s)}{a_{\pm} s} \end{pmatrix},$$

and so by the properties of $f_{\pm}(s)/s$, we just need to bound the off-diagonal term. For some $|\zeta|, |\xi| \in (0, s)$ we have $f_{\pm}(s) = a_{\pm}s + \frac{1}{2}s^2 f''_{\pm}(\zeta)$, and $f'_{\pm}(s) = a_{\pm} + s f''_{\pm}(\xi)$. Therefore, it is sufficient to show that

$$|t f''_{\pm}(\zeta)| + |t f''_{\pm}(\xi)| \leq C^* \quad (34)$$

for a constant C^* . Since $f'_{\pm}(x)$ is bounded, and the curvature of the two sides of $\Omega(\mathbf{w})$ is bounded from above by $\kappa^*/L(\mathbf{w})$, we obtain

$$|f''_{\pm}(x)| \leq C^* \kappa^*/L(\mathbf{w}).$$

Combining this with the upper bound on δ from (12), and since $|t| \leq \delta$, it implies (34) and completes the proof of the lemma. $\square$

We are now ready to state and prove the main bound on the number of corner nodal domains.

Proposition 5.1 (Number of corner nodal domains) *Let $\Omega(\mathbf{w})$ be a chain domain. There exists a constant $C_2^* > 0$ such that for all $0 < \varepsilon < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$ the following holds. If u is a Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue μ and $|\Omega(\mathbf{w})|\mu \geq (C_2^*)^{1/\beta}$, then*

$$v_2^{\delta}(\varepsilon, u) \leq C_2^* \varepsilon^{-4} (|\Omega(\mathbf{w})|\mu)^{3-6\beta},$$

for $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$.

Proof Throughout, we assume that $(|\Omega(\mathbf{w})|\mu)^{\beta}$ is sufficiently large so that δ satisfies (12), and in particular Lemmas 4.1 and 5.1 apply. The proof of the bound is divided into two cases, depending on whether we are counting nodal domains $D \in \mathcal{V}_2^{\delta}(\varepsilon, u)$ for which either

$$(A) \quad \|u_2\|_{L^2(D)}^2 \geq \frac{1}{4}\varepsilon \|u\|_{L^2(D)}^2 \quad \text{or} \quad (B) \quad \|u_4\|_{L^2(D \cap \mathcal{U}_\ell \mathcal{D}_\ell)}^2 \geq \frac{1}{8}\varepsilon \|u\|_{L^2(D)}^2.$$

In Case 1 below, we prove that

$$\#\{D \in \mathcal{V}_2^{\delta}(\varepsilon, u) : (A) \text{ holds}\} \leq C_2^* \varepsilon^{-4} (|\Omega(\mathbf{w})|\mu)^{3-6\beta}, \quad (35)$$

and we explain how to do the same with (A) replaced by (B) in Case 2 (see (44)). The bound on $v_2^{\delta}(\varepsilon, u)$ then follows from these two estimates.

Case 1 : $D \in \mathcal{V}_2^\delta(\varepsilon, u)$ is such that (A) holds.

First, we claim that there exist constants c^* , C_2^* , and a vertex v_0 , such that, if $|\Omega(\mathbf{w})|\mu \geq (C_2^*)^{1/\beta}$, then

$$\text{Area}(D \cap B_\delta(v_0)) \geq c^* \varepsilon^4 |\Omega(\mathbf{w})| (|\Omega(\mathbf{w})|\mu)^{4\beta-3}. \quad (36)$$

Here $B_\delta(v_0)$ is the disc of radius δ centered at the vertex v_0 . Since $B_\delta(v_0)$ has area $\pi\delta^{-2} = \pi|\Omega(\mathbf{w})|^{1-2\beta}\mu^{-2\beta}$, the bound in (36) yields (35) as claimed.

We next proceed to prove the claim in (36). Let v, V be as in Lemma 5.1. By the lower bound in (33), there exists x^* , with $|x^*| < \delta$ such that

$$\int_{y_1}^{y_2} v(x^*, y)^2 dy \geq \frac{1}{2} c^* \varepsilon \delta^{-1}, \quad (37)$$

where $y_1 = \inf J(x^*)$, $y_2 = \sup J(x^*)$, with $J(x^*) = \{y \in \mathbb{R} : (x^*, y) \in V\}$. We now extend v identically by zero outside of V . Note that this does not give a function in $H^1(\mathbb{R}^2)$, because v does not satisfy Dirichlet boundary conditions on the part of ∂V coinciding with the sector $\theta = \pm \frac{1}{2}\theta_0$. However, denoting this extension of v by w , for each x^* such that (x^*, y) is in $\bar{V}$, we do have

$$w(\cdot, y) \in H^1([x^*, x^* + \delta]), \quad (38)$$

with

$$\int_{y_1}^{y_2} \left(\int_{x^*}^{x^* + \delta} |\nabla w|^2 dx \right) dy \leq C^*(\mu + \delta^{-2}). \quad (39)$$

This is because for all such y , the interval $(x^*, x^* + \delta)$ is contained in the interior of the infinite sector $|\theta| \leq \frac{1}{2}\theta_0$. Therefore, since v vanishes on the portion of ∂V in the interior of the infinite sector, its extension by 0 satisfies (38), $w(\cdot, y)$ is continuous on $(x^*, x^* + \delta)$, and (39) follows from the second estimate in (33). In particular, $\partial_t w(t, y)$ is integrable for $t \in (x^*, x^* + \delta)$, and this means that by the fundamental theorem of calculus

$$w(x, y) = w(x^*, y) + \int_{x^*}^x \partial_t w(t, y) dt$$

for $x \in (x^*, x^* + \delta)$. We can write

$$\left| \int_{x^*}^x \partial_t w(t, y) dt \right| \leq |x - x^*|^{1/2} \left(\int_{x^*}^x |\partial_t w(t, y)|^2 dt \right)^{1/2},$$

and so

$$\begin{aligned}
w(x, y)^2 &= w(x^*, y)^2 + 2w(x^*, y) \int_{x^*}^x \partial_t w(t, y) dt + \left(\int_{x^*}^x \partial_t w(t, y) dt \right)^2 \\
&\geq \frac{1}{2} w(x^*, y)^2 - \left(\int_{x^*}^x \partial_t w(t, y) dt \right)^2 \\
&\geq \frac{1}{2} w(x^*, y)^2 - |x - x^*| \int_{x^*}^x |\partial_t w(t, y)|^2 dt.
\end{aligned}$$

Integrating in y and using (37) this implies that

$$\int_{y_1}^{y_2} w(x, y)^2 dy \geq \frac{1}{4} \varepsilon c^* \delta^{-1} - |x - x^*| \int_{y_1}^{y_2} \left(\int_{x^*}^x |\partial_t w(t, y)|^2 dt \right) dy. \quad (40)$$

