

Wall crossing for K-moduli spaces of plane curves

Kenneth Ascher¹ | Kristin DeVleming²  | Yuchen Liu³

¹Department of Mathematics, University of California, Irvine, California, USA

²Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts, USA

³Department of Mathematics, Northwestern University, Evanston, Illinois, USA

Correspondence

Kristin DeVleming, Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-9305, USA.

Email: kdevleming@umass.edu

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Abstract

We construct proper good moduli spaces parametrizing K-polystable $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs (X, cD) , where X is a Fano variety and D is a rational multiple of the anticanonical divisor. We then establish a wall-crossing framework of these K-moduli spaces as c varies. The main application in this paper is the case of plane curves of degree $d \geq 4$ as boundary divisors of $\mathbb{P}^2$. In this case, we show that when the coefficient c is small, the K-moduli space of these pairs is isomorphic to the GIT moduli space. We then show that the first wall crossing of these K-moduli spaces are weighted blow-ups of Kirwan type. We also describe all wall crossings for degree 4, 5, 6 and relate the final K-moduli spaces to Hacking's compactification and the moduli of K3 surfaces.

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1 | INTRODUCTION

Constructing compactifications of moduli spaces of varieties is a fundamental question in algebraic geometry. While it is well known that singular varieties will occur on the boundary of the moduli space, an interesting question to investigate is how different compactifications yield different singularities on the boundary. The classical approach of Deligne and Mumford for genus ≥ 2 curves only allows stable curves (i.e., nodal singularities) in the boundary. Later on, it was realized that there are meaningful alternate compactifications allowing curves with worse singularities that can be obtained from running the minimal model program (MMP) on the Deligne–Mumford compactification, often going by the name the *Hassett–Keel program*.

If we consider plane curves, there are two well-known compactifications of the moduli space. Classically, Mumford’s geometric invariant theory (GIT) yields a projective variety $\overline{P}_d^{\text{GIT}}$ parametrizing S-equivalent classes of GIT semistable plane curves of degree d . Moreover, in each S-equivalence class, there exists a unique closed orbit whose representative is a GIT polystable plane curve. In the philosophy of Alper, this gives a *good* moduli space morphism $\overline{P}_d^{\text{GIT}} \rightarrow \overline{P}_d^{\text{GIT}}$ where $\overline{P}_d^{\text{GIT}}$ is the GIT quotient stack. An approach due to Hacking (following ideas from Kollár–Shepherd–Barron and Alexeev) is via *stable pairs* and the MMP [49, 50]. Roughly speaking, each smooth plane curve C of degree $d \geq 4$ can be viewed as a boundary divisor in $\mathbb{P}^2$, and so, one can study certain pairs of degenerations of the plane with a curve subject to some conditions on positivity and singularities of the pair coming from the MMP. Hacking’s moduli stack $\overline{P}_d^{\text{H}}$ is a proper Deligne–Mumford stack whose coarse moduli space $\overline{P}_d^{\text{H}}$ is a projective variety.

One notable feature of these two different approaches is that although GIT semistable curves all lie in $\mathbb{P}^2$, they can be quite singular; on the other hand, although the surfaces in the boundary of $\overline{P}_d^{\text{H}}$ can be quite singular (possibly nonnormal), the singularities of the degenerate curve are usually mild. As these two moduli spaces are birational, it is a natural question to ask how to interpolate between them.

For example, we illustrate the simplest nontrivial case: $\deg d = 4$. The six-dimensional GIT quotient $\overline{P}_4^{\text{GIT}}$ generically parametrizes curves that are at worst cuspidal in $\mathbb{P}^2$, has a curve parametrizing tacnodal curves (i.e., locally $y^2 + x^4 = 0$), and a point parametrizing the double conic. On the other hand, Hacking’s space $\overline{P}_4^{\text{H}}$ (see also [52]) also generically parametrizes cuspidal curves in $\mathbb{P}^2$, but has a divisor parametrizing curves on $\mathbb{P}(1, 1, 4)$ (which are at worst cuspidal away from the singular point, and at worst nodal at the singularity), and has a codimension two locus parametrizing curves on the nonnormal surface $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$. Here, the curves are snc at the double locus and at worst cuspidal elsewhere. In particular, one can see directly the trade-off between having very singular curves that still are in $\mathbb{P}^2$, and having mildly singular curves on singular surfaces. It is natural to ask how to relate the two spaces in a modular way.

In this article, we investigate a new family of compactifications of the moduli space of smooth plane curves using K-stability and (conical) Kähler–Einstein metrics. For any smooth plane curve C of degree $d \geq 4$, the celebrated work [28] and [120] implies that $(\mathbb{P}^2, cC)$ admits a conical Kähler–Einstein metric for any $0 < c < \frac{3}{d}$ hence is K-stable. Thus, it is natural to construct K-stability compactifications of these moduli spaces. Recently, Li, Wang, and Xu [92] showed that there exist proper good moduli spaces parametrizing K-polystable $\mathbb{Q}$ -Gorenstein smoothable Fano varieties (see also [104] and [115]). Based on [92] and the very recent work by Tian and Wang [121] on the solution of the log smooth Yau–Tian–Donaldson conjecture, we construct K-moduli stacks and spaces for all $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs (see Definition 2.21). In particular, this

implies that the K-stability compactification of the moduli space of log Fano pairs $(\mathbb{P}^2, cC)$ with $0 < c < \frac{3}{d}$ a rational number exists as a proper good moduli space.

Theorem 1.1 (= Theorem 3.1). *Let χ_0 be the Hilbert polynomial of an anticanonically polarized Fano manifold. Fix $r \in \mathbb{Q}_{>0}$ and a rational number $c \in (0, \min\{1, r^{-1}\})$. Then there exists a reduced Artin stack $\mathcal{KM}_{\chi_0, r, c}$ of finite type over $\mathbb{C}$ parametrizing all K-semistable $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs (X, cD) with Hilbert polynomial $\chi(X, \mathcal{O}_X(-mK_X)) = \chi_0(m)$ for sufficiently divisible m and $D \sim_{\mathbb{Q}} -rK_X$. Moreover, the Artin stack $\mathcal{KM}_{\chi_0, r, c}$ admits a good moduli space $KM_{\chi_0, r, c}$ as a proper reduced scheme of finite type over $\mathbb{C}$ whose closed points parametrize K-polystable pairs.*

It is thus natural to ask how the moduli spaces depend on the coefficient c . In this setting, we prove the following wall-crossing type result. In the statement below, VGIT refers to variation of GIT as in [6] and CM stands for Chow-Mumford (see Definition 2.17).

Theorem 1.2 (= Theorem 3.2). *There exist rational numbers*

$$0 = c_0 < c_1 < c_2 < \cdots < c_k = \min\{1, r^{-1}\}$$

such that c -K-(poly/semi)stability conditions do not change for $c \in (c_i, c_{i+1})$. For each $1 \leq i \leq k-1$ and $0 < \epsilon \ll 1$, we have open immersions

$$\mathcal{KM}_{\chi_0, r, c_i - \epsilon} \hookrightarrow \Phi_i^- \mathcal{KM}_{\chi_0, r, c_i} \hookrightarrow \Phi_i^+ \mathcal{KM}_{\chi_0, r, c_i + \epsilon},$$

which induce projective morphisms

$$KM_{\chi_0, r, c_i - \epsilon} \xrightarrow{\phi_i^-} KM_{\chi_0, r, c_i} \xleftarrow{\phi_i^+} KM_{\chi_0, r, c_i + \epsilon}.$$

Moreover, all the above wall crossing morphisms have local VGIT presentations as in [6, (1.2)], and the CM $\mathbb{Q}$ -line bundles on $KM_{\chi_0, r, c_i \pm \epsilon}$ are $\phi_i^{\pm}$ -ample.

While the two above results hold for any K-moduli stack and space of $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs, the remainder of this paper will focus on $\overline{\mathcal{P}}_{d, c}^K$ and $\overline{\mathcal{P}}_{d, c}^K$, that is, the K-moduli stack and space parametrizing K-semistable and K-polystable limits of $(\mathbb{P}^2, cC)$, respectively, where C is a smooth plane curve of degree d and $c \in (0, \frac{3}{d})$ is a rational number.

Since K-semistable log Fano pairs are always Kawamata log terminal (klt) by [102], as c increases the surface X in $\overline{\mathcal{P}}_{d, c}^K$ must become more singular, while the divisor D becomes less singular; this is a general version of the phenomenon seen in the degree 4 example. In particular, it is reasonable to expect that these moduli spaces provide the proper framework for interpolating between $\overline{\mathcal{P}}_d^{\text{GIT}}$ and $\overline{\mathcal{P}}_d^{\text{H}}$. Our next results characterize the behavior of the wall crossings for K-moduli spaces of plane curves.

First, we give a complete understanding of the first wall crossing in all degrees. The K-moduli space corresponding to $0 < c \ll 1$ is isomorphic to the GIT moduli space, and the first wall crossing is a Kirwan-type blow-up of the GIT quotient. Note that since the K-moduli stacks and spaces for $d \leq 3$ are well known (see Example 4.5), we usually assume $d \geq 4$.

Theorem 1.3 (First wall crossing). *Let $d \geq 4$ be an integer, $c \in (0, \frac{3}{d})$ be a rational number, let Q be a smooth conic in $\mathbb{P}^2$, let L be a line in $\mathbb{P}^2$ transverse to Q , and let x, y, z be coordinates of $\mathbb{P}(1, 1, 4)$. Let*

$$c_1 = \begin{cases} \frac{3}{2d} & d \text{ is even} \\ \frac{3}{2d-3} & d \text{ is odd} \end{cases} \quad Q_d = \begin{cases} \frac{d}{2}Q & d \text{ is even} \\ \frac{d-1}{2}Q + L & d \text{ is odd} \end{cases} \quad Q'_d = \begin{cases} z^{d/2} = 0 & d \text{ is even} \\ xyz^{(d-1)/2} = 0 & d \text{ is odd} \end{cases}$$

- (1) *For any $0 < c < c_1$, a plane curve C of degree d is GIT (poly/semi)stable if and only if the log Fano pair $(\mathbb{P}^2, cC)$ is K-(poly/semi)stable. Moreover, there is an isomorphism of Artin stacks $\overline{\mathcal{P}}_{d,c}^K \cong \overline{\mathcal{P}}_d^{\text{GIT}}$.*
- (2) *There is an open immersion $\Phi^- : \overline{\mathcal{P}}_d^{\text{GIT}} = \overline{\mathcal{P}}_{d,c_1-\epsilon}^K \hookrightarrow \overline{\mathcal{P}}_{d,c_1}^K$ that descends to an isomorphism of good moduli spaces.*
- (3) *If $\Phi^+ : \overline{\mathcal{P}}_{d,c_1+\epsilon}^K \rightarrow \overline{\mathcal{P}}_{d,c_1}^K$ denotes the latter morphism in the first wall crossing, then there exists a stacky weighted blow-up morphism $\rho : \overline{\mathcal{P}}_{d,c_1+\epsilon}^K \rightarrow \overline{\mathcal{P}}_{d,c_1-\epsilon}^K = \overline{\mathcal{P}}_d^{\text{GIT}}$ along $\{[Q_d]\}$ (see Definition 5.10) such that $\Phi^+ = \Phi^- \circ \rho$. In particular, we have*
 - (a) *The descent morphism $\varphi = (\phi^-)^{-1} \circ \phi^+ : \overline{\mathcal{P}}_{d,c_1+\epsilon}^K \rightarrow \overline{\mathcal{P}}_{d,c_1-\epsilon}^K = \overline{\mathcal{P}}_d^{\text{GIT}}$ of ρ between good moduli spaces is a weighted blow-up of the point $[Q_d]$.*
 - (b) *If d is even, then φ is a partial desingularization of Kirwan type.*

Before preceding, we note that Gallardo, Martinez-Garcia, and Spotti independently showed in [47, Theorem 1.2] that a similar result to Theorem 1.3 (1) holds for all hypersurfaces in $\mathbb{P}^n$ assuming the Gap Conjecture [114, Conjecture 5.5] that is true when $n \leq 3$ by [82, Proposition 4.10] and [96, Theorem 1.3]. The following result removes this assumption.

Theorem 1.4 (= Theorem 9.24). *Let n and $d \geq 2$ be positive integers. Then there exists a positive rational number $c_1 = c_1(n, d)$ such that for any fixed $0 < c < c_1$, a hypersurface $S \subset \mathbb{P}^n$ of degree d is GIT (poly/semi)stable if and only if the log Fano pair $(\mathbb{P}^n, cS)$ is K-(poly/semi)stable.*

It is natural to ask what happens beyond the first wall crossing for plane curves. When d is small ($d \leq 6$), we explicitly determine all K-moduli wall crossings. In fact, if d is 4 or 6, we can relate our moduli spaces to Baily–Borel compactifications of moduli spaces of K3 surfaces (see Section 6). Let $\overline{\mathcal{P}}_4^*$ denote the Baily–Borel compactification of the moduli space of ADE K3 surfaces of degree 4 with $\mathbb{Z}/4\mathbb{Z}$ symmetry constructed by Kondō [74], and let $\overline{\mathcal{P}}_6^*$ denote the Baily–Borel compactification of the moduli space of K3 surfaces of degree 2.

Theorem 1.5 ($d = 4, 6$, see Theorem 6.1 and Section 6.2). *If $d = 4$ (resp. 6), there is only one wall crossing for K-moduli spaces given by the weighted blow-up $\widehat{\mathcal{P}}_4^{\text{GIT}} \rightarrow \overline{\mathcal{P}}_4^{\text{GIT}}$ (resp. $\widehat{\mathcal{P}}_6^{\text{GIT}} \rightarrow \overline{\mathcal{P}}_6^{\text{GIT}}$) at the double conic (resp. triple conic). Furthermore, the ample model of the Hodge line bundle on $\widehat{\mathcal{P}}_4^{\text{GIT}}$ (resp. $\widehat{\mathcal{P}}_6^{\text{GIT}}$) is $\overline{\mathcal{P}}_4^*$ (resp. $\overline{\mathcal{P}}_6^*$).*

In fact, in the degree 4 case, we can say more using Hyeon and Lee’s results on the log MMP for moduli of genus three curves [54]; see Section 9.3.1.

TABLE 1 Wall crossings for K-moduli spaces of plane quintics

i	c_i	E_i^-	E_i^+
1	$\frac{3}{7}$	$(\mathbb{P}^2, Q_5)$	$(\mathbb{P}(1, 1, 4), (xyz^2 + (ax^6 + by^6)z + g(x, y) = 0))$
2	$\frac{8}{15}$	$(\mathbb{P}^2, A_{12}\text{-quintic})$	$(X_{26}, (w = g(x, y)))$
3	$\frac{6}{11}$	$(\mathbb{P}^2, A_{11}\text{-reducible quintics})$	$(\mathbb{P}(1, 1, 4), (x^2z^2 + y^6z + g(x, y) = 0))$
4	$\frac{63}{115}$	$(\mathbb{P}^2, A_{11}\text{-irreducible quintics})$	$(\mathbb{P}(1, 4, 25), (z^2 + x^2y^{12} + x^{10}g(x, y) = 0))$
5	$\frac{54}{95}$	$(\mathbb{P}^2, A_{10}\text{-quintics})$	$(\mathbb{P}(1, 4, 25), (z^2 + x^6y^{11} + x^{14}g(x, y) = 0))$

Theorem 1.6 ($d = 5$, see Theorems 5.9, 7.1 and Section 8). *If $d = 5$, then there are five wall crossings for K-moduli spaces of plane quintics. Among them, the first two are weighted blow-ups, while the last three are flips.*

In Table 1, we summarize the behavior of all wall crossings for plane quintics (see also Figure 1 on Page 60). Here, we denote by $E_i^\pm$ the exceptional loci of $\phi_i^\pm : \overline{P}_{5, c_i \pm \epsilon}^K \rightarrow \overline{P}_{5, c_i}^K$ where general pairs parametrized by them are described in the table. The full description of $E_i^\pm$ will be presented in Theorems 5.8, 7.1, and Section 8. We use $[x, y, z]$ for projective coordinates of weighted projective planes $\mathbb{P}(1, 1, 4)$ and $\mathbb{P}(1, 4, 25)$. The surface X_{26} is the degree 26 weighted hypersurface $(xw = y^{13} + z^2)$ in $\mathbb{P}(1, 2, 13, 25)$ with projective coordinates $[x, y, z, w]$.

In general, it is expected that K-moduli spaces are projective with ample CM line bundles (see [27, 29]). Using recent work of Codogni and Patakfalvi [29] and Posva [107], we show the following.

Theorem 1.7 (= Theorem 9.1). *The K-moduli spaces $\overline{P}_{d, c}^K$ are projective when $d \in \{4, 5, 6\}$ with ample CM line bundles.*

During the review process of this paper, we learned that the ampleness of CM line bundles on K-moduli spaces of log Fano pairs is proved in [98, 125] using purely algebraic methods (see Remark 1.9). In particular, the CM line bundle is ample on $\overline{P}_{d, c}^K$ for all degrees and all coefficients (see Theorem 9.2).

The remainder of our paper is devoted to some further questions that will serve as motivation for our future work. For example, in Section 7.3, we discuss the second weighted blow-up for $d \geq 7$ (see Theorem 7.10). Another consequence of our work is that the birational maps $\overline{P}_{d, c'}^K \dashrightarrow \overline{P}_{d, c}^K$ are birational contractions for $0 < c < c' < \frac{3}{d}$ whenever $3 \mid d$ or $d < 13$ (see Theorem 9.5). If this is true for $3 \nmid d$ and $d \geq 13$, then together with the ampleness of the CM line bundle (Theorem 9.2), this would imply that the wall crossing of K-moduli spaces exhibit similar behavior to the Hassett–Keel program for $\overline{M}_g$ (see Theorem 9.4).

We show in Theorem 9.18 that the only difference between the K-moduli space $\overline{P}_{d, \frac{3}{d} - \epsilon}^K$ and $\overline{P}_d^H$ are the maximally lc pairs in the K-moduli space, and the nonnormal pairs in the Hacking moduli space. We conjecture that there is a proper good moduli space of log Calabi–Yau pairs, which relates to the K-moduli and Hacking moduli spaces via the following wall crossing.

Conjecture 1.8 (Log Calabi–Yau wall crossings, see Conjecture 9.19). *There exists a proper good moduli space $\overline{P}_d^{\text{CY}}$ that parametrizes S-equivalence classes of semistable log Calabi–Yau pairs $(X, \frac{3}{d}D)$ where X admits a $\mathbb{Q}$ -Gorenstein smoothing to $\mathbb{P}^2$. Moreover, we have a log Calabi–Yau wall*

crossing diagram

$$\overline{P}_{d, \frac{3}{d}-\epsilon}^K \xrightarrow{\phi_{CY}^-} \overline{P}_d^{CY} \xleftarrow{\phi_{CY}^+} \overline{P}_d^H$$

where $\overline{P}_d^{CY}$ is the common ample model of the Hodge line bundles on $\overline{P}_{d, \frac{3}{d}-\epsilon}^K$ and $\overline{P}_d^H$.

We partially verify this conjecture in degree 4,5,6, and will investigate this space in forthcoming work.

Remark 1.9 (Postscript). Since the first version of this article appeared on the arXiv, there has been much progress on the study of K-stability and K-moduli spaces of log Fano pairs and wall crossings. We list a few related works below.

- (1) The K-moduli spaces for log Fano pairs are shown to exist as a projective good moduli space where the CM line bundle is ample. This is a combination of many recent works [1, 19, 25, 27, 29, 60, 93, 98, 124–126]. As a result, the construction of our moduli spaces $\mathcal{KM}_{\chi_0, r, c}$ is generalized to all log Fano pairs using purely algebraic methods.
- (2) The Yau–Tian–Donaldson conjecture for all (possibly singular) log Fano pairs is solved as a combination of [14, 80, 91, 98].
- (3) The wall crossing framework from Section 3, except the local VGIT presentation from Section 3.7, is generalized to all log Fano pairs (X, D) satisfying $D \sim_{\mathbb{Q}} -rK_X$ for some $r \in \mathbb{Q}_{>0}$ in [128] using purely algebraic methods.
- (4) The wall crossing framework of this paper has been applied to the study of moduli of quartic K3 surfaces in [3, 4]. In particular, the paper [4] verifies Laza–O’Grady’s conjecture on the Hassett–Keel–Looijenga program for quartic K3 surfaces (see [85–87] for backgrounds).
- (5) For moduli of stable pairs in terms of Kollár–Shepherd–Barron–Alexeev, the wall crossing framework was established in [2]. The main difference of these two wall crossing frameworks is that wall crossing maps in [2] are always morphisms (after normalization), while our wall crossing maps may be flips.

Organization

This paper is organized as follows. In Section 2, we collect preliminary materials on K-stability, normalized volumes, CM line bundles, good moduli spaces, and Hacking’s moduli spaces. In Section 3, we give a detailed construction of K-moduli stacks and spaces of $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs which is largely based on [92] with new inputs from [27, 60, 121]. We prove Theorem 3.1 that is a generalization of [92, Theorem 1.3]. Our main new result is Theorem 3.2 that characterizes fundamental behaviors of K-moduli wall crossings when varying the coefficient. Our construction heavily relies on the solution of Yau–Tian–Donaldson conjecture for log smooth log Fano pairs by Tian and Wang [121] that is a generalization of [28, 120]. Hence, our approach is a mixture of algebraic and analytic methods.

In Section 4, we study the general properties of K-moduli stacks $\overline{P}_{d, c}^K$ and spaces $\overline{P}_{d, c}^K$ of degree d plane curves with coefficient c . We describe the well-known K-moduli stacks and spaces for degree at most 3 in Example 4.5. Using normalized volumes, we prove a result on bounding local

Gorenstein indices of singular surfaces appearing in $\overline{\mathcal{P}}_{d,c}^K$ (see Theorem 4.8). This is crucial in the detailed study of our K-moduli spaces.

Section 5 is devoted to studying the first wall crossing in all degrees. We prove parts (1) and (2) of Theorem 1.3 in Section 5.1 by applying the index bound (Theorem 4.8) and the Paul–Tian criterion (Theorem 2.22). In Section 5.2, we show that the K-moduli stack $\overline{\mathcal{P}}_{d,c_1+\epsilon}^K$ is a weighted blow-up of the GIT moduli stack of Kirwan type, hence confirming part (3) of Theorem 1.3. This is done by a careful analysis of GIT of curves on $\mathbb{P}(1, 1, 4)$ (see Definition 5.7 and Theorem 5.8) and the index bound (Theorem 4.8).

In Section 6, we show that there is only one log Fano K-moduli wall crossing in degree $d = 4$ or 6 (see Theorem 6.1). This is proven by computing the log canonical thresholds of GIT polystable curves on $\mathbb{P}^2$ and $\mathbb{P}(1, 1, 4)$ (see Propositions 6.3 and 6.4) and applying an interpolation result on K-stability (see Proposition 2.13). In Section 6.2, we relate the final K-moduli spaces $\overline{\mathcal{P}}_{d,\frac{3}{d}-\epsilon}^K$ for $d = 4$ or 6 to the Baily–Borel compactification of moduli spaces of K3 surfaces as cyclic covers (see Theorems 6.5 and 6.6).

Sections 7 and 8 are devoted to studying all wall crossings in degree 5. In Section 7, we show that the second wall crossing of plane quintics precisely replaces the plane quintic with a unique A_{12} -singularity by curves on X_{26} (see Theorem 7.1). Its proof involves a valuative criterion computation (see Proposition 7.3), an explicit construction of a special degeneration (see Proposition 7.4), and verifying the K-polystability of this degeneration using techniques of Ilten and Süß [59] on T -varieties of complexity one (see Proposition A.2). In Section 8, we use similar strategy to further study the rest wall crossings of plane quintics where the auxiliary computations are collected in Appendix A. In Section 7.3, we apply these results for quintics to get more information on the second weighted blow-up of K-moduli spaces in higher degrees (see Theorem 7.10).

In Section 9, we discuss further questions regarding our K-moduli spaces. In Section 9.1, we show that $\overline{\mathcal{P}}_{d,c}^K$ is projective for degree $d \in \{4, 5, 6\}$ by proving the ampleness of CM line bundles using work of Codogni and Patakfalvi [29] and Posva [107]. In Section 9.2, we study the question of whether the birational map $\overline{\mathcal{P}}_{d,c'}^K \dashrightarrow \overline{\mathcal{P}}_{d,c}^K$ is a birational contraction when $0 < c < c' < \frac{3}{d}$ (see Question 9.3). We give affirmative answers when $d \leq 13$ or d is divisible by 3 (see Theorem 9.5). In Sections 9.3 and 9.4, we provide evidence supporting Conjecture 9.19 on the log Calabi–Yau wall crossing when $d \in \{4, 5, 6\}$. In degree 4, we relate our wall crossings to the log MMP for $\overline{\mathcal{M}}_3$ (see Section 9.3.1). We give a set-theoretic description of the conjectural log Calabi–Yau moduli spaces of plane quintics in Section 9.4. Finally, we prove Theorem 1.4 in Section 9.5 as an application of our machinery developed in Sections 2 and 3.

2 | PRELIMINARIES

2.1 | K-stability of log Fano pairs

In this section, we give a review of K-stability of log Fano pairs.

Definition 2.1. Let X be a normal variety. Let D be an effective $\mathbb{Q}$ -divisor on X . Then, (X, D) is called a *log pair*. If in addition X is projective and $-(K_X + D)$ is $\mathbb{Q}$ -Cartier ample, then we say that (X, D) is a *log Fano pair*. If a log Fano pair (X, D) is klt, then we say that it is a *klt log Fano pair*. We say that X is a *$\mathbb{Q}$ -Fano variety* if $(X, 0)$ is a klt log Fano pair.

We first recall the definition of a test configuration.

Definition 2.2 [36, 118]. Let X be a projective variety. Let L be an ample line bundle on X .

- (a) A *test configuration* $(\mathcal{X}; \mathcal{L})/\mathbb{A}^1$ of $(X; L)$ consists of the following data:
- a variety $\mathcal{X}$ together with a flat projective morphism $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$;
 - a π -ample line bundle $\mathcal{L}$ on $\mathcal{X}$;
 - a $\mathbb{G}_m$ -action on $(\mathcal{X}; \mathcal{L})$ such that π is $\mathbb{G}_m$ -equivariant with respect to the standard action of $\mathbb{G}_m$ on $\mathbb{A}^1$ via multiplication;
 - $(\mathcal{X} \setminus \mathcal{X}_0; \mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_0})$ is $\mathbb{G}_m$ -equivariantly isomorphic to $(X; L) \times (\mathbb{A}^1 \setminus \{0\})$.
- (b) Let w_m be the weight of the $\mathbb{G}_m$ -action on the determinant line $\det H^0(X_0, L_0^{\otimes m})$, and $N_m := h^0(X, L^{\otimes m})$. Then we have an asymptotic expansion

$$\frac{w_m}{mN_m} = F_0 + m^{-1}F_1 + m^{-2}F_2 + \dots$$

with $F_i \in \mathbb{Q}$. The *generalized Futaki invariant* of $(\mathcal{X}; \mathcal{L})/\mathbb{A}^1$ is defined as $\text{Fut}(\mathcal{X}; \mathcal{L}) = -2F_1$. More precisely, if we write

$$N_m = a_0 m^n + a_1 m^{n-1} + O(m^{n-2}), \quad w_m = b_0 m^{n+1} + b_1 m^n + O(m^{n-1}),$$

$$\text{then } \text{Fut}(\mathcal{X}; \mathcal{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0^2}.$$

Definition 2.3 [36, 77, 94, 105, 118]. Let $(X, D = \sum_{i=1}^k c_i D_i)$ be a projective log pair. Let L be an ample line bundle on X .

- (a) A *test configuration* $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ of $(X, D; L)$ consists of the following data:
- a test configuration $(\mathcal{X}; \mathcal{L})/\mathbb{A}^1$ of $(X; L)$;
 - a formal sum $D = \sum_{i=1}^k c_i D_i$ of codimension one closed integral subschemes D_i of $\mathcal{X}$ such that D_i is the Zariski closure of $D_i \times (\mathbb{A}^1 \setminus \{0\})$ under the identification between $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{A}^1 \setminus \{0\})$.

It is clear that $(D_i; \mathcal{L}|_{D_i})/\mathbb{A}^1$ is a test configuration of $(D_i; L|_{D_i})$.

- (b) For each $1 \leq i \leq k$, let $\tilde{w}_{i,m}$ be the weight of the $\mathbb{G}_m$ -action on the determinant line $\det H^0(D_{i,0}, L_{i,0}^{\otimes m})$, and $\tilde{N}_{i,m} := h^0(D_i, L_i^{\otimes m})$. Then we have an asymptotic expansion

$$\tilde{N}_{i,m} = \tilde{a}_{i,0} m^{n-1} + O(m^{n-2}), \quad \tilde{w}_{i,m} = \tilde{b}_{i,0} m^n + O(m^{n-1}).$$

We define $\tilde{a}_0 = \sum_{i=1}^k c_i \tilde{a}_{i,0}$ and $\tilde{b}_0 = \sum_{i=1}^k c_i \tilde{b}_{i,0}$. The *relative Chow weight* of $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ is defined as $\text{CH}(\mathcal{X}, D; \mathcal{L}) := \frac{a_0 \tilde{b}_0 - b_0 \tilde{a}_0}{a_0^2}$. The *generalized Futaki invariant* of $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ is defined as $\text{Fut}(\mathcal{X}, D; \mathcal{L}) = \text{Fut}(\mathcal{X}; \mathcal{L}) + \text{CH}(\mathcal{X}, D; \mathcal{L})$.

- (c) A test configuration $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ is called a *normal* test configuration if $\mathcal{X}$ is normal. A normal test configuration is called a *product* test configuration if

$$(\mathcal{X}, D; \mathcal{L}) \cong (X \times \mathbb{A}^1, D \times \mathbb{A}^1; \text{pr}_1^* L \otimes \mathcal{O}_{\mathcal{X}}(k\mathcal{X}_0))$$

for some $k \in \mathbb{Z}$. A product test configuration is called a *trivial* test configuration if the above isomorphism is $\mathbb{G}_m$ -equivariant with respect to the trivial $\mathbb{G}_m$ -action on X and the standard $\mathbb{G}_m$ -action on $\mathbb{A}^1$ via multiplication.

- (d) Let (X, D) be a log Fano pair. Let L be an ample line bundle on X such that for some $l \in \mathbb{Q}_{>0}$, we have $L \sim_{\mathbb{Q}} -l(K_X + D)$. Then the log Fano pair (X, D) is said to be:
- (i) *K-semistable* if $\text{Fut}(\mathcal{X}, D; \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ and any $l \in \mathbb{Q}_{>0}$ such that L is Cartier;
 - (ii) *K-stable* if it is K-semistable and $\text{Fut}(\mathcal{X}, D; \mathcal{L}) = 0$ for a normal test configuration $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ if and only if it is a trivial test configuration; and
 - (iii) *K-polystable* if it is K-semistable and $\text{Fut}(\mathcal{X}, D; \mathcal{L}) = 0$ for a normal test configuration $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ if and only if it is a product test configuration.
- (e) Let (X, D) be a klt log Fano pair. Let L be an ample line bundle on X such that $L \sim_{\mathbb{Q}} -l(K_X + D)$ for some $l \in \mathbb{Q}_{>0}$. Then a normal test configuration $(\mathcal{X}, D; \mathcal{L})/\mathbb{A}^1$ is called a *special test configuration* if $\mathcal{L} \sim_{\mathbb{Q}} -l(K_{\mathcal{X}/\mathbb{A}^1} + D)$ and $(\mathcal{X}, D + \mathcal{X}_0)$ is plt. In this case, we say that (X, D) *specially degenerates to* $(\mathcal{X}_0, D_0)$ that is necessarily a klt log Fano pair.

Remark 2.4. We give some useful remarks toward the above definition.

- (1) We provide an intersection formula for the generalized Futaki invariant (cf. [101, 123]). Let (X, D) be a log Fano pair. Let L be an ample line bundle on X such that $L \sim_{\mathbb{Q}} -l(K_X + D)$ for some $l \in \mathbb{Q}_{>0}$. Assume $\pi : (\mathcal{X}, D; \mathcal{L}) \rightarrow \mathbb{A}^1$ is a normal test configuration of $(X, D; L)$. Let $\bar{\pi} : (\bar{\mathcal{X}}, \bar{D}; \bar{\mathcal{L}}) \rightarrow \mathbb{P}^1$ be the natural $\mathbb{G}_m$ -equivariant compactification of π . Then we have the intersection formula

$$\text{Fut}(\mathcal{X}, D; \mathcal{L}) := \frac{1}{(-(K_X + D))^n} \left(\frac{n}{n+1} \cdot \frac{(\bar{\mathcal{L}}^{n+1})}{l^{n+1}} + \frac{(\bar{\mathcal{L}}^n \cdot (K_{\bar{\mathcal{X}}/\mathbb{P}^1} + \bar{D}))}{l^n} \right).$$

- (2) By the work of Odaka [100], any K-semistable log Fano pair is klt. By the work of Li and Xu [94], we know that to test K-(poly/semi)stability of a klt log Fano pair, it suffices to test only on special test configurations.
- (3) A test configuration is called *almost trivial* (resp. *almost product*) if its normalization is trivial (resp. product). By [17, Proposition 3.15], we know that the generalized Futaki invariant never increases under normalization.

Definition 2.5. Let X be a $\mathbb{Q}$ -Fano variety. Let $D \sim_{\mathbb{Q}} -K_X$ be an effective $\mathbb{Q}$ -divisor on X . We say that (X, D) is *K-semistable* if $(X, D; L)$ is K-semistable for some Cartier divisor $L \sim_{\mathbb{Q}} -lK_X$ and some $l \in \mathbb{Z}_{>0}$. From [102], this is equivalent to saying that (X, D) is log canonical.

2.2 | Valuative criteria for K-stability

In this section, we recall the *valuative criteria* for K-stability due to [45, 78] with a slight improvement from [27]. For this, we need to make a few definitions.

Definition 2.6. Let X be a normal variety of dimension n . We say that E is a *prime divisor over* X if E is a divisor on a normal variety Y where $f : Y \rightarrow X$ is a proper birational morphism. Let L be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on X . Take $m \in \mathbb{Z}_{>0}$ such that mL is Cartier and let $x \in \mathbb{R}_{\geq 0}$. If X is projective, we define the *volume* of $L - xE$ on X as

$$\text{vol}_X(L - xE) := \text{vol}_Y(f^*L - xE) = \limsup_{\substack{m \rightarrow \infty \\ mL \text{ is Cartier}}} \frac{h^0(X, \mathcal{O}_X(mL - \lceil mx \rceil E))}{m^n/n!}.$$

Remark 2.7. By [45, Definition 1.1, Remark 1.2], the above lim sup is actually a limit, the function $\text{vol}_X(L - xE)$ is a monotonically decreasing continuous function that vanishes for x sufficiently large, and the definition does not depend on the choice of $f : Y \rightarrow X$.

Definition 2.8. Let (X, D) be a log pair such that $K_X + D$ is $\mathbb{Q}$ -Cartier. Let E be a prime divisor over X . Assume that E is a divisor on Y where $f : Y \rightarrow X$ is a proper birational morphism from a normal variety Y . We define the *log discrepancy* of E with respect to (X, D) as

$$A_{(X,D)}(\text{ord}_E) := 1 + \text{ord}_E(K_Y - f^*(K_X + D)),$$

where ord_E is the divisorial valuation measuring order of vanishing along E . If (X, D) is a log Fano pair, we also define the following functional:

$$S_{(X,D)}(\text{ord}_E) := \frac{1}{\text{vol}_X(-K_X - D)} \int_0^\infty \text{vol}_X(-K_X - D - tE) dt.$$

Sometimes, we also use the notation $A_{(X,D)}(E)$ and $S_{(X,D)}(E)$ for $A_{(X,D)}(\text{ord}_E)$ and $S_{(X,D)}(\text{ord}_E)$, respectively.

The following theorem summarizes the valuative criteria of uniform K-stability [45], K-semistability [45, 78], and K-stability [27]. We will view part (2) of this theorem as an alternative definition of uniform K-stability of log Fano pairs.

Theorem 2.9 [27, 45, 78]. *Let (X, D) be a log Fano pair.*

(1) *(X, D) is K-semistable (resp. K-stable) if and only if for any prime divisor E over X ,*

$$A_{(X,D)}(\text{ord}_E) \geq (\text{resp. } >) S_{(X,D)}(\text{ord}_E).$$

(2) *(X, D) is uniformly K-stable if and only if there exists $\epsilon > 0$ such that*

$$A_{(X,D)}(\text{ord}_E) \geq (1 + \epsilon) S_{(X,D)}(\text{ord}_E)$$

for any prime divisor E over X .

From Theorem 2.9, we see that uniform K-stability implies K-stability for log Fano pairs. Moreover, it follows from a recent result [98, Theorem 1.6] that these two stability notions are indeed equivalent to each other for log Fano pairs.

Definition 2.10 [20, 39]. The *stability threshold* $\delta(X, D)$ of a klt log Fano pair (X, D) is defined as

$$\delta(X, D) := \inf_E \frac{A_{(X,D)}(\text{ord}_E)}{S_{(X,D)}(\text{ord}_E)},$$

where the infimum is taken over all prime divisors E over X .

Theorem 2.11 [20, 39]. *A klt log Fano pair (X, D) is K-semistable (resp. uniformly K-stable) if and only if $\delta(X, D) \geq 1$ (resp. > 1).*

Definition 2.12. Let X be a $\mathbb{Q}$ -Fano variety. Let $D \sim_{\mathbb{Q}} -rK_X$ be an effective $\mathbb{Q}$ -divisor. For a rational number $0 < c < r^{-1}$, we say that (X, D) is c - K -(poly/semi)stable (resp. uniformly c - K -stable) if (X, cD) is K -(poly/semi)stable (resp. uniformly K -stable).

Next we provide a useful result on interpolation of K -stability (see, e.g., [31, Lemma 2.6]). These kinds of interpolation results were known before in the smooth case via analytic arguments (see, e.g., [90]).

Proposition 2.13. Let X be a $\mathbb{Q}$ -Fano variety. Let D and Δ be effective $\mathbb{Q}$ -divisors on X satisfying the following properties.

- Both D and Δ are proportional to $-K_X$ under $\mathbb{Q}$ -linear equivalence.
- $-K_X - D$ is ample, and $-K_X - \Delta$ is nef.
- The log pairs (X, D) and (X, Δ) are K -(poly/semi)stable and K -semistable, respectively.

Then, we have

- (1) If $D \neq 0$, then $(X, tD + (1-t)\Delta)$ is K -(poly/semi)stable for any $t \in (0, 1]$.
- (2) If $D = 0$, then $(X, (1-t)\Delta)$ is K -semistable for any $t \in (0, 1]$.
- (3) If $\Delta \sim_{\mathbb{Q}} -K_X$ and (X, Δ) is klt, then $(X, tD + (1-t)\Delta)$ is uniformly K -stable for any $t \in (0, 1)$.

Proof. Parts (1) and (2) follow directly from the linearity of generalized Futaki invariants in terms of the coefficient. For part (3), we use the valuative criterion (Theorem 2.9). Simple computation shows that

$$S_{(X, tD + (1-t)\Delta)}(E) = tS_{(X, D)}(E), \quad A_{(X, tD + (1-t)\Delta)}(E) = tA_{(X, D)}(E) + (1-t)A_{(X, \Delta)}(E).$$

By [20, Theorem A], we know that $\delta_0 := \delta(X, \Delta; -K_X - D) = \inf_E A_{(X, \Delta)}(E)/S_{(X, D)}(E)$ is strictly positive. Since (X, D) is K -semistable, Theorem 2.9 implies $A_{(X, D)}(E) \geq S_{(X, D)}(E)$. Hence,

$$A_{(X, tD + (1-t)\Delta)}(E) \geq tS_{(X, D)}(E) + (1-t)\delta_0 S_{(X, D)}(E) = \left(1 + \frac{(1-t)\delta_0}{t}\right) S_{(X, tD + (1-t)\Delta)}(E).$$

This implies the uniform K -stability of $(X, tD + (1-t)\Delta)$ for any $t \in (0, 1)$ by Theorem 2.9. $\square$

2.3 | Normalized volumes

We give a brief review of normalized volume of valuations introduced by Chi Li [79]. See [83] for a survey about recent developments on this subject.

Definition 2.14. Let (X, D) be a klt log pair of dimension n . Let $x \in X$ be a closed point. A valuation v on X centered at x is a valuation of $\mathbb{C}(X)$ such that $v \geq 0$ on $\mathcal{O}_{X, x}$ and $v > 0$ on $\mathfrak{m}_x$. The set of such valuations is denoted by $\text{Val}_{X, x}$. The volume is a function $\text{vol}_{X, x} : \text{Val}_{X, x} \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$\text{vol}_{X, x}(v) := \lim_{k \rightarrow \infty} \frac{\dim_{\mathbb{C}} \mathcal{O}_{X, x} / \{f \in \mathcal{O}_{X, x} \mid v(f) \geq k\}}{k^n / n!}.$$

The *log discrepancy* is a function $A_{(X,D)} : \text{Val}_{X,x} \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$ defined in [15, 61]. Note that if $v = a \cdot \text{ord}_E$ where $a \in \mathbb{R}_{>0}$ and E is a prime divisor over X centered at x , then

$$A_{(X,D)}(v) = a(1 + \text{ord}_E(K_Y - \pi^*(K_X + D))),$$

where $\pi : Y \rightarrow X$ is a birational model of X containing E as a divisor.

The *normalized volume* is a function $\widehat{\text{vol}}_{(X,D),x} : \text{Val}_{X,x} \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$ defined as

$$\widehat{\text{vol}}_{(X,D),x}(v) := \begin{cases} A_{(X,D)}(v)^n \cdot \text{vol}_{X,x}(v) & \text{if } A_{(X,D)}(v) < +\infty \\ +\infty & \text{if } A_{(X,D)}(v) = +\infty. \end{cases}$$

The *local volume* of a klt singularity $x \in (X, D)$ is defined as

$$\widehat{\text{vol}}(x, X) := \min_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}_{(X,D),x}(v).$$

Note that the existence of a $\widehat{\text{vol}}$ -minimizer is proven in [24].

The following theorem from [82] generalizing [43, Theorem 1.1] and [81, Theorem 1.2] is crucial in the study of explicit K-moduli spaces.

Theorem 2.15 [82, Proposition 4.6]. *Let (X, D) be a K-semistable log Fano pair of dimension n . Then for any closed point $x \in X$, we have*

$$(-K_X - D)^n \leq \left(1 + \frac{1}{n}\right)^n \widehat{\text{vol}}(x, X, D).$$

The following useful result is proved in [22, Corollary 4] and independently in [95, Proposition 2.36] as an application of the lower semicontinuity of local volumes.

Theorem 2.16. *Let (X, D) be a klt log Fano pair. If (X, D) specially degenerates to a K-semistable log Fano pair (X_0, D_0) , then (X, D) is also K-semistable.*

2.4 | CM line bundles

Let us start with the original definition of CM line bundles due to Paul and Tian [109, 110] (see also [40]).

Definition 2.17. Let $f : \mathcal{X} \rightarrow T$ be a proper flat morphism of schemes of finite type over $\mathbb{C}$. Let $\mathcal{L}$ be an f -ample line bundle on $\mathcal{X}$. We assume that the fibers $(\mathcal{X}_t, \mathcal{L}_t)$ of f have constant pure dimension $n \geq 1$ and constant Hilbert polynomial χ . A result of Mumford–Knudsen [68] said that there exists line bundles $\lambda_i = \lambda_i(\mathcal{X}, \mathcal{L})$ on T such that for all k ,

$$\det f_!(\mathcal{L}^k) = \lambda_{n+1}^{\binom{k}{n+1}} \otimes \lambda_n^{\binom{k}{n}} \otimes \cdots \otimes \lambda_0.$$

By flatness, the Hilbert polynomial $\chi(\mathcal{X}_t, \mathcal{L}_t^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$. We set $\mu = \mu(\mathcal{X}, \mathcal{L}) := \frac{2a_1}{a_0}$. Then the *CM line bundle* is defined as

$$\lambda_{\text{CM}, f, \mathcal{L}} := \lambda_{n+1}^{\mu+n(n+1)} \otimes \lambda_n^{-2(n+1)}.$$

The *Chow line bundle*[†] is defined as

$$\lambda_{\text{Chow}, f, \mathcal{L}} := \lambda_{n+1}.$$

Next, we recall the definition of log CM line bundles.

Definition 2.18 (log CM line bundle). Assume $f : \mathcal{X} \rightarrow T$ and $\mathcal{L}$ satisfy the conditions in Definition 2.17. For $i = 1, \dots, k$, let D_i be a closed subscheme of $\mathcal{X}$ such that $f|_{D_i} : D_i \rightarrow T$ is of pure dimension $n-1$, and either $f|_{D_i}$ is flat whose fibers have constant Hilbert polynomial, or $f|_{D_i}$ is a well-defined family of cycles over a seminormal base scheme T . Let $c_i \in [0, 1]$ be rational numbers. We define the log CM $\mathbb{Q}$ -line bundle of the data $(f : \mathcal{X} \rightarrow T, \mathcal{L}, D := \sum_{i=1}^k c_i D_i)$ to be

$$\lambda_{\text{CM}, f, D, \mathcal{L}} := \lambda_{\text{CM}, f, \mathcal{L}} - \frac{n(\mathcal{L}_t^{n-1} \cdot D_t)}{(\mathcal{L}_t^n)} \lambda_{\text{Chow}, f, \mathcal{L}} + (n+1) \lambda_{\text{Chow}, f|_{D, \mathcal{L}|_D}},$$

where

$$(\mathcal{L}_t^{n-1} \cdot D_t) := \sum_{i=1}^k c_i (\mathcal{L}_t^{n-1} \cdot D_{i,t}), \quad \lambda_{\text{Chow}, f|_{D, \mathcal{L}|_D}} := \bigotimes_{i=1}^k \lambda_{\text{Chow}, f|_{D_i}, \mathcal{L}|_{D_i}}^{\otimes c_i}.$$

For any $\mathbb{G}_m$ -linearized line bundle over $\mathbb{A}^1$ equipped with the standard $\mathbb{G}_m$ -action, we denote by $\text{wt}(\cdot)$ the corresponding $\mathbb{G}_m$ -weight of the central fiber. The following result from [109] is a fundamental property of (log) CM line bundles.

Proposition 2.19. Let $(X, D = \sum_{i=1}^k c_i D_i)$ be an n -dimensional projective log pair. Let L be an ample line bundle on X . Let $\pi : (\mathcal{X}, D; \mathcal{L}) \rightarrow \mathbb{A}^1$ be a test configuration of $(X, D; L)$. Then, $\lambda_{\text{CM}, \pi, \mathcal{L}}$, $\lambda_{\text{Chow}, \pi, \mathcal{L}}$ and $\lambda_{\text{Chow}, \pi|_{D_i}, \mathcal{L}|_{D_i}}$ are all $\mathbb{G}_m$ -linearized line bundles over $\mathbb{A}^1$. Then, we have

$$\begin{aligned} \text{Fut}(\mathcal{X}; \mathcal{L}) &= \frac{1}{(n+1)(L^n)} \text{wt}(\lambda_{\text{CM}, \pi, \mathcal{L}}), \\ \text{CH}(\mathcal{X}, D; \mathcal{L}) &= \frac{1}{(n+1)(L^n)} \left(-\frac{n(L^{n-1} \cdot D)}{(L^n)} \text{wt}(\lambda_{\text{Chow}, \pi, \mathcal{L}}) + (n+1) \text{wt}(\lambda_{\text{Chow}, \pi|_{D, \mathcal{L}|_D}) \right). \end{aligned}$$

In particular,

$$\text{Fut}(\mathcal{X}, D; \mathcal{L}) = \frac{1}{(n+1)(L^n)} \text{wt}(\lambda_{\text{CM}, \pi, D, \mathcal{L}}).$$

[†] This can also be defined when f is a well-defined family of n -dimensional cycles over a seminormal base scheme T (see [69, Section I.3]).

Next, we will introduce the concept of $\mathbb{Q}$ -Gorenstein flat families of log Fano pairs. In order to adapt this concept to our moduli problems, we need to include cases when the base is a normal Deligne–Mumford stack.

Definition 2.20.

- (a) Let $f : \mathcal{X} \rightarrow T$ be a proper flat morphism between reduced schemes. Let $D = \sum_{i=1}^k c_i D_i$ be a finite $\mathbb{Q}_{\geq 0}$ -linear combination of reduced closed subschemes of $\mathcal{X}$. We say $f : (\mathcal{X}, D) \rightarrow T$ is a $\mathbb{Q}$ -Gorenstein flat family of log Fano pairs if the following conditions hold:
- f has normal, geometrically connected fibers of the same dimension n ;
 - D_i is a relative Mumford divisor on $\mathcal{X}$ over T for every i (see [73, Definition 1]);
 - $-(K_{\mathcal{X}/T} + D)$ is $\mathbb{Q}$ -Cartier and f -ample.
- We define the CM $\mathbb{Q}$ -line bundle of $f : (\mathcal{X}, D) \rightarrow T$ as $\lambda_{\text{CM},f,D} := l^{-n} \lambda_{\text{CM},f,D,\mathcal{L}}$ where $\mathcal{L} := -l(K_{\mathcal{X}/T} + D)$ is an f -ample Cartier divisor on $\mathcal{X}$ for some $l \in \mathbb{Z}_{>0}$.
- (b) Let $\mathcal{X}$ and $\mathcal{T}$ be normal separated Deligne–Mumford stacks that are finite type over $\mathbb{C}$. Let D be an effective $\mathbb{Q}$ -divisor on $\mathcal{X}$. We say $f : (\mathcal{X}, D) \rightarrow \mathcal{T}$ is a $\mathbb{Q}$ -Gorenstein flat family of log Fano pairs if for some (or equivalently, any) étale cover $u : U \rightarrow \mathcal{T}$ from a normal scheme U , the base change $f \times_{\mathcal{T}} u : (\mathcal{X}, D) \times_{\mathcal{T}} U \rightarrow U$ is a $\mathbb{Q}$ -Gorenstein flat family of log Fano pairs.

In our moduli problems, we mainly consider the following class of log Fano pairs.

Definition 2.21. Let c, r be positive rational numbers such that $cr < 1$. A log Fano pair (X, cD) is $\mathbb{Q}$ -Gorenstein smoothable if there exists a $\mathbb{Q}$ -Gorenstein flat family of log Fano pairs $\pi : (\mathcal{X}, cD) \rightarrow B$ over a pointed smooth curve $(0 \in B)$ such that the following holds:

- Both $-K_{\mathcal{X}/B}$ and D are $\mathbb{Q}$ -Cartier, π -ample and $D \sim_{\mathbb{Q},\pi} -rK_{\mathcal{X}/B}$;
- Both π and $\pi|_D$ are smooth morphisms over $B \setminus \{0\}$;
- $(\mathcal{X}_0, cD_0) \cong (X, cD)$.

A $\mathbb{Q}$ -Gorenstein flat family of log Fano pairs $f : (\mathcal{X}, cD) \rightarrow T$ is called a $\mathbb{Q}$ -Gorenstein smoothable log Fano family if all fibers are $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs and D is $\mathbb{Q}$ -Cartier.

For application purposes, it is always convenient to work with a smaller family rather than the whole Hilbert scheme. Thus, the next criterion is important when checking K-stability in explicit families. It is a partial generalization of [109, Theorem 1] and [106, Theorem 3.4].

Theorem 2.22. Let $f : (\mathcal{X}, D) \rightarrow \mathcal{T}$ be a $\mathbb{Q}$ -Gorenstein flat family of log Fano pairs over a normal proper Deligne–Mumford stack $\mathcal{T}$ that is finite type over $\mathbb{C}$. Denote by T the coarse moduli space of $\mathcal{T}$. Let G be a reductive group acting on $\mathcal{X}$ and $\mathcal{T}$ such that D is G -invariant and f is G -equivariant. Assume in addition that

- (a) if $\text{Aut}(\mathcal{X}_t, D_t)$ is finite for $t \in T$, then the stabilizer subgroup G_t is also finite;
- (b) if $(\mathcal{X}_t, D_t) \cong (\mathcal{X}_{t'}, D_{t'})$ for $t, t' \in T$, then $t' \in G \cdot t$;
- (c) $\lambda_{\text{CM},f,D}$ descends to an ample $\mathbb{Q}$ -line bundle $\Lambda_{\text{CM},f,D}$ on T .

Then, $t \in T$ is GIT (poly/semi)stable with respect to the G -linearized $\mathbb{Q}$ -line bundle $\Lambda_{\text{CM},f,D}$ if $(\mathcal{X}_t, D_t)$ is a K -(poly/semi)stable log Fano pair.

Proof. We first show that K-semistability implies GIT-semistability. Denote by $\mathcal{L} := -l(K_{\mathcal{X}/T} + D)$ a Cartier divisor on $\mathcal{X}$ for $l \in \mathbb{Z}_{>0}$. Let $t \in T$ be a closed point such that $(\mathcal{X}_t, D_t)$ is K-semistable. Then t induces a morphism $\tau : \text{Spec } \mathbb{C} \rightarrow \mathcal{T}$ that is unique up to isomorphism. Let $\sigma : \mathbb{G}_m \rightarrow G$ be a 1-PS of G . Then, we have a morphism $\rho : \mathbb{G}_m \rightarrow \mathcal{T}$ as the composition of $\tau \times \sigma : \mathbb{G}_m = \mathbb{G}_m \times \text{Spec } \mathbb{C} \rightarrow G \times \mathcal{T}$ and the G -action morphism $G \times \mathcal{T} \rightarrow \mathcal{T}$. Since $\mathcal{T}$ is proper, ρ extends to a $\mathbb{G}_m$ -equivariant morphism $\bar{\rho} : \mathbb{A}^1 \rightarrow \mathcal{T}$. Pulling back the morphism $f : (\mathcal{X}, D) \rightarrow \mathcal{T}$ and $\mathcal{L}$ to $\mathbb{A}^1$ under $\bar{\rho}$ yields a test configuration $(\mathcal{X}_{t,\sigma}, D_{t,\sigma}; \mathcal{L}_{t,\sigma})/\mathbb{A}^1$ of $(\mathcal{X}_t, D_t)$. By Proposition 2.19, we know that $\text{Fut}(\mathcal{X}_{t,\sigma}, D_{t,\sigma}; \mathcal{L}_{t,\sigma})$ is a positive multiple of the GIT weight $\mu^{\Lambda_{\text{CM},f,D}}(t, \sigma)$ (see [99, Definition 2.2]) that implies that both are nonnegative by K-semistability of $(\mathcal{X}_t, D_t)$. Thus, the Hilbert-Mumford criterion implies that $t \in T$ is GIT semistable.

Next, assume that $(\mathcal{X}_t, D_t)$ is K-polystable. From the above discussion, we see that t is GIT semistable. Let $\sigma : \mathbb{G}_m \rightarrow G$ be a 1-PS such that $t_0 := \lim_{s \rightarrow 0} \sigma(s) \cdot t$ is GIT polystable. Then, the GIT weight $\mu^{\Lambda_{\text{CM},f,D}}(t, \sigma)$ is zero, which implies that $\text{Fut}(\mathcal{X}_{t,\sigma}, D_{t,\sigma}; \mathcal{L}_{t,\sigma})$ vanishes as well. Thus, we have $(\mathcal{X}_t, D_t) \cong (\mathcal{X}_{t_0}, D_{t_0})$ which implies that $t \in G \cdot t_0$ is GIT polystable by assumption (b). The stable part is a consequence of the polystable part and assumption (a). $\square$

The following proposition provides an intersection formula for log CM line bundles. For the case without divisors, this was proven by Paul and Tian [109]. The current statement is a consequence of [29, Proposition 3.7]. We provide a proof here for readers' convenience.

Proposition 2.23. *Let $f : (\mathcal{X}, D) \rightarrow T$ be a $\mathbb{Q}$ -Gorenstein flat family of n -dimensional log Fano pairs over a normal proper variety T . Then,*

$$c_1(\lambda_{\text{CM},f,D}) = -f_*((-K_{\mathcal{X}/T} - D)^{n+1}). \quad (2.1)$$

Proof. Since both sides of (2.1) are functorial under pull-backs, by passing to a resolution, we may assume that T is smooth and projective. Then, [29, Lemma A.2] implies that

$$c_1(f_* \mathcal{L}^{\otimes q}) = \frac{f_*(\mathcal{L}^{n+1})}{(n+1)!} q^{n+1} - \frac{f_*(K_{\mathcal{X}/T} \cdot \mathcal{L}^n)}{2 \cdot n!} q^n + O(q^{n-1}),$$

where $\mathcal{L} := -l(K_{\mathcal{X}/T} + D)$ is Cartier and f -ample. Hence, we have $c_1(\lambda_{\text{Chow},f,\mathcal{L}}) = f_*(\mathcal{L}^{n+1})$ and $c_1(\lambda_{n,f,\mathcal{L}}) = \frac{1}{2} f_*(n(\mathcal{L}^{n+1}) - (K_{\mathcal{X}/T} \cdot \mathcal{L}^n))$. It is clear that $\mu(\mathcal{X}, \mathcal{L}) = -\frac{n(K_{\mathcal{X}_t} \cdot \mathcal{L}_t^{n-1})}{(\mathcal{L}_t^n)}$. Hence,

$$c_1(\lambda_{\text{CM},f,\mathcal{L}}) = -\frac{n(K_{\mathcal{X}_t} \cdot \mathcal{L}_t^{n-1})}{(\mathcal{L}_t^n)} f_*(\mathcal{L}^{n+1}) + (n+1) f_*(K_{\mathcal{X}/T} \cdot \mathcal{L}^n).$$

We also know that $c_1(\lambda_{\text{Chow},f|_D,\mathcal{L}|_D}) = f_*(\mathcal{L}^n \cdot D)$. Thus,

$$\begin{aligned} c_1(\lambda_{\text{CM},f,D,\mathcal{L}}) &= c_1(\lambda_{\text{CM},f,\mathcal{L}}) - \frac{n(K_{\mathcal{X}_t} \cdot \mathcal{L}_t^{n-1})}{(\mathcal{L}_t^n)} c_1(\lambda_{\text{Chow},f,\mathcal{L}}) + (n+1) c_1(\lambda_{\text{Chow},f|_D,\mathcal{L}|_D}) \\ &= \frac{n((-K_{\mathcal{X}_t} - D_t) \cdot \mathcal{L}_t^{n-1})}{(\mathcal{L}_t^n)} f_*(\mathcal{L}^{n+1}) - (n+1) f_*((-K_{\mathcal{X}/T} - D) \cdot \mathcal{L}^n) \\ &= -l^n f_*((-K_{\mathcal{X}/T} - D)^{n+1}). \end{aligned}$$

$\square$

Next, we recall the definition of Hodge line bundles.

Definition 2.24. Let $f : \mathcal{X} \rightarrow T$ be a $\mathbb{Q}$ -Gorenstein flat family of $\mathbb{Q}$ -Fano varieties over a normal base. Let D be an effective $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on $\mathcal{X}$ not containing any fiber of f such that $D \sim_{\mathbb{Q},f} -rK_{\mathcal{X}/T}$ for some $r \in \mathbb{Q}_{>0}$. The Hodge $\mathbb{Q}$ -line bundle $\lambda_{\text{Hodge},f,r^{-1}D}$ is defined as the $\mathbb{Q}$ -linear equivalence class of $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisors on T such that

$$K_{\mathcal{X}/T} + r^{-1}D \sim_{\mathbb{Q}} f^* \lambda_{\text{Hodge},f,r^{-1}D}.$$

Proposition 2.25. With the notation of Definition 2.24, for any rational number $0 \leq c < r^{-1}$, we have

$$(1 - cr)^{-n} \lambda_{\text{CM},f,cD} = (1 - cr) \lambda_{\text{CM},f} + cr(n+1)(-K_{\mathcal{X}_t})^n \lambda_{\text{Hodge},f,r^{-1}D}. \quad (2.2)$$

Proof. For simplicity, we denote $\lambda_c := \lambda_{\text{CM},f,cD}$ and $\lambda_{\text{Hodge}} := \lambda_{\text{Hodge},f,r^{-1}D}$. Let $\mathcal{L} := -lK_{\mathcal{X}/T}$ be an ample Cartier divisor on $\mathcal{X}$ for some $l \in \mathbb{Z}_{>0}$. Since CM line bundles are invariant by twisting pull-back of a line bundle on the base, we know that $\lambda_c = l^{-n}(1 - cr)^n \lambda_{\text{CM},f,cD,\mathcal{L}}$. We also know that

$$\lambda_{\text{CM},f,cD,\mathcal{L}} = \lambda_{\text{CM},f,\mathcal{L}} - c \left(\frac{n(\mathcal{L}_t^{n-1} \cdot D_t)}{(\mathcal{L}_t^n)} \lambda_{\text{Chow},f,\mathcal{L}} - (n+1) \lambda_{\text{Chow},f|_{D,\mathcal{L}}|_D} \right).$$

Hence, to show (2.2), it suffices to show that $\lambda_{\text{CM},f,r^{-1}D,\mathcal{L}} = (n+1)l^n(-K_{\mathcal{X}_t})^n \lambda_{\text{Hodge}}$. Since both sides are functorial under pull-backs, we may assume that T is smooth and quasi-projective. By taking closure in the relative Hilbert scheme of $(\mathcal{X}_t, D_t; \mathcal{L}_t)$, passing to a resolution of the base, and taking normalization of the total family, we can find a smooth projective closure $\bar{T}$ of T , an extension $\bar{f} : (\bar{\mathcal{X}}, \bar{D}) \rightarrow \bar{T}$ of f , and an $\bar{f}$ -ample Cartier divisor $\bar{\mathcal{L}}$ on $\bar{\mathcal{X}}$ such that $\bar{\mathcal{X}}$ is normal projective, $\bar{D}$ is an effective $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on $\bar{\mathcal{X}}$, $\bar{f}$ and $\bar{f}|_{\bar{D}}$ are pure dimensional, and $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}} = \mathcal{L}$. Although $\bar{f}$ is not necessarily flat, the CM line bundle $\lambda_{\text{CM},\bar{f},\bar{c}\bar{D},\bar{\mathcal{L}}}$ can still be defined by [29, Lemma A.2] such that its restriction to T is $\lambda_{\text{CM},f,cD,\mathcal{L}}$. By similar argument to the proof of Proposition 2.23, we have that

$$\begin{aligned} c_1(\lambda_{\text{CM},\bar{f},r^{-1}\bar{D},\bar{\mathcal{L}}}) &= \frac{n((-K_{\mathcal{X}_t} - r^{-1}D_t) \cdot \mathcal{L}_t^{n-1})}{(\mathcal{L}_t^n)} \bar{f}_*(\bar{\mathcal{L}}^{n+1}) - (n+1) \bar{f}_*((-K_{\bar{\mathcal{X}}/\bar{T}} - r^{-1}\bar{D}) \cdot \bar{\mathcal{L}}^n) \\ &= (n+1) \bar{f}_*((K_{\bar{\mathcal{X}}/\bar{T}} + r^{-1}\bar{D}) \cdot \bar{\mathcal{L}}^n). \end{aligned}$$

Since $K_{\mathcal{X}/T} + r^{-1}D = f^* \lambda_{\text{Hodge}}$, we know that

$$\lambda_{\text{CM},f,r^{-1}D,\mathcal{L}} = (n+1)(\mathcal{L}_t^n) \lambda_{\text{Hodge}} = (n+1)l^n(-K_{\mathcal{X}_t})^n \lambda_{\text{Hodge}}.$$

The proof is finished. $\square$

2.5 | Good moduli spaces in the sense of Alper

We recall the definition and some notions regarding *good moduli spaces* from [10], as these objects naturally appear in the construction of moduli spaces in K-stability.

