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• ARTICLES •

A note on Kollár valuations

In memory of Gang Xiao

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Abstract We prove the set of Kollár valuations in the dual complex of a klt singularity with a fixed complement is path connected. We also classify the case when the dual complex is one-dimensional.

Keywords klt singularities, Kollár valuations, finite generation, dual complex

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1 Introduction

One new question with a birational geometric feature, arising from algebraic K-stability theory, is to ask what kind of quasi-monomial valuation v over a polarized projective varieties (X, L) satisfies that $\mathrm{Gr}_v \bigoplus_{m\in\mathbb{N}} H^0(X, mL)$ is finitely generated. For now the only main case we have some knowledge is in the Fano setting, i.e., the following question.

Question 1.1 (Global version). Let (X, Δ) be a klt log Fano pair, and v be an lc place of a $\mathbb{Q}$-complement. Let r satisfy that $r(K_X + \Delta)$ is Cartier. Then what condition implies that

$$\bigoplus_{m\in r\cdot\mathbb{N}} \mathrm{Gr}_v H^0(X, -m(K_X + \Delta))$$

is finitely generated?

There is a local version which implies the global version by taking the cone.

Question 1.2 (Local version). Let $(X = \mathrm{Spec}(R), \Delta)$ be a klt singularity, and $v \in \mathrm{Val}_{X,x}$ be an lc place of a $\mathbb{Q}$-complement. Then what condition implies that $\mathrm{Gr}_v R$ is finitely generated?

When v is a divisorial valuation, i.e., $v = \mathrm{ord}_E$, then it follows from [2] and our assumption that E is an lc place of a $\mathbb{Q}$-complement that $\mathrm{Gr}_E R$ is finitely generated. However, for v with rational rank $\mathrm{rank}_{\mathbb{Q}}(v) \geqslant 2$, the question is quite unclear. Built on [16], in [21], a smaller class is sorted out.

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Definition 1.3. Let $x \in (X, \Delta)$ be a klt singularity. We say $v \in \mathrm{Val}_{X,x}$ is a *Kollár valuation* if there exists a birational model $\mu\colon Y \to X$ with $\mathrm{Ex}(\mu) = \sum_{i=1}^{p} E_i$ and $D \geqslant 0$ on Y such that

(1) $(Y, E + D)$ is q-dlt with $\lfloor E + D \rfloor = E$ and $v \in \mathrm{QM}(Y, E)$;

(2) $K_Y + E + D \geqslant \mu^*(K_X + \Delta)$ and $-K_Y - E - D$ is ample over X.

We say that $(Y, E + D)$ is a *Kollár model* over X, extracting $E_1, \dots, E_p$.

We have the following theorem.

Theorem 1.4 (See [21]). *If v is a Kollár valuation, then $\mathrm{Gr}_v R$ is finitely generated.*

In fact, it is proved that v is a Kollár valuation if and only if the degeneration of (X, Δ) to $X_0 = \mathrm{Spec}(\mathrm{Gr}_v R)$ yields a klt pair. Fix a $\mathbb{Q}$-complement D such that x is the only lc center. It is known that the set of lc places

$$\mathrm{LCP}(X, \Delta + D) = \{\text{nontrivial valuations } v \text{ such that } A_{X,\Delta}(v) = v(D)\}$$

is a cone over the dual complex $\mathcal{D}(X, \Delta + D)$, which is a collapsible pseudo-manifold with boundary (see [6, 10]). Based on the discussion above, it is also worth looking at the set of all Kollár valuations $\mathrm{KV}(X, \Delta + D) \subseteq \mathrm{LCP}(X, \Delta + D)$, which is a cone over a space, denoted by $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$. In our note, we will identify a valuation with its non-zero rescaling, and in this sense we talk about a valuation in the dual complex.

Question 1.5. Let $x \in (X, \Delta)$ be a klt singularity, and D be a $\mathbb{Q}$-complement such that x is the only lc center, i.e., D is an effective $\mathbb{Q}$-divisor such that $(X, \Delta + D)$ is klt outside x, and lc but not klt at x. How does $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D) \subseteq \mathcal{D}(X, \Delta + D)$ look like?

In general, we know little about $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$. The following summarizes what has been proved.

Theorem 1.6. *We know*

(1) (*See* [17]) $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D) \neq \emptyset$.

(2) (*See* [14]) *We fix a log resolution of $(X, \Delta + D)$, which yields a triangulation of $\mathcal{D}(X, \Delta + D)$. If a point $x \in \mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$, then there exists a neighborhood U of x in the smallest affine linear subspace V defined over $\mathbb{Q}$ containing x, such that $U \subseteq \mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$.*

One easily sees the conclusion in (2) does not depend on the choice of the log resolution.

Our first main result proves the path connectedness of the locus of Kollár valuations.

Theorem 1.7. *Let $x \in (X, \Delta)$ be a klt singularity, and let D be a $\mathbb{Q}$-complement such that $\{x\}$ is the only lc center of $(X, \Delta + D)$. Then $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$ is path connected.*

We make the following conjectures.

Conjecture 1.8. We conjecture $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$ satisfies the following:

(1) (Local closedness, weak version) There is a rational triangulation of $\mathcal{D}(X, \Delta + D)$ such that for each open simplex C°, $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D) \cap C^\circ$ is open in its closure $\overline{\mathcal{D}^{\mathrm{KV}}(X, \Delta + D) \cap C^\circ} \subset C^\circ$.