Using the estimate from (39) in (40) then gives

$$\int_{y_1}^{y_2} w(x, y)^2 dy \geq \frac{1}{4} \varepsilon c^* \delta^{-1} - C^* |x - x^*| (\mu + \delta^{-2}).$$

So,

$$\int_{y_1}^{y_2} v(x, y)^2 dy = \int_{y_1}^{y_2} w(x, y)^2 dy \geq \frac{1}{8} c^* \varepsilon \delta^{-1} \quad (41)$$

for all $x \geq x^*$ with $|x - x^*| \leq \frac{1}{8} c^* \varepsilon \delta^{-1} (C^*)^{-1} (\mu + \delta^{-2})^{-1}$. For simplicity, we define the new parameter

$$\tilde{\delta} = c^* (C^*)^{-1} \delta^{-1} (\mu + \delta^{-2})^{-1}.$$

Note that we have set $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$, and so $\tilde{\delta} \sim \delta^{-1} \mu^{-1} = |\Omega(\mathbf{w})|^{1/2} (|\Omega(\mathbf{w})| \mu)^{\beta-1}$ for large $|\Omega(\mathbf{w})| \mu$ and $0 < \beta < \frac{1}{2}$. In particular, $\tilde{\delta} < \delta$ for large $|\Omega(\mathbf{w})| \mu$.

Now let $\chi = \chi(x, y)$ be a smooth cut-off function, equal to 1 on the set

$$\left[x^* + \frac{1}{16} \varepsilon \tilde{\delta}, x^* + \frac{1}{16} \varepsilon \tilde{\delta} + c_0^* \varepsilon \tilde{\delta} \right] \times [y_1, y_2]$$

and vanishing outside a $c_0^* \varepsilon \tilde{\delta}$ neighborhood of this set, with first derivatives bounded by $C_0^* \varepsilon^{-1} \tilde{\delta}^{-1}$. Here $c_0^* > 0$ and C_0^* are chosen, depending only on the interior angle θ_0 at the vertex (which is bounded from below by a constant which only depends on δ^*), so that the support of χ is contained within the interior of the sector. Using (41), this will ensure that

$$\int_V \chi^2 w^2 \geq c^* \varepsilon^2 \tilde{\delta}^{-1}$$

for a constant $c^* > 0$. Here, and from now on, the constants c^* , C^* may change from line-to-line (but will depend only on the five geometric constants of $\Omega(\mathbf{w})$). Let W_χ be the support of χ . Then, the definition of χ and (33) also ensures that

$$\int_{V \cap W_\chi} |\nabla(\chi w)|^2 \leq C^*(\mu + \delta^{-2} + \tilde{\delta}^{-2} \varepsilon^{-2}).$$

Since W_χ is contained in the interior of the sector, χw vanishes on the boundary of $V \cap W_\chi$. This implies that the first Dirichlet eigenvalue of $V \cap W_\chi$ is bounded by a multiple of

$$\frac{\int_{V \cap W_\chi} |\nabla(\chi w)|^2}{\int_{V \cap W_\chi} \chi^2 w^2} \leq C^* \varepsilon^{-2} \tilde{\delta}^{-1} \delta (\mu + \delta^{-2} + \tilde{\delta}^{-2} \varepsilon^{-2}). \quad (42)$$

Since $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$, for $0 < \beta < \frac{1}{2}$, and for $(|\Omega(\mathbf{w})|\mu)^\beta \geq C^*$ sufficiently large, the right hand side of (42) can be bounded from above by

$$C^* \varepsilon^{-4} \delta^4 \mu^3 = C^* \varepsilon^{-4} |\Omega(\mathbf{w})|^{-1} (|\Omega(\mathbf{w})|\mu)^{3-4\beta}. \quad (43)$$

Therefore, using the Faber-Krahn Theorem as in (30),

$$\text{Area}(V \cap W_\chi) \geq c^* \varepsilon^4 |\Omega(\mathbf{w})| (|\Omega(\mathbf{w})|\mu)^{4\beta-3},$$

completing the proof of the claim in (36).

Case 2 : $D \in \mathcal{V}_2^\delta(\varepsilon, u)$ is such that (B) holds.

In this case, some of the mass of u is contained in the intersection of the support of χ_4 with a particular domain $\mathcal{D}_\ell$ (near where $\mathcal{D}_\ell$ and a neck $\mathcal{N}_{ij,k}(\mathbf{w})$ are joined). Then, we straighten the part of $\partial \mathcal{D}_\ell$ using Lemma 4.1 to again obtain, after a rotation, a Lipschitz set V contained in the half-plane $\{x > 0\}$, and a function $v \in H^1(V)$ with

$$\text{Area}(V) \leq C^* \text{Area}(\{x \in D \cap \mathcal{D}_\ell : \chi_4^\delta u \neq 0\}),$$

and

$$c^* \varepsilon \leq \int_V v^2 \leq C^*, \quad \int_V |\nabla v|^2 \leq C^* (\mu + \delta^{-2}).$$

Moreover, v can be taken to vanish on $\partial V \cap \{x > 0\}$. This then allows us to replicate the proof of Case 1) and show that

$$\#\{D \in \mathcal{V}_2^\delta(\varepsilon, u) : (B) \text{ holds}\} \leq C_2^* \varepsilon^{-4} (|\Omega(\mathbf{w})|\mu)^{3-6\beta}, \quad (44)$$

as needed to finish the proof of the proposition. $\square$

6 Estimates on Neck Nodal Domains

In this section, we will obtain an upper bound on the number of neck nodal domains. Let $\Omega(\mathbf{w})$ be a chain domain, $\delta, \varepsilon > 0$, and u be a Neumann eigenfunction for $\Omega(\mathbf{w})$.

We recall that a neck nodal domain, $D \in \mathcal{V}_3^\delta(\varepsilon, u)$, is one for which (17) holds. As introduced in (18), we write $\nu_3^\delta(\varepsilon; u)$ for the number of neck nodal domains of u .

This section is divided into three parts. Given $D \in \mathcal{V}_3^\delta(\varepsilon, u)$, in Sect. 6.1 we explain how to find a neck $\mathcal{N}$ for which there is a lower bound on the area of $\mathcal{N} \cap D$. This lower bound will be given in the form of $\text{Area}(V)$ where V is a subset of a flat cylinder or a strip into which the neck has been straightened. Then, in Sect. 6.2 we explain how to find a lower bound on $\text{Area}(V)$, when V is a subset of a cylinder, in terms of its first Dirichlet eigenvalue. Finally, in Sect. 6.3 we state and prove the bound on $\nu_3^\delta(\varepsilon; u)$.

