Published for SISSA by Springer

Received: December 18, 2023
Revised: February 18, 2024
Accepted: March 12, 2024
Published: April 5, 2024

Holographic transport beyond the supergravity approximation

Alex Buchel,[a,b] **Sera Cremonini**[c] **and Laura Early**[c]

[a]*Department of Physics and Astronomy, University of Western Ontario, London, Ontario N6A 5B7, Canada*
[b]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J 2W9, Canada*
[c]*Department of Physics, Lehigh University, Bethlehem, PA, 18015, U.S.A.*

E-mail: abuchel@uwo.ca, cremonini@lehigh.edu, lte219@lehigh.edu

Abstract: We set up a unified framework to efficiently compute the shear and bulk viscosities of strongly coupled gauge theories with gravitational holographic duals involving higher derivative corrections. We consider both Weyl^4 corrections, encoding the finite 't Hooft coupling corrections of the boundary theory, and Riemann^2 corrections, responsible for non-equal central charges $c \neq a$ of the theory at the ultraviolet fixed point. Our expressions for the viscosities in higher derivative holographic models are extracted from a radially conserved current and depend only on the horizon data.

Keywords: AdS-CFT Correspondence, Gauge-Gravity Correspondence, Holography and Hydrodynamics

ArXiv ePrint: 2312.05377

https://doi.org/10.1007/JHEP04(2024)032

JHEP04(2024)032

Contents

1 Introduction and summary

For well over two decades there has been an ongoing program to apply the holographic gauge/gravity duality to gain insights into the dynamics of strongly coupled quantum phases of matter [1]. Indeed, the techniques of holography have been adopted to probe the transport properties of a wide spectrum of strongly correlated systems, ranging from the QCD quark gluon plasma (QGP) to high temperature superconductors, strange metals and a variety of electronic materials. Within this program, the most elegant result to date remains the universality [2, 3] of the shear viscosity η to entropy s ratio,

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}\,, \tag{1.1}$$

which holds in strongly coupled gauge theories in the limit of an infinite number of colors, $N \to \infty$, and infinite 't Hooft coupling, $\lambda \to \infty$. These theories are dual to Einstein gravity coupled to an arbitrary matter sector, with the result (1.1) relying on the additional assumption that rotational invariance is preserved.[1]

The significance of (1.1) can be traced not only to its universal nature, but also to the fact that its value is remarkably close to the experimental range extracted from the QGP data at RHIC and at the LHC. This led to the compelling KSS proposal [5, 6] that the shear viscosity might obey a fundamental lower bound in nature, $\frac{\eta}{s} \geq \frac{1}{4\pi}$ (from now on we take $\hbar = k_B = 1$). Despite its appeal, it is now well understood that the KSS bound can be violated in a number of ways, either by relaxing symmetries or by introducing

[1] For a recent discussion of η/s in anisotropic theories we refer the reader to [4].

higher derivative *curvature* corrections to the low-energy gravitational action (see i.e. [7] for a review). Indeed, notable early examples of the effects of higher derivatives include Weyl^4 corrections to Einstein gravity [8], which encode finite λ effects in the dual gauge theory, and Riemann^2 corrections [9, 10], which describe finite N effects in the dual theory.[2] Holographic models involving a more complicated matter sector have been used to study the temperature dependence of η/s [11, 12] and have direct applications to the QGP, where the temperature variations of η are expected to play an important role (see e.g. the recent review [13]). A key lesson that has emerged from these studies is that the universality of η/s is generically lost once we move away from the limits $\lambda, N \to \infty$ (or appropriately break symmetries). Moreover, other transport coefficients have failed to exhibit the simple universal behavior encoded in (1.1).

In addition to η, the bulk viscosity ζ has also attracted considerable attention in holography (see e.g. [14–23]), largely because of its relevance to the physics of the QGP near the deconfinement transition, where ζ is expected to rise dramatically. As non-zero bulk viscosity requires theories with broken conformal symmetry, holographic model building has typically involved adding bulk scalars with non-trivial profiles. Since the latter also yields a non-trivial temperature dependence for η/s, such holographic models have played a prominent role in the attempts to build realistic models of QCD [24, 25].