(1′) (Local closedness, strong version) There is a rational triangulation of $\mathcal{D}(X, \Delta + D)$ such that for each open simplex C°, $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D) \cap C^\circ \subset C^\circ$ is open.

(2) (Finiteness, weak version) If $\mathcal{D}(X, \Delta + D) = \mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$, i.e., $\mathcal{D}(X, \Delta + D)$ consists of Kollár valuations, then there is a finite triangulation of $\mathcal{D}(X, \Delta + D)$, such that each simplex can be realized on a Kollár model.

(2′) (Finiteness, strong version) If $C \subseteq \mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$ is a closed cell of $\mathcal{D}(X, \Delta + D)$ after a triangulation given by a log resolution, then there is a finite triangulation of $C = \bigcup_{i=1}^{N} C_i$, such that each C_i is realized in a Kollár model.

See Example 3.4 for sharpness of our formulation.

In Section 3, we study this conjecture when the dual complex is one-dimensional, and completely address the question in this case.

Theorem 1.9. *Assume $\mathcal{D}(X, \Delta + D)$ is homeomorphic to $[0, 1]$. We denote by v_t the valuation corresponding to $t \in [0, 1]$. The set of Kollár valuations $\mathcal{D}^{\mathrm{KV}}(X, \Delta + D)$ is*

(1) *either precisely one of E_0 or E_1;*

(2) *or* v_t *for all* $t \in [0,1]$.

Notation and conventions: We follow the notation and conventions in [11, 12, 19]. By a singularity $x \in X$, we mean $X = \mathrm{Spec}(R)$ for a local ring R which is essentially of finite type, x is the closed point. If v is a valuation on $K(X)$, whose center on X is x, then we denote by $\mathrm{Gr}_v R$ the associated graded ring, i.e., $\mathrm{Gr}_v R = \bigoplus_{\lambda \in \mathbb{R}} \mathfrak{a}_{\geqslant \lambda}/\mathfrak{a}_{>\lambda}$, where

$$\mathfrak{a}_{\geqslant \lambda} \text{ (resp. } \mathfrak{a}_{>\lambda}) = \{f \in R \mid v(f) \geqslant \text{(resp. } >) \ \lambda\}.$$

When $v = \mathrm{ord}_E$ for some divisor E over X, we also write $\mathrm{Gr}_E R$ for $\mathrm{Gr}_{\mathrm{ord}_E} R$. Let (Y, E) be a log smooth model or more generally a toroidal model such that components of E are $\mathbb{Q}$-Cartier, we denote by $\mathrm{QM}(Y, E)$ the set of quasi-monomial valuations (see, e.g., [21, Definition 2.8]). A q-dlt pair is defined as [6, Definition 35].

2 Connectedness

In this section, we aim to prove Theorem 1.7.

Lemma 2.1. *Fix a klt singularity* $x \in (X, \Delta)$ *and a* $\mathbb{Q}$*-complement* D *such that* $(X, \Delta + D)$ *is klt outside* x*. There exists a positive* ε *depending only on* $\dim(X)$, $\mathrm{Coeff}(\Delta)$ *and* $\mathrm{Coeff}(D)$*, such that any lc place* E *of a lc pair* $(X, \Delta + (1-\varepsilon)D + G)$ *for an effective* $\mathbb{Q}$*-divisor* G *belongs to* $\mathcal{D}(X, \Delta + D)$*.*

Proof. This follows directly from [23, Lemma 5.5] by applying ACC of log canonical thresholds [9] (see also [13, Proof of Proposition 6.9]). □

Proof of Theorem 1.7. Replacing $x \in (X, \Delta)$ and D by $x \in (X, \Delta + (1-\varepsilon)D)$ and εD respectively as in Lemma 2.1, we may assume that lc places of any $\mathbb{Q}$-complement of (X, Δ) are contained in $\mathcal{D}(X, \Delta + D)$. Using Theorem 1.6, we can replace two Kollár valuations by nearby Kollár components. Thus we may assume two valuations are Kollár components E_0 and E_1. Let $\mu_i \colon Y_i \to X$ $(i = 0, 1)$ be the model extracting the Kollár component E_i, and H_i the pushforward of a general effective $\mathbb{Q}$-divisor in $|-K_{Y_i} - E_i - \mu_{i*}^{-1}\Delta|_{\mathbb{Q}}$. So H_i is a $\mathbb{Q}$-complement such that E_i are the only lc places of $(X, \Delta + H_i)$. Applying Lemma 2.1 to E_i, there exists ε_i such that E_i are the only lc places of any $\mathbb{Q}$-complement of $(X, \Delta + (1-\varepsilon_i)H_i)$. In particular, E_i is the minimizer of $\widehat{\mathrm{vol}}_{X, \Delta + (1-\varepsilon_i)H_i}$.