Definition 2.26. A quasi-compact morphism $\phi : \mathcal{X} \rightarrow Y$ from an Artin stack to an algebraic space is a *good moduli space* if

- (1) the push-forward functor on quasi-coherent sheaves is exact, and
- (2) the induced morphism on sheaves $\mathcal{O}_Y \rightarrow \phi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Definition 2.27. Let $\phi : \mathcal{X} \rightarrow Y$ be a good moduli space. An open substack $\mathcal{U} \subseteq \mathcal{X}$ is *saturated* for ϕ if $\phi^{-1}(\phi(\mathcal{U})) = \mathcal{U}$.

This is useful for the following reasons.

Remark 2.28 [10, Remark 6.2, Lemma 6.3].

- (1) If $\mathcal{U}$ is saturated for ϕ , then $\phi(\mathcal{U})$ is open and $\phi|_{\mathcal{U}} : \mathcal{U} \rightarrow \phi(\mathcal{U})$ is a good moduli space.
- (2) If $\psi : \mathcal{X} \rightarrow Z$ is a morphism to a scheme Z and $V \subseteq Z$ is an open subscheme, then $\psi^{-1}(V)$ is saturated for ϕ .

2.6 | Hacking's compact moduli of plane curves

Hacking constructed a proper moduli stack $\overline{\mathcal{P}}_d^{\text{H}}$ of plane curves of degree $d \geq 4$ using tools from the MMP [49, 50]. It is a special case of the moduli theory of log canonically polarized pairs developed by Kollár, Shepherd-Barron, and Alexeev. Roughly speaking the parametrized elements are demi-normal pairs (X, D) , where X is a $\mathbb{Q}$ -Gorenstein deformation of $\mathbb{P}^2$ and D is a degeneration of plane curves such that the pair satisfies some natural properties that will be reviewed below. First, we recall the normal surfaces parametrized by $\overline{\mathcal{P}}_d^{\text{H}}$.

Definition 2.29. A *Manetti surface* is a klt projective surface that admits a $\mathbb{Q}$ -Gorenstein smoothing to $\mathbb{P}^2$.

Proposition 2.30 [50, Theorems 8.2 & 8.3]. *A surface X is a Manetti surface if and only if it is a $\mathbb{Q}$ -Gorenstein deformation of the weighted projective plane $\mathbb{P}(a^2, b^2, c^2)$ where $a^2 + b^2 + c^2 = 3abc$. Moreover, all such X have unobstructed $\mathbb{Q}$ -Gorenstein deformations.*

We now give the definition of the surface pairs parametrized by $\overline{\mathcal{P}}_d^{\text{H}}$.

Definition 2.31. Let X be a demi-normal surface and let D be an effective $\mathbb{Q}$ -Cartier divisor on X . Let $d \geq 4$ be an integer. The pair (X, D) is a *Hacking stable pair* of degree d if:

- (1) the pair $(X, (\frac{3}{d} + \epsilon)D)$ is slc, and the divisor $K_X + (\frac{3}{d} + \epsilon)D$ is ample for any $0 < \epsilon \ll 1$,
- (2) $dK_X + 3D \sim 0$, and
- (3) there is a $\mathbb{Q}$ -Gorenstein deformation of (X, D) to $(\mathbb{P}^2, C_t)$ where C_t is a family of plane curves of degree d .

We can now define the stack $\overline{\mathcal{P}}_d^{\text{H}}$.

Definition 2.32. Let $d \geq 4$ be an integer. We define the *Hacking moduli stack* $\overline{\mathcal{P}}_d^H$ to be the reduced stack representing the following moduli pseudofunctor over a reduced base S :

$$\overline{\mathcal{P}}_d^H(S) = \{(\mathcal{X}, D)/S \mid (\mathcal{X}, D)/S \text{ is a } \mathbb{Q}\text{-Gorenstein family of Hacking stable pairs of degree } d\}.$$

Theorem 2.33 [50, Theorem 4.4, 7.2] and [9]. *The stack $\overline{\mathcal{P}}_d^H$ is a reduced proper Deligne–Mumford stack of finite type over $\mathbb{C}$. Its coarse moduli space $\overline{P}_d^H$ is a reduced projective variety that compactifies the moduli space of smooth plane curves of degree d . Furthermore, if $3 \nmid d$, then*

- (1) *the stack $\overline{\mathcal{P}}_d^H$ is smooth, and*
- (2) *the underlying surface of a Hacking stable pair of degree d is either a Manetti surface, or the slt^+ union of two normal surfaces glued along a smooth rational curve.*

3 | CONSTRUCTION OF K-MODULI SPACES OF SMOOTHABLE LOG FANO PAIRS

In this section, we construct K-moduli stacks and spaces of $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs (see Definition 2.21). Our construction is largely based on [92] with new input from [27, 60, 121]. Results from this section can be applied to the study of many explicit K-moduli spaces, including the K-moduli spaces of plane curves as the main subject of this paper. We will investigate other applications in forthcoming work.

The following theorems are the main results of this section. The first theorem is a natural generalization of [92, Theorem 1.3].

Theorem 3.1. *Let χ_0 be the Hilbert polynomial of an anticanonically polarized Fano manifold. Fix $r \in \mathbb{Q}_{>0}$ and a rational number $c \in (0, \min\{1, r^{-1}\})$. Then there exists a reduced Artin stack $\mathcal{KM}_{\chi_0, r, c}$ of finite type over $\mathbb{C}$ parametrizing all K-semistable $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs (X, cD) with Hilbert polynomial $\chi(X, \mathcal{O}_X(-mK_X)) = \chi_0(m)$ for sufficiently divisible m and $D \sim_{\mathbb{Q}} -rK_X$. Moreover, the Artin stack $\mathcal{KM}_{\chi_0, r, c}$ admits a good moduli space $KM_{\chi_0, r, c}$ as a proper reduced scheme of finite type over $\mathbb{C}$.*

Indeed, in Section 3.6 we show that the K-moduli stack $\mathcal{KM}_{\chi_0, r, c}$ represents the moduli pseudofunctor of $\mathbb{Q}$ -Gorenstein smoothable K-semistable log Fano families with certain numerical invariants over reduced base schemes.

The second theorem provides a wall crossing principle for these K-moduli spaces when varying the coefficient c .

Theorem 3.2. *There exist rational numbers $0 = c_0 < c_1 < c_2 < \dots < c_k = \min\{1, r^{-1}\}$ such that c -K-(poly/semi)stability conditions do not change for $c \in (c_i, c_{i+1})$. For each $1 \leq i \leq k-1$, we have open immersions*

$$\mathcal{KM}_{\chi_0, r, c_i - \epsilon} \hookrightarrow \Phi_i^- \mathcal{KM}_{\chi_0, r, c_i} \hookrightarrow \Phi_i^+ \mathcal{KM}_{\chi_0, r, c_i + \epsilon},$$

which induce projective morphisms

$$KM_{\chi_0, r, c_i - \epsilon} \xrightarrow{\phi_i^-} KM_{\chi_0, r, c_i} \xleftarrow{\phi_i^+} KM_{\chi_0, r, c_i + \epsilon}.$$

[†] Recall that a demi-normal pair (X, D) is semi-log terminal (slt) if it is slc and its normalization is plt.

Moreover, all the above wall crossing morphisms have local VGIT presentations as in [6, (1.2)], and the CM $\mathbb{Q}$ -line bundles on $KM_{\chi_0, r, c_i \pm \epsilon}$ are $\phi_i^\pm$ -ample (see Theorems 3.33 and 3.36 for the precise statements).

Remark 3.3. Recently, there has been tremendous progress on constructing K-moduli spaces and stacks using purely algebraic methods. We point the reader to Remark 1.9 for a further discussion and citations. Our construction is mostly based on the analytic works [92, 104, 115] and algebraic works [27, 60, 93]. Since many of the works mentioned in Remark 1.9(1) appeared simultaneously to or after the preparation of this paper, our constructions do not rely on those results, though it is likely some of our arguments can be simplified using those results.

Meanwhile, a suitable condition for families of log pairs over nonreduced bases was discovered in [73] as the K-flatness condition. Since we only study families of $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs, in this paper, we restrict the bases of such families to be reduced.

3.1 | Foundations

We will fix an arbitrary rational number $\epsilon_0 \in (0, 1)$. For technical reasons, we will concentrate on constructing the K-moduli space of $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs (X, cD) where $D \sim_{\mathbb{Q}} -rK_X$ and $c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\})$ a rational number. As we will see in Theorem 3.20, the K-(poly/semi)stability conditions will not change for c sufficiently close to r^{-1} . Besides, all numbers except β , β_* , and $\mathfrak{B}$ are assumed to be rational.

The first boundedness result is a generalization of work in [92] that was a consequence of [28, 120].

Theorem 3.4. Fix n a positive integer, r a positive rational number, and $\epsilon_0 \in (0, 1)$ a rational number. Then the following collection of $\mathbb{Q}$ -Gorenstein smoothable pairs

$$\{(X, D) \mid \dim X = n, (X, cD) \text{ is K-semistable for some } c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\})\}.$$

is log bounded. In particular, there exists $m_1 = m_1(n, r, \epsilon_0) \in \mathbb{Z}_{>0}$ such that $m_1 K_X$ is Cartier whenever (X, D) belongs to the above collection.

Proof. Since $D \sim_{\mathbb{Q}} -rK_X$, it suffices to bound the variety X . From the assumption that (X, cD) is K-semistable for some $c < (1 - \epsilon_0)r^{-1}$, by [23, Theorem 7.2], we conclude that $\delta(X) \geq \epsilon_0$. Since the volume of X is a positive integer, X belongs to a bounded family by [60]. $\square$

Proposition 3.5. Fix $n, m \in \mathbb{Z}_{>0}$, $r \in \mathbb{Q}_{>0}$ and $\epsilon_0 \in (0, 1)$. Let X be a $\mathbb{Q}$ -Fano variety with mK_X Cartier. Let $D \sim_{\mathbb{Q}} -rK_X$ be a Weil divisor. Let $c < (1 - \epsilon_0)r^{-1}$ be a rational number such that (X, cD) is a log Fano pair. Then,

- (1) there exists a positive integer $q = q(n, r, \epsilon_0, m)$ and a Cartier divisor $\Delta \in |-qK_X|$ such that $(X, \text{lct}(X; D) \cdot D + \Delta)$ is a log canonical pair.
- (2) there exists $\gamma_0 = \gamma_0(n, r, \epsilon_0, m)$ such that either $c > \text{lct}(X; D) - \frac{\epsilon_0 r^{-1}}{n+1}$ which implies $\alpha(X, cD) < \frac{1}{n+1}$, or $(X, cD + \frac{(1-\beta)(1-cr)}{q} \Delta)$ is uniformly K-stable for any $\beta \in (0, \gamma_0]$.

Proof.

- (1) By [58], such X form a bounded family. This follows from the Bertini theorem for bounded families.
- (2) Let us assume that $\alpha(X, cD) \geq \frac{1}{n+1}$. We know that $\alpha(X, cD; -K_X) = (1 - cr)\alpha(X, cD) \geq \frac{\epsilon_0}{n+1}$. Thus, $(X, cD + \frac{\epsilon_0 r^{-1}}{n+1} D)$ is log canonical that implies $c \leq \text{lct}(X; D) - \frac{\epsilon_0 r^{-1}}{n+1}$. It is clear that $(X, (\text{lct}(X; D) - \frac{\epsilon_0 r^{-1}}{n+1})D + \frac{1}{q}\Delta)$ form a bounded family of klt pairs. Hence, [23] implies that there exists $\alpha_1 = \alpha_1(n, r, \epsilon_0, m) > 0$ such that

$$\alpha\left(X, cD + \frac{1-cr}{q}\Delta; -K_X\right) \geq \alpha\left(X, (\text{lct}(X; D) - \frac{\epsilon_0 r^{-1}}{n+1})D + \frac{1}{q}\Delta; -K_X\right) \geq \alpha_1.$$

Hence,

$$\begin{aligned} \alpha\left(X, cD + \frac{(1-\beta)(1-cr)}{q}\Delta\right) &= \frac{1}{\beta(1-cr)}\alpha\left(X, cD + \frac{(1-\beta)(1-cr)}{q}\Delta; -K_X\right) \\ &\geq \frac{1}{\beta(1-cr)}\left(\beta\alpha(X, cD; -K_X) + (1-\beta)\alpha\left(X, cD + \frac{1-cr}{q}\Delta; -K_X\right)\right) \\ &\geq \alpha(X, cD; -K_X) + (\beta^{-1} - 1)\alpha\left(X, cD + \frac{1-cr}{q}\Delta; -K_X\right) \\ &\geq \frac{\epsilon_0}{n+1} + (\beta^{-1} - 1)\alpha_1. \end{aligned}$$

Let us take $\gamma_0 := \frac{\alpha_1}{1+\alpha_1}$, then for any $\beta \in (0, \gamma_0]$, we have $\alpha(X, cD + \frac{(1-\beta)(1-cr)}{q}\Delta) \geq 1$. Thus, [39] implies that $(X, cD + \frac{(1-\beta)(1-cr)}{q}\Delta)$ is uniformly K-stable. $\square$

Next, we use an important result obtained in the solution of Yau–Tian–Donaldson conjecture for log smooth log Fano pairs [121]. It plays a crucial role in proving openness and properness of the K-moduli conjecture in our setting. We are very grateful to Feng Wang for kindly providing a proof. For the case where $D_i = 0$, that is, the boundary is a single smooth pluricanonical divisor, see [16, 28, 120] or [92, Theorem 4.1].

Theorem 3.6 [121]. *Fix $n, q \in \mathbb{Z}_{>0}$, $r \in \mathbb{Q}_{>0}$, and $\epsilon_0, \gamma_0 \in (0, 1)$. Let X_i be a sequence of n -dimensional Fano manifolds with a fixed Hilbert polynomial χ_0 . Let $D_i \sim_{\mathbb{Q}} -rK_{X_i}$ be smooth divisors on X_i . Let Δ_i be smooth divisors in $|-qK_{X_i}|$ that are transversal to D_i . Let c_i and β_i be a sequence converging, respectively, to c_∞ and β_∞ with $c_\infty < \min\{1, (1 - \epsilon_0)r^{-1}\}$ and $0 < \gamma_0 \leq \beta_i \leq 1$. Suppose that each X_i admits a conical Kähler–Einstein metric $\omega(\beta_i)$ solving:*

$$\text{Ric}(\omega(\beta_i)) = \beta_i(1 - c_i r)\omega(\beta_i) + c_i[D_i] + \frac{(1 - \beta_i)(1 - c_i r)}{q}[\Delta_i]. \quad (3.1)$$

Then, the Gromov–Hausdorff limit of any subsequence of $\{(X_i, \omega(\beta_i))\}_i$ is homeomorphic to a $\mathbb{Q}$ -Fano variety Y . Furthermore, there are unique Weil divisors $E, \Gamma \subset Y$ such that

- (1) $(Y, c_\infty E + \frac{(1-\beta_\infty)(1-c_\infty r)}{q}\Gamma)$ is a klt log Fano pair;

(2) Y admits a weak conical Kähler–Einstein metric $\omega(\beta_\infty)$ solving

$$\mathrm{Ric}(\omega(\beta_\infty)) = \beta_\infty(1 - c_\infty r)\omega(\beta_\infty) + c_\infty[E] + \frac{(1 - \beta_\infty)(1 - c_\infty r)}{q}[\Gamma].$$

In particular, $\mathrm{Aut}(Y, E + \Gamma)$ is reductive and the pair $(Y, c_\infty E + \frac{(1 - \beta_\infty)(1 - c_\infty r)}{q}\Gamma)$ is K -polystable;

(3) there exists a positive integer $m_2 = m_2(\chi_0, r, q, \epsilon_0, \gamma_0)$, such that possibly after passing to a subsequence, there are Tian's embeddings $T_i : X_i \rightarrow \mathbb{P}^N$ and $T_\infty : Y \rightarrow \mathbb{P}^N$, defined by taking a suitable orthonormal basis of the complete linear system $| -mK_{X_i} |$ and $| -mK_Y |$ with respect to $\omega(\beta_i)$ and $\omega(\beta_\infty)$, respectively, such that for any multiple m of m_2 and $N + 1 = \chi(X_i, \mathcal{O}_{X_i}(-mK_{X_i}))$, we have that $T_i(X_i)$ converge to $T_\infty(Y)$ as projective varieties, and $T_i(D_i)$ (respectively, $T_i(\Delta_i)$) converge to $T_\infty(E)$ (respectively, $T_\infty(\Gamma)$) as algebraic cycles.

Proof. It is a combination of [121, Proposition 4.18, Corollary 4.19, and Lemma 5.4]. By taking subsequences, we can assume that $(X_i; \omega(\beta_i))$ converges to a metric space $(X; d)$ in the Gromov–Hausdorff topology. Since the divisors are ample, Cheeger–Colding–Tian's theory applies. From [121, Proposition 4.18], there exists a positive integer m_2 such that the partial C^0 holds for $(X_i; \omega(\beta_i); -mK_{X_i})$ for any multiple m of m_2 . Then, we get a sequence of Tian's embeddings $T_i : X_i \rightarrow \mathbb{P}^N$ using an orthonormal basis of $H^0(X_i, \mathcal{O}_{X_i}(-mK_{X_i}))$. By taking subsequences again, we can assume that $T_i(X_i)$ converges to Y as cycles. Since T_i are uniform Lipschitz, we get a map T_∞ from $(X; d)$ to Y . By [121, Corollary 4.19], T_∞ is a homeomorphism and Y is a normal projective variety. Moreover, if we define the Gromov–Hausdorff limit of D_i, Δ_i as D_∞, Δ_∞ , then $E = T_\infty(D_\infty)$ and $\Gamma = T_\infty(\Delta_\infty)$ are divisors on Y such that $(Y, c_\infty E + \frac{(1 - \beta_\infty)(1 - c_\infty r)}{q}\Gamma)$ is a klt log Fano pair, and it admits a weak conical Kähler–Einstein metric $\omega(\beta_\infty)$:

$$\mathrm{Ric}(\omega(\beta_\infty)) = \beta_\infty(1 - c_\infty r)\omega(\beta_\infty) + c_\infty[E] + \frac{(1 - \beta_\infty)(1 - c_\infty r)}{q}[\Gamma].$$

Since both D_i and Δ_i are proportional to $-K_{X_i}$, we have that both E and Γ are also proportional to $-K_Y$ as cycle limits. Thus, Y is a $\mathbb{Q}$ -Fano variety, and both E and Γ are $\mathbb{Q}$ -Cartier divisors on Y . By [121, Lemma 5.4], $\mathrm{Aut}_0(Y, c_\infty E + \frac{(1 - \beta_\infty)(1 - c_\infty r)}{q}\Gamma)$ is reductive. $\square$

We now introduce the relevant Hilbert schemes.

Definition 3.7. Let $\mathbb{H}^{\chi;N} := \mathrm{Hilb}_\chi(\mathbb{P}^N)$ denote the Hilbert scheme of closed subschemes of $\mathbb{P}^N$ with Hilbert polynomial χ . Given a closed subscheme $X \subset \mathbb{P}^N$ with Hilbert polynomial $\chi(X, \mathcal{O}_{\mathbb{P}^N}(k)|_X) = \chi(k)$, let $\mathrm{Hilb}(X) \in \mathbb{H}^{\chi;N}$ denote its Hilbert point.

Let χ_0 be a Hilbert polynomial of an anticanonically polarized Fano manifold. Let m be a positive integer. Denote $\chi(k) := \chi_0(mk)$, $\tilde{\chi}(k) := \chi_0(mk) - \chi_0(mk - r)$, and $N = \chi_0(m) - 1$. Let $\chi = (\chi, \tilde{\chi})$ be the Hilbert polynomials of $(X, D) \hookrightarrow \mathbb{P}^N$. Denote by $\mathbb{H}^{\chi;N} = \mathbb{H}^{\chi;N} \times \mathbb{H}^{\tilde{\chi};N}$. We define

$$Z := \left\{ \mathrm{Hilb}(X, D) \in \mathbb{H}^{\chi;N} \left| \begin{array}{l} X \text{ is a Fano manifold, } D \sim_{\mathbb{Q}} -rK_X \text{ is a smooth divisor,} \\ \mathcal{O}_{\mathbb{P}^N}(1)|_X \cong \mathcal{O}_X(-mK_X), \\ \text{and } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{\cong} H^0(X, \mathcal{O}_X(-mK_X)). \end{array} \right. \right\}.$$

Then, Z is a locally closed subscheme of $\mathbb{H}^{X:N}$. Let $\bar{Z}$ be the Zariski closure of Z . We also define

$$Z^{\text{klt}} := \left\{ \text{Hilb}(X, D) \in \bar{Z} \mid \begin{array}{l} X \text{ is a } \mathbb{Q}\text{-Fano variety, } D \sim_{\mathbb{Q}} -rK_X \text{ is an effective Weil divisor,} \\ -m_1K_X \text{ is Cartier, } \mathcal{O}_{\mathbb{P}^N}(1)|_X \cong \mathcal{O}_X(-mK_X), \\ \text{and } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{\cong} H^0(X, \mathcal{O}_X(-mK_X)), \end{array} \right\}.$$

and

$$Z_c^\circ := \{\text{Hilb}(X, D) \in Z^{\text{klt}} \mid (X, cD) \text{ is K-semistable}\}.$$

It is clear that Z^{klt} is a Zariski open subset of $\bar{Z}$. We will see in Theorem 3.15 that Z_c° is a Zariski open subset of Z^{klt} . Let Z^{red} and Z_c^{red} be reduced schemes supported on Z and Z_c° , respectively. In the cases when we keep track of m , we use the notation $Z_m, \bar{Z}_m, Z_m^{\text{klt}}, Z_{c,m}^\circ, Z_m^{\text{red}}$, and $Z_{c,m}^{\text{red}}$ instead of $Z, \bar{Z}, Z^{\text{klt}}, Z_c^\circ, Z^{\text{red}}$, and Z_c^{red} , respectively.

We define the K-moduli stacks and spaces as follows.

Definition 3.8. Let χ_0 be a Hilbert polynomial of an anticanonically polarized Fano manifold. Fix $r \in \mathbb{Q}_{>0}$ and $0 < \epsilon_0 \ll 1$. Let $c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\})$ be a rational number. We denote by $\chi(k) := \chi_0(mk)$, $\tilde{\chi}(k) := \chi_0(mk) - \chi_0(mk - r)$, and $N_m := \chi_0(m) - 1$. As we will see in Theorem 3.24, the Artin stacks $[Z_{c,m}^{\text{red}}/\text{PGL}(N_m + 1)]$ stabilize for m sufficiently divisible which we simply denote by $\mathcal{KM}_{\chi_0, r, c}$. Moreover, according to Theorem 3.1, the Artin stack $\mathcal{KM}_{\chi_0, r, c}$ admits a proper reduced scheme $KM_{\chi_0, r, c}$ as its good moduli space. We define the *K-moduli stack* (resp. *K-moduli space*) with respect to the triple (χ_0, r, c) as the reduced Artin stack $\mathcal{KM}_{\chi_0, r, c}$ (resp. reduced proper scheme $KM_{\chi_0, r, c}$).

3.2 | Continuity method

In this section, we will use Theorem 3.6 and the continuity method [92, Section 4.2 and 6] to prove the following theorem.

Theorem 3.9. Fix $n \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Q}_{>0}$. Fix $\epsilon_0 \in (0, 1)$. Fix a Hilbert polynomial χ_0 of an n -dimensional anticanonically polarized Fano manifold. Then there exists $m_3 = m_3(\chi_0, r, \epsilon_0)$ such that for any multiple m of m_3 , any rational number $c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\})$, and any log pair (X, D) with $\text{Hilb}(X, D) \in Z_m^{\text{klt}}$, the following holds:

- (1) If (X, cD) is K-unstable, then either $c > \text{lct}(X; D) - \frac{\epsilon_0 r^{-1}}{n+1}$ or there exists a 1-PS in $\text{SL}(N_m + 1)$ that destabilizes (X, cD) .
- (2) If (X, cD) is K-semistable but not K-polystable, then there exists a 1-PS in $\text{SL}(N_m + 1)$ that provides a special degeneration to a K-polystable log Fano pair (X', cD') . In addition, $\text{Hilb}(X', D') \in Z_m^{\text{klt}}$.
- (3) If (X, cD) is K-polystable, then it admits a weak conical Kähler–Einstein metric. In particular, $\text{Aut}(X, D)$ is reductive.
- (4) If (X, cD) is K-stable, then it is uniformly K-stable.

Before presenting the proof of Theorem 3.9, we introduce the following notation and prove several preliminary results.

Notation 3.10. Let us fix $n \in \mathbb{Z}_{>0}$, $r \in \mathbb{Q}_{>0}$, $\epsilon_0 \in (0, 1)$, and χ_0 a Hilbert polynomial of an n -dimensional Fano manifold. Let $m_1 = m_1(n, r, \epsilon_0) \in \mathbb{Z}_{>0}$ be chosen as in Theorem 3.4. Let $q = q(n, r, \epsilon_0, m_1) \in \mathbb{Z}_{>0}$ and $\gamma_0 = \gamma_0(n, r, \epsilon_0, m_1) \in (0, 1)$ be chosen as in Proposition 3.5. Let $m_2 = m_2(\chi_0, r, q, \epsilon_0, \gamma_0)$ be chosen as in Theorem 3.6. Let us take $m_3 := \text{lcm}(m_1, m_2)$. For $m \in \mathbb{Z}_{>0}$ a multiple of m_3 , let us pick an arbitrary pair (X, D) with $\text{Hilb}(X, D) \in Z_m^{\text{kl}}$. We also fix $c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\})$ such that $c \leq \text{lct}(X; D) - \frac{\epsilon_0 r^{-1}}{n+1}$. To avoid bulky notation, let $a := \frac{1-cr}{q}$.

According to Proposition 3.5, let $\Delta \in |-qK_X|$ be chosen such that $(X, cD + (1 - \beta)a\Delta)$ is uniformly K-stable for any $0 < \beta \leq \gamma_0$. Let us choose a smoothing $(\mathcal{X}, D + \tilde{\Delta}) \rightarrow B$ over a pointed curve $0 \in B$ such that all fibers over $B \setminus \{0\}$ are log smooth, and $(\mathcal{X}_0, D_0, \tilde{\Delta}_0) \cong (X, D, \Delta)$. Denote by

$$\mathfrak{B} := \sup\{\beta \in (0, 1) \mid (X, cD + (1 - \beta)a\Delta) \text{ is uniformly K-stable}\}.$$

By [44], we know that $\gamma_0 < \mathfrak{B} \leq 1$. Since the pair $(X, cD + (1 - \beta)a\Delta)$ is uniformly K-stable for any $\beta \in (0, \mathfrak{B})$, by [23, 121], we know that there exists a Zariski neighborhood B_β of 0 in B such that $(\mathcal{X}_b, cD_b + (1 - \beta)a\tilde{\Delta}_b)$ is uniformly K-stable hence admits conical Kähler–Einstein metrics for any $b \in B_\beta^\circ := B_\beta \setminus \{0\}$. Consider the triple Hilbert scheme $\mathbb{H}^{\mathcal{X}, q; N}$ of $\mathbb{P}^N$ with the same Hilbert polynomials as (X, D, Δ) . Let $\text{Hilb}(\mathcal{X}_b, cD_b + (1 - \beta)a\tilde{\Delta}_b) \in \mathbb{H}^{\mathcal{X}, q; N}$ be the Hilbert point of $(\mathcal{X}_b, cD_b + (1 - \beta)a\tilde{\Delta}_b)$ via Tian’s embedding.

Proposition 3.11. *With Notation 3.10, the log Fano pair $(X, cD + (1 - \beta)a\Delta)$ admits a weak conical Kähler–Einstein metric for any $\beta \in [\gamma_0, \mathfrak{B})$. Moreover, for any sequence of points $b_i \rightarrow 0$ in B_β° , there exists a sequence of matrices $g_i \in \text{U}(N + 1)$ such that*

$$g_i \cdot \text{Hilb}(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta)a\tilde{\Delta}_{b_i}) \rightarrow \text{Hilb}(X, cD + (1 - \beta)a\Delta) \in \mathbb{H}^{\mathcal{X}, q; N} \quad \text{as } i \rightarrow \infty.$$

Proof. By Theorem 3.6, we know that after choosing suitable $g_i \in \text{U}(N + 1)$, the Hilbert points $g_i \cdot \text{Hilb}(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta)a\tilde{\Delta}_{b_i})$ converge as $i \rightarrow \infty$ to $\text{Hilb}(Y, cE + (1 - \beta)a\Gamma)$ that is the Hilbert point of a log Fano pair $(Y, cE + (1 - \beta)a\Gamma)$ via Tian’s embedding of its weak conical Kähler–Einstein metric. Then [16] (see also [82, Section 3.1]) implies that $(Y, cE + (1 - \beta)a\Gamma)$ is K-polystable. By Lemma 3.12, after possibly replacing $(0 \in B)$ by its quasi-finite cover $(0' \in B')$, the log Fano pair $(Y, cE + (1 - \beta)a\Gamma)$ is a K-polystable fill-in of the family $(\mathcal{X}, cD + (1 - \beta)a\tilde{\Delta}) \times_B (B' \setminus \{0'\})$. Since a K-polystable fill-in is always unique by [27], we know that

$$(Y, cE + (1 - \beta)a\Gamma) \cong (X, cD + (1 - \beta)a\Delta).$$

The proof is finished. $\square$

Lemma 3.12. *Let G be an algebraic group acting on $\mathbb{P}^M$. Let $z : B \rightarrow \mathbb{P}^M$ be a morphism from a smooth pointed curve $(0 \in B)$. Denote by $B^\circ := B \setminus \{0\}$. Suppose that z_0 is a point in $\mathbb{P}^M$ satisfying that there exists $g_i \in G$ and $B^\circ \ni b_i \rightarrow 0$ for $i \in \mathbb{Z}_{>0}$ such that $g_i \cdot z(b_i) \rightarrow z_0$ as $i \rightarrow \infty$. Then, there exists a quasi-finite morphism $\pi : (0' \in B') \rightarrow (0 \in B)$ from a smooth pointed curve $(0' \in B')$ with*

$\{0'\} = \pi^{-1}(\{0\})$ and two morphisms $\tau : B'^{\circ} := B' \setminus \{0\} \rightarrow G$ and $z' : B' \rightarrow \mathbb{P}^M$ such that $z'(b') = \tau(b') \cdot z(\pi(b'))$ for any $b' \in B'^{\circ}$ and $z'(0') = z_0$.

Proof. Let $\phi : G \times B^{\circ} \rightarrow \mathbb{P}^M \times B$ be the morphism defined as $\phi(g, b) := (g \cdot z(b), b)$. Let $\bar{G}$ be a normal proper variety that compactifies G . Let Γ be the normalized graph of the rational map $\phi : \bar{G} \times B \rightarrow \mathbb{P}^M \times B$. Hence, we have a proper birational morphism $\psi : \Gamma \rightarrow \bar{G} \times B$ that is an isomorphism over $G \times B^{\circ}$, and a proper B -morphism $\tilde{\phi} : \Gamma \rightarrow \mathbb{P}^M \times B$. From the assumption, we know that $(z_0, 0) \in \phi(G \times B^{\circ}) = \tilde{\phi}(\Gamma)$. Let us take a point $\tilde{z}_0 \in \tilde{\phi}^{-1}(z_0, 0) \subset \Gamma$. Then we may choose a smooth pointed curve $(0' \in B')$ together with a finite morphism $f : B' \rightarrow \Gamma$ such that $f(0') = \tilde{z}_0$ and $f(B') \cap (G \times B^{\circ}) \neq \emptyset$. After possibly shrinking $(0' \in B')$, we may assume that $f(B'^{\circ}) \subset G \times B^{\circ}$. Then, by defining $\pi := \text{pr}_2 \circ \psi \circ f$, $\tau := \text{pr}_1 \circ f|_{B'^{\circ}}$, and $z' := \text{pr}_1 \circ \tilde{\phi} \circ f$, it is easy to check that the conclusion is satisfied. $\square$

A priori $\mathfrak{B}$ might only be a real number. Nevertheless, the following result shows that $\mathfrak{B}$ has to be rational and we can find a destabilizing test configuration in $\mathbb{P}^{N_m}$.

Proposition 3.13. *With Notation 3.10, if $\mathfrak{B} < 1$, then $(X, cD + (1 - \mathfrak{B})a\Delta)$ does not admit a weak conical Kähler–Einstein metric. There exists a 1-PS λ of $\text{SL}(N_m + 1)$ inducing a nonproduct special test configuration of $(X, cD + (1 - \mathfrak{B})a\Delta)$ such that the central fiber $(X', cD' + (1 - \mathfrak{B})a\Delta')$ admits a weak conical Kähler–Einstein metric. Moreover, the generalized Futaki invariant of this special test configuration vanishes. In particular, $\mathfrak{B}$ is a rational number.*

Proof. We first show that $(X, cD + (1 - \mathfrak{B})a\Delta)$ does not admit a weak conical Kähler–Einstein metric. Since $(X, cD + (1 - \gamma_0)a\Delta)$ is uniformly K-stable, we know that (X, D, Δ) has finite automorphism group by [27, Corollary 3.5]. Assume to the contrary that $(X, cD + (1 - \mathfrak{B})a\Delta)$ admits a weak conical Kähler–Einstein metric. Then, from [13, 30, 35], we know that the Mabuchi energy is proper. This indeed implies that $(X, cD + (1 - \mathfrak{B})a\Delta)$ is uniformly K-stable by [14, 18]. However, by [44] we know that $(X, cD + (1 - \mathfrak{B} - \epsilon)a\Delta)$ is also uniformly K-stable for $0 < \epsilon \ll 1$ which contradicts our definition of $\mathfrak{B}$.

Next, let us choose $\gamma_0 \leq \beta_i \nearrow \mathfrak{B}$ as $i \rightarrow \infty$. Then by Proposition 3.11, we may choose $B_{\beta_i}^{\circ} \ni b_i \rightarrow 0$ as $i \rightarrow \infty$ such that both $(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\tilde{\Delta}_{b_i})$ and $(X, cD + (1 - \beta_i)a\Delta)$ admit a (weak) conical Kähler–Einstein metric for any i , and

$$\lim_{i \rightarrow \infty} \text{dist}_{\mathbb{H}^{X, q; N}} \left(\text{Hilb}(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\tilde{\Delta}_{b_i}), U(N + 1) \cdot \text{Hilb}(X, cD + (1 - \beta_i)a\Delta) \right) = 0.$$

By the results of Theorem 3.6, there exists a sequence of matrices $g_i \in U(N + 1)$ and a log Fano pair $(X', cD' + (1 - \mathfrak{B})a\Delta')$ admitting a weak conical Kähler–Einstein metric such that

$$g_i \cdot \text{Hilb}(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\tilde{\Delta}_{b_i}) \rightarrow \text{Hilb}(X', cD' + (1 - \mathfrak{B})a\Delta') \in \mathbb{H}^{X, q; N} \quad \text{as } i \rightarrow \infty.$$

Since $\text{Hilb}(X, cD + (1 - \beta_i)a\Delta) \in \text{SL}(N + 1) \cdot \text{Hilb}(X, D, \Delta)$, we know that

$$\text{Hilb}(X', cD' + (1 - \mathfrak{B})a\Delta') \in \overline{\text{SL}(N + 1) \cdot \text{Hilb}(X, D, \Delta)} \subset \mathbb{H}^{X, q; N}.$$

On the other hand, we know that $(X', cD' + (1 - \mathfrak{B})a\Delta')$ is not isomorphic to $(X, cD + (1 - \mathfrak{B})a\Delta)$ since the latter does not admit a conical Kähler–Einstein metric. Since $(X', cD' + (1 - \mathfrak{B})a\Delta')$

admits a weak conical Kähler–Einstein metric, its automorphism group is reductive by [13, Theorem 5.2]. Hence, by [37, Proposition 1], there exists a 1-PS λ of $\mathrm{SL}(N+1)$ that induces a special test configuration $(\mathcal{X}_\lambda, cD_\lambda + (1 - \mathfrak{B})a\Delta_\lambda)$ of $(X, cD + (1 - \mathfrak{B})a\Delta)$ with central fiber $(X', cD' + (1 - \mathfrak{B})a\Delta')$. Let $\mathcal{L}_\lambda$ be a sufficiently divisible multiple of $-(K_{\mathcal{X}_\lambda/\mathbb{A}^1} + cD_\lambda)$. It is clear from the definition that $\beta \mapsto \mathrm{Fut}(\mathcal{X}_\lambda, cD_\lambda + (1 - \beta)a\Delta_\lambda; \mathcal{L}_\lambda)$ is a degree 1 polynomial in β with rational coefficients. Since $(X, cD + (1 - \beta)a\Delta)$ is uniformly K-stable when $\beta < \mathfrak{B}$, we know that

$$\mathrm{Fut}(\mathcal{X}_\lambda, cD_\lambda + (1 - \mathfrak{B})a\Delta_\lambda; \mathcal{L}_\lambda) \geq 0.$$

On the other hand, the 1-PS λ^{-1} of $\mathrm{Aut}(X', D', \Delta') \subset \mathrm{SL}(N+1)$ induces a product test configuration $(\mathcal{X}'_{\lambda^{-1}}, cD'_{\lambda^{-1}} + (1 - \mathfrak{B})a\Delta'_{\lambda^{-1}})$ of $(X', cD' + (1 - \mathfrak{B})a\Delta')$ such that

$$\mathrm{Fut}(\mathcal{X}'_{\lambda^{-1}}, cD'_{\lambda^{-1}} + (1 - \mathfrak{B})a\Delta'_{\lambda^{-1}}; \mathcal{L}'_{\lambda^{-1}}) + \mathrm{Fut}(\mathcal{X}_\lambda, cD_\lambda + (1 - \mathfrak{B})a\Delta_\lambda; \mathcal{L}_\lambda) = 0.$$

Again, by [16], we know that $(X', cD' + (1 - \mathfrak{B})a\Delta')$ is K-polystable. Hence, both generalized Futaki invariants in the above equation are zero. Thus, $\beta = \mathfrak{B}$ is the solution of the equation $\mathrm{Fut}(\mathcal{X}_\lambda, cD_\lambda + (1 - \beta)a\Delta_\lambda; \mathcal{L}_\lambda) = 0$ which implies that $\mathfrak{B}$ is a rational number. $\square$

Now we are ready to prove Theorem 3.9.

Proof of Theorem 3.9. We follow Notation 3.10.

- (1) Assume (X, cD) is K-unstable and $c \leq \mathrm{lct}(X; D) - \frac{\varepsilon_0 r^{-1}}{n+1}$. Thus, we have $\mathfrak{B} < 1$. Then, by Proposition 3.13, there exists a 1-PS λ of $\mathrm{SL}(N_m + 1)$ which induces a test configuration $(\mathcal{X}_\lambda, cD_\lambda + (1 - \beta)a\Delta_\lambda)$ of $(X, cD + (1 - \beta)a\Delta)$ such that

$$\mathrm{Fut}(\mathcal{X}_\lambda, cD_\lambda + (1 - \beta)a\Delta_\lambda; \mathcal{L}_\lambda) \geq 0 \quad \text{if and only if } \beta \leq \mathfrak{B}.$$

Therefore, $\mathrm{Fut}(\mathcal{X}_\lambda, cD_\lambda; \mathcal{L}_\lambda) < 0$ which implies that (X, cD) is destabilized by λ .

- (2) Assume that (X, cD) is K-semistable but not K-polystable. Let us choose $\gamma_0 \leq \beta_i \nearrow 1$ as $i \rightarrow \infty$. Following the proof of Proposition 3.13, there exists a log Fano pair $(X', cD' + 0 \cdot \Delta')$ in $\mathbb{P}^N$ admitting a weak conical Kähler–Einstein metric such that

$$\mathrm{Hilb}(X', cD' + 0 \cdot \Delta') \in \overline{\mathrm{SL}(N+1) \cdot \mathrm{Hilb}(X, D, \Delta)} \subset \mathbb{H}^{\mathcal{X}, q; N}.$$

In particular, (X', cD') is K-polystable by [16] hence is not isomorphic to (X, cD) . Then, by [37, Proposition 1], we obtain a special degeneration from (X, cD) to (X', cD') induced by a 1-PS in $\mathrm{SL}(N+1)$. In addition, since $\mathrm{Hilb}(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\Delta_{b_i})$ converges to $\mathrm{Hilb}(X', cD' + 0 \cdot \Delta')$ in $\mathbb{H}^{\mathcal{X}, q; N}$ as $i \rightarrow \infty$, we know that $\mathrm{Hilb}(\mathcal{X}_{b_i}, D_{b_i})$ converges to $\mathrm{Hilb}(X', D')$ in $\mathbb{H}^{\mathcal{X}; N}$ for suitable embeddings. Hence, $\mathrm{Hilb}(X', D') \in Z_m^{\mathrm{kl}}.$

- (3) Assume that (X, cD) is K-polystable. Similar to part (2), there exists a log Fano pair $(X', cD' + 0 \cdot \Delta')$ in $\mathbb{P}^N$ admitting a weak conical Kähler–Einstein metric such that

$$\mathrm{Hilb}(X', cD' + 0 \cdot \Delta') \in \overline{\mathrm{SL}(N+1) \cdot \mathrm{Hilb}(X, D, \Delta)} \subset \mathbb{H}^{\mathcal{X}, q; N}.$$

If $\mathrm{Hilb}(X', cD' + 0 \cdot \Delta') \in \mathrm{SL}(N+1) \cdot \mathrm{Hilb}(X, D, \Delta)$, then we are done. So, we may assume $\mathrm{Hilb}(X', cD' + 0 \cdot \Delta') \notin \mathrm{SL}(N+1) \cdot \mathrm{Hilb}(X, D, \Delta)$. Then, again, by [37, Proposition 1], we get a special degeneration from (X, cD) to (X', cD') induced by a 1-PS in $\mathrm{SL}(N_m + 1)$. By a similar argument as the proof of Proposition 3.13, the generalized Futaki invariant of this special

test configuration vanishes since both (X, cD) and (X', cD') are K-polystable. Hence, they are isomorphic.

- (4) Assume that (X, cD) is K-stable. Then it admits a weak conical Kähler–Einstein metric and $\text{Aut}(X, D)$ is reductive by (3). Hence, K-stability of (X, cD) implies that $\text{Aut}(X, D)$ is a finite group, which implies uniform K-stability of (X, cD) by [13, 14, 18, 30, 35]. The proof is finished. $\square$

3.3 | K-semistable thresholds are constructible

In this section, we show that the K-semistable thresholds are constructible functions satisfying certain semicontinuity conditions. In particular, this implies the openness of K-semistability in our setting. Our approach is based on [92, Section 7 and A.1].

Definition 3.14. Fix a rational number $\epsilon_0 \in (0, 1)$. For any (X, D) with $\text{Hilb}(X, D) \in Z^{\text{klt}}$, we define the *upper and lower K-semistable thresholds* as follows:

$$\text{kst}_{+, \epsilon_0}(X, D) := \sup\{c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\}) \mid (X, cD) \text{ is K-semistable}\};$$

$$\text{kst}_{-, \epsilon_0}(X, D) := \inf\{c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\}) \mid (X, cD) \text{ is K-semistable}\}.$$

Next, we will start the construction of the K-moduli stack $\mathcal{KM}_{\chi_0, r, c}$. In this section, we will focus on showing the openness of c -K-semistability and constructibility of K-semistable thresholds.

Theorem 3.15. *The functions $\text{kst}_{\pm, \epsilon_0}$ on Z^{klt} are constructible with rational values. Moreover, $\text{kst}_{+, \epsilon_0}$ (resp. $\text{kst}_{-, \epsilon_0}$) is lower (resp. upper) semicontinuous on Z^{klt} . In particular, Z_c° are Zariski open subsets of Z^{klt} whenever $c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\})$.*

Note that the semicontinuity properties of these types of functions (in relation to existence of conical KE metrics) were observed earlier in [92, 115].

Before presenting the proof of Theorem 3.15, we recall some results from [92, Section A.1] (see also [99, Chapter 2, Proposition 2.14] and [103, Proof of Lemma 2.11]).

Lemma 3.16 [92, Lemma A.3]. *Let Z be a projective variety. Let L, M be two G -linearized ample line bundles over Z . Let $T \subset G$ be a maximal torus. Then there is a finite set of linear functionals $l_1^L, \dots, l_{r_L}^L, l_1^M, \dots, l_{r_M}^M$ that are rational on $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ with the following property:*

For any $z \in Z$, there exist $I(z, L) \subset \{1, \dots, r_L\}$, $I(z, M) \subset \{1, \dots, r_M\}$ such that the λ -weight of $z \in Z$ with respect to the linearization of G on $L \otimes M^{-1}$ is given by

$$\mu^L(z, \lambda) - \mu^M(z, \lambda) = \max\{l_i^L(\lambda) \mid i \in I(z, L)\} - \max\{l_i^M(\lambda) \mid i \in I(z, M)\}$$

for all 1-PS λ of T . Moreover, the function

$$\begin{aligned} \psi^{L, M} : Z &\rightarrow 2^{\{1, \dots, r_L\}} \times 2^{\{1, \dots, r_M\}} \\ z &\mapsto (I(z, L), I(z, M)) \end{aligned}$$

is constructible.

We present the proof of Theorem 3.15 below which essentially follows from [92, Proposition A.4].

Proof of Theorem 3.15. Let us consider the c -K-stability on Z^{klt} for $c \in (0, \min\{1, (1 - \epsilon_0)r^{-1}\})$. Denote by $\pi : (\mathcal{X}, D) \rightarrow \bar{Z}$ the universal family with $\mathcal{L}$ representing the pull back of the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$. According to Definition 2.18, denote by

$$M_1 := \lambda_{\text{CM}, \pi, \mathcal{L}} \quad \text{and} \quad M_2 := \frac{n(\mathcal{L}_z^{n-1} \cdot D_z)}{(\mathcal{L}_z^n)} \lambda_{\text{Chow}, \pi, \mathcal{L}} - (n+1) \lambda_{\text{Chow}, \pi|_D, \mathcal{L}|_D}. \quad (3.2)$$

Hence, from Definition 2.18, we know that $\lambda_{\text{CM}, \pi, cD, \mathcal{L}} = M_1 - cM_2$. Notice that by flatness of the universal family, this function $z \in \bar{Z} \mapsto n(\mathcal{L}_z^{n-1} \cdot D_z)/(\mathcal{L}_z^n)$ does not depend on the choice of z . Let (X, D) be a log pair with $z := \text{Hilb}(X, D) \in Z^{\text{klt}}$. For simplicity, denote by $G := \text{SL}(N+1)$. Then every 1-PS $\lambda : \mathbb{G}_m \rightarrow G$ naturally induces a test configuration $(\mathcal{X}_\lambda, cD_\lambda; \mathcal{L}_\lambda)$ of $(X, cD; L)$. Moreover, Proposition 2.19 implies that

$$\text{Fut}(\mathcal{X}_\lambda, D_\lambda; \mathcal{L}_\lambda) = \frac{1}{(n+1)(L^n)} (\mu^{M_1}(z, \lambda) - c\mu^{M_2}(z, \lambda)).$$

Thus, Theorem 3.9 implies that (X, D) is c -K-semistable if and only if $c \leq \text{lct}(X; D) - \frac{\epsilon_0}{n+1}$ and

$$\mu^{M_1}(z, \lambda) - c\mu^{M_2}(z, \lambda) \geq 0 \quad \text{for any 1-PS } \lambda \text{ of } G.$$

Pick a sufficiently divisible positive integer k such that $M_1^{\otimes k}$ and $M_2^{\otimes k}$ are line bundles over $\bar{Z}$. Let M be a sufficiently ample $\text{SL}(N+1)$ -line bundle on $\bar{Z}$ such that $L_1 := M \otimes M_1^{\otimes k}$ and $L_2 := M \otimes M_2^{\otimes k}$ are both ample line bundles. Then we have

$$k(\mu^{M_1}(z, \lambda) - c\mu^{M_2}(z, \lambda)) = (\mu^{L_1}(z, \lambda) - \mu^M(z, \lambda)) - c(\mu^{L_2}(z, \lambda) - \mu^M(z, \lambda)).$$

We fix a maximal torus $T \subset G$. Hence, using Lemma 3.16, we know that there exists a decomposition $Z^{\text{klt}} = \sqcup_I S_I^T$ to constructible subsets S_I^T where I belongs to some finite index set, such that for any $z \in S_I^T$, the functions $\mu^{M_1}(z, \cdot)$ and $\mu^{M_2}(z, \cdot)$ are rational piecewise linear functions on $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ that are independent of the choice of z . We denote these two functions by $\mu_{1,I}(\cdot)$ and $\mu_{2,I}(\cdot)$, respectively. On the other hand, since any 1-PS λ of G is conjugate via $g \in G$ to a 1-PS $g\lambda g^{-1}$ of T , and $\mu^{M_i}(z, \lambda) = \mu^{M_i}(g \cdot z, g\lambda g^{-1})$ for $i = 1, 2$, we know that (X, D) is c -K-semistable if and only if $c \leq \text{lct}(X; D) - \frac{\epsilon_0}{n+1}$ and

$$\mu_{1,I}(\lambda) \geq c\mu_{2,I}(\lambda) \quad \text{for any 1-PS } \lambda \text{ of } T \text{ and any } I \text{ with } z \in S_I^G.$$

Here $S_I^G := G \cdot S_I^T$ is a constructible subset of Z^{klt} by Chevalley's Lemma [51, Exercise II.3.19]. Since $\mu_{i,I}$ is a rational piecewise linear function on $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$, the union $\cup_I S_I^G = Z^{\text{klt}}$, and $z \in Z^{\text{klt}} \mapsto \text{lct}(X; D)$ is a constructible function with rational values (see, e.g., [11, Corollary 2.10]), we know that $\text{kst}_{\pm, \epsilon_0}$ are constructible with rational values as well. Their semicontinuity follows from the very genericity of K-semistability [22, Theorem 3]. $\square$

Corollary 3.17. *Let $\pi : (\mathcal{X}, cD) \rightarrow T$ be a $\mathbb{Q}$ -Gorenstein smoothable log Fano family over a normal base T where $D \sim_{\pi} -rK_{\mathcal{X}/T}$. Then for any $c \in (0, \min\{1, r^{-1}\})$, the set*

$$\{t \in T \mid (\mathcal{X}_t, cD_t) \text{ is } K\text{-semistable}\}$$

is a Zariski open subset of T .

Proof. For each $c \in (0, \min\{1, r^{-1}\})$, we may choose $0 < \epsilon_0 \ll 1$ such that $c < (1 - \epsilon_0)r^{-1}$. Then the result follows from Theorems 3.4 and 3.15. $\square$

We finish this section with a useful result on K -polystability. See [90, Theorem 1.1] for a related result in the smooth case.

Proposition 3.18. *Let (X, c_0D) be a $\mathbb{Q}$ -Gorenstein smoothable K -polystable log Fano pair. Then the set*

$$\{c \in (0, \min\{1, r^{-1}\}) : (X, cD) \text{ is } K\text{-polystable}\}$$

is either $\{c_0\}$ or an open interval containing c_0 .

Proof. By choosing $0 < \epsilon_0 \ll 1$, we may assume that $c_0 < \min\{1, (1 - \epsilon_0)r^{-1}, \text{lct}(X; D) - \frac{\epsilon_0}{n+1}\}$. By Proposition 2.13, we know that the set

$$J := \{c \in (0, \min\{1, r^{-1}\}) : (X, cD) \text{ is } K\text{-polystable}\}$$

is either $\{c_0\}$ or an interval containing c_0 . Hence, it suffices to show that J contains an open neighborhood of c_0 . Denote by $z_0 = \text{Hilb}(X, D) \in Z^{\text{klt}}$. Let $G := \text{SL}(N + 1)$ and T be a maximal torus of G .

Assume to the contrary that J does not contain any neighborhood of c_0 and $J \neq \{c_0\}$. Then there exists $0 < |\epsilon| \ll 1$ such that $(X, (c_0 + \epsilon)D)$ is K -polystable, and $(X, (c_0 - \epsilon')D)$ is K -unstable whenever $0 < \epsilon'/\epsilon \leq 1$. From the proof of Theorem 3.15, we know that

$$\mu_{1,I}(\lambda) < (c_0 - \epsilon')\mu_{2,I}(\lambda) \quad \text{for some 1-PS } \lambda \text{ of } T \text{ and some } I \text{ with } z \in S_I^G.$$

A priori λ and I may depend on the choice of ϵ' . Nevertheless, since I belongs to a finite index set, and $\mu_{i,I}$ is a rational piecewise linear function on $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ for $i = 1, 2$, there exist λ and I that are independent of the choice of ϵ' satisfying

$$\mu_{1,I}(\lambda) < (c_0 - \epsilon')\mu_{2,I}(\lambda) \quad \text{whenever } 0 < \epsilon'/\epsilon \leq 1. \quad (3.3)$$

In particular, we know that $\mu_{1,I}(\lambda) \leq c_0\mu_{2,I}(\lambda)$. Since $z_0 \in S_I^G = G \cdot S_I^T$, we choose $g \in G$ such that $g \cdot z_0 \in S_I^T$. Then, since (X, c_0D) is K -polystable, we have

$$\mu_{1,I}(\lambda) = \mu^{M_1}(z_0, g^{-1}\lambda g) \geq c_0\mu^{M_2}(z_0, g^{-1}\lambda g) = c_0\mu_{2,I}(\lambda). \quad (3.4)$$

Combining (3.3) and (3.4), we have that $\mu^{M_1}(z_0, g^{-1}\lambda g) = c_0\mu^{M_2}(z_0, g^{-1}\lambda g) \neq 0$. This together with the K-polystability of (X, c_0D) implies that $g^{-1}\lambda g$ induces an almost product test configuration of (X, c_0D) . Since $(X, (c_0 + \epsilon)D)$ is also K-polystable, we have

$$\mu^{M_1}(z_0, g^{-1}\lambda g) = (c_0 + \epsilon)\mu^{M_2}(z_0, g^{-1}\lambda g),$$

which implies $\mu^{M_i}(z_0, g^{-1}\lambda g) = 0$ for $i = 1, 2$. However, this contradicts to (3.3). Thus, the proof is finished. $\square$

3.4 | Properness

In this section, we prove the valuative criterion of properness of K-moduli spaces. Recall that Blum and Xu [27] proved separatedness of K-moduli spaces (if they exist) for log Fano pairs. Hence, we only need to show compactness of K-moduli spaces, that is, the existence of a K-semistable filling for a K-semistable family over a punctured smooth curve.

Theorem 3.19. *Let $0 \in B$ be a smooth pointed curve. Let $\pi^\circ : (\mathcal{X}^\circ, cD^\circ) \rightarrow B^\circ$ be a $\mathbb{Q}$ -Gorenstein smoothable log Fano family over $B^\circ := B \setminus \{0\}$ where $D^\circ \sim_{\pi^\circ} -rK_{\mathcal{X}^\circ/B^\circ}$ and $c \in (0, \min\{1, r^{-1}\})$. If all fibers of π° are K-semistable, then there exists a quasi-finite morphism $(0' \in B') \rightarrow (0 \in B)$ from a smooth pointed curve $0' \in B'$ and a $\mathbb{Q}$ -Gorenstein smoothable log Fano family $\pi' : (\mathcal{X}', cD') \rightarrow B'$ such that $(\mathcal{X}', D') \times_{B'} B'^\circ \cong (\mathcal{X}^\circ, D^\circ) \times_{B^\circ} B'^\circ$ where $B'^\circ := B' \setminus \{0'\}$ and $(\mathcal{X}'_{0'}, cD'_{0'})$ is K-semistable (even K-polystable).*

Proof. By choosing $0 < \epsilon_0 \ll 1$, we may assume that $c < \min\{1, (1 - \epsilon_0)r^{-1}\}$. Let $b_i \rightarrow 0$ be a sequence of points in B° . Denote by $(\mathcal{X}_{b_i}, D_{b_i})$ the fiber of π° over b_i . Let χ_0 be the Hilbert polynomial of a smoothing of each fiber $\mathcal{X}_{b_i}$ that certainly does not depend on the choice of i or smoothing. Let us choose m_3, m, γ_0, q , and a as in Notation 3.10. Since $(\mathcal{X}_{b_i}, cD_{b_i})$ is K-semistable, we have $c \leq \text{lct}(\mathcal{X}_{b_i}; D_{b_i})$. Hence, Proposition 3.5 implies that there exists $\Delta_{b_i} \in |-qK_{\mathcal{X}_{b_i}}|$ such that $(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \gamma_0)a\Delta_{b_i})$ is uniformly K-stable. Let us choose $\gamma_0 \leq \beta_i \nearrow 1$. Then, Propositions 2.13 and 3.11 imply that $(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\Delta_{b_i})$ admits a weak conical Kähler–Einstein metric whose Hilbert point in $\mathbb{H}^{\mathcal{X}, q; N}$ is the limit of Hilbert points of conical Kähler–Einstein log smooth log Fano pairs. In particular, there exists a conical Kähler–Einstein log smooth log Fano pair $(Y_i, cE_i + (1 - \beta_i)a\Gamma_i)$ as a smoothing of $(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\Delta_{b_i})$ such that $\text{Hilb}(Y_i, E_i) \in Z_m, \Gamma_i \in |-qK_{Y_i}|$, and

$$\lim_{i \rightarrow \infty} \text{dist}_{\mathbb{H}^{\mathcal{X}, q; N}} \left(\text{Hilb}(Y_i, cE_i + (1 - \beta_i)a\Gamma_i), \text{Hilb}(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\Delta_{b_i}) \right) = 0. \quad (3.5)$$

By Theorem 3.6, there exists a sequence of matrices $g_i \in \text{U}(N + 1)$ and a log Fano pair $(Y, cE + 0 \cdot \Gamma)$ in $\mathbb{P}^N$ admitting a weak conical Kähler–Einstein metric such that

$$g_i \cdot \text{Hilb}(Y_i, cE_i + (1 - \beta_i)a\Gamma_i) \rightarrow \text{Hilb}(Y, cE + 0 \cdot \Gamma) \in \mathbb{H}^{\mathcal{X}, q; N} \quad \text{as } i \rightarrow \infty.$$

This together with (3.5) implies that

$$g_i \cdot \text{Hilb}(\mathcal{X}_{b_i}, cD_{b_i} + (1 - \beta_i)a\Delta_{b_i}) \rightarrow \text{Hilb}(Y, cE + 0 \cdot \Gamma) \in \mathbb{H}^{\mathcal{X}, q; N} \quad \text{as } i \rightarrow \infty.$$

Thus, there exists $g'_i \in \mathrm{SL}(N+1)$ such that $g'_i \cdot \mathrm{Hilb}(\mathcal{X}_{b_i}, \mathcal{D}_{b_i})$ converges to $\mathrm{Hilb}(Y, E)$ in $\bar{Z}_m$. Thus, by Lemma 3.12 after a quasi-finite base change of π° , we may fill in (Y, cE) as the K-polystable central fiber. The proof is finished. $\square$

3.5 | Almost log Calabi–Yau cases

Notice that Definition 3.8 depends on the choice of ϵ_0 . Indeed, if $r < 1$, then it suffices to choose $\epsilon_0 = 1 - r$. When $r \geq 1$, we will show that there exists $\epsilon_0 = \epsilon_0(n, r) \in (0, 1)$ such that the K-moduli spaces/stacks are the same for any $c \in [(1 - \epsilon_0)r^{-1}, r^{-1}]$.

