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Existence of an Effective Burning Velocity in a Cellular Flow for the Curvature G-Equation Proved Using a Game Analysis

Hongwei Gao[1] · Ziang Long[2] · Jack Xin[3] · Yifeng Yu[3]

Abstract

G-equation is a popular level set model in turbulent combustion, and becomes an advective mean curvature type evolution equation when curvature of a moving flame in a fluid flow is considered:

$$G_t + \left(1 - d \operatorname{div}\left(\frac{DG}{|DG|}\right)\right)_+ |DG| + V(x) \cdot DG = 0.$$

Here $d > 0$ is the Markstein number and the positive part $()_+$ is imposed to avoid a non-physical negative laminar flame speed. For simplicity of presentation, we focus mainly on the case when $V : \mathbb{R}^2 \to \mathbb{R}^2$ is the two dimensional cellular flow with Hamiltonian $H = \sin x_1 \sin x_2$ and amplitude A. Our main result is that for any unit vector $p \in \mathbb{R}^2$, there exists a positive number $\overline{H}(p)$ such that if $G(x, 0) = p \cdot x$, then

$$\left|G(x, t) - p \cdot x + \overline{H}(p)t\right| \leq C \quad \text{in } \mathbb{R}^2 \times [0, \infty)$$

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for a constant C depending only on on the Markstein number d and the cellular flow amplitude A. The number $\overline{H}(p)$ corresponds to the effective burning velocity in the physics literature. The non-coercivity encountered here is one of the major difficulties for homogenization of the mean curvature-type equations. To overcome it, we introduce a new approach that combines PDE methods with a dynamical analysis of the Kohn-Serfaty deterministic game characterization of the curvature G-equation utilizing the streamline structure of cellular flows. Extension to general two-dimensional incompressible flows is also discussed. In three dimensional incompressible flows, the existence of $\overline{H}(p)$ might fail when the flow intensity exceeds a bifurcation value even for simple shear flows Mitake et al. (Bifurcation of homogenization and nonhomogenization of the curvature G-equation with shear flows (2023)).

Keywords Front motion · Cellular flow · Curvature effect · Homogenization · Game theory

Mathematics Subject Classification 70H20 · 76M50 · 76M45 · 76N20

1 Introduction

G-equation is a well-known level-set model in turbulent combustion [39]. Precisely speaking, given a reference function $G(x, t)$, let the flame front be the zero level set $\{G(x, t) = 0\}$ at time t, where the burnt and unburnt regions are $\{G(x, t) < 0\}$ and $\{G(x, t) > 0\}$, respectively. See Fig. 1 below. The velocity of ambient fluid $V : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous and $\mathbb{Z}^n$-periodic. The propagation of flame front obeys a simple motion law:

$$v_{\vec{n}} = s_l + V(x) \cdot \vec{n},$$

i.e., the normal velocity is the laminar flame speed (s_l) plus the projection of V along the normal direction $\vec{n}$. This leads to the so–called G-equation, a level-set PDE [32, 34, 39]:

$$G_t + V(x) \cdot DG + s_l |DG| = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty).$$

In the simplest case when the laminar flame speed s_l is a positive constant, the above equation is called the inviscid G-equation, a non-coercive convex Hamilton-Jacobi equation.

However, in general, s_l is not constant along the flame front since the burning temperature varies in different locations. In order to quantify temperature difference, the curvature effect in turbulent combustion was first introduced by Markstein [30], which says that if the flame front bends toward the cold region (unburned area, point C in Fig. 2 below), the flame propagation slows down. If the flame front bends toward the hot spot (burned area, point B in Fig. 2), it burns faster.

There is a vast turbulent combustion literature discussing the impact of curvature effect. Below is the most recognized empirical linear relation proposed by Markstein
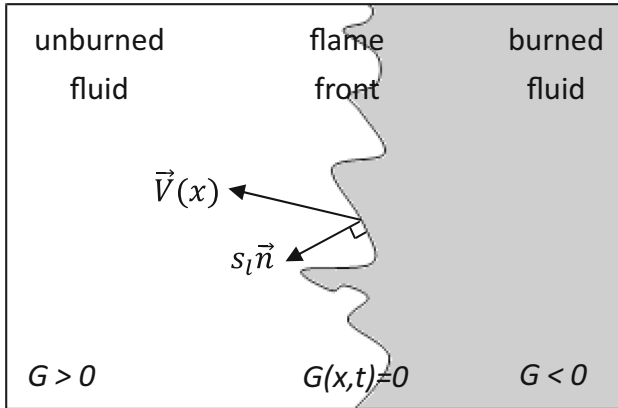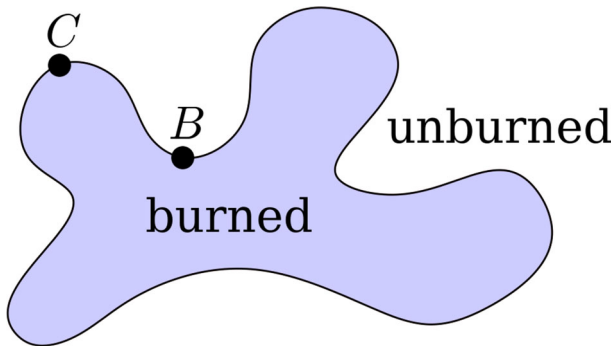
Fig. 1 Level-set formulation of front propagation

Fig. 2 Curvature effect

[30] to approximate the dependence of the laminar flame speed s_l on the curvature (see also [34]):

$$s_l = s_l^0 (1 - d\,\kappa)_+. \tag{1.1}$$

Here s_l^0, the mean value, is a positive constant. Hereafter we set $s_l^0 = 1$. The parameter $d > 0$ is the so called Markstein length which is proportional to the flame thickness. The mean curvature along the flame front is κ. The positive part $(a)_+ = \max\{a, 0\}$ is imposed to avoid negative laminar flame speed since materials cannot be "unburned". This correction usually is not explicitly mentioned in combustion literature since, by default, the curvature is always assumed to be small there. However, mathematically, large positive curvature could occur as time evolves. Therefore it is necessary to explicitly add this correction in theoretical study and numerical computations [44] if the physical validity is taken into consideration in the modeling of flame propagation.

Plugging the expression of the laminar flame speed (1.1) into the G-equation and normalizing the constant $s_l^0 = 1$, we obtain a mean curvature type equation with

advection:

$$G_t + \left(1 - d \ \mathrm{div}\left(\frac{DG}{|DG|}\right)\right)_+ |DG| + V(x) \cdot DG = 0. \qquad (1.2)$$

We would like to mention that curvature G-equations served as some of the first numerical examples in the development of level-set methods [32, 33, 36]. The rigorous mathematical foundation under the framework of viscosity solution was established in [10, 16].

In general, κ changes sign along a curved flame front. So a mathematically interesting and physically important question is:

Question 1 *How does the "averaged" flame propagation speed depend on the curvature term?*

As the first step, the main purpose of this paper is to establish the existence of a properly defined "averaged speed", which is basically to average fluctuations caused by both the flow and the curvature. The theory of homogenization provides such a rigorous mathematical framework in environments with microstructures.

Turbulent combustion usually involves small scales. As a simplified model, we rescale V as $V = V(\frac{x}{\epsilon})$ and replace d by $d \epsilon$. Here ϵ denotes the Kolmogorov scale (the small scale in the flow). The diffusivity constant $d > 0$ is called the Markstein number. We would like to point out that the dimensionless Markstein number is $d \cdot \frac{\delta_L}{\epsilon}$ with δ_L denoting the flame thickness [34]. In the thin reaction zone regime, $\delta_L = O(\epsilon)$, see Eq. (2.28) and Fig. 2.8 of [34]. Without loss of generality, let $\frac{\delta_L}{\epsilon} = 1$. Then (1.2) becomes

$$\frac{\partial G_\epsilon}{\partial t} + \left(1 - d \epsilon \ \mathrm{div}\left(\frac{DG_\epsilon}{|DG_\epsilon|}\right)\right)_+ |DG_\epsilon| + V\left(\frac{x}{\epsilon}\right) \cdot DG_\epsilon = 0. \qquad (1.3)$$

Since $\epsilon \ll 1$, it is natural to look at $\lim_{\epsilon \to 0} G_\epsilon$, i.e., the homogenization limit. If $G_\epsilon(x, 0) = p \cdot x$, the limit $\lim_{\epsilon \to 0} G_\epsilon(x, t)$ can be viewed as the flame propagation under the effective burning velocity (also called *"turbulent flame speed"* in physics literature) along a given direction p.

For general V, the effective burning velocity might not exist (see Remark 1.4). Moreover, a general V is unlikely to provide meaningful answers to delicate problems like Question 1. In this paper, we consider a two-dimensional (2D) cellular flow that is frequently used in the mathematics and physics literature. See [11] for more physical motivations. For clarity of presentation, we choose to work with the typical example

$$V(x) = A(DH)^\perp = A(-\cos x_2 \sin x_1, \ \cos x_1 \sin x_2).$$

Here $H(x_1, x_2) = \sin x_1 \sin x_2$ is the stream function and the positive constant A represents the flow intensity. The following is our main theorem.

Theorem 1.1 *Suppose that $V = A(DH)^\perp$ for $A > 0$. For each $\epsilon > 0$, let $G_\epsilon = G_\epsilon(x, t) \in C(\mathbb{R}^2 \times [0, \infty))$ be the unique viscosity solution to equation (1.3) on $\mathbb{R}^2 \times [0, \infty)$ subject to $G_\epsilon(x, 0) = p \cdot x$ for a given $p \in \mathbb{R}^2$. Then*

$$|G_\epsilon(x, t) - p \cdot x + \overline{H}_A(p)t| \leq C\epsilon \quad on \ \mathbb{R}^2 \times [0, \infty)$$

for a constant C depending only on d, V and |p|. Here $\overline{H}_A \in C(\mathbb{R}^2 \backslash \{0\}, (0, \infty))$ is a continuous positive homogeneous function of degree one.

Equivalently, if $G(x, t)$ is the unique viscosity solution to equation (1.2) *on $\mathbb{R}^2 \times [0, \infty)$ subject to $G(x, 0) = p \cdot x$, then*

$$\left| G(x, t) - p \cdot x + \overline{H}_A(p)t \right| \leq C \quad on \, \mathbb{R}^2 \times [0, \infty).$$

Remark 1.1 Our proof relies on establishing the existence of correctors to the cell problem for every $p \in \mathbb{R}^2$:

$$\left(1 - d \, \text{div} \left(\frac{p + Dv}{|p + Dv|} \right) \right)_+ |p + Dv| + V(x) \cdot (p + Dv) = \overline{H}(p).$$

In general, the cell problem is sufficient to obtain the full homogenization with any uniformly continuous initial data by Evans' perturbed test function method [15]. However, as it was first explicitly pointed out in [5], due to the discontinuity of the mean curvature operator, the standard perturbed test function method is not directly applicable. Roughly speaking, given a test function $\phi(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$, let $p = D_x\phi(x_0, t_0)$, it is not clear whether

$$\phi(x, t) + \epsilon v \left(\frac{x}{\epsilon} \right)$$

is an approximate solution of equation (1.3) near (x_0, t_0) due to possible vanishing gradients. The authors of [5] have introduced a mechanism to implement a modified version of the perturbed test function method. See also [2] for a different modified argument when the corresponding cell problem has a classical solution. To avoid a lengthy paper, we will address this subtle issue in a future publication. In view of our problem's physical origin, it would be an interesting project to study more detailed properties of $\overline{H}_A(p)$ (e.g., its anisotropy). See Remark 3.2.

Remark 1.2 We would like to point out that the small scales in the curvature G-equation help flame propagation. In fact, it is not hard to see that under equation (1.2), *the flame front is stagnant if the initial burned region is $\{x \in \mathbb{R}^2 | \, G(x, 0) < 0\} = \{x \in [0, \pi] \times [0, \pi] | \, H(x) > 1 - r\}$ and the initial flame front is*

$$\{x \in \mathbb{R}^2 | \, G(x, 0) = 0\} = \{x \in [0, \pi] \times [0, \pi] | \, H(x) = 1 - r\}$$

when r is small enough. Nevertheless, *after homogenization, the front moves forward with a positive normal speed $\overline{H}_A(p)$ along any given direction p.* The curvature term plays a subtle role of flame spreading when the cellular flow corrugates the level set. Such a flow induced geometric mechanism is absent in the invicid equation model (i.e. $d = 0$). The homogenization of the invicid G-equation has been established independently in [7] and [40] via completely different approaches for general incompressible flows in any dimension.

Remark 1.3 As pointed out by a referee, a natural question is whether the cut-off $(\cdot)_+$ in the governing equation is actually active (equal to zero) at some point (x, t) during the evolution in the cellular flow. A direct numerical simulation suggests that the answer is positive over $A \in (0, 45]$ at $d = 0.1$ where the $\overline{H}$ differs with and without the cutoff. That the cut-off is active is clear in the case of three dimensional shear flows [31] where the homogenization fails for large flow intensity with cut-off $(\cdot)_+$ and the homogenization holds for all flow intensity without cut-off $(\cdot)_+$. See Remark 1.4 for more details.

The major remaining question is to understand how the turbulent velocity $\overline{H}_A(p)$ depends on the Markstein number d and the flow intensity A. There is a consensus in combustion literature that the curvature effect slows down flame propagation [35]. Heuristically, this is because the curvature term smoothes out the flame front and reduces the total area of chemical reaction [36]. To the best of our knowledge, the first mathematically rigorous result in this direction was obtained in [29] for shear flows. It was proved that $\overline{H}$ is strictly decreasing with respect to the Markstein number d, which is consistent with the experimental observation (e.g., [9]).

For the more complicated cellular flow, it is probably more convenient to compare the growth law with respect to the flow intensity A. For the inviscid case, a sharp growth law was establish in [41]: when $d = 0$, for $p = (p_1, p_2)$,

$$\frac{A\,\pi(|p_1| + |p_2|)}{2 \log A + C_2} \leq \overline{H}_A(p) \leq \frac{A\,\pi(|p_1| + |p_2|)}{2 \log A + C_1}.$$

By refining the method and estimates in this paper, we conjecture that it might be possible to show that when $d > 0$,

$$\overline{H}_A(p) = O\left(\frac{A}{\log A}\right).$$

More interestingly, in future work we plan to investigate:

Question 2 *Does the presence of the curvature effect significantly reduce the prediction of the turbulent flame speed? More specifically, does there exist a unit direction p such that*

$$\limsup_{A \to +\infty} \frac{\overline{H}_A(p) \log A}{A} < \frac{\pi(|p_1| + |p_2|)}{2} \,?$$

If the curvature term is replaced by full diffusion (i.e. the Laplacian Δ without any correction), a dramatic slow-down ($\overline{H}_A(p)$ uniformly bounded in A) is proved in [28] for two dimensional (2D) cellular flows.

• **Other works.** In non-combustion practical suitations such as phase transition (e.g., crystal growth) in material science [12, 25], the motion law is given by (no drift term and $()_+$ correction)

$$v_{\vec{n}} = a(x) - d\kappa \tag{1.4}$$

for a continuous positive $\mathbb{Z}^n$-periodic function $a(x)$. The above formula is known as the Gibbs-Thomson relation. The $()_+$ is not needed in crystal growth since both freezing and melting could occur in the situation of ice formation [25]. See ([2, 5, 6, 8, 12, 14, 17–19, 26], etc) and reference therein for works regarding the homogenization or large time limit related to (1.4). When $a(x)$ satisfies a special coercivity condition, homogenization has been proved in [26] for all dimensions. When $n = 2$, homogenization was proved in [5] for all positive $a(x)$ by a geometric approach. Moreover, the authors have also constructed a counterexample in [5] when the dimension $n \geq 3$ for a positive $a(x)$. See [4] for more discussions about geometric approaches in homogenization. Due to the presence of the non-coercive transport term $V \cdot DG$, or $a = a(x, \vec{n}) = 1 + V(x) \cdot \vec{n}$, our situation and methods are very different from all these previous works. See Remark 3.3 for more discussions. Our result also holds when $(1 - d\kappa)_+$ is replaced by a more general form $(a(x) - d\kappa)_+$.