For a real number $t \in [0,1]$, define $\overline{H}_t := (1-t)(1-\varepsilon_0)H_0 + t(1-\varepsilon_1)H_1$. Thus $x \in (X, \Delta + \overline{H}_t)$ is a klt singularity with $\mathbb{R}$-boundary divisors. Let v_t be a minimizer of $\widehat{\mathrm{vol}}_{X, \Delta + \overline{H}_t}(\cdot)$ such that $A_{X,\Delta}(v_t) = 1$, whose existence and uniqueness follows from [3, 18, 20] and their generalization to $\mathbb{R}$-divisors [8, Theorems 3.3 and 3.4]. Moreover, by [21] and its generalization to $\mathbb{R}$-divisors [22, Theorem 2.19], we know that v_t is a Kollár valuation over $x \in (X, \Delta + \overline{H}_t)$ (hence over $x \in (X, \Delta)$). By [8, Theorem 2.20] (see also [15, Theorem 1.3]), there exists a sequence of Kollár components $(S_{t,j})_{j \in \mathbb{N}}$ over $x \in (X, \Delta + \overline{H}_t)$ such that $v_t = \lim_{j \to \infty} \frac{\mathrm{ord}_{S_{t,j}}}{A_{X,\Delta}(S_{t,j})}$. Hence $S_{t,j}$ is a Kollár component over $x \in (X, \Delta)$ which implies that it is an lc place of a $\mathbb{Q}$-complement of (X, Δ). By Lemma 2.1 we know that $S_{t,j} \in \mathcal{D}(X, \Delta + D)$, which implies that their rescaled limit $v_t \in \mathcal{D}(X, \Delta + D)$. Moreover, since $(v, t) \mapsto \widehat{\mathrm{vol}}_{X, \Delta + \overline{H}_t}(v)$ is a continuous function on $\mathcal{D}(X, \Delta + D) \times [0,1]$, the function

$$[0,1] \to \mathcal{D}(X, \Delta + D), \quad t \mapsto v_t$$

is continuous by Lemma 2.2. □

Lemma 2.2. *Let* $f : W \times [0,1] \to \mathbb{R}$ *be a continuous function where* W *is a compact topological space. Assume that for every* $t \in [0,1]$*, there is a unique minimizer* $v_t \in W$ *of the function* f_t *which is the restriction of* f *on* $W \times t$*. Then* $t \mapsto v_t$ *is a continuous function from* $[0,1]$ *to* W*.*

Proof. Consider the subset $\mathcal{U} \subset W \times [0,1]$ defined by

$$\mathcal{U} := \{(x,t) \in W \times [0,1] \mid \text{there exists } y \in W \text{ such that } f(x,t) > f(y,t)\}.$$

We claim that $\mathcal{U}$ is open in $W\times[0,1]$. Let $(x,t)\in\mathcal{U}$ be an arbitrary point. Then there exists $y\in W$ such that $f(x,t)>f(y,t)$. Since f is continuous, for $\varepsilon=\frac{f(x,t)-f(y,t)}{2}>0$ there exists open neighborhoods $x\in U_x\subset W$, $y\in U_y\subset W$, and $t\in U_t\subset[0,1]$, such that $f(x',t')>f(x,t)-\varepsilon$ and $f(y',t')<f(y,t)+\epsilon$ for any $x'\in U_x$, $y'\in U_y$ and $t'\in U_t$. In particular,

$$f(x',t')-f(y',t')>f(x,t)-f(y,t)-2\varepsilon=0.$$

Thus $(x',t')\in\mathcal{U}$ for any $(x',t')\in U_x\times U_t$, which implies that $\mathcal{U}$ is open.

Finally, by assumption we know that $(W\times[0,1])\backslash\mathcal{U}$ is precisely the graph of the function $\sigma:[0,1]\to W$ where $\sigma(t)=v_t$. Since $\mathcal{U}$ is open, the graph of σ is closed. Thus σ is continuous by the closed graph theorem as W is compact. □

3 One-dimensional dual complex

Let $x\in(X,\Delta)$ be a klt singularity, and D a $\mathbb{Q}$-complement such that $(X,\Delta+D)$ is klt outside x. For any collection of lc places $E_{t_1},\dots,E_{t_i}$ corresponding to points $t_1,\dots,t_i\in\mathcal{D}(X,\Delta+D)$, by [2, Corollary 1.4.3], there exists a model

$$\mu_{t_1t_2\cdots t_i}\colon Y_{t_1t_2\cdots t_i}\to X$$

which precisely extracts $E_{t_1},\dots,E_{t_i}$. Moreover, by running a $(\mu^{-1}_{t_1t_2\cdots t_i*}D)$-MMP over X, we may assume $-K_{Y_{t_1t_2\cdots t_i}}-\mu_{t_1t_2\cdots t_i*}(\Delta)-\sum_{j=1}^{i}E_{t_j}$ is nef, as the MMP sequence only has flips.

Such $Y_{t_1t_2\cdots t_i}$ is not unique, but any two models $Y_{t_1t_2\cdots t_i}$ and $Y'_{t_1t_2\cdots t_i}$ are crepant birationally equivalent. In particular, the notion of bigness of the restriction of $-K_{Y_{t_1t_2\cdots t_i}}-\mu_{t_1t_2\cdots t_i*}(\Delta)-\sum_{j=1}^{i}E_{t_j}$ on E_{t_j} is well defined for any $1\leqslant j\leqslant i$.

Therefore, for any $t\in\mathcal{D}(X,\Delta+D)$, E_t is a *Kollár component* if $(Y_t,\mu_{t*}(\Delta)+E_t)$ is plt; and $E_{t_1},\dots,E_{t_i}$ admits a Kollár model, if there exists $Y_{t_1t_2\cdots t_i}$ such that the pair $(Y_{t_1t_2\cdots t_i},\mu_{t_1t_2\cdots t_i*}(\Delta)+\sum_{j=1}^{i}E_{t_j})$ is q-dlt and $-K_{Y_{t_1t_2\cdots t_i}}-\mu_{t_1t_2\cdots t_i*}(\Delta)-\sum_{j=1}^{i}E_{t_j}$ is ample.