Theorem 3.20. *For any $n \in \mathbb{Z}_{>0}$ and any rational number $r \geq 1$, there exists $\epsilon_0 = \epsilon_0(n, r) \in (0, 1)$ such that for any $\mathbb{Q}$ -Gorenstein smoothable log Fano pair (X, cD) with $c \in [(1 - \epsilon_0)r^{-1}, r^{-1}]$, it is K-(poly/semi)stable if and only if $(X, (1 - \epsilon_0)r^{-1}D)$ is K-(poly/semi)stable.*

Proof. First, let us assume that $(X, (1 - \epsilon_0)r^{-1}D)$ is K-(poly/semi)stable hence klt. By ascending chain condition (ACC) of log canonical thresholds [55], there exists $\epsilon_1 = \epsilon_1(n, r)$ such that $(X, r^{-1}D)$ is log canonical whenever $(X, (1 - \epsilon_1)r^{-1}D)$ is log canonical. This is guaranteed for any $\epsilon_0 \in (0, \epsilon_1]$ since $(X, (1 - \epsilon_0)r^{-1}D)$ is klt. Thus, (X, cD) is K-(poly/semi)stable for any $c \in [(1 - \epsilon_0)r^{-1}, r^{-1}]$ provided $\epsilon_0 \in (0, \epsilon_1]$.

Next, let (X, cD) be a $\mathbb{Q}$ -Gorenstein smoothable log Fano pair for some $c \in (0, 1)$. We may choose a smoothing $\pi : (\mathcal{X}, D) \rightarrow B$ over a smooth pointed curve $(0 \in B)$ such that π is smooth over $B \setminus \{0\}$ and $(\mathcal{X}_0, D_0) \cong (X, D)$. By Lemma 3.21, we may choose $\epsilon_2 = \epsilon_2(n, r) \in (0, 1)$ such that $(\mathcal{X}_{b'}, c'D_{b'})$ is K-polystable for any $c' \in [(1 - \epsilon_2)r^{-1}, r^{-1}]$ and any $b \in B \setminus \{0\}$. For simplicity, let us assume $\epsilon_2 \leq \epsilon_1$. Then by Theorem 3.19, we know that there exists a K-polystable limit $(X', (1 - \epsilon_2)r^{-1}D')$ of $(\mathcal{X}_{b'}, (1 - \epsilon_2)r^{-1}D_{b'})$ after possibly passing to a finite cover of B . Since $\epsilon_2 \leq \epsilon_1$, we know that $(X', r^{-1}D')$ is log canonical. Then by Proposition 2.13, we know that $(X', c'D')$ is the K-polystable limit of $(\mathcal{X}_{b'}, c'D_{b'})$ whenever $c' \in [(1 - \epsilon_2)r^{-1}, r^{-1}]$. Let us choose $\epsilon_0 := \frac{\epsilon_2}{2}$. Assume that (X, cD) is K-(poly/semi)stable for some $c \in [(1 - \epsilon_0)r^{-1}, r^{-1}]$. Then by [27] we know that (X, cD) specially degenerates to the K-polystable pair (X', cD') . Hence, $(X, (1 - \epsilon_2)r^{-1}D)$ specially degenerates to the K-polystable pair $(X, (1 - \epsilon_2)r^{-1}D')$. In particular, $(X, (1 - \epsilon_2)r^{-1}D)$ is K-semistable by Theorem 2.16. Again, by Proposition 2.13, we know that $(X, (1 - \epsilon_0)r^{-1}D)$ is K-(poly/semi)stable. The proof is finished. $\square$

Lemma 3.21. *Let $n \in \mathbb{Z}_{>0}$ and $r \geq 1$ be a rational number. Then there exists $\epsilon_2 := \epsilon_2(n, r) \in (0, 1)$ such that for any pair (X, D) where X is an n -dimensional Fano manifold, D is a smooth prime divisor on X and $D \sim_{\mathbb{Q}} -rK_X$, we have that (X, cD) is K-polystable whenever $c \in [(1 - \epsilon_2)r^{-1}, r^{-1}]$.*

Proof. When $r > 1$, this is a consequence of [92, Theorem 5.2] based on ACC of log canonical thresholds [55]. When $r = 1$, we know that $D \sim -K_X$ since X is a Fano manifold. By boundedness of Fano manifolds, there exists a smooth proper morphism $\pi : (\mathcal{X}, D) \rightarrow T$ over a (possibly disconnected) normal base scheme T that parametrizes all pairs (X, D) where X is a Fano manifold and D is a smooth anticanonical divisor on X . For each $\epsilon \in (0, 1)$, let us consider the subset

$$T_\epsilon := \{t \in T \mid (\mathcal{X}_t, (1 - \epsilon)D_t) \text{ is K-semistable}\}.$$

By Corollary 3.17, we know that T_ϵ is an open subset of T . Since $(\mathcal{X}_t, D_t)$ is log canonical, by Proposition 2.13, we know that $T_\epsilon \subset T_{\epsilon'}$ whenever $0 < \epsilon' < \epsilon < 1$. Therefore, the noetherian property implies that (T_ϵ) stabilizes as $0 < \epsilon \ll 1$. By [62, Corollary 1], we know that for each $t \in T$, there exists $\beta_t \in (0, 1)$ such that $(\mathcal{X}_t, (1 - \beta_t)D_t)$ is K-polystable. In particular, we have $T_\epsilon = B$ for any $0 < \epsilon \ll 1$. Then, again by interpolation, we may choose $0 < \epsilon_2 \ll 1$ such that $(\mathcal{X}_t, (1 - \epsilon_2)D_t)$ is K-polystable for any $t \in T$. The proof is finished. $\square$

The following result on boundedness is an easy consequence of Theorems 3.4 and 3.20.

Corollary 3.22. *Fix $r > 1$ a positive rational number and n a positive integer. Then the following collection of $\mathbb{Q}$ -Gorenstein smoothable pairs*

$$\{(X, D) \mid \dim X = n, (X, cD) \text{ is K-semistable for some } c \in [0, r^{-1}]\}$$

is log bounded.

Next, we prove finiteness of K-moduli walls.

Proposition 3.23. *There exist rational numbers $0 = c_0 < c_1 < c_2 < \dots < c_k = \min\{1, r^{-1}\}$ such that for any $c, c' \in (c_i, c_{i+1})$ and any $0 \leq i \leq k - 1$, we have $Z_c^{\text{red}} = Z_{c'}^{\text{red}}$. Moreover, $Z_{c_i \pm \epsilon}^{\text{red}}$ are Zariski open subsets of $Z_{c_i}^{\text{red}}$ for each $1 \leq i \leq k - 1$.*

Proof. The first statement follows from combining Theorems 3.15 and 3.20. The second statement follows from the continuity of generalized Futaki invariants with respect to coefficients. $\square$

3.6 | Stabilization of quotient stacks

Next, we study the stabilization problem for the stacks $[Z_{c,m}^{\text{red}}/\text{PGL}(N_m + 1)]$.

Theorem 3.24. *Assume that m is sufficiently divisible. Then for each $k \in \mathbb{Z}_{>0}$, there exists a canonical isomorphism*

$$\Theta_k : [Z_{c,m}^{\text{red}}/\text{PGL}(N_m + 1)] \rightarrow [Z_{c,km}^{\text{red}}/\text{PGL}(N_{km} + 1)]$$

between reduced Artin stacks.

Proof. We first construct Θ_k as a morphism. For simplicity, denote by $T := Z_{c,m}^{\text{red}}$, $T' := Z_{c,km}^{\text{red}}$, $N := N_m$, and $N' := N_{km}$. Let $(\mathcal{X}, D) \subset \mathbb{P}^N \times T$ be the pull-back family of the universal family over the Hilbert scheme. Denote by $\pi : (\mathcal{X}, D) \rightarrow T$ the projection morphism. Then, $\pi_* \mathcal{O}_{\mathcal{X}}(k)$ is a rank $N' + 1$ vector bundle over T . Let $p : \mathcal{P} \rightarrow T$ be the $\text{PGL}(N' + 1)$ -torsor corresponding to projectivized basis of the vector bundle $\pi_* \mathcal{O}_{\mathcal{X}}(k)$. Then we will define a $\text{PGL}(N' + 1)$ -equivariant morphism $f : \mathcal{P} \rightarrow T'$ as follows. Since T' is a locally closed subscheme of the Hilbert scheme $\mathbb{H}_k^{\mathcal{X}}; N'$, we will first construct $f : \mathcal{P} \rightarrow \mathbb{H}_k^{\mathcal{X}}; N'$. Consider $\pi_{\mathcal{P}} : (\mathcal{X}_{\mathcal{P}}, D_{\mathcal{P}}) \rightarrow \mathcal{P}$ where $(\mathcal{X}_{\mathcal{P}}, D_{\mathcal{P}}) := (\mathcal{X}, D) \times_T \mathcal{P}$. Then we have a closed embedding $(\mathcal{X}_{\mathcal{P}}, D_{\mathcal{P}}) \hookrightarrow \mathbb{P}^{N'} \times \mathcal{P}$ given by the projectivized basis information encoded in $\mathcal{P}$. This gives a morphism $f : \mathcal{P} \rightarrow \mathbb{H}_k^{\mathcal{X}}; N'$.

Since T contains a Zariski dense open subset $T \cap Z_m$ parametrizing smooth log Fano pairs, we know that the restriction of f on $p^{-1}(T \cap Z_m)$ factors through Z_{km} . Thus, f factors through the scheme-theoretic closure $\bar{Z}_{km}$. It is clear that the image of f lies inside $\text{Supp}(T')$, so f factors as $\mathcal{P} \xrightarrow{f} T' \hookrightarrow \mathbb{H}^{X_k; N'}$ where the latter map is a locally closed embedding. It is clear that f is $\text{PGL}(N' + 1)$ -equivariant. Thus, f descends to a morphism $g : T \rightarrow [T'/\text{PGL}(N' + 1)]$. On the other hand, we may lift the $\text{PGL}(N + 1)$ -action on T to $\mathcal{P}$ via push forward sections. It is clear that f is $\text{PGL}(N + 1)$ -invariant that implies that g is also $\text{PGL}(N + 1)$ -invariant. Thus, we obtain Θ_k as the descent of g .

From the above arguments, it is clear that the actions of $\text{PGL}(N + 1)$ and $\text{PGL}(N' + 1)$ on $\mathcal{P}$ commute. Hence, to show Θ_k is an isomorphism, it suffices to show that $f : \mathcal{P} \rightarrow T'$ is a $\text{PGL}(N + 1)$ -torsor, from which Θ_k^{-1} can be constructed easily. Let us consider the pull back of the universal family $(\mathcal{X}', D') \subset \mathbb{P}^{N'} \times T'$ with $\pi' : (\mathcal{X}', D') \rightarrow T'$. Since all fibers of π' are klt with the same volume, by [72, Theorem 5.4], we know that $\mathcal{X}' \rightarrow T'$ is a locally stable family, in particular, $K_{\mathcal{X}'/T'}$ is $\mathbb{Q}$ -Cartier whose Cartier index is divisible by km . Since $-mK_{\mathcal{X}'_t}$ is Cartier for any fiber $\mathcal{X}'_t = \pi'^{-1}(t')$, we know that $-mK_{\mathcal{X}'/T'}$ is also Cartier. It is also clear from the construction that $t' \mapsto h^0(\mathcal{X}'_t, \omega_{\mathcal{X}'_t}^{[m]})$ is a constant function on T' . Hence, the coherent sheaf $\pi'_* \omega_{\mathcal{X}'/T'}^{[m]}$ is a vector bundle of rank $N + 1$ on T' . Thus, we may cover T' by Zariski open subsets T'_i which trivialize $\pi'_* \omega_{\mathcal{X}'/T'}^{[m]}$. Then over T'_i , a basis of sections of $\pi'_* \omega_{\mathcal{X}'/T'}^{[m]}$ gives us a Zariski local section $T'_i \rightarrow \mathcal{P}$ of f . These sections enable us to trivialize the map $f : \mathcal{P} \rightarrow T'$ over T'_i . $\square$

Remark 3.25. As a consequence of Theorem 3.24, we know that the K-moduli stack $\mathcal{KM}_{\chi_0, r, c}$ represents the following moduli pseudofunctor over reduced base S :

$$\mathcal{KM}_{\chi_0, r, c}(S) = \left\{ (\mathcal{X}, D)/S \left| \begin{array}{l} (\mathcal{X}, cD)/S \text{ is a } \mathbb{Q}\text{-Gorenstein smoothable log Fano family,} \\ D \sim_{S, \mathbb{Q}} -rK_{\mathcal{X}/S}, \text{ each fiber } (\mathcal{X}_s, cD_s) \text{ is K-semistable,} \\ \text{and } \chi(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}(-kK_{\mathcal{X}_s})) = \chi_0(k) \text{ for } k \text{ sufficiently divisible.} \end{array} \right. \right\}.$$

3.7 | Existence of good moduli spaces and local VGIT

In this section, we will show that the K-moduli stack $\mathcal{KM}_{\chi_0, r, c}$ admits a proper good moduli space $KM_{\chi_0, r, c}$ generalizing [92, Section 8]. Moreover, there are finitely many wall crossings when c varies in the interval $(0, \min\{1, r^{-1}\})$, and each wall crossing has a local VGIT presentation in the sense of [6, (1.2)].

We follow Notation 3.10. Throughout this section, we will assume that $c \leq \min\{1, (1 - \epsilon_0)r^{-1}\}$ thanks to Theorem 3.20. Let us fix two Plücker embeddings $\mathbb{H}^{X; N} \hookrightarrow \mathbb{P}^M$ and $\mathbb{H}^{X; N} \hookrightarrow \mathbb{P}^{\tilde{M}}$. Then we have an embedding $\mathbb{H}^{X; N} \hookrightarrow \mathbb{P}^M := \mathbb{P}^M \times \mathbb{P}^{\tilde{M}}$. Let (X, cD) be a K-polystable log Fano pair parametrized by a point in Z_c^{red} . Then by Theorem 3.9, it admits a weak conical Kähler–Einstein metric and $\text{Aut}(X, D) \subset \text{SL}(N + 1)$ is reductive. (Note here that in order to obtain a natural linearization on $\mathcal{O}_{\mathbb{P}^M}(1, 1)$, we always treat the automorphism group as a subgroup of $\text{SL}(N + 1)$.) Let us pick a $\text{U}(N + 1)$ -invariant metric on $\mathbb{P}^M$ coming from product of $\text{U}(N + 1)$ -invariant Fubini–Study metrics on $\mathbb{P}^M$ and $\mathbb{P}^{\tilde{M}}$. Let $z_0 = (z_{0,1}, z_{0,2}) := \text{Hilb}(X, cD) \in \mathbb{P}^M$ be the Hilbert point of (X, cD) via Tian’s embedding with respect to the weak conical Kähler–Einstein metric. Then we

may decompose the tangent space as $\text{Aut}(X, D)$ -invariant subspaces

$$T_{z_0} \mathbb{P}^M = W \oplus \mathfrak{aut}(X, D)^\perp.$$

Similarly, we have $\mathbb{P}^M = \mathbb{P}(W_1 \oplus \mathbb{C} \cdot z_{0,1} \oplus \mathfrak{aut}_1(X, D)^\perp)$ and $\mathbb{P}^{\tilde{M}} = \mathbb{P}(W_2 \oplus \mathbb{C} \cdot z_{0,2} \oplus \mathfrak{aut}_2(X, D)^\perp)$. Let us take $z_0^* := (z_{0,1}^*, z_{0,2}^*) \in (\mathbb{P}^M)^* \times (\mathbb{P}^{\tilde{M}})^*$ be the dual point of z_0 . Then the locus $(z_0^* \neq 0)$ gives an open immersion $\mathbb{A}^M := \mathbb{A}^M \times \mathbb{A}^{\tilde{M}} \hookrightarrow \mathbb{P}^M$ that maps the origin to z_0 . Hence, the image of W under the exponential map is a vector subspace of $\mathbb{A}^M$ which we also denote by W . Let $\overline{W}$ be the Zariski closure of W .

It is clear that z_0^* is an $\text{Aut}(X, D)$ -fixed point. Hence, $\mathbb{A}^M$ and W are both $\text{Aut}(X, D)$ -invariant. We have two induced representations

$$\rho_1 : \text{Aut}(X, D) \rightarrow \text{SL}(W_1 \oplus \mathbb{C} \cdot z_{0,1}) \quad \text{and} \quad \rho_2 : \text{Aut}(X, D) \rightarrow \text{SL}(W_2 \oplus \mathbb{C} \cdot z_{0,2}).$$

Let $\rho := \rho_1 \boxtimes \rho_2$ be the product representation that induces a linearization of $\mathcal{O}_{\overline{W}}(1, 1)$. Denote by $\rho_{(X, D)} : \text{Aut}(X) \rightarrow \mathbb{G}_m$ the character corresponding to the $\text{Aut}(X, D)$ -linearization of $\mathcal{O}_{\mathbb{P}^M}(1, 1)|_{z_0}$. Since the universal family $(\mathcal{X}, D) \rightarrow \overline{Z}$ is $\text{Aut}(X, D)$ -equivariant, it induces an $\text{Aut}(X, D)$ -linearization on M_2 which we denote by ρ_{M_2} (see (3.2) for the definition of M_2).

Definition 3.26. Let $z \in \overline{W} \cap \overline{Z}$ be a point.

- (1) We say that z is c -GIT (poly/semi)stable if it is GIT (poly/semi)-stable with respect to the $\text{Aut}(X, D)$ -action on $\overline{W}$ with linearization $\rho \otimes \rho_{(X, D)}^{-1}$ of $\mathcal{O}_{\overline{W}}(1, 1)$.
- (2) We say that z is $(c + \epsilon)$ -GIT (poly/semi)stable for some $0 < |\epsilon| \ll 1$ if it is GIT (poly/semi)-stable with respect to the $\text{Aut}(X, D)$ -action on $\overline{W} \cap \overline{Z}$ with linearization $\rho \otimes \rho_{X, D}^{-1} \otimes \rho_{M_2}^{\otimes -\epsilon}$ of $\mathcal{O}_{\overline{W} \cap \overline{Z}}(1, 1) \otimes M_2|_{\overline{W} \cap \overline{Z}}^{\otimes -\epsilon}$.

The next theorem on local GIT chart is a direct generalization of [92, Theorem 8.8].

Theorem 3.27. *There is an $\text{Aut}(X, D)$ -invariant saturated affine Zariski open neighborhood U_W of $z_0 = \text{Hilb}(X, cD)$ in $\overline{W} \cap \overline{Z}$ such that every point in U_W is c -GIT semistable whose corresponding log pair is c -K-semistable, and for any $\text{Hilb}(Y, E) \in U_W$, (Y, cE) is K -polystable if and only if $\text{Hilb}(Y, E)$ is c -GIT polystable.*

Moreover, for all c -GIT polystable point $\text{Hilb}(Y, E) \in U_W$, we have $\text{Aut}(Y, E) < \text{Aut}(X, D)$, that is, the local GIT presentation $U_W // \text{Aut}(X, D)$ is stabilizer preserving in the sense of [6, Definition 2.5].

Proof. By definition, z_0 is c -GIT polystable. Since saturated affine open neighborhoods of z_0 form a basis of all Zariski open neighborhoods of z_0 , we just need to find one U_W that is an $\text{Aut}(X, D)$ -invariant Zariski open neighborhood of z_0 . The semistable equivalence part of the statement follows from the openness of GIT semistability and openness of K -semistability in our setting (see Corollary 3.17). For the polystable equivalence part, the proof is the same as the proof of [92, Theorem 8.8], except that we replace [92, Lemma 8.10] by Lemma 3.28.

Next, we prove the stabilizer preserving for polystable points. First, we recall the $U(N + 1)$ -invariant slice Σ constructed in [92, Summary 8.6]. Let Σ be the subset of $\mathbb{H}^{\mathcal{X}:N}$ consisting of Hilbert points of K -polystable log Fano pairs in Z_c^{red} via Tian's embedding with respect to their weak

conical Kähler–Einstein metrics. By Lemma 3.28, we know that Σ satisfies [92, Assumption A.9]. Hence, we obtain stabilizer preserving for polystable points by [92, Theorem A.10]. This finishes the proof. $\square$

Lemma 3.28. *Let $z_i = \text{Hilb}(X_i, D_i) \in Z_c^{\text{red}}$ be a sequence of Hilbert points of c -K-semistable log pairs converging to $z_0 = \text{Hilb}(X, cD)$. Then each (X_i, cD_i) specially degenerates to a K-polystable log Fano pair $(Y_i, cE_i) \in Z_c^{\text{red}}$, such that*

$$\lim_{i \rightarrow \infty} \text{dist}_{\|\cdot\|_{\mathbb{K}^N}}(\text{Hilb}(Y_i, cE_i), U(N+1) \cdot z_0) = 0,$$

where $\text{Hilb}(Y_i, cE_i)$ is the Hilbert point corresponding to Tian's embedding of (Y_i, cE_i) with respect to the weak Kähler–Einstein metric.

Proof. Assume to the contrary. After passing to a subsequence, we know that $z'_i := \text{Hilb}(Y_i, cE_i)$ converges to $z' := \text{Hilb}(Y, cE)$ by taking Gromov–Hausdorff limits as Theorem 3.6, and (Y, cE) is not isomorphic to (X, cD) . Since $z'_i \in \overline{\text{SL}(N+1) \cdot z_i}$, there exists a sequence $g_i \in \text{SL}(N+1)$ such that $g_i \cdot z_i \rightarrow z'$ as $i \rightarrow \infty$. Thus, $z' \in \overline{BO}_{z_0} \cap Z_c^{\text{red}}$ where $\overline{BO}_{z_0}$ is the broken orbit of z_0 with respect to the action of $\text{SL}(N+1)$ on $\bar{Z}$ (see [92, Section 3] for the definition). By [27] on the uniqueness of K-polystable limit, we know that $(Y, cE) \cong (X, cD)$, a contradiction. $\square$

The next result gives stabilizer preserving property of semistable points near z_0 as a straightforward consequence of [92, Lemma 8.12].

Lemma 3.29. *After possibly shrinking U_W , we have that $\text{Aut}(Y, E) < \text{Aut}(X, D)$ for any point $\text{Hilb}(Y, E) \in U_W$.*

Proof. We follow the proof of [92, Lemma 8.12]. By the proof of [92, Theorem 3.1], we know that the connected component $\text{Aut}_0(Y, E)$ is a subgroup of $\text{Aut}(X, D)$. Let us pick a finite subgroup $H < \text{Aut}(Y, E)$ that meets every connected component of $\text{Aut}(Y, E)$. Hence, it suffices to show that $H < \text{Aut}(X, D)$. We also know from [92, Proof of Theorem 8.8] that the set

$$\{\text{Hilb}(Y, E) \in \bar{W} \cap \bar{Z} \mid \text{Aut}(Y, E) < \text{Aut}(X, D)\}$$

is constructible. Hence, it suffices to show that the statement for z lies inside an analytic open neighborhood U_W^{an} of z_0 in $\bar{W} \cap \bar{Z}$. We may assume that z degenerates via a 1-PS λ of $\text{Aut}(X)$ to a c -GIT polystable point $z' = \text{Hilb}(Y', E')$ in U_W^{an} . Hence, Theorem 3.27 implies that (Y', cE') is K-polystable.

Since (Y, E) belongs to a bounded family, we may assume that $|H|$ is uniformly bounded from above. Since (Y, cE) is klt and belongs to a bounded family, there exists a positive integer $q_1 = q_1(n, r, \epsilon_0) > 1$ and an H -invariant divisor $\Gamma \in |-q_1 K_Y|$ such that $(Y, cE + \Gamma)$ is log canonical. Hence, by Proposition 2.13, we know that $(Y, cE + (1 - \beta)a\Gamma)$ is uniformly K-stable where $a = \frac{1-cr}{q_1}$ and $\beta \in (0, 1)$. Let us take $m_4 := \text{lcm}(m_2(\chi_0, r, q_1, \epsilon_0, \gamma_0), m_3)$ where $m_2(\chi_0, r, q_1, \epsilon_0, \gamma_0)$ is chosen as in Theorem 3.6. From now on we assume that m is a multiple of m_4 . Then by Proposition 3.11, for each $\beta \in [\gamma_0, 1)$, there exists an H -invariant weak conical Kähler–Einstein metric $\omega_{Y, cE}(\beta)$ together with a Tian embedding $\text{Hilb}(Y, cE, \omega_{Y, cE}(\beta)) \in \mathbb{H}^{\mathbb{K}^N}$ via $\omega_{Y, cE}(\beta)$. Moreover,

from the uniqueness of K-polystable limits [27], we have

$$\mathrm{Hilb}(Y, cE, \omega_{Y, cE}(\beta)) \xrightarrow{\beta \rightarrow 1} \mathrm{U}(N+1) \cdot \mathrm{Hilb}(Y', cE') \subset \mathbb{H}^{\mathcal{X}:N}.$$

Hence, the proof proceeds the same as [92, Proof of Lemma 8.12]. $\square$

The following Luna slice-type result is also a straightforward consequence of [92, Lemmas 8.15 and A.15]. We omit the proof here because it is identical to the proof therein on verifying the finite distance property.

Lemma 3.30. *After possibly shrinking U_W , the morphism*

$$\psi : \mathrm{SL}(N+1) \times_{\mathrm{Aut}(X, D)} U_W \rightarrow U$$

is a finite strongly étale $\mathrm{SL}(N+1)$ -morphism onto a Zariski open subset U of Z_c^{red} .

Now we are ready to prove Theorem 3.1, the first main result of our construction of K-moduli stacks and spaces.

Proof of Theorem 3.1. We first show that the Artin stack $\mathcal{KM}_{\chi_0, r, c}$ admits a good moduli space $KM_{\chi_0, r, c}$ as a proper reduced algebraic space. By [6, Theorem 1.2], this boils down to proving the following: for any closed point $[z_0] \in [Z_c^{\mathrm{red}}/\mathrm{SL}(N+1)]$, there is a saturated affine neighborhood $z_0 := \mathrm{Hilb}(X, cD) \in U_W \subset \overline{U} \cap \overline{Z}$ (as in Lemma 3.30) such that

- (1) The morphism $f : [U_W/\mathrm{Aut}(X, D)] \rightarrow [Z_c^{\mathrm{red}}/\mathrm{SL}(N+1)]$ is a local quotient presentation in the sense of [6, Definition 2.1]. Moreover, f is stabilizer preserving and sends closed point to closed point, and
- (2) For any $\mathbb{C}$ -point $z \in Z_c^{\mathrm{red}}$ specializing to z_0 under the $\mathrm{SL}(N+1)$ -action, the closure $\{\overline{z}\}$ admits a good moduli space.

We have shown the $\mathrm{Aut}(X, D)$ is reductive, and z_0 is a c -GIT polystable point with stabilizer $\mathrm{Aut}(X, D)$. Since $\mathrm{SL}(N+1) \times_{\mathrm{Aut}(X, D)} U_W$ is the quotient of the affine scheme $\mathrm{SL}(N+1) \times U_W$ by the free action of the reductive group $\mathrm{Aut}(X, D)$, we know that $\mathrm{SL}(N+1) \times_{\mathrm{Aut}(X, D)} U_W$ is also affine. Hence, Lemma 3.30 implies that f is étale and affine, in particular $U = \mathrm{SL}(N+1) \cdot U_W$ is affine. Thus, f is a local quotient presentation according to [6, Definition 2.1]. In Lemma 3.29, we showed that f is stabilizer preserving. By Theorem 3.9.2, we know that a closed point in $[Z_c^{\mathrm{red}}/\mathrm{SL}(N+1)]$ corresponds to a c -K-polystable pair. Thus, f sends closed point to closed point by Theorem 3.27. Hence, part (1) is proved. For part (2), notice that the closure $\overline{\mathrm{SL}(N+1) \cdot z}$ in Z_c^{red} is a closed subset of U since $\overline{\mathrm{SL}(N+1) \cdot z} = \mathrm{SL}(N+1) \cdot \mathrm{Aut}(X, D) \cdot z$ and ψ is finite onto U . Since U is affine, we know that $\overline{\mathrm{SL}(N+1) \cdot z}$ is also affine. Thus, we finish the proof of part (2).

Indeed, we have shown that $U // \mathrm{SL}(N+1)$ is affine. Hence, the good moduli space $KM_{\chi_0, r, c}$ is a reduced scheme. Its properness follows from [27] and Theorem 3.19. $\square$

The next theorem provides a local VGIT chart of K-moduli wall crossing in the slice W . Before stating the theorem, we recall a lemma we will need.

Lemma 3.31.

- (1) ([63]) Let G be a reductive group acting on a polarized projective scheme (Z, L) . Let $z \in Z$ be a closed point. Let $\lambda : \mathbb{G}_m \rightarrow G$ be a 1-PS. Denote by $z' = \lim_{t \rightarrow 0} \lambda(t) \cdot z$. If z is GIT semistable and $\mu^L(z, \lambda) = 0$, then z' is also GIT semistable.
- (2) ([93, Lemma 3.1]) Let (X, Δ) be a log Fano pair. Let $(\mathcal{X}, \tilde{\Delta}; \mathcal{L})/\mathbb{A}^1$ be a normal test configuration of (X, Δ) . If (X, Δ) is K-semistable and $\text{Fut}(\mathcal{X}, \tilde{\Delta}; \mathcal{L}) = 0$, then $(\mathcal{X}, \tilde{\Delta}; \mathcal{L})/\mathbb{A}^1$ is a special test configuration and (X_0, Δ_0) is also K-semistable.

Theorem 3.32. *There is an $\text{Aut}(X, D)$ -invariant saturated affine Zariski open neighborhood U_W (as in Lemma 3.30) of $z_0 = \text{Hilb}(X, cD)$ such that for any $\text{Hilb}(Y, E) \in U_W$ and any $|\epsilon| \ll 1$, the log Fano pair $(Y, (c + \epsilon)E)$ is K-(poly/semi)stable if and only if $\text{Hilb}(Y, E)$ is $(c + \epsilon)$ -GIT (poly/semi)stable.*

Proof. Let (Y, E) be a pair with $z := \text{Hilb}(Y, E) \in U_W$. Suppose that $(Y, (c + \epsilon)E)$ is K-semistable. We will show that $\text{Hilb}(Y, E)$ is $(c + \epsilon)$ -GIT semistable. Assume to the contrary that $\text{Hilb}(Y, E)$ is $(c + \epsilon)$ -GIT unstable, then there exists a 1-PS $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(X, D)$ such that

$$\mu^{\mathcal{O}(1,1)}(z, \lambda) - \epsilon \mu^{M_2}(z, \lambda) < 0.$$

From the discussion in Section 3.3, we can choose λ so that $\mu^{\mathcal{O}(1,1)}(z, \lambda) = 0$ and $\epsilon \mu^{M_2}(z, \lambda) > 0$. Since $z \in \overline{W} \cap \overline{Z}$ is c -GIT semistable, by Lemma 3.31.1, we know that $z' := \lim_{t \rightarrow 0} \lambda(t) \cdot z$ is also c -GIT semistable. Since U_W is saturated, we know that $z' \in U_W$. Let (Y', E') be the pair such that $\text{Hilb}(Y', E') = z'$. Then, Theorem 3.27 implies that (Y', cE') is also K-semistable. Hence, $\mu^{M_1}(z, \lambda) - c \mu^{M_2}(z, \lambda) = 0$ since λ induces a c -K-semistable family over $\mathbb{A}^1$. Let $(\mathcal{Y}_\lambda, (c + \epsilon)\mathcal{E}_\lambda; \mathcal{L}_\lambda)$ be the test configuration of $(Y, (c + \epsilon)E)$ induced by λ . Thus,

$$\text{Fut}(\mathcal{Y}_\lambda, (c + \epsilon)\mathcal{E}_\lambda; \mathcal{L}_\lambda) = \mu^{M_1}(z, \lambda) - (c + \epsilon) \mu^{M_2}(z, \lambda) = -\epsilon \mu^{M_2}(z, \lambda) < 0.$$

This contradicts the assumption that $(Y, (c + \epsilon)E)$ is K-semistable.

Next, we want to show $(c + \epsilon)$ -GIT semistability implies $(c + \epsilon)$ -K-semistability in U_W . Suppose $z = \text{Hilb}(Y, E) \in U_W$ is $(c + \epsilon)$ -GIT semistable. Assume to the contrary that $(Y, (c + \epsilon)E)$ is K-unstable. From Theorem 3.9, we know that to test K-(poly/semi)stability of $(Y, (c + \epsilon)E)$, it suffices to test all 1-PS in $\text{SL}(N + 1)$. Hence, there exists a 1-PS $\lambda : \mathbb{G}_m \rightarrow \text{SL}(N + 1)$ such that $\mu^{M_1}(z, \lambda) - (c + \epsilon) \mu^{M_2}(z, \lambda) < 0$. Again, from the discussion in Section 3.3, we may choose λ so that $\mu^{M_1}(z, \lambda) - c \mu^{M_2}(z, \lambda) = 0$ and $\epsilon \mu^{M_2}(z, \lambda) > 0$. Since (Y, cE) is K-semistable, by [94] we know that $\mathcal{Y}_\lambda$ is normal in codimension 1 where $(\mathcal{Y}_\lambda, c\mathcal{E}_\lambda; \mathcal{L}_\lambda)$ is the test configuration of (Y, cE) induced by λ . If $\mathcal{Y}_\lambda$ is not normal, then we may take its normalization and reembed it into $\mathbb{P}^N$ without changing the generalized Futaki invariants. By doing so we may assume that λ induces a normal test configuration of (Y, cE) . Denote by $z' = \text{Hilb}(Y', E') := \lim_{t \rightarrow 0} \lambda(t) \cdot z \in \overline{Z}$. Since (Y, cE) is K-semistable, Lemma 3.31.2 implies that (Y', cE') is also a K-semistable log Fano pair. Let $z_1 := \text{Hilb}(Y_1, E_1)$ be the c -GIT polystable degeneration of z in U_W . Hence, (Y_1, cE_1) is the K-polystable degeneration of (Y, cE) by Theorem 3.27. From Theorem 3.9, we know that (Y', cE') specially degenerates to a K-polystable log Fano pair (Y'_1, cE'_1) in $\mathbb{P}^N$ via a 1-PS λ' of $\text{SL}(N + 1)$. Since K-polystable degenerations of S-equivalent K-semistable log Fano pairs are isomorphic by [93], we know that $(Y_1, cE_1) \cong (Y'_1, cE'_1)$. Hence, there exists $g \in \text{SL}(N + 1)$ such that

$$\lim_{t \rightarrow 0} \lambda'(t) \cdot z' = g \cdot z_1 = \text{Hilb}(Y'_1, cE'_1).$$

By Lemma 3.30, we know that $U = \mathrm{SL}(N+1) \cdot U_W$ is a Zariski open subset of Z_c^{red} . Hence, $z, z_1, g \cdot z_1$, and z' all belong to U . Hence,

$$\epsilon \mu^{\psi^* M_2}((\mathrm{id}, z), \lambda) = \epsilon \mu^{M_2}(z, \lambda) > 0.$$

This implies that (id, z) is GIT unstable on $\mathrm{SL}(N+1) \times_{\mathrm{Aut}(X,D)} U_W$ with respect to the $\mathrm{SL}(N+1)$ -linearized $\mathbb{Q}$ -line bundle $\psi^* M_2^{\otimes -\epsilon}$. Since $\psi^* M_2$ is induced from the $\mathrm{Aut}(X,D)$ -linearized $\mathbb{Q}$ -line bundle $M_2|_{U_W}$ on U_W , we know that z is $(c+\epsilon)$ -GIT unstable in U_W . Thus, we reach a contradiction.

Finally, we show that the polystability conditions coincide. Denote by $U_{W,\epsilon}^{\mathrm{ss}}$ the $(c+\epsilon)$ -GIT semistable locus in U_W , and U_ϵ^{ss} the $(c+\epsilon)$ -K-semistable locus in U . From Lemma 3.30 and the discussion above, we know that $\psi : \mathrm{SL}(N+1) \times_{\mathrm{Aut}(X,D)} U_{W,\epsilon}^{\mathrm{ss}} \rightarrow U_\epsilon^{\mathrm{ss}}$ is a finite surjective strongly étale $\mathrm{SL}(N+1)$ -morphism. We also see that U is a saturated affine open subset of Z_c^{red} from the proof of Theorem 3.1; hence, U_ϵ^{ss} is a saturated open subset of $Z_{c+\epsilon}^{\mathrm{red}}$. Thus, a point $z = \mathrm{Hilb}(Y, E) \in U_{W,\epsilon}^{\mathrm{ss}}$ is $(c+\epsilon)$ -GIT polystable if and only if its $\mathrm{Aut}(X,D)$ -orbit is closed in $U_{W,\epsilon}^{\mathrm{ss}}$. This is equivalent to $\mathrm{SL}(N+1) \cdot z$ is closed in U_ϵ^{ss} that is the same as saying $(Y, (c+\epsilon)E)$ is K-polystable by saturatedness of U_ϵ^{ss} . The proof is finished. $\square$

Theorem 3.33. *Given any closed point $[z_0]$ in $\mathcal{KM}_{\chi_0, r, c}$ and $0 < \epsilon \ll 1$, after possibly shrinking U_W , there exist a local quotient representation $f : [U_W/G_{z_0}] \rightarrow \mathcal{KM}_{\chi_0, r, c}$, a G_{z_0} -linearized line bundle L_{z_0} on U_W , and a Cartesian diagram*

$$\begin{array}{ccccc} [U_W^-/G_{z_0}] & \hookrightarrow & [U_W/G_{z_0}] & \hookleftarrow & [U_W^+/G_{z_0}] \\ \downarrow f^- & & \downarrow f & & \downarrow f^+ \\ \mathcal{KM}_{\chi_0, r, c-\epsilon} & \xrightarrow{\Phi^-} & \mathcal{KM}_{\chi_0, r, c} & \xleftarrow{\Phi^+} & \mathcal{KM}_{\chi_0, r, c+\epsilon} \end{array}$$

such that the following are true.

- (1) The quotient stacks $[U_W^\pm/G_{z_0}]$ are the VGIT chambers of $[U_W/G_{z_0}]$ with respect to L_{z_0} (see [6, Definition 2.4] for a definition).
- (2) All vertical arrows are finite strongly étale morphisms onto saturated open substacks of K-moduli stacks.

In particular, we have a Cartesian diagram

$$\begin{array}{ccccc} U_W^- // G_{z_0} & \longrightarrow & U_W // G_{z_0} & \longleftarrow & U_W^+ // G_{z_0} \\ \downarrow f^- & & \downarrow f & & \downarrow f^+ \\ KM_{\chi_0, r, c-\epsilon} & \xrightarrow{\phi^-} & KM_{\chi_0, r, c} & \xleftarrow{\phi^+} & KM_{\chi_0, r, c+\epsilon} \end{array} \quad (3.6)$$

where all vertical arrows are finite étale morphisms onto Zariski open subsets of K-moduli spaces, and all horizontal morphisms are projective.

Proof. We first look at the K-moduli stack parts. Let $U_W^\pm$ be the $(c \pm \epsilon)$ -GIT semistable locus of U_W . By Lemma 3.30, we know that f is a finite strongly étale morphism onto $[U/\mathrm{SL}(N+1)]$ that is a saturated open substack of $\mathcal{KM}_{\chi_0, r, c} = [Z_c^{\mathrm{red}}/\mathrm{SL}(N+1)]$. The diagram is Cartesian

by Theorem 3.32. Hence, $f^\pm$ are finite étale. From the proof of Theorem 3.32, we know that $U^\pm := \mathrm{SL}(N+1) \cdot U_W^\pm = Z_{c \pm \epsilon}^{\mathrm{red}} \cap U$. Hence their saturatedness follows from saturatedness of U . The stabilizer preserving property of $f^\pm$ follows from 3.29. The morphisms $f^\pm$ also send closed point to closed point by Theorem 3.32. Hence, we finish proving part (2). For part (1), we take $L_{z_0} := \mathcal{O}_{\overline{W}}(1,1)|_{U_W}$ together with the G_{z_0} -linearization $\rho \otimes \rho_{(X,D)}^{-1}$ (see Definition 3.26). After shrinking U_W if necessary we may assume that $M_2|_{U_W}$ is trivial. Hence, the G_{z_0} -representation $\rho_{M_2}^{-1}$ on M_2^{-1} corresponds to a G_{z_0} -character $\chi_{L_{z_0}} : G_{z_0} \rightarrow \mathbb{G}_m$. Then the VGIT chamber statement follows from Theorem 3.32. For the K-moduli spaces statements, the Zariski open part is clear by definition of saturatedness, the finite part follows from the definition of good moduli spaces, and the strongly étale part follows from descent property (see [6, Proposition 2.7]). $\square$

Definition 3.34. Let $\pi : (\mathcal{X}, D) \rightarrow \overline{Z}$ be the universal family over $\overline{Z}$. Let $\pi_c : (\mathcal{X}_c, D_c) \rightarrow Z_c^{\mathrm{red}}$ be the base change of π to Z_c^{red} . Let $0 \leq c' < r^{-1}$ be another rational number. Then the CM $\mathbb{Q}$ -line bundle $\lambda_{\mathrm{CM}, \pi_c, c' D_c}$ on Z_c^{red} descends to a $\mathbb{Q}$ -line bundle $\lambda_{c, c'}$ on $\mathcal{KM}_{\chi_0, r, c} = [Z_c^{\mathrm{red}}/\mathrm{PGL}(N+1)]$. We call $\lambda_{c, c'}$ the CM $\mathbb{Q}$ -line bundle on $\mathcal{KM}_{\chi_0, r, c}$ with coefficient c' . We simply denote $\lambda_c := \lambda_{c, c}$. We also denote $\lambda_{c, \mathrm{Hodge}} := \lambda_{\mathrm{Hodge}, \pi_c, r^{-1} D}$.

Proposition 3.35. *With the above notation, the CM $\mathbb{Q}$ -line bundle λ_c on $\mathcal{KM}_{\chi_0, r, c}$ descends to a $\mathbb{Q}$ -line bundle Λ_c on the K-moduli space $KM_{\chi_0, r, c}$. If in addition $Z_c^{\mathrm{red}} = Z_{c \pm \epsilon}^{\mathrm{red}}$ for any $0 < \epsilon \ll 1$, then the Hodge $\mathbb{Q}$ -line bundle $\lambda_{c, \mathrm{Hodge}}$ and the CM $\mathbb{Q}$ -line bundle $\lambda_{c, c'}$ on $\mathcal{KM}_{\chi_0, r, c}$ descends to $\mathbb{Q}$ -line bundles $\Lambda_{c, \mathrm{Hodge}}$ and $\Lambda_{c, c'}$ on $KM_{\chi_0, r, c}$ for any $0 \leq c' < r^{-1}$, respectively. Moreover, we have the following interpolation formula:*

$$(1 - c'r)^{-n} \Lambda_{c, c'} = (1 - c'r) \Lambda_{c, 0} + c'r(n+1)(-K_X)^n \Lambda_{c, \mathrm{Hodge}}. \quad (3.7)$$

Proof. First, we show the statements on descents. Let $[z_0] = [\mathrm{Hilb}(X, D)]$ be any closed point of $\mathcal{KM}_{\chi_0, r, c}$. By [10, Theorem 10.3], to show that $\lambda_{c, c'}$ on $\mathcal{KM}_{\chi_0, r, c}$ can descend to $KM_{\chi_0, r, c}$, it suffices to show that the group of stabilizers $\mathrm{Aut}(X, D)$ acts trivially on $\lambda_{c, c'}^{\otimes k}$ for k sufficiently divisible. Since $\mathrm{Aut}(X, D)$ is reductive, it suffices to show that any 1-PS σ in $\mathrm{Aut}(X, D)$ acting on $\lambda_{c, c'}|_{[z_0]}$ has weight zero. By Proposition 2.19, this weight is a nonzero multiple of the generalized Futaki invariant of the product test configuration $(\mathcal{X}_\sigma, c' D_\sigma; \mathcal{L}_\sigma)$ of $(X, c' D)$ induced by σ . Since (X, cD) is K-polystable, we know that the σ -weight of λ_c at $[z_0]$ always vanishes. If in addition that $(X, (c \pm \epsilon)D)$ is K-semistable, then by the linearity of generalized Futaki invariants in terms of coefficients, we know that $\mathrm{Fut}(\mathcal{X}_\sigma, c' D_\sigma; \mathcal{L}_\sigma) = 0$ for any $0 \leq c' < r^{-1}$. By Proposition 2.25, we know that

$$(1 - c'r)^{-n} \lambda_{c, c'} = (1 - c'r) \lambda_{c, 0} + c'r(n+1)(-K_X)^n \lambda_{c, \mathrm{Hodge}}.$$

Hence, $\lambda_{c, \mathrm{Hodge}}$ descends to $KM_{\chi_0, r, c}$ if $\lambda_{c, c'}$ descends for all c' . Thus, the interpolation formula (3.7) is just the descent of the above equation. $\square$

Theorem 3.36. *With the above notation, the $\mathbb{Q}$ -line bundles $\Lambda_{c \pm \epsilon}$ on $KM_{\chi_0, r, c \pm \epsilon}$ are $\phi^\pm$ -ample. In addition, we have $\lim_{\epsilon \rightarrow 0} \Lambda_{c \pm \epsilon} = \Lambda_{c \pm \epsilon, c} = (\phi^\pm)^* \Lambda_c$.*

Proof. By Theorem 3.32, it suffices to verify the statements over the local GIT chart $U_W // G_{z_0}$ for each closed point $[z_0] \in \mathcal{KM}_{\chi_0, r, c}$. Recall from (3.2) that for the universal family $\pi : (\mathcal{X}, D) \rightarrow \overline{Z}$,

we have a line bundle $\mathcal{L}$ on $\mathcal{X}$ as the pull back of $\mathcal{O}_{\mathbb{P}^N}(1)$ and $\mathbb{Q}$ -line bundles M_i on $\overline{Z}$ such that $\lambda_{\text{CM}, \pi, c'D, \mathcal{L}} = M_1 - cM_2$. Let $\mathcal{L}_c$ be the pull back of $\mathcal{L}$ to $\mathcal{X}_c$. Since

$$\mathcal{L}_c \sim_{\mathbb{Q}, \pi_c} -mK_{\mathcal{X}_c/Z_c^{\text{red}}} \sim_{\mathbb{Q}, \pi_c} m(1-cr)^{-1}(-K_{\mathcal{X}_c/Z_c^{\text{red}}} - cD_c),$$

we know that

$$m^n(1-cr)^{-n}\lambda_{\text{CM}, \pi_c, cD_c} = \lambda_{\text{CM}, \pi, cD, \mathcal{L}}|_{Z_c^{\text{red}}} = (M_1 - cM_2)|_{Z_c^{\text{red}}}.$$

Thus, $m^n(1-cr)^{-n}(f^\pm)^*\lambda_{c\pm\epsilon}$ on $[U_W^\pm/G_{z_0}]$ is the descent of $(M_1 - (c\pm\epsilon)M_2)|_{U_W^\pm}$. By Definition 3.26, we know that the descent of $\pm M_2$ on $U_W^\pm//G_{z_0}$ is antiample over $U_W//G_{z_0}$, hence $\Lambda_{c\pm\epsilon}$ is $\phi^\pm$ -ample. The second statement follows directly from the above computations. $\square$

Finally, we are able to prove Theorem 3.2 using the above results.

Proof of Theorem 3.2. The statements follow from combining Proposition 3.23 and Theorems 3.1, 3.32, and 3.36. $\square$

4 | GENERAL PROPERTIES OF K-MODULI OF PLANE CURVES

In this section, we use results from Section 3 to construct K-moduli stacks (resp. K-moduli spaces) of plane curves parametrizing K-semistable (resp. K-polystable) log Fano pairs (X, cD) where X is a $\mathbb{Q}$ -Gorenstein degeneration of $\mathbb{P}^2$, the curve D is a degeneration of smooth plane curves of degree d , and $0 < c < 3/d$. We also study properties of these K-moduli stacks and spaces. We note that similar computations and comparisons of K-moduli to GIT were carried out in [106] in the case of low degree del Pezzo surfaces.

4.1 | Definition and properties

We first recall the GIT moduli stacks and spaces of plane curves.

Definition 4.1. Let d be a positive integer. Let $\mathbf{P}_d := \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)))$ be the projective space of dimension $\binom{d+2}{2} - 1$ parametrizing all plane curves of degree d . It is clear that the natural $\text{PGL}(3)$ -action on $\mathbb{P}^2$ lifts up to an action on $\mathbf{P}_d$.

- (1) The line bundle $\mathcal{O}_{\mathbf{P}_d}(1)$ has a unique $\text{SL}(3)$ -linearization. Let $\mathbf{P}_d^{\text{ss}}$ be the GIT semistable locus of $\mathbf{P}_d$ with respect to the $\text{SL}(3)$ -linearized line bundle $\mathcal{O}_{\mathbf{P}_d}(1)$. We define the *GIT moduli stack* $\overline{\mathcal{P}}_d^{\text{GIT}}$ of plane curves of degree d to be $\overline{\mathcal{P}}_d^{\text{GIT}} := [\mathbf{P}_d^{\text{ss}}/\text{PGL}(3)]$. We define the *GIT moduli space* of plane curves of degree d to be $\overline{P}_d^{\text{GIT}} := \mathbf{P}_d^{\text{ss}}//\text{SL}(3)$.
- (2) Let $\mathbf{P}_d^{\text{sm}}$ be the Zariski open subset of $\mathbf{P}_d$ parametrizing smooth plane curves. Then the *moduli stack* $\mathcal{P}_d$ of smooth plane curves of degree d is defined as $\mathcal{P}_d := [\mathbf{P}_d^{\text{sm}}/\text{PGL}(3)]$. When $d \geq 2$, it is clear from [99, Chapter 4 §2] that $\mathbf{P}_d^{\text{sm}}$ is a saturated Zariski open subset of $\mathbf{P}_d^{\text{ss}}$. Hence, $\mathcal{P}_d$ admits a good moduli space P_d as a Zariski open subset of $\overline{P}_d^{\text{GIT}}$. We call P_d the *moduli space*

of smooth plane curves of degree d . Notice that when $d = 1$, the GIT moduli space $\overline{\mathcal{P}}_d^{\text{GIT}}$ is empty, and $\mathcal{P}_d$ does not admit a good moduli space due to the nonreductivity of stabilizers; when $d \geq 3$, the stack $\mathcal{P}_d$ is Deligne–Mumford.

Now we begin with the definition of K-moduli stacks and spaces of plane curves. Recall that in Definition 3.8, we define K-moduli stacks and spaces of $\mathbb{Q}$ -Gorenstein smoothable log Fano pairs. In what follows, we adapt this definition to define K-moduli stacks and spaces of plane curves.

Definition 4.2. Let d and m be positive integers. Let $c \in (0, \min\{1, \frac{3}{d}\})$ be a rational number. Denote by $\chi(k) := \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3mk))$, $\tilde{\chi}(k) = \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3mk)) - \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3mk - d))$, $\chi := (\chi, \tilde{\chi})$, and $N = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3m)) - 1$. Let $\mathbb{H}^{\chi;N}$ be the Hilbert schemes of pairs $(X, D) \hookrightarrow \mathbb{P}^N$ of Hilbert polynomial $(\chi, \tilde{\chi})$.

We define

$$Z := \left\{ \text{Hilb}(X, D) \in \mathbb{H}^{\chi;N} \left| \begin{array}{l} (X, D) \cong (\mathbb{P}^2, C) \text{ where } C \text{ is a smooth plane curve of degree } d, \\ \mathcal{O}_{\mathbb{P}^N}(1)|_X \cong \mathcal{O}_X(-mK_X), \\ \text{and } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{\cong} H^0(X, \mathcal{O}_X(-mK_X)). \end{array} \right. \right\}.$$

In other words, Z parametrizes Hilbert points of $(3m)$ th Veronese embedding of $(\mathbb{P}^2, C)$ into $\mathbb{P}^N$. Then Z is a locally closed subscheme of $\mathbb{H}^{\chi;N}$. Let $\overline{Z}$ be the Zariski closure of Z . We also define

$$Z_c^\circ := \left\{ \text{Hilb}(X, D) \in \overline{Z} \left| \begin{array}{l} X \text{ is a Manetti surface, } D \sim_{\mathbb{Q}} -\frac{d}{3}K_X \text{ is an effective Weil divisor,} \\ (X, cD) \text{ is K-semistable, } \mathcal{O}_{\mathbb{P}^N}(1)|_X \cong \mathcal{O}_X(-mK_X), \\ \text{and } H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{\cong} H^0(X, \mathcal{O}_X(-mK_X)). \end{array} \right. \right\}.$$

By Corollary 3.17, we know that Z_c° is a Zariski open subset of $\overline{Z}$. We denote by Z_c^{red} the reduced scheme supported on Z_c° .

Assume m is sufficiently divisible. We define the *K-moduli stack* $\overline{\mathcal{P}}_{d,c}^K$ of plane curves of degree d with coefficient c as the quotient stack

$$\overline{\mathcal{P}}_{d,c}^K := [Z_c^{\text{red}} / \text{PGL}(N+1)].$$

By Theorem 3.24, we know that $\overline{\mathcal{P}}_{d,c}^K$ does not depend on the choice of sufficiently divisible m . By Theorem 3.1, we know that $\overline{\mathcal{P}}_{d,c}^K$ admits a good moduli space $\overline{\mathcal{P}}_{d,c}^K$ as a reduced proper scheme of finite type over $\mathbb{C}$. We call $\overline{\mathcal{P}}_{d,c}^K$ the *K-moduli space* of plane curves of degree d with coefficient c .

Indeed, if we denote by $\chi_0(k) := \chi(\mathbb{P}^2, \mathcal{O}(3k))$, then $\overline{\mathcal{P}}_{d,c}^K = \mathcal{KM}_{\chi_0, d/3, c}$ and $\overline{\mathcal{P}}_{d,c}^K = \mathcal{KM}_{\chi_0, d/3, c}$ as in Definition 3.8 since $\mathbb{P}^2$ is the only smooth del Pezzo surface of degree 9. From the definition, we also know that a pair (X, cD) is parametrized by $\overline{\mathcal{P}}_{d,c}^K$ (resp. $\overline{\mathcal{P}}_{d,c}^K$) if and only if (X, cD) is K-semistable (resp. K-polystable), and it admits a $\mathbb{Q}$ -Gorenstein smoothing to $(\mathbb{P}^2, cC_t)$ with C_t a smooth plane curve of degree d .

The following useful proposition is an easy consequence of the Paul-Tian criterion Theorem 2.22.

Proposition 4.3. *Let C be a plane curve of degree d . Let $c \in (0, \min\{1, 3/d\})$ be a rational number. If $(\mathbb{P}^2, cC)$ is K -(poly/semi)stable, then C is GIT (poly/semi)stable.*

Proof. Consider the universal family $\pi : (\mathbb{P}^2 \times \mathbf{P}_d, cC) \rightarrow \mathbf{P}_d$ of plane curves of degree d . From Theorem 2.22, it suffices to show that the CM $\mathbb{Q}$ -line bundle $\lambda_{\text{CM}, \pi, cC}$ is ample on $\mathbf{P}_d$. Here we use the intersection formula as in Proposition 2.23. It is clear that $-K_{\mathbb{P}^2 \times \mathbf{P}_d / \mathbf{P}_d} \sim \mathcal{O}_{\mathbb{P}^2 \times \mathbf{P}_d}(3, 0)$ and $C \sim \mathcal{O}_{\mathbb{P}^2 \times \mathbf{P}_d}(d, 1)$. Denote by $p : \mathbb{P}^2 \times \mathbf{P}_d \rightarrow \mathbb{P}^2$ the projection to the first component. By computation,

$$\begin{aligned} \lambda_{\text{CM}, \pi, cC} &= -\pi_*((-K_{\mathbb{P}^2 \times \mathbf{P}_d / \mathbf{P}_d} - cC)^3) = -\pi_*(\mathcal{O}_{\mathbb{P}^2 \times \mathbf{P}_d}(3 - cd, -c)^3) \\ &= -\pi_*(p^*\mathcal{O}(3 - cd)^3 + 3p^*\mathcal{O}(3 - cd)^2 \cdot \pi^*\mathcal{O}(-c) + 3p^*\mathcal{O}(3 - cd) \cdot \pi^*\mathcal{O}(-c)^2 + \pi^*\mathcal{O}(-c)^3) \\ &= 3(3 - cd)^2 c \mathcal{O}_{\mathbf{P}_d}(1). \end{aligned}$$

Hence, $\lambda_{\text{CM}, \pi, cC}$ is ample whenever $c \in (0, \frac{3}{d})$. The proof is finished. $\square$

The following corollary was proved by Hacking [50, Propositions 10.2 and 10.4] and Kim and Lee [67, Theorem 2.3]. We give a proof using K -stability and CM line bundles.

Corollary 4.4. *Let C be a plane curve of degree d . If $\text{lct}(\mathbb{P}^2; C) \geq \frac{3}{d}$ (resp. $> \frac{3}{d}$), then C is GIT semistable (resp. GIT stable).*

Proof. If $\text{lct}(\mathbb{P}^2; C) \geq \frac{3}{d}$, then the log Calabi–Yau pair $(\mathbb{P}^2, \frac{3}{d}C)$ is K -semistable. Hence, $(\mathbb{P}^2, cC)$ is K -semistable for any $c \in (0, \frac{3}{d})$ by Proposition 2.13. Thus, Proposition 4.3 implies that C is GIT semistable. If $\text{lct}(\mathbb{P}^2; C) > \frac{3}{d}$, then again by Proposition 2.13, we know that $(\mathbb{P}^2, cC)$ is uniformly K -stable for any $c \in (0, \frac{3}{d})$. Hence, C is GIT stable by Proposition 4.3. $\square$

Example 4.5. We summarize the description of K -moduli stacks and spaces for $d \leq 3$.

- (1) $d = 1$. In this case, we know that $(\mathbb{P}^2, cC)$ is K -unstable for C a line and any $c \in (0, 1)$ by [90, Example 3.16]. Hence, $Z \cap Z_c^\circ$ is empty. Since K -semistability is an open property by Corollary 3.17, we know $Z_c^\circ = \emptyset$. Hence, both $\overline{\mathcal{P}}_{1,c}^K$ and $\overline{P}_{1,c}^K$ are empty for any $c \in (0, 1)$.
- (2) $d = 2$. Denote by C a smooth plane conic curve.
 - (a) If $c \in (0, \frac{3}{4})$, by [90, Theorem 1.5], we know that $(\mathbb{P}^2, cC)$ is K -polystable. By Proposition 4.3, we know that $(\mathbb{P}^2, cC')$ is K -unstable for any singular plane conic curve C' . Thus, the only K -semistable point in $\overline{\mathcal{P}}_{2,c}^K$ is $[(\mathbb{P}^2, cC)]$ which is indeed K -polystable. Hence $\overline{\mathcal{P}}_{2,c}^K \cong \overline{\mathcal{P}}_2^{\text{GIT}} \cong [\text{Spec } \mathbb{C} / \text{PGL}(2)]$ and $\overline{P}_{2,c}^K \cong \overline{P}_2^{\text{GIT}} \cong \text{Spec } \mathbb{C}$.
 - (b) If $c = \frac{3}{4}$, then by [90, Proof of Theorem 1.5], we know that $(\mathbb{P}^2, \frac{3}{4}C)$ is K -semistable and admits a special degeneration to the K -polystable pair $(\mathbb{P}(1, 1, 4), \frac{3}{4}C_0)$ where $C_0 = (z = 0)$ with $[x, y, z]$ the projective coordinates of $\mathbb{P}(1, 1, 4)$. If $[(X, \frac{3}{4}D)]$ is a K -semistable point in $\overline{\mathcal{P}}_{2,3/4}^K$ with X nonsmooth, then by [93] it admits a special degeneration to

$(\mathbb{P}(1, 1, 4), \frac{3}{4}C_0)$. Hence, the Gorenstein index of X is 2 and D is smooth which implies that $(X, D) \cong (\mathbb{P}(1, 1, 4), C_0)$ by [50, Theorem 8.3]. Thus, there are only two K-semistable points: $[(\mathbb{P}^2, \frac{3}{4}C)]$ and $[(\mathbb{P}(1, 1, 4), \frac{3}{4}C_0)]$ where the latter one is the only K-polystable point.

(c) If $c > \frac{3}{4}$, then by [90, Example 3.16], we know $(\mathbb{P}^2, cC)$ is K-unstable. Hence, similarly as in (1), both $\overline{\mathcal{P}}_{2,c}^K$ and $\overline{\mathcal{P}}_{2,c}^{\text{GIT}}$ are empty for any $c \in (\frac{3}{4}, 1)$.

(3) $d = 3$. We will show $\overline{\mathcal{P}}_{3,c}^K \cong \overline{\mathcal{P}}_3^{\text{GIT}}$ for any $c \in (0, 1)$. From [99, Page 80], we know that a plane cubic curve C is GIT semistable (resp. stable) if and only if it has at worst nodal singularities (resp. smooth). Thus, we know that $(\mathbb{P}^2, C)$ is a log canonical log Calabi–Yau pair whenever C is GIT semistable. Then, we know by Proposition 2.13 that $(\mathbb{P}^2, cC)$ is K-semistable for any $c \in (0, 1)$. If C is GIT stable, that is, smooth, then [62] implies $(\mathbb{P}^2, (1 - \epsilon)C)$ is K-polystable. Hence, $(\mathbb{P}^2, cC)$ is K-stable for any $c \in (0, 1)$ by Proposition 2.13. It is well known that $(xyz = 0)$ is the unique GIT polystable plane cubic curve up to a projective transformation. By [46], we know that $(\mathbb{P}^2, c(xyz = 0))$ is K-polystable for any $c \in (0, 1)$. If $[(X, cD)]$ is a point in $\overline{\mathcal{P}}_{3,c}^K$, then it is a K-semistable limit of K-semistable log Fano pairs $(\mathbb{P}^2, cC_t)$ where $\{C_t\}_{t \in T \setminus \{0\}}$ is a family of cubic curves over a punctured smooth curve $T \setminus \{0\}$. Since C_t is GIT semistable by Proposition 4.3, we know that (possibly after a finite base change of T) there exists an algebraic family $g_t \in \text{PGL}(3)$ and a GIT polystable plane cubic curve C_0 such that $g_t \cdot C_t \rightarrow C_0$ in $\mathbf{P}_3$. Therefore, $(\mathbb{P}^2, cC_0)$ is the K-polystable limit of $(\mathbb{P}^2, cC_t)$. By [27] we know that (X, cD) admits a special degeneration to $(\mathbb{P}^2, cC_0)$ which implies $X \cong \mathbb{P}^2$ and D is GIT semistable. Hence, by similar arguments to the last paragraph in the proof of Theorem 5.2, we have $\overline{\mathcal{P}}_{3,c}^K \cong \overline{\mathcal{P}}_3^{\text{GIT}}$ and hence $\overline{\mathcal{P}}_{3,c}^K \cong \overline{\mathcal{P}}_3^{\text{GIT}}$.

