

Imaging with thermal noise induced currents*

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Abstract. We use thermal noise induced currents to image the real and imaginary parts of the conductivity of a body. Covariances of the thermal noise currents measured at a few electrodes are shown to be related to a deterministic problem. We use the covariances obtained while selectively heating the body to recover the real power density in the body under known boundary conditions and at a known frequency. The resulting inverse problem is related to acousto-electric tomography, but where the conductivity is complex and only the real power is measured. We study the local solvability of this problem by determining where its linearization is elliptic. Numerical experiments illustrating this inverse problem are included.

Key words. Conductivity imaging, Hybrid inverse problems, Thermal noise, Johnson-Nyquist noise

MSC codes. 35R30, 35J25, 35Q61

1. Introduction. In an electrical conductor, the excitement of charge carriers due to heat produces random currents. This phenomenon is called Johnson-Nyquist noise and was first observed in the early 20th century [22, 36]. Given a single component with impedance $Z(\omega)$ (in Ohms) at an angular frequency ω (in 2π Hz), the power spectral density $\langle |J(\omega)|^2 \rangle$ (in A^2/Hz) of the random current $J(t)$ across the component is

$$(1.1) \quad \langle |J(\omega)|^2 \rangle = \frac{2\kappa T \operatorname{Re}(Z(\omega))}{\pi |Z(\omega)|^2}.$$

Here $\kappa \approx 1.36 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}$ is Boltzmann's constant and T is temperature (in Kelvin). We recall the power spectral density of a finite variance, ergodic random process $J(t)$ is given by

$$(1.2) \quad \langle |J(\omega)|^2 \rangle = \lim_{T \rightarrow \infty} \frac{1}{2\pi T} |\hat{J}_T(\omega)|^2, \text{ where } \hat{J}_T(\omega) = \int_{-T/2}^{T/2} dt J(t) \exp[-i\omega t],$$

where $i = \sqrt{-1}$. Although thermally induced noise is a nuisance in electrical circuits, we show one way to use it to image the conductive properties of a body. Johnson-Nyquist noise and its generalization to the Maxwell equations (see, e.g., [39]) are examples of a more general physical principle called the fluctuation dissipation theorem, which relates the variance of fluctuations of a linear system about an equilibrium to the dissipative properties of the system, see e.g. [24, 38, 40].

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To image the conductivity of a body, we propose heating the body while simultaneously measuring the variance of the thermal noise induced currents using electrodes that are connected to the ground. For instance, this can be done on a two-dimensional conductive body as illustrated in Figure 1.1, with electrodes on its boundary that are connected to the ground (zero voltage or potential). The electrical measurements are made while the body is heated at a known spatial location via an external source, e.g. a laser, and the process is repeated at different locations to scan the body. In our approach, we also need to subtract measurements of thermal noise induced currents at a known and constant background temperature.

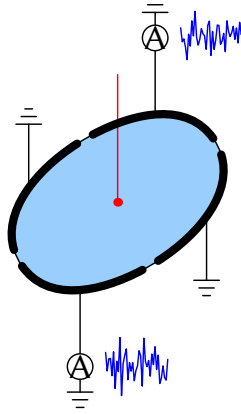


Figure 1.1. A two-dimensional conductive body is attached to the ground via electrodes which are used to measure the thermal noise currents resulting from heating the body at particular locations, e.g. the location depicted in red.

Our main contribution is to show that such thermal noise current measurements are equivalent to measuring the real power dissipated inside the conductive body, i.e.

$$(1.3) \quad \sigma'(x) |\nabla u(x)|^2,$$

where $\sigma'(x)$ is the real part of the conductivity and u solves an appropriate (deterministic) auxiliary problem which depends on electrodes configuration and the conductivity $\sigma(x)$ which can be complex (see section 3).

1.1. Related work. Recovering $\sigma'(x)$ from functionals of the form (1.3) is well-studied for the case of real σ in ultrasound modulated electrical impedance tomography or acousto-electric tomography [2], where the internal functional (1.3) is measured by locally perturbing a conductive body using ultrasound waves, while making electrical measurements on the body's surface. Various reconstruction approaches have since been studied for this problem [7, 10, 15, 25, 26, 34] as well as its well-posedness [4, 21, 27]. The problem of recovering an anisotropic real conductivity has also been studied [8, 31, 32, 33]. A similar problem using microwaves instead of ultrasound is discussed in [3]. Optical tomography can also be modulated by ultrasound, allowing measurements of a functional similar to (1.3), see [11, 37]. Hybrid inverse problems (including acousto-electric tomography) have been studied by formulating them as an overdetermined system of non-linear partial differential equations

and then studying their local uniqueness properties by linearizing, see [6, 27]. For reviews on hybrid inverse problems, see [1, 5].

The σ complex case is considered in [12], but this analysis only applies if the fields are known (as in elastography). Complex σ were also considered in the case of the Maxwell equations in [13]. However, to our knowledge, there is no study of the functional (1.3) where u depends on a complex σ , but only its real part σ' appears explicitly in the measurements.

1.2. Possible applications. The biggest challenge to the applicability of the method that we present here is that the thermally induced random currents are very small and this could introduce signal-to-noise issues. We envision two possible applications.

The first possible application would be to Atomic Force Microscopy (AFM). In this imaging modality, a height-map of a sample is obtained by measuring the deflections of a cantilever as its tip scans the sample. A heated cantilever tip can be used to heat the sample locally without touching it, see e.g. [23]. Moreover, electrical measurements of thermal noise induced currents can be done simultaneously with the AFM scan. An advantage of this approach is that one can measure height and conductivity of the sample without touching the sample, possibly making the cantilever tip last longer. We mention that conductivity variations can be measured in AFM by creating a voltage difference between the sample and the cantilever (assuming it is conductive). This method is known as Conductive Atomic Force Microscopy or CAFM, see e.g. the review in [29].

The second possible application is to monitoring of laser welding, see e.g. [18]. If two sheets of metal are been welded together, their temperature is raised significantly near the weld and the sheets also become electrically connected. We believe that by measuring thermal noise currents, one can monitor whether the weld was effective. We give an order of magnitude of the signals and background that would need to be measured in this situation in subsection 3.3.

1.3. Contents. We start in section 2 by deriving a quasi-static model from the Maxwell equations with a current source modeling the random currents. In section 3 we show how variances of the random currents are related to a deterministic problem. This is done for two different kinds of boundary conditions. Moreover we give rough magnitude estimates for the currents that would need to be measured to implement our approach. In section 4, we analytically and numerically analyze the linearized real problem (subsection 4.1) and linearized complex problem (subsection 4.2). This analysis is based on [6] where the ellipticity, in the Douglis-Nirenberg sense [19], is established for the real linearized problem with at least two distinct boundary conditions. We give a condition in Lemma 4.1 for the linearized problem with complex conductivity to be elliptic for at least three distinct boundary conditions. Still, it remains unclear if boundary conditions exist such that the fields associated with the auxiliary problem satisfy the condition of Lemma 4.1. Then in section 5 we present a simple numerical reconstruction approach based on a finite difference discretization of the problem (subsection 5.1). We solve the inverse problem using data that either comes directly from the internal functional (1.3) or from simulated realizations of random currents. In addition, we show reconstructions in the case that conductivity is real (subsection 5.2) or complex (subsection 5.3). Finally, we summarize our results in section 6.

2. The quasi-static model. In an isotropic medium, thermal fluctuations induce fluctuation of charge carriers near an equilibrium. For non-magnetic media, the thermal fluctuation currents can be modeled by a random external electric current j_e (A/m²) in the Maxwell equations [39], namely

$$(2.1) \quad \begin{aligned} \nabla \times H &= i\omega\varepsilon E + j_e \\ \nabla \times E &= -i\omega\mu H. \end{aligned}$$

Here E and H are the electric and magnetic fields and the angular frequency is ω . The convention for time harmonic fields here is that $\mathcal{E}(x, t) = \text{Re} [E(x, \omega) \exp[i\omega t]]$. The electric permittivity is ε and may be written as $\varepsilon = \varepsilon' - i\sigma'/\omega$, where $\varepsilon' \equiv \text{Re} \varepsilon$ and σ' is the real conductivity. The magnetic permeability μ is assumed real and equal to that of the vacuum. Note that if μ had an imaginary part (i.e. non-zero magnetic losses), then an analogous “random magnetic current” needs to be added to the Maxwell equations. The fluctuation dissipation theorem (see e.g. [40, Chapter 1] and the particular application to the Maxwell equations in [39]) states that the random current field j_e has zero mean $\langle j_e \rangle = 0$ and its power spectral density (at a fixed frequency) is

$$(2.2) \quad \langle j_e(x) j_e^*(x') \rangle = \frac{\kappa}{\pi} T(x) \sigma'(x) \delta(x - x') \mathbb{I},$$

where $T(x)$ is the temperature in Kelvin at a point x and $\mathbb{I}$ is the identity matrix. We emphasize that (2.2) depends only on the temperature and the real part of the conductivity. The real part of the electrical permittivity (which is associated with lossless behavior) does not directly appear in (2.2). In general (2.2) should use the energy of a quantum oscillator [39] instead of κT , namely

$$(2.3) \quad \Theta(T, \omega) = \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2\kappa T},$$

where $\hbar \approx 1.05 \times 10^{-34} \text{J} \cdot \text{s}$ is Planck’s constant. Here we assume we work with relatively small frequencies so that $\kappa T \gg \hbar\omega$ and we can make the approximation $\Theta(T, \omega) \approx \kappa T$, see also [28]. In particular, this approximation is valid at room temperature and frequencies of the order of 1kHz or 1MHz.