6.1 Straightening Lemmas

Given a nodal domain $D \in \mathcal{V}_3^\delta(\varepsilon, u)$, we know that one of the following two inequalities hold:

$$(\tilde{A}) \quad \|u_3\|_{L^2(D)}^2 \geq \frac{1}{4}\varepsilon\|u\|_{L^2(D)}^2 \quad \text{or} \quad (\tilde{B}) \quad \|u_4\|_{L^2(D \cup \mathcal{N}_{ij,k}(\mathbf{w}))}^2 \geq \frac{1}{8}\varepsilon\|u\|_{L^2(D)}^2 \quad (45)$$

In the following lemmas, we explain how to find a neck $\mathcal{N}$ for which there is a lower bound on the area of $\mathcal{N} \cap D$ in both the $(\tilde{A})$ and $(\tilde{B})$ cases.

Lemma 6.1 *Let $\Omega(\mathbf{w})$ be a chain domain. There exist constants c^*, C^* such that the following holds. Let $\delta > 0$ satisfy (12), $\varepsilon > 0$, and u be a Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue μ . Let $D \in \mathcal{V}_3^\delta(\varepsilon, u)$ such that $(\tilde{A})$ holds. Then, there exist a neck $\mathcal{N}_{ij,k}(\mathbf{w})$, a set V with a Lipschitz boundary that is a subset of a flat cylinder of circumference $4w_{ij,k}$, and a function $v \in H_0^1(V)$ with the following properties:*

$$\text{Area}(V) \leq C^* \text{Area}(\{x \in D \cap \mathcal{N}_{ij,k}(\mathbf{w}) : \chi_3^\delta u \neq 0\}) \quad (46)$$

and

$$c^*\varepsilon \leq \int_V v^2 \leq C^*, \quad \int_V |\nabla v|^2 \leq C^*(\mu + \delta^{-2}). \quad (47)$$

Proof As $1/\delta^*$ is an upper bound on the number of vertices of $\Omega(\mathbf{w})$, and each neck contributes four vertices, $\Omega(\mathbf{w})$ has at most $1/(4\delta^*)$ necks. Since $\|u_3\|_{L^2(D)}^2 \geq \frac{1}{4}\varepsilon\|u\|_{L^2(D)}^2$, there therefore exists a neck $\mathcal{N}_{ij,k}(\mathbf{w})$ such that

$$\int_{D \cap \mathcal{N}_{ij,k}(\mathbf{w})} u_3^2 \geq \varepsilon\delta^* \int_D u^2.$$

We note that the support of the neck cut-off function χ_3^δ only intersects a neck $\mathcal{N}_{ij,k}(\mathbf{w})$ when $w_{ij,k} \leq 4\delta$ (see Definition 3.1). Recall that $w_{ij,k}$ is the minimum width of the neck $\mathcal{N}_{ij,k}(\mathbf{w})$.

As we commented in Remark 3.1, by the definition of the neck $\mathcal{N}_{ij,k}(\mathbf{w})$, there exists a diffeomorphism $F_{ij,k}$,

$$F_{ij,k} : [0, L_{ij,k}] \times (-w_{ij,k}, w_{ij,k}) \rightarrow \mathcal{N}_{ij,k}(\mathbf{w})$$

$$F_{ij,k}(s, t) = \mathcal{G}_{ij,k}(s, t_1 + (t + w_{ij,k})|I_{ij,k}|/(2w_{ij,k})).$$

The Jacobian of $F_{ij,k}$ is bounded from above and below by a constant depending only on the neck-width constant w^* . Then, we define the function $\tilde{u} := u_3 \circ F_{ij,k}$ and set $\tilde{V}$ given by

$$F_{ij,k}^{-1}(D \cap \{u_3 \neq 0\}) \subset [0, L_{ij,k}] \times (-w_{ij,k}, w_{ij,k}).$$

Note that $\tilde{V}$ has Lipschitz boundary and meets the lines $t = \pm w_{ij,k}$ at non-zero angles, by Remark 3.2. We now reflect $\tilde{u}$ and $\tilde{V}$ across the line $t = w_{ij,k}$, and glue across the line $t = -w_{ij,k}$. This gives a subset V of a cylinder of circumference $4w_{ij,k}$, with Lipschitz boundary, and a function $v \in H_0^1(V)$, with the required properties. $\square$

Lemma 6.2 *Let $\Omega(\mathbf{w})$ be a chain domain. There exist constants c^* , C^* such that the following holds. Let $\delta > 0$ satisfy (12), $\varepsilon > 0$, and u be a Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue μ . Let $D \in \mathcal{V}_3^\delta(\varepsilon, u)$ such that $(\tilde{B})$ holds. Then, there exist a neck $\mathcal{N}_{ij,k}(\mathbf{w})$, a set V with a Lipschitz boundary that is a subset of a strip of width $2w_{ij,k}$ and length $C^*\delta$, and a function $v \in H^1(V)$ with the following properties:*

$$\text{Area}(V) \leq C^* \text{Area}(\{x \in D \cap \mathcal{N}_{ij,k}(\mathbf{w}) : \chi_4^\delta u \neq 0\})$$

and

$$c^* \varepsilon \leq \int_V v^2 \leq C^*, \quad \int_V |\nabla v|^2 \leq C^*(\mu + \delta^{-2}). \quad (48)$$

Proof This lemma follows using the same idea as for the proof of Lemma 6.1: There exists a neck $\mathcal{N}_{ij,k}(\mathbf{w})$ such that the lower bound

$$\int_{D \cap \mathcal{N}_{ij,k}(\mathbf{w})} u_4^2 dx \geq \frac{1}{2} \varepsilon \delta^* \int_D u^2 dx$$

holds. We can use the same diffeomorphism $F_{ij,k}$ to transform this neck to an exact strip of width $2w_{ij,k}$, with a bounded Jacobian. Since the cut-off function χ_4^δ is supported in a δ -neighborhood of the ends of the neck $\mathcal{G}_{ij,k}(\{0\} \times I_{ij,k})$ and $\mathcal{G}_{ij,k}(\{L_{ij,k}\} \times I_{ij,k})$, the function $v = u_4 \circ F_{ij,k}$ and set $V = F_{ij,k}^{-1}(D \cap \mathcal{N}_{ij,k}(\mathbf{w}) \cap \{u_4 \neq 0\})$ satisfy the estimates given in the statement of the lemma. In particular, as we commented in Remark 3.2, this ensures that the set V has Lipschitz boundary. $\square$

6.2 Controlling the Area of a Subset of a Cylinder

Given $D \in \mathcal{V}_3^\delta(\varepsilon, u)$ satisfying $(\tilde{A})$, Lemma 6.1 gives the existence of a neck $\mathcal{N}_{ij,k}(\mathbf{w})$ so that the area of $\mathcal{N}_{ij,k}(\mathbf{w}) \cap D$ can be bounded below by the area $\text{Area}(V)$ where V is a subset of a flat cylinder. In this section we explain how to obtain a lower bound on $\text{Area}(V)$ in terms of the first Dirichlet eigenvalue for V .