We will study the case where $\mathcal{D}(X,\Delta+D)$ is homeomorphic to a one-dimensional interval $[0,1]$. For any $t\in[0,1]$, up to rescaling, it corresponds to a valuation v_t, and for $t\in\mathbb{Q}$, it corresponds to a divisorial valuation E_t.

Lemma 3.1. *At least one of the endpoints corresponds to a Kollár component.*

Proof. Assume one ending point, say E_1, is not a Kollár component. Then we know that we can construct a model $\mu_1\colon Y_1\to X$ which precisely extracts E_1. From our assumption, $\mu_*^{-1}(D)$ does not contain the log canonical center of $(Y_1,E_1+\mu^{-1}_{1*}(\Delta))$ properly contained in E_1. Therefore, $A_{Y_1,E_1+\mu^{-1}_{1*}(\Delta)}(E_0)=0$, so we can get a model $\mu_{10}\colon Y_{10}\to Y_1\to X$ extracting E_0, such that $-(K_{Y_{10}}+E_0+E_1+\mu^{-1}_{10*}(\Delta))$ is nef, as it is the pull-back of $-(K_{Y_1}+E_1+\mu^{-1}_{1*}(\Delta))$.

Similarly, if E_0 is not a Kollár component, we can get $\mu_{01}\colon Y_{01}\to X$ and $-(K_{Y_{01}}+E_0+E_1+\mu^{-1}_{01*}(\Delta))$ is nef. In particular, $(Y_{10},E_0+E_1+\mu^{-1}_{10*}(\Delta))$ and $(Y_{01},E_0+E_1+\mu^{-1}_{01*}(\Delta))$ are crepant birationally equivalent.

However, the restriction of $-(K_{Y_{10}}+E_0+E_1+\mu^{-1}_{10*}(\Delta))$ on E_1 is big and on E_0 is not big, while the restriction of $-(K_{Y_{01}}+E_0+E_1+\mu^{-1}_{01*}(\Delta))$ on E_0 is big and on E_1 is not big. A contradiction. □

By Lemma 3.1, we can always assume E_0 is a Kollár component. Lemma 3.1 also follows from the standard tie breaking argument, but the above proof sheds more light on our approach.

Proposition 3.2. *Assume $\mathcal{D}(X,\Delta+D)$ is homeomorphic to $[0,1]$, and E_0, E_1 are Kollár components. Then there exists a Kollár model which precisely extracts E_0 and E_1.*

Proof. Let $\mu_{01}\colon Y_{01}\to X$ be a $\mathbb{Q}$-factorial model which extracts E_0 and E_1 such that $-K_{Y_{01}}-E_0-E_1-\mu^{-1}_{01*}\Delta$ is nef. In particular, $(Y_{01},E_0+E_1+\mu^{-1}_{01*}\Delta)$ is q-dlt (see [6, Proposition 34]). We claim $(-K_{Y_{01}}-E_0-E_1-\mu^{-1}_{01*}\Delta)|_{E_i}$ is big for $i=0,1$. If not, say for $i=0$ this is not true, then let $f\colon Y_{01}\to Y'_1$ be the ample model of $-K_{Y_{01}}-E_0-E_1-\mu^{-1}_{01*}\Delta$ over X which contracts E_0. Then $(Y'_1,f_*(E_1+\mu^{-1}_{01*}\Delta))$ is not plt as it is crepant birationally equivalent to $(Y_{01},E_0+E_1+\mu^{-1}_{01*}\Delta)$. On the other hand, Y'_1 is isomorphic to Y_1 in codimension one. Thus by the negativity lemma, $(Y_1', f_*(E_1+\mu_{01*}^{-1}\Delta))$ is crepant birationally equivalent to $(Y_1, E_1+\mu_{1*}^{-1}(\Delta))$, which contradicts to the assumption that E_1 is a Kollár component.

So if we take the anti-canonical model $h\colon Z\to X$ of Y_{01} over X for $-K_{Y_{01}}-E_0-E_1-\mu_{01*}^{-1}\Delta$, it contains the birational transform F_i of E_i for $i=0,1$.

We claim a component W of $F_0\cap F_1$ is of codimension two in Z. In fact, for sufficiently divisible m, the effective divisor $F=m(A_{X,\Delta}(F_0)F_0+A_{X,\Delta}(F_1)F_1)$ is Cartier and supported on $F_0\cup F_1$, hence $\mathcal{O}_F$ is Cohen-Macaulay by [11, Corollary 5.25]. If we localize at the generic point η of W, as $\operatorname{Spec}\mathcal{O}_{F,\eta}\setminus\{\eta\}$ is disconnected, $\dim(\operatorname{Spec}\mathcal{O}_{F,\eta})=1$ by [7, Proposition 2.1]. By the classification of surface log canonical singularities ([12, Subsection 3.3]), the pair $(Z, F_1+F_2+h_*^{-1}\Delta)$ is toriodal in a neighborhood U of the generic point $\eta(W)$ of W, and $Y\to Z$ is isomorphic over $\eta(W)$, from the generic point of $E_0\cap E_1$. Then for any divisor E exceptional over Z whose center contained $Z\setminus U$, we have

$$A_{Z,F_1+F_2+h_*^{-1}\Delta}(E)=A_{Y_{01},E_0+E_1+\mu_{01*}^{-1}\Delta}(E)>0.$$

Thus $(Z,F_1+F_2+h_*^{-1}\Delta)$ is q-dlt. In particular, it is a Kollár model. $\square$

Proof of Theorem 1.9. By Lemma 3.1, we may assume one of E_0 or E_1, say E_0, is a Kollár component. By Proposition 3.2, if E_1 is a Kollár component, then Case (2) happens. So it remains to prove if E_1 is not a Kollár component, then for any $s\in(0,1]\cap\mathbb{Q}$, E_s is not a Kollár component.