From now on, we will always assume $d \geq 4$. We now mention some basic properties satisfied by the loci Z, Z_c° as well as the K-moduli stacks and spaces we just defined.

Proposition 4.6. *With notation as above, the following properties hold for any $c \in (0, \frac{3}{d})$:*

- (1) $\text{Hilb}(X, D) \in Z$ if and only if (X, D) is isomorphic to $(\mathbb{P}^2, C)$ where C is a smooth plane curve of degree d . Moreover, the locus Z is a saturated open subset of Z_c° .
- (2) The locus Z_c° is smooth for any $c \in (0, \frac{3}{d})$. In particular, $Z_c^\circ = Z_c^{\text{red}}$, and $\overline{\mathcal{P}}_{d,c}^K$ is a smooth Artin stack.
- (3) The open immersion $\mathcal{P}_d \hookrightarrow \overline{\mathcal{P}}_{d,c}^K$ induces an open immersion between their good moduli spaces $P_d \hookrightarrow \overline{P}_{d,c}^K$. Furthermore $\overline{P}_{d,c}^K$ is a normal proper variety, and P_d has only quotient singularities.

Proof. For part (1), it suffices to show that $(\mathbb{P}^2, cC)$ is K-stable for any $c \in (0, 3/d)$ and any smooth plane curve C of degree $d \geq 4$. This follows from Proposition 2.13 since $(\mathbb{P}^2, \frac{3}{d}C)$ is klt. Since Z lies inside the K-stable locus of Z_c° , we know that Z is saturated in Z_c° by the uniqueness of K-polystable degeneration [93].

For part (2), recall that for any point $\text{Hilb}(X, D) \in Z_c^\circ$, the surface X is a Manetti surface. Hence, X has unobstructed $\mathbb{Q}$ -Gorenstein deformations by Proposition 2.30. Since K-semistability is an open condition by Corollary 3.17, it suffices to show that the $\mathbb{Q}$ -Gorenstein deformations of the pair (X, D) are also unobstructed. Let $\pi : (\mathcal{X}, D) \rightarrow T$ be a $\mathbb{Q}$ -Gorenstein smoothing of (X, D) over a smooth pointed curve $0 \in T$, that is, $(\mathcal{X}_0, D_0) \cong (X, D)$ and π is smooth over

$T^\circ := T \setminus \{0\}$. Denote by $(\mathcal{X}^\circ, D^\circ) := \pi^{-1}(T^\circ)$. Then it is clear that $dK_{\mathcal{X}^\circ} + 3D^\circ \sim_\pi 0$. By taking Zariski closure, we know that $dK_{\mathcal{X}} + 3D \sim_\pi 0$ which implies $dK_X + 3D \sim 0$ by adjunction. In addition, if $3 \mid d$, then we get $\frac{d}{3}K_X + D \sim 0$. Hence, the statement of [50, Lemma 3.13] holds for (X, D) . Since X is klt and $D - K_X \sim_{\mathbb{Q}} -\frac{d-3}{3}K_X$ is ample, Kawamata–Viehweg vanishing implies $H^1(X, \mathcal{O}_X(D)) = 0$. Hence, the statement of [50, Lemma 3.14] also holds for (X, D) . Therefore, we may apply [50, Theorem 3.12] to deduce that (X, D) has unobstructed $\mathbb{Q}$ -Gorenstein deformations.

For part (3), the first statement follows from part (1) and [10, Remark 6.2]. The normality of $\overline{P}_{d,c}^{\overline{K}}$ follows from part (2) and a result of Alper [10, Theorem 4.16 (viii)]. Since any smooth plane curve of degree $d \geq 4$ has finite automorphism group, we know that $\mathcal{P}_d$ is a smooth Deligne–Mumford stack. Hence, P_d has only quotient singularities. $\square$

There are certain open subsets of $\overline{P}_{d,c}^{\overline{K}}$ that remain unchanged under subsequential wall crossings. Let $P_{d,c}^{\text{klt}}$ and $P_{d,c}^{\text{lc}}$ be the subsets of $\overline{P}_{d,c}^{\overline{K}}$ parametrizing c -K-polystable curves with $\text{lct} > \frac{3}{d}$ and $\text{lct} \geq \frac{3}{d}$, respectively. By the constructibility and lower semicontinuity of log canonical thresholds in bounded families, we know that both $P_{d,c}^{\text{klt}}$ and $P_{d,c}^{\text{lc}}$ are Zariski open subsets of $\overline{P}_{d,c}^{\overline{K}}$. Denote by $\mathcal{P}_{d,c}^{\text{klt}}$ and $\mathcal{P}_{d,c}^{\text{lc}}$ the preimage of $P_{d,c}^{\text{klt}}$ and $P_{d,c}^{\text{lc}}$ under the quotient map $\overline{\mathcal{P}}_{d,c}^{\overline{K}} \rightarrow \overline{P}_{d,c}^{\overline{K}}$, respectively.

Proposition 4.7.

- (1) *There exist open immersions $\mathcal{P}_{d,c}^{\text{klt}} \hookrightarrow \mathcal{P}_{d,c}^{\text{lc}} \hookrightarrow \overline{\mathcal{P}}_{d,c'}^{\overline{K}}$ that descend to open immersions $P_{d,c}^{\text{klt}} \hookrightarrow P_{d,c}^{\text{lc}} \hookrightarrow \overline{P}_{d,c'}^{\overline{K}}$ for any $0 < c \leq c' < 3/d$. Moreover, there exists an open immersion $P_{d,c}^{\text{klt}} \hookrightarrow \overline{P}_d^{\text{H}}$ for any $c \in (0, 3/d)$.*
- (2) *Assume $c_0 \in (0, 3/d)$ satisfies the following: for any K -polystable point $[(X, c_0C)] \in \overline{\mathcal{P}}_{d,c_0}^{\overline{K}}$, we have $\text{lct}(X; C) \geq 3/d$ (or equivalently, $P_{d,c_0}^{\text{lc}} = \overline{P}_{d,c_0}^{\overline{K}}$). Then $\overline{\mathcal{P}}_{d,c}^{\overline{K}} \cong \overline{\mathcal{P}}_{d,c_0}^{\overline{K}}$ for any $c_0 < c < 3/d$. In other words, there are no wall crossings among K -moduli spaces in the region $c \in (c_0 - \epsilon, 3/d)$ for $0 < \epsilon \ll 1$.*

Proof. For part (1), if (X, cD) is log K -semistable and $\text{lct}(X; D) \geq \frac{3}{d}$, then $(X, c'D)$ is K -semistable for any $c \leq c' < \frac{3}{d}$ by Proposition 2.13. Hence, we have open immersions $\mathcal{P}_{d,c}^{\text{klt}} \hookrightarrow \mathcal{P}_{d,c}^{\text{lc}} \hookrightarrow \overline{\mathcal{P}}_{d,c'}^{\overline{K}}$. To show that they descend to open immersions among the good moduli spaces, it suffices to show that the larger open substack $\mathcal{P}_{d,c}^{\text{lc}}$ is a saturated open substack of $\overline{\mathcal{P}}_{d,c'}^{\overline{K}}$. Let $[(X, cD)] \in \mathcal{P}_{d,c}^{\text{lc}}$ be a point. Then (X, cD) admits a K -polystable degeneration (X_0, cD_0) in $\mathcal{P}_{d,c}^{\text{lc}}$ such that $\text{lct}(X_0, D_0) \geq \frac{3}{d}$. Thus, by Proposition 2.13, we know that $(X_0, c'D_0)$ is K -polystable. Hence, $\mathcal{P}_{d,c}^{\text{lc}}$ is saturated in $\overline{\mathcal{P}}_{d,c'}^{\overline{K}}$.

By definition, we know that $P_{d,c}^{\text{klt}}$ admits an injective map to $\overline{P}_d^{\text{H}}$. To show that $P_{d,c}^{\text{klt}}$ admits an open immersion to $\overline{P}_d^{\text{H}}$, by [23] it suffices to show that a Hacking stable pair (X, D) belongs to $P_{d,c}^{\text{klt}}$ if and only if it is uniformly c - K -stable. The “if” part is clear from the definition. For the “only if” part, if (X, D) is both Hacking stable and c - K -polystable, then by Theorem 3.9, it admits a weak conical Kähler–Einstein metric, and its automorphism group is finite. Hence, (X, cD) is uniformly K -stable by Theorem 3.9(4). This finishes the proof of part (1).

For part (2), notice from part (1) that $\mathcal{P}_{d,c_0}^{\text{lc}}$ is a saturated open substack of $\overline{\mathcal{P}}_{d,c}^K$ for $c \in (c_0, \frac{3}{d})$ which induces an open immersion $\varphi : \overline{\mathcal{P}}_{d,c_0}^K = \mathcal{P}_{d,c_0}^{\text{lc}} \hookrightarrow \overline{\mathcal{P}}_{d,c}^K$. Since the K-moduli spaces are normal proper varieties by Proposition 4.6, we know that φ is an isomorphism by [10, Proposition 6.4]. Hence, $\overline{\mathcal{P}}_{d,c_0}^K = \mathcal{P}_{d,c_0}^{\text{lc}} \cong \overline{\mathcal{P}}_{d,c}^K$ whenever $c \in (c_0, \frac{3}{d})$. The $c \in (c_0 - \epsilon, c_0)$ part follows from Proposition 3.18. $\square$

4.2 | Index bounds

In this section, we prove the following theorem on bounding local Gorenstein indices of singular surfaces appearing in the boundary of K-moduli spaces. It is a K-stability analog of Hacking's result [50, Theorem 4.5] and [49, Theorem 2.22]. As in Hacking's work, it is crucial in the study of singular objects in K-moduli spaces of plane curves.

Theorem 4.8. *Let (X, cD) be a K-semistable log Fano pair that admits a $\mathbb{Q}$ -Gorenstein smoothing to $(\mathbb{P}^2, cC_t)$ with $c \in (0, 3/d)$ and $\deg C_t = d$. Let $x \in X$ be any singular point with Gorenstein index $\text{ind}(x, K_X)$, then*

$$\text{ind}(x, K_X) \leq \begin{cases} \min \left\{ \lfloor \frac{3}{3-cd} \rfloor, d \right\} & \text{if } 3 \nmid d, \\ \min \left\{ \lfloor \frac{3}{3-cd} \rfloor, \frac{2d}{3} \right\} & \text{if } 3 \mid d. \end{cases}$$

Proof. Let $\beta := 1 - cd/3 \in (0, 1)$. By [50, Propositions 6.1, 6.2, & Theorem 7.1], we know that a Gorenstein index n point $x \in X$ is a cyclic quotient singularity of type $\frac{1}{n^2}(1, an - 1)$ where $\gcd(a, n) = 1$ and $3 \nmid n$.

We first show that $n \leq \lfloor \frac{3}{3-cd} \rfloor$. By Theorem 2.15, we know that

$$\widehat{\text{vol}}(x, X, cD) \geq \frac{4}{9}(-K_X - cD)^2 = 4\beta^2.$$

On the other hand, we have $\widehat{\text{vol}}(x, X, cD) \leq \widehat{\text{vol}}(x, X) = \frac{4}{n^2}$ by [82, Proposition 4.10]. Combining these two inequalities, we get $n \leq \lfloor \beta^{-1} \rfloor = \lfloor \frac{3}{3-cd} \rfloor$.

Next, we show the inequality $n \leq d$ or $\frac{2d}{3}$ depending on divisibility of d by 3. We know that $dK_X + 3D \sim 0$, so if $x \notin D$, then $n \mid d$ hence $n \leq d$ (in fact $n \leq d/3$ if $3 \mid d$). From now on, let us assume $x \in D$. Let $(\tilde{x} \in \tilde{X})$ be the smooth cover of $(x \in X)$, with $\tilde{D}$ being the preimage of D . Since the finite degree formula for local volumes is true in dimension 2 by [96, Theorem 2.7(3)] [83, Theorem 4.15], we have

$$\widehat{\text{vol}}(\tilde{x}, \tilde{X}, c\tilde{D}) = n^2 \cdot \widehat{\text{vol}}(x, X, cD).$$

On the other hand, Theorem 2.15 implies that $\widehat{\text{vol}}(x, X, cD) \geq 4\beta^2$, so we have

$$n \leq \frac{\sqrt{\widehat{\text{vol}}(\tilde{x}, \tilde{X}, c\tilde{D})}}{2\beta} \leq \frac{2 - \text{cord}_{\tilde{x}} \tilde{D}}{2\beta}. \quad (4.1)$$

In particular, we have $n < \beta^{-1}$. We know that $\text{lct}_{\tilde{X}}(\tilde{X}; \tilde{D}) > c$, and Skoda's estimate [113] implies $\text{lct}_{\tilde{X}}(\tilde{X}; \tilde{D}) \leq \frac{2}{\text{ord}_{\tilde{X}} \tilde{D}}$, so we have $\text{ord}_{\tilde{X}} \tilde{D} < \frac{2}{c}$. Assume $\tilde{x} \in \tilde{X}$ has local coordinates (u, v) where the cyclic group action is scaling on each coordinate. Let $u^i v^j$ be a monomial appeared in the equation on $\tilde{D}$ with minimal $i + j = \text{ord}_{\tilde{X}} \tilde{D}$. Then $dK_X + 3D \sim 0$ implies $3(i + (na - 1)j) \equiv dna \pmod{n^2}$, in particular $i \equiv j \pmod{n}$.

Case 1. Assume $3 \nmid d$. If $\beta \geq \frac{1}{d+1}$, then $n < \beta^{-1} \leq d + 1$. Thus, we may assume $\beta < \frac{1}{d+1}$. Then $i + j = \text{ord}_{\tilde{X}} \tilde{D} < \frac{2}{c} < \frac{2(d+1)}{3}$. Assume to the contrary that $n \geq d + 1$. Then $i \equiv j \pmod{n}$ and $i + j < n$ implies that $i = j$. Hence, $3(i + (na - 1)j) \equiv dna \pmod{n^2}$ implies $3i \equiv d \pmod{n}$. But since $i < \frac{d+1}{3}$, we know that $3i = d$ which is a contradiction.

Case 2. Assume $3 \mid d$. If $\beta \geq \frac{3}{2d+3}$, then $n < \beta^{-1} \leq \frac{2d}{3} + 1$. Thus, we may assume $\beta < \frac{3}{2d+3}$. Then $i + j = \text{ord}_{\tilde{X}} \tilde{D} < \frac{2}{c} < \frac{2d}{3} + 1$. Assume to the contrary that $n \geq \frac{2d}{3} + 1$. Then $i \equiv j \pmod{n}$ and $i + j < n$ implies $i = j$. Hence, $3(i + (na - 1)j) \equiv dna \pmod{n^2}$ implies $3i \equiv d \pmod{n}$. Hence, $i = j = \frac{d}{3}$ and $\text{ord}_{\tilde{X}} \tilde{D} = \frac{2d}{3}$. Then (4.1) implies

$$n \leq \frac{2 - c \cdot \frac{2d}{3}}{2\beta} = \frac{2 - 2(1 - \beta)}{2\beta} = 1,$$

a contradiction! □

5 | THE FIRST WALL CROSSING

The goal of this section is to prove Theorem 1.3, which completely describes the first wall crossing of K-moduli spaces of plane curves for all degrees. We show that K-moduli and GIT coincide for small weights (see Theorems 5.2 and 5.5) and describe the explicit birational modification on the GIT moduli space occurring while crossing the first wall (see Theorem 5.6).

5.1 | Before the first wall

In this section, we will show that the K-moduli space for small coefficient is isomorphic to the GIT moduli space. We prove two results, Theorems 5.2 and 5.5, which correspond to parts (1) and (2) of Theorem 1.3.

Before we start, let us fix some notation for the discussion of the first wall crossing.

Notation 5.1. Let $d \geq 4$ be an integer. Let $c \in (0, \frac{3}{d})$ be a rational number. Let Q be a smooth conic in $\mathbb{P}^2$, let L be a line in $\mathbb{P}^2$ transverse to Q , and let x, y, z be coordinates of $\mathbb{P}(1, 1, 4)$. Let

$$c_1 = \begin{cases} \frac{3}{2d} & d \text{ is even} \\ \frac{3}{2d-3} & d \text{ is odd} \end{cases} \quad Q_d = \begin{cases} \frac{d}{2}Q & d \text{ is even} \\ \frac{d-1}{2}Q + L & d \text{ is odd} \end{cases} \quad Q'_d = \begin{cases} z^{d/2} = 0 & d \text{ is even} \\ xyz^{(d-1)/2} = 0 & d \text{ is odd} \end{cases}$$

We are ready to prove part (1) of Theorem 1.3.

Theorem 5.2 (First wall crossing 1). *We follow Notation 5.1. For any $0 < c < c_1$, a plane curve C of degree d is GIT (poly/semi)stable if and only if the log Fano pair $(\mathbb{P}^2, cC)$ is K-(poly/semi)stable. Moreover, there is an isomorphism of Artin stacks $\overline{\mathcal{P}}_{d,c}^K \cong \overline{\mathcal{P}}_d^{\text{GIT}}$.*

Proof. We first show that if (X, cD) is a K-semistable point in $\overline{\mathcal{P}}_{d,c}^K$ for $0 < c < c_1$, then $X \cong \mathbb{P}^2$. From Theorem 4.8, we know that the local Gorenstein indices of X are at most $\lfloor \beta^{-1} \rfloor$ where $\beta = 1 - \frac{cd}{3}$. If $c < \frac{3}{2d}$, then we have $\beta > \frac{1}{2}$. This implies that X is a Gorenstein Manetti surface so $X \cong \mathbb{P}^2$. Hence, we may assume that $d \geq 5$ is odd and $\frac{3}{2d} \leq c < \frac{3}{2d-3}$. By the same argument as above, the local Gorenstein indices of X are at most $\lfloor \beta^{-1} \rfloor < \frac{2d-3}{d-3} \leq \frac{7}{2}$. Hence, X has local Gorenstein index at most 3, which implies $X \cong \mathbb{P}^2$ or $\mathbb{P}(1, 1, 4)$. We shall show that the $\mathbb{P}(1, 1, 4)$ case is impossible under the assumption $c < \frac{3}{2d-3}$.

Assume to the contrary that $(X = \mathbb{P}(1, 1, 4), cD)$ is a K-semistable point in $\overline{\mathcal{P}}_{d,c}^K$. Then D is of degree $2d$ in $\mathbb{P}(1, 1, 4)$. Write $d = 2l + 1$, then the equation of D is

$$z^l f_2(x, y) + z^{l-1} f_6(x, y) + \cdots + f_{4l+2}(x, y) = 0,$$

where f_i is a homogeneous polynomial of degree i in (x, y) . Let E be the (-4) -curve over the singular point $[0, 0, 1]$ of type $\frac{1}{4}(1, 1)$. Then, from the defining equation of D , we see that $\text{ord}_E(D) \geq \frac{1}{2}$. Thus,

$$A_{(X, cD)}(\text{ord}_E) = A_X(\text{ord}_E) - c \text{ord}_E(D) \leq \frac{1-c}{2}.$$

On the other hand, $-K_X - cD \sim_{\mathbb{Q}} \mathcal{O}(6 - 2dc)$, and $\text{vol}_X(\mathcal{O}(1) - tE) = \max\{\frac{1}{4} - 4t^2, 0\}$. Hence,

$$S_{(X, cD)}(\text{ord}_E) = \frac{(6 - 2dc)}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tE) dt = 1 - \frac{dc}{3}.$$

Since $c < \frac{3}{2d-3}$, we know that $A_{(X, cD)}(\text{ord}_E) \leq \frac{1-c}{2} < 1 - \frac{dc}{3} = S_{(X, cD)}(\text{ord}_E)$. Hence, (X, cD) is K-unstable by the valuative criterion (Theorem 2.9).

So far we have shown that any K-semistable point (X, cD) in $\overline{\mathcal{P}}_{d,c}^K$ is isomorphic to $(\mathbb{P}^2, cC)$ where C is a plane curve of degree d . By Proposition 4.3, we know that K-(poly/semi)stability of $(\mathbb{P}^2, cC)$ implies GIT (poly/semi)stability of C . Hence, we just need to show the converse to deduce the equivalence between K-stability and GIT stability. Suppose that C is a GIT semistable plane curve. Take $\{C_t\}_{t \in T}$ a family of plane curves over a smooth pointed curve $(0 \in T)$ such that $C_0 = C$ and C_t is smooth for $t \in T \setminus \{0\}$. Then by properness of K-moduli spaces (Theorem 3.19), we have a K-polystable limit (X, cD) of $(\mathbb{P}^2, cC_t)$ as $t \rightarrow 0$ after a possible finite base change of T . Hence, $(X, cD) \cong (\mathbb{P}^2, C'_0)$ where C'_0 is a GIT polystable plane curve. By the separatedness of GIT quotients, we know that C specially degenerates to $g \cdot C_0$ for some $g \in \text{PGL}(3)$. Thus, $(\mathbb{P}^2, cC)$ is K-semistable by Theorem 2.16. If in addition that C is GIT polystable, then $(\mathbb{P}^2, cC)$ has a K-polystable limit $(\mathbb{P}^2, cC'_0)$. In particular, by Proposition 4.3, we know that C'_0 is a GIT polystable point S-equivalent to C . Hence, $C = g \cdot C_0$ for some $g \in \text{PGL}(3)$ and $(\mathbb{P}^2, cC)$ is K-polystable.

From the equivalence between c -K-semistability and GIT semistability, we obtain a morphism of Artin stacks $\varphi : \overline{\mathcal{P}}_d^{\text{GIT}} \rightarrow \overline{\mathcal{P}}_{d,c}^K$. It suffices to show that φ is an isomorphism of stacks. Consider the morphism of Artin stacks $\psi : \overline{\mathcal{P}}_{d,c}^K \rightarrow \text{BPGL}(3)$ sending $[(X, D) \rightarrow S]$ to $[X \rightarrow S]$, where

$B\text{PGL}(3)$ is the classifying stack of $\mathbb{P}^2$ -bundles. Clearly, ψ is representable as the group homomorphism $\text{Aut}(X, D) \rightarrow \text{Aut}(X)$ is injective for $[(X, D)] \in \overline{\mathcal{P}}_{d,c}^K$. We look at the base change of φ under the natural quotient map $\text{Spec } \mathbb{C} \rightarrow B\text{PGL}(3)$, in which we obtain a $\text{PGL}(3)$ -equivariant morphism of algebraic spaces $\tilde{\varphi} : \mathbf{P}_d^{\text{ss}} \rightarrow Z$ where $Z = \overline{\mathcal{P}}_{d,c}^K \times_{B\text{PGL}(3)} \text{Spec } \mathbb{C}$. Thus, for any reduced scheme S of finite type over $\mathbb{C}$, the set $Z(S)$ is given by $\{(\mathcal{X}, D; f)\} / \cong$ where $[(\mathcal{X}, cD) \rightarrow S] \in \overline{\mathcal{P}}_{d,c}^K$ and $f : \mathcal{X} \rightarrow \mathbb{P}_S^2$ is an isomorphism. From the equivalence between K -semistability and GIT semistability, we know that $\tilde{\varphi}(S)$ is a bijection for every reduced S , which implies that φ is an isomorphism between reduced algebraic spaces. Hence, φ is an isomorphism between Artin stacks. This finishes the proof. $\square$

Next, we discuss the K -moduli stack and space when $c = c_1$.

Lemma 5.3. *We follow Notation 5.1. Then the log Fano pair $(\mathbb{P}^2, c_1 Q_d)$ is K -semistable with K -polystable degeneration $(\mathbb{P}(1, 1, 4), c_1 Q'_d)$. Moreover, the only c_1 - K -polystable curve on $\mathbb{P}(1, 1, 4)$ of degree $2d$ is Q'_d .*

Proof. By taking the degeneration of $\mathbb{P}^2$ to the normal cone of Q , the pair $(\mathbb{P}^2, Q)$ specially degenerates to $(\mathbb{P}(1, 1, 4), (z = 0))$. Since L intersects Q transversally, it degenerates to the union of two distinct rulings of $\mathbb{P}(1, 1, 4)$. Hence, a suitable choice of projective coordinates of $\mathbb{P}(1, 1, 4)$ yields that $(\mathbb{P}^2, Q, L)$ specially degenerates to $(\mathbb{P}(1, 1, 4), (z = 0), (xy = 0))$.

Next, we show that $(\mathbb{P}(1, 1, 4), c_1 Q'_d)$ is K -polystable. When d is even, we have $c_1 Q'_d = \frac{3}{4}(z = 0)$. Hence, by [90, Proof of Theorem 1.5], we know that $(\mathbb{P}(1, 1, 4), \frac{3}{4}(z = 0))$ is K -polystable. When d is odd, we have $c_1 Q'_d = \frac{3}{2d-3}(xy = 0) + \frac{3(d-1)}{2(2d-3)}(z = 0)$. Hence, $(\mathbb{P}(1, 1, 4), c_1 Q'_d)$ is a projective cone over $(\mathbb{P}^1, \frac{3}{2d-3}([0] + [\infty]))$ with polarization $\mathcal{O}_{\mathbb{P}^1}(4) \sim_{\mathbb{Q}} \frac{2d-3}{d-3}(-K_{\mathbb{P}^1} - \frac{3}{2d-3}([0] + [\infty]))$. Since $\frac{3(d-1)}{2(2d-3)} = 1 - \frac{1}{2} \cdot (\frac{2d-3}{d-3})^{-1}$ and $(\mathbb{P}^1, \frac{3}{2d-3}([0] + [\infty]))$ admits a conical Kähler–Einstein metric, by [82, Proposition 3.3], we know that $(\mathbb{P}(1, 1, 4), c_1 Q'_d)$ is conical Kähler–Einstein hence K -polystable.

Finally, we show that Q'_d is the only c_1 - K -polystable curve on $\mathbb{P}(1, 1, 4)$ of degree $2d$. Suppose $(X := \mathbb{P}(1, 1, 4), c_1 D)$ is K -polystable with $\deg D = 2d$. Let E be the (-4) -curve over the singular point $x := [0, 0, 1]$ of type $\frac{1}{4}(1, 1)$. Then, by Theorem 2.15, we have

$$4(K_X + c_1 D)^2 \leq 9A_{(X, c_1 D)}(\text{ord}_E)^2 \text{vol}_{X,x}(\text{ord}_E). \quad (5.1)$$

By computation, we know that

$$A_{(X, c_1 D)}(\text{ord}_E) = A_X(\text{ord}_E) - c_1 \text{ord}_E(D) = \frac{1}{2} - c_1 \text{ord}_E(D), \quad \text{vol}_{X,x}(\text{ord}_E) = 4.$$

When d is even, we know that $(K_X + c_1 D)^2 = \frac{9}{4}$. Hence, (5.1) implies that $9 \leq 9(1 - 2c_1 \text{ord}_E(D))^2$, that is, D does not pass through x . Thus, the equation of D is given by

$$z^{d/2} + f_4(x, y)z^{(d-2)/2} + \dots + f_{2d}(x, y) = 0.$$

By taking the 1-PS $\lambda : \mathbb{G}_m \rightarrow \text{Aut}(X)$ as $\lambda(t)([x, y, z]) = [x, y, tz]$ for $t \in \mathbb{G}_m$, we see that $\lim_{t \rightarrow 0} \lambda(t) \cdot D = Q'_d$. Thus, $(X, c_1 D) \cong (\mathbb{P}(1, 1, 4), c_1 Q'_d)$ since they are both K -polystable. When

d is odd, we know that $(K_X + c_1 D)^2 = \frac{9(d-3)^2}{(2d-3)^2}$. Hence, (5.1) implies that $\frac{36(d-3)^2}{(2d-3)^2} \leq 9(1 - \frac{6}{2d-3} \text{ord}_E(D))^2$, that is, $\text{ord}_E(D) \leq \frac{1}{2}$. Since the equation of D is given by

$$z^l f_2(x, y) + z^{l-1} f_6(x, y) + \cdots + f_{4l+2}(x, y) = 0,$$

where $l := \frac{d-1}{2}$, we have $\text{ord}_E(D) \geq \frac{1}{2}$ with equality holds if and only if $f_2 \neq 0$. Hence, $\text{ord}_E(D) = \frac{1}{2}$, $f_2 \neq 0$ and the equality of (5.1) holds. Then by [81, Lemma 33], we know that E minimizes the normalized volume function at the singularity $x \in (X, c_1 D)$. So, [97, Theorem 1.2] implies that $(E, \Delta_E) \cong (\mathbb{P}_{[x,y]}^1, c_1(f_2(x, y) = 0))$ is a K-semistable Kollár component. Thus, f_2 is a non-degenerate quadratic form in (x, y) , and after a suitable choice of projective coordinates of $\mathbb{P}(1, 1, 4)$, we may assume that $f_2(x, y) = xy$. By taking the same 1-PS λ as before, we see $(X, c_1 D)$ specially degenerates to $(\mathbb{P}(1, 1, 4), c_1 Q'_d)$. Thus, $(X, c_1 D) \cong (\mathbb{P}(1, 1, 4), c_1 Q'_d)$ since they are both K-polystable. We finish the proof. $\square$

Corollary 5.4. *The plane curve Q_d is GIT polystable.*

Proof. If d is even, we know that $(\mathbb{P}^2, \epsilon Q_d = \frac{d\epsilon}{2} Q)$ is K-polystable by [90, Theorem 1.5]. Thus, Q_d is GIT polystable by Proposition 4.3. If d is odd, we know that $(\mathbb{P}^2, c_1 Q_d)$ is K-semistable by Lemma 5.3. Hence, Q_d is GIT semistable by Proposition 4.3. Assume to the contrary that Q_d is not GIT polystable. Denote by C_0 the GIT polystable plane curve that is S-equivalent to Q_d . If $\text{Supp}(C_0)$ contains a smooth conic Q , then $C_0 = \frac{d-1}{2} Q + L'$ where L' is a tangent line of Q . But then $\text{Aut}(\mathbb{P}^2, C_0) \cong \text{Aut}(\mathbb{P}^1, [0])$ is nonreductive, a contradiction. Hence, $\text{Supp}(C_0)$ is a union of lines. By Theorem 5.2, we know that $(\mathbb{P}^2, \epsilon C_0)$ is K-polystable. This implies that $(\mathbb{P}^2, \frac{3}{d} C_0)$ is log canonical by [46, Theorem 1.5]. But $\frac{3}{d} \cdot \frac{d-1}{2} > 1$ since $d \geq 5$, so $(\mathbb{P}^2, \frac{3}{d} C_0)$ cannot be log canonical, a contradiction. The proof is finished. $\square$

We present the proof of part (2) of Theorem 1.3 as follows.

Theorem 5.5 (First wall crossing 2). *We follow Notation 5.1. A log Fano pair $(X, c_1 D)$ is a K-polystable point of $\overline{P}_{d, c_1}^K$ if and only if either $X \cong \mathbb{P}^2$ and D is a GIT polystable plane curve not projectively equivalent to Q_d , or $(X, D) \cong (\mathbb{P}(1, 1, 4), Q'_d)$. Moreover, there is an open immersion $\Phi^- : \overline{P}_d^{\text{GIT}} = \overline{P}_{d, c_1 - \epsilon}^K \hookrightarrow \overline{P}_{d, c_1}^K$ which descends to an isomorphism of good moduli spaces $\phi^- : \overline{P}_d^{\text{GIT}} = \overline{P}_{d, c_1 - \epsilon}^K \xrightarrow{\cong} \overline{P}_{d, c_1}^K$.*

Proof. We first show that ϕ^- is an isomorphism. It is clear that ϕ^- is a birational morphism between normal proper varieties since P_d is a common open subset of $\overline{P}_d^{\text{GIT}}$ and $\overline{P}_{d, c_1}^K$ by Proposition 4.6. Indeed, we will show that the Picard number $\rho(\overline{P}_d^{\text{GIT}})$ is one. Since $\text{SL}(3)$ has no nontrivial characters, there are injections

$$\text{Pic}(\mathbf{P}_d^{\text{ss}} // \text{SL}(3)) \hookrightarrow \text{Pic}_{\text{SL}(3)}(\mathbf{P}_d^{\text{ss}}) \hookrightarrow \text{Pic}(\mathbf{P}_d^{\text{ss}})$$

by [66, Proposition 4.2 and §2.1]. Since we have a surjection from $\text{Pic}(\mathbf{P}_d) \cong \mathbb{Z}$ to $\text{Pic}(\mathbf{P}_d^{\text{ss}})$, we know that $\rho(\overline{P}_d^{\text{GIT}}) = 1$. Thus, ϕ^- is an isomorphism between good moduli spaces.

Let C be a GIT polystable plane curve not projectively equivalent to Q_d . Let (X, cD) be the K-polystable degeneration of $(\mathbb{P}^2, cC)$. From the index estimate in the proof of Theorem 5.2, we know that X is isomorphic to either $\mathbb{P}^2$ or $\mathbb{P}(1, 1, 4)$. If X is isomorphic to $\mathbb{P}(1, 1, 4)$, then D has to be Q_d' by Lemma 5.3. Thus, $\phi^-([C]) = \phi^-([Q_d])$ which contradicts to the injectivity of ϕ^- . Thus, $X \cong \mathbb{P}^2$ hence $(X, cD) \cong (\mathbb{P}^2, cC)$ by GIT polystability of C . The proof is finished by Lemma 5.3. $\square$

5.2 | After the first wall

In this section, we will show that the K-moduli stack $\overline{\mathcal{P}}_{d, c_1 + \epsilon}^K$ is isomorphic to a Kirwan-type weighted blow-up of the GIT moduli stack.

Theorem 5.6 (First wall crossing 3). *Let $\Phi^+ : \overline{\mathcal{P}}_{d, c_1 + \epsilon}^K \rightarrow \overline{\mathcal{P}}_{d, c_1}^K$ be the latter morphism in the first wall crossing. Then there exists a stacky weighted blow-up morphism $\rho : \overline{\mathcal{P}}_{d, c_1 + \epsilon}^K \rightarrow \overline{\mathcal{P}}_{d, c_1 - \epsilon}^K = \overline{\mathcal{P}}_d^{\text{GIT}}$ along $\{[Q_d]\}$ (see Definition 5.10) such that $\Phi^+ = \Phi^- \circ \rho$. In particular, we have*

- (1) *The descent morphism $\varphi = (\phi^-)^{-1} \circ \phi^+ : \overline{\mathcal{P}}_{d, c_1 + \epsilon}^K \rightarrow \overline{\mathcal{P}}_{d, c_1 - \epsilon}^K = \overline{\mathcal{P}}_d^{\text{GIT}}$ of ρ between good moduli spaces is a weighted blow-up of the point $[Q_d]$.*
- (2) *If d is even, then φ is a partial desingularization of Kirwan type.*

The proof of this theorem will be split up into a few parts. Before we analyze the stack structure of $\overline{\mathcal{P}}_{d, c_1 + \epsilon}^K$, we first give a complete description of its closed points.

Definition 5.7. We define GIT stability for certain curves on $\mathbb{P}(1, 1, 4)$.

- (1) Assume d is even. Given a curve D in $\mathbb{P}(1, 1, 4)$ of degree $2d$ with equation

$$z^{d/2} + f_8(x, y)z^{(d-4)/2} + f_{12}(x, y)z^{(d-6)/2} + \dots + f_{2d}(x, y) = 0, \quad (5.2)$$

we identify D to a point $(f_8, f_{12}, \dots, f_{2d})$ in the vector space $\mathbf{A}'_d \cong \bigoplus_{j=2}^{d/2} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4j))$. Consider the $\mathbb{G}_m$ -action σ on $\mathbf{A}'_d$ with weight j in each direct summand $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4j))$. Let $\mathbf{P}'_d$ be the weighted projective space as the coarse moduli space of the quotient stack $[(\mathbf{A}'_d \setminus \{0\})/\mathbb{G}_m]$. There is a natural $\text{SL}(2)$ -action on $\mathbf{A}'_d$ induced by the usual $\text{SL}(2)$ -action on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{C}x \oplus \mathbb{C}y$. Since this $\text{SL}(2)$ -action commutes with the previous $\mathbb{G}_m$ -action, it descends to an $\text{SL}(2)$ -action on $(\mathbf{P}'_d, \mathcal{O}_{\mathbf{P}'_d}(1))$. We say $[D] \in \mathbf{P}'_d$ is *GIT (poly/semi)stable* if it is GIT (poly/semi)stable with respect to this $\text{SL}(2)$ -action on $(\mathbf{P}'_d, \mathcal{O}_{\mathbf{P}'_d}(1))$.

- (2) Assume that d is odd. Suppose that D is a curve in $\mathbb{P}(1, 1, 4)$ of degree $2d$ with equation

$$xyz^{(d-1)/2} + f_6(x, y)z^{(d-3)/2} + f_{10}(x, y)z^{(d-5)/2} + \dots + f_{2d}(x, y) = 0,$$

where $f_6(x, y)$ contains no monomial divisible by xy . Then, we identify D to a point $(f_6, f_{10}, \dots, f_{2d})$ in the vector space

$$\mathbf{A}'_d := V_1 \oplus \bigoplus_{j=2}^{(d-1)/2} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4j + 2)),$$

where $V_1 := \mathbb{C}x^6 \oplus \mathbb{C}y^6$ is a sub vector space of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6))$. Consider the $\mathbb{G}_m$ -action σ on $\mathbf{A}'_d$ with weight 1 on V_1 and weight j on each direct summand $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4j + 2))$. Let $\mathbf{P}'_d$

be the weighted projective space that is the coarse moduli space of the quotient stack $[(\mathbf{A}'_d \setminus \{0\})/\mathbb{G}_m]$. Consider another $\mathbb{G}_m$ -action σ' on $\mathbf{A}'_d$ induced by

$$\sigma'(t) \cdot f_{4j+2}(x, y) = f_{4j+2}(tx, t^{-1}y)$$

for $t \in \mathbb{G}_m$ and $1 \leq j \leq \frac{d-1}{2}$. Since σ' commutes with σ , it descends to a $\mathbb{G}_m$ -action on $(\mathbf{P}'_d, \mathcal{O}_{\mathbf{P}'_d}(1))$ which we also denote by σ' . We say $[D] \in \mathbf{P}'_d$ is *GIT (poly/semi)stable* if it is GIT (poly/semi)stable with respect to the $\mathbb{G}_m$ -action σ' on $(\mathbf{P}'_d, \mathcal{O}_{\mathbf{P}'_d}(1))$.

Care is taken in the above definition to define GIT stability for curves on $\mathbb{P}(1, 1, 4)$ because the automorphism groups of weighted projective spaces are in general nonreductive. Recently, the theory of nonreductive GIT and variation of nonreductive GIT has been developed (see, e.g., [21, 34]) which may be useful in similar endeavors.

Theorem 5.8. *Let D be a curve on $\mathbb{P}(1, 1, 4)$ of degree $2d$ such that $[D] \in \mathbf{P}'_d$. Then the pair $(\mathbb{P}(1, 1, 4), (c_1 + \epsilon)D)$ is K -(poly/semi)stable if and only if $[D]$ is GIT (poly/semi)stable in the sense of Definition 5.7.*

Proof. We first prove the “only if” part. Let $\pi : (\mathbb{P}(1, 1, 4) \times \mathbf{A}'_d, D) \rightarrow \mathbf{A}'_d$ be the universal family of pairs over $\mathbf{A}'_d$ where the fiber of π over each point D of $\mathbf{A}'_d$ is $(\mathbb{P}(1, 1, 4), D)$. Then the $\mathbb{G}_m$ -action σ on $\mathbf{A}'_d$ has a natural lifting to the universal family, which we also denote by σ , namely, $\sigma(t) \cdot ([x, y, z], D) = ([x, y, tz], \sigma(t) \cdot D)$. Hence, by quotienting out σ , we obtain a $\mathbb{Q}$ -Gorenstein family of log Fano pairs over the Deligne–Mumford stack $[(\mathbf{A}'_d \setminus \{0\})/\mathbb{G}_m]$. The CM $\mathbb{Q}$ -line bundle $\lambda_{\text{CM}, \pi, cD}$ on $\mathbf{A}'_d$ also descends to a $\mathbb{Q}$ -line bundle on $\mathbf{P}'_d$ which we denote by Λ_c . By Theorem 2.22, it suffices to show that $\Lambda_{c_1 + \epsilon}$ is ample. Since $\lambda_{\text{CM}, \pi, cD}$ is a trivial $\mathbb{Q}$ -line bundle over $\mathbf{A}'_d$, the degree of Λ_c is equal to the σ -weight of the central fiber $\lambda_{\text{CM}, \pi, cD} \otimes \mathbb{C}(0)$. By Proposition 2.19, we know that

$$\deg \Lambda_c = \text{Fut}((\mathbb{P}(1, 1, 4), cQ_d; \mathcal{O}_{\mathbb{P}(1, 1, 4)}(4)) \times \mathbb{A}^1),$$

where the product test configuration $(\mathbb{P}(1, 1, 4), cQ_d; \mathcal{O}_{\mathbb{P}(1, 1, 4)}(4)) \times \mathbb{A}^1$ is induced from the $\mathbb{G}_m$ -action σ . From the definition, easy computation, and K -polystability of $(\mathbb{P}(1, 1, 4), c_1 Q_d)$ we know that $\text{Fut}((\mathbb{P}(1, 1, 4), cQ_d; \mathcal{O}_{\mathbb{P}(1, 1, 4)}(4)) \times \mathbb{A}^1)$ is linear in c , is negative when $c = 0$, and zero when $c = c_1$. Hence, it is positive when $c = c_1 + \epsilon$. As a result, the CM $\mathbb{Q}$ -line bundle $\Lambda_{c_1 + \epsilon}$ is ample on $\mathbf{P}'_d$. Note that when d is odd, the action σ' on $\mathbf{A}'_d$ has zero weight on the central fiber $\lambda_{\text{CM}, \pi, cD} \otimes \mathbb{C}(0)$ by a straightforward computation of the generalized Futaki invariant. Thus, the $\mathbb{G}_m$ -linearization of a suitable positive power of $\Lambda_{c_1 + \epsilon}$ coincides with the $\mathbb{G}_m$ -linearization on $\mathcal{O}_{\mathbf{P}'_d}(1)$ of σ' . This completes the proof of the “only if” part.

Next, we prove the “if” part. Since each curve $[D] \in \mathbf{P}'_d$ admits a special degeneration to Q_d , we know $(\mathbb{P}(1, 1, 4), c_1 D)$ is K -semistable by Lemma 5.3 and Theorem 2.16. By Bertini’s theorem, it is clear that a general curve D in $\mathbf{P}'_d$ has at worst a nodal point at the unique singularity of $\mathbb{P}(1, 1, 4)$ and smooth elsewhere. Thus, for a general curve D , we know that $(\mathbb{P}(1, 1, 4), \frac{3}{d}D)$ is klt. This implies that $(\mathbb{P}(1, 1, 4), (c_1 + \epsilon)D)$ is K -stable by Proposition 2.13. Let $[D_0] \in \mathbf{P}'_d$ be a GIT polystable point. From the above argument, we can find a family of curves $[D_t] \in \mathbf{P}'_d$ parametrized by a punctured smooth curve $t \in T \setminus \{0\}$ such that $(\mathbb{P}(1, 1, 4), (c_1 + \epsilon)D_t)$ is K -stable and $\lim_{t \rightarrow 0} [D_t] = [D_0]$. Then by properness of K -moduli spaces (i.e., Theorem 3.19), after a possible finite base change of T ,

we obtain a K-polystable limit $(X, (c_1 + \epsilon)D')$ of $(\mathbb{P}(1, 1, 4), (c_1 + \epsilon)D_t)$ as t goes to 0. By the index estimate Theorem 4.8, we know that $X \cong \mathbb{P}(1, 1, 4)$. From the continuity of generalized Futaki invariants in c , we know $(\mathbb{P}(1, 1, 4), c_1 D')$ is K-semistable. Thus, D' specially degenerates to Q_d by Lemma 5.3. After a suitable change of coordinates of $\mathbb{P}(1, 1, 4)$, we may assume $[D'] \in \mathbf{P}'_d$. Thus, the “only if” part implies that $[D']$ is a GIT polystable limit of $g_t \cdot [D_t]$ for $g_t \in \mathrm{SL}(2)$ (for even d) or $\mathbb{G}_m$ (for odd d). By separatedness of the GIT quotient, we know that $[D']$ and $[D_0]$ lie in the same orbit. Thus, $(\mathbb{P}(1, 1, 4), (c_1 + \epsilon)D_0)$ is K-polystable. The proof regarding K-semistability and K-stability follows from similar arguments as in the proof of Theorem 5.2. $\square$

Theorem 5.9. *Let $[(X, (c_1 + \epsilon)D)]$ be a K-polystable point in $\overline{P}_{d, c_1 + \epsilon}^K$. Then either $(X, D) \cong (\mathbb{P}^2, C)$ where C is a GIT polystable plane curve not projectively equivalent to Q_d , or $(X, D) \cong (\mathbb{P}(1, 1, 4), D')$ where $[D'] \in \mathbf{P}'_d$ is GIT polystable. Conversely, any such pair $(\mathbb{P}^2, C)$ or $(\mathbb{P}(1, 1, 4), D')$ is $(c_1 + \epsilon)$ -K-polystable.*

Proof. We first prove that K-polystability implies GIT polystability. Let $[(X, (c_1 + \epsilon)D)]$ be a point in $\overline{P}_{d, c_1 + \epsilon}^K$. Then from the index estimate in the proof of Theorem 5.2, we know that X is isomorphic to either $\mathbb{P}^2$ or $\mathbb{P}(1, 1, 4)$. If $(X, D) \cong (\mathbb{P}^2, C)$, then C is GIT polystable by Proposition 4.3. It suffices to show that $(\mathbb{P}^2, (c_1 + \epsilon)Q_d)$ is K-unstable. In fact, if $(\mathbb{P}^2, (c_1 + \epsilon)Q_d)$ were K-semistable, then Proposition 2.13 together with K-polystability of $(\mathbb{P}^2, \epsilon Q_d)$ (see Theorem 5.2) implies that $(\mathbb{P}^2, c_1 Q_d)$ is K-polystable as well. But this contradicts Lemma 5.3, so $(\mathbb{P}^2, (c_1 + \epsilon)Q_d)$ is K-unstable. If $X \cong \mathbb{P}(1, 1, 4)$, then the statement follows from Theorem 5.8.

Next, we prove that GIT polystability implies K-polystability. If $C \subset \mathbb{P}^2$ is a GIT polystable plane curve not projectively equivalent to Q_d , then Theorem 5.5 implies that $(\mathbb{P}^2, cC)$ is K-polystable for any $c \in (0, c_1]$. Hence, $(\mathbb{P}^2, (c_1 + \epsilon)C)$ is also K-polystable by Proposition 3.18. The $\mathbb{P}(1, 1, 4)$ case follows from Theorem 5.8. $\square$

So far we have shown that $\overline{P}_d^{\mathrm{GIT}} \setminus \{[Q_d]\} \hookrightarrow \overline{P}_{d, c_1 + \epsilon}^K$ is an open immersion. Before proving Theorem 5.6, we set up some notation.

Definition 5.10. Let $Z \subset X$ be a smooth closed subvariety of a smooth quasi-projective variety X . We define $Bl_{\mathbf{w}, Z}X$, the *stacky weighted blow-up* of Z in X with weight $\mathbf{w}$ (see also [12, Section 3]). The standard weighted blow-up $Bl_{\mathbf{w}, Z}X$ will be the coarse space of $Bl_{\mathbf{w}, Z}X$.

Let $\mathcal{N}_{Z/X}$ be the normal bundle of $Z \subset X$ and consider a group G acting on X . Suppose that $\mathcal{N}_{Z/X}$ has a decomposition with respect to representations of G

$$\mathcal{N}_{Z/X} = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \cdots \oplus \mathcal{N}_k.$$

Let $\mathbf{w} = (w_1, w_2, \dots, w_k)$ be a weight vector with $w_i \in \mathbb{Z}_{>0}$. This gives a monomial valuation v of $\mathbb{C}(X)$ centered at Z of weights $(w_1, w_2, \dots, w_k)$ with respect to the decomposition of the normal bundle $\mathcal{N}_{Z/X}$.

Define $R := \bigoplus_{m=0}^{\infty} \mathfrak{a}_m(v)t^m$, where $\mathfrak{a}_m(v) = \{s \in \mathcal{O}_X : v(s) \geq m\}$. The *standard weighted blow-up* of Z in X is defined by $Bl_{\mathbf{w}, Z}X := \mathrm{Proj}_X R$. Define $Y := \mathrm{Spec}_X R$. Then, we have the zero section $X \hookrightarrow Y$ whose defining ideal is given by $I = \bigoplus_{m=1}^{\infty} \mathfrak{a}_m(v)t^m$. We define the *stacky weighted blow-up* of Z in X to be

$$Bl_{\mathbf{w}, Z}X = [(Y \setminus X)/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts as $t \mapsto \lambda t$, for $\lambda \in \mathbb{G}_m$.

Remark 5.11. The stack $Bl_{\mathbf{w},Z}X$ is a smooth Deligne–Mumford stack, its coarse space is indeed $Bl_{\mathbf{w},Z}X$, and the exceptional divisor E has weighted projective stacks as fibers over Z .

We now proceed with our construction of the partial desingularization of Kirwan type. First, let us recall some representation theory of $SL(2)$. Consider the standard action of $SL(2)$ on $\mathbb{P}^1 = \mathbb{P}(V^\vee)$. Then we have the dual $SL(2)$ -action on $V = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. We have a natural $SL(2)$ -action on $\mathbf{V} := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) = \text{Sym}^2 V$ so that the second Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2 = \mathbb{P}(\mathbf{V}^\vee)$ is $SL(2)$ -equivariant. Denote by $Q = (q = 0)$ the image of this Veronese embedding. Then we have

$$\text{Sym}^d \mathbf{V} = \text{Sym}^d(\text{Sym}^2 V) \cong \begin{cases} \bigoplus_{i=0}^{d/2} \text{Sym}^{4i} V & \text{if } d \text{ is even,} \\ \bigoplus_{i=0}^{(d-1)/2} \text{Sym}^{4i+2} V & \text{if } d \text{ is odd.} \end{cases} \quad (5.3)$$

Since q is $SL(2)$ -invariant, we have an injection between $SL(2)$ -representations $\text{Sym}^{d-2} \mathbf{V} \hookrightarrow \text{Sym}^d \mathbf{V}$ by multiplying with q . Let $\mathbf{V}_d$ be the cokernel of this injection. Then from (5.3), we know that the restriction map $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \rightarrow H^0(Q, \mathcal{O}_Q(d))$ induces an isomorphism between $\mathbf{V}_d$ and $\text{Sym}^{2d} V$. As a summary, we have

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = \bigoplus_{j=0}^{\lfloor d/2 \rfloor} q^j \cdot \mathbf{V}_{d-2j} \cong \bigoplus_{j=0}^{\lfloor d/2 \rfloor} \text{Sym}^{2d-4j} V. \quad (5.4)$$

Lemma 5.12. Let d be even and let Q_d denote the nonreduced curve defined by $(q^{d/2} = 0)$ where $(q = 0)$ is a smooth plane conic. Then a Luna slice to $SL(3) \cdot [Q_d] \subset \mathbf{P}_d^{\text{ss}}$ at $[Q_d]$ is given by the locally closed subset

$$W := \left\{ (q^{d/2} + f_4 q^{d/2-2} + f_6 q^{d/2-3} + \dots + f_d = 0) \right\},$$

where $f_{2i} \in \mathbf{V}_{2i} \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2i))$ for $2 \leq i \leq d/2$.

Proof. Denote by $G := SL(3)$. Since Q_d is GIT polystable by Corollary 5.4, we know that the orbit $G[Q_d]$ is closed in $\mathbf{P}_d^{\text{ss}}$. The tangent space of $\mathbf{P}_d^{\text{ss}}$ at $[Q_d]$ is given by $\mathcal{T}_{\mathbf{P}_d^{\text{ss}}, [Q_d]} = (\text{Sym}^d \mathbf{V}) / \mathbb{C} q^{d/2}$. If q_t is a small deformation of $q_0 = q$ in $\text{Sym}^2 \mathbf{V}$, then we have $\frac{d}{dt}(q_t^{d/2})|_{t=0} = \frac{d}{2} q^{d/2-1} \frac{dq_t}{dt}|_{t=0}$. Thus, the tangent space of $G[Q_d]$ at $[Q_d]$ is given by $T_{G[Q_d], [Q_d]} = (q^{d/2-1} \cdot \text{Sym}^2 \mathbf{V}) / \mathbb{C} q^{d/2}$. Hence, by (5.4), the normal space of $G[Q_d] / \mathbf{P}_d^{\text{ss}}$ at the point $[Q_d]$ satisfies

$$\mathcal{N}_{G[Q_d] / \mathbf{P}_d^{\text{ss}}, [Q_d]} = \mathcal{T}_{\mathbf{P}_d^{\text{ss}}, [Q_d]} / \mathcal{T}_{G[Q_d], [Q_d]} \cong \text{Sym}^d \mathbf{V} / (q^{d/2-1} \cdot \text{Sym}^2 \mathbf{V}) \cong \bigoplus_{i=2}^{d/2} \mathbf{V}_{2i}. \quad (5.5)$$

Therefore, taking the exponential map of the normal space yields a Luna slice

$$W := \left\{ (q^{d/2} + f_4 q^{d/2-2} + f_6 q^{d/2-3} + \dots + f_d = 0) \right\},$$

where $f_{2i} \in \mathbf{V}_{2i}$ for $2 \leq i \leq d/2$. □

Lemma 5.13. Let d be odd and let Q_d denote the curve $(l q^{(d-1)/2} = 0)$, where $(q = 0)$ is a smooth plane conic and $(l = 0)$ is a line transverse to q . Then a Luna slice to $SL(3) \cdot [Q_d] \in \mathbf{P}_d^{\text{ss}}$ at $[Q_d]$ is

given by the locally closed set

$$W := \{lq^{(d-1)/2} + f_3q^{(d-3)/2} + f_5q^{(d-5)/2} + \dots + f_d = 0\},$$

where $f_3 \in \mathbf{V}_3^\circ$ and $f_{2i+1} \in \mathbf{V}_{2i+1}$ for any $2 \leq i \leq (d-1)/2$. Here, $\mathbf{V}_3^\circ$ is a subspace of $\mathbf{V}_3$ such that it pulls back to $\mathbb{C}u^6 + \mathbb{C}v^6 \subset H^0(\mathbb{P}^1_{[u,v]}, \mathcal{O}(6))$ under the isomorphism from $(\mathbb{P}^1_{[u,v]}, (uv=0))$ to $((q=0), (l=q=0))$.

Proof. Again, we consider the orbit $G[Q_d]$ under the action of $G := \mathrm{SL}(3)$. It is clear that $\mathcal{T}_{\mathbf{P}_d^{\mathrm{ss}}, [Q_d]} = \mathrm{Sym}^d \mathbf{V} / \mathbb{C}lq^{(d-1)/2}$. If l_t and q_t are small deformations of $l_0 = l$ and $q_0 = q$ in $\mathbf{V}$ and $\mathrm{Sym}^2 \mathbf{V}$, respectively, then we have $\frac{d}{dt}(l_t q_t^{(d-1)/2})|_{t=0} = \frac{d-1}{2} l q^{(d-3)/2} \frac{dq_t}{dt}|_{t=0} + q^{(d-1)/2} \frac{dl_t}{dt}|_{t=0}$. Thus, the tangent space of $G[Q_d]$ at $[Q_d]$ is given by $T_{G[Q_d], [Q_d]} = (lq^{(d-3)/2} \cdot \mathrm{Sym}^2 \mathbf{V} + q^{(d-1)/2} \cdot \mathbf{V}) / \mathbb{C}lq^{(d-1)/2}$. Hence, the normal space of $G[Q_d] / \mathbf{P}_d^{\mathrm{ss}}$ at the point $[Q_d]$ satisfies

$$\mathcal{N}_{G[Q_d] / \mathbf{P}_d^{\mathrm{ss}}, [Q_d]} = \mathcal{T}_{\mathbf{P}_d^{\mathrm{ss}}, [Q_d]} / \mathcal{T}_{G[Q_d], [Q_d]} \cong \mathrm{Sym}^d \mathbf{V} / (lq^{(d-3)/2} \cdot \mathrm{Sym}^2 \mathbf{V} + q^{(d-1)/2} \cdot \mathbf{V}).$$

It is clear that $l \cdot \mathrm{Sym}^2 \mathbf{V}$ and $q \cdot \mathbf{V}$ are both contained in $\mathrm{Sym}^3 \mathbf{V}$. Let $G_1 \cong \mathbb{G}_m \rtimes \mathbb{Z}/2$ be the subgroup of $\mathrm{SL}(2)$ preserving both l and q . Then both $l \cdot \mathrm{Sym}^2 \mathbf{V}$ and $q \cdot \mathbf{V}$ are sub G_1 -representation of $\mathrm{Sym}^3 \mathbf{V}$. Since $\mathrm{Sym}^3 \mathbf{V} / (q \cdot \mathbf{V}) \cong \mathbf{V}_3$ by (5.4), we denote by $\mathbf{V}_3^\circ$ the sub G_1 -representation of $\mathbf{V}_3$ complementary to $(l \cdot \mathrm{Sym}^2 \mathbf{V} + q \cdot \mathbf{V}) / (q \cdot \mathbf{V})$.

By (5.4), we have

$$\begin{aligned} \mathcal{N}_{G[Q_d] / \mathbf{P}_d^{\mathrm{ss}}, [Q_d]} &\cong \mathrm{Sym}^3 \mathbf{V} / (l \cdot \mathrm{Sym}^2 \mathbf{V} + q \cdot \mathbf{V}) \oplus \mathrm{Sym}^d \mathbf{V} / (q^{(d-3)/2} \mathrm{Sym}^3 \mathbf{V}) \\ &\cong \mathbf{V}_3^\circ \oplus \bigoplus_{i=2}^{(d-1)/2} \mathbf{V}_{2i+1}. \end{aligned} \quad (5.6)$$

Therefore, taking the exponential map of the normal space yields a Luna slice

$$W := \{lq^{(d-1)/2} + f_3q^{(d-3)/2} + f_5q^{(d-5)/2} + \dots + f_d = 0\},$$

where $f_3 \in \mathbf{V}_3^\circ$ and $f_{2i+1} \in \mathbf{V}_{2i+1}$ for $2 \leq i \leq (d-1)/2$.

Finally, we verify that $\mathbf{V}_3^\circ$ corresponds to $\mathbb{C}u^6 + \mathbb{C}v^6$ under the pull back to $\mathbb{P}^1$. Since the pull-back of l on $\mathbb{P}^1$ is uv , we know that $\mathbf{V}_3^\circ$ is complementary to $uv \cdot H^0(\mathbb{P}^1, \mathcal{O}(4))$ in $H^0(\mathbb{P}^1, \mathcal{O}(6))$. Since the identity component of G_1 acts on $H^0(\mathbb{P}^1, \mathcal{O}(1)) = \mathbb{C}u + \mathbb{C}v$ diagonally, we know that $\mathbf{V}_3^\circ$ corresponds to $\mathbb{C}u^6 + \mathbb{C}v^6$ under the pull back to $\mathbb{P}^1$. This finishes the proof. $\square$

We now proceed to the proof of Theorem 5.6.

Theorem 5.14. *Let $U = \mathbf{P}_d^{\mathrm{ss}}$, and consider the universal family $(\mathbb{P}_U^2, C_U) \rightarrow U$, where each fiber is c -K-polystable for $c \in (0, c_1)$. After base change to a stacky weighted blow-up $\hat{U} \rightarrow U$ along the orbit $G[Q_d]$ with stacky exceptional divisor $\mathcal{E}$ where $G = \mathrm{SL}(3)$, a blow-up along the conic component over $\mathcal{E}$, and a divisorial contraction, there exists a family $(\mathcal{X}, C_{\mathcal{X}}) \rightarrow \hat{U}$ which:*

- (1) *is isomorphic to $(\mathbb{P}_{U \setminus G[Q_d]}^2, C_{U \setminus G[Q_d]}) \rightarrow U \setminus G[Q_d]$ over the complement of $\mathcal{E}$,*
- (2) *and whose fibers over $\mathcal{E}$ are curves on $\mathbb{P}(1, 1, 4)$.*

- (3) Let $\mathcal{E}_W$ be the exceptional divisor of the stacky weighted blow-up $\widehat{\mathcal{W}} := \widehat{\mathcal{U}} \times_U W \rightarrow W$ over the Luna slice W as in Lemmas 5.12 and 5.13. Then the family $(\mathcal{Y}, C_{\mathcal{Y}}) \times_{\widehat{\mathcal{U}}} \mathcal{E}_W$ over $\mathcal{E}_W$ is isomorphic to the universal family over $[(\mathbf{A}'_d \setminus \{0\})/\mathbb{G}_m]$ as in Definition 5.7.