Lemma 6.3 *Let $\mathcal{C}_p$ be the flat infinite cylinder with circumference $P > 0$, and let V be a Lipschitz set on $\mathcal{C}_p$ with first Dirichlet eigenvalue equal to λ . Then,*

$$\text{Area}(V) \geq \min\{\pi\lambda_1(\mathbb{D})\lambda^{-1}, P\lambda_1(\mathbb{D})^{1/2}\lambda^{-1/2}\}.$$

Proof To prove the lemma, we use the classical proof of the Faber-Krahn Theorem in the case of constant sectional curvature $\kappa = 0$ (see [5]). However, instead of using the isoperimetric inequality in $\mathbb{R}^2$, we use the following inequality on the flat cylinder [13, Theorem 6]: Let W be a region on the cylinder $\mathcal{C}_p$, enclosing area A and of perimeter L . Then,

$$1) \quad L \geq 2\sqrt{\pi A} \quad \text{if} \quad A \leq \frac{1}{\pi}P^2, \quad 2) \quad L \geq 2P \quad \text{if} \quad A \geq \frac{1}{\pi}P^2, \quad (49)$$

with equality only when W is an isometric embedding of a round disc on the cylinder in the first case, and the region between two cross-sections of $\mathcal{C}_p$ in the second case.

Let $v \in H_0^1(V)$ be a non-negative first Dirichlet eigenfunction of V , with eigenvalue λ . We then build a comparison function with circular level sets as follows: Let $\mathbb{D}(t) \subset \mathbb{R}^2$ be the disc of radius $r(t)$, such that $\pi r(t)^2 = |\mathbb{D}(t)| = \text{Area}(\{v > t\})$. Note that $r : [0, t_0] \rightarrow [0, r(0)]$, with $t_0 = \max v$ and $r(0) = \sqrt{\text{Area}(V)/\pi}$, is continuous and strictly decreasing.

Next, let $\Psi = r^{-1}$, and define $F : \mathbb{D}(0) \rightarrow \mathbb{R}$ by $F(p) = \Psi(|p|)$. Note that by the co-area formula

$$\int_{\{v=t\}} \frac{1}{|\nabla v|} d\sigma_t = -\frac{d}{ds}(\text{Area}(\{v > s\}))\bigg|_{s=t} = -2\pi r(t)r'(t), \quad (50)$$

where $d\sigma_t$ is the measure on the level set $\{v = t\} \subset V$. Therefore, since $t = \Psi(r(t))$,

$$\begin{aligned} \int_D v^2 dv_g &= \int_0^{t_0} t^2 \int_{\{v=t\}} \frac{1}{|\nabla v|} d\sigma_t dt \\ &= -2\pi \int_0^{t_0} (\Psi(r(t)))^2 r(t)r'(t) dt = \int_{\mathbb{D}(0)} |F|^2 dv. \end{aligned} \quad (51)$$

Here we have used dv_g and dv to denote the area measure on V and $\mathbb{R}^2$ respectively. Next, notice that by Cauchy-Schwartz $\left(\int_{\{v=t\}} \frac{1}{|\nabla v|} d\sigma_t\right) \left(\int_{\{v=t\}} |\nabla v| d\sigma_t\right) \geq \text{Length}(\{v = t\})^2$. Thus, by (50) we have

$$\int_{\{v=t\}} |\nabla v| d\sigma_t \geq -\frac{\text{Length}(\{v = t\})^2}{2\pi r(t)r'(t)}.$$

Using again the co-area formula

$$\int_D |\nabla v|^2 dv_g = \int_0^{t_0} \int_{\{v=t\}} |\nabla v| d\sigma_t dt \geq -\int_0^{t_0} \frac{\text{Length}(\{v = t\})^2}{2\pi r(t)r'(t)} dt. \quad (52)$$

We now split into two cases depending on the relative size of $\text{Area}(\{v > t\})$ and the circumference P of the cylinder $\mathcal{C}_P$.

Case 1. Suppose that the area of V satisfies

$$\text{Area}(V) \leq \frac{1}{\pi} P^2.$$

Then, since $\text{Area}(\{v > t\}) \leq \text{Area}(V)$, case 1) in (49) implies that

$$\text{Length}(\{v = t\}) \geq 2\sqrt{\pi A(t)} = 2\pi r(t).$$

Thus, using that $\Psi'(r(t))r'(t) = 1$, the bound in (52) yields

$$\begin{aligned} \int_D |\nabla v|^2 \, dv_g &\geq - \int_0^{t_0} \frac{2\pi r(t)}{r'(t)} dt \\ &= - \int_0^{t_0} 2\pi r(t) (\Psi'(r(t)))^2 r'(t) dt = \int_{\mathbb{D}(0)} |\nabla F|^2 \, dv. \end{aligned} \quad (53)$$

From (51) and (53) we have

$$\lambda \geq \lambda_1(\mathbb{D}(0)) = \frac{\pi}{\text{Area}(V)} \lambda_1(\mathbb{D}), \quad (54)$$

which can be rearranged to give $\text{Area}(V) \geq \pi \lambda_1(\mathbb{D}) \lambda^{-1}$.

Case 2. Suppose now that the area of V instead satisfies

$$\text{Area}(V) \geq \frac{1}{\pi} P^2.$$

Set $A(t) = \text{Area}(\{v > t\})$. Then, the isoperimetric inequality in (49) implies that

$$\begin{aligned} \text{Length}(\{v = t\}) &\geq \min \left\{ 2P, 2\sqrt{\pi A(t)} \right\} = \min \left\{ \frac{P}{\pi r(t)}, 1 \right\} 2\pi r(t) \\ &\geq \frac{P}{\sqrt{\pi \text{Area}(V)}} 2\pi r(t). \end{aligned}$$

To get the last inequality we used that $\frac{P}{\pi r(t)} \geq \frac{P}{\pi r(0)} = \frac{P}{\sqrt{\pi \text{Area}(V)}}$, together with $\frac{P}{\sqrt{\pi \text{Area}(V)}} \leq 1$. Thus, using that $\Psi'(r(t))r'(t) = 1$, the bound in (52) yields

$$\begin{aligned} \int_D |\nabla v|^2 \, dv_g &\geq - \frac{P^2}{\pi \text{Area}(V)} \int_0^{t_0} \frac{2\pi r(t)}{r'(t)} dt \\ &= - \frac{P^2}{\pi \text{Area}(V)} \int_0^{t_0} 2\pi r(t) (\Psi'(r(t)))^2 r'(t) dt \\ &= \frac{P^2}{\pi \text{Area}(V)} \int_{\mathbb{D}(0)} |\nabla F|^2 \, dv. \end{aligned} \quad (55)$$