Let $\mu_1\colon Y_1\to X$ be the model extracting E_1. From our assumption, $\mu_{1*}^{-1}D$ does not pass through the lc center of $(Y_1,E_1+\mu_{1*}^{-1}\Delta)$ properly contained in E_1. As a result, E_0 and therefore any E_s are lc places of $(Y_1,E_1+\mu_{1*}^{-1}\Delta)$. So we can extract E_s over Y_1 to get a $\mathbb{Q}$-factorial model $\mu_{1s}\colon Y_{1s}\to Y_1\to X$. Since

$$\begin{aligned}\mathcal{D}(X,\Delta+D)&\simeq\mathcal{D}(Y_1,E_1+\mu_{1*}^{-1}(\Delta+D))\\&\simeq\mathcal{D}(Y_1,E_1+\mu_{1*}^{-1}\Delta)\simeq\mathcal{D}(Y_{1s},E_1+E_s+\mu_{1s*}^{-1}\Delta)\simeq[0,1],\end{aligned}$$

$W_1:=E_1\cap E_s$ is the only lc center properly contained in E_1 with $\dim(W_1)=\dim(X)-2$. In particular, E_1 and W_1 are normal and W_1 does not properly contain any lc center of $(Y_{1s},E_1+E_s+\mu_{1s*}^{-1}\Delta)$. If we denote by $\nu\colon E_s^{\mathrm{n}}\to E_s$ the normalization, and write

$$(K_{Y_{1s}}+E_1+E_s+\mu_{1s*}^{-1}\Delta)|_{E_s^{\mathrm{n}}}=K_{E_s^{\mathrm{n}}}+\Delta_{E_s^{\mathrm{n}}},$$

then $\mathcal{D}(E_s^{\mathrm{n}},\Delta_{E_s^{\mathrm{n}}})$ are two points corresponding to two disjoint lc centers W_1' and W_0'. Since (Y_{1s},E_1+E_s) is toroidal at the generic point $\eta(W_1)$ by the classification of surface log canonical singularities (see [12, Section 3.3]), $\nu\colon W_1'\to W_1$ is birational and $W_1\neq\nu(W_0')$. Moreover, $W_0:=\nu(W_0')$ does not meet W_1 as W_1 does not properly contain any lc center of $(Y_{1s},E_1+E_s+\mu_{1s*}^{-1}\Delta)$. So $(Y_{1s},E_1+E_s+\mu_{1s*}^{-1}\Delta)$ has two disjoint lc centers W_0 and W_1 properly contained in E_s.

We can run an E_1-minimal model program of Y_{1s} over X, obtaining a birational model $Y_{1s}\dashrightarrow Y_s'$, which terminates by contracting E_1. This minimal model program is isomorphic outside E_1. Therefore, if we denote $\mu_s'\colon Y_s'\to X$, then the divisorial part of $\mathrm{Ex}(\mu_s')$ is the birational transform E_s' of E_s, and $(Y_s',E_s'+{\mu_{s*}'}^{-1}\Delta)$ has an lc center (isomorphic to W_0) properly contained in E_s'.

So if we extract E_s to get a model $\mu_s\colon Y_s\to X$, such that $-(K_{Y_s}+E_s+\mu_{s*}^{-1}\Delta)$ is ample, then by the negativity lemma, $(Y_s,E_s+\mu_{s*}^{-1}\Delta)$ is not plt, i.e., E_s is not a Kollár component. $\square$

The following result also illuminates the situation when E_s is a Kollár component for $s\in(0,1)$ by taking affine cone over a klt log Fano pair with complement containing only two lc places.

Theorem 3.3. *Let (X,Δ) be a klt log Fano pair. Let D be a $\mathbb{Q}$-complement of (X,Δ) such that $\mathcal{D}(X,\Delta+D)$ contains only two isolated points E_1 and E_2. Then $\mathrm{Aut}^0(X,\Delta+D)\cong\mathbb{G}_m$, and both E_1 and E_2 induce product test configurations of $(X,\Delta+D)$ that are opposite to each other up to positive scaling.*