Instead of working with the Maxwell equations, we use a quasi-static approximation that is used in electrical impedance tomography, see e.g. [14, 16]. In this approximation, it is convenient to define the complex conductivity σ by

$$(2.4) \quad \sigma(x) = \sigma'(x) + i\omega\varepsilon'(x).$$

If we assume that $\omega\mu|\sigma|L^2 \ll 1$, where L is the characteristic length of the problem, then one can use the approximation $\nabla \times E \approx 0$. In other words, we may assume that the electric field comes from a potential $E = -\nabla\phi$. By taking divergence on both sides of the first equation in (2.1) we get

$$(2.5) \quad \nabla \cdot [\sigma \nabla \phi] = \nabla \cdot j_e.$$

Remark 1. As noted in [16], the quasi-static approximation holds for conductivities consistent with human tissues (see, e.g., [14]). For example if we take $L = 10$ cm, $\sigma' = 2$ cm⁻¹k Ω ⁻¹, $\omega = 2\pi 10$ kHz and $\varepsilon' = 1\mu\text{F}/\text{m}$, we get $\omega\mu|\sigma|L^2 \approx 1.7 \times 10^{-4} \ll 1$.

3. From the stochastic to the deterministic problem. Let Ω be a smooth simply connected open domain of $\mathbb{R}^3$ and let $\sigma \in C^1(\overline{\Omega})$ with its real part satisfying $\sigma' > c$ for some positive constant c . We assume a potential ϕ satisfies

$$(3.1) \quad \begin{aligned} \nabla \cdot [\sigma \nabla \phi] &= \nabla \cdot j_e, \text{ in } \Omega, \\ \phi &= 0, \text{ on } \partial\Omega. \end{aligned}$$

Here j_e is the random current term with $\langle j_e \rangle = 0$ and $\langle |j_e|^2 \rangle$ given in (2.2). We assume that j_e is $C^1(\overline{\Omega})$ and hence ϕ is $C^2(\overline{\Omega})$ [20, Ch. 6.3].

We assume we measure currents flowing out of the domain Ω at n “electrodes” by the complex vector with n entries

$$(3.2) \quad J = \int_{\partial\Omega} dS(x) \begin{bmatrix} e_1(x) \\ \vdots \\ e_n(x) \end{bmatrix} \sigma(x) \nabla \phi(x) \cdot \nu(x),$$

where $\nu(x)$ is the unit outward pointing normal to $\partial\Omega$ at some $x \in \partial\Omega$, and the function $e_i(x)$ are possibly complex $C^1(\partial\Omega)$ “electrode functions” defined on $\partial\Omega$. For example they could be a continuously differentiable approximation of the characteristic function of electrodes at the boundary. Note that a particular electrode function may be used to represent an experiment where currents are collected from a particular combination of distinct physical electrodes on $\partial\Omega$. Thus, n could also be thought of as a number of experiments.

In the following result we prove that the $n \times n$ covariance matrix¹ $\langle JJ^* \rangle$ of such measurements can be related to solutions to deterministic auxiliary problems.

Theorem 3.1. The covariance of the vector of measurements J are given by

$$(3.3) \quad [\langle JJ^* \rangle]_{ij} = \frac{\kappa}{\pi} \int_{\Omega} dy \operatorname{Re}(\sigma(y)) T(y) \nabla u_i(y) \cdot \nabla \overline{u_j(y)},$$

where the functions u_i are solutions to the Dirichlet problems

$$(3.4) \quad \begin{aligned} \nabla \cdot [\sigma \nabla u_i] &= 0, \text{ in } \Omega \\ u_i &= e_i, \text{ on } \partial\Omega, \quad i = 1, \dots, n. \end{aligned}$$

Proof. First, note that we can write the solution to (3.1) as

$$(3.5) \quad \phi(x) = \int_{\Omega} dy G(x, y) \nabla_y \cdot j_e(y),$$

where $G(x, y)$ is the Green function $G(x, y)$ satisfying the equation

$$(3.6) \quad \begin{aligned} \nabla_x \cdot [\sigma(x) \nabla_x G(x, y)] &= \delta(x - y), \quad x, y \in \Omega \\ G(x, y) &= 0, \quad x \in \partial\Omega \text{ or } y \in \partial\Omega. \end{aligned}$$

¹This is a slight abuse of terminology, as $\langle JJ^* \rangle$ is a cross-spectral power density.

158 To double check (3.5), it is clear that $\phi(x) = 0$ for $x \in \partial\Omega$ because $G(x, y) = 0$ for $x \in \partial\Omega$.
 159 Also:

$$\begin{aligned} \nabla_x \cdot [\sigma(x) \nabla_x \phi] &= \int_{\Omega} dy \nabla_x \cdot [\sigma(x) \nabla_x G(x, y)] \nabla_y \cdot j_e(y) \\ 160 \quad &= \int_{\Omega} dy \delta(x - y) \nabla_y \cdot j_e(y) \\ &= \nabla_x \cdot j_e(x). \end{aligned}$$

161 Now it is helpful to use integration by parts to get

$$\begin{aligned} \phi(x) &= \int_{\Omega} dy G(x, y) \nabla_y \cdot j_e(y) \\ 162 \quad (3.7) \quad &= \int_{\partial\Omega} dS(y) G(x, y) j_e(y) \cdot \nu(y) - \int_{\Omega} dy \nabla_y G(x, y) \cdot j_e(y) \\ &= - \int_{\Omega} dy \nabla_y G(x, y) \cdot j_e(y). \end{aligned}$$

163 The ij -th entry of the covariance $\langle JJ^* \rangle$ can be written from (3.2) as

$$164 \quad (3.8) \quad \langle JJ^* \rangle_{ij} = \left\langle \int_{\partial\Omega} dS(x) \int_{\partial\Omega} dS(x') e_i(x) \sigma(x) \nabla_x \phi(x) \cdot \nu(x) \overline{e_j(x') \sigma(x') \nabla_{x'} \phi(x')} \cdot \nu(x') \right\rangle.$$

165 By (3.7), ϕ is in turn given as a linear functional of the random currents $j_e(y)$, which introduces
 166 two integrals over Ω in the above expression. The linearity of the mean can be used to get
 167 the covariance of j_e , which by (2.2) is delta correlated in space. This reduces the number of
 168 integrals over Ω by one. More precisely, we obtain:

$$\begin{aligned} \langle JJ^* \rangle_{ij} &= \int_{\partial\Omega} dS(x) \int_{\partial\Omega} dS(x') \int_{\Omega} dy \frac{\kappa}{\pi} \text{Re}(\sigma(y)) T(y) e_i(x) \sigma(x) \overline{e_j(x') \sigma(x') \nu(x)^T} \\ &\quad \nabla_x \nabla_y G(x, y) \nabla_{x'} \nabla_y \overline{G(x', y)} \nu(x') \\ 169 \quad (3.9) \quad &= \frac{\kappa}{\pi} \int_{\Omega} dy \text{Re}(\sigma(y)) T(y) \left[\int_{\partial\Omega} dS(x) e_i(x) \sigma(x) \nabla_x \nabla_y G(x, y) \nu(x) \right]^T \\ &\quad \left[\int_{\partial\Omega} dS(x') e_j(x') \sigma(x') \nabla_{x'} \nabla_y \overline{G(x', y)} \nu(x') \right] \\ &= \frac{\kappa}{\pi} \int_{\Omega} dy \text{Re}(\sigma(y)) T(y) \nabla_y u_i(y) \cdot \nabla_y \overline{u_j(y)}, \end{aligned}$$

170 where u_i solves the problem (3.4) and we use that $\nabla_x \nabla_y G(x, y)$ is symmetric. The last equality

171 follows by doing integration by parts twice:

$$\begin{aligned}
 \int_{\partial\Omega} dS(x) e_i(x) \sigma(x) \nabla_x G(x, y) \cdot \nu(x) &= \int_{\Omega} dx \nabla_x \cdot [\sigma(x) \nabla_x G(x, y) u_i(x)] \\
 &= \int_{\Omega} dx \nabla_x \cdot [\sigma(x) \nabla_x G(x, y)] u_i(x) \\
 &\quad + \int_{\Omega} dx \sigma(x) \nabla_x G(x, y) \cdot \nabla_x u_i(x) \\
 172 \quad (3.10) \quad &= u_i(y) + \int_{\Omega} dx \nabla_x \cdot [\sigma(x) G(x, y) \nabla_x u_i(x)] \\
 &\quad - \int_{\Omega} dx G(x, y) \nabla_x \cdot [\sigma(x) \nabla_x u_i(x)] \\
 &= u_i(y) + \int_{\partial\Omega} dS(x) G(x, y) \sigma(x) \nabla_x u_i(x) \cdot \nu(x) \\
 &= u_i(y).
 \end{aligned}$$

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174 It may be possible to loosen the regularity assumptions on j_e and e_i and derive a similar result
 175 to [Theorem 3.1](#). Since the scope of this work is focused on establishing the relation between
 176 the stochastic and deterministic problems, we leave this for future work. While [\(3.3\)](#) gives
 177 the entire covariance matrix $\langle JJ^* \rangle$, we only need its diagonal entries.