Remark 1.4 Like the invisicd G-equation case ($d = 0$), it is easy to construct a smooth periodic V such that the effective speed does not exist, e.g. let $V(x) = K(x - P_0)$ near $P_0 = (\frac{\pi}{2}, \frac{\pi}{2})$ for a large number $K > 0$ and equal to the cellular flow away from P_0. We conjecture that our method could be extended to cover general 2D smooth periodic incompressible flows ($\text{div}(V) = 0$). See Sect. 4 for more detailed discussions. When $n \geq 3$, it was proved in [31] that, for the shear flow $V(x_1, x_2, x_3) = (0, 0, Af(x_1, x_2))$, the effective burning velocity ceases to exist when the flow intensity A exceeds a bifurcation value. Note that when $n = 2$, the existence of effective burning velocity can be established very easily for all shear flows since the cell problem is reduced to an ODE and maximum principle immediately leads to uniform bound of derivatives.

Question 3 *Does Theorem 1.1 hold if V is a 3D physically meaningful flow that possesses turbulent or chaotic structures, e.g. the Arnold-Beltrami-Childress flow [11, 43]?*

• **Outline of the paper.** In Sect. 2, for reader's convenience, we will go over the definition and comparison principle of viscosity solutions of curvature type equations. The associated game theoretic interpretation from [24] will be reviewed as well. In Sect. 3, we prove Theorem 1.1 by a novel approach combining Lagrangian (game dynamics) and Eulerian approaches (PDE techniques). In Sect. 4, we discuss how to extend our results and methods to general 2D incompressible flows. Our approach also suggests a possible *general framework* to tackle non-periodic settings (see Remark 4.1). To help derive a reachability property in our proof, we show rigorously in the Appendix a useful fact related to the consistency of viscosity and classical solutions in front propagation.

2 Preliminary

In this section, $V : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous and $\mathbb{Z}^n$-periodic. Write

$$F(A, p) = \left(|p| - d \left(\text{tr} A - \frac{p \cdot A \cdot p}{|p|^2} \right) \right)_+$$

for $(A, p) \in S^{n \times n} \times (\mathbb{R}^n \setminus \{0\})$. Here $S^{n \times n}$ is the space of $n \times n$ real symmetric matrices. Following the definition in [13], let

$$\underline{F}(A, p) = \begin{cases} F(A, p) & \text{if } |p| \neq 0 \\ 0 & \text{if } |p| = 0 \end{cases} \quad \text{and} \quad \overline{F}(A, p) = \begin{cases} F(A, p) & \text{if } |p| \neq 0 \\ 2dn\|A\| & \text{if } |p| = 0. \end{cases}$$

Here $\|A\|$ is the largest absolute value of eigenvalues of A. Note that $\underline{F}$ and $\overline{F}$ are lower semicontinuous and upper semicontinuous respectively. As it was pointed out in [13], the definition at $p = 0$ is not really important as long as the lower and upper continuity hold. This is because A will be zero when $p = 0$ with the proper test function (see the proof of Theorem 2.1 for instance). When $n = 2$,

$$\text{tr} A - \frac{p \cdot A \cdot p}{|p|^2} = \frac{p^\perp \cdot A \cdot p^\perp}{|p|^2}.$$

Here $p^\perp = (-p_2, p_1)$ if $p = (p_1, p_2) \in \mathbb{R}^2 \setminus \{0\}$.

2.1 Definition of Viscosity Solutions and Comparison Principle

We first introduce several definitions and terminologies.
Let

$$LSC(\Omega) : \text{ the set of lower semicontinuous functions defined on } \Omega,$$
$$USC(\Omega) : \text{ the set of upper semicontinuous functions defined on } \Omega,$$

and

$$UC(\Omega) : \text{ the set of uniformly continuous functions defined on } \Omega.$$

Definition 2.1 Assume that $G = G(x, t) \in USC(\mathbb{R}^n \times (0, \infty))$. G is called a viscosity subsolution of equation (1.2) provided that given $\phi(x, t) \in C^2(\mathbb{R}^n \times (0, \infty))$, if

$$G(x_0, t_0) - \phi(x_0, t_0) = \max_{\mathbb{R}^n \times (0, \infty)} (G(x, t) - \phi(x, t)),$$

then

$$\phi_t(x_0, t_0) + \underline{F}\left(D^2\phi(x_0, t_0), D\phi(x_0, t_0)\right) + V(x_0) \cdot D\phi(x_0, t_0) \leq 0.$$

Definition 2.2 Assume that $G = G(x, t) \in LSC(\mathbb{R}^n \times (0, \infty))$. G is called a viscosity supersolution of equation (1.2) provided that given $\phi(x, t) \in C^2(\mathbb{R}^n \times (0, \infty))$, if

$$G(x_0, t_0) - \phi(x_0, t_0) = \min_{\mathbb{R}^n \times (0, \infty)} (G(x, t) - \phi(x, t)),$$

then

$$\phi_t(x_0, t_0) + \overline{F}\left(D^2\phi(x_0, t_0), D\phi(x_0, t_0)\right) + V(x_0) \cdot D\phi(x_0, t_0) \geq 0.$$

Definition 2.3 $G = G(x, t) \in C(\mathbb{R}^n \times (0, \infty))$ is called a viscosity solution of equation (1.2) if it is both a viscosity subsolution and a viscosity supersolution. If the initial data $G(x, 0) = g(x)$ is given, then we require that

$$\lim_{t \to 0} G(x, t) = g(x) \quad \text{locally uniformly for } x \in \mathbb{R}^n.$$

The following comparison principle for solutions can be proved by standard approaches ([5, 10, 13, 16]), which is a special case of Theorem 3.3 in [5] . For the reader's convenience, we provide a sketch of the proof. See the proof of Theorem 3.3 in [5] (the long version) for more general operators.

Theorem 2.1 *Assume that* $V \in W^{1,\infty}(\mathbb{R}^n)$, $g_1, g_2 \in UC(\mathbb{R}^n)$ *and* $g_1 \leq g_2$. *Suppose that for some* $T > 0$, $G_1 = G_1(x, t) \in USC(\mathbb{R}^n \times (0, T))$ *is a viscosity subsolution of equation* (1.2) *and* $G_2 = G_2(x, t) \in LSC(\mathbb{R}^n \times (0, T))$ *is a viscosity supersolution of equation* (1.2). *If both* G_1 *and* G_2 *are subject to (for* $i = 1, 2$):

$$\lim_{t \to 0} |G_i(x, t) - g_i(x)| = 0 \quad \text{locally uniformly in } \mathbb{R}^n,$$

then

$$G_1(x, t) \leq G_2(x, t) \quad \text{for all } (x, t) \in \mathbb{R}^n \times [0, T).$$

Proof By considering $\frac{2}{\pi} \arctan(G_1)$ and $\frac{2}{\pi} \arctan(G_2)$, we may assume that for $i = 1, 2$,

$$-1 \leq G_i \leq 1.$$

We argue by contradiction. If not, then

$$\sup_{(x,t) \in \mathbb{R}^n \times [0,T)} (G_1(x, t) - G_2(x, t)) > 0.$$

Let $M = 4(||V||_{W^{1,\infty}(\mathbb{R}^n)} + 1)$ and

$$\tilde{G}_i = e^{-Mt} G_i.$$

Then $\tilde{G}_1$ is a viscosity subsolution of

$$\frac{\partial \tilde{G}_1}{\partial t} + M\tilde{G}_1 + \underline{F}(D^2\tilde{G}_1, D\tilde{G}_1) + V(x) \cdot D\tilde{G}_1 = 0$$

and $\tilde{G}_2$ is a viscosity supersolution of

$$\frac{\partial \tilde{G}_2}{\partial t} + M\tilde{G}_2 + \overline{F}(D^2\tilde{G}_2, D\tilde{G}_2) + V(x) \cdot D\tilde{G}_2 = 0$$

subject to $\lim_{t \to 0} |\tilde{G}_i(x, t) - g_i(x)| = 0$ locally uniformly in $\mathbb{R}^n$.

We also have that

$$r = \sup_{(x,t) \in \mathbb{R}^n \times [0,T)} (\tilde{G}_1(x, t) - \tilde{G}_2(x, t)) > 0.$$

Now let

$$w(x, y, t) = \tilde{G}_1(x, t) - \tilde{G}_2(y, t) - \Phi_{\rho, K, \delta}(x, y)$$

for some $K, \rho, \delta > 0$ and

$$\Phi_{\rho, K, \delta}(x, y, t) = \rho\left(\sqrt{|x|^2 + 1} + \sqrt{|y|^2 + 1}\right) + K|x - y|^4 + \frac{\delta}{T - t}.$$

Step 1: It is easy to see that there exist $\rho_0, \delta_0 \in (0, 1)$ such that for all $\rho \leq \rho_0$, $\delta \leq \delta_0$ and $K \in \mathbb{N}$,

$$\max_{\mathbb{R}^n \times \mathbb{R}^n \times [0,T)} w(x, y, t) \geq \frac{1}{2} \sup_{(x,t) \in \mathbb{R}^n \times [0,T)} (\tilde{G}_1 - \tilde{G}_2) = \frac{r}{2}.$$

Choose $(\bar{x}, \bar{y}, \bar{t}) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T)$ such that

$$w(\bar{x}, \bar{y}, \bar{t}) = \max_{\mathbb{R}^n \times \mathbb{R}^n \times [0,T)} w(x, y, t).$$

For simplicity, we omit the dependence of $(\bar{x}, \bar{y}, \bar{t})$ on ρ, K and δ.

Step 2 (avoiding the boundary $t = 0$): Due to $|\tilde{G}_i| \leq 1$, we have that

$$\Phi_{\rho, K, \delta}(\bar{x}, \bar{y}, \bar{t}) \leq 2.$$

Since for $i = 1, 2$, $g_i \in UC(\mathbb{R}^n)$, there exists $K_0 \in \mathbb{N}$ independent of ρ and δ such that $\bar{t} > 0$ when $K \geq K_0$.

Step 3 (plugging into the equation): Now we fix $K = K_0$ and $\delta = \delta_0$. Write $\gamma(x, y) = \rho\left(\sqrt{|x|^2 + 1} + \sqrt{|y|^2 + 1}\right)$. Owing to Theorem 8.3 and Remark 3.8 in [13], there exist $a, b \in \mathbb{R}$, two $n \times n$ symmetric matrices X and Y such that

$$a - b = \frac{\delta}{(T - t)^2}, \quad X \leq Y, \quad ||X|| + ||Y|| \leq CK_0|\bar{x} - \bar{y}|^2$$

and at point $(\bar{x}, \bar{y}, \bar{t})$

$$a + M\tilde{G}_1(\bar{x}, \bar{t}) + \underline{F}(X + D_x^2\gamma, \bar{p} + D_x\gamma) + V(\bar{x}) \cdot (\bar{p} + D_x\gamma) \leq 0$$
$$b + M\tilde{G}_2(\bar{y}, \bar{t}) + \overline{F}(Y - D_y^2\gamma, \bar{p} - D_y\gamma) + V(\bar{y}) \cdot (\bar{p} - D_y\gamma) \geq 0$$

for $\bar{p} = 4K_0(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^2$. Since $|D_x^2\gamma| + |D_y^2\gamma| + |D_x\gamma| + |D_y\gamma| \leq C\rho$,

$$\tilde{G}_1(\bar{x}, \bar{t}) - \tilde{G}_2(\bar{y}, \bar{t}) \geq \frac{r}{2} + K_0|\bar{x} - \bar{y}|^4,$$

$$|\bar{p}| \leq 4K_0|\bar{x} - \bar{y}|^3 \quad \text{and} \quad |V(\bar{x}) - V(\bar{y})| \leq \left(\frac{M}{4} - 1\right)|\bar{x} - \bar{y}|,$$

we derive that

$$\underline{F}(X + D_x^2\gamma, \bar{p} + D_x\gamma) - \overline{F}(Y - D_y^2\gamma, \bar{p} - D_y\gamma) \leq -\frac{Mr}{2} - \frac{\delta_0}{(T - \bar{t})^2} + C\rho. \quad (2.1)$$

Step 4: Let us look at the following two cases (up to a subsequence if necessary).

Case 1: $\lim_{\rho \to 0} |\bar{x} - \bar{y}| = 0$. Then both the gradient and the Hessian of the test function go to zero, i.e.,

$$\lim_{\rho \to 0} \bar{p} = 0$$
$$\lim_{\rho \to 0} X = \lim_{\rho \to 0} Y = 0.$$

By (2.1), this implies $0 < -\frac{Mr}{2}$ after sending $\rho \to 0$, which is absurd.

Case 2: $\lim_{\rho \to 0} |\bar{x} - \bar{y}| \neq 0$. Without loss of generality, we may assume that

$$\lim_{\rho \to 0} \bar{p} = p_0 \neq 0, \quad \lim_{\rho \to 0} X = X_0 \quad \text{and} \quad \lim_{\rho \to 0} Y = Y_0.$$

Since $F(A, p)$ is monotonically decreasing with respect to the symmetric matrix A and $X_0 \leq Y_0$, taking limit $\rho \to 0$ on (2.1) leads to

$$0 \leq F(X_0, p_0) - F(Y_0, p_0) \leq -\frac{Mr}{2}.$$

This is impossible.   □

Corollary 2.1 *Assume that $g \in UC(\mathbb{R}^n)$ and $|g(x)| \leq C(|x| + 1)$ for some positive constant C. Suppose that $G = G(x, t) \in C(\mathbb{R}^n \times [0, \infty))$ is a viscosity subsolution of equation (1.2) subject to $G(x, 0) = g(x)$. Then there exists a constant $\tilde{C}$ depending only on C, V and d such that*

$$|G(x, t)| \leq \tilde{C}(1 + |x| + t) \quad \text{for all } (x, t) \in \mathbb{R}^n \times [0, \infty).$$

Proof Let $\phi(x) = 2C\sqrt{|x|^2 + 1}$. Let

$$C_1 = \max_{x \in \mathbb{R}^n} \left| \overline{F}(D^2\phi(x), D\phi(x)) + V(x) \cdot D\phi(x) \right|.$$

Then $G_1(x, t) = \phi(x) + C_1 t$ is a viscosity supersolution to equation (1.2). Hence by the comparison principle Theorem 2.1, we have that

$$G(x, t) \leq G_1(x, t) \leq \tilde{C}(1 + |x| + t)$$

for $\tilde{C} = \max(2C, C_1)$. A similar inequality holds in the other direction. $\qquad \square$

2.2 Game Theory Interpretation From [24]

In this section, let $n = 2$ and $x \in \mathbb{R}^2$. Fix the game step size parameter $\tau > 0$ and the number of steps $N \in \mathbb{N}$.

For $n \geq 1$, consider the discrete dynamical system $\{x_n\}_{n=1}^N \subset \mathbb{R}^2$ associated with the game starting from $x_0 = x$: for $n = 0, 1, 2, .., N - 1$

$$\begin{cases} x_{n+1} = x_n + \tau\sqrt{2d}b_n\eta_n + \tau^2\eta_n^\perp - \tau^2 V(x_n) \\ x_0 = x, \end{cases}$$

where $\eta_n \in \overline{B}_1(0) = \{v \in \mathbb{R}^2 | \ |v| \leq 1\}$ and $b_n \in \{-1, 1\}$. Here for convenience, compared to [24], we reverse the time that leads to $-V$ instead of V. Moreover, $\eta^\perp = (-c, a)$ if $\eta = (a, c) \in \mathbb{R}^2$.