Proof. Let $Y\to X$ be the extraction of E_1 and E_2 such that each E_i is anti-ample over X. Let $(\mathcal{X},\Delta_{\mathcal{X}}+\mathcal{D}_{\mathcal{X}})\to\mathbb{A}^1_s$ be the weakly special test configuration induced by E_1 in the sense of boundary polarized CY pairs as in [1,4]. Thus $\mathcal{X} \to \mathbb{A}^1$ is of Fano type. Let $\mathcal{E}_i$ $(i = 1, 2)$ be the divisor over $\mathcal{X}$ corresponding to $E_i \times \mathbb{A}^1_s$. Let $\mathcal{Y} \to \mathcal{X}$ be the extraction of $\mathcal{E}_1$ and $\mathcal{E}_2$ such that each $\mathcal{E}_i$ is anti-ample over $\mathcal{X}$. We claim that $(\mathcal{Y}, \Delta_{\mathcal{Y}} + \mathcal{D}_{\mathcal{Y}} + \mathcal{Y}_0 + \mathcal{E}_1 + \mathcal{E}_2)$ is dlt. Clearly it is log canonical as it is crepant to $(\mathcal{X}, \Delta_{\mathcal{X}} + \mathcal{D}_{\mathcal{X}} + \mathcal{X}_0)$ which is log canonical. In addition, it is plt away from $\mathcal{Y}_0$ as it is isomorphic to $(Y, \Delta_Y + \mathcal{D}_Y + E_1 + E_2) \times (\mathbb{A}^1 \setminus \{0\})$. Denote by $\mathcal{E}_{i,0} := \mathcal{E}_i|_{\mathcal{Y}_0}$. Then each $\mathcal{E}_{i,0}$ is connected as the general fiber of $\mathcal{E}_i \to \mathbb{A}^1$ is connected. Since $(\mathcal{Y}_0, \Delta_{\mathcal{Y}_0} + \mathcal{D}_{\mathcal{Y}_0} + \mathcal{E}_{1,0} + \mathcal{E}_{2,0})$ is crepant birational to $(\mathcal{X}_0, \Delta_{\mathcal{X}_0} + \mathcal{D}_{\mathcal{X}_0})$, we know that $\mathcal{E}_{i,0}$ is reduced whose irreducible components are lc places of $(\mathcal{X}_0, \Delta_{\mathcal{X}_0} + \mathcal{D}_{\mathcal{X}_0})$. By [1, Proposition 8.7], we know that $\mathcal{D}(\mathcal{X}_0, \Delta_{\mathcal{X}_0} + \mathcal{D}_{\mathcal{X}_0})$ has dimension 0. Thus $\mathcal{E}_{1,0}$ and $\mathcal{E}_{2,0}$ are disjoint prime divisors. By [12, Proposition 4.37] we know that $\mathcal{E}_{1,0}$ and $\mathcal{E}_{2,0}$ are the only lc places of $(\mathcal{X}_0, \Delta_{\mathcal{X}_0} + \mathcal{D}_{\mathcal{X}_0})$. By inversion of adjunction, we know that $\mathcal{E}_{1,0}$ and $\mathcal{E}_{2,0}$ are the only minimal lc centers of $(\mathcal{Y}, \Delta_{\mathcal{Y}} + \mathcal{D}_{\mathcal{Y}} + \mathcal{Y}_0 + \mathcal{E}_1 + \mathcal{E}_2)$. Since $\mathcal{Y}_0$ is regular at the generic point $\eta_{i,0}$ of $\mathcal{E}_{i,0}$, we have that $\mathcal{Y}$ is regular at $\eta_{i,0}$ as $\mathcal{Y}_0$ is Cartier in $\mathcal{Y}$. As a result, $(\mathcal{Y}, \Delta_{\mathcal{Y}} + \mathcal{D}_{\mathcal{Y}} + \mathcal{Y}_0 + \mathcal{E}_1 + \mathcal{E}_2)$ is dlt at $\eta_{i,0}$ which implies that it is dlt everywhere.

Next, we show that $(\mathcal{X}_0, \Delta_{\mathcal{X}_0})$ is klt. From the above arguments we know that $\mathcal{E}_{1,0}$ and $\mathcal{E}_{2,0}$ are the only lc places of the slc pair $(\mathcal{X}_0, \Delta_{\mathcal{X}_0} + \mathcal{D}_{\mathcal{X}_0})$. Moreover, since $\mathrm{ord}_{E_i}(D) > 0$, we have $\mathrm{ord}_{\mathcal{E}_i}(\mathcal{D}_{\mathcal{X}}) > 0$ which implies that $\mathrm{ord}_{\mathcal{E}_{i,0}}(\mathcal{D}_{\mathcal{X}_0}) > 0$. Thus $(\mathcal{X}_0, \Delta_{\mathcal{X}_0})$ is klt.

So far, we have shown that E_1 is a special divisor over (X, Δ). By symmetry, so is E_2. Moreover, since $(\mathcal{X}_0, \Delta_{\mathcal{X}_0} + \mathcal{D}_0)$ admits a $\mathbb{G}_m$-action, we know that each $\mathcal{E}_{i,0}$ induces a product test configuration of $(\mathcal{X}_0, \Delta_{\mathcal{X}_0} + \mathcal{D}_0)$ by [5] (see also [1, Theorem 4.8]). In particular, $\mathcal{E}_{2,0}$ is a special divisor over $(\mathcal{X}_0, \Delta_{\mathcal{X}_0})$. Thus $\mathcal{E}_2$ provides a family of special divisors over $(\mathcal{X}, \Delta_{\mathcal{X}})$. By the proof of [21, Proposition 4.5] (also see [19, Theorem 5.7]), there exists a family of special test configurations $(\mathfrak{X}, \Delta_{\mathfrak{X}}) \to \mathbb{A}^2_{s,t}$ of $\mathcal{X}$ induced by $\mathcal{E}_2$ where

$$(\mathfrak{X}, \Delta_{\mathfrak{X}}) \times_{\mathbb{A}^2} (\mathbb{A}^2 \setminus (t = 0)) \cong (\mathcal{X}, \Delta_{\mathcal{X}}) \times (\mathbb{A}^1_t \setminus \{0\}).$$