178 **3.1. Boundary conditions modeling electrodes with insulating gaps.** The setup using
 179 Dirichlet boundary conditions [\(3.1\)](#) assumes that $\phi|_{\partial\Omega} = 0$, which would likely be hard to
 180 realize in practice because we expect to have a few electrodes connected to the ground with
 181 insulating gaps between them. This corresponds to a boundary condition of mixed type:
 182 homogeneous Dirichlet on the electrodes and homogeneous Neumann (zero flux) on the gaps
 183 between the electrodes. To be more precise, let $\Gamma = \text{supp } e_1 \cup \dots \cup \text{supp } e_n$ then we replace
 184 [\(3.1\)](#) with

$$\begin{aligned}
 \nabla \cdot [\sigma \nabla \phi] &= \nabla \cdot j_e, \text{ in } \Omega, \\
 185 \quad (3.11) \quad \sigma \nabla \phi \cdot \nu &= 0, \text{ on } \partial\Omega - \Gamma, \\
 \phi &= 0, \text{ on } \Gamma.
 \end{aligned}$$

186 Then [Theorem 3.1](#) holds in the same fashion, but we assume that the u_i are solutions to the
 187 following mixed boundary problem replacing [\(3.4\)](#) with

$$\begin{aligned}
 \nabla \cdot [\sigma \nabla u_i] &= 0, \text{ in } \Omega, \\
 188 \quad (3.12) \quad \sigma \nabla u_i \cdot \nu &= 0, \text{ on } \partial\Omega - \Gamma, \\
 u_i &= e_i, \text{ on } \Gamma.
 \end{aligned}$$

189 The proof follows by noting that integration by parts now yields

$$190 \quad \phi(x) = \int_{\partial\Omega - \Gamma} dS(y) G(x, y) j_e(y) \cdot \nu(y) - \int_{\Omega} dy \nabla_y G(x, y) \cdot j_e(y),$$

resulting in four terms when $\phi(x)$ is substituted in $\langle JJ^* \rangle_{ij}$. One of the terms is similar to the case of the Dirichlet boundary conditions, and all of the others contain integrals over the zero flux part of the boundary. By invoking the zero flux boundary conditions, these terms can be easily seen to disappear, leaving us with the same formula for $\langle JJ^* \rangle_{ij}$.

3.2. Differential temperature measurements. Utilizing [Theorem 3.1](#), we can now relate the differential temperature measurements as described in [section 1](#) to measurements of the internal functional [\(1.3\)](#). Concretely, we take a set of measurements of the covariance of the currents in a body at temperatures $T = T_0$ and $T = T_0 + \delta T(x)$, where $\delta T(x)$ is a prescribed heating pattern. Then the differential temperature measurements give

$$(3.13) \quad [\langle J_{T_0+\delta T} J_{T_0+\delta T}^* \rangle - \langle J_{T_0} J_{T_0}^* \rangle]_{ii} = \frac{\kappa}{\pi} \int_{\Omega} dx \delta T(x) \operatorname{Re}(\sigma(x)) |\nabla u_i(x)|^2,$$

considering only the diagonal elements of the covariance matrix. As previously noted, only measurements of the diagonal elements are used for our reproduction approach. By taking a sufficiently rich set of heating patterns we ideally get an estimate for the internal functional

$$(3.14) \quad H_{ii}(x) = \sigma'(x) |\nabla u_i(x)|^2, \text{ for } x \in \Omega, i \in \{1, \dots, n\}.$$

For real conductivities ($\sigma = \sigma'$), the internal functional [\(3.14\)](#) corresponds to the power dissipated inside the domain.

Remark 2. *As an example of a heating pattern localized about x' , we can take $\delta T(x; x') = c \exp[-|x - x'|^2/(2a)]$, where $c > 0$ and $a > 0$ is sufficiently small. In this case, we get from [\(3.13\)](#), the convolution of H_{ii} with a Gaussian, evaluated at x' . These are precisely the heating patterns we use in our numerical experiments (see [section 5](#)). Notice that such heating patterns $\delta T(x)$ are proportional to time snapshots of the heat equation Green function in a homogeneous medium. The heating patterns can be achieved by a spatially and temporally localized heating of Ω (e.g. with a laser beam). Of course, this is assuming that the electrical measurements occur at a much faster time-scale than heat propagation and sufficient time is left for the medium to cool down before moving to another location x' . The interplay between electrical measurements and the heat equation is left for future studies. Heating patterns need not be spatially localized and one may consider other patterns such as cosines and sines. Spatially extended patterns may be advantageous in terms of signal to noise ratio, but may not be practical to realize.*

3.3. Rough estimation of thermal noise induced currents. The thermal noise induced currents are very small and may limit the application of this approach. To get an idea of the magnitude of the signals that need to be measured to obtain H_{ii} in [\(3.13\)](#), we need to distinguish between current measurements with the background temperature T_0 and with perturbed temperature $T_0 + \delta T$. We make rough estimates of these currents in two situations: the first is consistent with the numerical experiments and the second one is consistent with laser welding.

Conductivities used in the numerical experiments. For the background temperature measurements, recall that Boltzmann's constant is on the order of $10^{-23} \text{ J} \cdot \text{K}^{-1}$. For our

numerical experiments we chose $\Delta z = 0.1$ cm and a domain with area 10 cm^2 . The conductivity is assumed to be $10^{-3} \text{ cm}^{-1} \Omega^{-1}$ and constant on a band of width $\Delta\omega/(2\pi) = 10 \text{ kHz}$ centered at $\omega/(2\pi) = 10 \text{ kHz}$. Then at temperature $T_0 = 300 \text{ K}$ and accounting for the $1/\pi$ factor, the variance of the random currents is on the order of 10^{-20} A^2 . To reach this estimate we assumed the squared gradient of the auxiliary fields is constant and equal to 10^{-2} cm^{-2} . For the differential measurements we may further assume a $\delta T = 10 \text{ K}$ on area of $(0.2)^2 \text{ cm}^2$. This gives a current variance of the order 10^{-25} A^2 and a signal to noise ratio of 10^{-5} .

Conductivities consistent with welding. The conductivity of gold is much higher than what we used in the numerical experiments and is on the order of $4.5 \times 10^7 \text{ m}^{-1} \Omega^{-1}$. For instance consider a sheet of gold of dimensions $1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ mm}$ and a bandwidth and central frequencies on the order of 100 Hz . For this choice of frequencies, the quasi-static approximation (section 2) is not well satisfied. Nevertheless, if the conductivity is assumed constant on this frequency band and $T_0 = 300 \text{ K}$, the variance of the random currents is on the order of 10^{-14} A^2 . For the differential measurements we may further assume a $\Delta T = 1300 \text{ K}$ (which is close to the melting point of gold) on an area of $(0.1)^2 \text{ mm}^2$. This gives a current variance of the order 10^{-17} A^2 and a signal to noise ratio of 10^{-3} .

3.4. The inverse problem for real conductivities. The inverse problem for a real conductivity $\sigma = \sigma'$ consists of the measurement equation (3.14) and the auxiliary problem (3.4). To be more precise, we seek to recover u_i and σ given H_{ii} and e_i from the real non-linear system of partial differential equations, for $i = 1, \dots, n$,

$$\begin{aligned} \nabla \cdot [\sigma \nabla u_i] &= 0, & x \in \Omega, \\ u_i - e_i &= 0, & x \in \partial\Omega, \\ H_{ii} - \sigma |\nabla u_i|^2 &= 0, & x \in \Omega. \end{aligned} \quad (3.15)$$

We call the model associated with measurements H_{ii} given by the expectation in Theorem 3.1 the *deterministic model* and the model associated with measurements given by realizations of randomly induced currents the *stochastic model*.

3.5. The inverse problem for complex conductivities. We write the complex problem by separating the real and complex parts of (3.14) and (3.4). We use a single prime (resp. double prime) to denote the real part (resp. imaginary part) of a complex quantity, e.g. $\sigma = \sigma' + i\sigma''$, $u_j = u'_j + iu''_j$, and $e_j = e'_j + ie''_j$. Then the problem is to find σ' , σ'' , u'_j and u''_j given H_{jj} , e'_j , and e''_j from the non-linear system of partial differential equations, for $j = 1, \dots, n$,

$$\begin{aligned} \nabla \cdot [\sigma' \nabla u'_j] - \nabla \cdot [\sigma'' \nabla u''_j] &= 0, & x \in \Omega, \\ \nabla \cdot [\sigma' \nabla u''_j] + \nabla \cdot [\sigma'' \nabla u'_j] &= 0, & x \in \Omega, \\ u'_j - e'_j &= 0, & x \in \partial\Omega, \\ u''_j - e''_j &= 0, & x \in \partial\Omega, \\ H_{jj} - \sigma' (|\nabla u'_j|^2 + |\nabla u''_j|^2) &= 0, & x \in \Omega. \end{aligned} \quad (3.16)$$

An equivalent formulation of (3.16) can be found using the conjugates of u_j and the e_j instead of their real and imaginary components separately. Both the system (3.15) and (3.16) can be modified to instead use the experimental boundary conditions (3.12).

Remark 3. The non-linear system of equations that would be obtained by allowing the conductivity to be complex in ultrasound modulated EIT (see e.g. [5]) is similar to (3.16) with two real measurement equations per boundary condition instead of a single one, i.e. for $x \in \Omega$:

$$\begin{aligned} H'_{jj} - \sigma'(|\nabla u'_j|^2 + |\nabla u''_j|^2) &= 0, \text{ and} \\ H''_{jj} - \sigma''(|\nabla u'_j|^2 + |\nabla u''_j|^2) &= 0. \end{aligned}$$

We did not consider this problem because the form of the measurements we consider (3.16) is a direct result of using thermal induced random currents, see Theorem 3.1.