There are two players: player I and player II. In each step,

- Player I: First choose the direction η_n;
- Player II: Then choose the sign b_n.

The sequence of positions $\{x_k\}_{k=0}^N$ is called a game trajectory associated with (τ, N). The goal of player I is to minimize $g(x_N)$ and player II aims to maximize $g(x_N)$.

A strategy Γ_1 of player I refers to how player I chooses η_n based on $\{x, \eta_i, b_i | \ 1 \leq i \leq n - 1\}$ for $n \geq 1$. Similarly, a strategy Γ_2 of player II refers to how player II chooses the sign b_n based on $\{x, \eta_i, \eta_n, b_i, | \ 1 \leq i \leq n - 1\}$ for $n \geq 1$.

Assuming that both players play optimally. Write the value function of player I as

$$u_\tau(x, N\tau^2) = \text{infimum of } g(x_N). \tag{2.2}$$

Then we have the following dynamic principle: for $k \geq 1$,

$$u_\tau(x, k\tau^2) = \inf_{|\eta| \leq 1} \max_{b = \pm 1} u_\tau\left(x + \tau b\sqrt{2d}\eta + \tau^2\eta^\perp - \tau^2 V(x), \ (k - 1)\tau^2\right). \tag{2.3}$$

For the standard mean curvature motion, player I can only choose unit vectors. Here we basically allow player I to also choose $\eta_n = 0$, which excludes the possibility of

"unburning". The consistency holds: for small τ,

$$\min_{\eta \in \overline{B}_1(0)} \max_{b=\pm 1} \{\tau b \sqrt{2d}\eta \cdot D\phi(x) + d\tau^2 \eta \cdot D^2\phi(x) \cdot \eta + \tau^2 \eta^\perp \cdot D\phi\}$$

$$\in -\tau^2 [\underline{F}(D^2\phi(x), D\phi(x)), \ \overline{F}(D^2\phi(x), D\phi(x))].$$

2.3 Convergence

Given $g(x) \in UC(\mathbb{R}^2)$, as in [24], we have that

$$\lim_{\tau \to 0, \ N\tau^2 \to t} u_\tau(x, N\tau^2) = u(x, t) \quad \text{locally uniformly on } \mathbb{R}^2 \times (0, \infty). \quad (2.4)$$

Here $u \in C(\mathbb{R}^2 \times [0, \infty))$ is the unique viscosity solution to equation (1.2) subject to

$$\lim_{t \to 0} u(x, t) = g(x) \quad \text{uniformly in } \mathbb{R}^2.$$

As it was mentioned in [23], this game theory approach is essentially a semi-discrete approximation scheme (continuous in space, discrete in time) for curvature-type equations. See [37] for an analogous game in combinatorics.

The proof is standard under the framework of viscosity solution. For the reader's convenience, we provide an outline of the proof.

Step 1: Define

$$\overline{u}(x, t) := \limsup_{y \to x, \ \tau \to 0, \ N\tau^2 \to t} u_\tau(y, N\tau^2)$$

$$\underline{u}(x, t) := \liminf_{y \to x, \ \tau \to 0, \ N\tau^2 \to t} u_\tau(y, N\tau^2).$$

Then $\overline{u}$ is upper semicontinuous and $\underline{u}$ is lower semicontinuous.

Lemma 2.1 *For $t \in [0, 1]$,*

$$\min_{|y-x| \leq C\sqrt{t}} g(y) \leq \underline{u}(x, t) \leq \overline{u}(x, t) \leq \max_{|y-x| \leq C\sqrt{t}} g(y).$$

Here C is a constant depending only on d and $\max_{\mathbb{R}^2} |V|$.

Proof Fix $t > 0$. Consider the game starting from $x_0 \in \mathbb{R}^2$ with (τ, N), $N\tau^2 \in [\frac{t}{2}, 2t]$.

On the one hand, if player I adopts the following strategy similar to the exit strategy in [24]: at each step n, player I chooses the direction η_n such that

$$\eta_n \cdot (x_n - x_0) = 0,$$

then regardless of how player II plays, if $|x_n - x_0| \geq \sqrt{t}$, then

$$|q + w| - |q| = \frac{2w \cdot q + |w|^2}{|q + w| + |q|} \implies |x_{n+1} - x_0| \leq |x_n - x_0| + C\tau^2 \left(1 + \frac{1}{\sqrt{t}}\right).$$

Hence

$$|x_N - x_0| \leq C\sqrt{t} \quad \text{for } t \in [0, 1].$$

Then by the definition of $\overline{u}(x, t)$,

$$\overline{u}(x_0, t) \leq \max_{|y-x_0| \leq C\sqrt{t}} g(y).$$

On the other hand, if player II employs the strategy as: at each step n, player II chooses the sign b_n such that

$$b_n \eta_n \cdot (x_n - x_0) \leq 0$$

after player I picks the direction η_n, then regardless what kind of strategy player I uses, similarly, we have that

$$|x_N - x_0| \leq C\sqrt{t}.$$

This implies that

$$\underline{u}(x_0, t) \geq \min_{|y-x_0| \leq C\sqrt{t}} g(y).$$

$\square$

Since $g \in UC(\mathbb{R}^2)$,

$$\lim_{y \to x, t \to 0} \overline{u}(y, t) = \lim_{y \to x, t \to 0} \underline{u}(y, t) = g(x) \quad \text{uniformly for } x \in \mathbb{R}^2.$$

Step 2: As in [24], we have that

Lemma 2.2 $\overline{u}(x, t)$ *is a viscosity subsolution of*

$$\overline{u}_t + \underline{F}(D^2\overline{u}, D\overline{u}) + V(x) \cdot D\overline{u} = 0 \quad on \, \mathbb{R}^2 \times (0, \infty)$$

and $\underline{u}(x, t)$ *is a viscosity supersolution of*

$$\underline{u}_t + \overline{F}(D^2\underline{u}, D\underline{u}) + V(x) \cdot D\underline{u} = 0 \quad on \, \mathbb{R}^2 \times (0, \infty).$$

$\square$

Step 3: Finally, the comparison principle Theorem 2.1 implies that $\underline{u} \geq \overline{u}$. Therefore, $\underline{u} = \overline{u}$ and (2.4) holds. $\square$

Remark 2.1 By the uniqueness of solution and Lemma 2.1, if $n = 2$ and $G(x, t)$ is the unique viscosity to equation (1.2) subject to $G(x, 0) = g(x)$, Then for $t \in [0, 1]$

$$\min_{|y-x| \leq C\sqrt{t}} g(y) \leq G(x, t) \leq \max_{|y-x| \leq C\sqrt{t}} g(y).$$

Of course, this can also be proved by pure PDE methods through comparison principle.

Remark 2.2 Although the game is formulated in a deterministic way, it has some intrinsic stochastic features due to the different scales τ and τ^2 so that the game trajectory is practically hard to analyze. Having the positive part (equivalently, allowing player I to choose $\eta_n = 0$) renders the game more deterministic. As mentioned in [23], one of the two main questions about the game is whether it can be used to prove new results about PDE, which requires finding the right structure of PDE to work with. The curvature G-equation with integrable ambient fluid flows turns out to be a good candidate. The game formulation does allow us to make the best use of the underlying structure that is not naturally accessible by the relatively rough PDE approaches. See [27] and [20] for other works related to applications of the game.

2.4 Stationary Equation and Reachability

As in [24], we may also consider stationary equations

$$\left(1 - d \, \text{div}\left(\frac{Du}{|Du|}\right)\right)_+ |Du| + V(x) \cdot Du = \alpha. \tag{2.5}$$

for a constant $\alpha \geq 0$.

Definition 2.4 $u \in USC(\Omega)$ is called a viscosity subsolution of equation (2.5) provided that given $\phi(x) \in C^2(\mathbb{R}^n)$, if for $x_0 \in \Omega$

$$u(x_0) - \phi(x_0) = \max_{x \in \Omega}(u(x) - \phi(x)),$$

then

$$\underline{F}\left(D^2\phi(x_0), D\phi(x_0)\right) + V(x_0) \cdot D\phi(x_0) \leq \alpha.$$

Definition 2.5 $u \in LSC(\Omega)$ is called a viscosity supersolution of equation (2.5) provided that given $\phi(x) \in C^2(\mathbb{R}^n)$, if for $x_0 \in \Omega$

$$u(x_0) - \phi(x_0) = \min_{x \in \Omega}(u(x) - \phi(x)),$$

then

$$\overline{F}\left(D^2\phi(x_0), D\phi(x_0)\right) + V(x_0) \cdot D\phi(x_0) \geq \alpha.$$

$u \in C(\Omega)$ is called a viscosity solution of (2.5) if it is both a viscosity subsolution and a viscosity supersolution. Unlike the Cauchy problem, the stationary problem might not have a solution with a prescribed boundary data.

Similar to the proof of Theorem 2.1, we can prove the following comparison principle.

Theorem 2.2 *Assume that* $\alpha > 0$ *and* Ω *is a bounded open set. Suppose that* $u_1 \in USC(\Omega) \cap L^\infty(\Omega)$ *is a viscosity subsolution of equation* (2.5) *on* Ω *and* $u_2 \in LSC(\Omega) \cap L^\infty(\Omega)$ *is a viscosity supersolution of equation* (2.5) *on* Ω. *If for every point* $x \in \partial\Omega$,

$$\limsup_{y \in \Omega \to x} u_1(y) \leq \liminf_{y \in \Omega \to x} u_2(y),$$

then

$$u_1(x) \leq u_2(x) \quad for\, x \in \Omega.$$

Lemma 2.3 *Suppose* $\alpha > 0$ *and* $u \in C(\overline{\Omega})$ *is a viscosity supersolution of equation* (2.5), *then*

$$\min_{x \in \overline{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x).$$

Proof By the definition of viscosity supersolution, u cannot attain minimum at $x_0 \in \Omega$. Otherwise, we can use the constant function $\phi \equiv \min_{\overline{\Omega}} u$ as the test function and obtain

$$0 = \overline{F}(D^2 0, D0) + V(x_0) \cdot 0 \geq \alpha > 0,$$

which is absurd. Hence our conclusion holds. $\qquad\square$

Definition 2.6 Consider the game dynamics introduced in Sect. 2.2. Let S be subset of $\mathbb{R}^2$. We say that S is reachable from x (or x can reach S) within time T if for every open set U satisfying $S \subset U$, there exist a sequence of positive numbers $\{\tau_m\}_{m \geq 1}$ such that $\lim_{m \to +\infty} \tau_m = 0$ and for each fixed τ_m, player I has a U-oriented strategy Γ_{m1} such that regardless of how player II chooses his strategy Γ_2, under strategy Γ_{m1}, player I can force the associated game trajectory starting from x to enter U at $N(m, \Gamma_{m1}, \Gamma_2)$-th step (i.e., $x_{N(m,\Gamma_{m1},\Gamma_2)} \in U$), where $N(m, \Gamma_{m1}, \Gamma_2)$ is a positive integer depending on $\tau_m, \Gamma_{m1}, \Gamma_2$ and satisfying $N(m, \Gamma_{m1}, \Gamma_2)\tau_m^2 \leq T$.

Remark 2.3 To simplify notations, we usually omit the dependence of the game trajectory (in particular, the terminal point x_N) on τ, N and the strategy of player II. Moreover, by the above definition, it is clear that if a point x can reach an open set U within time t_1 and every point on U can reach a set S within t_2. Then the point x can reach S within time $t_1 + t_2$.

Hereafter, for any set $E \subset \mathbb{R}^n$ and $t \geq 0$, we denote by E_t the image of E under the $-V$ flow $\dot{\xi} = -V(\xi)$ at time t, i.e.,

$$E_t = \{\xi_x(t) | x \in E, \ \dot{\xi}_x(s) = -V(\xi_x(s)), \ \xi_x(0) = x\}.$$

Lemma 2.4 *Suppose that $u \in UC(\mathbb{R}^2)$ is a viscosity subsolution of equation* (2.5) *on $\mathbb{R}^2$. For $x_0 \in \mathbb{R}^2$, if there exists $T_0 > 0$ such that a bounded set S is reachable from x_0 within time T_0, then*

$$u(x_0) \leq \max_{y \in \cup_{t \in [0, T_0]} \bar{S}_t} u(y) + \alpha T_0.$$

In particular, if the set S is $-V$ flow invariant (i.e., $S_t = S$ for all $t \geq 0$), then

$$u(x_0) \leq \max_{y \in \bar{S}} u(y) + \alpha T_0.$$

Proof let $w(x, t) = u(x) - \alpha t$. Then $w(x, t)$ is a viscosity subsolution to equation (1.2) with initial data $w(x, 0) = u(x)$. Now let $G(x, t) \in C(\mathbb{R}^2 \times [0, \infty))$ be the unique viscosity solution to equation (1.2) with initial data $G(x, 0) = u(x)$. Owing to the comparison principle Theorem 2.1,

$$w(x, t) \leq G(x, t).$$

For $r > 0$ and $E \subset \mathbb{R}^2$, let $D_r(E) = \{x \in \mathbb{R}^2 | d(x, E) < r\}$. Fix $r > 0$. Due to the game formulation of $G(x, t)$ and the Definition 2.6 of reachability, there are a sequence of positive numbers $\{\tau_m\}_{m \geq 1}$ and a sequence of positive integers $\{N_m\}_{m \geq 1}$ such that $\lim_{m \to +\infty} \tau_m = 0$ and for each τ_m, player I has a $D_r(S)$-oriented strategy to drive the game trajectory to enter $D_r(S)$ at the N_m-th step for $N_m \tau_m^2 \leq T_0$ regardless of how player II plays. Here for convenience, we omit the dependence of N_m on the concrete strategy of player II since what really matters here is the upper bound $N_m \tau_m^2 \leq T_0$.

Next, starting from $x_{N_m} \in D_r(S)$, player I chooses $\eta = 0$ for k more steps. Here k is the first whole number such that $T_0 \geq (N_m + k)\tau_m^2 \geq T_0 - \tau_m^2$.

Let J_m be the integer part of $\frac{T_0}{\tau_m^2}$. Then the above argument says that player I has a strategy such that regardless of how player II plays, player I can steer the game trajectory into

$$D_{C\tau_m^2}(E_r).$$

at the J_m-th step for $E_r = \cup_{0 \leq t \leq T_0} (D_r(S))_t$.