In particular, $(\mathfrak{X}_{(0,0)}, \Delta_{\mathfrak{X}_{(0,0)}})$ is a klt log Fano pair. Denote by $\mathfrak{D}_{\mathfrak{X}}$ the closure of $\mathcal{D} \times (\mathbb{A}^1_t \setminus \{0\})$ in $\mathfrak{X}$. Then by [1, Theorem 6.3] we know that $(\mathfrak{X}, \Delta_{\mathfrak{X}} + \mathfrak{D}_{\mathfrak{X}}) \to \mathbb{A}^2$ is a family of boundary polarized CY pairs. Since $\mathcal{E}_2$ is $\mathbb{G}_m$-equivariant for the natural $\mathbb{G}_m$-action on $\mathcal{X}$, we know that $\mathfrak{X} \to \mathbb{A}^2_{s,t}$ is $\mathbb{G}^2_m$-equivariant with the standard $\mathbb{G}^2_m$-action on $\mathbb{A}^2$. Moreover, the above arguments implies that the central fiber $(\mathfrak{X}_{(0,0)}, \Delta_{\mathfrak{X}_{(0,0)}} + \mathfrak{D}_{\mathfrak{X}_{(0,0)}})$ has only two lc places $\mathfrak{E}_{1,(0,0)}$ and $\mathfrak{E}_{2,(0,0)}$ which induce 1-PS's of $\mathbb{G}^2_m$ of weights $(1,0)$ and $(0,1)$ respectively. On the other hand, by [1, Proof of Proposition 8.11] we know that

$$\mathrm{Aut}^0(\mathfrak{X}_{(0,0)}, \Delta_{\mathfrak{X}_{(0,0)}} + \mathfrak{D}_{\mathfrak{X}_{(0,0)}}) \cong \mathbb{G}_m.$$

As a result, we know that there exists a non-trivial 1-PS $\sigma : \mathbb{G}_m \to \mathbb{G}^2_m$ of weight (a, b) such that σ acts trivially on $\mathfrak{X}_{(0,0)}$. Since the 1-PS's of weights $(1,0)$ and $(0,1)$ are induced by valuations with different centers, we know that a and b have the same sign, and we may assume $a > 0$ and $b > 0$. Thus by pulling back $\mathfrak{X} \to \mathbb{A}^2$ under σ we obtain a test configuration of $(X, \Delta + D)$ whose $\mathbb{G}_m$-action on the central fiber $(\mathfrak{X}_{(0,0)}, \Delta_{\mathfrak{X}_{(0,0)}} + \mathfrak{D}_{\mathfrak{X}_{(0,0)}})$ is trivial. Therefore, we must have

$$(X, \Delta + D) \cong (\mathfrak{X}_{(0,0)}, \Delta_{\mathfrak{X}_{(0,0)}} + \mathfrak{D}_{\mathfrak{X}_{(0,0)}}),$$

which implies that $\mathrm{Aut}^0(X, \Delta + D) \cong \mathbb{G}_m$, and the proof is finished by [5]. □

Example 3.4. Let $V = \mathbb{P}^2 = \mathbb{P}(x, y, z)$ and D_V be the nodal cubic $D_V = (zx^2 + zy^2 + y^3 = 0)$. The lc places of (V, D_V) is a cone over a circle. Let u_t $(0 < t < +\infty)$ be the quasi-monomial valuations with weight $(1, t)$ over the two branches of D_V at $[0 : 0 : 1]$. Then [16, Section 6] shows that u_t is a special valuation, i.e., $\mathrm{Proj}\,(\mathrm{Gr}_{u_t} k[x, y, z])$ is a klt Fano variety if and only if

$$t \in \left(\frac{7 - 3\sqrt{5}}{2}, \frac{7 + 3\sqrt{5}}{2}\right).$$

Now we consider the affine cone $(\mathbb{A}^3, D)$ of (V, D_V) with polarization $\mathcal{O}_V(1)$, so $o \in \mathbb{A}^3$ is the origin, and D is the divisor which is the cone over D_V whose affine equation is given by $D = (zx^2 + zy^2 + y^3 = 0) \subset \mathbb{A}^3$. Then the dense open subcomplex $\mathcal{D}(\mathbb{A}^3, D)^\circ \subset \mathcal{D}(\mathbb{A}^3, D)$ consisting of valuations centered at o is of the form $v_{s,t} = (\mathrm{ord}_o, s \cdot u_{\frac{t}{s}})$, where $s, t \in [0, +\infty)$ with $v_{s,0}$ and $v_{0,s}$ glued (corresponding to $(\mathrm{ord}_o, s \cdot \mathrm{ord}_D)$). By writing $v = (\mathrm{ord}_o, u)$ for $u \in \mathrm{Val}_V$ we mean that for $g = \sum_{m \geqslant 0} g_m \in \mathcal{O}_{\mathbb{A}^3,o}$ with g_m the homogeneous component of degree m of g, we have

$$v(g) := \min\{m + u(g_m) \mid g_m \neq 0\}.$$

From this expression, we know that $v_{s,t}$ is a quasi-monomial valuations with weights $(1, s, t)$ in the blow-up of $\mathbb{A}^3$ at o with respect to the exceptional divisor and the two branches of the strict transform of D. Let $\mathfrak{m}$ be the maximal ideal corresponding to o. So for $\mathfrak{a} = (f, \mathfrak{m}^\ell)$ with $f = zx^2 + zy^2 + y^3$ and $\ell \in \mathbb{N}$, we have