Proof. By (5.5) and (5.6), we have an $\text{Aut}(Q_d)$ -equivariant decomposition of the normal space

$$\mathcal{N}_{G[Q_d]/U, [Q_d]} \cong \begin{cases} \bigoplus_{i=2}^{d/2} \mathbf{V}_{2i} & \text{if } d \text{ is even,} \\ \mathbf{V}_3^{\circ} \oplus \bigoplus_{i=2}^{(d-1)/2} \mathbf{V}_{2i+1} & \text{if } d \text{ is odd.} \end{cases}$$

Thus, by translating the above decomposition via the G -action, we obtain a G -equivariant decomposition of the normal bundle $\mathcal{N}_{G[Q_d]/U}$ which we denote by $\bigoplus_{i=2}^{d/2} \mathcal{N}_i$ and $\bigoplus_{i=1}^{(d-1)/2} \mathcal{N}_i$ when d is even and odd, respectively. Let $\widehat{\mathcal{U}} \rightarrow U$ be the weighted stacky blow-up along the G -orbit of Q_d as in Definition 5.10 given by weight i on $\mathcal{N}_i$. Let $\mathcal{E}$ denote the corresponding stacky exceptional divisor, and let $\widehat{\mathcal{U}}$ and E denote the weighted projective blow-up coarse space and corresponding exceptional divisor, respectively. Note that $\mathcal{E}$ is a smooth Cartier divisor in $\widehat{\mathcal{U}}$ as we are doing a stacky weighted blow-up.

We note that this blow-up is $\text{SL}(3)$ -equivariant. Let $(\mathbb{P}_{\widehat{\mathcal{U}}}^2, C_{\widehat{\mathcal{U}}}) \rightarrow \widehat{\mathcal{U}}$ denote the pullback of this universal family via the blow-up, and denote by the pullback of the universal family to the stack by $(\mathbb{P}_{\widehat{\mathcal{U}}}^2, C_{\widehat{\mathcal{U}}}) \rightarrow \widehat{\mathcal{U}}$.

Consider the blow-up $\phi : (\mathcal{Y}, C_{\mathcal{Y}}) \rightarrow (\mathbb{P}_{\widehat{\mathcal{U}}}^2, C_{\widehat{\mathcal{U}}})$ of the universal family at the conic component $\mathcal{Q}$ of $C_{\widehat{\mathcal{U}}}|_{\mathbb{P}_{\widehat{\mathcal{U}}}^2}$ over the exceptional divisor $\mathcal{E} \subset \widehat{\mathcal{U}}$, where $C_{\mathcal{Y}}$ is the strict transform of $C_{\widehat{\mathcal{U}}}$. Let $\mathcal{F}$ denote the ϕ -exceptional divisor. Since $\mathcal{Q}$ is smooth of codimension 2 in $\mathbb{P}_{\widehat{\mathcal{U}}}^2$, and $\mathcal{E}$ is a smooth Cartier divisor of $\widehat{\mathcal{U}}$, we know that $\pi = \text{pr}_2 \circ \phi : \mathcal{Y} \rightarrow \widehat{\mathcal{U}}$ is a projective equidimensional morphism between smooth Deligne–Mumford stacks, which implies that $\mathcal{Y}$ is flat over $\widehat{\mathcal{U}}$ by miracle flatness. Furthermore, the codimension two locus $\mathcal{Q}$ we blow-up is $\text{SL}(3)$ -equivariant. In particular, we obtain the following diagram:

$$\begin{array}{ccccccc} \mathcal{F} & \hookrightarrow & (\mathcal{Y}, C_{\mathcal{Y}}) & \longrightarrow & (\mathbb{P}_{\widehat{\mathcal{U}}}^2, C_{\widehat{\mathcal{U}}}) & \longrightarrow & (\mathbb{P}_{\widehat{\mathcal{U}}}^2, C_{\widehat{\mathcal{U}}}) & \longrightarrow & (\mathbb{P}_U^2, C_U) \\ & & \searrow & & \downarrow & & \downarrow & & \downarrow \\ & & & & \widehat{\mathcal{U}} & \longrightarrow & \widehat{\mathcal{U}} & \longrightarrow & U \\ & & & & \uparrow & & \uparrow & & \uparrow \\ & & & & \mathcal{E} & \longrightarrow & E & \longrightarrow & G[Q_d] \end{array}$$

(A curved arrow also points from $\mathcal{F}$ to $\mathcal{E}$.)

Note that fibers of $(\mathcal{Y}, C_{\mathcal{Y}}) \rightarrow \widehat{\mathcal{U}}$ over $\widehat{\mathcal{U}} \setminus \mathcal{E}$ are unchanged, and the fibers over $\mathcal{E}$ are simple normal crossing surfaces of the form $\mathbb{P}^2 \cup \mathbb{F}_4$, where $\mathbb{F}_4$ denotes the fourth Hirzebruch surface, and they are glued along the conic component in $\mathbb{P}^2$ and the negative section in $\mathbb{F}_4$. Let $\mathcal{D}$ denote the strict transform of $\mathbb{P}_{\widehat{\mathcal{U}}}^2$ under ϕ .

We claim that there is a divisorial contraction $\psi : \mathcal{Y} \rightarrow \mathcal{X}$ over $\widehat{\mathcal{U}}$ that contracts $\mathcal{D}$ to a section over $\mathcal{E}$. Indeed, let $\mathcal{L} := \phi^* \mathcal{O}_{\mathbb{P}_{\widehat{\mathcal{U}}}^2}(1) + \frac{1}{2} \mathcal{D}$ be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on $\mathcal{Y}$. Then we know that $2\mathcal{L} = \phi^* \mathcal{O}_{\mathbb{P}_{\widehat{\mathcal{U}}}^2}(2) + \mathcal{D}$ is a Cartier divisor on $\mathcal{Y}$. First, we show that $2\mathcal{L}$ is nef and big over $\widehat{\mathcal{U}}$. Clearly, $2\mathcal{L}$ is ample over $\widehat{\mathcal{U}} \setminus \mathcal{E}$ where ϕ is isomorphic. Now, let us restrict to a fiber $\mathcal{Y}_{\mathcal{E}}$ over a geometric

point $e \in |\mathcal{E}|$. Since $F + D = \pi^* \mathcal{E}$, we have $2\mathcal{L} \sim_{\hat{U}} \phi^* \mathcal{O}_{\mathbb{P}^2}(2) - F$. Denote by $\mathcal{Y}_e = \mathbb{P}_e^2 \cup \mathbb{F}_{4,e}$. Let H_2 be an element of $\mathcal{O}_{\mathbb{P}^2}(2)$. Then $\phi^* H_2 - F$ is 0 over $\mathbb{P}_e^2$ and big and nef on $\mathbb{F}_{4,e}$, where F is the restriction of F to the fiber $\mathcal{Y}_e$. Indeed, if l is a line in $\mathbb{P}_e^2$, then $(\phi^* H_2 \cdot l) = 2 = (F \cdot l)$, since F came from blowing up a conic. Therefore, $((\phi^* H_2 - F) \cdot l) = 0$ for any line in $\mathbb{P}_e^2$. An adjunction calculation shows that $(\phi^* H_2 - F)|_{\mathbb{F}_{4,e}} = 4f + s$, where f is a fiber of the ruled surface $\mathbb{F}_{4,e}$ and s is the negative section. Therefore, $((\phi^* H_2 - F) \cdot f) = 1$ and $((\phi^* H_2 - F) \cdot s) = 0$ and $\phi^* H_2 - F$ is big and nef on $\mathbb{F}_{4,e}$.

In particular, we see that $2\mathcal{L}$ is a big and nef line bundle over $\hat{U}$ that is 0 on D over $\mathcal{E}$. In order to show that $2\mathcal{L}$ defines a divisorial contraction, we use cohomology and base change. In fact, since $\mathcal{Y} \rightarrow \hat{U}$ is a $\mathbb{Q}$ -Gorenstein flat family of slc Fano varieties (this can be checked fiberwise, as both $\mathbb{P}^2$ and $\mathbb{P}^2 \cup \mathbb{F}_4$ are slc with ample anticanonical divisors), by Fujino's vanishing theorem [42, Theorem 1.7], we know that $R^i \pi_* \mathcal{O}_{\mathcal{Y}}(2m\mathcal{L}) = 0$ and $\pi_* \mathcal{O}_{\mathcal{Y}}(2m\mathcal{L})$ is locally free for every $i > 0$ and $m > 0$. Moreover, both sheaves commute with base change. Note that this argument can be applied over the smooth Deligne–Mumford stack $\hat{U}$ as we can first pass to an étale cover of $\hat{U}$ by smooth schemes, apply Fujino's vanishing theorem there, then descend to $\hat{U}$. See also [26] for cohomology and base change for Deligne–Mumford stacks. Therefore, since $2\mathcal{L}$ is basepoint free on every geometric fiber, we know that $2\mathcal{L}$ is relatively basepoint free, and we obtain a divisorial contraction $\psi : \mathcal{Y} \rightarrow \mathcal{X}$ where

$$\mathcal{X} := \text{Proj}_{\hat{U}} \bigoplus_{m=0}^{\infty} \pi_* \mathcal{O}_{\mathcal{Y}}(2m\mathcal{L}).$$

Let $C_{\mathcal{X}} := \psi_* C_{\mathcal{Y}}$. Since the line bundle $2\mathcal{L}$ has a natural $\text{SL}(3)$ linearization, this contraction is also $\text{SL}(3)$ -equivariant. Thus, we obtain an $\text{SL}(3)$ -equivariant $\mathbb{Q}$ -Gorenstein flat family $(\mathcal{X}, C_{\mathcal{X}}) \rightarrow \hat{U}$.

Next, we verify conditions (1) and (2). Since $2\mathcal{L}$ is ample over $\hat{U} \setminus \mathcal{E}$, (1) follows directly from the construction. For (2), note that $2\mathcal{L}|_{\mathcal{Y}_e}$ is trivial on $\mathbb{P}_e^2$ and has class $4f + s$ on $\mathbb{F}_{4,e}$ for $e \in |\mathcal{E}|$. Therefore, the linear system $|2\mathcal{L}|_{\mathcal{Y}_e}|$ is basepoint free and contracts $\mathbb{P}_e^2 \cup \mathbb{F}_{4,e}$ to $\mathbb{P}(1, 1, 4)$. In particular, we know that the algebra $\bigoplus_{m=0}^{\infty} H^0(\mathcal{Y}_e, 2m\mathcal{L}|_{\mathcal{Y}_e})$ is generated in degree 1. Thus, by cohomology and base change, we know that the fiber $\mathcal{X}_e$ is isomorphic to $\mathbb{P}(1, 1, 4)$.

Finally, we discuss what happens to the family of curves over the Luna slice W . Let $\rho_W : \widehat{\mathcal{W}} \rightarrow W$ be the base change of the stacky weighted blow-up $\hat{U} \rightarrow U$. Denote by $(\mathcal{Y}_W, C_{\mathcal{Y}_W}) := (\mathcal{Y}, C_{\mathcal{Y}}) \times_{\hat{U}} \widehat{\mathcal{W}}$ and $(\mathcal{X}_W, C_{\mathcal{X}_W}) := (\mathcal{X}, C_{\mathcal{X}}) \times_{\hat{U}} \widehat{\mathcal{W}}$. Let $\phi_W : \mathcal{Y}_W \rightarrow \mathbb{P}_{\widehat{\mathcal{W}}}^2$ be the pullback of ϕ under the base change $\widehat{\mathcal{W}} \rightarrow \hat{U}$. Let $\pi_W : \mathcal{Y}_W \rightarrow \widehat{\mathcal{W}}$ be the projection. Then, we know that ϕ_W is the blow-up of the smooth conic component $\mathcal{Q}_W \subset \mathbb{P}_{\mathcal{E}_W}^2$. Let $\mathcal{D}_W := (\phi_W)_*^{-1} \mathbb{P}_{\mathcal{E}_W}^2$. Denote by $\mathcal{L}_W := \phi_W^* \mathcal{O}_{\mathbb{P}_{\widehat{\mathcal{W}}}^2}(1) + \frac{1}{2} \mathcal{D}_W$. Then from the above discussion, we know that $2\mathcal{L}_W$ is relatively basepoint free over $\widehat{\mathcal{W}}$. We shall define a morphism from $\mathcal{Y}_W$ to a $\mathbb{P}(1^3, 2)$ -bundle $\mathbb{P}\mathcal{V}$ over $\widehat{\mathcal{W}}$ that induces a closed embedding $\mathcal{X}_W \hookrightarrow \mathbb{P}\mathcal{V}$.

Since $[\mathcal{L}_W] = \phi_W^* \mathcal{O}_{\mathbb{P}_{\widehat{\mathcal{W}}}^2}(1)$, we know that $(\pi_W)_* \mathcal{O}_{\mathcal{Y}_W}([\mathcal{L}_W]) = \mathbf{V} \otimes \mathcal{O}_{\widehat{\mathcal{W}}} =: \mathcal{V}_1$ where $\mathbf{V} = H^0(\mathbb{P}^2, \mathcal{O}(1))$. Moreover, we have

$$(\pi_W)_* \mathcal{O}_{\mathcal{Y}_W}(2\mathcal{L}_W) = (\pi_W)_* (\phi_W^* \mathcal{O}_{\mathbb{P}_{\widehat{\mathcal{W}}}^2}(2) \otimes \mathcal{O}_{\mathcal{Y}_W}(\mathcal{D}_W)).$$

Recall that F denotes the exceptional divisor of ϕ . We know that $Q = (q = 0)$ where $q \in H^0(\mathbb{P}^2, \mathcal{O}(2))$ defines a divisor $(\phi_W)_*^{-1}(Q \times \widehat{\mathcal{W}}) - F_W$ which corresponds to a section

$$q_W \in H^0(\widehat{\mathcal{W}}, (\pi_W)_* (\phi_W^* \mathcal{O}_{\mathbb{P}_{\widehat{\mathcal{W}}}^2}(2) \otimes \mathcal{O}_{\mathcal{Y}_W}(-F_W))).$$

Let $\mathcal{V}_2 := \mathcal{O}_{\widehat{\mathcal{W}}}(\mathcal{E}_W)$. Then there is a surjection

$$(\mathrm{Sym}^2 \mathcal{V}_1) \oplus \mathcal{V}_2 \twoheadrightarrow (\pi_W)_* \mathcal{O}_{\mathcal{Y}_W}(2\mathcal{L}_W),$$

whose first component is given by the usual embedding $\phi_W^* \mathcal{O}_{\mathbb{P}^2}(2) \hookrightarrow \phi_W^* \mathcal{O}_{\widehat{\mathcal{W}}}^{\mathbb{P}^2}(2) \otimes \mathcal{O}_{\mathcal{Y}_W}(D_W)$, and whose second component is given by multiplying with q_W as $D_W = \pi_W^* \mathcal{E}_W - F_W$. We define $\mathcal{V} := \mathcal{V}_1 \oplus \mathcal{V}_2$ and define $\mathbb{P}\mathcal{V} := \mathrm{Proj}_{\widehat{\mathcal{W}}} \bigoplus_{m=0}^{\infty} \mathrm{Sym}^m \mathcal{V}$ where $\mathcal{V}_i$ has degree i . Since the ample model of $2\mathcal{L}$ is defined by $|2\mathcal{L}|$ on each fiber, by cohomology and base change from earlier, we know that the above surjection gives a closed embedding $\mathcal{X}_W \hookrightarrow \mathbb{P}\mathcal{V}$.

Next, we work out the defining equations of $\mathcal{X}_W$ and its family of curves $\mathcal{C}_{\mathcal{X}_W}$. We focus on the d even case as the d odd case is similar. Let x_0, x_1, x_2 be a basis of $\mathcal{V}_1$ induced by the standard basis of $\mathbf{V} = H^0(\mathbb{P}^2, \mathcal{O}(1))$. Let x_3 be a nonzero constant section of $\mathcal{V}_2 \otimes \mathcal{O}_{\widehat{\mathcal{W}}}(-\mathcal{E}_W) = \mathcal{O}_{\widehat{\mathcal{W}}}$. Then the equation of $\mathcal{X}_W$ is given by

$$\mathcal{X}_W = (sx_3 - q(x_0, x_1, x_2) = 0) \subset \mathbb{P}\mathcal{V},$$

where $s \in H^0(\widehat{\mathcal{W}}, \mathcal{O}_{\widehat{\mathcal{W}}}(\mathcal{E}_W))$ is some defining section of $\mathcal{E}_W$. Since d is even, W has affine coordinates $(f_4, f_6, \dots, f_{d-2}, f_d)$ where each $f_{2i} \in \mathbf{V}_{2i} \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2i))$. Since each f_{2i} is the coordinate of $\mathcal{N}_i$ with weight i under the stacky weighted blow-up ρ_W , we know that $\bar{f}_{2i} := s^{-i} \rho_W^* f_{2i}$ is a global section of $\mathcal{O}_{\widehat{\mathcal{W}}}(-i\mathcal{E}_W)$. Therefore, we have

$$\mathcal{C}_{\mathcal{X}_W} = \left(x_3^{d/2} + \sum_{i=2}^{d/2} \bar{f}_{2i}(x_0, x_1, x_2) x_3^{d/2-i} = 0 \right) |_{\mathcal{X}_W}.$$

Next, we pullback $\mathcal{X}_W \hookrightarrow \mathbb{P}\mathcal{V}$ under the quotient map $\mathbf{A}'_d \setminus \{0\} \rightarrow \mathcal{E}_W = [(\mathbf{A}'_d \setminus \{0\})/\mathbb{G}_m]$. Since $\mathcal{V}_1$ is a trivial vector bundle and $\mathcal{V}_2 = \mathcal{O}_{\mathcal{W}}(\mathcal{E}_W)$, we know that the pull-back $\mathbb{P}\mathcal{V} \times_{\widehat{\mathcal{W}}} (\mathbf{A}'_d \setminus \{0\})$ is $\mathbb{G}_m$ -equivariantly isomorphic to the product $\mathbb{P}(1^3, 2)_{[x_0, x_1, x_2, x_3]} \times (\mathbf{A}'_d \setminus \{0\})$ where $\mathbb{G}_m$ acts on $\mathbb{P}(1^3, 2)$ as $t \cdot [x_0, x_1, x_2, x_3] = [x_0, x_1, x_2, tx_3]$. Under an isomorphism $\mathbb{P}^1 \rightarrow Q = (q=0) \subset \mathbb{P}^2$, we may identify $\mathcal{X}_W \times_{\widehat{\mathcal{W}}} (\mathbf{A}'_d \setminus \{0\})$ with $\mathbb{P}(1, 1, 4)_{[x, y, z]} \times (\mathbf{A}'_d \setminus \{0\})$ where x_0, x_1, x_2 are a basis of quadratic forms in x, y such that $q(x_0, x_1, x_2) = 0$, and $x_3 = z$. Clearly, the $\mathbb{G}_m$ -action on $\mathbb{P}(1, 1, 4) \times (\mathbf{A}'_d \setminus \{0\})$ is $t \cdot [x, y, z] = [x, y, tz]$. Then the equation of $\mathcal{C}_{\mathcal{X}_W} \times_{\widehat{\mathcal{W}}} (\mathbf{A}'_d \setminus \{0\})$ becomes

$$\left(z^{d/2} + \sum_{i=2}^{d/2} g_{4i}(x, y) z^{d/2-i} = 0 \right) \subset \mathbb{P}(1, 1, 4) \times (\mathbf{A}'_d \setminus \{0\}),$$

where $g_{4i}(x, y) := \bar{f}_{2i}(x_0, x_1, x_2) \in \mathbf{V}_{2i} = H^0(\mathbb{P}^1, \mathcal{O}(4i))$. Thus, the pullback of the family $(\mathcal{X}, \mathcal{C}_{\mathcal{X}}) \rightarrow \widehat{\mathcal{U}}$ to $\mathcal{E}_W$ is isomorphic to the universal family over $[(\mathbf{A}'_d \setminus \{0\})/\mathbb{G}_m]$. This verifies (3). $\square$

Let us choose an ideal sheaf $\mathcal{I} \subset \mathcal{O}_U$ such that $\widehat{U} \cong Bl_{\mathcal{I}} U$. Let $\bar{\mathcal{I}} \subset \mathcal{O}_{\mathbf{P}_d}$ be an $\mathrm{SL}(3)$ -equivariant extension ideal sheaf of $\mathcal{I}$ that is cosupported on the Zariski closure of $G[Q_d]$ in $\mathbf{P}_d$. Let $\widehat{\mathbf{P}}_d$ be the normalization of $Bl_{\bar{\mathcal{I}}} \mathbf{P}_d$ with $\pi_{\mathbf{P}_d} : \widehat{\mathbf{P}}_d \rightarrow \mathbf{P}_d$ the projection morphism. Let $\bar{E}$ be the $\pi_{\mathbf{P}_d}$ -exceptional divisor on $\widehat{\mathbf{P}}_d$ such that $\mathcal{O}_{\widehat{\mathbf{P}}_d}(-\bar{E}) = \bar{\mathcal{I}} \cdot \mathcal{O}_{\widehat{\mathbf{P}}_d}$. Then, for $k \gg 1$, the line bundle $L_k := \pi_{\mathbf{P}_d}^* \mathcal{O}_{\mathbf{P}_d}(k) \otimes \mathcal{O}_{\widehat{\mathbf{P}}_d}(-\bar{E})$ is an $\mathrm{SL}(3)$ -linearized ample line bundle on $\widehat{\mathbf{P}}_d$. By [64] we know that the

GIT stability of $(\hat{\mathbf{P}}_d, L_k)$ is independent of the choice of $k \gg 1$, and the GIT semistable locus $\hat{\mathbf{P}}_d^{\text{ss}}$ is contained in $\hat{U} = \pi_{\mathbf{P}_d}^{-1}(\mathbf{P}_d^{\text{ss}})$. Denote by $\hat{U}^{\text{ss}} := \hat{\mathbf{P}}_d^{\text{ss}}$ and $\hat{U}^{\text{ss}} := \hat{U} \times_{\hat{U}} \hat{U}^{\text{ss}}$.

Theorem 5.15. *There is an isomorphism $\psi : [\hat{U}^{\text{ss}}/\text{PGL}(3)] \rightarrow \overline{\mathcal{P}}_{d,c_1+\epsilon}^{\text{K}}$.*

Proof. We start from the construction of a morphism $\psi : [\hat{U}^{\text{ss}}/\text{PGL}(3)] \rightarrow \overline{\mathcal{P}}_{d,c_1+\epsilon}^{\text{K}}$. Denote by $\hat{U}^{\text{ps}}$ and U^{ps} the GIT polystable locus in $\hat{\mathbf{P}}_d$ and $\mathbf{P}_d$, respectively. Let $\hat{U}^{\text{ps}}$ be the preimage of $\hat{U}^{\text{ps}}$ under the coarse moduli space morphism $\hat{U} \rightarrow \hat{U}$. For simplicity, denote $G := \text{SL}(3)$. Then by [64] we know that

$$\hat{U}^{\text{ps}} = \pi_{\mathbf{P}_d}^{-1}(U^{\text{ps}} \setminus G[Q_d]) \cup G \cdot E_W^{\text{ps}},$$

where E_W^{ps} is the GIT polystable locus in the exceptional divisor E_W of the weighted blow-up $\hat{W} \rightarrow W$. Then by Theorems 5.8, 5.9, and 5.14, we know that the fibers of $(\mathcal{Y}, (c_1 + \epsilon)\mathcal{C}_{\mathcal{Y}}) \rightarrow \hat{U}$ over $\hat{U}^{\text{ps}}$ are all K-polystable. Hence, by Theorem 2.16, we know that the fibers over $\hat{U}^{\text{ss}}$ are all K-semistable. Then ψ is constructed by the universality of the K-moduli stacks (see Section 3.6).

Next, we show that ψ is an isomorphism between Artin stacks. By Theorem 5.2, it is clear that ψ is birational. Since $\overline{\mathcal{P}}_{d,c_1+\epsilon}^{\text{K}}$ is normal, by Zariski's Main Theorem, it suffices to show that the above morphism is finite. To do so, we use [10, Proposition 6.4]. To use Alper's result, we must check that the morphism on the level of good moduli spaces is finite, and that the morphism ψ is representable, separated, quasi-finite, and sends closed points to closed points.

First, we note that the good moduli spaces are isomorphic since GIT-polystability and K-polystability coincide by the above argument and Theorem 5.9. In particular, we also see that the morphism ψ sends closed points to closed points, as the closed points are the same. Hence, it suffices to show that ψ is representable, separated, and quasi-finite. $\square$

Lemma 5.16. *There exists a morphism $\varphi : \overline{\mathcal{P}}_{d,c_1+\epsilon}^{\text{K}} \rightarrow [U/\text{PGL}(3)]$ such that the composition $\varphi \circ \psi : [\hat{U}^{\text{ss}}/\text{PGL}(3)] \rightarrow [U/\text{PGL}(3)]$ is induced from the stacky weighted blow-up $\hat{U} \rightarrow U$.*

Proof of the lemma. Recall from Section 3.1 that $Z_{m,c_1+\epsilon}^{\circ}$ is a locally closed subscheme of the relative Hilbert scheme $\mathbb{H}^{\mathcal{X};N} \times \mathbb{H}^{\tilde{\mathcal{X}};N}$. For simplicity, denote by $T := Z_{m,c_1+\epsilon}^{\circ}$. Let $\pi : (\mathcal{X}, D) \rightarrow T$ be the universal family. Let $T' := \text{pr}_1(T)$ be the projection in the Hilbert scheme $\mathbb{H}^{\mathcal{X};N}$. Let $\pi' : \mathcal{X}' \rightarrow T'$ be the universal family such that $\pi = \pi' \times_{T'} T$. Let $H' \subset T'$ (resp. $H \subset T$) be the divisor parametrizing $\mathbb{P}(1, 1, 4)$. By the proof of Proposition 4.6(2), we know that T and T' are both smooth, and $\text{pr}_1 : T \rightarrow T'$ is a smooth morphism. Moreover, since H' is a $\text{PGL}(N+1)$ -orbit in $\mathbb{H}^{\mathcal{X};N}$, we know that H' and H are smooth prime divisors in T' and T , respectively.

Since π' is a $\mathbb{P}^2$ -fibration over $T' \setminus H'$, there exists a dominant étale morphism $\tilde{T}^{\circ} \rightarrow T' \setminus H'$ such that $\pi' \times_{T'} \tilde{T}^{\circ}$ is a trivial $\mathbb{P}^2$ -bundle over $\tilde{T}^{\circ}$. By Zariski's main theorem, there exists an open immersion $\tilde{T}^{\circ} \hookrightarrow \tilde{T}$ to a smooth variety $\tilde{T}$ together with a quasi-finite morphism $\tilde{T} \rightarrow T'$ étale away from H' whose image contains the generic point of H' . In particular, $\tilde{T}$ is flat over T' by miracle flatness. Since both $T' \setminus H'$ and H' are $\text{PGL}(N+1)$ -orbits, there exists $T'_i = g_i \cdot \tilde{T}$ where $g_i \in \text{PGL}(N+1)$ such that $\sqcup_i T'_i \rightarrow T'$ is a fppf covering. Denote by H'_i the preimage of H' in T'_i . Then from the above discussion, we see that $\pi' \times_{T'} (T'_i \setminus H'_i) : \mathcal{X}'_{T'_i \setminus H'_i} \rightarrow T'_i \setminus H'_i$ is a trivial $\mathbb{P}^2$ -bundle. Let $\mathcal{L}'_i$ be the Weil divisorial sheaf on $\mathcal{X}'_{T'_i}$ as the Zariski closure of $\mathcal{O}(1)$ on $\mathcal{X}'_{T'_i \setminus H'_i}$. Since

the families here are $\mathbb{Q}$ -Gorenstein with integral fibers, we know that $\mathcal{L}'_i{}^{[-3]}$ is the same as $\omega_{\mathcal{X}'_{T'_i}/T'_i}$ twisted by the pull-back of some line bundle on the base T'_i . After replacing T'_i by its Zariski covering, we may assume that $\mathcal{L}'_i{}^{[-3]} \cong \omega_{\mathcal{X}'_{T'_i}/T'_i}$. By Kawamata–Viehweg vanishing, we know that $(\pi'_{T'_i})_* \mathcal{L}'_i$ is a rank 3 vector bundle over T'_i .

We claim that the $\mathrm{PGL}(3)$ -torsors $\{\mathcal{P}'_i/T'_i\}_i$, by taking a projectivized basis of $(\pi'_{T'_i})_* \mathcal{L}'_i$, is a descent datum. Indeed, from the above construction, we know that the cocycle condition of $\mathcal{L}'_i$ is off by a third root of unity. Hence, the cocycle condition of $(\pi'_{T'_i})_* \mathcal{L}'_i$ is off by a scalar, which implies that a projectivized basis of $(\pi'_{T'_i})_* \mathcal{L}'_i$ satisfies the cocycle condition. Hence, by the fpqc descent of G -torsors [116, Tag 04U1], the $\mathrm{PGL}(3)$ -torsors $\{\mathcal{P}'_i/T'_i\}_i$ descend to a $\mathrm{PGL}(3)$ -torsor $\mathcal{P}'/T'$. Pulling back to T , we get a $\mathrm{PGL}(3)$ -torsor $\mathcal{P}/T$ where over $T \setminus H$ it is obtained by taking projectivized basis of $\pi \times_T (T \setminus H)$. It is clear that $\mathcal{P} \rightarrow T$ is $\mathrm{PGL}(N+1)$ -equivariant. Hence, the morphism φ is induced by the $\mathrm{PGL}(3)$ -equivariant morphism $\mathcal{P} \rightarrow U$ where $(t, [s_0, s_1, s_2]) \mapsto [s_0, s_1, s_2](\mathcal{D}_t) \subset \mathbb{P}^2$. $\square$

From Lemma 5.16 and the separatedness of $\widehat{U}^{\mathrm{ss}} \rightarrow U$, we know that to check ψ is representable, separated, and quasi-finite, it suffices to show that the restriction of ψ on $[\mathcal{E}^{\mathrm{ss}}/\mathrm{PGL}(3)]$ maps isomorphically onto $[H/\mathrm{PGL}(N+1)]$ where $\mathcal{E}^{\mathrm{ss}} := \mathcal{E} \cap \widehat{U}^{\mathrm{ss}}$. To prove this, we will construct an inverse morphism $\psi^{-1} : [H/\mathrm{PGL}(N+1)] \rightarrow [\mathcal{E}^{\mathrm{ss}}/\mathrm{PGL}(3)]$. We will focus on the case when d is even, as the strategy for d odd is similar. By Theorem 5.14, we know that $\mathcal{E} \cong \mathcal{E}_W \times_{\mathrm{PGL}(2)} \mathrm{PGL}(3)$ where $\mathrm{PGL}(2)$ is identified with $\mathrm{Aut}(\mathbb{P}^2, Q_d)$ as a subgroup of $\mathrm{PGL}(3)$. Hence, we know that $[\mathcal{E}^{\mathrm{ss}}/\mathrm{PGL}(3)] \cong [\mathcal{E}^{\mathrm{ss}}_W/\mathrm{PGL}(2)]$. From Theorem 5.14, we know that $\mathcal{E}_W \cong [(\mathbf{A}'_d \setminus \{0\})/\mathbb{G}_m]$ is the weighted projective stack. Let us consider the action of $\mathrm{GL}(2) \times \mathbb{G}_m$ on $\mathbf{A}'_d$ where $\mathrm{GL}(2)$ acts on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4j))$ as the symmetric power of the standard $\mathrm{GL}(2)$ -action on $(\mathbb{P}^1, \mathcal{O}(1))$, and $\mathbb{G}_m$ acts as σ . Consider a 1-PS $\tau : \mathbb{G}_m \rightarrow \mathrm{GL}(2) \times \mathbb{G}_m$ defined as $\tau(t) = (\mathrm{diag}(t, t), t^4)$, then it is clear that τ acts as identity on $\mathbf{A}'_d$. Consider the group $\mathcal{G} := \mathrm{GL}(2) \times_{\tau} \mathbb{G}_m$. It is clear that the quotient $\bar{\sigma}$ of σ gives a 1-PS in $\mathcal{G}$ and $\mathcal{G}/_{\bar{\sigma}} \mathbb{G}_m \cong \mathrm{PGL}(2)$. Hence, we know that $[\mathcal{E}_W/\mathrm{PGL}(2)] \cong [(\mathbf{A}'_d \setminus \{0\})/\mathcal{G}]$.

Next, we will construct a morphism $[H/\mathrm{PGL}(N+1)] \rightarrow [(\mathbf{A}'_d \setminus \{0\})/\mathcal{G}]$ which directly induces ψ^{-1} . Indeed, this reduces to construct a $\mathrm{PGL}(N+1)$ -equivariant $\mathcal{G}$ -torsor $\mathcal{P}_H/H$ and a $\mathrm{PGL}(N+1)$ -invariant morphism $\mathcal{P}_H \rightarrow \mathbf{A}'_d \setminus \{0\}$. Let $\pi_H : (\mathcal{X}_H, \mathcal{D}_H) \rightarrow H$ be the universal family where each fiber is isomorphic to $\mathbb{P}(1, 1, 4)$ with a degree $2d$ curve. By similar argument to the proof of Lemma 5.16, there exists an étale covering $\sqcup_i H_i \rightarrow H$ and a Weil divisorial sheaf $\mathcal{L}_{H_i}$ on $\mathcal{X}_{H_i}$ (as fiberwise $\mathcal{O}(1)$ on $\mathbb{P}(1, 1, 4)$) such that $\mathcal{L}_{H_i}^{[-6]} \cong \omega_{\mathcal{X}_{H_i}/H_i}$. Let $\mathcal{F}_i := (\pi_{H_i})_* \mathcal{L}_i$ as a rank 2 vector bundle on H_i . Define $\mathcal{F}'_i := (\pi_{H_i})_* \mathcal{L}_i^{[4]}/\mathrm{Sym}^4 \mathcal{F}_i$ as a line bundle on H_i . Then taking basis (s_0, s_1) and s_2 of $\mathcal{F}_i$ and $\mathcal{F}'_i$ resp. gives a $\mathrm{GL}(2) \times \mathbb{G}_m$ -torsor $\widetilde{\mathcal{P}}_{H_i}/H_i$. By projectivizing $(s_0, s_1, s_2) \mapsto (ts_0, ts_1, t^4 s_2)$ under τ , we get a $\mathcal{G}$ -torsor $\mathcal{P}_{H_i}/H_i$. Since the descent datum of $\{\mathcal{L}_i\}$ is off by a sixth root of unity, it is easy to see that $\{\mathcal{P}_{H_i}/H_i\}_i$ form an étale descent datum hence descend to a $\mathcal{G}$ -torsor $\mathcal{P}_H/H$. As the $\sqcup_i H_i$ and $\mathcal{L}_{H_i}$ can be chosen $\mathrm{PGL}(N+1)$ -equivariantly, we know that $\mathcal{P}_H/H$ is also $\mathrm{PGL}(N+1)$ -equivariant. Notice that a projectivized basis $[s_0, s_1, s_2]$ in $\mathcal{P}_H$ is the same as an equivalent class of projective coordinates $[x, y, z]$ for $\mathbb{P}(1, 1, 4)$ under the equivalence relation $z \sim z + g(x, y)$. For a point $t \in H$ and a projectivized basis $[s_0, s_1, s_2]$ lying over t , there is a unique projective coordinates $[x, y, z]$ in the equivalent class such that $\mathcal{D}_t$ has the form (5.2). This gives the $\mathrm{PGL}(N+1)$ -invariant morphism $\mathcal{P}_H \rightarrow \mathbf{A}'_d \setminus \{0\}$ whose image is contained in the GIT semistable locus by Theorem 5.8. Thus, the proof is finished.

Proof of Theorem 5.6. The theorem follows from Theorems 5.8, 5.14, and 5.15. $\square$

We have thus completed the proof of Theorem 1.3.

Proof of Theorem 1.3. The proof follows from Theorems 5.2, 5.5, and 5.6. $\square$

6 | K-MODULI SPACES OF PLANE QUARTICS AND SEXTICS AND K3 SURFACES

In the first section, we show that the wall crossing discussed in Section 5 is the only wall crossing in the log Fano region for $d = 4$ and 6. Using that, we relate the K-moduli spaces to certain moduli spaces of K3 surfaces.

6.1 | K-moduli wall crossings

The goal of this section is to prove the following theorem.

Theorem 6.1. Assume $d = 4$ or 6. Then for any $\frac{3}{2d} < c < \frac{3}{d}$, we have $\overline{\mathcal{P}}_{d,c}^K \cong \overline{\mathcal{P}}_{d,\frac{3}{2d}+\epsilon}^K$ as Artin stacks. In other words, there is only one wall crossing in the log Fano region.

Remark 6.2. Note that the K-moduli space $\overline{\mathcal{P}}_{4,\frac{1}{2}}^K$ was previously described by Odaka, Spotti, and Sun [106] in their study of K-moduli spaces of del Pezzo surfaces of degree 2.

Recall in Theorem 1.3, we constructed $\overline{\mathcal{P}}_{d,c_1+\epsilon}^K$ as the partial Kirwan blow-up of $\overline{\mathcal{P}}_d^{\text{GIT}}$. Therefore, by Proposition 4.7, to prove the above theorem it suffices to show that $\text{lct}(X; C) \geq 3/d$ for any K-polystable point $[(X, (c_1 + \epsilon)C)] \in \overline{\mathcal{P}}_{d,c_1+\epsilon}^K$ for $d = 4$ or 6. Using an explicit description of such GIT polystable points, we will verify the lct inequality as follows. Here, we consider GIT of curves on $\mathbb{P}(1, 1, 4)$ in the sense of Definition 5.7.

Proposition 6.3 (Degree $d = 4$).

- (1) Any GIT polystable plane curve of degree 4 that is not the double conic has $\text{lct} \geq 3/4$.
- (2) Any GIT polystable curve of degree 8 in $\mathbb{P}(1, 1, 4)$ has $\text{lct} \geq 3/4$.

Proof.

- (1) Recall that a plane quartic curve is GIT stable if and only if it has type A_1 or A_2 singularities. Moreover, the only reduced strictly GIT polystable quartic curves are the “cat-eye” and “ox” which have type singularities of type A_3 and possibly A_1 (see [99, Table on Page 80] and [54, Proposition 7]).
- (2) Any GIT semistable curve C of degree 8 in $\mathbb{P}(1, 1, 4)$ is given by the equation $z^2 = f(x, y)$ where $f \neq 0$ is a degree 8 polynomial whose roots have multiplicities at most 4. Therefore, C has at worst singularities of type A_3 . $\square$

Proposition 6.4 (Degree $d = 6$).

- (1) Any GIT polystable plane curve of degree 6 that is not the triple conic has $\text{lct} \geq 1/2$.

(2) Any GIT polystable curve of degree 12 in $\mathbb{P}(1, 1, 4)$ has $\text{lct} \geq 1/2$.

Proof.

- (1) The GIT polystable plane sextics are classified in [112, Theorem 2.4]. In Shah's terminology, we only need to check that all curves in Group I, II, or III have $\text{lct} \geq 1/2$. Group I are ADE singularities, so they have $\text{lct} > 1/2$. Group II and III both have $\text{lct} = 1/2$.
- (2) This follows from [112, Theorem 4.3]. In Shah's terminology, Case 1(i) corresponds to ADE singularities so $\text{lct} > 1/2$. Case 1(ii) has equations $(z^3 + a(xy)^4z + b(xy)^6 = 0)$ where a, b are not simultaneously zero, and the $\text{lct} = 1/2$ in this case. Case 2 is similar. $\square$

Proof of Theorem 6.1. As mentioned above, the proof follows from Theorem 1.3, Proposition 4.7, Theorem 5.9, and the two above propositions explicitly calculating the lct of the GIT polystable curves. $\square$

6.2 | Relating degree 4 and 6 plane curves to K3 surfaces

In this section, we describe a relation between K-moduli spaces of plane curves of degree 4 and 6 and certain Baily–Borel compactifications of moduli spaces of K3 surfaces that already appear in the literature. In the case of quartics, we use work of Hyeon and Lee [54] and Kondō [74]. In the case of sextics, we use work of Shah [112] and Looijenga [89] (see also Laza [76]).

6.2.1 | Plane quartics

We recall that the GIT quotient $\overline{P}_4^{\text{GIT}}$ generically parametrizes curves in $\mathbb{P}^2$ with at worst cuspidal singularities. There is a curve parametrizing plane curves with a tacnode (locally $(x^2 + y^4 = 0)$), and there is a point on this curve parametrizing the double conic. In [54], the authors construct two GIT moduli spaces that do not coincide with the standard GIT quotient $\overline{P}_4^{\text{GIT}}$. In particular, they construct $\overline{M}_3^{\text{hs}} := \text{Hilb}_{3,2} // \text{SL}(6)$ and $\overline{M}_3^{\text{cs}} := \text{Chow}_{3,2} // \text{SL}(6)$, where $\text{Hilb}_{3,2}$ (resp. $\text{Chow}_{3,2}$) denotes the closure of the locus of bicanonical curves of genus three in the Hilbert scheme (resp. Chow scheme). They then show the existence of the following diagram (see [54, Theorem 1 and Page 4]):

$$\overline{M}_3^{\text{cs}} \xleftarrow{\Psi^+} \overline{M}_3^{\text{hs}} \xrightarrow{\Theta} \overline{P}_4^{\text{GIT}}$$

where Θ is a divisorial contraction corresponding to the blow-up of the point parametrizing the double conic in $\overline{P}_4^{\text{GIT}}$ and Ψ^+ is a small contraction identifying all tacnodal curves in $\overline{M}_3^{\text{hs}}$ to the same point in $\overline{M}_3^{\text{cs}}$. By Theorems 1.3 and 6.1, we have $\overline{M}_3^{\text{hs}} \cong \overline{P}_{4, \frac{3}{4}-\epsilon}^{\text{K}}$. By construction, the space $\overline{P}_{4, \frac{3}{4}-\epsilon}^{\text{K}}$ has a divisor parametrizing curves on $\mathbb{P}(1, 1, 4)$ (the exceptional divisor of the weighted blow-up of the double conic), along with a curve that still parametrizes the tacnodal curves. This is the curve that is contracted via Ψ^+ to a point in $\overline{M}_3^{\text{cs}}$.

In [74], Kondō constructs a moduli space of K3 surfaces by considering $\mathbb{Z}/4\mathbb{Z}$ -cover of $\mathbb{P}^2$ branched along a quartic curve. Kondō's moduli space $\overline{P}_4^*$ is a Baily–Borel compactification of

the moduli space M of ADE K3 surfaces of degree 4 with $\mathbb{Z}/4\mathbb{Z}$ -symmetry. The boundary $\overline{P}_4^* \setminus M$ is a single point. Hyeon and Lee prove (see [54, Proposition 21]) that $\overline{P}_4^* \cong \overline{M}_3^{\text{cs}}$ and identify the point in the boundary of $\overline{P}_4^*$ with the locus of tacnodal curves (i.e., the image of the tacnodal curves in $\overline{M}_3^{\text{hs}}$ under the small contraction ψ^+). We now prove that the moduli space $\overline{P}_4^*$ of K3 surfaces is the ample model of the Hodge line bundle (see Proposition 3.35) on $\overline{P}_{4,3/4-\epsilon}^{\text{K}}$, thus relating our K-moduli to a moduli space of K3 surfaces.

Theorem 6.5. *The moduli space $\overline{P}_4^*$ is the ample model of the Hodge line bundle on $\overline{P}_{4,3/4-\epsilon}^{\text{K}}$.*

Proof. For simplicity, denote by λ_{Hodge} the Hodge line bundle on $\overline{P}_{4,3/4-\epsilon}^{\text{K}}$. Let M be the open subset of $\overline{P}_4^*$ parametrizing ADE K3 surfaces. Then by taking $\mathbb{Z}/4\mathbb{Z}$ -quotient, it is clear that M also parametrizes quartic curves on $\mathbb{P}^2$ and degree 8 curves on $\mathbb{P}(1, 1, 4)$ with at worst A_2 -singularities. Indeed, the moduli stack $\mathcal{M}$ associated to M is a $(\mathbb{Z}/4\mathbb{Z})$ -gerbe over $\mathcal{P}_{4,3/4-\epsilon}^{\text{klt}}$. Thus, M can be identified with the open subset $\mathcal{P}_{4,3/4-\epsilon}^{\text{klt}}$ of $\overline{P}_{4,3/4-\epsilon}^{\text{K}}$ whose complement has codimension ≥ 2 . By [57, Section 6.2], $\lambda_{\text{Hodge}}|_M$ is pulled back from the Hodge line bundle on the relevant period domain $\mathbb{D}/\Gamma$, which is the descent of $\mathcal{O}(-1)$, and thus ample. Since $\overline{P}_4^*$ is the Baily–Borel compactification of M , we know that $\lambda_{\text{Hodge}}|_M$ extends to an ample $\mathbb{Q}$ -line bundle on $\overline{P}_4^*$. By [29] we know that λ_{Hodge} is nef on $\overline{P}_{4,3/4-\epsilon}^{\text{K}}$. Since M are big open subsets in both $\overline{P}_{4,3/4-\epsilon}^{\text{K}}$ and $\overline{P}_4^*$, we know that λ_{Hodge} is big and semiample, and $\overline{P}_4^* \cong \text{Proj}(\bigoplus_{k=0}^{\infty} H^0(\overline{P}_{4,3/4-\epsilon}^{\text{K}}, \lambda_{\text{Hodge}}^{\otimes k}))$ is the ample model of λ_{Hodge} . This finishes the proof.

Finally, we remark here that an alternative proof can be obtained using [41, Theorem 1.2] and functoriality of the Hodge line bundle. $\square$

We note that Hyeon–Lee’s paper also studies the spaces $\overline{M}_3^{\text{cs}}$, $\overline{M}_3^{\text{hs}}$, $\overline{P}_4^*$, and $\overline{P}_4^{\text{GIT}}$ in the context of the log MMP on $\overline{M}_3$ (i.e., the Deligne–Mumford compactification) as well as their relations with Hacking’s $\overline{P}_4^{\text{H}}$. We postpone discussing this viewpoint until Section 9.3.

6.2.2 | Plane sextic curves

For sextic curves, we recall that Shah constructed a partial Kirwan desingularization of the GIT quotient of plane sextic curves [112]. In particular, as above there is a divisorial contraction $\widehat{P}_6^{\text{GIT}} \rightarrow \overline{P}_6^{\text{GIT}}$ corresponding to a weighted blow-up of the triple conic. Shah also constructed a set-theoretic morphism $\widehat{P}_6^{\text{GIT}} \rightarrow \overline{P}_6^*$, where $\overline{P}_6^*$ denotes the Baily–Borel compactification of the space of polarized K3 surfaces of degree two. This map was shown to be algebraic by Looijenga [88, 89] (see also [76, Theorem 1.9]). In particular, we have a similar diagram in the sextic case.

$$\overline{P}_6^* \longleftarrow \widehat{P}_6^{\text{GIT}} \longrightarrow \overline{P}_6^{\text{GIT}}.$$

Again using Theorems 1.3 and 6.1, we can identify $\widehat{P}_6^{\text{GIT}} \cong \overline{P}_{6,1/2-\epsilon}^{\text{K}}$. The proof of Theorem 6.5 gives the following, noting that the codimension of the klt locus inside $\overline{P}_{6,1/2-\epsilon}^{\text{K}}$ is ≥ 2 .

Theorem 6.6. *The moduli space $\overline{P}_6^*$ is the ample model of the Hodge line bundle on $\overline{P}_{6, \frac{1}{2}-\epsilon}^K$.*

As in the case of quartics, we will discuss the relation of the above picture with Hacking's $\overline{P}_6^H$ and $\overline{P}_6^*$ in Section 9.3.

7 | THE SECOND WALL CROSSING FOR PLANE QUINTICS

In this section, we discuss the second wall crossing for K-moduli spaces of plane quintics. For simplicity, we abbreviate $\overline{P}_{5,c}^K$ and $\overline{P}_{5,c}^K$ to $\overline{P}_c^K$ and $\overline{P}_c^K$, respectively. The main result goes as follows.

Theorem 7.1 (Second wall crossing for plane quintics). *Let C_0 be a plane quintic curve with a singular point of type A_{12} . Denote by $X_{26} := (xw - y^{13} - z^2 = 0) \subset \mathbb{P}(1, 2, 13, 25)$. Let C'_0 be the curve $(w = 0)$ on X_{26} .*

- (1) *There is no wall crossing for K-moduli stacks $\overline{P}_c^K$ when $c \in (\frac{3}{7}, \frac{8}{15})$.*
- (2) *There is an isomorphism of good moduli spaces $\phi_2^- : \overline{P}_{\frac{8}{15}-\epsilon}^K \rightarrow \overline{P}_{\frac{8}{15}}^K$ which only replaces $[(\mathbb{P}^2, C_0)]$ by $[(X_{26}, C'_0)]$.*
- (3) *There is a weighted blow-up morphism $\phi_2^+ : \overline{P}_{\frac{8}{15}+\epsilon}^K \rightarrow \overline{P}_{\frac{8}{15}}^K$ at the point $[(X_{26}, C'_0)]$. The exceptional divisor of ϕ_2^+ parametrizes curves on X_{26} of the form $(w = g(x, y))$ where $g \neq 0$ and g does not contain the term xy^{12} .*

In particular, the second wall for K-moduli spaces of plane quintics is $c_2 = \frac{8}{15}$.

We will split the proof of Theorem 7.1 into several steps.

7.1 | K-polystable replacement of A_{12} -quintic curve

Let C_0 be a plane quintic curve with a singular point of type A_{12} . Then by [122, 127], we know that up to a projective transformation, the equation of C_0 is

$$C_0 = \left((y^2 - xz)^2 \left(\frac{1}{4}x + y + z \right) - x^2(y^2 - xz)(x + 2y) + x^5 = 0 \right).$$

In the affine patch $[x, y, 1]$, there is a unique 6-jet $x' = x - y^2 + y^5 - \frac{1}{2}y^6$ so that the equation of C in the coordinates (x', y) becomes

$$x'^2 = ay^{13} + \text{higher order terms, where } a \neq 0.$$

Here, we assign weights 13 and 2 to x' and y , respectively. Since C_0 has only one singularity that is a double point, we know that it is a GIT stable plane quintic curve by [99, Table on Page 80].

In this section, we will show that $\frac{8}{15}$ is the upper K-semistable threshold of $(\mathbb{P}^2, C_0)$ by constructing its K-polystable degeneration. The goal is to prove the following.

Theorem 7.2. *The log Fano pair $(\mathbb{P}^2, cC_0)$ is K -semistable if and only if $0 < c \leq \frac{8}{15}$. Moreover, $(X_{26}, \frac{8}{15}C'_0)$ is the K -polystable degeneration of $(\mathbb{P}^2, \frac{8}{15}C_0)$.*

We prove this in steps.

Proposition 7.3. *If the log Fano pair $(\mathbb{P}^2, cC_0)$ is K -semistable, then $0 < c \leq \frac{8}{15}$.*

Proof. Suppose $(\mathbb{P}^2, cC_0)$ is K -semistable. Let us perform the $(13,2)$ -weighted blow-up in the coordinates (x', y) , and denote the resulting surface and exceptional divisor by (X, E) . Let $\pi : X \rightarrow \mathbb{P}^2$ be the weighted blow-up morphism. Straightforward computation shows that

$$A_{(\mathbb{P}^2, cC_0)}(E) = 15 - 26c, \quad -K_{\mathbb{P}^2} - cC_0 \sim_{\mathbb{Q}} (3 - 5c)H,$$

where $H \sim \mathcal{O}(1)$ is the hyperplane divisor on $\mathbb{P}^2$. If $\bar{C}_0 := \pi_*^{-1}C_0 \subset X$, then we know that $\bar{C}_0$ is a smooth rational curve in the smooth locus of X . It is easy to see that $(E^2) = -\frac{1}{26}$, the curve $\bar{C}_0 \sim 5\pi^*H - 26E$, and $(\bar{C}_0^2) = -1$. Thus, the Mori cone of X is generated by E and $\bar{C}_0$. Hence, $\pi^*H - tE$ is ample if and only if $0 < t < 5$, and big if and only if $0 \leq t < \frac{26}{5}$. Then, by computations,

$$\text{vol}_X(\pi^*H - tE) = \begin{cases} 1 - \frac{t^2}{26} & \text{if } 0 \leq t \leq 5; \\ \frac{(26-5t)^2}{26} & \text{if } 5 \leq t \leq \frac{26}{5}. \end{cases}$$

Hence, $S_{(\mathbb{P}^2, cC_0)}(E) = \int_0^\infty \text{vol}_X(\pi^*H - tE)dt = \frac{17}{5}(3 - 5c)$. Since $(\mathbb{P}^2, cC_0)$ is K -semistable, by the valuative criterion (Theorem 2.9), we know that

$$15 - 26c = A_{(\mathbb{P}^2, cC_0)}(E) \geq S_{(\mathbb{P}^2, cC_0)}(E) = \frac{17}{5}(3 - 5c).$$

This is equivalent to $c \leq \frac{8}{15}$. □

Now we construct a special degeneration.

Proposition 7.4. *The log Fano pair $(\mathbb{P}^2, cC_0)$ admits a special degeneration to (X_{26}, cC'_0) where C'_0 is given by the equation $(w = 0)$ with $[x, y, z, w]$ being the projective coordinates of X_{26} .*

Proof. We construct the special degeneration. Consider the family $(\mathbb{P}^2, C_0) \times \mathbb{A}^1$ and perform the following birational transformations,

$$\begin{array}{ccccc} & \mathcal{X} & \overset{f}{\dashrightarrow} & \mathcal{X}^+ & \\ \swarrow \pi & & \searrow g & \swarrow h & \searrow \psi \\ \mathbb{P}^2 \times \mathbb{A}^1 & & \mathcal{Y} & & \mathcal{Z} \end{array}$$

where in the central fiber, we have

$$\begin{array}{ccccc}
 S \cup X & \xrightarrow{\quad f \quad} & \widehat{S} \cup X' & & \\
 \swarrow \pi & \searrow g & \swarrow h & \searrow \psi & \\
 \mathbb{P}^2 & & S \cup X' & & S'
 \end{array}$$

- (1) π is the (13,2,1)-weighted blow-up of $\mathbb{P}^2 \times \mathbb{A}^1$ in the coordinates (x', y, t) where t is the parameter of $\mathbb{A}^1$,
- (2) the surface $S = \mathbb{P}(1, 2, 13)$ is the exceptional divisor of π ,
- (3) the map g is the contraction of $\overline{C}_0$ in $X \subset \mathcal{X}_0$ where $\overline{C}_0$ is the strict transform of C_0 in X ,
- (4) the map f is the Atiyah flop of the curve $\overline{C}_0$ in $\mathcal{X}_0$ (by computation the normal bundle $\mathcal{N}_{\overline{C}_0/\mathcal{X}} \cong \mathcal{O}_{\overline{C}_0}(-1) \oplus \mathcal{O}_{\overline{C}_0}(-1)$), and
- (5) ψ is the divisorial contraction that contracts X' to a point.

Let us analyze the geometry of these birational maps. Suppose that S has projective coordinates $[x_1, x_2, x_3]$ of weights (1,2,13), respectively. Then $S \cap X = E = (x_1 = 0)$, and $\overline{C}_0 \cap E = \{p\}$ is a smooth point of S and X . So, $h : \widehat{S} = \text{Bl}_p S \rightarrow S$, the map $g : X \rightarrow X'$ contracts the (-1) -curve $\overline{C}_0$, and $\psi : \widehat{S} \rightarrow S'$ contracts $h_*^{-1}(E)$. A simple analysis of the singularity of S' shows that S' has only one singularity of type $\frac{1}{25}(1, 4)$.

Let F be the exceptional divisor of $h : \widehat{S} \rightarrow S = \mathbb{P}(1, 2, 13)$. We may look at the $\mathbb{Q}$ -divisor $D := F + \frac{26}{25}h_*^{-1}E$. It is clear that the ample model of D on $\widehat{S}$ is exactly S' . The projective coordinate ring $\bigoplus_{m \geq 0} H^0(\widehat{S}, \mathcal{O}_{\widehat{S}}(\lfloor mD \rfloor))$ has four distinguished generators in degree 1, 2, 13, and 25, corresponding to x_1, x_2, x_3 , and $x_2^2 + x_3^{13}$ on S . If we denote x, y, z, w as these four generators, then their relation is $xw = z^2 + y^{13}$ which shows $S' \cong X_{26}$. It is clear that $\pi_*^{-1}C_0 \times \mathbb{A}^1 \cap S$ has the equation $x_2^{13} + x_3^2 = 0$. Hence, the degeneration of C_0 on X_{26} is the strict transform of the curve $(x_2^{13} + x_3^2 = 0)$ which is exactly $(w = 0)$. This finishes the proof. $\square$

In the Appendix, we use techniques of Ilten and Süß [59] to show that $(X_{26}, \frac{8}{15}C'_0)$ is indeed K-polystable (see Proposition A.2). We now prove Theorem 7.2.

Proof of Theorem 7.2. By Proposition 7.4, the log Fano pair $(\mathbb{P}^2, \frac{8}{15}C_0)$ admits a special degeneration to $(X_{26}, \frac{8}{15}C'_0)$, which is K-polystable by Proposition A.2. Therefore, the pair $(\mathbb{P}^2, \frac{8}{15}C_0)$ is K-semistable by Theorem 2.16. We then conclude that $(\mathbb{P}^2, cC_0)$ is K-semistable for any $c \in (0, \frac{8}{15})$ using Propositions 2.13 and 7.3. $\square$

7.2 | Proof of second wall crossing

In this section, we prove Theorem 7.1. Before presenting its proof, we provide several results that are needed. First, we limit the surfaces that can appear.

Lemma 7.5. *If $[(X, cD)] \in \overline{\mathcal{P}}_c^K$ for some $c \in (0, \frac{3}{5})$, then X is isomorphic to one of the following surfaces: $\mathbb{P}^2$, $\mathbb{P}(1, 1, 4)$, X_{26} , or $\mathbb{P}(1, 4, 25)$.*

Proof. By the index estimate (Theorem 4.8), we know that the Gorenstein index of any singular point on X is at most 5. Thus, from the classification of Manetti surfaces, we know that X is either isomorphic to $\mathbb{P}^2$, $\mathbb{P}(1, 1, 4)$, $\mathbb{P}(1, 4, 25)$ or isomorphic to some of their partial smoothings. It is clear (e.g., from [50]) that X_{26} is the only new surface appearing which is a partial smoothing. $\square$

Next, we discuss K-stability of curves on X_{26} .

Proposition 7.6. *Let C be a curve on X_{26} of degree 25. If (X_{26}, cC) is K-semistable, then $c \geq \frac{8}{15}$. If in addition that C passes through the singular point of X_{26} , then (X_{26}, cC) is K-unstable for any $c \in (0, \frac{3}{5})$.*

Proof. For simplicity, we denote by $X := X_{26}$. Let us consider the unique singular point $[0, 0, 0, 1]$ on X . Denote by $[x, y, z, w]$ the projective coordinates where X is defined by $xw = y^{13} + z^2$. If we set $w = 1$, then we have a cyclic quotient map $\pi : \mathbb{A}_{(y,z)}^2 \rightarrow X$ defined by $\pi(y, z) = [y^{13} + z^2, y, z, 1]$. Let F be the exceptional divisor on $\mathbb{A}_{(y,z)}^2$ given by the $(2, 13)$ -weighted blow-up. Let E be the quotient of F over X . Then it is clear that $\text{ord}_E = \pi_* \text{ord}_F / 25$, $(F^2) = -\frac{1}{26}$ and $(E^2) = -\frac{25}{26}$. Let Γ be the curve $x = 0$ on X . Then $\text{ord}_E(\Gamma) = \text{ord}_F(\Gamma) / 25 = \frac{26}{25}$. Hence, on the blow-up $\mu : Y \rightarrow X$ extracting E , the proper transform $\bar{\Gamma}$ of Γ satisfies $\bar{\Gamma} = \mu^* \Gamma - \frac{26}{25}E$ and $(\bar{\Gamma}^2) = -1$. So, the Mori cone of Y is generated by E and $\bar{\Gamma}$. Computation shows

$$\text{vol}_X(\mathcal{O}(1) - tE) = \begin{cases} \frac{1}{25} - \frac{25}{26}t^2 & \text{if } 0 \leq t \leq \frac{1}{25} \\ \frac{1}{26} \left(\frac{26}{25} - t \right)^2 & \text{if } \frac{1}{25} \leq t \leq \frac{26}{25}. \end{cases}$$

Thus,

$$S_{(X, cC)}(\text{ord}_E) = \frac{15 - 25c}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tE) dt = \frac{9}{25}(15 - 25c).$$

Since $A_X(\text{ord}_E) = A_{\mathbb{A}^2}(\text{ord}_F) / 25 = 15/25 = 3/5$ and (X, cC) is K-semistable, the valuative criterion (Theorem 2.9) yields $\frac{3}{5} \geq A_{(X, cC)}(\text{ord}_E) \geq S_{(X, cC)}(\text{ord}_E) = \frac{9}{25}(15 - 25c)$ which implies $c \geq \frac{8}{15}$.

If C passes through $[0, 0, 0, 1]$, then its equation is given by $(f(x, y)z + g(x, y) = 0)$ where $\deg f = 12$ and $\deg g = 25$. Let $\tilde{C}$ be the preimage of C under π . Then, $\tilde{C}$ has equation

$$f(y^{13} + z^2, y)z + g(y^{13} + z^2, y) = 0.$$

Then, by simple calculation, we see that $\text{ord}_E(C) = \text{ord}_F(\tilde{C}) / 25 \geq 1$. Thus, we have

$$A_{(X, cC)}(\text{ord}_E) \leq \frac{1}{5}(3 - 5c) < \frac{9}{25}(15 - 25c) = S_{(X, cC)}(\text{ord}_E).$$

This implies that (X, cC) is always K-unstable for $c \in (0, \frac{3}{5})$ by the valuative criterion (Theorem 2.9). The proof is finished. $\square$

Proposition 7.7. *Let C be a curve on X_{26} of degree 25. Then $(X_{26}, \frac{8}{15}C)$ is K-semistable if and only if C does not pass through the unique singular point of X_{26} . Moreover, $(X_{26}, \frac{8}{15}C)$ is K-polystable if and only if $C \cong C'_0$ under an automorphism of X_{26} .*

Proof. We first look at the K-semistable statement. The “only if” part holds by Proposition 7.6. For the “if” part, suppose that C does not pass through $[0,0,0,1]$. Hence, the equation of C is given by $w = f(x, y)z + g(x, y)$. Consider the 1-PS in $\text{Aut}(X_{26})$ defined by $[x, y, z, w] \mapsto [t^{26}x, t^2y, t^{13}z, w]$. It is clear that C specially degenerates to $C'_0 = (w = 0)$ via this 1-PS as $t \rightarrow \infty$. Hence, $(X_{26}, \frac{8}{15}C)$ is K-semistable by Theorem 2.16 and the K-polystability of $(X_{26}, \frac{8}{15}C'_0)$ (see Proposition A.2). The K-polystable statement follows by uniqueness of K-polystable degenerations [93]. $\square$

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1.