From (51) and (55) we have

$$\lambda \geq \frac{P^2}{\pi \text{Area}(V)} \lambda_1(\mathbb{D}(0)) = \frac{P^2}{\text{Area}(V)^2} \lambda_1(\mathbb{D}). \quad (56)$$

This can be rearranged to give $\text{Area}(V) \geq P \lambda_1(\mathbb{D})^{1/2} \lambda^{-1/2}$. Since either (54) or (56) must hold, this completes the proof of the proposition. $\square$

Remark 6.1 We do not expect the second lower bound in Lemma 6.3 to be sharp: The first lower bound of $\text{Area}(V) \geq \pi \lambda_1(\mathbb{D}) \lambda^{-1}$ is the same as the lower bound from the Faber-Krahn Theorem for the disc. For the section S_A on the cylinder $\mathcal{C}_P$ of area A , the first Dirichlet eigenfunction is $\sin\left(\frac{P\pi x}{A}\right)$, with eigenvalue $\lambda = P^2 \pi^2 A^{-2}$. Therefore, in this case, we have the equality $\text{Area}(S_A) = P \pi \lambda^{-1/2}$.

Motivated by this, we conjecture that

$$\text{Area}(V) \geq \min\{\pi \lambda_1(\mathbb{D}) \lambda^{-1}, P \pi \lambda^{-1/2}\},$$

with the minimizer given by a disc on $\mathcal{C}_P$ if $\lambda \geq \lambda_1(\mathbb{D})^2 P^{-2}$, and a section of $\mathcal{C}_P$ if $\lambda \leq \lambda_1(\mathbb{D})^2 P^{-2}$. The second lower bound on $\text{Area}(V)$ from the proposition is off from this conjectured sharpest lower bound by a factor of

$$\pi \lambda_1(\mathbb{D})^{-1/2} = \pi j_{0,1}^{-1} \approx 1.306.$$

As this factor is independent of λ and P , Lemma 6.3 is sufficient for our nodal domain count estimate.

6.3 Number of Neck Nodal Domains

We are now ready to state and prove our main result for this section.

Proposition 6.1 (Number of neck nodal domains) *Let $\Omega(\mathbf{w})$ be a chain domain. There exists a constant $C_3^* > 0$ such that for all $0 < \varepsilon < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$ the following holds. If u is a Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue μ and $|\Omega(\mathbf{w})|_\mu \geq (C_3^*)^{1/\beta}$, then*

$$v_3^\delta(\varepsilon; u) \leq C_3^* \left[\varepsilon^{-1} (|\Omega(\mathbf{w})|_\mu)^{1-\beta} + \varepsilon^{-4} (|\Omega(\mathbf{w})|_\mu)^{3-6\beta} \right],$$

for $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$.

Proof Throughout, we assume that $(|\Omega(\mathbf{w})|_\mu)^\beta$ is sufficiently large so that δ satisfies (12), and in particular Lemmas 6.1 and 6.2 apply. The collection of neck nodal domains $\mathcal{V}_3^\delta(\varepsilon, u)$ is split into those who satisfy either $(\tilde{A})$ or $(\tilde{B})$ in (45). We proceed to prove that the upper bound we claim on $v_3^\delta(\varepsilon; u)$ holds in each case.

Case 1 : $D \in \mathcal{V}_3^\delta(\varepsilon, u)$ is such that $(\tilde{A})$ holds.

In this case, by Lemma 6.1, there exist a neck $\mathcal{N}(\mathbf{w}) := \mathcal{N}_{ij,k}(\mathbf{w})$, of minimum width $w := w_{ij,k}$ and a Lipschitz set V that is a subset of a flat cylinder of circumference $4w$ such that

$$\text{Area}(V) \leq C^* \text{Area}(\{x \in D \cap \mathcal{N}(\mathbf{w}) : \chi_3^\delta u \neq 0\}).$$

Here, to simplify notation, we have dropped the ij, k subscripts. Therefore, Lemma 6.3 yields

$$\min\{\pi\lambda_1(\mathbb{D})\lambda^{-1}, P\lambda_1(\mathbb{D})^{1/2}\lambda^{-1/2}\} \leq C^* \text{Area}(\{x \in D \cap \mathcal{N}(\mathbf{w}) : \chi_3^\delta u \neq 0\}),$$

where λ is the first Dirichlet eigenvalue for V . By (47), we have

$$\lambda \leq C^* \varepsilon^{-1}(\mu + \delta^{-2}) =: C^* \gamma,$$

and so

$$\text{Area}(\{x \in D \cap \mathcal{N}(\mathbf{w}) : \chi_3^\delta u \neq 0\}) \geq (C^*)^{-1} \min\{\pi\lambda_1(\mathbb{D})\gamma^{-1}, 4w\lambda_1(\mathbb{D})^{1/2}\gamma^{-1/2}\}$$

for a constant C^* that may increase from line-to-line. The area of the neck $\mathcal{N}(\mathbf{w})$ is bounded by a constant depending only on the neck-width constant w^* multiplied by $wL(\mathbf{w})$. Therefore,

$$\#\{D \in \mathcal{V}_3^\delta(\varepsilon, u) : (\tilde{A}) \text{ holds}\} \leq C^* wL(\mathbf{w}) \max\{\gamma, w^{-1}\gamma^{1/2}\}. \quad (57)$$

For $\delta = |\Omega(\mathbf{w})|^{1/2-\beta}\mu^{-\beta}$, we have

$$\gamma = \varepsilon^{-1}|\Omega(\mathbf{w})|^{-1} \left(|\Omega(\mathbf{w})|\mu + (|\Omega(\mathbf{w})|\mu)^{2\beta} \right).$$