4. Linearized problem. Before attempting to reconstruct conductivities numerically, we analyze the linearizations of the real (3.15) and complex (3.16) conductivity problems. Our goal is to find sufficient conditions for injectivity of the linearized problems, or in other words, if they admit a unique solution. Our analysis is based on [6], which includes a proof that the linearized real conductivity problem is elliptic in the sense of Douglis-Nirenberg under certain boundary conditions [19]. This was established in [27] for ultrasound modulated EIT and generalized to other hybrid inverse problem in [6].

The linearization of the real conductivity problem (3.15) around the solution (u_i, σ) in the variables $(\delta u_i, \delta \sigma)$ for $i = 1, \dots, n$ is given by

$$\begin{aligned} \nabla \cdot [\sigma \nabla \delta u_i] + \nabla \cdot [\delta \sigma \nabla u_i] &= 0, \quad x \in \Omega, \\ \delta u_i &= 0, \quad x \in \partial\Omega, \\ \delta H_{ii} - \delta \sigma |\nabla u_i|^2 - 2\sigma \nabla \delta u_i \cdot \nabla u_i &= 0, \quad x \in \Omega. \end{aligned} \tag{4.1}$$

The linearization of the complex conductivity problem (3.16) around the solution $(u'_j, u''_j, \sigma', \sigma'')$ in the variables $(\delta u'_j, \delta u''_j, \delta \sigma', \delta \sigma'')$ for $j = 1, \dots, n$ is given by

$$\begin{aligned} \nabla \cdot [\sigma' \nabla \delta u'_j] + \nabla \cdot [\delta \sigma' \nabla u'_j] - \nabla \cdot [\sigma'' \nabla \delta u''_j] - \nabla \cdot [\delta \sigma'' \nabla u''_j] &= 0, \quad x \in \Omega, \\ \nabla \cdot [\sigma' \nabla \delta u''_j] + \nabla \cdot [\delta \sigma' \nabla u''_j] + \nabla \cdot [\sigma'' \nabla \delta u'_j] + \nabla \cdot [\delta \sigma'' \nabla u'_j] &= 0, \quad x \in \Omega, \\ \delta u'_j &= 0, \quad x \in \partial\Omega, \\ \delta u''_j &= 0, \quad x \in \partial\Omega, \\ \delta H_{jj} - \delta \sigma'(|\nabla u'_j|^2 + |\nabla u''_j|^2) - 2\sigma' \nabla \delta u'_j \cdot \nabla u'_j - 2\sigma' \nabla \delta u''_j \cdot \nabla u'_j &= 0, \quad x \in \Omega. \end{aligned} \tag{4.2}$$

In [6], it is established (4.1) is elliptic using two boundary conditions if the gradients of the associated fields are nowhere orthogonal or parallel. We do not attempt to analyze how this condition might be satisfied in the case of the mixed boundary conditions (3.12). We note that this establishes that (4.1) is not elliptic in the case of one experiment with mixed boundary conditions. Instead, we attempt to analyze the problem numerically by estimating the conditioning of the symbol of the linearized problem. For the case of complex conductivity (4.2), we give a sufficient condition in Lemma 4.1 for ellipticity; however, we do not give boundary conditions that guarantee this is satisfied, nor do we prove that such boundary conditions exist. We note for elliptic linear systems, it is possible to obtain stability estimates by augmenting the system with boundary conditions satisfying the Lopatinskii condition, following [6].

To establish if (4.1) and (4.2) are elliptic, we first compute the principal symbol of their associated matrix-valued differential operators $\mathcal{A}(x, D)$ for $x \in \Omega$, where $D = (\partial_{x_1}, \dots, \partial_{x_d})$, and d is the dimension ($d = 2$ in our case). Since these are linearized systems, the entries $\mathcal{A}_{ij}(x, D)$ are polynomials in D for each $x \in \Omega$. We associate each row of $\mathcal{A}$ with an integer s_i and each column with an integer t_j , chosen such that the maximum degree of each polynomial $\mathcal{A}_{ij}(x, D)$ is $s_i + t_j$. The principal component $\mathcal{A}_0(x, D)$ is obtained from $\mathcal{A}(x, D)$ by keeping only the terms in $\mathcal{A}_{ij}(x, D)$ with order exactly $s_i + t_j$. If the principal symbol $\mathcal{A}_0(x, \xi)$ is injective for all $\xi \neq 0$, then the problem is elliptic in the Douglis-Nirenberg sense at $x \in \Omega$.

Remark 4. *Although the results in this section are derived for Dirichlet boundary conditions in the auxiliary problem (3.4), the same analysis can be carried out with minor modifications for mixed type boundary conditions corresponding to electrodes with zero-flux gaps (3.12).*

4.1. Injectivity of the linearized real problem. Letting $F_i = \nabla u_i$ the principal symbol of the real problem (4.1) is the $2n \times (n + 1)$ matrix

$$(4.3) \quad \mathcal{A}_0(x, \xi) = \begin{bmatrix} |F_1|^2 & 2\sigma F_1 \cdot \imath \xi & \cdots & 0 \\ F_1 \cdot \imath \xi & -\sigma |\xi|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ |F_n|^2 & 0 & \cdots & 2\sigma F_n \cdot \imath \xi \\ F_n \cdot \imath \xi & 0 & \cdots & -\sigma |\xi|^2 \end{bmatrix},$$

for $i = 1, \dots, n$ and where $\sigma = \sigma'$. The system is in Douglis-Nirenberg form where the row weights s_i are given by the $2n$ vector $(0, 1, 0, 1, \dots, 0, 1)$ and the column weights t_j are given by the $n + 1$ vector $(0, 1, 1, \dots, 1)$. As noted previously, this symbol is shown to be injective in two dimensions using two boundary conditions such that F_1 and F_2 are nowhere orthogonal, or parallel [6].

We consider the discretized problem on the square $[0, 10]^2$ using a uniform 200×200 grid. The conductivity used can be seen in Figure 5.1 (a). We numerically solve (3.12) to calculate the fields u_i for $i = 1, \dots, n$. The Dirichlet boundary conditions are defined on the set

$$(4.4) \quad \Gamma = ([0, 4.5] \cup [5.5, 10]) \times \{0, 10\} \cup (\{0, 10\} \times ([0, 4.5] \cup [5.5, 10])).$$

For $x \in \Gamma$ the boundary conditions are of the form

$$(4.5) \quad \begin{aligned} g_m &= 5 \sin(\theta) \left(\frac{r}{10} \right)^m, \\ h_m &= 5 \cos(\theta) \left(\frac{r}{10} \right)^m, \end{aligned}$$

where (r, θ) is the polar representation of the nodes in Γ . We enforce the no flux boundary condition on the gaps, i.e. $x \in \partial\Omega - \Gamma$.

To establish the ellipticity of the operator, we need to show its principal symbol is injective for all $\xi \neq 0$. As a numerical indication of ellipticity, we check injectivity numerically for $\xi \in \Xi$, where Ξ is a set of 100 vectors uniformly spaced on the unit circle (since ξ is two-dimensional in our simulations). At each point x in the grid we use to discretize Ω , we compute the

323 maximum condition number of the symbol along directions $\xi \in \Xi$, i.e.

324 (4.6)
$$\max_{\xi \in \Xi} \left[\frac{\sigma_{\max}(\mathcal{A}_0(x, \xi))}{\sigma_{\min}(\mathcal{A}_0(x, \xi))} \right],$$

325 where $\sigma_{\min}(A)$ (resp. $\sigma_{\max}(A)$) is the smallest (resp. largest) singular value of a matrix A .
 326 Notice that although $\mathcal{A}_0(x, \xi)$ depends quadratically on ξ , its condition number depends non-
 327 linearly on ξ , so we resort to a brute force approach. Of course, this strategy is a heuristic
 328 that does not replace checking for every $\xi \neq 0$ but that still seems to reveal lack of ellipticity
 329 of the symbol. The maximum condition number (4.6) of the symbol (4.3) can be seen in
 330 Figure 4.1. These numerical results are in line with the previously established theory: the
 331 maximum condition number is higher under one experiment than under two experiments,
 332 indicating that the problem with one experiment is worse than the one for two experiments.
 333 Indeed, the problem with one experiment is hyperbolic and may be solved locally (see e.g.
 334 [6, 7, 34]). Numerically, reconstructions can still be obtained with one experiment, albeit with
 335 severe artifacts.