Hence for the value function defined in (2.2), due to $G(x, 0) = u(x)$,

$$u_{\tau_m}(x_0, J_m \tau_m^2) \leq \max_{y \in D_{C\tau_m^2}(E_r)} u(y)$$

$$G(x_0, T_0) = \lim_{m \to +\infty} u_{\tau_m}(x_0, J_m \tau_m^2) \leq \max_{y \in \overline{E}_r} u(y).$$

So

$$u(x_0) - T_0\alpha = w(x_0, T_0) \le G(x_0, T_0) \le \max_{y \in \overline{E}_r} u(y).$$

Accordingly, our conclusion follows by sending $r \to 0$. $\square$

Lemma 2.5 *Suppose that $u \in UC(\mathbb{R}^n)$ is a viscosity subsolution of equation* (2.5) *on $\mathbb{R}^n$. Let $\xi : [0, \infty) \to \mathbb{R}^n$ satisfy $\dot{\xi}(s) = -V(\xi(s))$. Then for $t_1 < t_2$,*

$$u(\xi(t_1)) \le u(\xi(t_2)) + \alpha(t_2 - t_1).$$

Proof This follows directly from the two facts below:

(1) $w(x, t) = u - \alpha t$ is also a viscosity subsolution of the transport equation

$$w_t + V(x) \cdot Dw = 0 \quad \text{subject to } w(x, 0) = u(x),$$

(2) the unique solution to

$$\begin{cases} v_t + V(x) \cdot Dv = 0 \\ v(x, 0) = u(x) \end{cases}$$

is given by the representation formula: $v(x, t) = u(\xi_x(t))$ for $\dot{\xi}_x(s) = -V(\xi_x(s))$ subject to $\xi_x(0) = x$. $\square$

3 Proof of Theorem 1.1

Let $p \in \mathbb{R}^n$ be a fixed unit vector. By the standard Perron's method [13], for any given $\lambda > 0$, the following equation has a unique continuous $\mathbb{Z}^n$-periodic viscosity solution $v = v_\lambda \in C(\mathbb{R}^n)$.

$$\lambda v + \left(1 - d\,\text{div}\left(\frac{p + Dv}{|p + Dv|}\right)\right)_+ |p + Dv| + V(x) \cdot (p + Dv) = 0 \quad \text{in } \mathbb{R}^n.$$

The above equation also has a comparison principle whose proof is similar to that of Theorem 2.1. To prove Theorem 1.1, our main task will be to show that when $n = 2$ and $V = A(DH)^\perp$ is the 2D cellular flow, there exists a positive constant $\overline{H}(p)$ such that

$$\lim_{\lambda \to 0} \lambda v_\lambda(x) = -\overline{H}(p) \quad \text{uniformly on } \mathbb{R}^2.$$

Here we omit the dependence of $\overline{H}$ on the flow intensity A.

Throughout this section, C represents a constant depending only on d and V. By maximum principle, it is easy to see that

$$\max_{x \in \mathbb{R}^2} |\lambda v_\lambda(x)| \leq 1 + \max_{\mathbb{R}^2} |V|. \tag{3.1}$$

We believe that the equi-continuity of the family of functions $\{v_\lambda\}_{\lambda > 0}$ does not hold due to the degeneracy of the curvature term and the lack of coercivity. Instead, we will show that

$$\max_{x,y \in \mathbb{R}^2} |v_\lambda(x) - v_\lambda(y)| \leq C.$$

Our strategy is (1) establishing partial reachability from the associated game dynamics based on the special structure of the cellular flow to apply Lemma 2.4 and Lemma 2.5, then (2) using the minimum value principle Lemma 2.3 to compensate for the lack of full reachability. The proof can be viewed as a combination of Lagrangian and Eulerian approaches. The game trajectory under player I's strategy more or less mimics the reverse of the propagation route of flame.

By adopting the proof of Theorem 4 in [1], we first establish the connection between $G(x, t)$ and v_λ. Combining with partial reachability, this will lead to a negative upper bound of $\lambda v_\lambda(x)$ for small λ in Corollary 3.2 later, which allows us to apply the minimum value principle.

Lemma 3.1 *Let $G(x, t)$ be the unique solution of* (1.2) *with $G(x, 0) = p \cdot x$. Suppose that there exists $\beta > 0$ such that for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$*

$$G(x, t) - p \cdot x \leq -\beta t + C.$$

Then

$$\max_{x \in \mathbb{R}^n} \lambda v_\lambda(x) < -\frac{\beta}{2} + \lambda C$$

Here C represents a constant depending only on d and V.

Proof By comparing $G(x, t)$ and $p \cdot x \pm Mt$ for a suitable constant M depending only on d and V, the comparison principle Theorem 2.1 implies that

$$|G(x, t) - p \cdot x| \leq Mt \quad \text{for all } (x, t) \in \mathbb{R}^n \times [0, \infty).$$

Apparently, $\beta \leq M$.

In addition, by periodicity of V, $G(x + \vec{l}, t) - p \cdot \vec{l}$ is also a solution of (1.2) with intial data $p \cdot x$ for any $\vec{l} \in \mathbb{Z}^n$. Then uniqueness implies that $G(x + \vec{l}, t) - p \cdot \vec{l} = G(x, t)$, equivalently, $G(x, t) - p \cdot x$ is $\mathbb{Z}^n$-periodic for x. By the assumption, we may choose $T_0 > 0$ such that

$$G(x, t) - p \cdot x \leq -\frac{2\beta t}{3} \quad \text{for } (x, t) \in \mathbb{R}^n \times [T_0, \infty).$$

Step 1: Choose $f(t) \in C^{\infty}([0, \infty))$ such that

$$f'(t) \leq -\frac{\beta}{2}$$

and

$$f(t) = \begin{cases} -2Mt & \text{when } t \geq 2T_0 \\ -\frac{\beta t}{2} & \text{when } t \in [0, T_0]. \end{cases}$$

Define

$$h(x) = \min_{t>0}(G(x, t) - p \cdot x - f(t)) > -\infty.$$

Clearly, $h(x) < 0$. Hence the minimum is attained for some $t_x \in (0, 2T_0]$. Therefore, $h(x)$ is periodic, continuous and is a viscosity supersolution of

$$\left(1 - d\,\text{div}\left(\frac{p + Dh}{|p + Dh|}\right)\right)_+ |p + Dh| + V(x) \cdot (p + Dh) = \frac{\beta}{2} \quad \text{on } \mathbb{R}^n.$$

Step 2: Then

$$h_\lambda(x) = h(x) - \min_{x \in \mathbb{R}^n} h(x) - \frac{\beta}{2\lambda}$$

is a viscosity supersolution of

$$\lambda h_\lambda(x) + \left(1 - d\,\text{div}\left(\frac{p + Dh_\lambda}{|p + Dh_\lambda|}\right)\right)_+ |p + Dh_\lambda| + V(x) \cdot (p + Dh_\lambda) = 0.$$

Accordingly, by comparison principle,

$$v_\lambda(x) \leq h_\lambda(x),$$

which implies that

$$\max_{x \in \mathbb{R}^n} \lambda v_\lambda(x) \leq -\frac{\beta}{2} + 2\lambda \max_{x \in \mathbb{R}^n} |h(x)|.$$

Hence our conclusion holds.   □

Hereafter, we let $x = (x_1, x_2) \in \mathbb{R}^2$ and
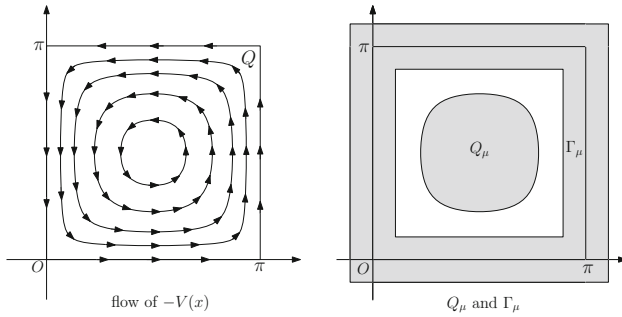
$$V(x) = A(DH)^{\perp} = A(-\cos(x_2)\sin(x_1), \cos(x_1)\sin(x_2)).$$

flow of $-V(x)$ | Q_μ and Γ_μ

Fig. 3 Flow of $-V$, two domains Q_μ and Γ_μ

for $A > 0$. Keep in mind that our game is using

$$-V(x) = A(\cos(x_2)\sin(x_1), -\cos(x_1)\sin(x_2)).$$

Write

$$Q = [0, \pi] \times [0, \pi], \quad Q_\mu = \{x \in Q \mid H(x) > \mu\}$$

and

$$\Gamma_\mu = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid \min\{|x_1|, |x_2|, |x_1 - \pi|, |x_2 - \pi|\} < \mu\}.$$

for $\mu \in (0, 1]$. Note that Q_μ is $-V$ flow invariant (See Fig. 3).

We will utilize the special structure of the cellular flow to establish reachability.

Lemma 3.2 *Suppose $\mu \in [0, 1)$. For $P_1 \in Q_\mu$, the level curve $\{x \in Q \mid H(x) = \mu\}$ is reachable from P_1 within time $3\sqrt{2}$. In particular, for $\mu > 0$, every point $P_2 \in \{x \in Q \mid H(x) = \mu\}$ is reachable from P_1 within time $\leq C(1 + |\log \mu|)$ for a constant C depending only on V.*

Proof First we prove the reachability to the level curve. Given the game step size parameter τ, recall that the game dynamics is

$$\begin{cases} X_{n+1} = X_n + \tau\sqrt{2d}b_n\eta_n + \tau^2\eta_n^\perp - \tau^2 V(X_n) \\ X_0 = P_1 \end{cases}$$

for $n \geq 0$. Here $\eta_n \in \overline{B}_1(0) = \{v \in \mathbb{R}^2 \mid |v| \leq 1\}$ and $b_n \in \{-1, 1\}$.

Case 1: $P_1 \neq (\frac{\pi}{2}, \frac{\pi}{2})$. Then $|DH(P_1)| > 0$. Player I chooses the strategy as follows: Let $X_0 = P_1$.

At each step $n \geq 0$, if $H(X_n) > H(P_2)$, player I chooses

$$\eta_n = \frac{V(X_n)}{|V(X_n)|}.$$

Then

$$\eta_n^\perp = -\frac{DH(X_n)}{|DH(X_n)|}.$$

Then regardless of how player II moves, we have that

$$
\begin{aligned}
H(X_{n+1}) - H(X_n) &= H\left(X_n + \tau b_n \sqrt{2d}\,\eta_n + \tau^2 \eta_n^\perp - \tau^2 V(X_n)\right) - H(X_n) \\
&= \left\langle DH, \tau b_n \sqrt{2d}\,\eta_n + \tau^2 \eta_n^\perp - \tau^2 V(X_n)\right\rangle \\
&\quad + \frac{1}{2}\left\langle D^2 H \tau b_n \sqrt{2d}\,\eta_n, \tau b_n \sqrt{2d}\,\eta_n\right\rangle + O(\tau^3) \\
&= \tau^2\left[-|DH| + d\left\langle D^2 H \frac{D^\perp H}{|DH|}, \frac{D^\perp H}{|DH|}\right\rangle\right] + O(\tau^3).
\end{aligned}
$$

Since $H(X_n) = \sin x_{1n} \sin x_{2n}$ for $X_n = (x_{1n}, x_{2n})$, we have

$$
\begin{aligned}
-\max_{\mathbb{R}^2} \|D^2 H\| &\leq \left\langle D^2 H \frac{D^\perp H}{|DH|}, \frac{D^\perp H}{|DH|}\right\rangle \\
&= \frac{\sin(x_{1n}) \sin(x_{2n})(\cos(2x_{1n}) + \cos(2x_{2n}) + 2)}{\cos(2x_{1n}) \cos(2x_{2n}) - 1} \leq 0.
\end{aligned}
$$

Here the negative sign is essentially due to the convexity of the level curve of H instead of the specific form of H. So

$$H(X_{n+1}) - H(X_n) \leq -|DH(X_n)|\tau^2 + O(\tau^3).$$

Let $a_n = H(X_n)$. Then by Lemma 3.3,

$$a_{n+1} \leq a_n - \sqrt{2(a_n - a_n^2)}\,\tau^2 + O(\tau^3).$$

This implies that the decreasing rate of $a_n = H(X_n)$ when n increases is no slower than the decreasing rate along the ODE

$$\dot{s}(t) = -\sqrt{2s(t)(1 - s(t))} \quad \text{with } s(0) = H(P_1) \in (0, 1),$$

which has a unique solution when $s(t) \in (0, 1)$. Assume that $s(t) > 0$ for $t \in (0, t_0)$ and $s(t_0) = 0$. Clearly $s(t)$ is strictly decreasing for $t \in [0, t_0)$. Since $\sqrt{2s(t)(1 - s(t))} \geq \sqrt{1 - s(t)}$ when $s(t) \in (\frac{1}{2}, 1)$ and $\sqrt{2s(t)(1 - s(t))} \geq \sqrt{s(t)}$ when $s(t) \in (0, \frac{1}{2}]$, we have that

$$t_0 \leq \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1 - s}}\, ds + \int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{s}}\, ds = 2\sqrt{2}.$$
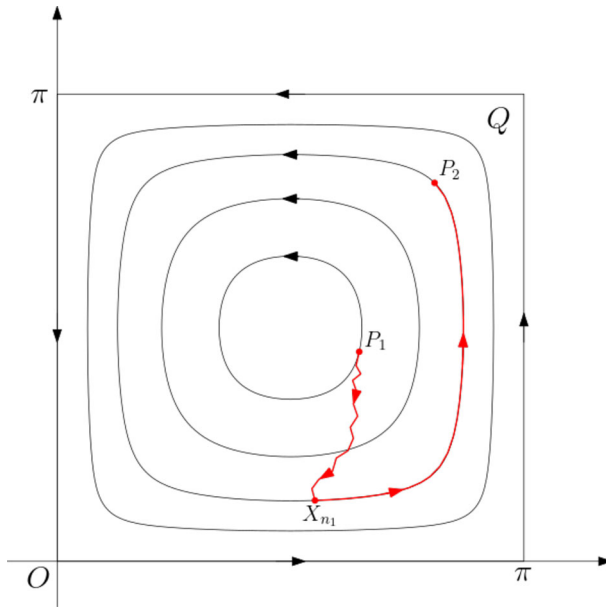
Fig. 4 Game trajectory of Lemma 3.2

In particular, the upper bound is independent of the value of $H(P_1)$. Hence a simple calculation shows that when τ is small enough, after at most $n_1 \leq \frac{3\sqrt{2}}{\tau^2}$ steps, we have that

$$H(X_{n_1}) \leq \mu$$

for the first time. In particular, $d(X_{n_1}, \{x \in Q | H = \mu\}) \leq C\tau$. Hence by Definition 2.6, the needed time to reach the level curve $T_1 \leq 3\sqrt{2}$.

Case 2: $P_1 = (\frac{\pi}{2}, \frac{\pi}{2})$. First travel a little bit away from P_1. Choose $r > 0$ such that $\max_{\overline{B}_r(P_1)} |V| \leq \frac{1}{4}$. Then player I can use the exit strategy as in the proof of Lemma 2.1. It is easy to see that player I could reach $\partial B_r(P_1)$ within time Cr^2. Then it goes to Case 1.

Next we show the reachability to every point P_2 on the level curve $\{x \in Q | H(x) = \mu\}$. After the game trajectory reaches the level curve , player I can just choose $\eta = 0$, i.e., follow the $-V$ flow to travel along the level curve to reach every point P_2 on the curve. When μ is close to 1, the traveling time around the level curve is near 2π (See Fig. 4). When μ is close to 0, the traveling time is bounded by $O(|\log \mu|)$. Hence the conclusion holds. □

Remark 3.1 The reachability established in the above lemma is only one way: P_1 is NOT reachable from P_2 if $H(P_1)$ is close to 1 since the curvature on the level curve will become very big and surpass 1. This is different from the invicid case ($d = 0$) where two points are mutually reachable.

Lemma 3.3 *For $x = (x_1, x_2) \in [0, \pi] \times [0, \pi]$,*

$$|DH(x)| \geq \sqrt{2(H(x) - H^2(x))}.$$

Proof A direction computation shows that

$$
\begin{aligned}
|DH(X)| &= \sqrt{\sin^2 x_1 + \sin^2 x_2 - 2\sin^2 x_1 \sin^2 x_2} \\
&\geq \sqrt{2\sin x_1 \sin x_2 - 2\sin^2 x_1 \sin^2 x_2} \\
&= \sqrt{2(H(x) - H^2(x))}.
\end{aligned}
$$

$\square$

We would like to point out that the convexity of level sets of H and the above inequality provide technical convenience in the proof, but they are not essential in obtaining the existence of $\overline{H}(p)$. See Sect. 4 for more details.