By Definition 3.1, as the support of χ_3^δ intersects $\mathcal{N}(\mathbf{w})$, we must have $w \leq 4\delta$. Therefore, since the isoperimetric ratio $L(\mathbf{w})^2/|\Omega(\mathbf{w})|$ is bounded from above by ρ^* , we have

$$\begin{aligned} wL(\mathbf{w})\gamma &\leq 4\delta L(\mathbf{w})\gamma = 4L(\mathbf{w})|\Omega(\mathbf{w})|^{-1/2} \\ &\quad (|\Omega(\mathbf{w})|\mu)^{-\beta} \varepsilon^{-1} \left(|\Omega(\mathbf{w})|\mu + (|\Omega(\mathbf{w})|\mu)^{2\beta} \right) \\ &\leq 4(\rho^*)^{1/2} \varepsilon^{-1} \left((|\Omega(\mathbf{w})|\mu)^{1-\beta} + (|\Omega(\mathbf{w})|\mu)^\beta \right) \end{aligned}$$

and

$$\begin{aligned} wL(\mathbf{w})w^{-1}\gamma^{1/2} &= L(\mathbf{w})|\Omega(\mathbf{w})|^{-1/2}\varepsilon^{-1/2} \left(|\Omega(\mathbf{w})|\mu + (|\Omega(\mathbf{w})|\mu)^{2\beta} \right)^{1/2} \\ &\leq (\rho^*)^{1/2} \varepsilon^{-1/2} \left(|\Omega(\mathbf{w})|\mu + (|\Omega(\mathbf{w})|\mu)^{2\beta} \right)^{1/2}. \end{aligned}$$

Using these bounds in (57), together with $0 < \beta < \frac{1}{2}$, implies that

$$\#\{D \in \mathcal{V}_3^\delta(\varepsilon, u) : (\tilde{A}) \text{ holds}\} \leq C^* \varepsilon^{-1} \left((|\Omega(\mathbf{w})|\mu)^{1-\beta} + (|\Omega(\mathbf{w})|\mu)^{1/2} \right). \quad (58)$$

Since $\frac{1}{2} < 1 - \beta$ for $\beta < \frac{1}{2}$, the quantity in (58) therefore satisfies the estimate in the statement of the proposition.

Case2 : $D \in \mathcal{V}_3^\delta(\varepsilon, u)$ is such that $(\tilde{B})$ holds.

In this case, by Lemma 6.2 there exist a neck $\mathcal{N}(\mathbf{w}) := \mathcal{N}_{ij,k}(\mathbf{w})$, of minimum width $w := w_{ij,k}$, a Lipschitz set V that is a subset of a strip of width $2w$ and length $C^*\delta$, and a function $v \in H^1(V)$ such that

$$\text{Area}(V) \leq C^* \text{Area}(\{x \in D \cap \mathcal{N}(\mathbf{w}) : \chi_4^\delta u \neq 0\})$$

and $\int_V v^2 \geq c^* \varepsilon$. In particular, we can find x^* such that

$$\int_{V \cap \{x=x^*\}} v(x^*, y)^2 dy \geq c^* (C^*)^{-1} \varepsilon \delta^{-1}. \quad (59)$$

We now integrate to the left or right of the strip, as in the proof of Proposition 5.1 from (37) to (43), using (48) and (59) in place of (33) and (37), this time applying a cut-off function in the x -variable. This provides a Lipschitz set W which is a subset of the infinite strip $\mathbb{R} \times [-w, w]$, and a function $\tilde{v} \in H^1(W)$ such that $\tilde{v}$ vanishes on the part of ∂W in the interior of the strip. Moreover, $\text{Area}(W) \leq \text{Area}(V)$, and, for $(|\Omega(\mathbf{w})|\mu)^\beta$ sufficiently large, the function $\tilde{v}$ satisfies

$$\frac{\int_W |\nabla \tilde{v}|^2}{\int_W \tilde{v}^2} \leq C^* \varepsilon^{-4} \delta^4 \mu^3,$$

for a constant C^* that may increase from line-to-line. By reflecting across the line $y = w$, we therefore get a set $\tilde{W}$ on a cylinder of circumference $4w$, with $\text{Area}(\tilde{W}) = 2 \text{Area}(W)$, and so that its first Dirichlet eigenvalue, $\lambda = \lambda_1(\tilde{W})$, satisfies

$$\lambda \leq C^* \varepsilon^{-4} \delta^4 \mu^3 =: C^* \gamma.$$

As

$$\text{Area}(\tilde{W}) \leq C^* \text{Area}(\{x \in D \cap \mathcal{N}(\mathbf{w}) : \chi_4^\delta u \neq 0\}),$$

by Lemma 6.3 we obtain

$$\text{Area}(\{x \in D \cap \mathcal{N}(\mathbf{w}) : \chi_4^\delta u \neq 0\}) \geq (C^*)^{-1} \min\{\pi \lambda_1(\mathbb{D}) \gamma^{-1}, 4w \lambda_1(\mathbb{D})^{1/2} \gamma^{-1/2}\}.$$

By Definition 3.1, and the definition of the neck-width constant w^* , the area of the part of the neck $\mathcal{N}(\mathbf{w})$ in the support of χ_4^δ is bounded by a constant C^* multiplied by $w\delta$. Therefore,

$$\#\{D \in \mathcal{V}_3^\delta(\varepsilon, u) : (\tilde{B}) \text{ holds}\} \leq C^* w \delta \max\{\gamma, w^{-1} \gamma^{1/2}\}. \quad (60)$$

Setting $\delta = |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}$ gives

$$\gamma = \varepsilon^{-4} |\Omega(\mathbf{w})|^{2-4\beta} \mu^{3-4\beta} = \varepsilon^{-4} |\Omega(\mathbf{w})|^{-1} (\Omega(\mathbf{w}) \mu)^{3-4\beta}.$$

Using $0 < \beta < \frac{1}{2}$, and $w \leq 4\delta$, the quantity in (60) therefore satisfies

$$\#\{D \in \mathcal{V}_3^\delta(\varepsilon, u) : (\tilde{B}) \text{ holds}\} \leq C^* \varepsilon^{-4} ((|\Omega(\mathbf{w})| \mu)^{3-6\beta} + (|\Omega(\mathbf{w})| \mu)^{3/2-3\beta}).$$

This satisfies the estimate given in the statement of the proposition, and finishes the proof. $\square$

7 Proof of Theorem 1.1

This section is dedicated to the proof of Theorem 1.1. The upper bound on $v(u_m(\mathbf{w}))$, the number of nodal domains for the m -th eigenfunction $u_m(\mathbf{w})$, will follow from the control on $\{v_j^\delta(\varepsilon, u_m(\mathbf{w}))\}_{j=0}^3$ that we developed in previous sections for appropriately chosen δ, ε . At the same time, when $u_m(\mathbf{w})$ is Courant sharp, we know that $v(u_m(\mathbf{w})) = m$ is bounded below by the number of Neumann eigenvalues under $\mu_m(\mathbf{w})$. Therefore, we prove Theorem 1.1 in Sect. 7.2 after first establishing a lower bound on the Neumann counting function in Sect. 7.1.