- (1) Assume to the contrary that there are wall crossings within the interval $(\frac{3}{7}, \frac{8}{15})$. Let $c_2 \in (\frac{3}{7}, \frac{8}{15})$ be the second wall. Then there exists a new K-polystable pair (X, c_2D) such that (X, cD) is K-unstable for any $c \neq c_2$ by Proposition 3.18. Thus, X is not isomorphic to X_{26} or $\mathbb{P}(1, 4, 25)$ by Propositions 7.6 and A.14 (the valuative criterion for curves on $\mathbb{P}(1, 4, 25)$) since $c_2 < \frac{8}{15}$. By Lemma 7.5, we are left with two possibilities, that is, X is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}(1, 1, 4)$. If $X \cong \mathbb{P}^2$, then Proposition 2.13 implies that $(X, \epsilon D)$ is also K-polystable that is a contradiction. Hence, we may assume $(X, D) \cong (\mathbb{P}(1, 1, 4), C)$.

Assume that the equation of C is given by $f(x, y)z^2 + g(x, y)z + h(x, y) = 0$. If $f(x, y)$ is a nondegenerate quadratic form, then it is clear that C specially degenerates to Q'_5 . Since $(\mathbb{P}(1, 1, 4), \frac{3}{7}Q'_5)$ is K-polystable by Lemma 5.3, we know that $(\mathbb{P}(1, 1, 4), \frac{3}{7}C)$ is K-semistable by Theorem 2.16. But this is a contradiction since $(X, \frac{3}{7}D)$ is K-unstable. Thus, $f(x, y)$ is a degenerate quadratic form. By Proposition A.13 (the valuative criterion for curves on $\mathbb{P}(1, 1, 4)$), we know that $c_2 \geq \frac{6}{11} > \frac{8}{15}$, a contradiction. This finishes the proof of part (1).

- (2) From Theorem 7.2, we know that ϕ_2^- replaces $[(\mathbb{P}^2, C_0)]$ with $[(X_{26}, C'_0)]$. By Proposition 7.7, we know that $(X_{26}, \frac{8}{15}C'_0)$ is the only new K-polystable pair appearing in $\overline{P}_{\frac{8}{15}}^K$. Hence, to show ϕ_2^- is an isomorphism, it suffices to show that the preimage of $[(X_{26}, C'_0)]$ under ϕ_2^- is exactly $[(\mathbb{P}^2, C_0)]$. Denote by $E_2^\pm$ the preimage of $[(X_{26}, C'_0)]$ under the morphisms $\phi_2^\pm$. Assume to the contrary that E_2^- contains at least two points. Since all K-moduli spaces of plane curves are normal by Proposition 4.6, we know that E_2^- is connected hence has positive dimension. It is clear that $\text{Aut}_0(X_{26}, C'_0) \cong \mathbb{G}_m$. Let U_W be the Luna slice at the point $z_0 = \text{Hilb}(X_{26}, \frac{8}{15}C'_0) \in Z_{\frac{8}{15}}^\circ$ satisfying Theorem 3.33. Hence, we know that z_0 is the only $\mathbb{G}_m$ -invariant point in U_W . The smoothness of Z_c° (Proposition 4.6) implies that U_W is also smooth. Applying [32, Theorem 0.2.5] or [117, Corollary 1.13] to the local VGIT presentation (3.6) near z_0 implies that

$$\dim(E_2^-) + \dim(E_2^+) + 1 = \dim\left(\overline{P}_{\frac{8}{15}}^K\right) = 12.$$

In particular, we know that the locus E_2^+ has codimension at least two in the K-moduli space. However, we will show that this is not true.

Let C be a curve on X_{26} of the form $(w = g(x, y))$ where $g \neq 0$ and g does not contain the term xy^{12} . Hence, C does not pass through the singular point $[0, 0, 0, 1]$ of X_{26} . It is clear that C is a smooth curve on X_{26} for a general choice of g . Hence, $(X_{26}, \frac{3}{5}C)$ is klt for a general choice of g . Since $(X_{26}, \frac{8}{15}C)$ is K-semistable by Proposition 7.7, we know that $(X_{26}, (\frac{8}{15} + \epsilon)C)$ is K-stable for a general g . In the affine chart $x = 1$, the equation of C becomes

$$z^2 = -y^{13} + a_{11}y^{11} + a_{10}y^{10} + \cdots + a_0. \quad (7.1)$$

Hence, C is a hyperelliptic curve of arithmetic genus six. Indeed, from the above discussion, we see that any smooth hyperelliptic curve C of genus six produces a K-stable pair $(X_{26}, (\frac{8}{15} + \epsilon)C)$ in the K-moduli space $\overline{P}_{\frac{8}{15} + \epsilon}^K$. Since the moduli space of smooth hyperelliptic curves of genus six has dimension 11, we know that E_2^+ has dimension at least 11 that contradicts to the assumption that ϕ_2^- is not an isomorphism. This finishes the proof of part (2).

- (3) From the local VGIT argument in part (2), we know that ϕ_2^+ is a weighted blow-up since ϕ_2^- is an isomorphism (see [32, Theorem 0.2.5] or [117, Corollary 1.13]). If $[(X, D)]$ is a point in E_2^+ , then it admits a special degeneration to (X_{26}, C'_0) . Thus, X is either $\mathbb{P}^2$ or X_{26} . However, if $X \cong \mathbb{P}^2$, then $(X, \frac{8}{15}D)$ is K-polystable as well by Proposition 2.13, a contradiction. Hence, $X \cong X_{26}$ and D does not pass through the singular point on X by Proposition 7.6. Thus, after a suitable change of coordinates, we can put D into the form in the statement. Note that $g \neq 0$ is because otherwise $(X, D) \cong (X_{26}, C'_0)$ is $(\frac{8}{15} + \epsilon)$ -K-unstable.

To prove the rest of part (3), it suffices to show that $(X_{26}, (\frac{8}{15} + \epsilon)C)$ is K-polystable for any curve C on X_{26} described in the statement. Given such a curve C , we may find a family of smooth hyperelliptic curves D_t on X_{26} of the same form over a punctured smooth curve $T \setminus \{0\}$ such that $\lim_{t \rightarrow 0} D_t = C$. Let $(X, (\frac{8}{15} + \epsilon)D)$ be the K-polystable limit of $(X_{26}, (\frac{8}{15} + \epsilon)D_t)$ using Theorem 3.19. Since the Gorenstein index of X is a multiple of X_{26} which is 5, we know that X can only be X_{26} or $\mathbb{P}(1, 4, 25)$. But $\mathbb{P}(1, 4, 25)$ is impossible by Proposition A.14. Hence, $X \cong X_{26}$ and D is a curve not passing through the singular point $[0, 0, 0, 1]$. By an automorphism of X_{26} , we may assume that D has the equation $(w = h(x, y))$ where h is a homogeneous polynomial of degree 25. Since $(X_{26}, (\frac{8}{15} + \epsilon)C'_0)$ is K-unstable by Proposition 3.18, after a further change of coordinates, we may assume that $h \neq 0$ and h does not contain the term xy^{12} . Thus, we conclude that $(X, D) \cong (X_{26}, C)$. The proof is finished. $\square$

The following result follows easily from the proof of Theorem 7.1 (see, e.g., (7.1)).

Corollary 7.8. *The moduli stack of smooth hyperelliptic curves C of genus six with a marked Weierstrass point p admits a locally closed embedding into the K-moduli stack $\overline{P}_{\frac{8}{15} + \epsilon}^K$. Moreover, this embedding is stabilizer preserving and sends closed points to closed points. In particular, the coarse moduli space of such pairs (C, p) admits a locally closed embedding into the K-moduli space $\overline{P}_{\frac{8}{15} + \epsilon}^K$ whose image closure is the exceptional divisor of ϕ_2^+ .*

Corollary 7.8 is a strengthening of an earlier result of [48]. Although not explicitly stated, it is a consequence of [48, Theorem 1, Theorem 1.A] that a smooth hyperelliptic curve C of genus six admits an embedding into X_{26} coming from the marked Weierstrass point p . Indeed, the author

computes an embedding $C \hookrightarrow \mathbb{P}(1, 1, 1, 2, 3, 3)$ by

$$C \cong \text{Proj}(R(C, \mathcal{O}(5p))) \cong \text{Proj } \mathbb{C}[x_1, x_2, x_3, y, z_1, z_2]/I,$$

where x_i has weight 1, y weight 2, and z_i weight 3 and I depends on a uniquely determined degree 5 polynomial Q . One can show that X_{26} admits an embedding into $\mathbb{P}(1, 1, 1, 2, 3, 3, 5)$ such that, if t is the variable of weight 5, the curve $C \subset X_{26}$ is cut out by the equation $t = Q$.

7.3 | Applications to higher degree

It is natural to ask what can be said about wall crossing beyond the first wall in higher degree, and we address that now. The key observation is the following proposition (see Definition 2.10 for the definition of δ).

Proposition 7.9. *We have $\delta(X_{26}) = \frac{1}{9}$.*

Proof. For simplicity, denote by $X := X_{26}$. We follow notation from Proposition 7.6. Consider the valuation ord_E centered over the unique singular point of X . Since $-K_X \sim_{\mathbb{Q}} \mathcal{O}_X(15)$, by Proposition 7.6, we have

$$A_X(\text{ord}_E) = \frac{3}{5}, \quad S_X(\text{ord}_E) = \frac{15}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tE) dt = \frac{27}{5}.$$

Hence, we have $\delta(X) \leq A_X(\text{ord}_E)/S(\text{ord}_E) = \frac{1}{9}$. Assume to the contrary that $\delta(X) < \frac{1}{9}$. From Proposition A.2, we know that $(X, \frac{8}{15}C'_0)$ is K-polystable where $C'_0 = (w = 0)$. On the other hand, [23, Theorem 7.2] implies that $(X, \frac{8}{15}C'_0)$ is K-unstable by taking $\beta = \frac{1}{9}$ and $D = \frac{3}{5}C'_0$. This is a contradiction. Therefore, $\delta(X) = \frac{1}{9}$. $\square$

Using the above proposition, we can prove the following.

Theorem 7.10. *Let $d \geq 4$ be an integer.*

- (1) *For any $c < \frac{8}{3d}$, the only surfaces appearing in the K-moduli stack $\overline{\mathcal{P}}_{d,c}^K$ are $\mathbb{P}^2$ or $\mathbb{P}(1, 1, 4)$.*
- (2) *Suppose $5 \mid d$, then we have the following wall crossing at $c = \frac{8}{3d}$:*

$$\overline{\mathcal{P}}_{d, \frac{8}{3d} - \epsilon}^K \xrightarrow{\phi^-} \overline{\mathcal{P}}_{d, \frac{8}{3d}}^K \xleftarrow{\phi^+} \overline{\mathcal{P}}_{d, \frac{8}{3d} + \epsilon}^K,$$

where ϕ^- is an isomorphism near $(\mathbb{P}^2, \frac{d}{5}C)$ whose replacement is $(X_{26}, \frac{d}{5}C'_0)$, and ϕ^+ is a weighted blow-up at $(X_{26}, \frac{d}{5}C'_0)$. In particular, $\frac{8}{3d}$ is the second smallest wall extracting a divisor.

Proof.

- (1) Let (X, cD) be a K-semistable pair appearing in $\overline{\mathcal{P}}_{d,c}^K$ for some $c < \frac{8}{3d}$. By Theorem 4.8, we know that any local Gorenstein index of X is at most 9. Then from the classification of Manetti sur-

faces, we know that X is isomorphic to one of $\mathbb{P}^2$, $\mathbb{P}(1, 1, 4)$, $\mathbb{P}(1, 4, 25)$ or X_{26} . If $X \cong \mathbb{P}(1, 4, 25)$, then $\delta(X) = \frac{1}{10}$ by [20, Section 7]. Since $1 - \frac{cd}{3} > \frac{1}{10}$, by [23, Theorem 7.2], the pair (X, cD) is K-unstable by taking $\beta = 1 - \frac{cd}{3}$. This is a contradiction. If $X \cong X_{26}$, then $\delta(X) = \frac{1}{9}$ by Proposition 7.9. Hence, the same argument implies that (X, cD) is K-unstable whenever $c < \frac{8}{3d}$, again a contradiction.

- (2) The statement essentially follows from the proof of Theorem 7.1. Indeed, if C is a general curve on X_{26} of degree $5d$, then C does not pass through the singular point $[0, 0, 0, 1]$ on X_{26} . Thus, the same argument as the proof of Proposition 7.7 implies that $(X, \frac{8}{3d}C)$ is K-semistable. Since a general C is smooth, by Proposition 2.13, we know that (X_{26}, cC) is K-stable for any $c \in (\frac{8}{3d}, \frac{3}{d})$. It is clear that $\dim \text{Aut}(X_{26}) = \dim \text{Aut}(\mathbb{P}^2) - 1$, hence such pairs (X_{26}, C) form a divisor in the K-moduli space $\overline{P}_{d, \frac{8}{3d} + \epsilon}^K$ (see Section 9.2). Hence, the proof is finished by [32, Theorem 0.2.5] or [117, Corollary 1.13]. $\square$

8 | LOG FANO WALL CROSSINGS FOR K-MODULI SPACES OF PLANE QUINTICS

In this section, we discuss all wall crossings of K-moduli spaces of plane quintics in the log Fano region $c \in (0, \frac{3}{5})$. For simplicity, we again abbreviate $\overline{P}_5^{\text{GIT}}$, $\overline{P}_{5,c}^K$, and $\overline{P}_{5,c}^K$ to $\overline{P}^{\text{GIT}}$, $\overline{P}_c^K$, and $\overline{P}_c^K$, respectively. Thanks to Sections 5 and 7, we have detailed descriptions of the first two wall crossings. The main results of this section, namely, Theorems 8.2–8.5 will show that there are three more wall crossings after the first two walls. We begin by a description of $\overline{P}^{\text{GIT}}$.

8.1 | GIT of plane quintics

By Theorem 1.3, we know that the K-moduli space $\overline{P}_c^K$ is isomorphic to the GIT quotient for plane quintics, so we begin with a description of the (classical) GIT quotient $\overline{P}^{\text{GIT}}$ for plane quintics. This was calculated by Mumford [99, Chapter 4, Section 5]. A detailed description also appears in [75]. Under the identification of $\overline{P}_c^K$ and $\overline{P}^{\text{GIT}}$, Proposition 4.7 provides open embeddings

$$P_c^{\text{klt}} \hookrightarrow P_c^{\text{lc}} \hookrightarrow \overline{P}^{\text{GIT}},$$

where P_c^{klt} and P_c^{lc} denote the loci in $\overline{P}^{\text{GIT}}$ parametrizing GIT polystable plane quintics with $\text{lct} > \frac{3}{5}$ and $\geq \frac{3}{5}$, respectively.

Lemma 8.1. *The boundary $\overline{P}^{\text{GIT}} \setminus P_c^{\text{klt}}$ is a disjoint union of the following locally closed strata:*

Zero-dimensional loci

- $\Sigma_1 = \{[Q_5]\}$, and
- Σ_2 parametrizing a plane quintic curve with an A_{12} singularity.

One-dimensional loci

- Σ_3 parametrizing a reducible plane quintic curve with an A_{11} singularity,
- Σ_4 parametrizing an irreducible plane quintic curve with an A_{11} singularity, and

- Σ_6 parametrizing the union of two conics tangent at two distinct points and a line through them (two D_6 singularities), that is,

$$(z(xy - z^2)(xy - az^2) = 0) \quad \text{where } a \neq 1.$$

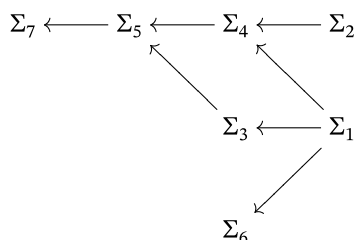
Two-dimensional locus

- Σ_5 parametrizing a plane quintic curve with an A_{10} singularity.

Three-dimensional locus

- Σ_7 parametrizing a plane quintic curve with an A_9 singularity.

Moreover, the incidence of such strata is as follows:



Here, $\Sigma_i \rightarrow \Sigma_j$ means that Σ_i is contained in the Zariski closure of Σ_j in $\bar{P}^{\text{GIT}}$. The closure of the stratum $\bar{\Sigma}_6 = \Sigma_6 \sqcup \Sigma_1$ is isomorphic to $\mathbb{P}^1$, and is the only strictly semistable stratum. The other strata are contained in the stable locus. In addition, $\bar{P}^{\text{GIT}} \setminus P_\epsilon^{\text{lc}} = \bigsqcup_{i=1}^5 \Sigma_i$.

Proof. By the GIT analysis in [75], we know that if C is a plane curve of degree 5, then

- C is GIT stable if and only if C is either smooth or has singularities of type A_k with $1 \leq k \leq 12$, D_4 and D_5 .
- C is GIT strictly semistable (i.e., semistable but not stable) if and only if it has a singularity of type D_k with $6 \leq k \leq 12$ such that if $k = 9$, then C is not the union of a nodal quartic and line.
- C is GIT strictly polystable (i.e., polystable but not stable) if and only if C is the union of double conic and a transverse line or the union of two tangent conics and a line passing through their tangent points.

Therefore, the GIT quotient $\bar{P}^{\text{GIT}}$ is the union of the GIT stable locus and a smooth rational curve parametrizing the GIT strictly polystable plane quintics. The statement follows by considering the log canonical thresholds of these singularities and the jet computations (see Proposition A.1). $\square$

8.2 | Explicit wall crossings

As we saw in Theorem 1.3, the GIT quotient $\bar{P}^{\text{GIT}}$ of plane quintics can be identified with the K-moduli space $\bar{\mathcal{P}}_c^K$ where $0 < c < 3/7$. In this section, we discuss the subsequent wall crossings among the K-moduli spaces of plane quintic curves. The following diagram gives an overview of the K-moduli spaces for plane quintics based on results from Sections 5, 7 and this section (see Table 1 for a summary).

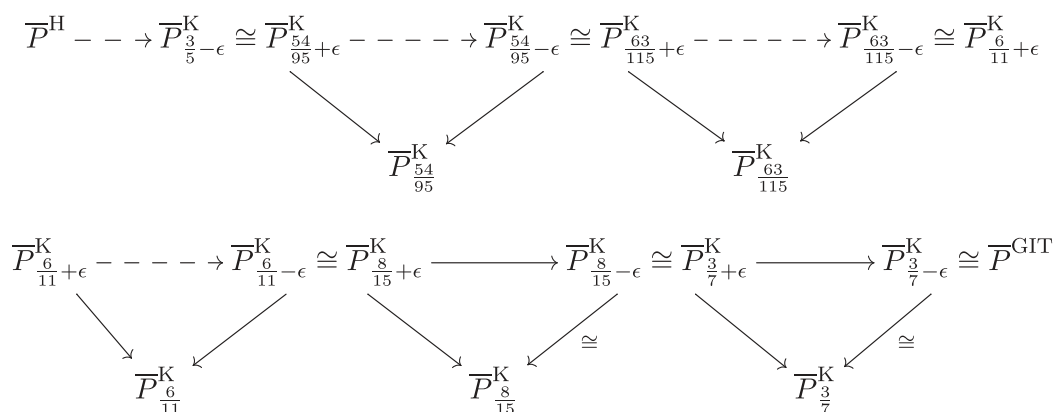


FIGURE 1 Log Fano wall crossings for K-moduli spaces of plane quintics.

The following results describe the remaining three walls that occur after the first two walls for K-moduli spaces of plane quintics. Their proofs will be presented in Section 8.3 with ingredients from calculations of Section A. If the birational map $\bar{P}^{\text{GIT}} \dashrightarrow \bar{P}_c^K$ is isomorphic at the generic point of a locus Σ_i , then we denote by $\bar{\Sigma}_{c,i}$ the Zariski closure of the proper transform of Σ_i in $\bar{P}_c^K$.

Theorem 8.2 (Third wall crossing). *The third wall is $c_3 = \frac{6}{11}$.*

- (1) The birational morphism $\phi_3^- : \bar{P}_{\frac{6}{11}-\epsilon}^K \rightarrow \bar{P}_{\frac{6}{11}}^K$ is an isomorphism away from the locus $\bar{\Sigma}_{\frac{6}{11}-\epsilon,3}$. Moreover, ϕ_3^- contracts $\bar{\Sigma}_{\frac{6}{11}-\epsilon,3}$ to a point $[(\mathbb{P}(1, 1, 4), \frac{6}{11}(x^2z^2 + y^6z = 0))]$.
- (2) The birational morphism $\phi_3^+ : \bar{P}_{\frac{6}{11}+\epsilon}^K \rightarrow \bar{P}_{\frac{6}{11}}^K$ is an isomorphism away from the point $[(\mathbb{P}(1, 1, 4), \frac{6}{11}(x^2z^2 + y^6z = 0))]$. Moreover, the exceptional locus E_3^+ of ϕ_3^+ is of codimension 2 and parametrizes curves on $\mathbb{P}(1, 1, 4)$ of the form $(x^2z^2 + y^6z + g(x, y) = 0)$ with $g \neq 0$.

Theorem 8.3 (Fourth wall crossing). *The fourth wall is $c_4 = \frac{63}{115}$.*

- (1) The birational morphism $\phi_4^- : \bar{P}_{\frac{63}{115}-\epsilon}^K \rightarrow \bar{P}_{\frac{63}{115}}^K$ is an isomorphism away from the locus $\bar{\Sigma}_{\frac{63}{115}-\epsilon,4}$. Moreover, ϕ_4^- contracts $\bar{\Sigma}_{\frac{63}{115}-\epsilon,4}$ to a point $[(\mathbb{P}(1, 4, 25), \frac{63}{115}(z^2 + x^2y^{12} = 0))]$.
- (2) The birational morphism $\phi_4^+ : \bar{P}_{\frac{63}{115}+\epsilon}^K \rightarrow \bar{P}_{\frac{63}{115}}^K$ is an isomorphism away from the point $[(\mathbb{P}(1, 4, 25), \frac{63}{115}(z^2 + x^2y^{12} = 0))]$. Moreover, the exceptional locus E_4^+ of ϕ_4^+ is of codimension 2 and parametrizes curves on $\mathbb{P}(1, 4, 25)$ of the form $(z^2 + x^2y^{12} + x^{10}g(x, y) = 0)$ with $g \neq 0$.

Theorem 8.4 (Fifth wall crossing). *The fifth wall is $c_5 = \frac{54}{95}$.*

- (1) The birational morphism $\phi_5^- : \bar{P}_{\frac{54}{95}-\epsilon}^K \rightarrow \bar{P}_{\frac{54}{95}}^K$ is an isomorphism away from the locus $\bar{\Sigma}_{\frac{54}{95}-\epsilon,5}$. Moreover, ϕ_5^- contracts $\bar{\Sigma}_{\frac{54}{95}-\epsilon,5}$ to a point $[(\mathbb{P}(1, 4, 25), \frac{54}{95}(z^2 + x^6y^{11} = 0))]$.
- (2) The birational morphism $\phi_5^+ : \bar{P}_{\frac{54}{95}+\epsilon}^K \rightarrow \bar{P}_{\frac{54}{95}}^K$ is an isomorphism away from the point $[(\mathbb{P}(1, 4, 25), \frac{54}{95}(z^2 + x^6y^{11} = 0))]$. Moreover, the exceptional locus E_5^+ of ϕ_5^+ is of

codimension 3 and parametrizes curves on $\mathbb{P}(1, 4, 25)$ of the form $(z^2 + x^6y^{11} + x^{14}g(x, y) = 0)$ with $g \neq 0$.

Theorem 8.5 (No other walls). *The five walls above are all walls occurring for K-moduli spaces of plane quintics in the log Fano region $0 < c < \frac{3}{5}$. In other words, for any $\frac{54}{95} < c < \frac{3}{5}$, we have an isomorphism $\overline{\mathcal{P}}_c^K \cong \overline{\mathcal{P}}_{\frac{54}{95}+\epsilon}^K$ between Artin stacks.*

8.3 | Proofs

In this section, we present proofs of Theorems 8.2–8.5. Our strategy is quite similar to the proof of Theorem 7.1.

Proof of Theorem 8.2.

- (1) We first show that there is no wall crossing when $c \in (\frac{8}{15}, \frac{6}{11})$. Assume to the contrary that the third wall $c_3 \in (\frac{8}{15}, \frac{6}{11})$. Then there exists a new K-polystable pair (X, c_3D) such that (X, cD) is K-unstable for any $c \neq c_3$ by Proposition 3.18. Thus, X is not isomorphic to $\mathbb{P}^2$ or $\mathbb{P}(1, 4, 25)$ by Propositions 2.13 and A.14. Hence, by Lemma 7.5, we are left with two possibilities, that is, X is isomorphic to $\mathbb{P}(1, 1, 4)$ or X_{26} . If $X \cong X_{26}$, then D does not pass through the singular point on X by Proposition 7.6. Hence, $(X, \frac{8}{15}D)$ is K-semistable by Proposition 7.7, but this is a contradiction since $c_3 \neq \frac{8}{15}$. Thus, we may assume $(X, D) \cong (\mathbb{P}(1, 1, 4), C)$ where the equation of C is given by $f(x, y)z^2 + g(x, y)z + h(x, y) = 0$. If $f(x, y)$ is nondegenerate, then $(\mathbb{P}(1, 1, 4), \frac{3}{7}C)$ is K-semistable since C specially degenerates to Q'_5 which is a contradiction. If $f(x, y)$ is degenerate, then Proposition A.13 implies that $(\mathbb{P}(1, 1, 4), c_3C)$ is K-unstable since $c_3 < \frac{6}{11}$. Thus, we have shown that no walls can appear in the interval $(\frac{8}{15}, \frac{6}{11})$.

Next, we show that $(\mathbb{P}(1, 1, 4), \frac{6}{11}(x^2z^2 + y^6z = 0))$ is the only new K-polystable pair in $\overline{\mathcal{P}}_{\frac{6}{11}}^K$. Clearly, this pair is K-polystable by Proposition A.6. If $(X, \frac{6}{11}D)$ is a new K-polystable pair, then from the argument above, we see that $(X, D) \cong (\mathbb{P}(1, 1, 4), C)$ and the defining equation of C has a degenerate quadratic form in (x, y) as the z^2 -term coefficient. Then, by Proposition A.13, we know that after a coordinate change, the equation of C has the form $x^2z^2 + y^6z + g(x, y) = 0$. Consider the 1-PS in $\text{Aut}(\mathbb{P}(1, 1, 4))$ defined by $[x, y, z] \mapsto [t^3x, ty, z]$. It is clear that C specially degenerates to the curve $C_1 := (x^2z^2 + y^6z = 0)$ via this 1-PS as $t \rightarrow 0$. Since $(\mathbb{P}(1, 1, 4), \frac{6}{11}C_1)$ is K-polystable, it follows that $(X, D) \cong (\mathbb{P}(1, 1, 4), C_1)$ by [93].

So far we have shown that $c_3 = \frac{6}{11}$. Next, we analyze the wall crossing morphisms $\phi_3^\pm$. Since $(\mathbb{P}(1, 1, 4), \frac{6}{11}C_1)$ is the only new K-polystable pair in $\overline{\mathcal{P}}_{\frac{6}{11}}^K$, we know by Proposition 3.18 that $\phi_3^\pm$ are isomorphic over $\overline{\mathcal{P}}_{\frac{6}{11}}^K \setminus \{[(\mathbb{P}(1, 1, 4), \frac{6}{11}C_1)]\}$. Denote by $E_3^\pm$ the preimage of $[(\mathbb{P}(1, 1, 4), \frac{6}{11}C_1)]$ under the morphisms $\phi_3^\pm$. It is clear that $\text{Aut}_0(\mathbb{P}(1, 1, 4), C_1) \cong \mathbb{G}_m$. Using the local VGIT presentation (3.6) and applying [32, Theorem 0.2.5] or [117, Corollary 1.13], we have

$$\dim(E_3^-) + \dim(E_3^+) + 1 = \dim\left(\overline{\mathcal{P}}_{\frac{6}{11}}^K\right). \quad (8.1)$$

Moreover, $E_3^\pm$ are weighted projective spaces quotient out some finite group action, so they are irreducible. By Theorem A.4, we know that $\bar{\Sigma}_{\frac{6}{11}-\epsilon,3} \subset E_3^-$ which implies $\dim E_3^- \geq 1$. Let C be a general curve on $\mathbb{P}(1, 1, 4)$ of the form $x^2z^2 + y^6z + g(x, y) = 0$. Then, it is clear that $\text{lct}(\mathbb{P}(1, 1, 4); C) \geq \frac{2}{3} > \frac{3}{5}$. Hence, $(\mathbb{P}(1, 1, 4), (\frac{6}{11} + \epsilon)C)$ is K-stable by Proposition 2.13. It is easy to see that such pairs $(\mathbb{P}(1, 1, 4), C)$ form a locally closed subset in the K-moduli space $\bar{P}_{\frac{6}{11}+\epsilon}^{\text{K}}$ of codimension two. Hence, $\dim(\bar{P}_{\frac{6}{11}}^{\text{K}}) - \dim(E_3^+) \leq 2$ which implies $\dim(E_3^-) \leq 1$ by (8.1). Therefore, we know $\dim(E_3^-) = 1$ and hence $\bar{\Sigma}_{\frac{6}{11}-\epsilon,3} = E_3^-$ by irreducibility of E_3^- . This finishes the proof of part (1).

- (2) The proof of this part is basically the same as the proof of Theorem 7.1(3). First, if $[(X, D)]$ is a point in E_3^+ , then (X, D) specially degenerates to $(\mathbb{P}(1, 1, 4), C_1)$. Hence, X is either $\mathbb{P}^2$ or $\mathbb{P}(1, 1, 4)$. If $X \cong \mathbb{P}^2$, then $(X, \frac{6}{11}D)$ is K-polystable by Proposition 2.13, a contradiction. Hence, $(X, D) \cong (\mathbb{P}(1, 1, 4), C)$. If C has the form $xyz^2 + f(x, y)z + h(x, y) = 0$ after a suitable change of coordinates, then it admits a special degeneration to Q_5 which implies that $(X, \frac{3}{7}D)$ is K-semistable by Theorem 2.16 and Lemma 5.3. Hence, $(X, \frac{6}{11}D)$ is K-polystable by Proposition 2.13, again a contradiction. Then by Proposition A.13, we know that C must have the form $x^2z^2 + y^6z + g(x, y) = 0$ after a suitable change of coordinates. Note that $g \neq 0$ because otherwise $(X, D) \cong (\mathbb{P}(1, 1, 4), C_1)$ is $(\frac{6}{11} + \epsilon)$ -K-unstable.

To prove the rest of part (2), it suffices to show that $(\mathbb{P}(1, 1, 4), (\frac{6}{11} + \epsilon)C)$ is K-polystable for any curve C on $\mathbb{P}(1, 1, 4)$ described in the statement. We omit the rest of the proof here since it is the same as the proof of Theorem 7.1(3) by using properness of K-moduli spaces (Theorem 3.19). $\square$

Proof of Theorem 8.3.

- (1) We first show that there is no wall crossing when $c \in (\frac{6}{11}, \frac{63}{115})$. Assume to the contrary that the fourth wall $c_4 \in (\frac{6}{11}, \frac{63}{115})$. Then there exists a new K-polystable pair (X, c_4D) such that (X, cD) is K-unstable for any $c \neq c_4$ by Proposition 3.18. Thus, similar argument to the proof of Theorem 8.2(1) implies that (X, D) has to be isomorphic to $(\mathbb{P}(1, 1, 4), C)$ where the equation of C is given by $f(x, y)z^2 + g(x, y)z + h(x, y) = 0$ with f degenerate. Then Proposition A.13 implies that the equation of C has to have the form $x^2z^2 + y^6z + h(x, y) = 0$, so C admits a special degeneration to the curve $C_1 = (x^2z^2 + y^6z = 0)$. Thus, by Proposition A.6 and Theorem 2.16, we know that $(\mathbb{P}(1, 1, 4), \frac{6}{11}C)$ is K-semistable, but this is a contradiction as $c_4 \neq \frac{6}{11}$. Thus, we have shown that no walls can appear in the interval $(\frac{6}{11}, \frac{63}{115})$.

Next, we show that $(\mathbb{P}(1, 4, 25), \frac{63}{115}(z^2 + x^2y^{12} = 0))$ is the only new K-polystable pair in $\bar{P}_{\frac{63}{115}}^{\text{K}}$. Clearly this pair is K-polystable by Proposition A.9. If $(X, \frac{63}{115}D)$ is a new K-polystable pair, then from the argument above, we see that $(X, D) \cong (\mathbb{P}(1, 4, 25), C)$. By Proposition A.14, the defining equation of C has the form $z^2 + x^2y^{12} + x^6g(x, y) = 0$. Consider the 1-PS in $\text{Aut}(\mathbb{P}(1, 4, 25))$ defined by $[x, y, z] \mapsto [x, ty, t^6z]$. It is clear that C specially degenerates to the curve $C_2 := (z^2 + x^2y^{12} = 0)$ via this 1-PS as $t \rightarrow 0$. Since $(\mathbb{P}(1, 4, 25), \frac{63}{115}C_2)$ is K-polystable, it follows that $(X, D) \cong (\mathbb{P}(1, 4, 25), C_2)$ by [93].

So far we have shown that $c_4 = \frac{63}{115}$. Next, we analyze the wall crossing morphisms $\phi_4^\pm$. Since $(\mathbb{P}(1, 4, 25), \frac{63}{115}C_2)$ is the only new K-polystable pair in $\bar{P}_{\frac{63}{115}}^{\text{K}}$, we know by Proposition 3.18 that $\phi_4^\pm$ are isomorphic over $\bar{P}_{\frac{63}{115}}^{\text{K}} \setminus \{(\mathbb{P}(1, 4, 25), \frac{63}{115}C_2)\}$. Denote by $E_4^\pm$ the preimage

of $[(\mathbb{P}(1, 4, 25), \frac{63}{115}C_2)]$ under the morphisms $\phi_4^\pm$. It is clear that $\text{Aut}_0(\mathbb{P}(1, 4, 25), C_2) \cong \mathbb{G}_m$. By Theorem A.7, we know that $\bar{\Sigma}_{\frac{63}{115}-\epsilon, 4} \subset E_4^-$ which implies $\dim(E_4^-) \geq 1$. Using a similar argument to the proof of Theorem 8.2(1), it suffices to show that E_4^+ has codimension at most 2 in the K-moduli space. Let C be a general curve on $\mathbb{P}(1, 4, 25)$ of the form $z^2 + x^2y^{12} + x^{10}g(x, y) = 0$. Then it is clear that $\text{lct}(\mathbb{P}(1, 4, 25); C) \geq 1 > \frac{3}{5}$. Hence, $(\mathbb{P}(1, 4, 25), (\frac{63}{115} + \epsilon)C)$ is K-stable by Proposition 2.13. It is easy to see that such pairs $(\mathbb{P}(1, 4, 25), C)$ form a locally closed subset in the K-moduli space $\bar{P}_{\frac{63}{115}+\epsilon}^K$ of codimension 2. Hence, $\bar{\Sigma}_{\frac{63}{115}-\epsilon, 4} = E_4^-$ by irreducibility of E_4^- and local VGIT presentation. This finishes the proof of part (1).

- (2) The proof of this part is basically the same as the proof of Theorem 7.1(3). First, if (X, D) is a point in E_4^+ , then (X, D) specially degenerates to $(\mathbb{P}(1, 4, 25), C_2)$. If X is isomorphic to $\mathbb{P}^2$ or X_{26} , then $(X, \frac{63}{115}D)$ is K-polystable by Proposition 2.13, a contradiction. If $(X, D) \cong (\mathbb{P}(1, 1, 4), C)$, then by Proposition A.13, we know that the equation of C has the form $xyz^2 + f(x, y)z + g(x, y) = 0$ or $x^2z^2 + y^6z + h(x, y)$. From the proof of Theorem 8.2, we know that the former curve is $\frac{3}{7}$ -K-semistable, while the latter curve is $\frac{6}{11}$ -K-semistable. Hence, $(X, \frac{63}{115}D)$ is K-polystable by Proposition 2.13, again a contradiction. Therefore, $(X, D) \cong (\mathbb{P}(1, 4, 25), C)$. By Proposition A.14, we know that the equation of C must be of the form $z^2 + x^2y^{12} + x^6h(x, y) = 0$. After a suitable change of coordinates, the equation of C can be even simplified to $z^2 + x^2y^{12} + x^{10}g(x, y) = 0$. Note that $g \neq 0$ because otherwise $(X, D) \cong (\mathbb{P}(1, 4, 25), C_2)$ is $(\frac{63}{115} + \epsilon)$ -K-unstable. The rest of part (2) also follows from similar argument to proof of Theorem 7.1(3) by using properness of K-moduli spaces (Theorem 3.19). $\square$

Proof of Theorem 8.4.

- (1) We first show that there is no wall crossing when $c \in (\frac{63}{115}, \frac{54}{95})$. Assume to the contrary that the fifth wall $c_5 \in (\frac{63}{115}, \frac{54}{95})$. Then there exists a new K-polystable pair (X, c_5D) such that (X, cD) is K-unstable for any $c \neq c_5$ by Proposition 3.18. Thus, similar argument to the proof of Theorems 8.2(1) and 8.3(1) implies that (X, D) has to be isomorphic to $(\mathbb{P}(1, 4, 25), C)$. By Proposition A.14, the equation of C must have the form $z^2 + x^2y^{12} + x^6g(x, y) = 0$ since $c_5 < \frac{54}{95}$. So, C admits a special degeneration to the curve $C_2 = (z^2 + x^2y^{12} = 0)$. Thus, by Proposition A.6 and Theorem 2.16, we know that $(\mathbb{P}(1, 4, 25), \frac{63}{115}C)$ is K-semistable, but this is a contradiction as $c_5 \neq \frac{63}{115}$. Thus, we have shown that no walls can appear in the interval $(\frac{63}{115}, \frac{54}{95})$.

Next, we show that $(\mathbb{P}(1, 4, 25), \frac{54}{95}(z^2 + x^6y^{11} = 0))$ is the only new K-polystable pair in $\bar{P}_{\frac{54}{95}}^K$. Clearly, this pair is K-polystable by Proposition A.12. If $(X, \frac{54}{95}D)$ is a new K-polystable pair, then from the argument above, we see that $(X, D) \cong (\mathbb{P}(1, 4, 25), C)$ such that C admits no special degeneration to C_2 . Thus, Proposition A.14 implies that the defining equation of C has the form $z^2 + x^6y^{11} + x^{10}g(x, y) = 0$. Consider the 1-PS in $\text{Aut}(\mathbb{P}(1, 4, 25))$ defined by $[x, y, z] \mapsto [x, t^2y, t^{11}z]$. It is clear that C specially degenerates to the curve $C_3 := (z^2 + x^6y^{11} = 0)$ via this 1-PS as $t \rightarrow 0$. Since $(\mathbb{P}(1, 4, 25), \frac{54}{95}C_3)$ is K-polystable, it follows that $(X, D) \cong (\mathbb{P}(1, 4, 25), C_3)$ by [93].

So far we have shown that $c_5 = \frac{54}{95}$. Next, we analyze the wall crossing morphisms $\phi_5^\pm$. Since $(\mathbb{P}(1, 4, 25), \frac{54}{95}C_3)$ is the only new K-polystable pair in $\bar{P}_{\frac{54}{95}}^K$, we know by Proposition 3.18 that $\phi_5^\pm$ are isomorphic over $\bar{P}_{\frac{54}{95}}^K \setminus \{(\mathbb{P}(1, 4, 25), \frac{54}{95}C_3)\}$. Denote by $E_5^\pm$ the preimage of $[(\mathbb{P}(1, 4, 25), \frac{54}{95}C_3)]$ under the morphisms $\phi_5^\pm$. It is clear that $\text{Aut}_0(\mathbb{P}(1, 4, 25), C_3) \cong \mathbb{G}_m$. By

Theorem A.10, we know that $\bar{\Sigma}_{\frac{54}{95}-\epsilon,5} \subset E_5^-$ which implies $\dim(E_5^-) \geq 2$. Using a similar argument to the proof of Theorem 8.2(1), it suffices to show that E_5^+ has codimension at most 3 in the K-moduli space. Let C be a general curve on $\mathbb{P}(1, 4, 25)$ of the form $z^2 + x^6y^{11} + x^{14}g(x, y) = 0$. Then it is clear that $\text{lct}(\mathbb{P}(1, 4, 25); C) \geq \frac{2}{3} > \frac{3}{5}$. Hence, $(\mathbb{P}(1, 4, 25), (\frac{54}{95} + \epsilon)C)$ is K-stable by Proposition 2.13. It is easy to see that such pairs $(\mathbb{P}(1, 4, 25), C)$ form a locally closed subset in the K-moduli space $\bar{P}_{\frac{54}{95}+\epsilon}^K$ of codimension 3. Hence, $\bar{\Sigma}_{\frac{54}{95}-\epsilon,5} = E_5^-$ by irreducibility of E_5^- and local VGIT presentation. This finishes the proof of part (1).

- (2) The proof of this part is basically the same as proofs of Theorems 7.1(3) and 8.3(2). First, if $[(X, D)]$ is a point in E_5^+ , then (X, D) specially degenerates to $(\mathbb{P}(1, 4, 25), C_3)$. By similar argument to the proof of Theorem 8.3(2), we know that $(X, D) \cong (\mathbb{P}(1, 4, 25), C)$. If the equation of C has the form $z^2 + x^2y^{12} + x^6g(x, y) = 0$, then it is $\frac{63}{115}$ -K-semistable since it specially degenerates to C_2 . Hence, $(X, \frac{54}{95}D)$ is K-polystable by Proposition 2.13, a contradiction. Hence, Proposition A.14 implies that the equation of C must be of the form $z^2 + x^6y^{11} + x^{10}h(x, y) = 0$. After a suitable change of coordinates, the equation of C can be even simplified to $z^2 + x^6y^{11} + x^{14}g(x, y) = 0$. Note that $g \neq 0$ because otherwise $(X, D) \cong (\mathbb{P}(1, 4, 25), C_3)$ is $(\frac{54}{95} + \epsilon)$ -K-unstable. The rest of part (2) also follows from similar argument to proof of Theorem 7.1(3) by using properness of K-moduli spaces (Theorem 3.19). $\square$

Proof of Theorem 8.5. Assume to the contrary that the sixth wall $c_6 \in (\frac{54}{95}, \frac{3}{5})$ exists. Then there exists a K-polystable log Fano pair (X, c_6D) in $\bar{P}_{c_6}^K$ such that (X, cD) is K-unstable for any $c \neq c_6$. Then X is isomorphic to $\mathbb{P}^2$, $\mathbb{P}(1, 1, 4)$, X_{26} , or $\mathbb{P}(1, 4, 25)$ by Lemma 7.5. If $X \cong \mathbb{P}^2$, then $(X, \epsilon D)$ is K-semistable by Proposition 2.13, a contradiction. If $X \cong \mathbb{P}(1, 1, 4)$, then Proposition A.13 implies that the equation of D has the form $xyz^2 + f(x, y)z + g(x, y) = 0$ or $x^2z^2 + y^6z + h(x, y) = 0$ after a suitable change of coordinates. Hence, D admits a special degeneration to either Q'_5 or $C_1 = (x^2z^2 + y^6z = 0)$. Thus, Lemma 5.3 and Proposition A.6 combined with Theorem 2.16 imply that either $(X, \frac{3}{7}D)$ or $(X, \frac{6}{11}D)$ is K-semistable, a contradiction. If $X \cong X_{26}$, then Proposition 7.6 implies that D does not pass through the singular point of X . Hence, $(X, \frac{8}{15}D)$ is K-semistable by Proposition 7.7, a contradiction. If $X \cong \mathbb{P}(1, 4, 25)$, then Proposition A.14 implies that D admits a special degeneration to either $C_2 = (z^2 + x^2y^{12} = 0)$ or $C_3 = (z^2 + x^6y^{11} = 0)$. Thus, Propositions A.9 and A.12 combined with Theorem 2.16 imply that either $(X, \frac{63}{115}D)$ or $(X, \frac{54}{95}D)$ is K-semistable, again a contradiction. Since we rule out all four possibilities, the proof is finished. $\square$

9 | PROJECTIVITY, BIRATIONAL CONTRACTIONS, AND THE LOG CALABI-YAU WALL CROSSING

In this final section, we discuss some questions with incomplete answers that are interesting for future study.

9.1 | Projectivity

In this section, we will show that for any $d \in \{4, 5, 6\}$ and any $c \in (0, \frac{3}{d})$, the CM $\mathbb{Q}$ -line bundle Λ_c (see Proposition 3.35) on the K-moduli space $\bar{P}_{d,c}^K$ is ample, which, in particular, implies the

projectivity of $\overline{P}_{d,c}^K$. Our main tools are the work of Codogni and Patakfalvi [29] and its generalization by Posva [107], as well as the relative ampleness of CM line bundles under wall crossing (see Theorem 3.36).

Theorem 9.1. *When $d \in \{4, 5, 6\}$, the CM $\mathbb{Q}$ -line bundle Λ_c on $\overline{P}_{d,c}^K$ is ample for any $c \in (0, \frac{3}{d})$.*

Proof. We first treat the cases when $d = 4$ or 6 . When $c < \frac{3}{2d}$, Theorem 5.2 implies that $\overline{P}_{d,c}^K \cong \overline{P}_d^{\text{GIT}}$. Moreover, Proposition 4.3 implies that Λ_c is the descent of $3(3 - cd)^2 c \mathcal{O}_{\mathbf{P}_d^{\text{ss}}}(1)$. Hence, Λ_c is ample when $c < \frac{3}{2d}$. By Theorem 5.5, we know that $\phi^- : \overline{P}_{d,3/(2d)}^K \rightarrow \overline{P}_d^{\text{GIT}}$ is an isomorphism. Hence, Theorem 3.36 implies that $\Lambda_{3/(2d)}$ is the ϕ^+ -pull back of the descent of $3(3 - c_1 d)^2 c_1 \mathcal{O}_{\mathbf{P}_d^{\text{ss}}}(1)$ with $c_1 = 3/(2d)$. Hence, $\Lambda_{3/(2d)}$ is ample. By Theorem 3.36, we know that $\Lambda_{3/(2d)+\epsilon}$ is ample for $0 < \epsilon \ll 1$. We know that $\overline{P}_{d,c}^K$ is independent of the choice of $c \in (\frac{3}{2d}, \frac{3}{d})$ by Theorem 6.1. Hence, the ampleness of Λ_c for $\frac{3}{2d} < c < \frac{3}{d}$ follows from the ampleness of $\Lambda_{3/(2d)+\epsilon}$, the nefness of $\Lambda_{c,\text{Hodge}}$ (see Theorems 6.5 and 6.6), and the interpolation formula (3.7).

Next, we consider the case when $d = 5$. For simplicity, we omit d in the subscript of K-moduli stacks and spaces. Similar to the above arguments, we know that Λ_c is ample for $c \leq \frac{3}{7} + \epsilon$ with $0 < \epsilon \ll 1$. Hence, we will assume $c \in (\frac{3}{7}, \frac{3}{5})$ in the rest of the proof. By [29, Theorem 1.13], we know that Λ_c is nef on $\overline{P}_c^K$. Denote by U_c the Zariski open subset of $\overline{P}_c^K$ that parametrizes K-stable pairs. Denote by $S_c := \overline{P}_c^K \setminus U_c$. Then by the Nakai–Moishezon criterion, it suffices to show the following statements:

- (i) $\Lambda_c|_{S_c}$ is ample.
- (ii) For any generically finite morphism $f : V \rightarrow \overline{P}_c^K$ from a normal proper variety V , the pull-back $f^* \Lambda_c$ is big on V whenever $f(V)$ intersects U_c .

For (i), from the description of $\overline{P}_c^K$ in Sections 7 and 8, we know that S_c is either $\overline{\Sigma}_{c,6}$ (when c does not lie on a wall) or $\overline{\Sigma}_{c,6}$ union an isolated point as the exceptional locus of a wall crossing (when c lies on a wall). Recall from Section 8 that $\overline{\Sigma}_{c,6}$ is a rational curve precisely parametrizing $2D_6$ -curves $\{(\mathbb{P}^2, (z(xy - z^2)(xy - az^2) = 0))\}_{a \neq 1}$ and $\{(\mathbb{P}(1, 1, 4), (xy(z^2 - x^4 y^4) = 0))\}$. In particular, $\overline{\Sigma}_{c,6}$ is not changed under every wall crossing for $c \in (\frac{3}{7}, \frac{3}{5})$. Hence, by the nefness of Λ_c , the ampleness of $\Lambda_{3/7+\epsilon}$, and the interpolation formula (3.7), we know that $\Lambda_c|_{S_c}$ is ample for any $c \in (\frac{3}{7}, \frac{3}{5})$.

The proof of (ii) is more involved. By Theorem 3.9(4), we know that U_c also parametrizes uniformly K-stable pairs in $\overline{P}_c^K$. Let $\mathcal{U}_c$ be the preimage of U_c as a saturated open substack of $\overline{P}_c^K$. By [38], there exists a birational morphism $\tilde{\mathcal{P}}_c \rightarrow \overline{P}_c^K$ from a smooth proper Deligne–Mumford stack $\tilde{\mathcal{P}}_c$ that is an isomorphism over $\mathcal{U}_c$. By [84, Chapter 16], there exists a finite surjective morphism $\tilde{P}'_c \rightarrow \tilde{\mathcal{P}}_c$ from a normal proper variety $\tilde{P}'_c$. Let $\tilde{V}$ be a projective resolution of the main component of $V \times_{\overline{P}_c^K} \tilde{P}'_c$. Denote by $\tau : \tilde{V} \rightarrow V$ and $\sigma : \tilde{V} \rightarrow \overline{P}_c^K$. Then from the above construction, we see that τ is generically finite, and $\tau^*(f^* \Lambda_c) = \sigma^* \lambda_c$. Since the σ -pull-back of the universal family over $\tilde{V}$ has K-semistable fibers where a general fiber is uniformly K-stable, and is of maximal variation, by [107] as a generalization of [29, Theorem 1.2(c)] to log Fano pairs, we know that $\sigma^* \lambda_c$ is nef and big on $\tilde{V}$. This implies that $f^* \Lambda_c$ is big on V . The proof is finished. $\square$

We expect Λ_c to be ample in any degree that is a special case of the projectivity part of the Fano K-moduli conjecture (see, e.g., [29, Conjecture 1.1(c)] and [27, (VI) on page 611]). During the review process, this expectation was proved in [98, 125] as part of the proof of the Fano K-moduli conjecture (see Remark 1.9).

Theorem 9.2 cf. [98, 126]. *The CM $\mathbb{Q}$ -line bundle Λ_c on $\overline{P}_{d,c}^K$ is ample for any $c \in (0, \frac{3}{d})$ and any $d \geq 3$.[†]*

9.2 | Birational contractions along wall crossings

As we saw in Sections 6 and 8, the birational map $\overline{P}_{d,c'}^K \dashrightarrow \overline{P}_{d,c}^K$ is always a birational contraction for $d \in \{4, 5, 6\}$ and $0 < c < c' < \frac{3}{d}$. It is natural to ask whether this holds true for all degrees.

Question 9.3. Is $\overline{P}_{d,c'}^K \dashrightarrow \overline{P}_{d,c}^K$ a birational contraction for all $0 < c < c' < \frac{3}{d}$ and all $d \geq 4$?

As the CM line bundles are always ample by Theorem 9.2, an affirmative answer to the above question would imply that the wall crossing of K-moduli spaces have similar behavior to the Hassett–Keel program for Deligne–Mumford moduli spaces $\overline{M}_g$.

Theorem 9.4. *Assume that Question 9.3 is true for some $d \geq 4$. Then for any $c \in (0, \frac{3}{d})$ and a sufficiently divisible $l \in \mathbb{Z}_{>0}$, we have*

$$\overline{P}_{d,c}^K = \text{Proj} \bigoplus_{j=0}^{\infty} H^0 \left(\overline{P}_{d, \frac{3}{d}-\epsilon}^K, l j \left(\Lambda_{\frac{3}{d}-\epsilon, \text{Hodge}} + \frac{3-cd}{27cd} \Lambda_{\frac{3}{d}-\epsilon, 0} \right) \right).$$

Proof. By Proposition 3.35, we know that $\Lambda_{\frac{3}{d}-\epsilon, c}$ is a positive multiple of $\Lambda_{\frac{3}{d}-\epsilon, \text{Hodge}} + \frac{3-cd}{27cd} \Lambda_{\frac{3}{d}-\epsilon, 0}$. Denote by $\varphi : \overline{P}_{d, \frac{3}{d}-\epsilon}^K \dashrightarrow \overline{P}_{d,c}^K$ the birational contraction. By the functoriality of CM line bundles, we know that $\Lambda_c = \varphi_* \Lambda_{\frac{3}{d}-\epsilon, c}$ as cycles. Since Λ_c is ample on $\overline{P}_{d,c}^K$ by Theorem 9.2, it suffices to show that for a common resolution $(p, q) : \tilde{P}_d \rightarrow \overline{P}_{d, \frac{3}{d}-\epsilon}^K \times \overline{P}_{d,c}^K$, we have $p^* \Lambda_{\frac{3}{d}-\epsilon, c} - q^* \Lambda_c \geq 0$ (see [65, Definition 2.3 and Remark 2.4]).

Let $0 = c_0 < c_1 < \dots < c_k = \frac{3}{d}$ be the walls of K-moduli spaces $\overline{P}_{d, \cdot}^K$. By passing to a higher birational model, we may assume that $\tilde{P}_d$ is a resolution of $\overline{P}_{d, c_i \pm \epsilon}^K$ for any $1 \leq i \leq k-1$ with birational morphisms $\psi_i : \tilde{P}_d \rightarrow \overline{P}_{d, c_i}^K$ and $\psi_i^{\pm} : \tilde{P}_d \rightarrow \overline{P}_{d, c_i \pm \epsilon}^K$. Since $\Lambda_{c_i+\epsilon}$ is ample by assumption, we know that $(\psi_i^+)^* \Lambda_{c_i+\epsilon} - (\psi_i^-)^* \Lambda_{c_i-\epsilon, c_i+\epsilon}$ is ψ_i^- -nef and ψ_i^- -exceptional. Hence, by negativity lemma, we know that $(\psi_i^+)^* \Lambda_{c_i+\epsilon} \leq (\psi_i^-)^* \Lambda_{c_i-\epsilon, c_i+\epsilon}$ is effective. By Theorem 3.36, we know that $(\psi_i^+)^* \Lambda_{c_i+\epsilon, c_i} = (\psi_i^-)^* \Lambda_{c_i-\epsilon, c_i}$. Since a positive rescaling of $\Lambda_{c, c'}$ is linear in c' , we know that $(\psi_i^+)^* \Lambda_{c_i+\epsilon, c} \geq (\psi_i^-)^* \Lambda_{c_i-\epsilon, c}$ whenever $c_i \geq c$. Hence, by the reverse induction on i for $c_i \in [c, \frac{3}{d})$, we conclude that $p^* \Lambda_{\frac{3}{d}-\epsilon, c} - q^* \Lambda_c \geq 0$. The proof is finished. $\square$

[†] If $d = 2$ and $c \in (0, \frac{3}{4}]$, then the theorem is also true for trivial reasons as $\overline{P}_{d,c}^K$ is a point (see Example 4.5).

The goal of this section is to verify Question 9.3 when either d is divisible by 3 or d is small.

Theorem 9.5. *Question 9.3 is true when either $3 \mid d$, or $3 \nmid d$ and $d < 13$.*

First, we rephrase the question in a form that is easier to verify.

Question 9.6. Suppose that X is a Manetti surface with the following properties:

- (1) $\dim \operatorname{Aut}(X) = \dim \operatorname{Aut}(\mathbb{P}^2) + 1 = 9$, and
- (2) (X, cD) is K-stable for some $0 < c < \frac{3}{d}$ and a general $D \in |- \frac{d}{3} K_X|$ (in particular, C has local indices $\leq d$ when $3 \nmid d$ or $\leq \frac{2d}{3}$ when $3 \mid d$).

Then, is $\operatorname{lct}(X; D) \geq \frac{3}{d}$?

Lemma 9.7. *Let X be a singular Manetti surface. Then we have $h^0(X, \mathcal{O}_X(-\frac{d}{3}K_X)) = h^0(\mathbb{P}^2, \mathcal{O}(d))$ for any $d \in \mathbb{Z}_{>0}$ and $\dim \operatorname{Aut}(X) > \dim \operatorname{Aut}(\mathbb{P}^2) = 8$.*

Proof. Let $\mathcal{X} \rightarrow B$ over a smooth pointed curve $0 \in B$ be a $\mathbb{Q}$ -Gorenstein smoothing of $X \cong \mathcal{X}_0$. Let $\mathcal{L}$ be the $\mathbb{Q}$ -Cartier Weil divisor on $\mathcal{X}$ such that $3\mathcal{L} \sim_B -dK_{\mathcal{X}/B}$. Since $\mathcal{X}$ has klt singularities, we know that $\mathcal{O}_{\mathcal{X}}(\mathcal{L}) \otimes \mathcal{O}_{\mathcal{X}_0} \cong \mathcal{O}_{\mathcal{X}_0}(\mathcal{L}_0)$. By Kawamata-Viehweg vanishing, we know that $H^i(\mathcal{X}_0, \mathcal{L}_0) = H^i(X, \mathcal{O}_X(-\frac{d}{3}K_X)) = 0$ for any $i > 0$. Hence, the equation of h^0 follows from the flatness of $\mathcal{L}$ over B . For the automorphism part, let p be a sufficiently large positive integer such that $|-pK_X|$ is base point free. Then for a general curve $D \in |-pK_X|$, we know that $(X, \frac{1}{p}D)$ is a klt log Calabi–Yau pair. By [92, Theorem 5.2] and Theorem 3.20, we know that $(X, (\frac{1}{p} - \epsilon)D)$ is uniformly K-stable for $0 < \epsilon \ll 1$. Let $U \subset |-pK_X|$ be the Zariski open locus parametrizing D such that $(X, (\frac{1}{p} - \epsilon)D)$ is uniformly K-stable for some (or any) $0 < \epsilon \ll 1$. Then $[U/\operatorname{Aut}(X)]$ is a Deligne–Mumford stack whose coarse moduli space $U/\operatorname{Aut}(X)$ admits an injection into $\overline{P}_{3p, \frac{1}{p} - \epsilon}^K$. Since a general point in the K-moduli space parametrize a smooth plane curve on $\mathbb{P}^2$, we know that $\dim(U/\operatorname{Aut}(X)) < \dim P_{3p} = \dim(\mathbf{P}_{3p}^{\operatorname{sm}}/\operatorname{Aut}(\mathbb{P}^2))$ that implies $\dim \operatorname{Aut}(X) > \dim \operatorname{Aut}(\mathbb{P}^2)$. $\square$

Proposition 9.8. *Questions 9.3 and 9.6 are equivalent to each other.*

Proof. For the “ $\Rightarrow$ ” direction, let X be a Manetti surface satisfying (1) and (2) of Question 9.6. Assume to the contrary that $\operatorname{lct}(X; D) < \frac{3}{d}$ for a general $D \in |- \frac{d}{3} K_X|$. Then $(X, (\frac{3}{d} - \epsilon)D)$ is K-unstable for $0 < \epsilon \ll 1$. Let c_i be the K-semistable threshold of (X, D) . Let U be the open subset of $|- \frac{d}{3} K_X|$ parametrizing D with $(X, (c_i - \epsilon)D)$ K-stable. Then $[U/\operatorname{Aut}(X)]$ is a Deligne–Mumford stack whose coarse space $U/\operatorname{Aut}(X)$ injects into $\overline{P}_{d, c_i - \epsilon}^K$ whose image closure E is a divisor by Lemma 9.7. Thus, the wall-crossing morphism $\phi_i^- : \overline{P}_{d, c_i - \epsilon}^K \rightarrow \overline{P}_{d, c_i}^K$ contracts E to a codimension ≥ 2 locus since the c_i -K-polystable pairs replacing (X, D) have continuous automorphisms. This contradicts the assumption on birational contractions.

For the “ $\Leftarrow$ ” direction, assume to the contrary that a wall-crossing morphism $\phi_i^- : \overline{P}_{d, c_i - \epsilon}^K \rightarrow \overline{P}_{d, c_i}^K$ contracts a divisor E . Since Manetti surfaces have no continuous moduli, a general point on E parametrizes (X, D) for the same Manetti surface X . For the same U as above, we know that $U/\operatorname{Aut}(X)$ injects into $\overline{P}_{d, c_i - \epsilon}^K$ with image closure E . Hence, $\dim \operatorname{Aut}(X) = \dim \operatorname{Aut}(\mathbb{P}^2) + 1$ by

Lemma 9.7. Then $\text{lct}(X; D) \geq \frac{3}{d}$ for a general $D \in |-\frac{d}{3}K_X|$ which implies that (X, cD) is K-stable for any $c_i < c < \frac{3}{d}$ by Proposition 2.13. This contradicts to the assumption that E is contracted under ϕ_i^- . The proof is finished. $\square$

Thanks to Proposition 9.8, we only need to verify Question 9.6 for either $3 \mid d$ or $d \leq 13$. The following lemma proves the case when $3 \mid d$.