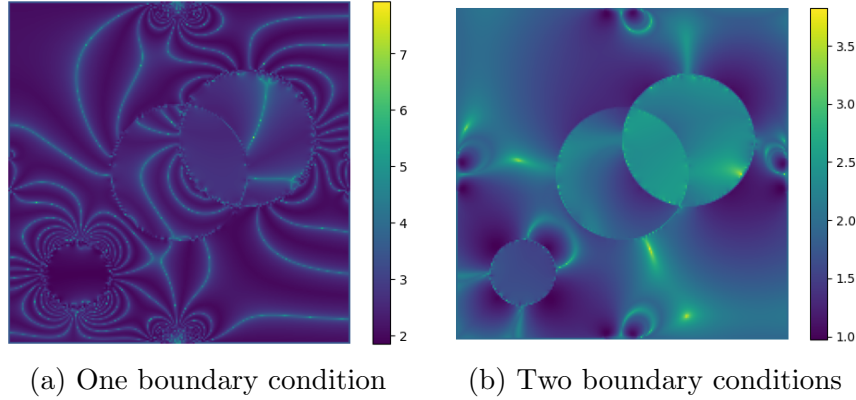


Figure 4.1. Maximum condition number (4.6) of the symbol of the linearized problem (4.1) on the square domain $[0, 10]^2$ with log 10 scaling. The left image (a) is the conditioning of the symbol with one boundary condition given by g_1 in (4.5). The right image (b) is the conditioning of the symbol with two boundary conditions given by g_1 and h_1 in (4.5).

336 **4.2. Injectivity of the linearized complex problem.** Letting $F'_j = \nabla u'_j$, $F''_j = \nabla u''_j$, and
 337 $F_j = F'_j + \imath F''_j$, the symbol for the complex linear system (4.2) is the $3n \times (2 + 2n)$ matrix

338 (4.7)
$$\tilde{\mathcal{A}}_0(x, \xi) = \begin{bmatrix} |F_1|^2 & 0 & 2\sigma' F'_1 \cdot \imath \xi & 2\sigma' F''_1 \cdot \imath \xi & \cdots & 0 & 0 \\ F'_1 \cdot \imath \xi & -F''_1 \cdot \imath \xi & -\sigma' |\xi|^2 & \sigma'' |\xi|^2 & \cdots & 0 & 0 \\ F''_1 \cdot \imath \xi & F'_1 \cdot \imath \xi & -\sigma'' |\xi|^2 & -\sigma' |\xi|^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ |F_n|^2 & 0 & \cdots & 0 & \cdots & 2\sigma' F'_n \cdot \imath \xi & 2\sigma' F''_n \cdot \imath \xi \\ F'_n \cdot \imath \xi & -F''_n \cdot \imath \xi & \cdots & 0 & \cdots & -\sigma' |\xi|^2 & \sigma'' |\xi|^2 \\ F''_n \cdot \imath \xi & F'_n \cdot \imath \xi & \cdots & 0 & \cdots & -\sigma'' |\xi|^2 & -\sigma' |\xi|^2 \end{bmatrix},$$

for $j = 1, \dots, n$. The system is in Douglas-Nirenberg form where the row weights s_i are given by the $3n$ vector $(0, 1, 1, 0, 1, 1, \dots, 0, 1, 1)$ and the column weights t_j are given by the $2 + 2n$ vector $(0, 0, 1, 1, \dots, 1, 1)$.

Lemma 4.1. Assume $|F_i| \geq c > 0$ and let $\hat{F}_i = F_i/|F_i|$, for $i = 1, \dots, n$. Assume also that $\sigma'(x), \sigma''(x) > 0$. Then the symbol $\tilde{\mathcal{A}}_0(x, \xi)$ of the system (4.2) is injective at x if there are three distinct indices i_1, i_2, i_3 such that

$$(4.8) \quad |\hat{F}_{i_1} \cdot \xi|^2 = |\hat{F}_{i_2} \cdot \xi|^2 = |\hat{F}_{i_3} \cdot \xi|^2 \text{ implies } \xi = 0.$$

In particular, the symbol is not injective for $n = 1$ or $n = 2$.

Proof. The symbol matrix when $n = 1$ is of size 3×4 and thus cannot be injective. If $n = 2$, the symbol matrix is square of size 6×6 and we write it such that the measurement equations are the first two rows

$$\begin{bmatrix} |F_1|^2 & 0 & 2\sigma' F_1' \cdot \imath \xi & 2\sigma' F_1'' \cdot \imath \xi & 0 & 0 \\ |F_2|^2 & 0 & 0 & 0 & 2\sigma' F_2' \cdot \imath \xi & 2\sigma' F_2'' \cdot \imath \xi \\ F_1' \cdot \imath \xi & -F_1'' \cdot \imath \xi & -\sigma' |\xi|^2 & \sigma'' |\xi|^2 & 0 & 0 \\ F_1'' \cdot \imath \xi & F_1' \cdot \imath \xi & -\sigma'' |\xi|^2 & -\sigma' |\xi|^2 & 0 & 0 \\ F_2' \cdot \imath \xi & -F_2'' \cdot \imath \xi & 0 & 0 & -\sigma' |\xi|^2 & \sigma'' |\xi|^2 \\ F_2'' \cdot \imath \xi & F_2' \cdot \imath \xi & 0 & 0 & -\sigma'' |\xi|^2 & -\sigma' |\xi|^2 \end{bmatrix}.$$

We consider this as a block matrix with the top left block being 2×2 , and then the bottom right block being 4×4 . The bottom right matrix is block diagonal with invertible diagonal 2×2 blocks, and we use this to compute the Schur complement

$$\begin{bmatrix} |F_1|^2 + \frac{2(\sigma')^2}{(|\xi|\sigma')^2 + (|\xi|\sigma'')^2} ((F_1' \cdot \imath \xi)^2 + (F_1'' \cdot \imath \xi)^2) & \frac{2\sigma''\sigma'}{(|\xi|\sigma')^2 + (|\xi|\sigma'')^2} ((F_1' \cdot \imath \xi)^2 + (F_1'' \cdot \imath \xi)^2) \\ |F_2|^2 + \frac{2(\sigma')^2}{(|\xi|\sigma')^2 + (|\xi|\sigma'')^2} ((F_2' \cdot \imath \xi)^2 + (F_2'' \cdot \imath \xi)^2) & \frac{2\sigma''\sigma'}{(|\xi|\sigma')^2 + (|\xi|\sigma'')^2} ((F_2' \cdot \imath \xi)^2 + (F_2'' \cdot \imath \xi)^2) \end{bmatrix}.$$

The determinant of the Schur complement is then

$$(4.9) \quad \frac{2\sigma''\sigma'}{(\sigma')^2 + (\sigma'')^2} \left(\frac{|F_1|^2}{|\xi|^2} [(F_2' \cdot \imath \xi)^2 + (F_2'' \cdot \imath \xi)^2] - \frac{|F_2|^2}{|\xi|^2} [(F_1' \cdot \imath \xi)^2 + (F_1'' \cdot \imath \xi)^2] \right),$$

which gives

$$(4.10) \quad \det \tilde{\mathcal{A}}_0(x, \xi) = 2\sigma'\sigma'' |\xi|^6 |\sigma|^2 (|F_1|^2 |F_2 \cdot \xi|^2 - |F_2|^2 |F_1 \cdot \xi|^2).$$

By the assumptions on σ', σ'', F_1 and F_2 , $\det \tilde{\mathcal{A}}_0(x, \xi) = 0$ is equivalent to $|\hat{F}_1 \cdot \xi|^2 = |\hat{F}_2 \cdot \xi|^2$. Thus, we now focus on finding the null cone of the (real) quadratic form $q(\xi) = |\hat{F}_1 \cdot \xi|^2 - |\hat{F}_2 \cdot \xi|^2 = \xi^T Q \xi$, where the symmetric matrix Q is given by

$$(4.11) \quad Q = \hat{F}_1 \hat{F}_1'^T + \hat{F}_1'' \hat{F}_1''^T - \hat{F}_2 \hat{F}_2'^T - \hat{F}_2'' \hat{F}_2''^T.$$

Notice that $\text{trace } Q = |\hat{F}_1|^2 - |\hat{F}_2|^2 = 0$. Hence, either Q is identically zero or it has one positive and one negative eigenvalue. Since the spectrum of Q is contained in $\{q(\xi) \mid |\xi| = 1\}$, there must be a unit length ξ such that $q(\xi) = 0$ and thus the symbol is not injective.

For $n \geq 3$, the symbol matrix has more rows than columns, thus if it is rank deficient, the determinants of all its largest square sub-matrices must be zero. We first compute the determinant of one such $(2 + 2n) \times (2 + 2n)$ matrix, which is associated with the first two experiments and where the rows associated with measurements for experiments $i \geq 3$ are deleted, i.e.

$$\begin{bmatrix}
 |F_1|^2 & 0 & 2\sigma' F_1' \cdot \iota\xi & 2\sigma' F_1'' \cdot \iota\xi & 0 & 0 & \dots & 0 & 0 \\
 |F_2|^2 & 0 & 0 & 0 & 2\sigma' F_2' \cdot \iota\xi & 2\sigma' F_2'' \cdot \iota\xi & \dots & 0 & 0 \\
 F_1' \cdot \iota\xi & -F_1'' \cdot \iota\xi & -\sigma' |\xi|^2 & \sigma'' |\xi|^2 & 0 & 0 & \dots & 0 & 0 \\
 F_1'' \cdot \iota\xi & F_1' \cdot \iota\xi & -\sigma'' |\xi|^2 & -\sigma' |\xi|^2 & 0 & 0 & \dots & 0 & 0 \\
 F_2' \cdot \iota\xi & -F_2'' \cdot \iota\xi & 0 & 0 & -\sigma' |\xi|^2 & \sigma'' |\xi|^2 & \dots & 0 & 0 \\
 F_2'' \cdot \iota\xi & F_2' \cdot \iota\xi & 0 & 0 & -\sigma'' |\xi|^2 & -\sigma' |\xi|^2 & \dots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
 F_n' \cdot \iota\xi & -F_n'' \cdot \iota\xi & 0 & 0 & 0 & 0 & \dots & -\sigma' |\xi|^2 & \sigma'' |\xi|^2 \\
 F_n'' \cdot \iota\xi & F_n' \cdot \iota\xi & 0 & 0 & 0 & 0 & \dots & -\sigma'' |\xi|^2 & -\sigma' |\xi|^2
 \end{bmatrix}.$$