7.1 A Lower Bound on the Neumann Counting Function

In this section, we obtain a lower bound on the Neumann counting function $\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu) = \#\{j : \mu_j(\mathbf{w}) < \mu\}$, using a Weyl remainder estimate

Proposition 7.1 *Let $\Omega(\mathbf{w})$ be a chain domain. There exists $C^* > 0$, such that*

$$\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu) - \frac{1}{4\pi} |\Omega(\mathbf{w})| \mu \geq -C^* (|\Omega(\mathbf{w})| \mu)^{3/4}$$

whenever $|\Omega(\mathbf{w})| \mu \geq C^*$.

Proof of Proposition 7.1 By the min-max characterization of eigenvalues $\mu_m(\mathbf{w}) \leq \lambda_m(\mathbf{w})$, where $\lambda_m(\mathbf{w})$ is the m -th Dirichlet eigenvalue of $\Omega(\mathbf{w})$. Therefore,

$$\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu) \geq \mathcal{N}_{\Omega(\mathbf{w})}^D(\mu),$$

where $\mathcal{N}_{\Omega(\mathbf{w})}^D(\mu)$ is the Dirichlet counting function. Next, let

$$R_{\Omega(\mathbf{w})}(\mu) = \frac{1}{4\pi} |\Omega(\mathbf{w})| \mu - \mathcal{N}_{\Omega(\mathbf{w})}^D(\mu).$$

By [26, Equation (13)], in dimension 2, we obtain, for any $\varepsilon > 0$,

$$R_{\Omega(\mathbf{w})}(\mu) \leq \frac{1}{4\pi} M_{\Omega(\mathbf{w})}(\sqrt{2\varepsilon}) \mu + |\Omega(\mathbf{w})| (\sqrt{2\varepsilon})^{-1} \mu^{1/2}. \quad (61)$$

For t satisfying

$$t \leq \frac{3}{4} L(\mathbf{w}) \min \left\{ \tau^* \delta^*, \frac{1}{\kappa^*} \right\}, \quad (62)$$

we can apply Lemma 4.2, so that

$$M_{\Omega(\mathbf{w})}(t) \leq C^* L(\mathbf{w}) t,$$

for a constant C^* . Therefore, setting $t = 2\sqrt{\varepsilon}$, from (61) we obtain the estimate

$$R_{\Omega(\mathbf{w})}(\mu) \leq C^* L(\mathbf{w}) t \mu + |\Omega(\mathbf{w})| t^{-1} \mu^{1/2}, \quad (63)$$

provided (62) continues to hold. Defining $t > 0$ by

$$t^2 = (C^*)^{-1} |\Omega(\mathbf{w})| / L(\mathbf{w}) \mu^{-1/2}, \quad (64)$$

in order to minimize the right hand side of (63), gives

$$\begin{aligned} R_{\Omega(\mathbf{w})}(\mu) &\leq 2(C^*)^{1/2} L(\mathbf{w})^{1/2} |\Omega(\mathbf{w})|^{1/2} \mu^{3/4} \\ &\leq 2(C^*)^{1/2} (\rho^*)^{1/4} (|\Omega(\mathbf{w})| \mu)^{3/4}. \end{aligned} \quad (65)$$

Note that from (64) and the isoperimetric inequality,

$$\begin{aligned} \left(\frac{t}{L(\mathbf{w})} \right)^4 &= \frac{(C^*)^{-2} (|\Omega(\mathbf{w})| / L(\mathbf{w})^2)^3}{|\Omega(\mathbf{w})| \mu} \\ &\leq \frac{(C^*)^{-2} (4\pi)^{-3}}{|\Omega(\mathbf{w})| \mu}. \end{aligned}$$

Therefore, there exists a constant $\tilde{C}^*$ such that, for $|\Omega(\mathbf{w})| \mu \geq \tilde{C}^*$, this choice of t satisfies (62). Since

$$\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu) - \frac{1}{4\pi} |\Omega(\mathbf{w})| \mu \geq \mathcal{N}_{\Omega(\mathbf{w})}^D(\mu) - \frac{1}{4\pi} |\Omega(\mathbf{w})| \mu = -R_{\Omega(\mathbf{w})}(\mu),$$

the estimate in (65) thus completes the proof of the proposition. $\square$

7.2 Proof of Theorem 1.1

We now combine the estimates on the nodal domain counts from the previous sections with Proposition 7.1 in order to prove Theorem 1.1.

Let $u_m(\mathbf{w})$ be the m -th Neumann eigenfunction of $\Omega(\mathbf{w})$ with eigenvalue $\mu_m(\mathbf{w})$. To prove the theorem, we can assume that $|\Omega(\mathbf{w})| \mu_m(\mathbf{w})$ is sufficiently large so that the estimates in Propositions 4.1, 4.2, 5.1, 6.1, and 7.1 all hold with $\mu = \mu_m(\mathbf{w})$. We will use these estimates to then show that $|\Omega(\mathbf{w})| \mu_m(\mathbf{w}) \leq C^*$ for a constant C^* .

Let

$$\beta := \frac{3}{8}, \quad \delta := |\Omega(\mathbf{w})|^{1/2-\beta} \mu^{-\beta}.$$

By Propositions 4.1, 4.2, 5.1, and 6.1, there exists a constant $\tilde{C}^*$ such that for each $0 < \varepsilon < \frac{1}{2}$, by (19) the number of nodal domains of $u_m(\mathbf{w})$ satisfies

$$v(u_m(\mathbf{w})) \leq \sum_{j=0}^3 v_j^\delta(\varepsilon, u_m(\mathbf{w})) \quad (66)$$

$$\leq \frac{1}{\pi \lambda_1(\mathbb{D})} \frac{1+\varepsilon}{1-\varepsilon} |\Omega(\mathbf{w})| \mu_m(\mathbf{w}) + \tilde{C}^* \varepsilon^{-4} (|\Omega(\mathbf{w})| \mu_m(\mathbf{w}))^{3/4}, \quad (67)$$

provided that $|\Omega(\mathbf{w})| \mu \geq \tilde{C}^*$. Since $\mathcal{N}_{\Omega(\mathbf{w})}^N(\mu_m(\mathbf{w})) \leq m-1$, by Proposition 7.1,

$$m-1 - \frac{1}{4\pi} |\Omega(\mathbf{w})| \mu_m(\mathbf{w}) \geq -C^* (|\Omega(\mathbf{w})| \mu_m(\mathbf{w}))^{3/4}, \quad (68)$$

provided that $|\Omega(\mathbf{w})| \mu \geq C^*$.