Lemma 3.4 *There exists $\mu_0 > 0$ and $T_0 > 0$ depending only on d and V such that the set $Q_{2\mu_0}$ is reachable from every point $x \in \Gamma_{\mu_0}$ within time T_0.*

Proof It is enough to prove the above conclusion when Γ_{μ_0} is replaced by the set

$$\mathcal{A}_{\mu_0} = \left[-\mu_0, \frac{2\pi}{3}\right] \times [-\mu_0, \mu_0]$$

since the proof for other pieces are similar and we can just choose the smallest μ_0.

Step 1: We first consider points near the corner $O = (0, 0)$. For $\alpha > 0$, set

$$\Omega_\alpha = (\alpha, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Note that $V(x) \cdot (1, 0) = 0$ for all $x \in \{0\} \times \mathbb{R}$. By applying Lemma 5.3 for $S = (0, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\Omega = \Omega_\alpha$, we have that there is $\alpha_1 > 0$ such that $\Omega_{2\alpha_1}$ is reachable from every point $x \in [-\alpha_1, \alpha_1]^2$ within time 1.

Step 2: Next we look at points on the line segment

$$L_1 = \left\{(x_1, 0) \mid x_1 \in \left[\frac{\alpha_1}{2}, \frac{2\pi}{3}\right]\right\}.$$

By applying Lemma 5.3 for $S = Q$ and $\Omega = Q_\mu$, we deduce that there is $\mu_0 \in (0, \frac{\alpha_1}{2})$ such that $Q_{2\mu_0}$ is reachable by every $x \in L_{1,\mu_0} = L_1 \times [-\mu_0, \mu_0]$ within time 1.

Step 3: Next, given a point $P_1 \in [-\alpha_1, \alpha_1]^2$, by step 1, player I has a strategy to push the game trajectory to some point $P_2 \in \Omega_{2\alpha_1}$ within time 1. If $P_2 \notin L_{1,\mu_0}$, then $|H(P_2)| \geq \sin \mu_0 \sin(2\alpha_1)$. Hence, by Lemma 3.2, P_2 can reach L_{1,μ_0} within time t_1. Thus by step 2, P_1 can reach $Q_{2\mu_0}$ within time $1 + 1 + t_1 = 2 + t_1$ (See Fig. 5).

Accordingly, every point on $\mathcal{A}_{\mu_0} \subset \left([-\alpha_1, \alpha_1]^2 \cup L_{1,\mu_0}\right)$ can reach $Q_{2\mu_0}$ within time $2 + t_1$. Note that μ_0 (hence t_1 as well) depends only on d and V. $\square$
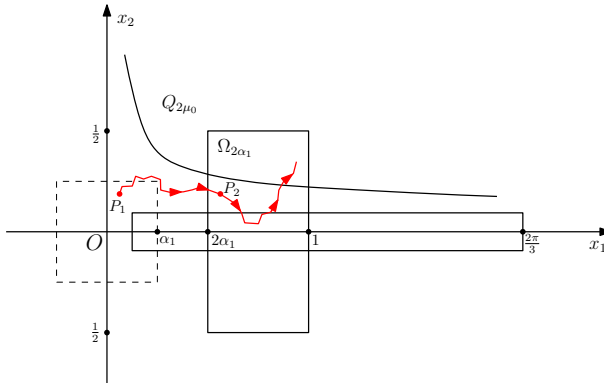
Fig. 5 Game trajectory of Lemma 3.4

Given a point $X_0 = (a_0, b_0)$ and $\tilde{\delta} > 0$, let $\gamma(s) \in C([0, 1], \mathbb{R}^2)$ be a continuous simple curve satisfying

$$\gamma((0, 1)) \in (a_0 - \tilde{\delta}, a_0 + \tilde{\delta}) \times (-\infty, b_0)$$

and $\gamma(0) \cdot (1, 0) = a_0 - \tilde{\delta}, \gamma(1) \cdot (1, 0) = a_0 + \tilde{\delta}$. Write

$$J_r = \left(a_0 - \frac{\tilde{\delta}}{2}, \ a_0 + \frac{\tilde{\delta}}{2}\right) \times \left(b_0 + r, \ b_0 + r + r'\right)$$

for some $r, r' > 0$. Note that the curve γ divides the strip $[a_0 - \tilde{\delta}, a_0 + \tilde{\delta}] \times \mathbb{R}$ into two connected components I_1 and I_2. Let I_1 be the lower component.

Lemma 3.5 *Given a point $x = (x_1, x_2) \in I_1$, if x can reach J_r within $t_0 \leq \frac{\tilde{\delta}}{3M}$, then x can also reach $\gamma([0, 1])$ within t_0. Here $M = 1 + \max_{\mathbb{R}^2} |V|$.*

Proof According to Definition 2.6 of the reachability, player I has a J_r-oriented strategy to move the game trajectory into J_r within time t_0 regardless of how player II responses. Let U be an open set such that $\gamma([0, 1]) \subset U$ and

$$\nu = d(\gamma([0, 1]), \partial U) < 1.$$

We claim that the corresponding game trajectory of player I's J_r-oriented strategy with a game step size parameter $\tau < \frac{\nu}{M+d}$ must enter U before it arrives at J_r. If not, then at some moment $t_1 < t_0$, the game trajectory arrives at

$$\{x = (x_1, x_2) | \ |x_1 - a_0| > \tilde{\delta}\}.$$

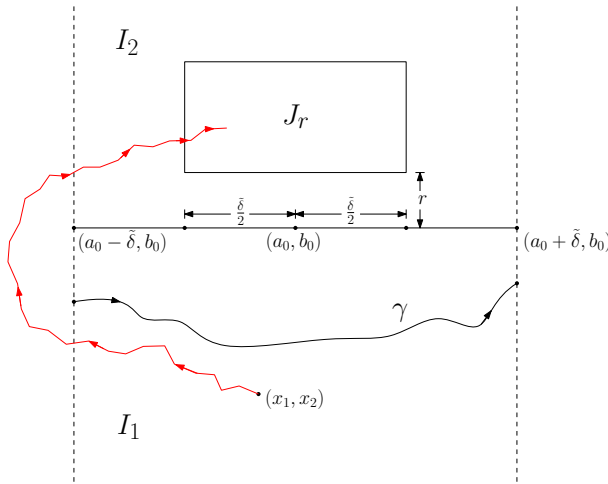Without loss generality, we assume that it arrives at $\{x_1 < a_0 - \tilde{\delta}\}$ (See Fig. 6).

$\underline{\textcircled{2}}$ Springer

Fig. 6 Game trajectory of Lemma 3.5

Then player II can simply employ the strategy: at each step n, after play I picks the direction η_n, player II chooses the sign b_n such that

$$b_n \eta_n \cdot (1, 0) \leq 0.$$

Then the only remaining force that can push the game trajectory to move along the positive horizontal direction is the mean burning speed 1 and the flow $-V$. Therefore, it will take the game at least $\frac{\tilde{\delta}}{2M}$ time to reach J_r. Since $t_0 \leq \frac{\tilde{\delta}}{3M} < \frac{\tilde{\delta}}{2M}$, this is a contradiction. □

Corollary 3.1 *For any $0 < \theta < \frac{\pi}{8}$, every point in the set*

$$Z_\theta = ((\theta, \pi - \theta) \times [0, \pi]) \cup ([0, \pi] \times (\theta, \pi - \theta))$$

is reachable from the center $P_0 = (\frac{\pi}{2}, \frac{\pi}{2})$ within time t_θ depending only on θ, V and d.

Proof Let $M = 1 + \max_{\mathbb{R}^2} |V|$. Due to symmetric structure and Lemma 3.2, it suffices to show that for some $\delta_\theta > 0$, every point in the set

$$S_\theta = \left([\theta, \pi - \theta] \times \left[0, \frac{\pi}{2}\right]\right) \cap \{0 \leq H < \delta_\theta\}$$

is reachable from P_0 within time t_θ depending only on θ, V and d.

By applying Lemma 5.3 for $S = \left(\frac{3\theta}{8}, \frac{5\theta}{8}\right) \times \left(-\frac{\theta}{4}, 0\right)$ and

$$\Omega_m = \left(\frac{3\theta}{8}, \frac{5\theta}{8}\right) \times \left(-\frac{1}{m} - \frac{\theta}{4}, -\frac{1}{m}\right)$$

we can show that there exist $m \in \mathbb{N}$ and $X_m = (\frac{\theta}{2}, \frac{1}{m})$ such that $\frac{1}{m} < \frac{\theta}{4}$ and every point $x \in B_{\frac{1}{4m}}(X_m)$ can reach Ω_m within time $\frac{\theta}{12M}$. Now fix m and set

$$\delta_\theta = \frac{1}{2} \min_{x \in \overline{B}_{\frac{1}{4m}}(X_m)} H(x).$$

Applying Lemma 3.5 with $\tilde{\delta} = \frac{\theta}{4}$ to level curves of the stream function H within the strip $\left[\frac{\theta}{4}, \frac{3\theta}{4}\right] \times \mathbb{R}$, we deduce that for any $\mu \in [0, \delta_\theta]$, every point on $B_{\frac{1}{4m}}(X_m)$ can reach the curve

$$\xi_\mu = \{H = \mu\} \cap \left(\left[\frac{\theta}{4}, \frac{3\theta}{4}\right] \times \left(-\infty, \frac{\pi}{2}\right]\right)$$

within time $\frac{\theta}{12M}$.

Next we show that every point on S_θ is reachable from P_0.

In fact, suppose that $P_1 \in S_\theta$. Let $\mu_1 = H(P_1) \in [0, \delta_\theta]$.

Step 1: Thanks to Lemma 3.2, P_0 can reach X_m within time t_1. By the definition of reachability, the game trajectory can arrive at some point in $Y_1 \in B_{\frac{1}{4m}}(X_m)$ within t_1.

Step 2: By the above discussion and the definition of reachability, starting from Y_1, for any $k \geq 2 \in \mathbb{N}$, the game trajectory can arrive at a point Y_k such that $d(Y_k, \xi_{\mu_1}) < \frac{1}{k}$ within time $t_2 = \frac{\theta}{12M}$.

Step 3: Starting from Y_k, at each step n, player I chooses $\eta_n = 0$ (i.e., just following the flow $-V$). Then within time t_3, the game trajectory will arrive at some point as close to P_1 as we want when $k \to +\infty$. Note that $|V|$ has a positive lower bound depending on θ on $\left(\left[\frac{\theta}{4}, \pi - \theta\right] \times \left[0, \frac{\pi}{2}\right]\right)$. So t_3 has a upper bound depending only on θ and V.

Hence P_0 can reach any neighborhood of P_1 within time $t_1 + t_2 + t_3$ (See Fig. 7). Note that each t_i depends only on θ, V and d.   □

Recall that $Q = [0, \pi]^2$. Consider the interior of four cells in $[-\pi, \pi]^2$: $U_1 = (0, \pi) \times (0, \pi)$, $U_2 = U_1 - (\pi, 0)$, $U_3 = U_1 - (0, \pi)$ and $U_4 = U_1 - (\pi, \pi)$. Clearly, by similar proofs, the corresponding versions of previous results on U_1 could also be established for U_i ($i = 2, 3, 4$).

Below is a transition property from one cell to another.

Lemma 3.6 *There exists $\beta > 0$ depending only on d and V such that every point $P \in U_1$ can reach the set U_i within time β for $i = 2, 3, 4$.*

Proof It suffices to prove this for U_2. The others are similar. Fix $P \in U_1$. Choose μ_0 from Lemma 3.4 that works for the corresponding statements for all four cells in $[-\pi, \pi]^2$. Choose the largest $\tilde{\mu} \in (0, \mu_0)$ such that

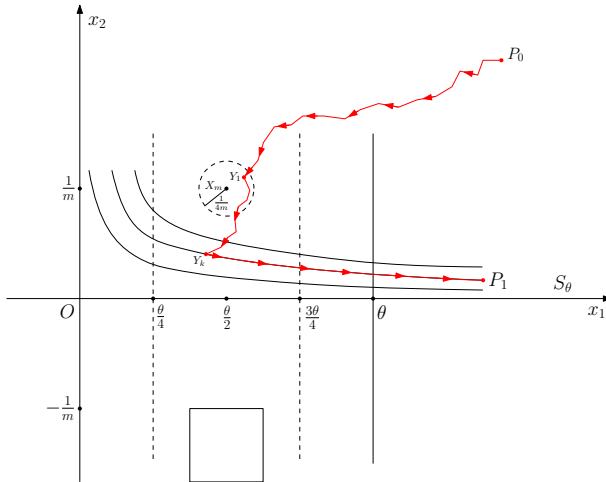$$\{x \in Q \mid H(x) \leq \tilde{\mu}\} \subset \overline{\Gamma}_{\mu_0}.$$

Fig. 7 Game trajectory of Corollary 3.1

Case 1: If $H(P) \geq \tilde{\mu}$, by Lemma 3.2, P can reach at some point

$$P_1 \in J = \left\{ x \in Q \mid \frac{\tilde{\mu}}{2} < H(x) < \tilde{\mu} \right\}$$

within time t_1. Then by following the flow $-V$, P_1 can reach a point $P_2 \in J \cap ((0, \mu_0) \times (0, \pi))$ within time t_2. Since

$$(0, \mu_0) \times (0, \pi) \subset \Gamma_{\mu_0} - (\pi, 0),$$

P_2 can reach U_2 within time T_0 by applying Lemma 3.4 to U_2 (See Fig. 8).

Case 2: If $H(P) < \tilde{\mu}$, then $P \in \Gamma_{\mu_0}$. Owing to Lemma 3.4, P can arrive at some $\tilde{P}_1 \in Q_{2\mu_0}$ within time t_3, which goes back to case 1 (See Fig. 9).

In the above, t_1, t_2 and t_3 depend only on d and V. Let $\beta = t_1 + t_2 + t_3 + T_0$.   □

Lemma 3.7 *Let $p \in \mathbb{R}^2$ be a unit vector and $G(x, t)$ be the unique solution of* (1.2) *with $G(x, 0) = p \cdot x$. Then there exists $\gamma > 0$ depending only on d and V such that*

$$G(x, t) - p \cdot x \leq -\gamma t + C.$$

Proof Since $|p| = 1$, without loss of generality, we may assume that $p \cdot (1, 0) > \frac{1}{2}$. Owing to the above Lemma 3.6, there exists $\beta > 0$ such that, starting from any point $x \in Q = [0, \pi]^2$, player I can design a strategy so that it takes at most 2β time to move the game trajectory into $U_1 + (-2\pi, 0)$. Using periodicity, staring from any point $x \in Q$, in time t, player I can design a strategy to move the game trajectory into $U_1 + (-2\pi k, 0)$ for some $k \geq \frac{t}{2\beta} - 1$ within time t. Since $U_1 + (-2\pi k, 0)$ is $-V$ flow invariant, when the game trajectory arrives at $U_1 + (-2\pi k, 0)$, if needed, player
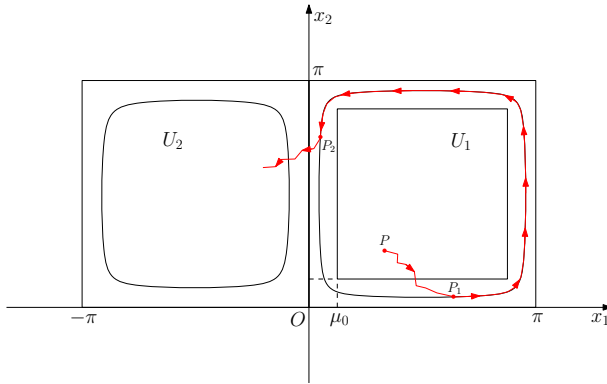
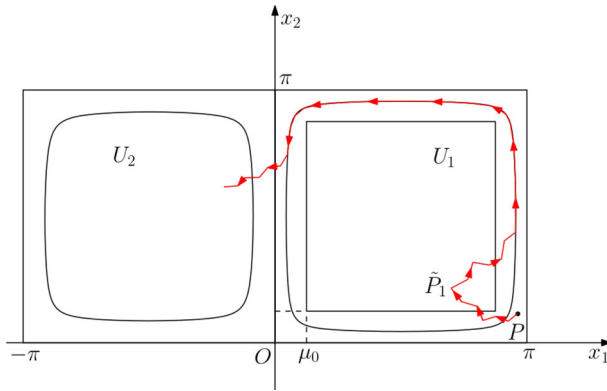**Fig. 8** Case 1 game trajectory of Lemma 3.6

Fig. 9 Case 2 game trajectory of Lemma 3.6

I can just choose $\eta = 0$ until time t. Accordingly, by the game theory formulation, we have that

$$G(x,t) \leq \max_{y \in Q + k(-2\pi, 0)} p \cdot y \leq -kp \cdot (2\pi, 0) + C \leq -\gamma t + C.$$

for $\gamma = \frac{\pi}{2\beta}$.   □

Combining with Lemma 3.1, we deduce that

Corollary 3.2 *When λ is small enough,*

$$\max_{x \in \mathbb{R}^2} \lambda v_\lambda(x) \leq -\frac{\gamma}{2}.$$

In particular, $v = v_\lambda$ is a viscosity supersolution of

$$\left(1 - ddiv\left(\frac{p + Dv}{|p + Dv|}\right)\right)_+ |p + Dv| + V(x) \cdot (p + Dv) = \frac{\gamma}{2} > 0, \quad in \, \mathbb{R}^2.$$

Here γ is from the previous Lemma 3.7.