We consider the top left 2×2 matrix as a block, and the bottom right $2n \times 2n$ matrix as a block. The bottom right matrix is still block diagonal with invertible 2×2 blocks. The determinant of the Schur complement is given by (4.9), and the determinant of the sub-matrix is given by

$$(4.12) \quad 2\sigma'' \sigma' |\xi|^{4n-2} |\sigma|^{2(n-1)} (|F_1|^2 |F_2 \cdot \xi|^2 - |F_2|^2 |F_1 \cdot \xi|^2).$$

We can generalize (4.12) to any two distinct indices i and j by considering the $(2n+2) \times (2n+2)$ sub-matrix obtained from $\tilde{\mathcal{A}}_0(x, \xi)$ by deleting all the rows corresponding to measurements for experiments other than i or j . This sub-matrix has determinant:

$$(4.13) \quad 2\sigma'' \sigma' |\xi|^{4n-2} |\sigma|^{2(n-1)} (|F_i|^2 |F_j \cdot \xi|^2 - |F_j|^2 |F_i \cdot \xi|^2).$$

If $\tilde{\mathcal{A}}_0(x, \xi)$ is rank deficient at some x , all the determinants (4.13) must vanish for any two distinct indices i and j . By the assumptions on σ' , σ'' and the F_i , this is equivalent to the following quadratic forms vanishing for any two distinct indices i and j :

$$(4.14) \quad q_{i,j}(\xi) = |\hat{F}_i \cdot \xi|^2 - |\hat{F}_j \cdot \xi|^2.$$

Since we assumed (4.8) holds, we conclude that $\xi = 0$ and that the symbol must be injective for $n \geq 3$. ■

Remark 5. We do not present a method for finding boundary conditions such that the condition in Lemma 4.1 is satisfied, nor do we know if such boundary conditions exist for all possible $\sigma'(x), \sigma''(x) > 0$. For a concrete example of three unit length vectors satisfying (4.8), take $\hat{F}_1 = e_1$, $\hat{F}_2 = e_2$ and $\hat{F}_3 = (e_1 + e_2)/\sqrt{2}$. In this particular case $q_{1,2}(\xi) = |\xi_1|^2 - |\xi_2|^2$ and $q_{1,3}(\xi) = |\xi_1|^2 - |\xi_1 + \xi_2|^2/2$. The null cone of $q_{1,2}$ is $\text{span}\{e_1 + e_2\}$ and that of $q_{1,3}$ is $\text{span}\{(1 + \sqrt{2})e_1 + e_2\}$, and clearly their intersection is $\{0\}$.

Numerically, we have found that the complex reconstruction is challenging under many combinations of boundary conditions. This can be expected from numerically computing the

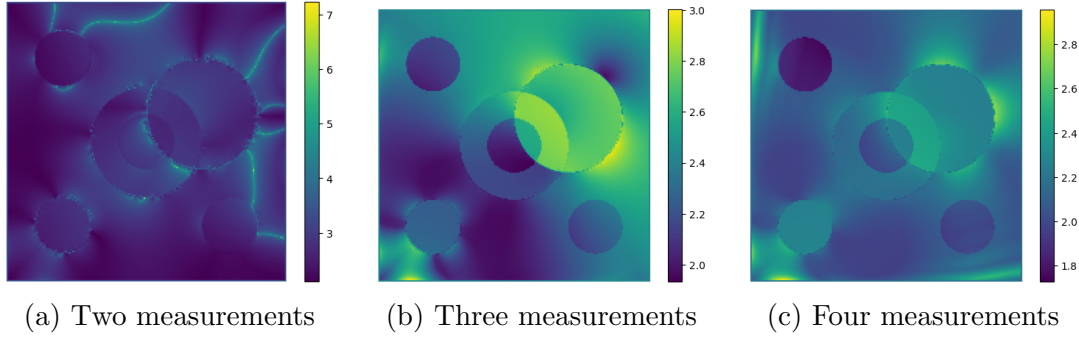


Figure 4.2. The maximum condition number (4.6) of the symbol of the complex linearized problem (4.2) on the discretized square domain with log10 scaling. The images from left to right are given by two, three, and four boundary conditions in the form of (4.15).

maximum condition number (4.6) as we illustrate in the numerical experiment appearing in Figure 4.2 and that we describe next.

The ground truth conductivity can be seen in Figure 5.3 with the real part in (a) and the imaginary part in (b). We only consider the linearized complex problem (4.2) with Dirichlet boundary conditions. The boundary conditions are of the form

$$(4.15) \quad \begin{aligned} \tilde{g}_m &= g_m + \frac{i}{2} h_m, \\ \tilde{h}_m &= h_m + \frac{i}{2} g_m, \end{aligned}$$

with h and g defined in (4.5). The scaling of the imaginary part by $1/2$ is to match the imaginary part of the background conductivity. In Figure 4.2, we can see the numerical condition of the symbol for two, three, and four boundary conditions. We begin this experiment with $n = 2$ since, with one measurement, the system is underdetermined. The boundary conditions for $n = 2$ are $\tilde{g}_1, \tilde{h}_1$, for $n = 3$ are $\tilde{g}_1, \tilde{g}_2, \tilde{h}_1$, and for $n = 4$ are $\tilde{g}_1, \tilde{g}_2, \tilde{h}_1, \tilde{h}_2$. We use the same domain and grid as in the real case.

The maximum condition number (4.6) improves significantly by moving from two to three measurements, but the improvement from three to four is modest. The high condition number for two boundary conditions (Figure 4.2 (a)) is consistent with Lemma 4.1, where two boundary conditions are insufficient for ellipticity. With more boundary conditions (Figure 4.2 (b) and (c)), the areas where the conditioning is high match up with the reconstruction artifacts for the complex case (Figure 5.3).

5. Numerical reconstructions. The following numerical reconstructions use values consistent with the quasi-static approximation, i.e., values such that $\omega\mu|\sigma|L^2 \ll 1$. In particular we let $L = 10$ cm and $\sigma' \in [1/3, 2]$ cm⁻¹ kΩ⁻¹. In the case of non-zero complex conductivity we let $\omega/(2\pi) = 10$ kHz and $\sigma'' = \omega\varepsilon' \in [1/2, 1]$ cm⁻¹ kΩ⁻¹. Our choice of parameters is near those in human tissues and satisfies the quasi-static approximation, see e.g. [14]. The examples we consider assume a thin plate that is homogeneous in the z direction with thickness $\Delta z = 0.1$ cm. If we consider $\Omega \subset \mathbb{R}^2$, then multiplying the measurements by Δz corresponds to the results in section 3.

5.1. Discrete model. We discretize the system (3.15) on a square domain $\Omega = [0, 10]^2$ using a uniform grid with n^2 nodes. We denote by N the set of nodes indexed with their integer coordinates (i_1, i_2) and the set of edges by $E \subset N \times N$. The nodes are partitioned into interior nodes I and boundary nodes B , which are the nodes that are on the boundary $\partial\Omega$. We use the forward difference operator $D \in \mathbb{R}^{|N| \times |E|}$ defined such that

$$D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

Here D_1 (resp. D_2) is the horizontal (resp. vertical) first order difference operator. Given a function ψ defined on the nodes N , the horizontal and vertical difference operators are defined by

$$(D_1\psi)(i_1, i_2) = \frac{\psi(i_1 + 1, i_2) - \psi(i_1, i_2)}{\Delta x_1},$$

$$(D_2\psi)(i_1, i_2) = \frac{\psi(i_1, i_2 + 1) - \psi(i_1, i_2)}{\Delta x_2},$$

where Δx_1 and Δx_2 are the horizontal and vertical discretization steps respectively.

If we use finite differences to discretize (3.15), we note that the gradient components in the x_1 and x_2 directions are defined on horizontal and vertical edges. Thus the norm of the discretized gradient is not defined at any particular edge. To obtain the gradient at a single spatial location in the discretized problem, we interpolate the gradient approximated values from their respective edges to the nodes and compute the gradient norm at the nodes. Thus it also makes sense to interpret the internal functional H_{ii} as a nodal based quantity and to completely determine the conductivity by interpolating a node based quantity. These interpolations between edges and nodes are achieved with the following matrices

$$\begin{aligned} N_1 &: \text{horizontal edges} \rightarrow \text{nodes}, \\ N_2 &: \text{vertical edges} \rightarrow \text{nodes}, \\ E_1 &: \text{nodes} \rightarrow \text{horizontal edges}, \\ E_2 &: \text{nodes} \rightarrow \text{vertical edges}, \\ E_{1,2} &: \text{nodes} \rightarrow \text{all edges}. \end{aligned}$$

To define these matrices we let φ be the matrix-valued function

$$\varphi(A) = (\text{Diag}(|A^T| \mathbf{1}))^{-1} |A^T|,$$

where $|\cdot|$ is the entry-wise absolute value, $\mathbf{1}$ is an appropriately sized vector of ones, and $\text{Diag}(v)$ denotes the matrix with the vector v on its diagonal. The matrix $\varphi(A)$ preserves constant vectors, more precisely, if c is an appropriate sized constant vector $\varphi(A)c = c$. The interpolation operators are then defined as

$$N_1 = \varphi(D_1^T), \quad N_2 = \varphi(D_2^T), \quad E_1 = \varphi(D_1), \quad E_2 = \varphi(D_2), \quad E_{1,2} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