In holography, transport coefficients such as η and ζ can be extracted in a number of complementary ways, e.g. computing correlators of the stress energy tensor and using Kubo formulas, or from linearized quasi-normal modes on black brane backgrounds, which in the hydrodynamic limit correspond to shear and sound modes of the dual field theory. The standard holographic dictionary instructs us to extract correlators from the *boundary* behavior of fluctuating bulk fields, appropriately supplemented with boundary conditions at the horizon. However, hydrodynamics is an effective description of the system at long wavelengths and small frequencies, and thus one would expect it to be encoded in properties of the geometry and its fluctuations in the IR, i.e. near the horizon. Thus, a natural question is to what extent the horizon of a black brane can *fully* capture the hydrodynamic behavior of a strongly coupled plasma, and its transport properties. Indeed, the diffusive modes can be understood [5, 26] as fluctuations of the black brane horizon using the membrane paradigm [27, 28], which identifies the horizon with a fictitious fluid. This approach was made more precise in [29] and led to various formulations for extracting transport coefficients entirely from the horizon geometry. However, these methods have been limited to special cases.

In this paper we revisit some of these questions, and set up a universal — and efficient — framework for extracting the shear and bulk viscosities of strongly coupled gauge theories with holographic duals involving higher derivative corrections. A crucial step in our analysis is the realization that the terms needed to compute both η and ζ can be extracted from *radially conserved currents*, even in the presence of higher derivatives. In turn, this implies that they can be evaluated at the black brane horizon. As we will see, one clear advantage of our framework is that it avoids having to compute dispersion relations. In our analysis we will consider both Weyl^4 corrections and Riemann^2 corrections, encoding, respectively, finite 't Hooft coupling and finite N effects. Moreover, our gravitational dual has an arbitrary number

[2]These are due to non-equal central charges $c \neq a$ of the gauge theory at the UV fixed point.

JHEP04(2024)032

of scalars, with an arbitrary interaction potential. Having such a framework is especially valuable for theories with higher derivatives, where the computations are intrinsically more cumbersome and a number of subtleties arise, related to the presence of, for instance, additional boundary terms and counterterms.

In closing we should mention that the recent paper [30] has also examined the connection between horizon and boundary data and has put forth an efficient method for computing transport coefficients directly from the horizon.[3] However, their analysis is restricted to two-derivative theories, and to matter sectors involving, in addition to Einstein gravity and a U(1) gauge field, only one scalar field. In our analysis, on the other hand, we have included an arbitrary number of scalars and allowed for curvature corrections to the leading gravitational action.

1.1 Summary of results

We will work with a five-dimensional theory of gravity in AdS coupled to an arbitrary number of scalars, described by

$$\begin{aligned} S_5 &= \frac{1}{16\pi G_N}\int_{\mathcal{M}_5} d^5x\sqrt{-g}\ L_5 \\ &\equiv \frac{1}{16\pi G_N}\int_{\mathcal{M}_5} d^5x\sqrt{-g}\left[R+12-\frac{1}{2}\sum_i(\partial\phi_i)^2-V\{\phi_i\}+\beta\cdot\delta\mathcal{L}\right], \end{aligned} \tag{1.2}$$

where $\delta\mathcal{L}$ denote terms involving higher derivative corrections to Einstein gravity. In particular, in this paper we consider two classes of models, to leading order[4] in β:

- four-derivative curvature corrections described by:

$$\delta\mathcal{L}_2 \equiv \alpha_1\ R^2+\alpha_2\ R_{\mu\nu}R^{\mu\nu}+\alpha_3 R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}\,; \tag{1.3}$$

- eight-derivative curvature corrections described by:

$$\delta\mathcal{L}_4 \equiv C^{hmnk}C_{pmnq}C_h{}^{rsp}C^q{}_{rsk}+\frac{1}{2}C^{hkmn}C_{pqmn}C_h{}^{rsp}C^q{}_{rsk}\,, \tag{1.4}$$

  where C is the Weyl tensor.