Lemma 9.9. If $3 \mid d$, then $\text{lct}(X; D) \geq \frac{3}{d}$ for X and D satisfying conditions in Question 9.6.

Proof. When $3 \mid d$, we can degenerate (X, D) to a weighted projective plane and the $(d/3)$ th multiple of the toric boundary divisor. Hence, the inequality $\text{lct}(X; D) \geq 3/d$ is obtained by lower semicontinuity of lct (see, e.g., [33]). $\square$

Next, we turn to the case $d < 13$ and $3 \nmid d$ that will be confirmed by careful study of the Manetti surfaces appearing in our K-moduli spaces. First, note that by Theorem 4.8 and Proposition 2.30, for $d < 13$ the only Manetti surfaces satisfying the conditions in Question 9.6 appearing in $\overline{P}_{d,c}^K$ for $0 < c < \frac{3}{d}$ are $\mathbb{P}(1, 1, 4)$ and X_{26} . For $X = \mathbb{P}(1, 1, 4)$, the canonical $\mathcal{O}_X(K_X) = \mathcal{O}_X(-6)$, so the linear system $|-\frac{d}{3}K_X|$ parametrizes elements of $\mathcal{O}_X(2d)$, hence we are interested in answering Question 9.6 only for curves of even degree on $\mathbb{P}(1, 1, 4)$.

As we can degenerate X_{26} to $\mathbb{P}(1, 4, 25)$, by semicontinuity of lct , we can study curves on $\mathbb{P}(1, 4, 25)$. Provided that the general curve on $\mathbb{P}(1, 4, 25)$ has the appropriate lct , we can reach the same conclusion for X_{26} . If $X = \mathbb{P}(1, 4, 25)$, then $\mathcal{O}_X(K_X) = \mathcal{O}_X(-30)$, so the linear system $|-\frac{d}{3}K_X|$ parametrizes elements of $\mathcal{O}_X(10d)$, hence we are interested in answering Question 9.6 only for curves of degree a multiple of 10 on $\mathbb{P}(1, 4, 25)$.

Lemma 9.10. For a general curve C of even degree on $\mathbb{P}(1, 1, 4)$, we have $\text{lct}(\mathbb{P}(1, 1, 4), C) = 1$.

Proof. If the degree d of the curve satisfies $d \equiv 0 \pmod{4}$, then the general curve C_d is smooth and contained in the smooth locus of $\mathbb{P}(1, 1, 4)$, hence $\text{lct}(\mathbb{P}(1, 1, 4); C_d) = 1$. Next, consider the case when $d \equiv 2 \pmod{4}$. If $d = 2$, the general curve $C_2 := (ax^2 + bxy + cy^2 = 0)$ passes through the singular point of $\mathbb{P}(1, 1, 4)$ and is nodal at that point. A computation shows that $\text{lct}(\mathbb{P}(1, 1, 4); C_2) = 1$. For any d such that $d \equiv 2 \pmod{4}$, the curve $C_{d-2} \cup C_2$ is in the linear system $|\mathcal{O}(d)|$, where C_{d-2} is a general curve of degree $d - 2$. As the general C_{d-2} is smooth, contained in the smooth locus of $\mathbb{P}(1, 1, 4)$, and intersects C_2 transversally away from the singular point of the surface, we have $\text{lct}(\mathbb{P}(1, 1, 4); C_{d-2} \cup C_2) = 1$. Therefore, by semicontinuity of lct , the general curve C_d of degree d also has $\text{lct}(\mathbb{P}(1, 1, 4); C_d) = 1$. $\square$

Remark 9.11. The previous statement is false without the assumption on even degree. If C is a general curve of degree 3 (or, more generally, degree d such that $d \equiv 3 \pmod{4}$), then $\text{lct}(\mathbb{P}(1, 1, 4); C) = \frac{2}{3}$.

Now, we mimic the previous argument for curves of degree 10 on $\mathbb{P}(1, 4, 25)$.

Lemma 9.12. For a general curve D of degree d such that $d \equiv 0, 30 \pmod{50}$ on $X = \mathbb{P}(1, 4, 25)$, $\text{lct}(\mathbb{P}(1, 4, 25), C') = 1$. If $d \equiv 10 \pmod{50}$, $\text{lct}(X; D) = \frac{1}{2}$. If $d \equiv 20 \pmod{50}$, $\text{lct}(X; D) = \frac{1}{4}$, and if $d \equiv 40 \pmod{50}$, $\text{lct}(X; D) = \frac{1}{3}$.

Proof. If the degree d of the curve is a multiple of 100, then the general curve C_d is smooth and contained in the smooth locus of $\mathbb{P}(1, 4, 25)$, hence $\text{lct}(\mathbb{P}(1, 4, 25); C_d) = 1$. If the degree is a multiple of 50, the general curve C_d has a nonzero z^2 term, so misses the $\frac{1}{25}(1, 4)$ singularity, and so the previous lemma implies the result.

For $d = 10$, consider the general curve $C = (ax^{10} + bx^6y + cx^2y^2 = 0)$. This has even degree through the $\frac{1}{4}(1, 1)$ singular point, so the lct in a neighborhood of that point is 1. In a neighborhood of the $\frac{1}{25}(1, 4)$ singular point, we compute the lct under the finite morphism $\pi : \mathbb{A}^2 \rightarrow \frac{1}{25}(1, 4)$ where π^*C is defined by the same equation. By [70, Lemma 8.12], $\text{lct}(\frac{1}{25}(1, 4); C) = \text{lct}(\mathbb{A}^2; \pi^*C)$, and we compute $\text{lct}(\mathbb{A}^2, \pi^*C) = \frac{1}{2}$ using [56, Example 4, 5]. For a general curve of degree $d \equiv 10 \pmod{50}$, there are nonzero terms of the form $z^n f_{10}(x, y)$, where $f_{10}(x, y)$ is a degree 10 polynomial, so for C_d a general curve of degree $d \equiv 10 \pmod{50}$, we have $\text{lct}(\frac{1}{25}(1, 4); C) = \frac{1}{2}$.

For $d \equiv 20 \pmod{50}$, consider a general curve C_d of degree d . Because this passes through the $\frac{1}{25}(1, 4)$ singular point with high multiplicity, we do not expect that $\text{lct}(\frac{1}{25}(1, 4); C_d) = 1$. Indeed, using the same method as above, we can compute $\text{lct}(\frac{1}{25}(1, 4), C_d) = \frac{1}{4}$.

For $d = 30$, the general curve has an xyz term, so is at worst nodal at each singular point of the surface, so actually has lct equal to 1. For $d \equiv 30 \pmod{50}$, the general curve has an xyz^{2k+1} term, so is nodal in a neighborhood of the $\frac{1}{25}(1, 4)$ singularity and has even degree in a neighborhood of the $\frac{1}{4}(1, 1)$ singularity, so still has lct equal to 1.

For $d = 40$, the general curve has a y^{10} term, and so, it misses the $\frac{1}{4}(1, 1)$ singular point. The general curve has a nonzero x^3y^3z term, so in a neighborhood of the $\frac{1}{25}(1, 4)$ singularity, a computation similar to that above shows that $\text{lct}(\frac{1}{25}(1, 4); C_d) = \frac{1}{3}$. For $d \equiv 40 \pmod{50}$, the general curve could be nodal through the $\frac{1}{4}(1, 1)$ singular point, but still has lct equal to 1 in a neighborhood of that point.

Finally, for $d = 50$, the general curve has a z^2 term, and so, it misses the $\frac{1}{25}(1, 4)$ singular point and passes through the $\frac{1}{4}(1, 1)$ singular point with even degree, so the discussion of $\mathbb{P}(1, 1, 4)$ above shows $\text{lct}(\mathbb{P}(1, 4, 25); C_{50}) = 1$. $\square$

Although the previous computation was done for $X = \mathbb{P}(1, 4, 25)$, the lct computation is local, so the same result holds for curves of the appropriate degree on $X = X_{26}$. The computation shows that, for $d = 4, 5, 8, 10$, and 11 , the answer to Question 9.6 is yes. For curves on $\mathbb{P}(1, 1, 4)$, by Lemma 9.10, we have that $\text{lct}(X; D) = 1 > \frac{3}{d}$.

For curves on X_{26} , we need a finer analysis. No curves on X_{26} appear for degree $d = 4$, but for $d = 5, 8$, and 10 , Lemma 9.12 shows $\text{lct}(X; D) = 1 > \frac{3}{d}$, and for $d = 11$, Lemma 9.10 shows $\text{lct}(X; D) = \frac{1}{2} > \frac{3}{11}$. This leaves four exceptional cases: $d = 6, 7, 9$, and 12 . For degrees $6, 9$, and 12 , we have $3 \mid d$, so Lemma 9.9 implies $\text{lct}(X; D) \geq \frac{3}{d}$. Therefore, we have contraction morphisms $\overline{P}_{d,c'}^K \dashrightarrow \overline{P}_{d,c}^K$ for all $0 < c < c' < \frac{3}{d}$.

This leaves only the case $d = 7$. The desired result (an affirmative answer to Question 9.6) will follow from the next surprising proposition.

Proposition 9.13. *For $d = 7$, curves on X_{26} or $\mathbb{P}(1, 4, 25)$ are K -unstable in $\overline{P}_{7,c}^K$ for all $c \in (0, \frac{3}{7})$. In other words, the only surfaces appearing in $\overline{P}_{7,c}^K$ for some $c \in (0, \frac{3}{7})$ are $\mathbb{P}^2$ and $\mathbb{P}(1, 1, 4)$.*

Proof. Let X be either X_{26} or $\mathbb{P}(1, 4, 25)$. Assume to the contrary that (X, cD) is K -semistable for some curve $D \in |-\frac{7}{3}K_X|$ and some $c \in (0, \frac{3}{7})$. From the above discussion, we know that

$\text{lct}(X; D) \leq \frac{1}{4}$ which implies $c < \frac{1}{4}$. Since X has a singularity of local Gorenstein index 5, by Theorem 4.8, we know that $5 \leq 3/(3 - 7c)$ hence $c \geq \frac{12}{35}$. This contradicts $c < \frac{1}{4}$. $\square$

Proof of Theorem 9.5. The result follows from Lemmas 9.9, 9.10, and 9.12 and Proposition 9.13. $\square$

Remark 9.14. For $d = 9$ or 12 , a valuative criterion computation shows that X_{26} cannot appear in the K-moduli spaces $\overline{P}_{d,c}^K$ for $0 < c < \frac{3}{d}$.

9.3 | Quartics and sextics revisited

Recall from Section 6, we interpreted the K-moduli spaces of quartic and sextic plane curves via K3 surfaces. In this section, we revisit these moduli spaces and study them via their relation to Hacking's moduli space $\overline{P}_d^H$, and give a log Calabi–Yau interpretation.

9.3.1 | Quartics

Recall that Hacking's space $\overline{P}_4^H$ generically parametrizes plane curves with at worst cuspidal singularities. There is a divisor parametrizing curves in $\mathbb{P}(1, 1, 4)$ which are at worst nodal at the singular point, and at worst cuspidal elsewhere. Finally, there is a codimension two locus parametrizing curves on the nonnormal surface $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ — the curves are snc at the double locus, and at worst cuspidal elsewhere.

Hyeon–Lee's original motivation was to complete the log MMP on $\overline{M}_3$. Let $\overline{M}_3(\alpha)$ denotes $\text{Proj } \bigoplus_{m \geq 0} \Gamma(\overline{M}_3, m(K_{\overline{M}_3} + \alpha\delta))$, where δ is the boundary divisor of $\overline{M}_3$. Hyeon–Lee produce the following diagram:

$$\begin{array}{ccccc}
 \overline{M}_3(1) \cong \overline{M}_3 & \xrightarrow{T} & \overline{M}_3\left(\frac{9}{11}\right) \cong \overline{M}_3^{\text{ps}} & \xrightarrow{\Theta} & \overline{M}_3\left(\frac{7}{10} - \epsilon\right) \cong \overline{M}_3^{\text{hs}} \\
 & & \searrow \Psi & \swarrow \Psi^+ & \searrow \Theta \\
 & & \overline{M}_3\left(\frac{7}{10}\right) \cong \overline{M}_3^{\text{cs}} \cong \overline{P}_4^* & & \overline{M}_3\left(\frac{17}{28}\right) \cong \overline{P}_4^{\text{GIT}}
 \end{array}$$

The main results of their work can be summarized in the following.

Theorem 9.15 (Birational geometry of the moduli space of genus three curves [54]).

- There is a contraction morphism $T : \overline{M}_3 \rightarrow \overline{M}_3(\frac{9}{11}) \cong \overline{M}_3^{\text{ps}}$ to Schubert's moduli space of pseudostable curves, given by contracting the locus of elliptic tails.
- There is a small contraction $\Psi : \overline{M}_3^{\text{ps}} \rightarrow \overline{M}_3^{\text{cs}}$, to the GIT quotient of the Chow variety of bicanonical curves $\text{Chow}_{3,2} // \text{SL}(6)$ given by contracting the locus of elliptic bridges.
- There is a flip $\Psi^+ : \overline{M}_3(\alpha) \rightarrow \overline{M}_3(\frac{7}{10})$ for $17/28 < \alpha < 7/10$, where $\overline{M}_3(\alpha) \cong \overline{M}_3^{\text{hs}}$, the GIT quotient of the Hilbert scheme of bicanonical curves $\text{Hilb}_{3,2} // \text{SL}(6)$.
- There is a divisorial contraction $\Theta : \overline{M}_3^{\text{hs}} \rightarrow \overline{P}_4^{\text{GIT}}$ to the GIT quotient of plane quartics given by $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4))) // \text{SL}(3)$.

Furthermore, $\overline{M}_3^{\text{ps}}$ can be identified with $\overline{P}_4^{\text{H}}$, Hacking's moduli space of plane quartics.

Remark 9.16.

- (1) The contraction at $\alpha = 9/11$ was originally discovered by Hassett–Hyeon [53].
- (2) In the case of degree $d = 4$, the space of Hacking was independently constructed by Hassett (see [52]).
- (3) As we saw in Section 6, we have an isomorphism $\overline{M}_3^{\text{hs}} \cong \overline{P}_{4, \frac{3}{4}-\epsilon}^{\text{K}}$.

The flip ϑ can be realized as flipping the codimension 2 locus in $\overline{P}_4^{\text{H}}$ parametrizing the curves on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ to the curve in $\overline{P}_{4, \frac{3}{4}-\epsilon}^{\text{K}}$ parametrizing tacnodal curves. These flipping and flipped loci of ϑ are contracted (via Ψ or Ψ^+) to a point as the unique 0-cusp in $\overline{P}_4^*$. It is thus natural to expect that $\overline{P}_4^*$ serves as the conjectural (good) moduli space $\overline{P}_4^{\text{CY}}$ that parametrizes certain log canonical log Calabi–Yau pairs that are $\mathbb{Q}$ -Gorenstein degenerations of $(\mathbb{P}^2, \frac{3}{4}C_4)$. Recall from Section 6 we showed that there was a large open set $M \subset \overline{P}_4^*$ whose codimension inside $\overline{P}_{4, \frac{3}{4}-\epsilon}^{\text{K}}$ is ≥ 2 . Using this, we proved (see Theorem 6.5) that the moduli space $\overline{P}_4^*$ was the ample model of the Hodge line bundle on $\overline{P}_{4, \frac{3}{4}-\epsilon}^{\text{K}}$. Noting that the codimension of M inside $\overline{P}_4^{\text{H}}$ is also ≥ 2 , the same proof gives the following.

Theorem 9.17. *The moduli space $\overline{P}_4^*$ is the ample model of the Hodge line bundle on $\overline{P}_4^{\text{H}}$.*

9.3.2 | Sextics

Recall from Section 6, we discussed the Kirwan desingularization of the GIT quotient of sextic curves (constructed by Shah), as well as the morphism from this moduli space to the Baily–Borel compactification of degree 2 K3 surfaces. By the work of [5], there is also a morphism $\overline{P}_6^{\text{H}} \rightarrow \overline{P}_6^*$, and so, we obtain the following diagram.

$$\begin{array}{ccc}
 \overline{P}_6^{\text{H}} & \dashrightarrow & \widehat{P}_6^{\text{GIT}} \cong \overline{P}_{6, \frac{1}{2}-\epsilon}^{\text{K}} \\
 \searrow & & \swarrow \\
 & \overline{P}_6^* & \\
 \swarrow & & \searrow \\
 & \overline{P}_6^{\text{GIT}} &
 \end{array}$$

Again, we argue that it is natural to believe that the candidate for $\overline{P}_6^{\text{CY}}$ is $\overline{P}_6^*$. The locus contracted from $\overline{P}_6^{\text{H}} \rightarrow \overline{P}_6^*$ is divisorial, so the proof of Theorem 6.6 does not imply the same result on the Hacking side as immediately as it did for degree four. In fact, [5] shows that there are actually several divisors contracted — these divisors parametrize pairs whose double covers give K3 surfaces of Type II or Type III, in the sense of Kulikov degenerations. Valery Alexeev has suggested to us that the result is still true and can be proven by looking at the Kulikov degenerations of the relevant K3 surfaces.

9.3.3 | Log Calabi–Yau wall crossing

In general, we can say the following.

Theorem 9.18. *Let $\overline{P}_d^{\text{H},\circ}$ denote the complement of the locus of nonnormal pairs in $\overline{P}_d^{\text{H}}$. Let $\overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K},\circ}$ denote the complement of the locus of pairs with $\text{lct} = \frac{3}{d}$ inside $\overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K}}$. Then $\overline{P}_d^{\text{H},\circ} \cong \overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K},\circ}$.*

Proof. We first show that $\overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K},\circ} \subseteq \overline{P}_d^{\text{H},\circ}$. If (X, D) is a pair parametrized by $\overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K},\circ}$, then $\text{lct} > \frac{3}{d}$ and so by definition (X, D) is a pair parametrized by $\overline{P}_d^{\text{H},\circ}$. Conversely, by [92, Theorem 5.2], if (X, D) is parametrized by $\overline{P}_d^{\text{H},\circ}$, then (X, D) is parametrized by $\overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K},\circ}$. $\square$

In particular, the above result says that if one looks at the K-moduli space for coefficient $\frac{3}{d} - \epsilon$, then the only difference between this space and the Hacking moduli space are the maximally lct pairs in the K-moduli space and the nonnormal pairs in the Hacking moduli space. In particular, we conjecture that there is a proper good moduli space of log Calabi–Yau pairs which relates to the K-moduli and Hacking moduli spaces via the following conjectural picture.

Conjecture 9.19 (Log Calabi–Yau wall crossings). *There exists a proper good moduli space $\overline{P}_d^{\text{CY}}$ that parametrizes S-equivalence classes of semistable log Calabi–Yau pairs $(X, \frac{3}{d}D)$ where X admits a $\mathbb{Q}$ -Gorenstein smoothing to $\mathbb{P}^2$. Moreover, we have a log Calabi–Yau wall crossing diagram*

$$\overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K}} \xrightarrow{\phi_-^{\text{CY}}} \overline{P}_d^{\text{CY}} \xleftarrow{\phi_+^{\text{CY}}} \overline{P}_d^{\text{H}},$$

where $\overline{P}_d^{\text{CY}}$ is the common ample model of the Hodge line bundles on $\overline{P}_{d,\frac{3}{d}-\epsilon}^{\text{K}}$ and $\overline{P}_d^{\text{H}}$.

9.4 | Log Calabi–Yau quintics

Recall that Hacking's moduli space $\overline{P}_5^{\text{H}}$ parametrizes $\mathbb{Q}$ -Gorenstein deformations of pairs $(\mathbb{P}^2, cC)$ where $c = \frac{3}{5} + \epsilon$ for ϵ sufficiently small and $\deg(C) = 5$. From Theorem 9.18, we know that there is a birational map

$$\overline{P}_5^{\text{H}} \dashrightarrow \overline{P}_c^{\text{K}},$$

where $c \in (\frac{54}{95}, \frac{3}{5})$, which is an isomorphism over the locus of pairs (X, D) where X is normal and $\text{lct}(X, D) > \frac{3}{5}$. Here we omit the degree 5 in the subscript of K-moduli spaces and stacks. In other words, the pairs that will become unstable when increasing the weight from $\frac{3}{5} - \epsilon$ to $\frac{3}{5} + \epsilon$ are precisely the pairs with $\text{lct} = \frac{3}{5}$ — that is, A_9 and D_6 singularities. Note that the A_9 singularity can

occur either at a smooth point of the surface X or at a $\frac{1}{4}(1, 1)$ singularity. In this setting, we have the following results:

- (1) the stable replacement in $\overline{P}_5^H$ of a curve $C \subset X$ with A_9 singularity is either a curve on $\mathbb{P}(1, 1, 5) \cup X_6$ or a curve on $\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5)$, and
- (2) the stable replacement in $\overline{P}_5^H$ of a curve $C \subset X$ with a D_6 singularity is a curve on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$.

These are precisely the nonnormal pairs that appear in $\overline{P}_5^H$. We conjecture that there is a projective variety $\overline{P}_5^{\text{CY}}$ parametrizing S-equivalence classes of slc log Calabi–Yau pairs $(X, \frac{3}{5}D)$.

Conjecture 9.20. *The ample models of the Hodge line bundles on $\overline{P}_5^H$ and $\overline{P}_{\frac{3}{5}-\epsilon}^K$ exist and coincide. We denote this common ample model by $\overline{P}_5^{\text{CY}}$. It parametrizes S-equivalence classes of slc log Calabi–Yau pairs that are $\mathbb{Q}$ -Gorenstein deformations of $(\mathbb{P}^2, \frac{3}{5}C)$.*

In particular, $\overline{P}_5^{\text{CY}}$ should serve as the base of a flip

$$\begin{array}{ccc} \overline{P}_5^H & \dashrightarrow & \overline{P}_{\frac{3}{5}-\epsilon}^K \\ & \searrow \quad \swarrow & \\ & \overline{P}_5^{\text{CY}} & \end{array}$$

which realizes the rational map $\overline{P}_5^H \dashrightarrow \overline{P}_{\frac{3}{5}-\epsilon}^K$.

In the following section, we provide some evidence for Conjecture 9.20.

9.4.1 | Evidence for the log Calabi–Yau conjecture

First, we verify that any curve with an A_9 singularity admits a common degeneration to a unique curve on $\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5)$ with an A_9 singularity in each component. Similarly, any curve with a D_6 singularity admits a common degeneration to a unique curve on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ with a D_6 singularity in each component.

Proposition 9.21. *All curves in $\overline{P}_{\frac{3}{5}-\epsilon}^K$ with an A_9 singularity and all curves in $\overline{P}_5^H$ on $X_6 \cup \mathbb{P}(1, 1, 5)$ or $\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5)$ admit a common degeneration to a unique curve on $\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5)$ with log canonical threshold exactly $\frac{3}{5}$.*

Proof. By Proposition A.1, a plane quintic curve C with an A_9 singularity has the equation

$$(x - y^2)((x - y^2)(1 + sx) - x^2(2py + rx)) + ux^5 = 0$$

in the affine coordinates $[x, y, 1]$ for some choice of $(s, r, p, u) \in \mathbb{A}^4$ satisfying $p^2 \neq u$. We will first construct a weakly special degeneration of $(\mathbb{P}^2, \frac{3}{5}C)$ to a pair $(X_6 \cup \mathbb{P}(1, 1, 5), \frac{3}{5}C_0)$.

Take our family $(\mathbb{P}^2, C) \times \mathbb{A}^1$. We perform the following birational transformations:

$$\mathbb{P}^2 \times \mathbb{A}^1 \xleftarrow{\pi} \mathcal{X} \xrightarrow{g} \mathcal{Y},$$

where in the central fiber, we have

$$\mathbb{P}^2 \xleftarrow{\pi} S \cup X \xrightarrow{g} S \cup X'$$

Here, π is the (5,1,1)-weighted blow-up of $\mathbb{P}^2 \times \mathbb{A}^1$ in the local coordinates (x', y, t) with $x' := x - y^2 - py^5$, and $S = \mathbb{P}(1, 1, 5)$ is the exceptional divisor of π . Let $Q = (x - y^2 = 0)$ be a smooth conic in $\mathbb{P}^2$. Then it is clear that $\pi^*Q = \bar{Q} + 5E$ where E is the exceptional curve of $\pi : X \rightarrow \mathbb{P}^2$ and $\bar{Q} := \pi_*^{-1}Q$. Since $(E^2) = -\frac{1}{5}$, we know that $(\bar{Q}^2) = -1$ and $\bar{Q}$ is a smooth rational curve contained in the smooth locus of X . By computation, we know that $\mathcal{N}_{\bar{Q}/X} \cong \mathcal{O}_{\bar{Q}}(-1) \oplus \mathcal{O}_{\bar{Q}}(-1)$. Hence, $g : \mathcal{X} \rightarrow \mathcal{Y}$ is the small flopping contraction of $\bar{Q}$.

Next, we show that $(X', \pi_*E) \cong (X_6, (x_1 = 0))$ where $X_6 = (x_0x_3 = x_1^3 + x_2^2)$ is a weighted hypersurface in $\mathbb{P}(1, 2, 3, 5)_{x_0, x_1, x_2, x_3}$. Let $L := \pi^*\mathcal{O}(1) - 2E$ be a divisor on X . Then L induces a morphism $g' : X \rightarrow \mathbb{P}(1, 2, 3, 5)$ defined by

$$x_0 = \pi^*x - 2E \in H^0(X, L), \quad x_1 = \pi^*(xz - y^2) - 4E \in H^0(X, 2L),$$

$$x_2 = \pi^*(y(xz - y^2)) - 6E \in H^0(X, 3L), \quad x_3 = \pi^*(z(xz - y^2)^2) - 10E \in H^0(X, 5L).$$

It is easy to show that g' is a birational map that only contracts $\bar{Q}$, and the image of g' is exactly X_6 . Thus, $g' = g$ and $\pi_*E = \pi_*(\bar{Q} + E) = (x_1 = 0)$. Let $[y_0, y_1, y_2]$ be the projective coordinates of $\mathbb{P}(1, 1, 5)$ as the projectivization of (t, y, x') . Thus, $S \cup X \cong \mathbb{P}(1, 1, 5) \cup X_6$ where the double locus π_*E is $(y_0 = 0) \subset \mathbb{P}(1, 1, 5)$ and $(x_1 = 0) \subset X_6$. The degeneration C_0 of C has equations

$$(y_2^2 = (p^2 - u)y_1^{10}) \subset \mathbb{P}(1, 1, 5) \quad \text{and} \quad (x_3 + x_0x_1^2 - 2px_0^2x_2 - rx_0^3x_1 + ux_0^5 = 0) \subset X_6.$$

Next, we show that $(\mathbb{P}(1, 1, 5) \cup X_6, \frac{3}{5}C_0)$ admits a weakly special degeneration to an slc log Calabi–Yau pair $(\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5), \frac{3}{5}C'_0)$ that is unique up to isomorphism. Consider the 1-PS $\sigma : \mathbb{G}_m \rightarrow \text{Aut}(\mathbb{P}(1, 2, 3, 5))$ defined as $\sigma(t) \cdot [x_0, x_1, x_2, x_3] = [x_0, t^{-1}x_1, x_2, x_3]$. Then we know that $\sigma(t) \cdot X_6$ has the equation $(x_0x_3 = t^3x_1^3 + x_2^2)$ in $\mathbb{P}(1, 2, 3, 5)$. Hence, $\lim_{t \rightarrow 0} \sigma(t) \cdot X_6 = (x_0x_3 = x_2^2)$. Indeed, we have an embedding from $\mathbb{P}(1, 4, 5)_{z_0, z_1, z_2}$ to $\mathbb{P}(1, 2, 3, 5)$ as $[z_0, z_1, z_2] \mapsto [z_0^2, z_1, z_0z_2, z_2^2]$ whose image has equation $x_0x_3 = x_2^2$. Hence, $\lim_{t \rightarrow 0} \sigma(t) \cdot X_6 \cong \mathbb{P}(1, 4, 5)$. By taking limit of the equation of $C_0 \cap X_6$ under the action of σ , we know that $C'_0 \cap \mathbb{P}(1, 4, 5)$ has equation $(z_2^2 - 2pz_0^5z_2 + uz_0^{10} = 0)$. Since $p^2 \neq u$, after a suitable projective coordinate change, the equation of C'_0 becomes $(y_2^2 = y_1^{10})$ and $(z_2^2 = z_0^{10})$ in $\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5)$. It is clear that C'_0 has an A_9 -singularity in the smooth locus of $\mathbb{P}(1, 1, 5)$ and an A_3 -singularity at the $\frac{1}{4}(1, 1)$ -singularity of $\mathbb{P}(1, 4, 5)$. Thus, $\text{lct}(\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5); C'_0) = \frac{3}{5}$. Since σ fixes π_*E pointwisely, we may compose the two degenerations as in [93, Proof of Lemma 3.1] to obtain a weakly special degeneration of $(\mathbb{P}^2, \frac{3}{5}C_0)$ to $(\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5), \frac{3}{5}C'_0)$.

The computation above can be extended to include the case of curves with A_9 -singularities on $\mathbb{P}(1, 1, 4)$, $\mathbb{P}(1, 4, 25)$ and X_{26} in $\bar{P}_{\frac{3}{5}-\epsilon}^K$. For the Hacking moduli space, the construction is sim-

ilar to the second step in the above degenerations with minor difference to consider 1-PS in $\text{Aut}(\mathbb{P}(1, 1, 5))$ and $\text{Aut}(\mathbb{P}(1, 4, 5))$ as well. Thus, the proof is finished. $\square$

Proposition 9.22. *All curves in $\overline{P}_{3/5-\epsilon}^K$ with a D_6 singularity and all curves in $\overline{P}_5^H$ lying on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ admit a common degeneration to a unique curve on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ with log canonical threshold exactly $\frac{3}{5}$.*

Proof. Consider the family $\pi : \mathcal{X} \rightarrow \mathbb{A}^2$ given by

$$\mathcal{X} := (x_0x_2 = rx_1^2 + sx_3) \subset \mathbb{P}(1, 1, 1, 2)_{x_0, x_1, x_2, x_3} \times \mathbb{A}_{r, s}^2.$$

The fiber of π above $r = 0, s = 0$ is isomorphic to $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ with projective coordinates $[x_0, x_1, x_3]$ and $[x_1, x_2, x_3]$, respectively. For $s = 0$ but $r \neq 0$, the fiber of π is isomorphic to $\mathbb{P}(1, 1, 4)_{y_0, y_1, y_2}$ where the isomorphism is given by $[y_0, y_1, y_2] \mapsto [ry_0^2, y_0y_1, y_1^2, y_2]$. For $s \neq 0$, the fiber of π is isomorphic to $\mathbb{P}_{x, y, z}^2$ where the isomorphism is given by $[x, y, z] \mapsto [x, y, z, s^{-1}(xz - ry^2)]$. Let $\sigma : \mathbb{G}_m \rightarrow \text{Aut}(\mathcal{X})$ be a 1-PS defined as

$$\sigma(t) \cdot ([x_0, x_1, x_2, x_3], (r, s)) := ([tx_0, x_1, x_2, x_3], (tr, ts)).$$

Then π is $\mathbb{G}_m$ -equivariant with respect to σ and the $\mathbb{G}_m$ -action $(r, s) \mapsto (tr, ts)$ on $\mathbb{A}^2$. Consider a divisor D on $\mathcal{X}$ defined by

$$D := (x_1(x_3 - x_1^2)(x_3 + x_1^2) = 0).$$

It is clear that D is $\mathbb{G}_m$ -invariant under the action of σ . We will show that suitable restrictions of the family $\pi : (\mathcal{X}, \frac{3}{5}D) \rightarrow \mathbb{A}^2$ give the desired weakly special degenerations of curves in $\overline{P}_{3/5-\epsilon}^K$ with D_6 -singularities.

From Section 8, we know that the locus of curves with D_6 -singularities in $\overline{P}_{3/5-\epsilon}^K$ is $\overline{\Sigma}_{6, \frac{3}{5}-\epsilon}$ that is isomorphic to $\mathbb{P}^1$. The pairs parametrized by $\overline{\Sigma}_{6, \frac{3}{5}-\epsilon}$ consist of the following form:

$$(\mathbb{P}^2, (y(xz - (1+a)y^2)(xz + (1-a)y^2) = 0)) \text{ where } a \in \mathbb{A}^1, \text{ and } (\mathbb{P}(1, 1, 4), (y_0y_1(y_2^2 - y_0^4y_1^4) = 0)).$$

Then one can check that the restriction of $\pi : (\mathcal{X}, \frac{3}{5}D) \rightarrow \mathbb{A}^2$ to the affine line $\{(as, s) \mid s \in \mathbb{A}^1\}$ (resp. $\{(r, 0) \mid r \in \mathbb{A}^1\}$) gives weakly special degeneration of D_6 -curves on $\mathbb{P}^2$ (resp. $\mathbb{P}(1, 1, 4)$).

A similar computation can be done for curves on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ in the Hacking moduli space. Note that the common degeneration has equation $(x_1(x_3^2 - x_1^4) = 0)$ in $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$. $\square$

These two propositions essentially show that if the log Calabi–Yau moduli space $\overline{P}_5^{\text{CY}}$ exists, then there must be two distinct points in $\overline{P}_5^{\text{CY}}$ of S-equivalence classes of slc log Calabi–Yau degenerations of pairs $(\mathbb{P}^2, \frac{3}{5}C)$ parametrizing curves with A_9 or D_6 singularities, respectively. In other words, the conjectural map $\overline{P}_{3/5-\epsilon}^K \rightarrow P$ must contract the disjoint loci $\overline{\Sigma}_{6, \frac{3}{5}-\epsilon}$ and $\overline{\Sigma}_{7, \frac{3}{5}-\epsilon}$ to two distinct points. Denote by $\overline{\Sigma}_6^H$ (resp. $\overline{\Sigma}_7^H$) the disjoint loci in $\overline{P}_5^H$ parametrizing

curves on $\mathbb{P}(1, 1, 2) \cup \mathbb{P}(1, 1, 2)$ (resp. $\mathbb{P}(1, 1, 5) \cup X_6$ or $\mathbb{P}(1, 1, 5) \cup \mathbb{P}(1, 4, 5)$). Then similarly the conjectural map $\bar{P}_5^H \rightarrow P$ must contract the disjoint loci $\bar{\Sigma}_6^H$ and $\bar{\Sigma}_7^H$ to the same set of two points.

To form the projective variety $\bar{P}_5^{CY}$, we expect that the Hodge line bundles are semiample on $\bar{P}_{\frac{3}{5}-\epsilon}^K$ and $\bar{P}_5^H$ and $\mathbb{Q}$ -trivial exactly on these contracted loci. Indeed, the Hodge line bundle is the limit of the CM line bundle by Proposition 3.35. Moreover, the CM line bundle is known to be ample on $\bar{P}_5^H$ by [111] and big and nef (and conjecturally ample) on $\bar{P}_c^K$ by [29, 93, 107]. Therefore, we expect some positivity properties of the Hodge line bundle.

As further evidence, we verify that the Hodge line bundle is trivial on the locus $\bar{\Sigma}_6^H \sqcup \bar{\Sigma}_7^H$ parametrizing curves on nonnormal surfaces in $\bar{P}_5^H$. Denote by L_+ and L_- the Hodge $\mathbb{Q}$ -line bundles over $\bar{P}_5^H$ and $\bar{P}_{\frac{3}{5}-\epsilon}^K$, respectively.

Proposition 9.23. *The restriction $L_+|_{\bar{\Sigma}_i^H}$ is $\mathbb{Q}$ -linearly trivial for $i = 6, 7$.*

Proof. We first look at the case of $\bar{\Sigma}_7^H$. Each Hacking stable pair in $\bar{\Sigma}_7^H$ is uniquely determined by gluing two plt pairs $(\mathbb{P}(1, 1, 5), (y = 0) + \frac{3}{5}C_1)$ and (X, D) where

- C_1 has the equation $z^2 = x^{10} + a_2x^8y^2 + a_3x^7y^3 + \dots + a_{10}y^{10}$ where $(a_2, a_3, \dots, a_{10}) \in \mathbb{A}^9 \setminus \{0\}$;
- X is a weighted hypersurface in $\mathbb{P}(1, 2, 3, 5)$ defined by the equation $(xw = ty^3 + z^2)$;
- $D = (y = 0) + \frac{3}{5}C_2$ where C_2 has the equation $w = x^5 + b_1x^3y + b_2xy^2$ such that $(b_1, b_2, t) \in \mathbb{A}^3 \setminus \{0\}$.

The double locus on both components is a $\mathbb{P}^1$ with three marked points: one of them is the index 5 singularity and the other two are intersections with curve C_i . Hence, the gluing is unique up to a μ_2 -action, and so, we have $\bar{\Sigma}_7^H \cong (\mathbb{P}(2, 3, \dots, 10) \times \mathbb{P}(1, 2, 3))/G$ where G is a finite group acting on the weighted biprojective space identifying isomorphic fibers. Therefore, to show that $L_+|_{\bar{\Sigma}_7^H}$ is $\mathbb{Q}$ -linearly trivial, it suffices to show that the $\mathbb{Q}$ -line bundle L on each weighted projective space is $\mathbb{Q}$ -linearly trivial.

Let us start with the weighted projective space $\mathbb{P} := \mathbb{P}(2, 3, \dots, 10)$. Denote $T := \mathbb{A}^9 \setminus \{0\}$. Consider the family $\pi_T : (\mathcal{X}_T, \mathcal{D}_T) \rightarrow T$ where $\mathcal{X}_T = \mathbb{P}(1, 1, 5) \times T$ and $\mathcal{D}_T = (y = 0) + \frac{3}{5}C_1$ with $C_1 = (z^2 = x^{10} + a_2x^8y^2 + a_3x^7y^3 + \dots + a_{10}y^{10})$. We have a $\mathbb{G}_m$ action on T given by $(a_i) \mapsto (\lambda^i a_i)$. This action lifts to $\mathcal{X}_T$ as $([x, y, z], (a_i)) \mapsto ([x, \lambda^{-1}y, z], (\lambda^i a_i))$ so that π_T is $\mathbb{G}_m$ -equivariant. It is clear that π_T descends to a universal family of plt log CY pairs over $[T/\mathbb{G}_m]$ whose coarse space is $\mathbb{P}$. Hence, it suffices to show that the $\mathbb{G}_m$ -linearized line bundle $\mathcal{L}_T^{\otimes 5} := (\pi_T)_* \mathcal{O}_{\mathcal{X}_T}(5(K_{\mathcal{X}_T/T} + \mathcal{D}_T))$ on T descends to a trivial line bundle on $\mathbb{P}$. We pick a nowhere zero section τ of $\mathcal{L}_T^{\otimes 5}$ that has the expression $\tau(a_2, \dots, a_{10}) = y^{-5}(1 - \sum_{i=2}^{10} a_i x^{10-i} y^i)^{-3} (dx \wedge dy)^{\otimes 5}$ in the affine chart $z = 1$. Hence, for any $\lambda \in \mathbb{G}_m$, we have

$$(\lambda_* \tau)(\lambda^i a_i) = (\lambda y)^{-5} \left(1 - \sum_{i=2}^{10} a_i x^{10-i} (\lambda y)^i \right)^{-3} (dx \wedge \lambda dy)^{\otimes 5} = \tau(\lambda^i a_i).$$

Thus, the $\mathbb{G}_m$ -linearization on $\mathcal{L}_T^{\otimes 5}$ is trivial.

Next, we analyze the weighted projective space $\mathbb{P}(1, 2, 3)$. Since it has Picard number 1, it suffices to show that L_+ restricting to the curve $\mathbb{P}(1, 2)$ is $\mathbb{Q}$ -linearly trivial where $\mathbb{P}(1, 2)$ corresponds to $t = 0$, that is, the surface component being $\mathbb{P}(1, 4, 5)$. Using the projective coordinates $[x, y, z]$ of $\mathbb{P}(1, 4, 5)$, the divisor $D = (y = 0) + \frac{3}{5}C_2$ where $C_2 = (z^2 = x^{10} + b_1x^6y + b_2x^2y^2)$. Then similar computations as in the case of $\mathbb{P}(1, 1, 5)$ imply that $L_+|_{\mathbb{P}(1,2)}$ is $\mathbb{Q}$ -linearly trivial. This finishes the proof of $\mathbb{Q}$ -linear triviality of $L_+|_{\bar{\Sigma}_7^H}$.

The case of $\bar{\Sigma}_6^H$ is similar to $\bar{\Sigma}_7^H$. Since each surface component is $\mathbb{P}(1, 1, 2)$, we may just consider one component given by the plt pair $(\mathbb{P}(1, 1, 2), (y = 0) + \frac{3}{5}C)$ where C has the equation $(xz^2 - x^5 = ay^3z + b_1x^4y + b_2x^3y^2 + \dots + b_5y^5)$ for $(a, b_1, \dots, b_5) \in \mathbb{A}^6 \setminus \{0\} =: T$. Then the $\mathbb{G}_m$ -action on $\mathcal{X}_T = \mathbb{P}(1, 1, 2) \times T$ is given by $([x, y, z], a, b_i) \mapsto ([x, \lambda^{-1}y, z], \lambda^3a, \lambda^ib_i)$. Hence, the moduli space parametrizing $(\mathbb{P}(1, 1, 2), (y = 0) + \frac{3}{5}C)$ is given by $\mathbb{P} := \mathbb{P}(1, 2, 3, 3, 4, 5)$ and $\bar{\Sigma}_6^H \cong (\mathbb{P} \times \mathbb{P})/G$ for some finite group G . Then similar computations show that the $\mathbb{G}_m$ -linearization on $\mathcal{L}_T^{\otimes 5}$ is trivial, hence $L_+|_{\bar{\Sigma}_6^H}$ is $\mathbb{Q}$ -linearly trivial. $\square$

To verify Conjecture 9.20, one must first show that the analogous statement of Proposition 9.23 holds for L_- and that $L_{\pm}$ is, in fact, ample away from the loci $\bar{\Sigma}_7^H$ and $\bar{\Sigma}_6^H$. If this holds, the ample models of the Hodge line bundles on $\bar{P}_5^H$ and $\bar{P}_{\frac{3}{5}-\epsilon}^K$ would coincide set-theoretically with our notion of S-equivalence classes. However, putting a natural modular structure from the K-stability viewpoint on the log Calabi–Yau space and determining the right class of objects to parameterize prevent us from defining a moduli stack of these pairs in this paper. We will pursue that this is forthcoming work.

9.5 | Higher dimensional applications

In this section, we give some applications of our machinery developed in Sections 2 and 3. The following result improves [47, Theorem 1.2] by removing their assumption on the Gap Conjecture and allowing small degree.

Theorem 9.24. *Let n and $d \geq 2$ be positive integers. Then there exists a positive rational number $c_1 = c_1(n, d)$ such that for any fixed $0 < c < c_1$, a hypersurface $S \subset \mathbb{P}^n$ of degree d is GIT (poly/semi)stable if and only if the log Fano pair $(\mathbb{P}^n, cS)$ is K-(poly/semi)stable.*

Proof. Let χ_0 be the Hilbert polynomial of $(\mathbb{P}^n, \mathcal{O}(n+1))$. Let $r := \frac{d}{n+1}$. We consider the K-moduli stack $\mathcal{KM}_{\chi_0, r, c}$ as in Definition 3.8. By Theorem 3.2, there exists a positive rational number $c_1 = c_1(n, d)$ such that $\mathcal{KM}_{\chi_0, r, c}$ remains constant for any $c \in (0, c_1)$. Thus, for any K-semistable pair $([X, cD]) \in \mathcal{KM}_{\chi_0, r, c}$, we have that X is K-semistable and $(-K_X)^n = (-K_{\mathbb{P}^n})^n = (n+1)^n$. Hence, by [81, Theorem 36], we know that $X \cong \mathbb{P}^n$ and $D \subset X$ is a hypersurface of degree d . By the Paul–Tian criterion Theorem 2.22 and computations on CM line bundles similar to the proof of Proposition 4.3, we know that K-(poly/semi)stability of $(\mathbb{P}^n, cS)$ implies GIT (poly/semi)stability of S .

We first show that there is a morphism of Artin stacks $\varphi : \mathcal{KM}_{\chi_0, r, c} \rightarrow \mathcal{H}_d^n$ where $\mathcal{H}_d^n$ is the GIT quotient stack $[\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))^{\text{ss}}/\text{PGL}(n+1)]$ of degree d hypersurfaces in $\mathbb{P}^n$. Indeed, the K-moduli stack $\mathcal{KM}_{\chi_0, r, c}$ is defined to be the quotient $[Z_{c, m}^{\text{red}}/\text{PGL}(N_m+1)]$ for every m sufficiently

divisible. Let

$$Z_m^{\text{GIT}} := \{(X, D) \in Z_m^{\text{klt}} \mid (X, D) \cong (\mathbb{P}^n, D') \text{ where } [D'] \in \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))^{\text{ss}}\}.$$

Since fiber being $\mathbb{P}^n$ and GIT semistability are both Zariski open conditions, we know that Z_m^{GIT} is a Zariski open subset of Z_m^{klt} . We equip Z_m^{GIT} with the reduced scheme structure. Thus, there is a morphism φ_m from the K-moduli stack to the GIT stack $\mathcal{H}_{d,m}^n := [Z_m^{\text{GIT}}/\text{PGL}(N_m + 1)]$. By Theorem 3.24, the K-moduli stacks stabilize. By a similar argument to the last paragraph of the proof of Theorem 5.2, the morphisms φ_m stabilize to $\varphi : \mathcal{KM}_{\chi_0, r, c} \rightarrow \mathcal{H}_d^n$ for m sufficiently divisible.

To show that the morphism φ is representable, we just need to show that the fiber product of a scheme and the K-moduli stack over the GIT stack is a scheme. For simplicity, denote by $G := \text{PGL}(N_m + 1)$. A map $T \rightarrow \mathcal{H}_{d,m}^n$ from a scheme T to the GIT stack is equivalent to the data of a G -torsor $\psi : P_T \rightarrow T$ together with a G -equivariant morphism $P_T \rightarrow Z_m^{\text{GIT}}$. Since $Z_{c,m}^{\text{red}} \hookrightarrow Z_m^{\text{GIT}}$ is a G -equivariant open immersion, we know that $P' := P_T \times_{Z_m^{\text{GIT}}} Z_{c,m}^{\text{red}}$ is a G -invariant open subscheme of P_T . Since $\psi : P_T \rightarrow T$ is flat, we know that $T' := \psi(P')$ is an open subscheme of T . Thus, $P' \rightarrow T'$ is a G -torsor as P' admits a G -action and this map is surjective. As a result, we have $T' \cong T \times_{\mathcal{H}_{d,m}^n} \mathcal{KM}_{\chi_0, r, c}$, which implies that φ is representable and an open immersion of Artin stacks. So, now it suffices to show that φ is an isomorphism.

Next, we verify that $\mathcal{KM}_{\chi_0, r, c}$ is nonempty. If $d \geq n + 1$, then any smooth hypersurface S satisfies $(\mathbb{P}^n, \frac{n+1}{d}S)$ is a log canonical log Calabi–Yau pair that implies $[(\mathbb{P}^n, cS)] \in \mathcal{KM}_{\chi_0, r, c}$ by Proposition 2.13. If $d \leq n$, then we know that there exists a smooth hypersurface S that admits Kähler–Einstein metrics (see, e.g., [119, Page 85–87] or [7]). By degeneration to normal cone of S , we know that $(\mathbb{P}^n, cS)$ special degenerates to (X_0, cS_∞) where $X_0 = C_p(S, \mathcal{O}_S(d))$ is the projective cone (see [71, Section 3.1] for a definition) and S_∞ is the section at infinity. By [82, Proposition 3.3], we know that $(X_0, (1 - \frac{r^{-1}-1}{n})S_\infty)$ admits a conical Kähler–Einstein metric. Hence, $(\mathbb{P}^n, (1 - \frac{r^{-1}-1}{n})S)$ is K-semistable by Theorem 2.16. Since $1 - \frac{r^{-1}-1}{n} = \frac{(d-1)(n+1)}{dn} > 0$, we know that $(\mathbb{P}^n, \epsilon S)$ is K-polystable for $0 < \epsilon \ll 1$ by Proposition 2.13. Hence, $(\mathbb{P}^n, cS)$ is K-polystable for any $0 < c < c_1$.

Finally, we show that φ is an isomorphism. By taking good moduli spaces, let $\varphi' : \mathcal{KM}_{\chi_0, r, c} \rightarrow H_d^n$ be the descent of φ where $H_d^n := \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}(d)))^{\text{ss}} // \text{PGL}(n + 1)$. It follows that φ' is an injective proper morphism that implies that φ' is finite. Hence, φ is finite by [10, Proposition 6.4]. This together with φ being an open immersion implies that φ is an isomorphism by Zariski's main theorem. $\square$

APPENDIX A: CALCULATIONS OF K-SEMISTABLE THRESHOLDS AND K-POLYSTABLE REPLACEMENTS

In this appendix, we calculate the K-semistable thresholds and K-polystable replacements of K-semistable pairs to determine the location of the walls for $d = 5$. By Proposition 4.7, to understand the wall crossings occurring after the first wall, we must understand the curves parametrized by $\overline{P}_5^{\text{GIT}} \setminus P^{\text{klt}}$. In Section 8.1, we showed that the curves parametrized by this space (aside from the nonreduced conic that was discussed at length in Section 5) have A_{12} , A_{11} , A_{10} , A_9 , and D_6 singularities. Therefore, this section contains the relevant calculations used in Section 8.2, and the subsections are organized by the singularity type.

Before proceeding, we state a result that will be used throughout. A standard jet computation shows that if a quintic curve has a double point of type A_k with $k \geq 9$ (not including ∞), then after a suitable projective coordinate change, we obtain the following equation in the affine coordinate:

$$(x - y^2)((x - y^2)(1 + sx) - x^2(2py + rx)) + ux^5 = 0. \quad (\text{A.1})$$

Here the double point is $(0,0)$ for any parameters $(s, r, p, u) \in \mathbb{A}^1 \times (\mathbb{A}^3 \setminus \{0\})$. Indeed, in the analytic coordinates (x', y) where $x' := x - y^2 - py^5$, the above equation (A.1) becomes

$$x'^2 = (p^2 - u)y^{10} + \text{higher order terms},$$

where (x', y) has weight $(5, 1)$. When r, p, u all vanish, we recover the curve Q_5 . If we rescale the coordinates as $(x, y) \mapsto (\lambda^{-2}x, \lambda^{-1}y)$, then the coordinates change as $(s, r, p, u) \mapsto (\lambda^2s, \lambda^4r, \lambda^3p, \lambda^6u)$. Note that the two curves defined by $(s, r, \pm p, u)$ are projectively equivalent by $y \mapsto -y$. Therefore, we may take the parameter space $\mathbb{A}^4 \setminus \{0\}$ and quotient out by projective equivalence to describe the locus of curves with an A_k singularity ($k \geq 9$) including Q_5 .

Similarly, a standard computation shows that GIT polystable quintics with a D_6 singularity, up to a projective coordinate change, can be written as

$$y(x - t_1y^2)(x - t_2y^2) = 0,$$

where $t_1 \neq t_2$. Since t_1 and t_2 are symmetric, we may take the coordinate change given by $(s_1, s_2) := (t_1 + t_2, (t_1 - t_2)^2)$. Then Σ_6 corresponds to $s_2 \neq 0$. The $\mathbb{G}_m$ -action scales the coordinates as $(t_1, t_2) \mapsto (\lambda t_1, \lambda t_2)$ and $(s_1, s_2) \mapsto (\lambda s_1, \lambda^2 s_2)$. It is clear that two curves with D_6 -singularities are projectively equivalent if and only if their (s_1, s_2) belong to the same $\mathbb{G}_m$ -orbit. Thus, we may take the parameter space $\mathbb{A}_{s_1, s_2}^2 \setminus \{0\}$ and quotient by this projective equivalence.

Combining the previous statements, following the notation of Lemma 8.1, we obtain the following description of the loci Σ_i in $\overline{P}_5^{\text{GIT}}$.

Proposition A.1. *The Zariski closure $\overline{\Sigma}_7$ of the A_9 locus in $\overline{P}_5^{\text{GIT}}$ is isomorphic to $\mathbb{P}(1, 2, 3, 3)$ with projective coordinates $[s, r, h, u]$ where $h := p^2 - u$. Moreover,*

- Σ_1 corresponds to the point $[1, 0, 0, 0]$;
- Σ_2 corresponds to the point $[0, 0, 0, 1]$;
- $\overline{\Sigma}_3$ corresponds to $h = u = 0$;
- $\overline{\Sigma}_4$ corresponds to $r = 0$ and $h = 0$;
- $\overline{\Sigma}_5$ corresponds to $h = 0$.

The Zariski closure $\overline{\Sigma}_6$ of the D_6 locus in $\overline{P}_5^{\text{GIT}}$ is isomorphic to $\mathbb{P}(1, 2) \cong \mathbb{P}^1$ with projective coordinates $[s_1, s_2]$. Moreover,

- Σ_1 corresponds to the point $[1, 0]$ and
- $\overline{\Sigma}_6$ and $\overline{\Sigma}_7$ intersect only at the point Σ_1 .

A.1 | A_{12}

Recall that X_{26} is given by $(xw - y^{13} - z^2 = 0) \subset \mathbb{P}(1, 2, 13, 25)$. In this section, we verify the K-polystability of $(X_{26}, \frac{8}{15}C'_0)$, where $C'_0 = (w = 0)$, using techniques of Ilten and Süß [59].

Proposition A.2. *The log Fano pair $(X_{26}, \frac{8}{15}C'_0)$ is K-polystable where $C'_0 = (w = 0)$.*

Proof. Consider the projective coordinate ring A of X_{26} , where

$$A = \mathbb{C}[x, y, z, w]/(xw - y^{13} - z^2).$$

Consider the action of $\mathbb{G}_m^2$ on $Y := \text{Spec } A$ given by $(x, y, z, w) \mapsto (\lambda^{26}\mu x, \lambda^2\mu^2y, \lambda^{13}\mu^{13}z, \mu^{25}w)$. It is clear that this action descends to a $\mathbb{G}_m$ -action on $(X_{26}, \frac{8}{15}C'_0)$. Thus, by [93, Theorem 1.4], it suffices to show $\mathbb{G}_m$ -equivariant K-polystability of $(X_{26}, \frac{8}{15}C'_0)$.

Denote by $R := \mathbb{C}[x, y, z, w]$, and $F = xw - y^{13} - z^2$, so $A = R/(F)$. The character lattice $M = \mathbb{Z}\langle(26, 1), (1, 1)\rangle \subset \mathbb{Z}^2$, and for every $(\alpha, \beta) \in M$, we know

$$R_{(\alpha, \beta)} = \langle x^a y^b z^c w^d \mid a(26, 1) + b(2, 2) + c(13, 13) + d(0, 25) = (\alpha, \beta) \rangle.$$

In the ring A , if we want to determine $A_{(\alpha, \beta)}$, it suffices to assume $c \in \{0, 1\}$. We also notice that the parities of c and α are the same, so if we assume α even, then we only need to take $c = 0$. Then the equation becomes

$$A_{(\alpha, \beta)} = \langle x^a y^b w^d \mid a(26, 1) + b(2, 2) + d(0, 25) = (\alpha, \beta) \rangle.$$

By passing to an even larger multiple, we may assume that $(\alpha, \beta) = e(26, 1) + f(0, 25)$. Then $a(26, 1) + b(2, 2) + d(0, 25) = e(26, 1) + f(0, 25)$ implies

$$a = e - g, \quad d = f - g, \quad b = 13g \text{ where } 0 \leq g \leq \min\{e, f\}.$$

The weight cone $\omega \subset M_{\mathbb{Q}}$ is generated by $(26, 1)$ and $(0, 25)$.

Next, we follow the setup by Altmann and Hausen on polyhedral divisors [8]. Recall that $M \subset \mathbb{Z}^2$ is the sublattice generated by $(25, 0)$ and $(1, 1)$. Let $N \supset \mathbb{Z}^2$ be the dual lattice of M generated by $\frac{1}{25}(1, -1)$ and $(0, 1)$. For each $u = (\alpha, \beta) \in \omega \cap M$, we decompose it as $u = e(26, 1) + f(0, 25)$. Here $e, f \in \frac{1}{26}\mathbb{Z}_{\geq 0}$ and $e - f \in \mathbb{Z}$. Then the polyhedral divisor $\mathfrak{D}$ on $\mathbb{P}^1$ is given by

$$\mathfrak{D}(u) = \begin{cases} 13f[1] - 12f[0] & \text{if } e \geq f \geq 0 \\ 13f[1] - 12f[0] + (e - f)[\infty] & \text{if } f \geq e \geq 0. \end{cases}$$

The polyhedrals are given by

$$\mathfrak{D}_{[0]} = \frac{6}{325}(1, -26) + \sigma, \quad \mathfrak{D}_{[1]} = \frac{1}{50}(-1, 26) + \sigma, \quad \mathfrak{D}_{[\infty]} = \text{conv}\left((0, 0), \frac{1}{25}(1, -1)\right) + \sigma.$$

Here, $\sigma = \omega^\vee \subset N_{\mathbb{Q}}$ is spanned by $(1, 0)$ and $(-1, 26)$. Denote the four vertices by $x_0 = \frac{6}{325}(1, -26)$, $x_1 = \frac{1}{50}(-1, 26)$, $x_2 = (0, 0)$, and $x_3 = (\frac{1}{25}(1, -1))$. Then these vertical divisors correspond to

$$D_{[0], x_0} = (y = 0), \quad D_{[1], x_1} = (z = 0), \quad D_{[\infty], x_2} = (w = 0), \quad D_{[\infty], x_3} = (x = 0).$$

We also denote the extremal ray by $\rho_1 = \langle(1, 0)\rangle$ and $\rho_2 = \langle(-1, 26)\rangle$. Notice that

$$\mu(x_0) = 13, \quad \mu(x_1) = 2, \quad \mu(x_2) = 1, \quad \mu(x_3) = 1.$$

Then, for any presentation $K_{\mathbb{P}^1} = a_0[0] + a_1[1] + a_2[\infty]$, we have

$$K_Y = (13a_0 + 12)D_{[0], x_0} + (2a_1 + 1)D_{[1], x_1} + a_2(D_{[\infty], x_2} + D_{[\infty], x_3}).$$

For simplicity, let us choose $a_0 = a_1 = -1$, $a_2 = 0$. Then $K_Y = -D_{[0],x_0} - D_{[1],x_1}$. Hence,

$$-(K_Y + cD_{[\infty],x_3}) = D_{[0],x_0} + D_{[1],x_1} - cD_{[\infty],x_2}.$$

Next, we will try to use T -varieties to study test configurations. We follow the notation of [59]. Let us choose a point $Q \in \mathbb{P}^1$, a natural number $m \in \mathbb{Z}_{>0}$, and a vector $v \in \frac{1}{m}N$. Consider the lattices $\tilde{M} := M \times \mathbb{Z}$ and $\tilde{N} := N \times \mathbb{Z}$. Define $\tilde{\sigma} \in \tilde{N}_{\mathbb{Q}}$ as

$$\tilde{\sigma} := \left\langle \left(v + \sum_{P \in \mathbb{P}^1 \setminus \{Q\}} \mathfrak{D}_P, \frac{1}{m} \right), \left(\sum_{P \in \mathbb{P}^1} \mathfrak{D}_P, 0 \right) \right\rangle.$$

Define the polyhedral divisor $\tilde{\mathfrak{D}}$ by

$$\tilde{\mathfrak{D}} := \left(\text{conv} \left(\left(v, \frac{1}{m} \right), (\mathfrak{D}_Q, 0) \right) + \tilde{\sigma} \right) \otimes Q + \sum_{P \in \mathbb{P}^1 \setminus \{Q\}} ((\mathfrak{D}_P, 0) + \tilde{\sigma}) \otimes P.$$

Then, we have a $\tilde{M}$ -graded algebra

$$\mathcal{A} := \bigoplus_{\tilde{u} \in \tilde{\sigma}^\vee \cap \tilde{M}} H^0(\mathbb{P}^1, \mathcal{O}([\tilde{\mathfrak{D}}(\tilde{u})])).$$

Let $\mathcal{Y}(Q, v, m) := \text{Spec } \mathcal{A}$. Then we see $\tilde{\mathfrak{D}}(0, k) = 0$, so we have a subring $\mathcal{A}_0$ of $\mathcal{A}$ that consists of $(0, k)$ -graded pieces. Moreover, $\mathcal{A}_0 = \mathbb{C}[t]$ where t is the canonical section of $\mathcal{O}([\tilde{\mathfrak{D}}(0, 1)]) = \mathcal{O}$. Thus, we get a $T \times \mathbb{G}_m$ -equivariant morphism $\mathcal{Y} \rightarrow \mathbb{A}^1$. It is clear that

$$\tilde{\mathfrak{D}}(u, k) = \min \left\{ (v, u) + \frac{k}{m}, \mathfrak{D}_Q(u) \right\} \cdot Q + \sum_{P \in \mathbb{P}^1 \setminus \{Q\}} \mathfrak{D}_P(u) \cdot P.$$

Hence, when $k \gg 0$, we see $\tilde{\mathfrak{D}}(u, k) = \mathfrak{D}(u)$. Thus, the localization $\mathcal{A}_t = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}]$.