Given H_{ii} (defined at the nodes) and e_i (defined at the boundary nodes) the discrete inverse problem for real conductivity is then to find s and u_i (defined at the nodes) such that for $i = 1, \dots, n$,

$$\begin{aligned} (5.1) \quad & D^T[E_{1,2}s \odot (Du_i)]_I = 0, \\ & u_i|_B - e_i = 0, \\ & H_{ii} - [N_1(E_1s \odot |D_1u_i|^2) + N_2(E_2s \odot |D_2u_i|^2)]_I = 0, \end{aligned}$$

where $\odot$ is the Hadamard or componentwise product. We note that in the first equation of (5.1), we have a graph Laplacian with edge weights given by $E_{1,2}s$, see e.g. [17]. This system is modified slightly under the assumption that the conductivity is known in a small neighborhood of the boundary. The modified system is solved using Gauss-Newton iteration. However, the interpolation process introduces a null space into the Jacobian. We use Tikhonov regularization with a parameter γ to prevent this null space from interfering when solving for the Gauss-Newton step. The parameter corresponds to adding a penalty term of $\gamma\|w\|^2$, when solving the least squares problem for finding a Gauss-Newton step w . An Armijo line search is used as globalization strategy (see e.g. [35]).

Given H_{jj} and e_j , the complex inverse problem is to recover s' , s'' , u'_j , and u''_j for $j = 1, \dots, n$,

$$\begin{aligned} (5.2) \quad & D^T[E_{1,2}s' \odot (Du'_j)]_I - D^T[E_{1,2}s'' \odot (Du''_j)]_I = 0, \\ & D^T[E_{1,2}s' \odot (Du''_j)]_I + D^T[E_{1,2}s'' \odot (Du'_j)]_I = 0, \\ & u'_j|_B - e'_j = 0, \\ & u''_j|_B - e''_j = 0, \\ & H_{jj} - [N_1(E_1s' \odot |D_1u'_j|^2) + N_2(E_2s' \odot |D_2u'_j|^2)]_I \\ & \quad + [N_1(E_1s' \odot |D_1u''_j|^2) + N_2(E_2s' \odot |D_2u''_j|^2)]_I = 0. \end{aligned}$$

A similar Gauss-Newton procedure was used to solve (5.2).

Heating Patterns: Recall that to obtain the measurements H_{ii} ; we need to locally heat a region of the conducting plate. This requires numerically approximating (3.13) at both the background temperature T_0 and when the plate is locally heated according to $\delta T(x)$. For our measurements we let $\delta T(x)$ be the Gaussian heating pattern,

$$g(x, a) = (2\pi a)^{-1} \exp(-|x|^2(2a)^{-1})$$

where $|\cdot|$ denotes the 2-norm. The heating pattern $g(x, a)$ can be considered an approximate Dirac since it integrates in x to one. This is also similar to the heating pattern from a laser covering an area of roughly πa or it could also be interpreted as a time snapshot of a Green function for the heat equation in a homogeneous medium.

Deterministic simulations: Continuous measurements from the deterministic model using this heating pattern can be written as

$$(5.3) \quad H_{ii}(x) \approx \langle \sigma |\nabla u_i|^2, T_0 + g(\cdot - x, a) \rangle_{L_2(\Omega)} - \langle \sigma |\nabla u_i|^2, T_0 \rangle_{L_2(\Omega)}.$$

We approximate $H_{ii}(x)$ by evaluating the heating pattern at each node x in the discrete model. Then to approximate the inner products, we use a uniform fine grid with $\tilde{n}^2$ nodes such that $\tilde{n} > n$. The number of fine grid nodes $\tilde{n}$ is chosen such that there are at least four fine grid nodes per effective area of the heating pattern, i.e. πa . Owing to (3.13), the right hand side of (5.3) is the convolution of H_{ii} with a Gaussian kernel.

Stochastic simulations: Ideally we would like to simulate acquiring the data H_{ii} through an empirical measurement of the power spectral density (1.2) of the currents J (see (3.2)) that leak to the ground, resulting from stochastic source term j_e in (3.1), which has a power spectral density (2.2) dictated by the fluctuation dissipation theorem. This could be achieved by generating a realization of j_e over a long time interval $[-T/2, T/2]$ and then calculating the time average of $|J_i|^2$ as in (1.2). The procedure is to be repeated for each heating pattern to approximate H_{ii} through differential temperature measurements (3.13). Since this could be computationally expensive, we used instead the ergodicity of j_e to calculate many realizations of a discretization of j_e satisfying (2.2).

We discretize a realization of j_e with one value per edge of the fine grid. For each edge $\mathbf{e}$, we take $j_e(\mathbf{e})$ sampled from a mean zero random normal distribution with variance $\kappa(T_0 + g(x, a))s(\mathbf{e})/\pi$ (heated) or $\kappa T_0 s(\mathbf{e})/\pi$ (unheated), where $s(\mathbf{e})$ is the conductivity of edge $\mathbf{e}$. These variances are determined by the fluctuation dissipation theorem as in (2.2) and depend linearly on the temperature and on the conductivity. The measurements from the simulated random current model are then given by approximating

$$(5.4) \quad H_{ii}(x) \approx \left\langle \langle \sigma |\nabla u_i|^2, T_0 + g(\cdot - x, a) \rangle_{L_2(\Omega)} \right\rangle - \left\langle \langle \sigma |\nabla u_i|^2, T_0 \rangle_{L_2(\Omega)} \right\rangle,$$

where the outer angular brackets denote ensemble averaging. The average is approximated empirically with M realizations of random currents where the inner product for each realization is approximated using a uniform fine grid. The realizations of the background temperature measurements (the rightmost term in (5.4)) are not recalculated for each heating pattern.

Remark 6. We believe the simulation method used for realizations of random currents creates a more challenging problem than experimental data. In practice, measurements could be taken over a time interval and then averaged over time. This gives temporal structure to the data that is not reflected by our simulations which ignore the temporal correlation of the random currents.

5.2. Real conductivity. First, we consider problems of purely real conductivity ($\sigma = \sigma'$) in both the case of measurements from simulated random currents (stochastic model) and measurements using their variances (deterministic model). For both problems we consider the Dirichlet boundary conditions (3.4) using $e_1 = (x_1 + x_2)/10$ and $e_2 = (1 + x_1 - x_2)/10$. For the more challenging problem of the experimental boundary conditions (3.12), we only consider the deterministic model of measurements.

We show in Figure 5.1 numerical reconstructions for a purely real conductivity in the deterministic and stochastic models. The deterministic measurements H_{ii} in (5.3) are taken with $T_0 = 300$ and $a = 0.01$ on a 60×60 coarse grid using a 120×120 fine grid to approximate the integrals. The same conditions were used for the stochastic model, where in addition we took $T_0 = 0.01$ using 1000 realizations of random currents. We chose a particularly low background temperature T_0 to get clean enough data with the number of realizations we chose.

As can be expected from subsection 3.3, the signal to noise ratio for the differential temperature measurements worsens for large T_0 . The reconstructions in the stochastic simulations Figure 5.1(c) are comparable to those in the deterministic case Figure 5.1(b), which is remarkable given the significant errors that are introduced in the data for the stochastic simulations. We mention that our choice of discretization ensures that each approximate Dirac heating pattern ($a = 0.01$) covers a minimum of four grid points.

The conductivity is assumed to be known for nodes that are 0.5cm, or less, away from the boundary. The Tikhonov regularization parameter is $\gamma = 5^{-4}$, and iterations are run until the 2-norm of the step is less than 0.1. The initial guess is the solution to (5.1) with a constant conductivity.

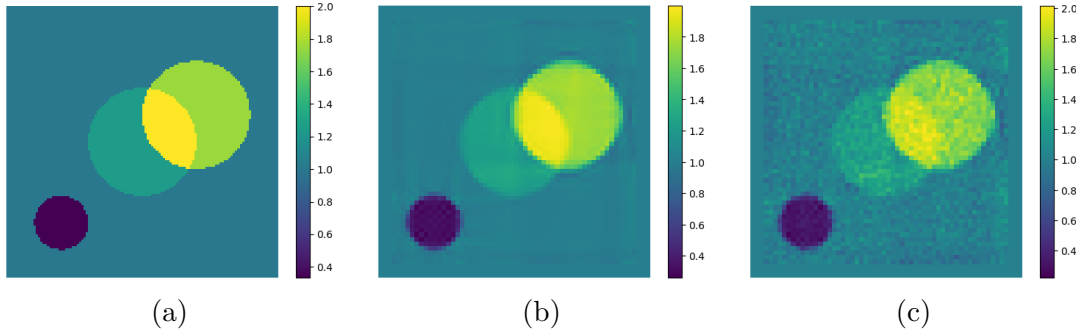


Figure 5.1. Reconstructions of purely real conductivity values ($\text{cm}^{-1} \text{ k}\Omega^{-1}$) on a $10\text{cm} \times 10\text{cm}$ square domain. The ground truth conductivity in (a) is evaluated on the fine grid. Both the reconstructions using the deterministic model (b) and a stochastic model (c) are evaluated on the coarse grid.