Let $O_1 = (0, 0)$, $O_2 = (1, 0)$, $O_3 = (1, 1)$ and $O_4 = (0, 1)$.

Lemma 3.8 *Let $u_\lambda(x) = p \cdot x + v_\lambda(x)$. Then*

$$\max_{(x,y)\in[-\pi,\pi]^2} |u_\lambda(x) - u_\lambda(y)| \leq C$$

for a constant C depending only on V and d.

Proof Owing to (3.1), u_λ is always a viscosity subsolution of

$$\underline{F}(D^2 u_\lambda, Du_\lambda) + V(x) \cdot Du_\lambda = 1 + \max_{\mathbb{R}^2} |V(x)| \quad \text{in } \mathbb{R}^2.$$

Throughout the proof, C_μ represents various constants depending only on d, V and a given parameter μ.

Step 1: We first establish the difference bound within Q_μ for any given $\mu \in (0, 1)$. Note that the level curve $\{x \in Q \mid H(x) = \mu\}$ is flow invariant. When μ is close to 1, the traveling time around the level curve is near 2π. When μ is close to 0, the traveling time is bounded by $O(|\log \mu|)$.

Owing to Lemma 3.2, Lemma 2.4 and Lemma 2.5, we have that

1. For every point $x \in \partial Q_\mu$ and every point $y \in \overline{Q}_\mu$, $u_\lambda(x) \geq u_\lambda(y) - C_\mu$. Accordingly,

$$\min_{x\in\partial Q_\mu} u_\lambda(x) \geq \max_{x\in\overline{Q}_\mu} u_\lambda(x) - C_\mu.$$

2. By Corollary 3.2 and the minimum value principle Lemma 2.3,

$$\min_{x\in\partial Q_\mu} u_\lambda(x) = \min_{x\in\overline{Q}_\mu} u_\lambda(x).$$

Accordingly,

$$\max_{x\in\overline{Q}_\mu} u_\lambda(x) - \min_{x\in\overline{Q}_\mu} u_\lambda(x) = \max_{x,y\in\overline{Q}_\mu} |u_\lambda(x) - u_\lambda(y)| \leq C_\mu. \tag{3.2}$$

Step 2: Recall the four cells in $[-\pi, \pi]^2$: $U_1 = (0, \pi) \times (0, \pi)$, $U_2 = U_1 - (\pi, 0)$, $U_3 = U_1 - (0, \pi)$ and $U_4 = U_1 - (\pi, \pi)$. Let $P_0 = (\frac{\pi}{2}, \frac{\pi}{2})$, $q_1 = (0, 0)$, $q_2 = -(\pi, 0)$, $q_3 = -(0, \pi)$ and $q_4 = -(\pi, \pi)$. Then the above (3.2) also holds when Q_μ is replaced by $Q_\mu + q_i$ for $1 \leq i \leq 4$.

Let μ_0 be the number from Lemma 3.4 that works for all four cells. Combining with corresponding versions of Corollary 3.1 in all four cells, $Q_{2\mu_0} + q_i$ is reachable by $P_0 + q_j$ for $1 \leq i, j \leq 4$ within time T_0 depending only on d and V. Since $Q_{2\mu_0} + q_i$ are flow invariant, owing to Lemma 2.4 and (3.2) from Step 1, we deduce that for $1 \leq i, j \leq 4$

$$|u_\lambda(P_0 + q_i) - u_\lambda(P_0 + q_j)| \leq C. \tag{3.3}$$
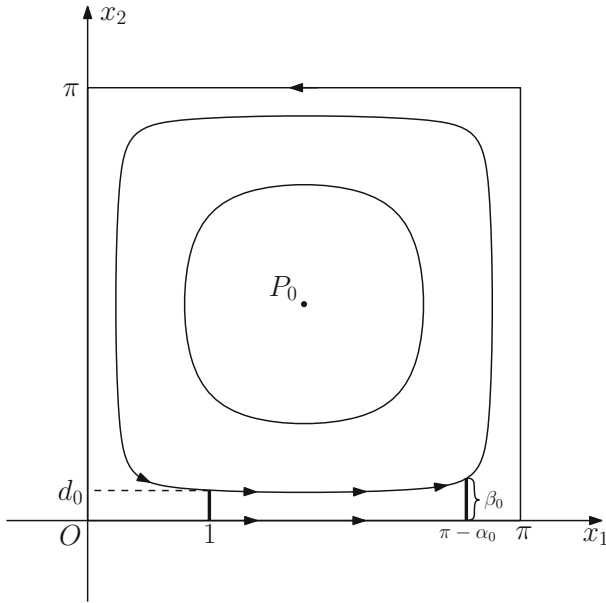
Fig. 10 d_0, β_0 and α_0

By employing (3.2) in all cells, we actually have that for any $\mu \in (0, 1]$,

$$|u_\lambda(x) - u_\lambda(y)| \le C_\mu \tag{3.4}$$

for $x, y \in \cup_{1 \le i \le 4}(Q_\mu + q_i)$.

Step 3: It remains to take care of regions near $\{H = 0\}$ since $C_\mu \to +\infty$ as $\mu \to 0$. Owing to Corollary 3.1, there exists $\tilde{t}_1$ such that every point $(1, d)$ for $d \in [0, 1]$ is reachable from P_0 within time $\tilde{t}_1$.

For $d \in [0, 1]$, let $\xi_d : [0, \infty) \to \mathbb{R}^2$ be the $-V$ flow starting from $(1, d)$, i.e., $\dot{\xi}_d(s) = -V(\xi_d(s))$ subject to $\xi_d(0) = (1, d)$. Write $\xi_0(\tilde{t}_1) = (\pi - 2v_0, 0)$. Fix $0 < \alpha_0 \le \min\{\mu_0, v_0\}$. Clearly, there exists $d_0 \in (0, \mu_0)$ such that for all $d \le d_0$,

$$\xi_d([0, \infty)) \cap (\{\pi - \alpha_0\} \times [0, 1]) = \xi_d(s_d)$$

for some $s_d \ge \tilde{t}_1$. Let

$$(\pi - \alpha_0, \beta_0) = \xi_{d_0}(s_{d_0}).$$

By choose d_0 small enough, we may assume that $\beta_0 \le \alpha_0$ (See Fig. 10).

Due to Lemma 2.4, for all $d \le d_0$,

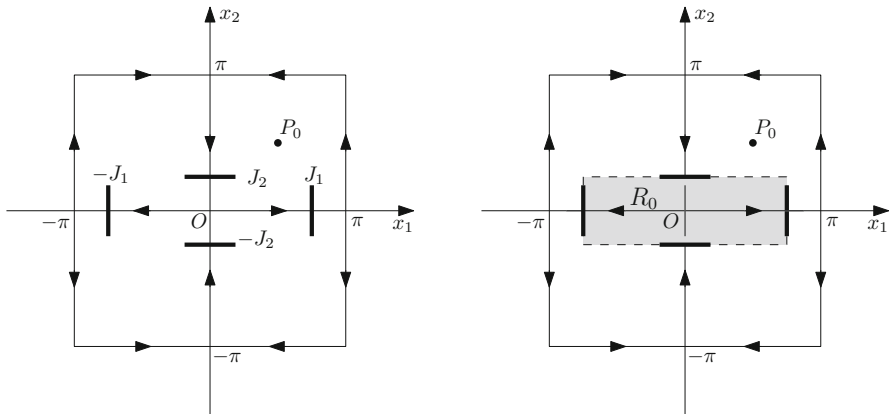$$u_\lambda(P_0) \le \max_{t \in [0, \tilde{t}_1]} u_\lambda(\xi_d(t)) + C.$$

Fig. 11 J_i and R_0

Combining with Lemma 2.5, for all $s \in [0, 1]$,

$$u_\lambda(P_0) \leq u_\lambda(\pi - \alpha_0, s\beta_0) + C.$$

By looking at other cells and using (3.3), we can find common $\beta_0 \leq \alpha_0 \in (0, \mu_0)$ such that

$$u_\lambda(P_0) \leq \min\{u_\lambda(x) \mid x \in J_1 \cup (-J_1) \cap J_2 \cap (-J_2)\} + C.$$

Here $J_1 = \{\pi - \alpha_0\} \times [-\beta_0, \beta_0]$ and $J_2 = [-\beta_0, \beta_0] \times \{\alpha_0\}$ (See Fig. 11).

Combining with Lemma 3.4, Lemma 2.4, the flow-invariance of Q_μ and (3.4), we have that

$$u_\lambda(P_0) \geq \max\{u_\lambda(x) \mid x \in J_1 \cup (-J_1) \cap J_2 \cap (-J_2)\} - C.$$

Now consider the rectangle $R_0 = [-\pi + \alpha_0, \pi - \alpha_0] \times [-\alpha_0, \alpha_0]$. Thanks to (3.4) and the above argument, we have that

$$\max_{x \in \partial R_0} |u_\lambda(P_0) - u_\lambda(x)| \leq C.$$

Again, combining with Lemma 3.4, Lemma 2.4, the flow-invariance of Q_μ and (3.4),

$$u_\lambda(P_0) \geq \max_{x \in R_0} u_\lambda(x) - C.$$

Finally, applying the minimum value principle Lemma 2.3 on R_0, we have that

$$\min_{x \in R_0} u_\lambda(x) = \min_{x \in \partial R_0} u_\lambda(x) \geq u_\lambda(P_0) - C.$$
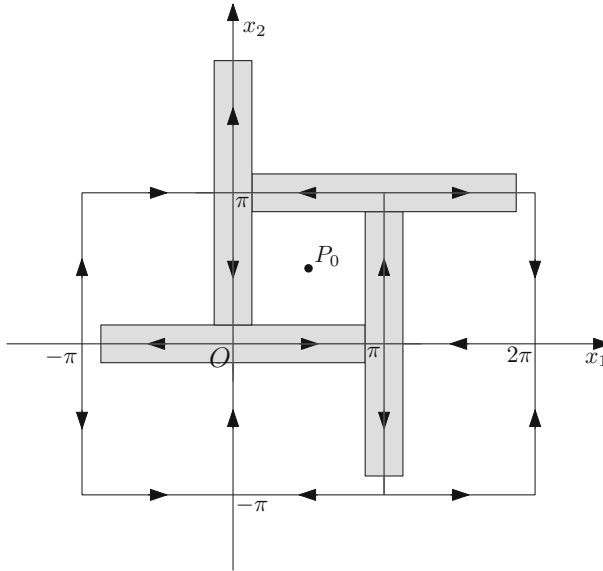
Fig. 12 Corresponding versions of R_0 on other edges

Accordingly, combining all the inequalities above, we derive that

$$\max_{x \in R_0} |u_\lambda(P_0) - u_\lambda(x)| \leq C.$$

By applying this to other cells and edges (see Fig. 12), we have that

$$\max_{x \in \Gamma_{\alpha_0}} |u_\lambda(P_0) - u_\lambda(x)| \leq C.$$

Similar conclusion also holds in other cells. Combining with (3.4), the lemma holds.
$\square$

According to the above Lemma 3.8, standard arguments lead to

Corollary 3.3 *For any unit vector p, there exists $\overline{H}(p) \in \mathbb{R}$ such that*

$$\lim_{\lambda \to 0} \lambda v_\lambda(x) = -\overline{H}(p) \quad \text{uniformly in } \mathbb{R}^2.$$

In particular, there exists a $\mathbb{Z}^2$-periodic $\overline{v} \in USC(\mathbb{R}^2)$ that is a viscosity subsolution of

$$\underline{F}(D^2\overline{v}, p + D\overline{v}) + V(x) \cdot (p + D\overline{v}) = \overline{H}(p)$$

and a $\mathbb{Z}^2$-periodic $\underline{v} \in LSC(\mathbb{R}^2)$ that is a viscosity supersolution of

$$\overline{F}(D^2\underline{v}, p + D\underline{v}) + V(x) \cdot (p + D\underline{v}) = \overline{H}(p).$$

Also,

$$\sup_{x \in \mathbb{R}^2} \{|\overline{v}(x)|, \ |\underline{v}(x)|\} < C_0$$

for a constant C_0 depending only on d and V.