For the shear viscosity to the entropy ratio we find, respectively:

- $$\frac{\eta}{s}\bigg|_{\delta\mathcal{L}_2} = \frac{1}{4\pi}\left(1+\beta\cdot\frac{2}{3}\alpha_3\,(V-12)\right); \tag{1.5}$$
- $$\frac{\eta}{s}\bigg|_{\delta\mathcal{L}_4} = \frac{1}{4\pi}\left(1-\beta\cdot\frac{1}{72}(V-12)\left[3\sum_i(\partial_i V)^2+5(V-12)^2\right]\right), \tag{1.6}$$

where $\partial_i V \equiv \frac{\partial V}{\partial\phi_i}$.

[3]See also [31–33] for related earlier treatment.

[4]Throughout the paper we keep the subscripts $_2$ or $_4$ in reference to models (1.3) and (1.4). The parameter β is assumed to be perturbatively small. The asymptotic AdS radius is set $L = 1$, in the absence of the higher derivative corrections.

All the quantifies in (1.5) and (1.6) are to be evaluated at the horizon of the dual black brane solution. Since the $\mathcal{O}(\beta^0)$ results are universal [3], it is sufficient to evaluate the scalar potential and its derivatives to leading $\mathcal{O}(\beta^0)$ order only.

For the bulk viscosity to the entropy ratio we find, respectively:

- $$\begin{split} 9\pi\frac{\zeta}{s}\bigg|_{\delta\mathcal{L}_2} =& \left(1-\frac{2}{3}(V-12)(5\alpha_1+\alpha_2-\alpha_3)\beta\right)\sum_i z_{i,0}^2 \\ &+\beta\cdot\frac{4(5\alpha_1+\alpha_2-\alpha_3)}{3(V-12)}\ \sum_i (z_{i,0}\cdot\partial_i V)^2\,; \end{split} \tag{1.7}$$

- $$9\pi\frac{\zeta}{s}\bigg|_{\delta\mathcal{L}_4} = \left(1+\frac{5}{144}\beta(V-12)^3\right)\sum_i z_{i,0}^2 - \beta\cdot\frac{5}{24}(V-12)\ \sum_i (z_{i,0}\cdot\partial_i V)^2\,. \tag{1.8}$$

Once again, all the quantifies in (1.7) and (1.8) are to be evaluated at the horizon of the dual black brane solution. Here $z_{i,0}$ are the values of the gauge invariant scalar fluctuations, at zero frequency, evaluated at the black brane horizon, see section A.3 and in particular (A.69). While the scalar potential and its derivatives can be evaluated to the leading $\mathcal{O}(\beta^0)$ order of the background black brane solution, the horizon values of the scalars $z_{i,0}$ must be evaluated including $\mathcal{O}(\beta)$ corrections.

The rest of the paper is organized as follows. In section 2 we present the analysis for some specific models, and provide extensive checks on the general formalism. We conclude in section 3 and highlight future directions. The formal proofs of the main results — the final expressions (1.5), (1.6) for the shear viscosity, and (1.7), (1.8) for the bulk viscosity are delegated to appendix A. We discuss the background black brane geometry in A.1, $\frac{\eta}{s}$ is computed in section A.2, and $\frac{\zeta}{s}$ is computed in section A.3.

2 Applications

In this section we use simple toy models to validate the general formulas reported in (1.5)–(1.8). First and foremost, note that if the boundary gauge theory is a CFT with

$$V \equiv 0\,, \tag{2.1}$$

we find[5] from (1.5) and (1.6)

$$\frac{\eta}{s}\bigg|_{\delta\mathcal{L}_2}^{\rm CFT} = \frac{1}{4\pi}\left(1-\beta\cdot 8\alpha_3\right)\,, \qquad \frac{\eta}{s}\bigg|_{\delta\mathcal{L}_4}^{\rm CFT} = \frac{1}{4\pi}\left(1+\beta\cdot 120\right)\,, \tag{2.2}$$

reproducing [9] and [8, 35, 36] correspondingly.