$$v_{s,t}(\mathfrak{a}) = \begin{cases} v_{s,t}(f) = 3 + s + t & \text{if } \ 3 + s + t \leqslant \ell, \\ \ell & \text{if } \ 3 + s + t \geqslant \ell, \end{cases}$$

and the log discrepancy $A_{\mathbb{A}^3}(v_{s,t}) = 3 + s + t$. For any fixed $\ell > 3$, we choose a set of general generators $h_1, h_2, \ldots, h_m \in \mathfrak{a}$, and let $D' := \frac{1}{m}(H_1 + H_2 + \cdots + H_m)$ with $H_i := (h_i = 0)$. Then the dual complex

$$\mathcal{D}(\mathbb{A}^3, D') = \{v_{s,t} \in \mathcal{D}(\mathbb{A}^3, D)^\circ \mid s, t \geqslant 0 \text{ and } s + t \leqslant \ell - 3\}.$$

Moreover, the set of Kollár valuations $\mathcal{D}^{\mathrm{KV}}(\mathbb{A}^3, D')$ consist of $\mathrm{ord}_o = v_{0,0}$ and $v_{s,t}$ with

$$s, t > 0, \quad \frac{t}{s} \in \left(\frac{7 - 3\sqrt{5}}{2}, \frac{7 + 3\sqrt{5}}{2}\right) \text{ and } \quad s + t \leqslant \ell - 3.$$

As a result, we see that $\mathcal{D}^{\mathrm{KV}}(\mathbb{A}^3, D')$ is neither open in $\mathcal{D}(\mathbb{A}^3, D')$, nor a finite union of open simplicies in any rational triangulation of $\mathcal{D}(\mathbb{A}^3, D')$. On the other hand, it is not hard to see that Conjecture 1.8 holds in this example.

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References

1 Ascher K, Bejleri D, Blum H, et al. Moduli of boundary polarized Calabi-Yau pairs. arXiv:2307.06522, 2023
2 Birkar C, Cascini P, Hacon C D, et al. Existence of minimal models for varieties of log general type. J Amer Math Soc, 2010, 23: 405–468
3 Blum H. Existence of valuations with smallest normalized volume. Compos Math, 2018, 154: 820–849
4 Blum H, Liu Y, Xu C. Openness of K-semistability for Fano varieties. Duke Math J, 2022, 171: 2753–2797
5 Chen G, Zhou C. Weakly special test configurations of log canonical Fano varieties. Algebra Number Theory, 2022, 16: 2415–2432
6 de Fernex T, Kollár J, Xu C. The dual complex of singularities. In: Higher Dimensional Algebraic Geometry—In Honour of Professor Yujiro Kawamata's Sixtieth Birthday. Advanced Studies in Pure Mathematics, vol. 74. Tokyo: Math Soc Japan, 2017, 103–129
7 Hartshorne R. Complete intersections and connectedness. Amer J Math, 1962, 84: 497–508
8 Han J, Liu Y, Qi L. ACC for local volumes and boundedness of singularities. J Algebraic Geom, 2023, 32: 519–583
9 Hacon C D, McKernan J, Xu, C. ACC for log canonical thresholds. Ann of Math (2), 2014, 180: 523–571
10 Kollár J, Kovács S J. Log canonical singularities are Du Bois. J Amer Math Soc, 2010, 23: 791–813
11 Kollár J, Mori S. Birational Geometry of Algebraic Varieties. Cambridge Tracts in Mathematics, vol. 134. Cambridge: Cambridge Univ Press, 1998
12 Kollár J. Singularities of the Minimal Model Program. Cambridge Tracts in Mathematics, vol. 200. Cambridge: Cambridge Univ Press, 2013
13 Li C, Liu Y, Xu C. A guided tour to normalized volume. In: Geometric Analysis—In Honor of Gang Tian's 60th Birthday. Progress in Mathematics, vol. 333. Cham: Birkhäuser/Springer, 2020, 167–219
14 Li C, Xu C. Stability of valuations: Higher rational rank. Peking Math J, 2018, 1: 1–79
15 Li C, Xu C. Stability of valuations and Kollár components. J Eur Math Soc (JEMS), 2020, 22: 2573–2627

16 Liu Y, Xu C, Zhuang Z. Finite generation for valuations computing stability thresholds and applications to K-stability. Ann of Math (2), 2022, 196: 507–566
17 Xu C. Finiteness of algebraic fundamental groups. Compos Math, 2014, 150: 409–414
18 Xu C. A minimizing valuation is quasi-monomial. Ann of Math (2), 2020, 191: 1003–1030
19 Xu C. K-stability of Fano varieties. Http://web.math.princeton.edu/∼chenyang/Kstabilitybook.pdf, 2024
20 Xu C, Zhuang Z. Uniqueness of the minimizer of the normalized volume function. Camb J Math, 2021, 9: 149–176
21 Xu C, Zhuang Z. Stable degenerations of singularities. arXiv:2205.10915, 2022
22 Zhuang Z. On boundedness of singularities and minimal log discrepancies of Kollár components, II. Geom Topol, to appear. arXiv:2302.03841, 2023
23 Zhuang Z. On boundedness of singularities and minimal log discrepancies of Kollár components. J Algebraic Geom, 2024, 33: 521–565