Proof The argument of the existence of the limit is similar to the proof of (3) in the following lemma. Let

$$\overline{v}(x) = \limsup_{\lambda \to 0, y \to x} (v_\lambda(y) - v_\lambda(0))$$

and

$$\underline{v}(x) = \liminf_{\lambda \to 0, y \to x} (v_\lambda(y) - v_\lambda(0)).$$

It is easy to see that our conclusion holds. □

Lemma 3.9 *Below are several properties of $\overline{H}$.*

1. for $\lambda > 0$,

$$\overline{H}(\lambda p) = \lambda \overline{H}(p);$$

2.
$$\overline{H}(p) > 0 \ \ for \ p \neq 0;$$

3.
$$\overline{H}(p) \in C(\mathbb{R}^2).$$

Proof (1) is obvious from the definition of v_λ. (2) follows from Corollary 3.2. (3) follows from the standard stability of viscosity solutions. For the reader's convenience, we present details of the proof. Fixed a unit vector p. We argue by contradiction. If not, without loss of generality, we may assume that there exist a sequence of unit vectors $\{p_m\}_{m \geq 1}$ such that for $\lim_{m \to +\infty} p_m = p$ and for all $m \geq 1$

$$\overline{H}(p_m) \geq \overline{H}(p) + r$$

for some $r > 0$. Owing to Corollary 3.3, for each $m \geq 1$, let $v_m \in LSC(\mathbb{R}^2)$ be a $\mathbb{Z}^2$-periodic viscosity supersolution of

$$\overline{F}(D^2 v_m, p_m + Dv_m) + V(x) \cdot (p_m + Dv_m) = \overline{H}(p) + r$$

and

$$\sup_{x \in \mathbb{R}^2} |v_m(x)| \leq C_0$$

with the constant C_0 from Corollary 3.3. Let

$$\underline{v}(x) = \liminf_{m \to +\infty, y \to x} v_m(y).$$

Then $\underline{v}(x) \in LSC(\mathbb{R}^2)$ is a $\mathbb{Z}^2$-periodic viscosity supersolution of

$$\overline{F}(D^2\underline{v}, p + D\underline{v}) + V(x) \cdot (p + D\underline{v}) = \overline{H}(p) + r.$$

Also by Corollary 3.3, there is $\overline{v} \in USC(\mathbb{R}^2)$ that is a $\mathbb{Z}^2$-periodic viscosity subsolution to

$$\overline{F}(D^2\overline{v}, p + D\overline{v}) + V(x) \cdot (p + D\overline{v}) = \overline{H}(p).$$

Write $\underline{u} = p \cdot x + \underline{v}$ and $\overline{u} = p \cdot x + \overline{v}$. Then $\underline{u}$ is a viscosity supersolution of

$$\overline{F}(D^2\underline{u}, D\underline{u}) + V(x) \cdot D\underline{u} = \overline{H}(p) + r$$

and $\overline{u}$ is a viscosity subsolution of

$$\overline{F}(D^2\overline{u}, D\overline{u}) + V(x) \cdot D\overline{u} = \overline{H}(p).$$

Let

$$w_\delta(x, y) = \overline{u}(x) - \underline{u}(y) - \frac{|x - y|^4}{\delta}.$$

Due to the periodicity of $\overline{v}$ and $\underline{v}$, we can find $(x_\delta, y_\delta) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that

$$w_\delta(x_\delta, y_\delta) = \max_{x, y \in \mathbb{R}^2} w_\delta(x, y).$$

It is easy to see that

$$\lim_{\delta \to 0} w_\delta(x_\delta, y_\delta) = \max_{x \in \mathbb{R}^2} (\overline{v} - \underline{v}) \quad \text{and} \quad \lim_{\delta \to 0} \frac{|x_\delta - y_\delta|^4}{\delta} = 0. \tag{3.5}$$

Owing to Theorem 3.2 and Remark 3.8 in [13], there are two 2×2 symmetric matrices X and Y such that

$$X \le Y, \quad ||X|| + ||Y|| \le C\frac{|x_\delta - y_\delta|^2}{\delta}$$

and for $\bar{p} = 4(x_\delta - y_\delta)\frac{|x_\delta - y_\delta|^2}{\delta}$

$$\underline{F}(X, \bar{p}) + V(x_\delta) \cdot \bar{p} \le \overline{H}(p)$$
$$\overline{F}(Y, \bar{p}) + V(y_\delta) \cdot \bar{p} \ge \overline{H}(p) + r.$$

Also,

$$|(V(x_\delta) - V(y_\delta)) \cdot \overline{p}| \le \frac{C|x_\delta - y_\delta|^4}{\delta}.$$

Accordingly,

$$\underline{F}(X, \bar{p}) - \overline{F}(Y, \bar{p}) \le \frac{C|x_\delta - y_\delta|^4}{\delta} - r.$$

Case 1: If $x_\delta - y_\delta = 0$, then $X = Y = 0$. We have that

$$0 = 0 - 0 \le -r,$$

which is impossible.

Case 2: If $x_\delta - y_\delta \ne 0$. Then

$$\underline{F}(X, \bar{p}) - \overline{F}(Y, \bar{p}) = F(X, \bar{p}) - F(Y, \bar{p}) \ge 0,$$

we obtain

$$0 \le \frac{C|x_\delta - y_\delta|^4}{\delta} - r.$$

Owing to (3.5), as $\delta \to 0$, we have

$$0 \le -r,$$

a contradiction.   □

Proof of Theorem 1.1 Let $\overline{v}$ and $\underline{v}$ be functions from Corollary 3.3. Then by comparison principle Theorem 2.1,

$$p \cdot x + \epsilon \left(\overline{v}\left(\frac{x}{\epsilon}\right) - C_0\right) - \overline{H}(p)t \le G_\epsilon(x, t) \le p \cdot x + \epsilon \left(\underline{v}\left(\frac{x}{\epsilon}\right) + C_0\right) - \overline{H}(p)t$$

and

$$p \cdot x + \overline{v}(x) - C_0 - \overline{H}(p)t \le G(x, t) \le p \cdot x + \underline{v}(x) + C_0 - \overline{H}(p)t.$$

for the constant C_0 from Corollary 3.3. The conclusion follows immediately.   □

Remark 3.2 For application purposes, people are interested in deriving explicit formulas of the effective burning velocity under the G-equation model (see [22] for instance). Although a simple formula is mathematically not available, it might be practically interesting to investigate more detailed properties of $\overline{H}(p)$ (e.g., its anisotropy due to the presence of the fluid) in addition to its dependence on physical parameters. These kind of problems often require methods deeper than those standard PDE approaches
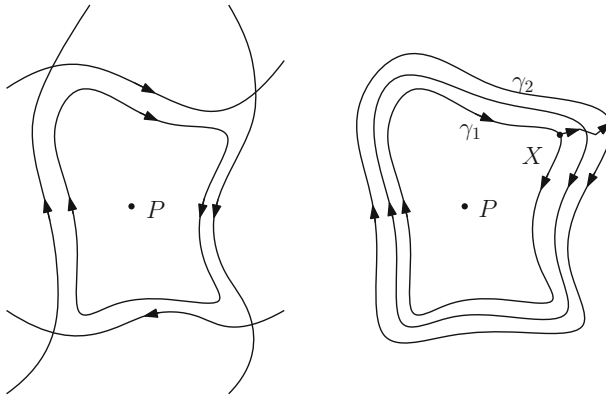
Fig. 13 Move forward within a non-convex cell

in Lemma 3.9. For instance, questions in this aspect have been studied in [21] and [38] for the case $v_{\vec{n}} = a(x)$ (e.g., phase transition in an inhomogeneous medium without considering curvature effect) using tools from dynamical systems. A conclusion from results there is that in 2D, a polygon could be an effective front if and only if it is centrally symmetric with rational vertices and nonempty interior. So different distributions of defects and heterogeneities can change the evolution significantly, which could have an important practical implication [12].

Remark 3.3 A natural question is whether our result can be proved by pure PDE or geometric approaches as in previous literature. For this aspect, Theorem 10.2 in [5] seems relevant. Checking Assumption B" there (if it holds) might need to evolve Q_μ by properly combining the motion law and the V flow. It is not clear to us how to arrange the motions to reach a stationary supersolution (basically a large V-flow invariant set in our context) within finite time. Game theoretical method provides more flexibility to handle detailed local structures. For instance, the non-divergence inviscid example in section 11.2 of [5] relying on Theorem 10.2 for 2D can be proved for all dimensions via control formulation using full reachability or partially reachability + minimum value principle as in this paper. See [42] for more applications of game theory in homogenization of non-coercive non-convex G-equations.

4 Extension to General 2D Incompressible Flows

In this section, we will briefly explain how to possibly modify our methods to cover more general 2D incompressible flows. Assume that $V = (-H_{x_2}, H_{x_1})$ for a general periodic stream function H. For simplicity, we just discuss two representative scenarios and the corresponding adjustments of our methods.

Case 1: Non-convex cells. In general, the level curves of H might consist of convex and concave parts (relative to a fixed cell).

- Movement within a cell (extension of Lemma 3.2). Starting from a level curve γ_1, the game trajectory can move forward on the convex part (or near flat part)
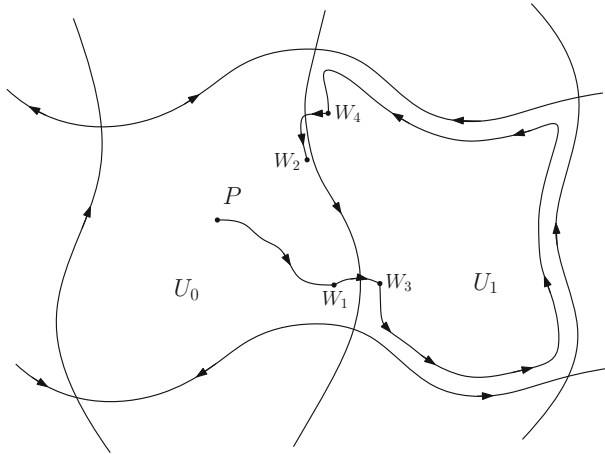
Fig. 14 Move to points near the boundary and to an adjacent cell

of γ_1 using a modified version of Lemma 5.3 to arrive at another level curve γ_2, then travel along γ_2 to its convex parts (or near flat parts) in order to further move forward (See Fig. 13).

- Movement to reach the boundary and adjacent cells (extension of Lemma 3.4 and Corollary 3.1). Starting from the center critical point P in the cell U_0, player I aims to drive the game trajectory to travel to a point W_2 that is very close to a concave portion of the boundary. As in this paper, we need to control the amount of travel time. Since the game trajectory might not be able to cross the boundary on the concave portion, it can first move to a point W_1 near (but not very close to) the convex part of the boundary. Next use a modified version of Lemma 5.3 to reach W_3 in the adjacent cell U_1. Then follow the flow $-V$ to a point W_4 that is close (but not too close) to the concave part of the boundary of U_0 and move back to U_0 based on a modified version of Lemma 5.3, and finally travel to W_2 along the flow $-V$ (See Fig. 14).

Other lemmas and conclusions in this paper can be extended similarly.

Case 2: Cat's-eye types. For simplicity, we look at the following representative example

$$H(x_1, x_2) = \sin x_1 \sin x_2 + \delta \cos x_1 \cos x_2$$

for $\delta \in (0, 1)$. The picture consists of islands (e.g. shaded regions I_1, I_2 in figure below) and unbounded periodic orbits of $-V$ flow (e.g., regions J_1, J_2 in the figure below) (See Fig. 15).

Step 1: By similar arguments in this paper and possible extensions as in the above Case 1 together with periodicity, there exist constants $\bar{c}_1$, $\bar{c}_2$, c_1 and c_2 such that for $i = 1, 2$, $\lim_{\lambda \to 0} \lambda u_\lambda = \bar{c}_i$ in the unbounded region J_i and $\lim_{\lambda \to 0} \lambda u_\lambda = c_i$ within I_i.
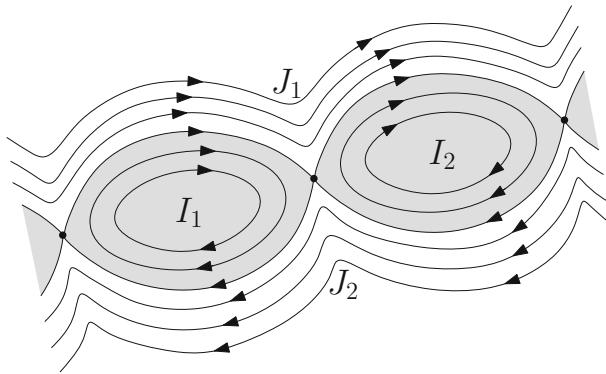
Fig. 15 Cat's-eye type flows

Step 2: By partial reachability (from I_1, I_2 to J_1 and J_2 due to convexity of the boundary of I_i), we have that $c_1, c_2 \leq \bar{c}_1, \bar{c}_2$.

Step 3: Note that for (x_1, x_2) near $(0, 0)$,

$$H(x_1, x_2) = x_1 x_2 + \delta \left(1 - \frac{x_1^2}{2} - \frac{x_2^2}{2} \right) + O(x_1^4 + x_2^4).$$

It is not hard to see that the curvature of $H(x_1, x_2) = \delta$ (the boundary of islands) tends 0 when approaching saddle points (inflection points). Therefore, starting from some regions of J_1 or J_2, player I can move the game trajectory into those islands through portions of the boundary near saddle points. Accordingly, we have that $c_1 = c_2 = \bar{c}_1 = \bar{c}_2$.

For other cat's-eye type flows, it could happen that one island I is uniformly convex and, hence, game trajectories might not be able to enter it from unbounded domains when the Markstein diffusivity d is large. For this case, game trajectories can still enter islands adjacent to I and the minimum value principle will lead to the same conclusion. One such example is

$$H(x_1, x_2) = x_1^2 - x_2^2 - 2x_1^3 + x_1^4 \quad \text{for } (x_1, x_2) \text{ near } (0, 0).$$

Here $H = 0$ implies $x_2 = x_1 - x_1^2$ or $x_2 = -x_1 + x_1^2$ near $(0, 0)$.

We expect that the effective burning velocity should exist for quite general 2D incompressible flows, at least if all critical points H are non-degenerate where the flow structure is well-understood and essentially the combination of case 1 and case 2 (see [3]). The main challenge is how to find an efficient systematic proof without examining all possible scenarios. For non-smooth flows ($H \in C^{1,1}$, or equivalently V is only Lipschitz continuous) or flows with degenerate stagnation points, the analysis appears more complicated. We plan to investigate this issue in the future.

Remark 4.1 Our proof of Theorem 1.1. suggests a *general framework* based on the one sided reachability of game trajectories and the minimum value principle of suitable

stationary problems. The former is a separate dynamical system issue that has to be verified for a given flow field V, be it periodic, almost periodic or random. If it is true or almost surely true, the remaining PDE argument extends and the existence of effective front speed might be established in interesting non-periodic settings.

Data Availability This paper does not generate any data.

5 Appendix

It is well known that the front propagation under the viscosity solution framework is consistent with the classical meaning when the smooth solutions exist. See section 6 in [16] for instance. The following conclusion is a special case in our context, which is needed to derive a reachability property. For the reader's convenience, we present its proof here in 2D, which is sufficient for our purpose.

Throughout this section, we only assume that $V \in W^{1,\infty}(\mathbb{R}^2)$ and

$$V(x) \cdot (1, 0) = 0 \quad \text{for } x \in \{0\} \times [0, 1]. \tag{5.1}$$

Let $S = (0, 1)^2$ and

$$g_S(x) = \begin{cases} -\frac{2}{\pi} \arctan(d(x, \partial S)) & \text{for } x \in S \\ \frac{2}{\pi} \arctan(d(x, \partial S)) & \text{otherwise.} \end{cases}$$

Here $d(x, \partial S)$ is the distance from x to ∂S.

Lemma 5.1 *Suppose that $G \in C(\mathbb{R}^2 \times [0, \infty))$ is the unique viscosity solution to equation* (1.2) *subject to*

$$G(x, 0) = g_S(x).$$

Then for a given $\delta \in (0, \frac{1}{2})$, there exists $t_\delta > 0$ depending only on d, V and δ such that

$$G((0, \theta), t) < 0 \quad \text{for } (\theta, t) \in [\delta, 1 - \delta] \times (0, t_\delta].$$

Proof Intuitively, this conclusion is obvious since the speed along the normal direction $\vec{n} = (-1, 0)$ at the point $(0, \theta)$ is

$$v_{\vec{n}} = 1 - d\kappa + V(x) \cdot \vec{n} = 1.$$

To make this rigorous, we need to build smooth supersolutions and employ comparison principle. It suffices to prove this at a fixed $\theta \in [\delta, 1 - \delta]$.
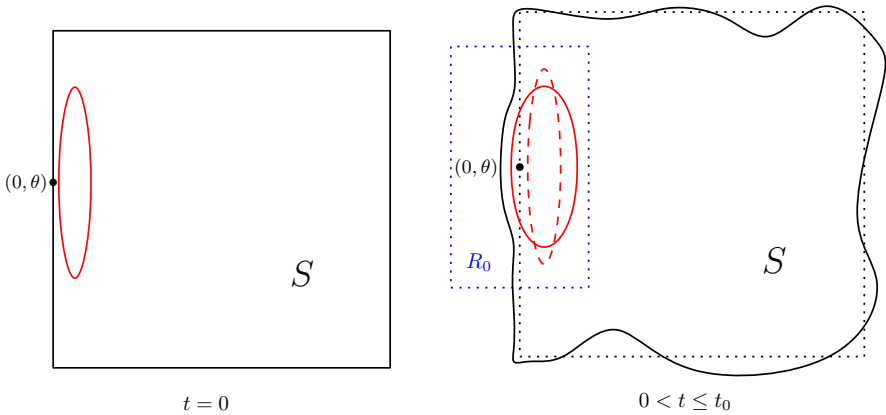
Fig. 16 Propagation of the ellipse

Step 1: Choose an ellipse. Consider the ellipse

$$E_\theta(t) : \frac{(x_1 - a_0 - v)^2}{a^2(t)} + \frac{(x_2 - \theta)^2}{b^2(t)} = 1.$$

Here $v \in (0, a_0)$ is added for technical convenience and will be sent to zero later (See Fig. 16).