Next, we analyze the central fiber $\mathcal{Y}_0(Q, v, m) = \text{Spec } \mathcal{A}/(t)$. It is clear that

$$\mathcal{Y}_0(Q, v, m) = \text{Spec } \bigoplus_{(u, k) \in \tilde{\sigma}^\vee \cap \tilde{M}} H^0(\mathbb{P}^1, \mathcal{O}([\tilde{\mathfrak{D}}(u, k)])) / H^0(\mathbb{P}^1, \mathcal{O}([\tilde{\mathfrak{D}}(u, k-1)])).$$

For computational purposes, consider the lattice automorphism $\phi : \tilde{N} \rightarrow \tilde{N}$ given by

$$\phi(v', m') := (v' - m'mv, m').$$

The dual automorphism $\phi^\vee : \tilde{M} \rightarrow \tilde{M}$ is given by $\phi^\vee(u, k) = (u, k - m(v, u))$. Hence,

$$\phi_* \tilde{\mathfrak{D}}(u, k) = \tilde{\mathfrak{D}}(\phi^\vee(u, k)) = \min \left\{ \frac{k}{m}, \mathfrak{D}_Q(u) \right\} \cdot Q + \sum_{P \in \mathbb{P}^1 \setminus \{Q\}} \mathfrak{D}_P(u) \cdot P.$$

In order for the $\phi^\vee(u, k)$ -graded piece of $\mathcal{Y}_0$ to be nonzero, we require two conditions:

- (1) $\lfloor \min\{\frac{k}{m}, \mathfrak{D}_Q(u)\} \rfloor > \lfloor \min\{\frac{k-1}{m}, \mathfrak{D}_Q(u)\} \rfloor$;
- (2) $\deg[\phi_* \tilde{\mathfrak{D}}(u, k)] \geq 0$.

Conditions (1) and (2) together are equivalent to $k \in m\mathbb{Z}$ and $\mathfrak{D}_Q(u) \geq \frac{k}{m}$, and

$$\sum_{P \in \mathbb{P}^1 \setminus \{Q\}} [\mathfrak{D}_P(u)] + \frac{k}{m} \geq 0.$$

Denote by $\tau \subset \tilde{N}_{\mathbb{Q}}$ the cone generated by $(\mathfrak{D}_Q, -\frac{1}{m})$ and $(\sum_{P \in \mathbb{P}^1 \setminus \{Q\}} \mathfrak{D}_P, \frac{1}{m})$. Consider the sublattice of $\tilde{M}$ of index m , namely, $\tilde{M}_m := M \times m\mathbb{Z} \subset \tilde{M}$. Consider the semigroup

$$S := \left\{ (u, k) \in \tau^\vee \cap \tilde{M}_m \mid \sum_{P \in \mathbb{P}^1 \setminus \{Q\}} [\mathfrak{D}_P(u)] + \frac{k}{m} \geq 0 \right\}.$$

Then, we have

- (1) The central fiber $\mathcal{Y}_0$ is isomorphic to the affine toric variety $\text{Spec } \mathbb{C}[S]$;
- (2) The normalization of $\mathcal{Y}_0$ is isomorphic to the affine toric variety $\text{Spec } \mathbb{C}[\tau^\vee \cap \tilde{M}_m]$.

Hence, $\mathcal{Y}_0$ is normal if and only if $S = \tau^\vee \cap \tilde{M}_m$, which is equivalent of saying that the collection $\{\mathfrak{D}_P\}_{P \neq Q}$ is admissible.

Next, we want to study the limit of boundary divisor ($w = 0$) in the central fiber $\mathcal{Y}_0$. We first realize ($w = 0$) in Y as a Cartier divisor. Let f be a rational function on $\mathbb{P}^1$ such that $\text{div}(f) = 12[0] - 13[1] + [\infty]$. Then we can check by [108, Proposition 3.14] that

$$\text{div}(f \cdot \chi^{(0,25)}) = D_{[\infty], x_2} = (w = 0).$$

By the admissibility condition, the only choice of Q will be $Q = [0]$ or $[1]$.

Case 1: $Q = [0]$. For simplicity, we also set $v = 0$ at the moment. Then, in our test configuration $\mathcal{Y}([0], 0, m)$, we can compute similarly that

$$\text{div}(f \cdot \chi^{(0,25,k)}) = (12m + k)D_{[0], (0, \frac{1}{m})} + D_{[\infty], (x_2, 0)}.$$

Also, $\text{div}(\chi^{(0,0,1)}) = D_{[0], (0, \frac{1}{m})} = \mathcal{Y}_0$. Hence, we know that $D_{[\infty], (x_2, 0)} = \text{div}(f \cdot \chi^{(0,25,-12m)})$. By carefully checking the quotient map $\mathcal{A} \rightarrow \mathcal{A}/(t)$, we find out that the restriction of $f \cdot \chi^{(0,25,-12m)}$ is exactly the function $\chi^{(0,25,-12m)}$ on $\mathcal{Y}_0$. Hence, we have $(w = 0)|_{\mathcal{Y}_0} = \text{div}(\chi^{(0,25,-12m)})$. So, the computations are about the toric variety $\mathcal{Y}_0$ and its boundary divisor $\Delta_0 := c \cdot (w = 0)|_{\mathcal{Y}_0}$. For simplicity, we may assume $m = 1$. Then $\mathcal{Y}_0 = \text{Spec } \mathbb{C}[\tau^\vee \cap \tilde{M}]$. The primitive vectors of τ in $\tilde{N}$ are given by $n_1 = (\frac{6}{25}(1, -26), -13)$, $n_2 = (\frac{1}{25}(-1, 26), 2)$, and $n_3 = (\frac{1}{25}(1, 24), 2)$. Let $\tilde{u}_0$ be the vector in $\tilde{M}_{\mathbb{Q}}$ representing the anticanonical divisor $-K_{\mathcal{Y}_0}$, then $(\tilde{u}_0, n_i) = 1$. Let $\tilde{u}_1$ be the vector in $\tilde{M}_{\mathbb{Q}}$ representing the divisor $(w = 0)|_{\mathcal{Y}_0}$. Then computation shows

$$\tilde{u}_0 = (15, 15, -7), \quad \tilde{u}_1 = (0, 25, -12).$$

Thus, for any toric valuation v_ξ of $\mathbb{C}(\mathcal{Y}_0)$ with $\xi \in \tau$, we have

$$A_{(\mathcal{Y}_0, \Delta_0)}(v_\xi) = (\tilde{u}_0 - c\tilde{u}_1, \xi), \quad \text{vol}_{\mathcal{Y}_0, o}(v_\xi) = 6\text{vol}(\tau^\vee \cap (\xi \leq 1)).$$

From [95], we know that

$$\text{Fut}(\mathcal{Y}, \Delta, \xi_0; \eta) = \frac{d}{dt} \Big|_{t=0} \widehat{\text{vol}}_{(\mathcal{Y}_0, \Delta_0), o} (v_{\xi_0 - t\eta}).$$

In our setting, $\xi_0 = (0, 1, 0)$ is induced by the quotient map $Y \setminus \{o\} \rightarrow X_{26}$, and $\eta = (-mv, m)$. We know that all such η satisfy $(\eta, \eta_0^*) > 0$ where $\eta_0^* = (0, 0, 1) \in \tilde{M}$. We always have $(\xi_0, \eta_0^*) = 0$. $\square$

Proposition A.3. *Under the above notation, we have $\text{Fut}(\mathcal{Y}, \Delta, \xi_0; \eta) > 0$ for any η satisfying $(\eta, \eta_0^*) > 0$ if and only if the centroid $\tilde{u}_2$ of $\tau^\vee \cap (\xi_0 = 1)$ satisfies*

$$\tilde{u}_2 = a(\tilde{u}_0 - c\tilde{u}_1) + b\eta_0^*$$

for some $a, b > 0$.

Proof. We first determine a by $(\tilde{u}_2, \xi_0) = a(\tilde{u}_0 - c\tilde{u}_1, \xi_0)$. Then the vector $a(\tilde{u}_0 - c\tilde{u}_1) \in (\xi_0 = 1)$. Then we may choose η in a way such that $(\tilde{u}_0 - c\tilde{u}_1, \eta) = 0$ and $(\eta, \eta_0^*) > 0$ since $\widehat{\text{vol}}$ is invariant under rescaling. Then $A_{(\mathcal{Y}_0, \Delta_0)}(v_{\xi_0 - t\eta}) = a^{-1} > 0$. Moreover, computation shows

$$\frac{d}{dt} \Big|_{t=0} \text{vol}(\tau^\vee \cap ((\xi_0 - t\eta) \leq 1)) = \text{vol}(\tau^\vee \cap (\xi_0 = 1)) \cdot (a(\tilde{u}_0 - c\tilde{u}_1) - \tilde{u}_2, \eta).$$

Hence, $\text{Fut}(\mathcal{Y}, \Delta, \xi_0; \eta) > 0$ is equivalent to $(a(\tilde{u}_0 - c\tilde{u}_1) - \tilde{u}_2, \eta) > 0$ for any η satisfying $(\tilde{u}_0 - c\tilde{u}_1, \eta) = 0$ and $(\eta, \eta_0^*) > 0$. This is equivalent to $a(\tilde{u}_0 - c\tilde{u}_1) - \tilde{u}_2 = a'(\tilde{u}_0 - c\tilde{u}_1) + b\eta_0^*$ for some $a' \in \mathbb{R}$ and $b > 0$. Since $(\xi, \eta_0^*) = 0$, we get $a' = 0$. Hence, the proof is finished. $\square$

By computation, we have

$$\tilde{u}_2 = \left(9, 1, -\frac{49}{150}\right), \quad \tilde{u}_0 - c\tilde{u}_1 = (15, 15 - 25c, -7 + 12c), \quad \eta_0^* = (0, 0, 1).$$

Hence, the only c satisfying the condition of Proposition A.3 is $c = \frac{8}{15}$.

Case 2: $Q = [1]$. As always, we want to first determine the polyhedral divisors. We know that

$$\mathfrak{D}_{[0]} + \mathfrak{D}_{[\infty]} = \text{conv}\left(\frac{6}{325}(1, -26), \frac{1}{325}(19, -169)\right) + \sigma,$$

$$\mathfrak{D}_{[0]} + \mathfrak{D}_{[1]} + \mathfrak{D}_{[\infty]} = \text{conv}\left(\frac{1}{650}(-1, 26), \frac{1}{26}(1, 0)\right) + \sigma.$$

It is clear that $\mathcal{Y}([1], 0, m)$ has five distinguished vertical divisors:

$$D_{[0], (x_0, 0)}, D_{[1], (x_1, 0)}, D_{[1], \left(0, \frac{1}{m}\right)}, D_{[\infty], (x_2, 0)}, D_{[\infty], (x_3, 0)}.$$

We can compute that

$$\text{div}(f \cdot \chi^{(0, 25, k)}) = (-13m + k)D_{[1], \left(0, \frac{1}{m}\right)} + D_{[\infty], (x_2, 0)}.$$

We also have $\operatorname{div}(\chi^{(0,0,1)}) = D_{[1],(0,\frac{1}{m})} = \mathcal{Y}_0$. Hence, we have $D_{[\infty],(x_2,0)} = \operatorname{div}(f \cdot \chi^{(0,25,13m)})$. Next, we will analyze the cone $\tau \subset \tilde{N}_{\mathbb{Q}}$. Assume $m = 1$ for simplicity. Then, the primitive vectors of τ in $\tilde{N}$ are $n_1 = (\frac{1}{25}(-1, 26), -2)$, $n_2 = (\frac{6}{25}(1, -26), 13)$, and $n_3 = (\frac{1}{25}(19, -169), 13)$. Let $\tilde{u}_0$ and $\tilde{u}_1$ be vectors in $\tilde{M}_{\mathbb{Q}}$ representing $-K_{\mathcal{Y}_0}$ and $(w = 0)|_{\mathcal{Y}_0}$, respectively. Then

$$\tilde{u}_0 = (15, 15, 7), \quad \tilde{u}_1 = (0, 25, 13).$$

We still have $\xi_0 = (0, 1, 0)$ and $\eta_0^* = (0, 0, 1)$. Hence,

$$\tilde{u}_2 = \left(9, 1, \frac{319}{975}\right), \quad \tilde{u}_0 - c\tilde{u}_1 = (15, 15 - 25c, 7 - 13c), \quad \eta_0^* = (0, 0, 1).$$

Hence, the only c satisfying the condition of Proposition A.3 is $c = \frac{8}{15}$.

A.2 | A_{11}

For plane quintic curves with A_{11} -singularities, we have two cases: reducible and irreducible curves. We begin with the reducible case.

A_{11} reducible

Let C be a reducible plane quintic curve with an A_{11} -singularity. Then after a projective transformation, in the affine coordinates $[x, y, 1]$, we can write the equation of C as (see Proposition A.1)

$$C = ((x - y^2)((x - y^2)(1 + sx) - x^3) = 0).$$

In other words, we have $p = 0$, $u = 0$, and $r = 1$ in (A.1). Let us choose a 6-jet (x', y) at the origin by $x' := x - y^2 - \frac{1}{2}y^6$. Then, the equation of C in (x', y) becomes

$$x'^2 = \frac{1}{4}y^{12} + \text{higher order terms},$$

where (x', y) has weight $(6, 1)$. The only parameter of C here is $s \in \mathbb{A}^1$. All these curves are GIT stable. When s goes to infinity, the unique GIT polystable limit will be Q_5 , that is, the double conic union a transversal line.

Theorem A.4. Suppose $C \subset \mathbb{P}^2$ is a reducible quintic curve with an A_{11} singularity. Then the log Fano pair $(\mathbb{P}^2, cC)$ is K -semistable if and only if $0 < c \leq \frac{6}{11}$. Moreover, $(\mathbb{P}(1, 1, 4), \frac{6}{11}C_0)$ is the K -polystable degeneration of $(\mathbb{P}^2, \frac{6}{11}C)$ where $C_0 = (x^2z^2 + y^6z = 0)$.

Proof. We first prove the “only if” part. Suppose that $(\mathbb{P}^2, cC)$ is K -semistable, and we want to show $c \leq \frac{6}{11}$. Let us perform the $(6, 1)$ -weighted blow-up of $\mathbb{P}^2$ in the coordinates (x', y) , and denote the resulting surface and exceptional divisor by (X, E) , with $\pi : X \rightarrow \mathbb{P}^2$ the weighted blow-up morphism. Let $Q = (x = y^2)$ be a smooth conic in $\mathbb{P}^2$. We know that the weight of $x - y^2 = x' + \frac{1}{2}y^6$ is 6, hence $\bar{Q} := \pi_*^{-1}Q \sim 2\pi^*H - 6E$ is effective on X . It is easy to see $(E^2) = -\frac{1}{6}$ and $(\bar{Q}^2) = -2$, hence the Mori cone of X is generated by E and $\bar{Q}$. It is clear that

$$A_{(\mathbb{P}^2, cC)}(E) = 7 - 12c, \quad -K_{\mathbb{P}^2} - cC \sim_{\mathbb{Q}} (3 - 5c)H.$$

We also have $\pi^*H - tE$ is ample if and only if $0 < t < 2$, and big if and only if $0 \leq t < 3$. Then, by computation, we have

$$\mathrm{vol}_X(\pi^*H - tE) = \begin{cases} 1 - \frac{t^2}{6} & \text{if } 0 \leq t \leq 2; \\ \frac{(3-t)^2}{3} & \text{if } 2 \leq t \leq 3. \end{cases}$$

Hence, $S_{(\mathbb{P}^2, cC)}(E) = (3 - 5c) \int_0^\infty \mathrm{vol}_X(\pi^*H - tE) = (3 - 5c) \frac{5}{3}$. So, the valuative criterion (Theorem 2.9) implies

$$7 - 12c = A_{(\mathbb{P}^2, cC)}(E) \geq S_{(\mathbb{P}^2, cC)}(E) = (3 - 5c) \frac{5}{3},$$

which implies $c \leq \frac{6}{11}$.

We now begin showing the “if” part. Similar to the proof of Theorem 7.2, we construct a special degeneration then later on use techniques of Ilten and Süß [59] to show K-polystability of the degeneration. $\square$

Proposition A.5. *The log Fano pair $(\mathbb{P}^2, cC)$ admits a special degeneration to $(\mathbb{P}(1, 1, 4), cC_0)$ where C_0 is given by the equation $x^2z^2 + y^6z = 0$.*

Proof. Here is the construction of the special degeneration. Take our family $(\mathbb{P}^2, C) \times \mathbb{A}^1$. We perform the following birational transformations:

$$\begin{array}{ccccc} & \mathcal{X} & \overset{f}{\dashrightarrow} & \mathcal{X}^+ & \\ \swarrow \pi & \searrow g & & \swarrow h & \searrow \psi \\ \mathbb{P}^2 \times \mathbb{A}^1 & & \mathcal{Y} & & \mathcal{Z} \end{array}$$

where in the central fiber, we have

$$\begin{array}{ccccc} & S \cup X & \overset{f}{\dashrightarrow} & \widehat{S} \cup X' & \\ \swarrow \pi & \searrow g & & \swarrow h & \searrow \psi \\ \mathbb{P}^2 & & S \cup X' & & S' \end{array}$$

Here π is the $(6,1,1)$ -weighted blow-up of $\mathbb{P}^2 \times \mathbb{A}^1$ in the local coordinates (x', y, t) , $S = \mathbb{P}(1, 1, 6)$ is the exceptional divisor of π , g is the contraction of $\overline{Q}$ in $X \subset \mathcal{X}_0$, f is the flip of the curve $\overline{Q}$ in $\mathcal{X}_0$ (since by computation the normal bundle $\mathcal{N}_{\overline{Q}/\mathcal{X}} \cong \mathcal{O}_{\overline{Q}}(-2) \oplus \mathcal{O}_{\overline{Q}}(-1)$), and ψ is the divisorial contraction that contracts X' to a point.

Let us analyze the geometry of these birational maps. Suppose that S has projective coordinates $[x_1, x_2, x_3]$ of weights $(1,1,6)$, respectively. Then $S \cap X = E = (x_1 = 0)$, and $\overline{Q} \cap E = \{p\}$ is a smooth point of S and X . Since $\overline{Q}$ has normal bundle $\mathcal{O}(-2) \oplus \mathcal{O}(-1)$ in $\mathcal{X}$, the surface $\widehat{S}$ is a $(2,1)$ -weighted blow-up of S at p . Let $\overline{Q}^+$ be the flipped curve in $\widehat{S}$, then $\widehat{S}$ has an A_1 -singularity at the

unique intersection of $h_*^{-1}E$ and $\overline{Q}^+$. Then $\psi : \widehat{S} \rightarrow S'$ contracts $h_*^{-1}E$ and creates a singularity of type $\frac{1}{4}(1, 1)$. Thus, S' is $\mathbb{P}(1, 1, 4)$.

For the degeneration C_0 of C in S' , note that $\pi_*^{-1}(C \times \mathbb{A}^1) \cap S$ is the curve $C'_0 = (x_3^2 = \frac{1}{4}x_2^{12})$. In addition, we know that p has coordinate $[0, 1, -\frac{1}{2}]$ which is contained in C'_0 . It is clear that C'_0 has an A_{11} -singularity at $[1, 0, 0]$ near where the birational map $S \dashrightarrow S'$ is an isomorphism. Thus, C_0 has an A_{11} -singularity at a smooth point of S' as well. Since Q is contained in C , we know that $\overline{Q}' := \psi_* \overline{Q}^+$ is contained in the degeneration C_0 of C . By a toric computation, we know that $\overline{Q}' = (z = 0)$ in $S' = \mathbb{P}(1, 1, 4)$. This implies that up to a projective transformation C_0 has the equation $(x^2z^2 + y^6z = 0)$ in $\mathbb{P}(1, 1, 4)$. $\square$

Now we return to the proof of the theorem. By Propositions A.5 and A.6 and Theorem 2.16, we know that $(\mathbb{P}^2, \frac{6}{11}C)$ is K-semistable. Hence, the proof is finished by Proposition 2.13.

We verify K-polystability of $(\mathbb{P}(1, 1, 4), \frac{6}{11}C_0)$ below.

Proposition A.6. *The log Fano pair $(\mathbb{P}(1, 1, 4), \frac{6}{11}C_0)$ is K-polystable where $C_0 = (x^2z^2 + y^6z = 0)$.*

Proof. It is clear that the pair $(\mathbb{P}(1, 1, 4), \frac{6}{11}C_0)$ admits a $\mathbb{G}_m$ -action, which can be lifted to a $\mathbb{G}_m^2$ -action on $Y := C(\mathbb{P}(1, 1, 4), \mathcal{O}_{\mathbb{P}(1,1,4)}(1)) \cong \mathbb{A}_{(x,y,z)}^3$ as

$$(x, y, z) \mapsto (\mu x, \lambda \mu y, \lambda^6 \mu^4 z).$$

Thus, by [93, Theorem 1.4], it suffices to show $\mathbb{G}_m$ -equivariant K-polystability of $(\mathbb{P}(1, 1, 4), \frac{6}{11}C_0)$.

Denote by N the lattice of 1-PS's in $\mathbb{G}_m^2$. Since Y is a toric variety with the standard $\mathbb{G}_m^3$ -action, we denote by N_Y the lattice of 1-PS's in $\mathbb{G}_m^3$. By [8, Section 11], we have an embedding of lattices $F : N \rightarrow N_Y$ and a (noncanonical) surjective map $s : N_Y \rightarrow N$ with $s \circ F = \text{id}$. Then the maps F and s can be chosen as

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 6 & 4 \end{bmatrix}, \quad s = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Computation shows $\sigma \subset N_{\mathbb{Q}}$ is spanned by $(-2, 3)$ and $(1, 0)$. The nontrivial polyhedral divisors are

$$\mathfrak{D}_{[0]} = \text{conv}\left((0, 0), \left(-\frac{1}{2}, \frac{1}{2}\right)\right) + \sigma, \quad \mathfrak{D}_{[\infty]} = \left(\frac{1}{6}, 0\right) + \sigma.$$

Denote by $x_0 = (0, 0)$, $x_1 = (-\frac{1}{2}, \frac{1}{2})$, $x_2 = (\frac{1}{6}, 0)$. Then computation shows

$$\text{div}(\chi^{(0,1)}) = D_{[0], x_1} = (x = 0),$$

$$\text{div}(\chi^{(1,1)}) = D_{[\infty], x_2} = (y = 0),$$

$$\text{div}(f \cdot \chi^{(6,4)}) = D_{[0], x_0} = (z = 0).$$

Here, f is a rational function on $\mathbb{P}^1$ with $\operatorname{div}(f) = [0] - [\infty]$. Let us choose a rational function g on $\mathbb{P}^1$ such that $\operatorname{div}(g) = [1] - [\infty]$. Then in a suitable coordinate of $\mathbb{P}^1$, we have

$$\operatorname{div}\left(g \cdot \chi^{(6,6)}\right) = (x^2 z = y^6) = D_{[1],0}.$$

Hence, the boundary divisor is given by $\Delta = c \cdot \operatorname{div}(f g \cdot \chi^{(12,10)}) = c(D_{[0],x_0} + D_{[1],0})$.

Next, we will look at the test configuration picture.

Case 1: $Q = [0]$. Computation shows that there are four distinguished vertical divisors on $\mathcal{Y}$:

$$D_{[0],(x_1,0)} = (x = 0), \quad D_{[\infty],(x_2,0)} = (y = 0), \quad D_{[0],(x_0,0)} = (z = 0), \quad D_{[0],(0,1)} = \mathcal{Y}_0.$$

By computation, we have

$$\operatorname{div}(f g \cdot \chi^{(12,10,k)}) = D_{[0],(x_0,0)} + D_{[1],(0,0)} + (k+1)D_{[0],(0,1)}.$$

Hence, $(D_{[0],(x_0,0)} + D_{[1],(0,0)})|_{\mathcal{Y}_0} = \operatorname{div}(\chi^{(12,10,-1)})$. We know that τ is spanned by $n_1 = (0, 0, -1)$, $n_2 = (-1, 1, -2)$, and $n_3 = (1, 0, 6)$. We still have $\xi_0 = (0, 1, 0)$ and $\eta_0^* = (0, 0, 1)$. Hence,

$$\tilde{u}_0 = (7, 6, -1), \quad \tilde{u}_1 = (12, 10, -1), \quad \tilde{u}_2 = \frac{1}{12}(10, 12, -1).$$

To satisfy the condition of Proposition A.3, the only c is $c = \frac{6}{11}$.

Case 2: $Q = [\infty]$. Computation shows that there are four distinguished vertical divisors on $\mathcal{Y}$:

$$D_{[0],(x_1,0)} = (x = 0), \quad D_{[\infty],(x_2,0)} = (y = 0), \quad D_{[0],(x_0,0)} = (z = 0), \quad D_{[\infty],(0,1)} = \mathcal{Y}_0.$$

By computation, we have

$$\operatorname{div}(f g \cdot \chi^{(12,10,k)}) = D_{[0],(x_0,0)} + D_{[1],(0,0)} + (k-2)D_{[0],(0,1)}.$$

Hence, $(D_{[0],(x_0,0)} + D_{[1],(0,0)})|_{\mathcal{Y}_0} = \operatorname{div}(\chi^{(12,10,2)})$. We know that τ is spanned by $n_1 = (0, 0, 1)$, $n_2 = (-1, 1, 2)$, and $n_3 = (1, 0, -6)$. We still have $\xi_0 = (0, 1, 0)$ and $\eta_0^* = (0, 0, 1)$. Hence,

$$\tilde{u}_0 = (7, 6, 1), \quad \tilde{u}_1 = (12, 10, 2), \quad \tilde{u}_2 = \frac{1}{12}(10, 12, 1).$$

To satisfy the condition of Proposition A.3, the only c is $c = \frac{6}{11}$. □

A_{11} irreducible

Let C be an irreducible plane quintic curve with an A_{11} -singularity. Then after a projective transformation, in the affine coordinate $[x, y, 1]$, we can write the equation of C as (see Proposition A.1)

$$C = ((x - y^2)((x - y^2)(1 + sx) + 2x^2y) + x^5 = 0).$$

In other words, we have $p = -1$, $u = 1$, and $r = 0$ in (A.1). Let us choose a 6-jet (x', y) at the origin by $x' := x - y^2 + y^5$. Then the equation of C in (x', y) becomes

$$x'^2 = -sy^{12} + \text{higher order terms},$$

where (x', y) has weight $(6, 1)$. The only parameter here is $s \in \mathbb{A}^1 \setminus \{0\}$. All these curves are GIT stable. If $s = 0$, we recover the GIT stable A_{12} quintic curve discussed earlier. When s goes to infinity, the unique GIT polystable limit will be Q_5 , that is, the double conic union a transversal line.

Theorem A.7. *Suppose $C \subset \mathbb{P}^2$ is an irreducible quintic curve with an A_{11} singularity. Then the log Fano pair $(\mathbb{P}^2, cC)$ is K-semistable if and only if $0 < c \leq \frac{63}{115}$. Moreover, $(\mathbb{P}(1, 4, 25), \frac{63}{115}C_0)$ is the K-polystable degeneration of $(\mathbb{P}^2, \frac{63}{115}C)$ where $C_0 = (z^2 + x^2y^{12} = 0)$.*

Proof. We first prove the “only if” part. Suppose that $(\mathbb{P}^2, cC)$ is K-semistable, and we want to show $c \leq \frac{63}{115}$. Let us perform the $(6, 1)$ -weighted blow-up of $\mathbb{P}^2$ in the coordinates (x', y) , and denote the resulting surface and exceptional divisor by (X, E) , with $\pi : X \rightarrow \mathbb{P}^2$ the weighted blow-up morphism. Let $Q = (x = y^2)$ be a smooth conic in $\mathbb{P}^2$. We know that the weight of $x - y^2 = x' - y^5$ is 5, hence $\bar{Q} := \pi_*^{-1}Q \sim 2\pi^*H - 5E$ is effective on X . It is easy to see $(E^2) = (\bar{Q}^2) = -\frac{1}{6}$; hence, the Mori cone of X is generated by E and $\bar{Q}$. We again use the valuative criterion of K-semistability by Fujita and Li (Theorem 2.9), then

$$A_{(\mathbb{P}^2, cC)}(E) = 7 - 12c, \quad -K_{\mathbb{P}^2} - cC \sim_{\mathbb{Q}} (3 - 5c)H.$$

We also have $\pi^*H - tE$ is ample if and only if $0 < t < \frac{12}{5}$, and big if and only if $0 \leq t < \frac{5}{2}$. Then by computation, we have

$$\text{vol}_X(\pi^*H - tE) = \begin{cases} 1 - \frac{t^2}{6} & \text{if } 0 \leq t \leq \frac{12}{5}; \\ (5 - 2t)^2 & \text{if } \frac{12}{5} \leq t \leq \frac{5}{2}. \end{cases}$$

Hence, $S_{(\mathbb{P}^2, cC)}(E) = (3 - 5c) \int_0^\infty \text{vol}_X(\pi^*H - tE) = (3 - 5c) \frac{49}{30}$. Since $(\mathbb{P}^2, cC)$ is K-semistable, the valuative criterion (Theorem 2.9) implies

$$7 - 12c = A_{(\mathbb{P}^2, cC)}(E) \geq S_{(\mathbb{P}^2, cC)}(E) = (3 - 5c) \frac{49}{30},$$

which implies $c \leq \frac{63}{115}$.

We now begin showing the “if” part. Similar to the proof of Theorems 7.2 and A.4, we construct a special degeneration then later on use techniques of Ilten and Süß [59] to show K-polystability of the degeneration. $\square$

Proposition A.8. *The log Fano pair $(\mathbb{P}^2, cC)$ admits a special degeneration to $(\mathbb{P}(1, 4, 25), cC_0)$ where C_0 is given by the equation $z^2 + x^2y^{12} = 0$.*

Proof. We follow notation from the first two diagrams of the proof of Proposition A.5. Here, π is the $(6, 1, 1)$ -weighted blow-up of $\mathbb{P}^2 \times \mathbb{A}^1$ in the local coordinates (x', y, t) , $S = \mathbb{P}(1, 1, 6)$ is the exceptional divisor of π , g is the contraction of $\bar{Q}$ in $X \subset \mathcal{X}_0$, f is the flip of the curve $\bar{Q}$ in $\mathcal{X}_0$, and ψ is the divisorial contraction that contracts X' to a point.

Let us analyze the geometry of these birational maps. Suppose that S has projective coordinates $[x_1, x_2, x_3]$ of weights $(1, 1, 6)$, respectively. Then, $S \cap X = E = (x_1 = 0)$, and $\bar{Q} \cap E = \{p\}$ is the unique singular point of S (type $\frac{1}{6}(1, 1)$) and X (type A_5). Inside the surface X , we have two

smooth rational curves E and $\bar{Q}$ intersecting at p , such that $(E^2) = (\bar{Q}^2) = -\frac{1}{6}$. Hence, contracting $\bar{Q}$ in X yields a smooth surface X' that has to be $\mathbb{P}^2$ by degree computation. On the other hand, $\hat{S}$ and X' intersect along the proper transform $\hat{E}$ of E that becomes a conic curve in $X' \cong \mathbb{P}^2$. Hence, $(\hat{E}^2) = -4$ in $\hat{S}$. Moreover, $h : \hat{S} \rightarrow S \cong \mathbb{P}(1, 1, 6)$ is a partial resolution of p that only extracts the flipped curve $\bar{Q}^+$. By some combinatorial computation, we know that $\hat{S}$ is a toric blow-up of S at p that creates a singularity of type $\frac{1}{25}(1, 4)$ away from $\hat{E}$. Hence, S' is a toric surface carrying two singularities of types $\frac{1}{25}(1, 4)$ and $\frac{1}{4}(1, 1)$. Thus, $S' \cong \mathbb{P}(1, 4, 25)$.

For the degeneration C_0 of C on S' , note that $\pi_*(C \times \mathbb{A}^1) \cap S$ is the curve $C'_0 = (x_3^2 + sx_2^{12} = 0)$. In addition, we know that p has coordinate $[0, 0, 1]$ which is not contained in C'_0 . It is clear that C'_0 has an A_{11} -singularity at the point $[1, 0, 0]$ which is not p and does not lie on E . Since $S \dashrightarrow S'$ is isomorphic around $[1, 0, 0]$, we know that C'_0 has an A_{11} -singularity at a smooth point of $\mathbb{P}(1, 4, 25)$. Since the $\frac{1}{25}(1, 4)$ singularity on $\hat{S}$ does not lie on $\hat{E}$, we know that C_0 does not pass through $[0, 0, 1]$ on $S' \cong \mathbb{P}(1, 4, 25)$. Hence, after a projective coordinates change, we may write $C_0 = (z^2 + x^2y^{12} = 0)$. $\square$

Now we return to the proof of the theorem. By Propositions A.8 and A.9 and Theorem 2.16, we know that $(\mathbb{P}^2, \frac{63}{115}C)$ is K-semistable. Hence, the proof is finished by Proposition 2.13.

We verify K-polystability of $(\mathbb{P}(1, 4, 25), \frac{63}{115}C_0)$ below.

Proposition A.9. *The log Fano pair $(\mathbb{P}(1, 4, 25), \frac{63}{115}C_0)$ is K-polystable where $C_0 = (z^2 + x^2y^{12} = 0)$.*

Proof. It is clear that the pair $(\mathbb{P}(1, 4, 25), \frac{63}{115}C_0)$ admits a $\mathbb{G}_m$ -action, which can be lifted to a $\mathbb{G}_m^3$ -action on $Y := C(\mathbb{P}(1, 4, 25), \mathcal{O}_{\mathbb{P}(1, 4, 25)}(1)) \cong \mathbb{A}_{(x, y, z)}^3$ as

$$(x, y, z) \mapsto (\mu x, \lambda \mu^4 y, \lambda^6 \mu^{25} z).$$

Thus, by [93, Theorem 1.4], it suffices to show $\mathbb{G}_m$ -equivariant K-polystability of $(\mathbb{P}(1, 4, 25), \frac{63}{115}C_0)$.

Denote by N the lattice of 1-PS's in $\mathbb{G}_m^2$. Since Y is a toric variety with the standard $\mathbb{G}_m^3$ -action, we denote by N_Y the lattice of 1-PS's in $\mathbb{G}_m^3$. By [8, Section 11], we have an embedding of lattices $F : N \rightarrow N_Y$ and a (noncanonical) surjective map $s : N_Y \rightarrow N$ with $s \circ F = \text{id}$. Then the maps F and s can be chosen as

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 4 \\ 6 & 25 \end{bmatrix}, \quad s = \begin{bmatrix} -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Computation shows $\sigma \subset N_{\mathbb{Q}}$ is spanned by $(-4, 1)$ and $(1, 0)$. The only nontrivial polyhedral divisor is

$$\mathfrak{D}_{[\infty]} = \text{conv}\left((-4, 1), \left(\frac{1}{6}, 0\right)\right) + \sigma.$$

Denote by $x_0 = (\frac{1}{6}, 0)$ and $x_1 = (-4, 1)$. Then, we have

$$(x = 0) = \text{div}(\chi^{(0,1)}) = D_{[\infty], x_0}, \quad (y = 0) = \text{div}(\chi^{(1,4)}) = D_{[\infty], x_1}.$$

Moreover, for any $P \neq [\infty]$, we may take the function f_P on $\mathbb{P}^1$ such that $\operatorname{div}(f) = P - [\infty]$. Then, we may compute out that

$$\operatorname{div}(f_P \cdot \chi^{(6,25)}) = D_{P,0} \sim D_{[\infty],x_0} + 6D_{[\infty],x_1}.$$

This implies that the curve C induces a divisor on Y which corresponding to $\operatorname{div}(f_{P_1} f_{P_2} \cdot \chi^{(12,50)})$ for some $P_1 \neq P_2 \in \mathbb{P}^1 \setminus \{\infty\}$.

Next, we will look at the test configuration picture.

Case 1: $Q = [\infty]$. Computation shows that on $\mathcal{Y}$ there are three distinguished vertical divisors:

$$D_{[\infty],(x_0,0)} = (x = 0), \quad D_{[\infty],(x_1,0)} = (y = 0), \quad D_{[\infty],(0,1)} = \mathcal{Y}_0.$$

Again, by computation, we have

$$\operatorname{div}(f_P \cdot \chi^{(6,25,k)}) = D_{P,(0,0)} + (k-1)D_{[\infty],(0,1)}.$$

Hence, we need to take $k = 1$, and we get $D_{P,(0,0)}|_{\mathcal{Y}_0} = \operatorname{div}(\chi^{(6,25,1)})$. This implies that $\Delta_0 = c \cdot \operatorname{div}(\chi^{(12,50,2)})$. The central fiber $\mathcal{Y}_0 = \operatorname{Spec} \mathbb{C}[\tau^\vee \cap \tilde{M}]$, where τ is spanned by $n_1 = (-4, 1, -1)$, $n_2 = (1, 0, -6)$, and $n_3 = (0, 0, 1)$. We still have $\xi_0 = (0, 1, 0)$ and $\eta_0^* = (0, 0, 1)$. Hence, we have

$$\tilde{u}_0 = (7, 30, 1), \quad \tilde{u}_1 = (12, 50, 2), \quad \tilde{u}_2 = \frac{1}{300}(49, 300, 4).$$

To satisfy the condition of Proposition A.3, the only c is $c = \frac{63}{115}$.

Case 2: $Q \neq [\infty]$. Computation shows that on $\mathcal{Y}$ there are three distinguished vertical divisors:

$$D_{[\infty],(x_0,0)} = (x = 0), \quad D_{[\infty],(x_1,0)} = (y = 0), \quad D_{Q,(0,1)} = \mathcal{Y}_0.$$

Again, by computation, we have

$$\operatorname{div}(f_P \cdot \chi^{(6,25,k)}) = \begin{cases} D_{P,(0,0)} + kD_{Q,(0,1)} & \text{if } P \neq Q \\ D_{Q,(0,0)} + (k+1)D_{Q,(0,1)} & \text{if } P = Q. \end{cases}$$

Hence, we have

$$D_{P,(0,0)}|_{\mathcal{Y}_0} = \begin{cases} \operatorname{div}(\chi^{(6,25,0)}) & \text{if } P \neq Q \\ \operatorname{div}(\chi^{(6,25,-1)}) & \text{if } P = Q. \end{cases}$$

This implies that $\Delta_0 = c \cdot \operatorname{div}(\chi^{(12,50,\delta)})$ where $\delta = -1$ if $Q \in \{P_1, P_2\}$ and $\delta = 0$ otherwise. The central fiber $\mathcal{Y}_0 = \operatorname{Spec} \mathbb{C}[\tau^\vee \cap \tilde{M}]$, where τ is spanned by $n_1 = (-4, 1, 1)$, $n_2 = (1, 0, 6)$, and $n_3 = (0, 0, -1)$. We still have $\xi_0 = (0, 1, 0)$ and $\eta_0^* = (0, 0, 1)$. Hence, we have

$$\tilde{u}_0 = (7, 30, -1), \quad \tilde{u}_1 = (12, 50, \delta), \quad \tilde{u}_2 = \frac{1}{300}(49, 300, -4).$$

To satisfy the condition of Proposition A.3, the only c is $c = \frac{63}{115}$. □

A.3 | A_{10}

Let C be an irreducible plane quintic curve with an A_{10} -singularity. Then after a projective transformation, in the affine coordinate $[x, y, 1]$, we can write the equation of C as (see Proposition A.1)

$$C = ((x - y^2)((x - y^2)(1 + sx) + 2x^2y - rx^3) + x^5 = 0).$$

In other words, we have $p = -1$ and $u = 1$ in (A.1). Let us choose a 5-jet (x', y) at the origin by $x' := x - y^2 + y^5$. Then the equation of C in (x', y) becomes

$$x'^2 = -ry^{11} + \text{higher order terms},$$

where (x', y) has weight $(11, 2)$. The parameter here is $(s, r) \in \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$. All these curves are GIT stable. If $r = 0$, we recover the GIT stable irreducible A_{11} quintic curve discussed earlier.

Theorem A.10. *Suppose $C \subset \mathbb{P}^2$ is a quintic curve with an A_{10} singularity. Then the log Fano pair $(\mathbb{P}^2, cC)$ is K-semistable if and only if $0 < c \leq \frac{54}{95}$. Moreover, $(\mathbb{P}(1, 4, 25), \frac{54}{95}C_0)$ is the K-polystable degeneration of $(\mathbb{P}^2, \frac{54}{95}C)$ where $C_0 = (z^2 + x^6y^{11} = 0)$.*

Proof. We first prove the “only if” part. Suppose that $(\mathbb{P}^2, cC)$ is K-semistable, and we want to show $c \leq \frac{54}{95}$. Let us perform the $(11, 2)$ -weighted blow-up of $\mathbb{P}^2$ in the coordinates (x', y) , and denote the resulting surface and exceptional divisor by (X, E) , with $\pi : X \rightarrow \mathbb{P}^2$ the weighted blow-up morphism. Let $Q = (x = y^2)$ be a smooth conic in $\mathbb{P}^2$. We know that the weight of $x - y^2 = x' - y^5$ is 10, hence $\bar{Q} := \pi_*^{-1}Q \sim 2\pi^*H - 10E$ is effective on X . It is easy to see $(E^2) = -\frac{1}{22}$ and $(\bar{Q}^2) = -\frac{6}{11}$, hence the Mori cone of X is generated by E and $\bar{Q}$. It is clear that $A_{(\mathbb{P}^2, cC)}(E) = 13 - 22c$ and $-K_{\mathbb{P}^2} - cC \sim_{\mathbb{Q}} (3 - 5c)H$. We also have $\pi^*H - tE$ is ample if and only if $0 < t < \frac{22}{5}$, and big if and only if $\frac{22}{5} \leq t < 5$. Then, by computation, we have

$$\text{vol}_X(\pi^*H - tE) = \begin{cases} \frac{1-t^2}{22} & \text{if } 0 \leq t \leq \frac{22}{5}; \\ \frac{(10-2t)^2}{12} & \text{if } \frac{22}{5} \leq t < 5. \end{cases}$$

Hence, $S_{(\mathbb{P}^2, cC)}(E) = (3 - 5c) \int_0^\infty \text{vol}_X(\pi^*H - tE) = (3 - 5c) \frac{47}{15}$. Since $(\mathbb{P}^2, cC)$ is K-semistable, the valuative criterion (Theorem 2.9) implies

$$13 - 22c = A_{(\mathbb{P}^2, cC)}(E) \geq S_{(\mathbb{P}^2, cC)}(E) = (3 - 5c) \frac{47}{15},$$

which implies $c \leq \frac{54}{95}$.

We now begin showing the “if” part. Similar to the proof of Theorems 7.2, A.4, and A.7, we construct a special degeneration then later on use techniques of Ilten and Süß [59] to show K-polystability of the degeneration. $\square$

Proposition A.11. *The log Fano pair $(\mathbb{P}^2, cC)$ admits a special degeneration to $(\mathbb{P}(1, 4, 25), cC_0)$ where C_0 is given by the equation $(z^2 + x^6y^{11} = 0)$.*

Proof. We follow notation from the first two diagrams of the proof of Proposition A.5. Here, π is the (11,2,1)-weighted blow-up of $\mathbb{P}^2 \times \mathbb{A}^1$ in the local coordinates (x', y, t) , $S = \mathbb{P}(1, 2, 11)$ is the exceptional divisor of π , g is the contraction of $\bar{Q}$ in $X \subset \mathcal{X}_0$, f is the flip of the curve $\bar{Q}$ in $\mathcal{X}_0$, and ψ is the divisorial contraction that contracts X' to a point.

Let us analyze the geometry of these birational maps. Suppose that S has projective coordinates $[x_1, x_2, x_3]$ of weights (1,2,11), respectively. Then $S \cap X = E = (x_1 = 0)$, and $\bar{Q} \cap E = \{p\}$ is a singular point of S (type $\frac{1}{11}(1, 2)$) and X (type $\frac{1}{11}(1, 9)$). Inside the surface X , we have two smooth rational curves E and $\bar{Q}$ intersecting at p , such that $(E^2) = -\frac{1}{22}$ and $(\bar{Q}^2) = -\frac{6}{11}$. Hence, contracting $\bar{Q}$ in X yields a surface X' with two singularities of types A_1 and $\frac{1}{6}(1, 5)$. On the other hand, $\hat{S}$ and X' intersect along the proper transform $\hat{E}$ of E with $(\hat{E}^2) = \frac{1}{3}$. Hence, $(\hat{E}^2) = -\frac{1}{3}$ in $\hat{S}$. Moreover, $h : \hat{S} \rightarrow S \cong \mathbb{P}(1, 2, 11)$ is a partial resolution of $p = [0, 0, 1]$ which only extracts the flipped curve $\bar{Q}^+$. By some combinatorial computation, we know that $\hat{S}$ is a toric blow-up of S at p that creates a singularity of type $\frac{1}{25}(1, 4)$ away from $\hat{E}$. Moreover, $\hat{E}$ passes through two singularities of $\hat{S}$ of types A_1 and $\frac{1}{6}(1, 1)$. Hence, S' is a toric surface carrying two singularities of types $\frac{1}{25}(1, 4)$ and $\frac{1}{4}(1, 1)$ (coming from contracting $\hat{E}$). Thus, $S' \cong \mathbb{P}(1, 4, 25)$.

For the degeneration C_0 of C on S' , note that $\pi_*(C \times \mathbb{A}^1) \cap S$ is the curve $C'_0 = (x_3^2 + rx_2^{11} = 0)$. In addition, we know that p has coordinate $[0, 0, 1]$ that is not contained in C'_0 . It is clear that C'_0 has an A_{10} -singularity at the point $[1, 0, 0]$ that is not p and does not lie on E . Since $S \dashrightarrow S'$ is isomorphic around $[1, 0, 0]$, we know that C'_0 has an A_{10} -singularity at a smooth point of $\mathbb{P}(1, 4, 25)$. Since the $\frac{1}{25}(1, 4)$ singularity on $\hat{S}$ does not lie on $\hat{E}$, we know that C_0 does not pass through $[0, 0, 1]$ on $S' \cong \mathbb{P}(1, 4, 25)$. Hence, after a projective coordinates change, we may write $C_0 = (z^2 + x^2(x^4 - ay)y^{11} = 0)$. Since $\text{Aut}(\mathbb{P}(1, 4, 25), C_0)$ is not discrete, we conclude that $a = 0$ which finishes the proof. $\square$

Now we return to the proof of the theorem. By Propositions A.11, A.12, and Theorem 2.16, we know that $(\mathbb{P}^2, \frac{54}{95}C)$ is K-semistable. Hence, the proof is finished by Proposition 2.13.

We verify K-polystability of $(\mathbb{P}(1, 4, 25), \frac{54}{95}C_0)$ below.

Proposition A.12. *The log Fano pair $(\mathbb{P}(1, 4, 25), \frac{54}{95}C_0)$ is K-polystable where $C_0 = (z^2 + x^6y^{11} = 0)$.*

Proof. It is clear that the pair $(\mathbb{P}(1, 4, 25), \frac{54}{95}C_0)$ admits a $\mathbb{G}_m$ -action, which can be lifted to a $\mathbb{G}_m^2$ -action on $Y := C(\mathbb{P}(1, 4, 25), \mathcal{O}_{\mathbb{P}(1, 4, 25)}(1)) \cong \mathbb{A}_{(x, y, z)}^3$ as

$$(x, y, z) \mapsto (\mu x, \lambda^2 \mu^4 y, \lambda^{11} \mu^{25} z).$$

Thus, by [93, Theorem 1.4], it suffices to show $\mathbb{G}_m$ -equivariant K-polystability of $(\mathbb{P}(1, 4, 25), \frac{54}{95}C_0)$.

Denote by N the lattice of 1-PS's in $\mathbb{G}_m^2$. Since Y is a toric variety with the standard $\mathbb{G}_m^2$ -action, we denote by N_Y the lattice of 1-PS's in $\mathbb{G}_m^3$. By [8, Section 11], we have an embedding of lattices $F : N \rightarrow N_Y$ and a (noncanonical) surjective map $s : N_Y \rightarrow N$ with $s \circ F = \text{id}$. Then the maps F and s can be chosen as

$$F = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 11 & 25 \end{bmatrix}, \quad s = \begin{bmatrix} 1 & 6 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Computation shows $\sigma \subset N_{\mathbb{Q}}$ is spanned by $(-2, 1)$ and $(1, 0)$. The polyhedral divisor is

$$\mathfrak{D}_{[0]} = \left(-\frac{1}{2}, 0\right) + \sigma, \quad \mathfrak{D}_{[\infty]} = \text{conv}\left(\frac{1}{6}(1, 1), \left(\frac{6}{11}, 0\right)\right) + \sigma.$$

Denote by $x_0 = (-\frac{1}{2}, 0)$, $x_1 = \frac{1}{6}(1, 1)$, and $x_2 = (\frac{6}{11}, 0)$. Then computation shows

$$\begin{aligned} \text{div}(\chi^{(0,1)}) &= D_{[\infty], x_1} = (x = 0), \\ \text{div}(f \cdot \chi^{(2,4)}) &= D_{[\infty], x_2} = (y = 0), \\ \text{div}(f^6 \cdot \chi^{(11,25)}) &= D_{[0], x_0} = (z = 0). \end{aligned}$$

Here, f is a rational function on $\mathbb{P}^1$ such that $\text{div}(f) = [0] - [\infty]$. Let us choose a rational function g on $\mathbb{P}^1$ such that $\text{div}(g) = [1] - [\infty]$. Then in a suitable coordinate of $\mathbb{P}^1$, we have

$$\text{div}(f^{11} g \cdot \chi^{(22,50)}) = (z^2 - x^6 y^{11} = 0) = D_{[1], 0}.$$

Next we will look at the test configuration picture.

Case 1: $Q = [0]$. Computation shows that there are four distinguished vertical divisors on $\mathcal{Y}$:

$$D_{[\infty], (x_1, 0)} = (x = 0), \quad D_{[\infty], (x_2, 0)} = (y = 0), \quad D_{[0], (x_0, 0)} = (z = 0), \quad D_{[0], (0, 1)} = \mathcal{Y}_0.$$

By computation, we have

$$\text{div}(f^{11} g \cdot \chi^{(22,50,k)}) = D_{[1], (0, 0)} + (k + 11)D_{[0], (0, 1)}.$$

Hence, $D_{[1], (0, 0)}|_{\mathcal{Y}_0} = \text{div}(\chi^{(22,50,-11)})$. We know that τ is spanned by $n_1 = (-1, 0, -2)$, $n_2 = (1, 1, 6)$, and $n_3 = (6, 0, 11)$. We still have $\xi_0 = (0, 1, 0)$ and $\eta_0^* = (0, 0, 1)$. Hence,

$$\tilde{u}_0 = (13, 30, -7), \quad \tilde{u}_1 = (22, 50, -11), \quad \tilde{u}_2 = \frac{1}{300}(94, 300, -49).$$

To satisfy the condition of Proposition A.3, the only c is $c = \frac{54}{95}$.

Case 2: $Q = [\infty]$. Computation shows that there are four distinguished vertical divisors on $\mathcal{Y}$:

$$D_{[\infty], (x_1, 0)} = (x = 0), \quad D_{[\infty], (x_2, 0)} = (y = 0), \quad D_{[0], (x_0, 0)} = (z = 0), \quad D_{[\infty], (0, 1)} = \mathcal{Y}_0.$$

By computation, we have

$$\text{div}(f^{11} g \cdot \chi^{(22,50,k)}) = D_{[1], (0, 0)} + (k - 12)D_{[\infty], (0, 1)}.$$

Hence, $D_{[1], (0, 0)}|_{\mathcal{Y}_0} = \text{div}(\chi^{(22,50,12)})$. We know that τ is spanned by $n_1 = (-1, 0, 2)$, $n_2 = (1, 1, -6)$, and $n_3 = (6, 0, -11)$. We still have $\xi_0 = (0, 1, 0)$ and $\eta_0^* = (0, 0, 1)$. Hence,

$$\tilde{u}_0 = (13, 30, 7), \quad \tilde{u}_1 = (22, 50, 12), \quad \tilde{u}_2 = \frac{1}{300}(94, 300, 49).$$

To satisfy the condition of Proposition A.3, the only c is $c = \frac{54}{95}$. □

A.4 | Valuative criterion computations

In this section, we provide several consequences of the valuative criterion (Theorem 2.9). These results will be useful in proving wall crossings for K-moduli spaces of plane quintics.

Proposition A.13. *Let C be a curve on $\mathbb{P}(1, 1, 4)$ of degree 10.*

- (1) *Assume that the equation of C has the form $x^2z^2 + y^6z + g(x, y) = 0$. If $(\mathbb{P}(1, 1, 4), cC)$ is K-semistable, then $c \geq \frac{6}{11}$.*
- (2) *Assume that the equation of C has the form $x^2z^2 + axy^5z + g(x, y) = 0$ or $f(x, y)z + g(x, y) = 0$. Then $(\mathbb{P}(1, 1, 4), cC)$ is K-unstable for any $c \in (0, \frac{3}{5})$.*

Proof. For simplicity, we denote by $X := \mathbb{P}(1, 1, 4)$.

- (1) Let us consider the point $[0, 0, 1]$ on X . If we set $z = 1$, then we have a cyclic quotient map $\pi : \mathbb{A}_{(x,y)}^2 \rightarrow X$ defined by $\pi(x, y) = [x, y, 1]$. Let $v := \pi_* \bar{v}$ where $\bar{v}$ is the monomial valuation on $\mathbb{A}_{(x,y)}^2$ of weights $(3, 1)$. Since the equation of C is given by $ax^2 + g(x, y) + h(x, y) = 0$ where $\deg g = 6$ and $\deg h = 10$, we have $v(C) \geq 6$. Since v is a toric valuation, computation shows that

$$\text{vol}_X(\mathcal{O}(1) - tv) = \begin{cases} \frac{1}{4} \left(1 - \frac{t^2}{3}\right) & \text{if } 0 \leq t \leq 1 \\ \frac{1}{24} (3 - t)^2 & \text{if } 1 \leq t \leq 3. \end{cases}$$

Hence, $S_{(X, cC)}(v) = \frac{6-10c}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tv) dt = \frac{4}{3}(6 - 10c)$. Since (X, cC) is K-semistable, by valuative criterion (Theorem 2.9), we have

$$4 - 6c \geq A_{(X, cC)}(v) \geq S_{(X, cC)}(v) = \frac{4}{3}(6 - 10c),$$

which implies $c \geq \frac{6}{11}$.

- (2) First, we assume that the equation of C has the form $x^2z^2 + y^6z + g(x, y) = 0$. We again consider the affine chart $z = 1$ and the cyclic quotient map π . Let $v' := \pi_* \bar{v}'$ where $\bar{v}'$ is the monomial valuation on $\mathbb{A}_{(x,y)}^2$ of weights $(5, 1)$. By the equation of C , we have $v'(C) \geq 10$. Since v' is also a toric valuation, computation shows that

$$\text{vol}_X(\mathcal{O}(1) - tv') = \begin{cases} \frac{1}{4} \left(1 - \frac{t^2}{5}\right) & \text{if } 0 \leq t \leq 1 \\ \frac{1}{80} (5 - t)^2 & \text{if } 1 \leq t \leq 5. \end{cases}$$

Hence, $S_{(X, cC)}(v') = \frac{6-10c}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tv') dt = 2(6 - 10c)$. Since (X, cC) is K-semistable, valuative criterion (Theorem 2.9) implies that

$$6 - 10c \geq A_{(X, cC)}(v') \geq S_{(X, cC)}(v') = 2(6 - 10c).$$

Thus, $c \geq \frac{3}{5}$ but this is a contradiction. So, (X, cC) is always K-unstable for any $c \in (0, \frac{3}{5})$.

Finally, we treat the case where C has equation $f(x, y)z + g(x, y) = 0$. It is clear that the log canonical threshold of $(X; C)$ at $[0, 0, 1]$ is at least $\frac{1}{3}$ since $\deg f = 6$ and $\deg g = 10$. Thus, (X, cC) is K-unstable whenever $c \geq \frac{1}{3}$. On the other hand, by Theorem 5.2, we know that the surface $X = \mathbb{P}(1, 1, 4)$ never appears in the K-moduli stack $\overline{\mathcal{P}}_{5,c}^K$ when $c < \frac{3}{7}$. Thus, (X, cC) is always K-unstable for any $c \in (0, \frac{3}{5})$. This finishes the proof. $\square$

Next, we will state results about curves on $\mathbb{P}(1, 4, 25)$.

Proposition A.14. *Let C be a curve on $\mathbb{P}(1, 4, 25)$ of degree 50.*

- (1) *Suppose that C is defined by $z^2 + x^2y^{12} + x^6g(x, y) = 0$. If $(\mathbb{P}(1, 4, 25), cC)$ is K-semistable, then $c \geq \frac{63}{115}$.*
- (2) *Suppose that C is defined by $z^2 + x^6y^{11} + x^{10}g(x, y) = 0$. If $(\mathbb{P}(1, 4, 25), cC)$ is K-semistable, then $c \geq \frac{54}{95}$.*
- (3) *Suppose that C is defined by $z^2 + x^{10}g(x, y) = 0$ or $f(x, y)z + g(x, y) = 0$. Then $(\mathbb{P}(1, 4, 25), cC)$ is K-unstable for any $0 < c < 3/5$.*

Proof. For simplicity, we denote by $X := \mathbb{P}(1, 4, 25)$.

- (1) Let us consider the point $[0, 1, 0]$ corresponding to the $\frac{1}{4}(1, 1)$ singularity in $\mathbb{P}(1, 4, 25)$. If we set $y = 1$, then we have a cyclic quotient map $\pi : \mathbb{A}_{(x,z)}^2 \rightarrow X$ defined by $\pi(x, z) = [x, 1, z]$. Let $v := \pi_* \text{ord}_0$, then $A_X(v) = 2$, and

$$\text{vol}_X(\mathcal{O}(1) - tv) = \begin{cases} \frac{1}{100}(1 - 25t^2) & \text{if } 0 \leq t \leq \frac{1}{25} \\ \frac{1}{96}(1 - t)^2 & \text{if } \frac{1}{25} \leq t \leq 1. \end{cases}$$

Hence, computation shows $S_{(X,cC)}(v) = \frac{30-50c}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tv) dt = \frac{26}{75}(30 - 50c)$. Since C is of degree 50, we have $v(C) \geq 2$ because the lowest degree terms of C at $[0, 1, 0]$ are x^2y^{12} , xzy^6 , and z^2 . Since (X, cC) is K-semistable, by valuative criterion (Theorem 2.9), we have

$$2 - 2c \geq A_{(X,cC)}(v) \geq S_{(X,cC)}(v) = \frac{26}{75}(30 - 50c),$$

which implies $c \geq \frac{63}{115}$.

- (2) We follow the similar setup as (1), except that we consider the valuation $v' = \pi_* \tilde{v}'$ where $\tilde{v}'$ is the monomial valuation of weights (1, 3) in coordinates (x, z) . Thus, $A_X(v') = 4$, and

$$\text{vol}_X(\mathcal{O}(1) - tv') = \begin{cases} \frac{1}{100} - \frac{t^2}{12} & \text{if } 0 \leq t \leq \frac{3}{25} \\ \frac{1}{88}(1 - t)^2 & \text{if } \frac{3}{25} \leq t \leq 1. \end{cases}$$

Hence, computation shows $S_{(X,cC)}(v') = \frac{30-50c}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tv') dt = \frac{28}{75}(30 - 50c)$. From the equation of C , it is clear that $v'(C) \geq 6$. Since (X, cC) is K-semistable, by valuative criterion (Theorem 2.9), we have $4 - 6c \geq A_{(X,cC)}(v') \geq S_{(X,cC)}(v') = \frac{28}{75}(30 - 50c)$, which implies $c \geq \frac{54}{95}$.

- (3) First, we consider the case where C has equation $z^2 + x^{10}g(x, y) = 0$. We follow the similar setup as (1), except that we consider the valuation $v'' = \pi_* \tilde{v}''$ where $\tilde{v}''$ is the monomial valuation of weights (1, 5) in coordinates (x, z) . Thus, $A_X(v'') = 6$, and

$$\text{vol}_X(\mathcal{O}(1) - tv'') = \begin{cases} \frac{1}{100}(1 - 5t^2) & \text{if } 0 \leq t \leq \frac{1}{5} \\ \frac{1}{80}(1 - t)^2 & \text{if } \frac{1}{5} \leq t \leq 1. \end{cases}$$

Hence, computation shows $S_{(X, cC)}(v'') = \frac{30-50c}{\text{vol}_X(\mathcal{O}(1))} \int_0^\infty \text{vol}_X(\mathcal{O}(1) - tv'') dt = \frac{2}{5}(30 - 50c)$. From the equation of C , it is clear that $v''(C) \geq 10$. If (X, cC) is K-semistable, then by valutive criterion (Theorem 2.9), we have $6 - 10c \geq A_{(X, cC)}(v'') \geq S_{(X, cC)}(v'') = \frac{2}{5}(30 - 50c)$, which implies $c \geq \frac{3}{5}$. Hence, (X, cC) is always K-unstable for any $c \in (0, \frac{3}{5})$.

Finally, we consider the case where C has equation $f(x, y)z + g(x, y) = 0$. We consider the singular point $[0, 0, 1]$ of type $\frac{1}{25}(1, 4)$. Let $\pi' : \mathbb{A}_{(x, y)}^2 \rightarrow X$ be the cyclic quotient map. Denote by $\tilde{C}'$ the preimage of C under π' . Then it is easy to see that $\text{ord}_0 \tilde{C}' \geq 7$, hence $\text{lct}(X; C) \leq \text{lct}(\mathbb{A}^2; \tilde{C}') \leq \frac{2}{7}$. Hence, we have $c < \frac{2}{7}$ if (X, cC) is K-semistable. However, by Theorem 5.2, we know that $\mathbb{P}(1, 4, 25)$ never occurs in the K-moduli stack $\overline{\mathcal{P}}_c^K$ when $c < \frac{3}{7}$. Hence, (X, cC) is always K-unstable for any $c \in (0, \frac{3}{5})$. The proof is finished. $\square$

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ORCID

Kristin DeVleming  <https://orcid.org/0000-0001-6098-4804>

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