The numerical example in Figure 5.2 uses data from experimental boundary conditions in (3.12). The set Γ defined in (4.4), corresponds to the electrode functions. On Γ , we use electrode functions g_1 and h_1 (4.5) for the boundary conditions. This set has gaps of 1cm at the center of each side of the square with no flux conditions. The no flux conditions are enforced by using centered approximations to the nodes on $\partial\Omega - \Gamma$ (see, e.g., [30, sec 2.12]). The ground truth conductivity is given in Figure 5.2 (a). The reconstructions are evaluated on a 100×100 coarse grid, and a 200×200 fine grid is used to evaluate the measurements (5.3). A minimum of twelve fine grid points are in the effective area of each approximate Dirac heating pattern.

The Gauss-Newton iteration is regularized with $\gamma = 3^{-3}$, and iterations are run until the 2-norm of the step size is less than 0.1. The reconstructions in Figure 5.2 are close to the original conductivity, although there are some numerical artifacts due to the gaps between the electrodes, which are superficially similar to those caused by corner type singularities in the conductivity as studied in [9].

5.3. Complex conductivity. An example of a complex conductivity reconstruction using the deterministic model of measurements can be seen in Figure 5.3. Four experiments are used with the Dirichlet boundary conditions $\tilde{g}_1, \tilde{g}_2, \tilde{h}_1, \tilde{h}_2$ given in (4.15). The reconstructions are evaluated on a 100×100 coarse grid, and a 200×200 fine grid is used to evaluate the measurements (5.3). The Gauss-Newton iteration now uses $\gamma = 10^{-4}$, and iterations are run

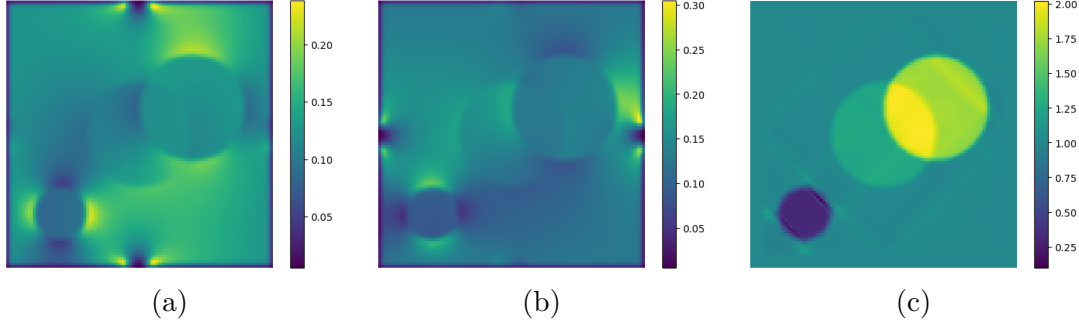


Figure 5.2. Conductivity values ($\text{cm}^{-1} \text{ k}\Omega^{-1}$) for a numerical reconstruction (c) of the same conductivity as Figure 5.1 using the experimental boundary conditions in (3.12). The data $H_{ii} = \text{Re}(\sigma(x))|\nabla u_i(x)|^2$ used for the reconstructions is shown in (a) for g_1 and in (b) for h_1 , with units $\text{cm}^{-3} \text{ k}\Omega^{-1}$.

until the 2-norm of the step size is less than 0.1. A constant complex conductivity and the corresponding solutions to (5.2) are used for the initial guess.

The numerical artifacts in the complex reproduction are consistent with the areas in Figure 4.2 (c), where the maximum condition number (4.6) of the symbol is largest. Intuitively, reconstructing the imaginary conductivity may be more challenging as it does not explicitly appear in the measurements. In the discrete system (5.2) it appears only when coupled with a gradient of the real or imaginary auxiliary field. When the gradient of the real part is large it may overwhelm the contribution of the complex conductivity. This is the exact behavior we see in our numerical experiments. When the real conductivity is high, the gradient of the real field is large, and the reconstructed complex conductivity (Figure 5.3 (d)) is lower than the true value.

6. Summary and perspectives. We propose a new hybrid inverse problem for recovering the conductivity of a body using thermal noise. The fluctuation dissipation theorem for electrodynamic media allows us to relate the variance of thermal noise currents taken with different temperature patterns to the real part of the conductivity of a body. By taking a sufficiently rich set of measurements we can estimate an internal functional that depends on this real conductivity and the solution to an associated auxiliary problem. We show this relation holds for both Dirichlet boundary conditions and mixed Neumann/Dirichlet boundary conditions, the latter of which is a more realistic description of an experimental setup where such measurements might be used. For purely real conductivities, these are power density measurements. This problem of recovering a real conductivity from power density measurements also appears in acousto-electric tomography.

Before attempting numerical reconstructions we try and determine if the linearized problems are elliptic in the Douglis-Nirenberg sense. The linearized real problem has previously been shown to be elliptic, given the auxiliary fields are nowhere orthogonal or parallel [6]. For the real problem with mixed boundary conditions, we make no effort to show our boundary conditions satisfy this condition. Instead, we numerically evaluate the worst conditioning of the principal symbol at each point in space. The numerical results are consistent with the previous theoretical work in [6]; using more boundary conditions improves the numerical con-

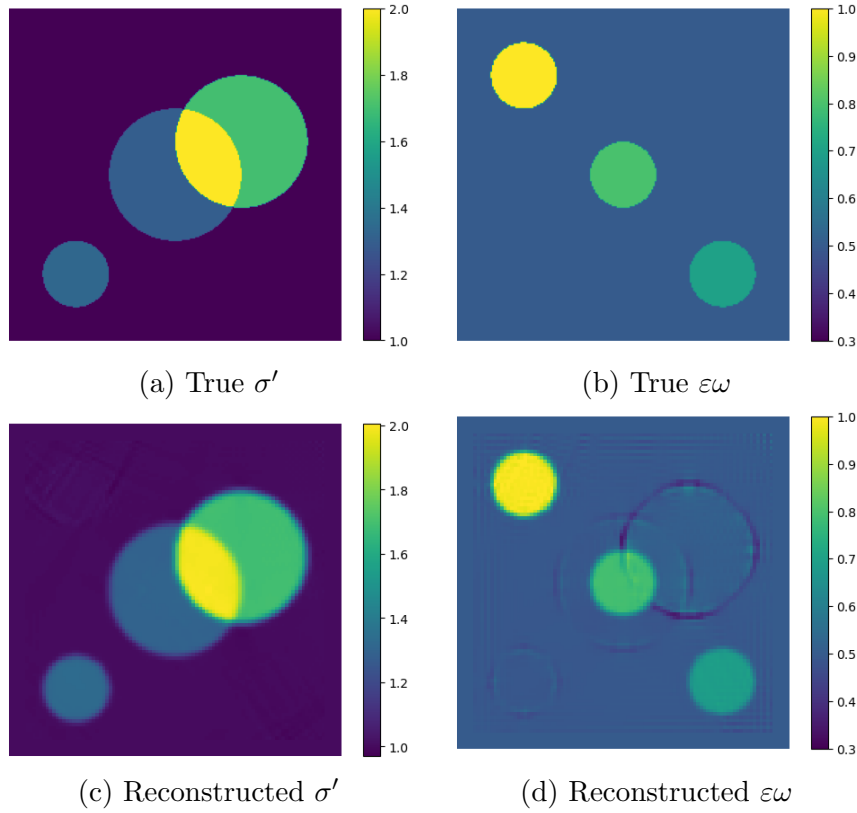


Figure 5.3. Real and imaginary conductivity values ($\text{cm}^{-1} \text{ k}\Omega^{-1}$) for a numerical reconstruction (c) & (d) of a complex conductivity on a $10\text{cm}\times 10\text{cm}$ square domain. The ground truth (a) & (b) is evaluated on the fine grid and the reconstruction is evaluated on the coarse grid.

ditioning of the symbol. For the complex symbol, we give a sufficient condition on the auxiliary fields for the problem to be elliptic, without proving that the auxiliary fields can be generated. In contrast with the real case where two boundary conditions are sufficient for ellipticity, the complex case requires three or more boundary conditions. We perform a similar numerical evaluation of the conditioning of the symbol under a number of different experiments. This evidence indicates that the complex conductivity problem with two boundary conditions is not elliptic. This numerical approach to classification may find use in similar problems, especially in problems with complicated boundary conditions or principal symbols. The clear limitation of this method is that it does not inform the choice of boundary conditions. We note that the conditioning in these problems would not normally be seen as high for other applications.

Finally, we present a simple discrete model for numerical reconstructions. Our numerical reconstructions are consistent with the linearization study. We also present results using simulated random thermal currents for the case of a purely real conductivity. These simulations ignore temporal correlations, making the problem more challenging. In this case, we can get an accurate, if noisy, reconstruction of the conductivity at a low temperature.

This method of thermal noise imaging may find applications in e.g. Atomic Force Mi-

croscopy, laser weld monitoring. A challenge in using our approach is that the relative size of the measurements due to the background temperature and the heating pattern (see [subsection 3.3](#)) results in currents that may be hard to measure reliably in practice.

Our results are for a fixed frequency ω , and removing this limitation may allow for more accurate reconstructions in the complex case. Considering multiple frequencies could take into account any frequency dependency of σ' or ε' in $\sigma = \sigma' + i\omega\varepsilon'$, but requires a different analysis. Additionally, the relative scale of the variables of interest, ω and ε , changes which may introduce other challenges to the reconstructions.

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