Let $b_0 = \frac{\delta}{2}$. Then choose $a_0 \in (0, \frac{b_0}{4})$ small enough such that

$$|V(x) \cdot (1, 0)| < \frac{1}{8} \quad \text{if } x \in [-4a_0, 4a_0] \times [0, 1]$$

and

$$\frac{d64a_0}{b_0^2} + 4M_0\sqrt{3}\frac{a_0}{b_0} < \frac{1}{8}.$$

Here $M_0 = \max_{x \in \mathbb{R}^2} |V(x)|$.

Then we define $a(t) = a_0 + \frac{1}{2}t$ and $b(t) = b_0 - Lt$ for $L > 0$ satisfying

$$\frac{1}{2} - \frac{3La_0}{4b_0} < 1 - \frac{db_0}{a_0^2} - M_0.$$

Hereafter we require $0 \le t \le t_\delta$ for $t_\delta = \min\left\{2a_0, \frac{b_0}{2L}\right\}$, which implies that

$$(a(t), b(t)) \in [a_0, 2a_0] \times \left[\frac{b_0}{2}, b_0\right]$$

and

$$E_\theta(t) \subset R_0 = [-4a_0, 4a_0] \times [0, 1]$$

for $t \in [0, t_\delta]$. Also, $4a_0 < b_0$ leads to

$$a(t) < b(t) \quad \text{for all } t \in [0, t_\delta].$$

For convenience, we drop the dependence of a and b on t. Hereafter $t \in [0, t_\delta]$ unless specified otherwise. Let $\vec{n}$ be the outward unit normal vector along $E_\theta(t)$ that has the following parameterization: for $\phi \in [0, 2\pi]$

$$\begin{cases} x_1 = a_0 + v + a \cos \phi \\ x_2 = \theta + b \sin \phi. \end{cases}$$

Then

$$V \cdot \vec{n} = V \cdot \left(\frac{b \cos \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}}, \frac{a \sin \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \right).$$

If $|\sin \phi| < \frac{\sqrt{3}}{2}$,

$$
\begin{aligned}
|V \cdot \vec{n}| &\le |V(x) \cdot (1, 0)| + M_0 \cdot \frac{a \sin \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \\
&\le |V(x) \cdot (1, 0)| + M_0 \cdot \frac{a}{b} \cdot |\tan \phi| \\
&\le |V(x) \cdot (1, 0)| + M_0 \cdot \frac{2a_0}{b_0/2} \cdot \sqrt{3} \\
&= |V(x) \cdot (1, 0)| + 4\sqrt{3} M_0 \cdot \frac{a_0}{b_0} < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.
\end{aligned}
\tag{5.2}
$$

Step 2: Evolution of an elliptic boundary. Let us recall some basic facts. Given a C^1 function $f(x, t)$ and the family of level curves

$$C(t) = \{x \in \mathbb{R}^2 \mid f(x, t) = 0\},$$

if $D_x f \ne 0$, the propagation speed of $C(t)$ along the outward normal direction $\vec{n} = \frac{D_x f}{|D_x f|}$ is given by

$$v_{\vec{n}} = -\frac{f_t}{|D_x f|},$$

which can be easily derived through the chain rule. Moreover, the corresponding mean curvature along $C(t)$ is

$$\kappa = \operatorname{div}_x(\vec{n}).$$

Now let us verify that for $t \in [0, t_\delta]$, the propagation of $E_\theta(t)$ obeys the following inequality:

$$v_{\vec{n}} < 1 - d\kappa + V \cdot \vec{n}, \tag{5.3}$$

which will be used to construct a supersolution. Fix any $a, b > 0$ and an ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1,$$

direct compuations show that its curvature at point $P(a \cos \phi, b \sin \phi), 0 \le \phi < 2\pi$, is

$$\kappa = \frac{ab}{\left(a^2 \sin^2 \phi + b^2 \cos^2 \phi\right)^{\frac{3}{2}}}$$

and the normal velocity at $P(a \cos \phi, b \sin \phi)$ is

$$v_{\vec{n}} = \frac{a'b \cos^2 \phi + ab' \sin^2 \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}}.$$

Case 1. If $|\sin \phi| \le \frac{\sqrt{3}}{2}$, then

$$d\kappa = \frac{dab}{\left(a^2 \sin^2 \phi + b^2 \cos^2 \phi\right)^{\frac{3}{2}}} \le \frac{dab}{\left(b^2 \cos^2 \phi\right)^{\frac{3}{2}}} \le \frac{dab}{(b/2)^3} \le \frac{d64a_0}{b_0^2} < \frac{1}{8}.$$

Note that

$$v_{\vec{n}} = \frac{\frac{b}{2} \cos^2 \phi - La \sin^2 \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \le \frac{\frac{b}{2} \cos^2 \phi}{\sqrt{b^2 \cos^2 \phi}} \le \frac{1}{2}.$$

Since for $t \in [0, t_\delta]$,

$$E_\theta(t) \subset [-4a_0, 4a_0] \times [0, 1],$$

(5.2) implies that $v_{\vec{n}} \le \frac{1}{2} < 1 - \frac{1}{8} - \frac{1}{4} < 1 - d\kappa + V \cdot \vec{n}$.

Case 2. If $|\sin \phi| \ge \frac{\sqrt{3}}{2}$, then

$$v_{\vec{n}} = \frac{\frac{b}{2} \cos^2 \phi - La \sin^2 \phi}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} \le \frac{\frac{b}{2} \cos^2 \phi}{\sqrt{b^2 \cos^2 \phi}} - \frac{La \sin^2 \phi}{b}$$

$$\le \frac{1}{2} - \frac{3La}{4b} \le \frac{1}{2} - \frac{3La_0}{4b_0}.$$

Meanwhile,

$$d\kappa = \frac{dab}{\left(a^2 \sin^2 \phi + b^2 \cos^2 \phi\right)^{\frac{3}{2}}} \le \frac{db}{a^2} \le \frac{db_0}{a_0^2}.$$

Therefore, due to the choice of L, we have $v_{\vec{n}} < 1 - d\kappa + V \cdot \vec{n}$.

Combining case 1 and case 2, we see that (5.3) holds, i.e., the evolution of $E_\theta(t)$ satisfies

$$v_{\vec{n}} < 1 - d\kappa + V \cdot \vec{n}, \text{ for } 0 \le t \le t_\delta.$$

Step 3: Comparison. Let

$$\Psi(x,t) = \frac{(x_1 - a_0 - v)^2}{a^2(t)} + \frac{(x_2 - \theta)^2}{b^2(t)} - 1.$$

Owing to (5.3), we may choose $\mu_0 \in (0, \frac{1}{2})$ such that (5.3) also holds along the curve $\{x \in \mathbb{R}^2 | \Psi(x,t) = \mu\}$ for all $\mu \in [-\mu_0, \mu_0]$ and $t \in [0, t_\delta]$. Equivalently, for

$$D_\theta = \left\{ (x,t) \in \mathbb{R}^2 \times [0, t_\delta] \, | -\mu_0 \leq \Psi(x,t) \leq \mu_0 \right\},$$

$$\Psi_t + \left(1 - d \operatorname{div}\left(\frac{D\Psi}{|D\Psi|}\right)\right)|D\Psi| + V(x) \cdot D\Psi \geq 0 \quad \text{on } D_\theta. \quad (5.4)$$

Let $\{h_k\}_{k \geq 1} \in C^\infty(\mathbb{R})$ be a sequence of functions such that

$$0 < h_k' \leq 1 \quad \text{in } I_k = \left(-\mu_0 + \frac{1}{k}, \, \mu_0 - \frac{1}{k}\right), \quad h_k' = 0 \quad \text{in } \mathbb{R} \backslash I_k$$

and $\lim_{k \to +\infty} h_k(s) = h(s)$ uniformly in $\mathbb{R}$, where

$$h(s) = \begin{cases} \mu_0 & \text{for } s \geq \mu_0 \\ s & \text{for } s \in [-\mu_0, \mu_0] \\ -\mu_0 & \text{for } s \leq -\mu_0. \end{cases}$$

Apparently, $\Psi_k(x,t) = h_k(\Psi(x,t))$ satisfies

$$\frac{\partial \Psi_k}{\partial t} + \left(1 - d \operatorname{div}\left(\frac{D\Psi_k}{|D\Psi_k|}\right)\right)|D\Psi_k| + V(x) \cdot D\Psi_k \geq 0 \quad \text{on } \mathbb{R}^2 \times (0, t_\delta). \quad (5.5)$$

By stability, we have that $G_1(x,t) = h(\Psi(x,t)) \in W^{1,\infty}(\mathbb{R}^2 \times [0, \infty))$ is a viscosity supersolution of

$$\frac{\partial G_1}{\partial t} + \left(1 - d \operatorname{div}\left(\frac{DG_1}{|DG_1|}\right)\right)|DG_1| + V(x) \cdot DG_1 \geq 0 \quad \text{on } \mathbb{R}^2 \times (0, t_\delta). \quad (5.6)$$

Since $(a)_+ \geq a$, $G_1 = G_1(x,t)$ is also a viscosity supersolution of equation (1.2) on $\mathbb{R}^2 \times (0, t_\delta)$.

Because for fixed $v > 0$

$$\{G_1(x,0) \leq 0\} = \{\Psi(x,0) \leq 0\} \subset S,$$

we can choose a function $\xi \in C^\infty(\mathbb{R})$ such that $\dot{\xi} > 0$, $\xi(0) = 0$, $\sup_{s \in \mathbb{R}} |\xi(s)| < \infty$ and

$$g(x) \leq \xi(G_1(x,0)).$$

Since $\xi(G_1(x, 0))$ is also a viscosity supersolution of equation (1.2) on $\mathbb{R}^2 \times (0, t_\delta)$, thanks to Theorem 2.1, we have that

$$G(x, t) \leq \xi(G_1(x, t)) \quad \text{for } (x, t) \in \mathbb{R}^2 \times [0, t_\delta].$$

In particular, this implies that

$$\{x \in \mathbb{R}^2 | G_1(x, t) < 0\} = \{x \in \mathbb{R}^2 | \xi(G_1(x, t)) < 0\} \subset \{x \in \mathbb{R}^2 | G(x, t) < 0\}.$$

Note that $\{x \in \mathbb{R}^2 | G_1(x, t) < 0\} = \{x \in \mathbb{R}^2 | \Psi(x, t) < 0\}$. Sending $\nu \to 0$, we have that for $t \in [0, t_\delta]$,

$$\left\{ x = (x_1, x_2) \in \mathbb{R}^2 | \frac{(x_1 - a_0)^2}{a^2(t)} + \frac{(x_2 - \theta)^2}{b^2(t)} < 1 \right\} \subset \{x \in \mathbb{R}^2 | G(x, t) < 0\}.$$

Then

$$G((0, \theta), t) < 0 \quad \text{for } t \in (0, t_\delta],$$

which finishes the proof. Note that t_δ only depends on δ and V. $\qquad \square$

Let $\Omega \subset \mathbb{R}^2$ be an open convex set. Denote by $G_\Omega(x, t)$ the unique viscosity solution to equation (1.2) subject to $G_\Omega(x, 0) = g_\Omega(x)$ where

$$g_\Omega(x) = \begin{cases} -\frac{2}{\pi} \arctan(d(x, \partial\Omega)) & \text{for } x \in \Omega \\ \frac{2}{\pi} \arctan(d(x, \partial\Omega)) & \text{otherwise.} \end{cases}$$

Given two sets E_1 and E_2, their Hausdorff distance

$$d_H(E_1, E_2) = \max\{\max_{x \in E_1} d(x, E_2), \ \max_{x \in E_2} d(x, E_1)\}.$$

Also, for $\alpha > 0$ and $\delta \in (0, \frac{1}{2})$, we write

$$W_{\alpha, \delta} = [-\alpha, \alpha] \times [\delta, 1 - \delta].$$

Lemma 5.2 *Let $S = (0, 1)^2$. For given $\delta \in (0, \frac{1}{2})$ and $n \in \mathbb{N}$, there exists $\sigma_{\delta, n} > 0$ such that if*

$$d_H(S, \Omega) \leq \sigma_{\delta, n} \quad \text{and} \quad \alpha \leq \sigma_{\delta, n},$$

then

$$G_\Omega(x, t) < 0 \quad \text{for } (x, t) \in W_{\alpha, \delta} \times \left[\frac{t_\delta}{n}, \ t_\delta \right].$$
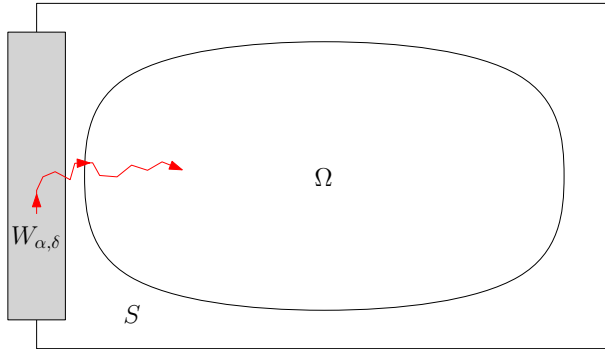
Fig. 17 Reach Ω from $W_{\alpha,\delta}$

Here t_δ is from the previous Lemma 5.1.

Proof We argue by contradiction. If not, then there exist a sequence of convex open sets $\{\Omega_m\}_{m \geq 1}$ such that

$$d_H(S, \Omega_m) \leq \frac{1}{m}$$

and for some $(x_m, t_m) \in W_{\frac{1}{m},\delta} \times \left[\frac{t_\delta}{n}, \ t_\delta\right]$

$$G_{\Omega_m}(x_m, t_m) \geq 0.$$

Since

$$\lim_{m \to +\infty} g_{\Omega_m}(x) = g_S(x) \quad \text{uniformly on } \mathbb{R}^2,$$

due to Remark 2.1, the uniqueness of viscosity solutions, we have that

$$\lim_{m \to +\infty} G_{\Omega_m}(x, t) = G(x, t) \quad \text{locally uniformly on } \mathbb{R}^2 \times [0, \infty).$$

Here G is from Lemma 5.1. The proof is similar to that of (2.4). Also, up to a subsequence if necessary, we may assume that $\lim_{m \to +\infty}(x_m, t_m) = ((0, \theta), \bar{t})$ for $(\theta, \bar{t}) \in [\delta, 1 - \delta] \times \left[\frac{t_\delta}{n}, \ t_\delta\right]$. Then we have that

$$G((0, \theta), \bar{t}) = \lim_{m \to +\infty} G_{\Omega_m}(x_m, t_m) \geq 0.$$

This is a contradiction.   □

As an immediate corollary, we have the following reachability.

Lemma 5.3 *Consider the game in section 2.2. Under the assumption of Lemma* 5.2, *every point on $W_{\alpha,\delta}$ can reach Ω within time $\frac{t_\delta}{n}$. Also, it is easy to see that if we replace*

S by an arbitrary rectangle, all previous results are still true in the corresponding forms (See Fig. 17).

Remark 5.1 Due to the hidden stochastic nature of the game trajectory, it is not clear to us how to use pure game dynamics to prove the above reachability conclusion. An interesting analog in random walk (or Brownian motion) is to use strong maximum principle of the Laplace equation to show that a particle has a positive probability to exit from any small window of the boundary.

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