When $u_m(\mathbf{w})$ is Courant sharp, we have $v(u_m(\mathbf{w})) = m$. Therefore, combining (66) and (68) implies, in the Courant sharp case, that $\mu_m(\mathbf{w})$ satisfies

$$\begin{aligned} \frac{1}{4\pi} |\Omega(\mathbf{w})| \mu_m(\mathbf{w}) - C^* (|\Omega(\mathbf{w})| \mu_m(\mathbf{w}))^{3/4} &\leq \frac{1}{\pi \lambda_1(\mathbb{D})} \frac{1+\varepsilon}{1-\varepsilon} |\Omega(\mathbf{w})| \mu_m(\mathbf{w}) \\ &\quad + \tilde{C}^* \varepsilon^{-4} (|\Omega(\mathbf{w})| \mu_m(\mathbf{w}))^{3/4}. \end{aligned} \quad (69)$$

We fix a small absolute constant $\varepsilon > 0$ so that

$$\frac{1}{\pi \lambda_1(\mathbb{D})} \frac{1+\varepsilon}{1-\varepsilon} < \frac{1}{4\pi},$$

which we can do since $4/\lambda_1(\mathbb{D}) < 1$. Then, we can rearrange (69) to guarantee that

$$|\Omega(\mathbf{w})| \mu_m(\mathbf{w}) \leq C_1^*$$

for a constant depending only on C^* , $\tilde{C}^*$ and this value of ε . This establishes the second part of the theorem. To prove the first part of the theorem, we see from (66) that for any eigenfunction $u_m(\mathbf{w})$,

$$\frac{v(u_m(\mathbf{w}))}{m} \leq \frac{1}{\pi \lambda_1(\mathbb{D})} \frac{1+\varepsilon}{1-\varepsilon} \frac{|\Omega(\mathbf{w})| \mu_m(\mathbf{w})}{m} + \tilde{C}^* \varepsilon^{-4} \frac{(|\Omega(\mathbf{w})| \mu_m(\mathbf{w}))^{3/4}}{m}.$$

Using (68), we therefore have

$$\limsup_{m \rightarrow \infty} \frac{\nu(u_m(\mathbf{w}))}{m} \leq \frac{4\pi}{\pi \lambda_1(\mathbb{D})} \frac{1 + \varepsilon}{1 - \varepsilon}$$

for any $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$ then completes the proof of the theorem.

Acknowledgements T.B. was supported through NSF Grant DMS-2042654. Y.C. was supported by the Alfred P. Sloan Foundation, NSF CAREER Grant DMS-2045494, and NSF Grant DMS-1900519. JLM acknowledges support from the NSF through NSF grant DMS-1909035 and NSF FRG grant DMS-2152289.

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

1. Arrieta, J.M.: Rates of eigenvalues on a dumbbell domain. Simple eigenvalue case. *Trans. Am. Math. Soc.* **347**(9), 3503–3531 (1995)
2. Beck, T., Gupta, M., Marzuola, J.L.: Nodal set openings on perturbed rectangular domains. [arXiv:2204.12009](https://arxiv.org/abs/2204.12009) (2022)
3. Bérard, P., Helffer, B.: The weak Pleijel theorem with geometric control. *J. Spectr. Theory* **6**(4), 717–733 (2016)
4. Bérard, P., Meyer, D.: Inégalités isopérimétriques et applications. *Ann. Sci. École Norm. Sup.* (4) **15**(3), 513–541 (1982)
5. Chavel, I.: *Eigenvalues in Riemannian Geometry*. Series in Pure and Applied Mathematics. Academic Press, USA (1984)
6. Courant, R., Hilbert, D.: *Methods of mathematical physics: partial differential equations*. John Wiley & Sons, Bristol (2008)
7. Daners, D.: Dirichlet problems on varying domains. *J. Differ. Equ.* **188**(2), 591–624 (2003)
8. De Ponti, N., Farinelli, S., Yuri Violo, I.: Pleijel nodal domain theorem in non-smooth setting. [arXiv:2307.13983](https://arxiv.org/abs/2307.13983) (2023)
9. Gittins, K., Léna, C.: Upper bounds for Courant-sharp Neumann and Robin eigenvalues. *Bull. Soc. Math. France* **148**(1), 99–132 (2020)
10. Grisvard, P.: *Elliptic problems in nonsmooth domains*. SIAM, Philadelphia (2011)
11. Hassannezhad, A., Sher, D.: On Pleijel’s nodal domain theorem for the Robin problem. [arXiv:2303.08094](https://arxiv.org/abs/2303.08094) (2023)
12. Helffer, B., Hoffmann-Ostenhof, T., Terracini, S.: Nodal domains and spectral minimal partitions. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**(1), 101–138 (2009)
13. Howards, H., Hutchings, M., Morgan, F.: The isoperimetric problem on surfaces. *Am. Math. Month.* **106**(5), 430–439 (1999)
14. Jimbo, S.: The singularly perturbed domain and the characterization for the eigenfunctions with Neumann boundary condition. *J. Differential Equ.* **77**(2), 322–350 (1989)
15. Jimbo, S.: Perturbation formula of eigenvalues in a singularly perturbed domain. *J. Math. Soc. Japan* **45**(2), 339–356 (1993)
16. Jimbo, S., Morita, Y.: Remarks on the behavior of certain eigenvalues on a singularly perturbed domain with several thin channels. *Comm. Partial Differ. Equ.* **17**(3–4), 523–552 (1992)
17. Kac, M.: Can one hear the shape of a drum? *Am. Math. Month.* **73**(4P2), 1–23 (1966)
18. Léna, Corentin: Pleijel’s nodal domain theorem for Neumann and Robin eigenfunctions. *Ann. Inst. Fourier (Grenoble)* **69**(1), 283–301 (2019)
19. Peetre, J.: A generalization of Courant’s nodal domain theorem. *Math. Scand.* **5**, 15–20 (1957)
20. Pleijel, Å.: Remarks on Courant’s nodal line theorem. *Comm. Pure Appl. Math.* **9**, 543–550 (1956)
21. Polterovich, I.: Pleijel’s nodal domain theorem for free membranes. *Proc. Amer. Math. Soc.* **137**(3), 1021–1024 (2009)
22. Singer, A., Hau-Tieng, W.: Spectral convergence of the connection Laplacian from random samples. *Inf. Inference. J. IMA* **6**(1), 58–123 (2017)
23. Trillos, N.G., Slepčev, D.: A variational approach to the consistency of spectral clustering. *Appl. Comput. Harm. Anal.* **45**(2), 239–281 (2018)

24. Uhlenbeck, K.: The Morse index theorem in Hilbert space. *J. Differential Geomet.* **8**, 555–564 (1973)
25. Uhlenbeck, K.: Generic properties of eigenfunctions. *Amer. J. Math.* **98**(4), 1059–1078 (1976)
26. van den Berg, M., Gittins, K.: On the number of Courant-sharp Dirichlet eigenvalues. *J. Spectr. Theory* **6**(4), 735–745 (2016)
27. Wigley, N.: Mixed boundary value problems in plane domains with corners. *Math. Z.* **115**, 33–52 (1970)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.