We discuss the following models:

- ($\mathcal{A}_{2,\Delta}$): $\delta\mathcal{L}_2$ model with
$$\{\alpha_1\,,\,\alpha_2\,,\,\alpha_3\} = \{0\,,\,0\,,\,1\}\,, \tag{2.3}$$
with
$$V = \frac{m^2}{2}\,\phi^2\,, \qquad m^2\left(1+\frac{2}{3}\beta\right) = \Delta(\Delta-4)\,. \tag{2.4}$$

[5] The bulk viscosity of a CFT plasma vanishes; see [34] for the original analysis of $\mathcal{N}=4$ SYM.

Note the $\mathcal{O}(\beta)$ modification of the relation between the mass of the bulk scalar and the dimension Δ of the dual boundary operator.[6] We consider $\Delta = \{2, 3\}$. The bulk viscosity in these models was not discussed in the literature before.

Models $\mathcal{A}_{2,\Delta}$ are interesting in that the gravitational holographic bulk is higher derivative *and* the black brane horizon Wald entropy differs from its Bekenstein entropy, see (A.28).

- ($\mathcal{B}_{2,\Delta}$): $\delta\mathcal{L}_2$ model with

$$\{\alpha_1\,,\,\alpha_2\,,\,\alpha_3\} = \{1\,,-4\,,\,1\}\,, \tag{2.5}$$

with

$$V = \frac{m^2}{2}\,\phi^2\,, \qquad m^2\Big(1+2\beta\Big) = \Delta(\Delta-4)\,. \tag{2.6}$$

The $\mathcal{O}(\beta)$ modification of the relation between the mass of the bulk scalar and the dimension Δ of the dual boundary operator is precisely as reported in [37]. We consider $\Delta = \{2, 3\}$. The bulk viscosity in these models was considered in [37], but only to leading order in the non-normalizable coefficient of the scalar ϕ (albeit to all orders in β). Here we consider leading perturbative in β corrections to transport in these models, but to all orders in the conformal symmetry breaking parameter, i.e., non-perturbatively in the non-normalizable coefficient of the bulk scalar ϕ.

Models $\mathcal{B}_{2,\Delta}$ are interesting in that the gravitational holographic bulk represents a two-derivative model: the coefficients in (2.5) assemble Riemann squared terms into the Gauss-Bonnet combination. Notice that for the black branes in these models the Wald entropy is identical to their Bekenstein entropy, see (A.28).

- ($\mathcal{C}_{2,\Delta}$): $\delta\mathcal{L}_2$ model with

$$\{\alpha_1\,,\,\alpha_2\,,\,\alpha_3\} = \{0\,,\,1\,,\,1\}\,, \tag{2.7}$$

with

$$V = \frac{m^2}{2}\,\phi^2\,, \qquad m^2\Big(1+2\beta\Big) = \Delta(\Delta-4)\,. \tag{2.8}$$

The $\mathcal{O}(\beta)$ modification of the relation between the mass of the bulk scalar and the dimension Δ of the dual boundary operator is identical to the one in models ($\mathcal{B}_{2,\Delta}$). We consider $\Delta = \{2, 3\}$. The bulk viscosity in these models was not discussed in the literature before.

Models $\mathcal{C}_{2,\Delta}$ are interesting in that the gravitational holographic bulk is higher-derivative, but the horizon physics is effectively two-derivative: as in the case above, in these models there is no difference between the Wald and the Bekenstein entropies of the dual black brane horizon.

- ($\mathcal{D}_{4,\Delta}$): $\delta\mathcal{L}_4$ model with

$$V = \frac{m^2}{2}\,\phi^2\,, \qquad m^2 = \Delta(\Delta-4)\,. \tag{2.9}$$

[6] See [37] where the need for such a modification was first pointed out.

Notice that here the Weyl[4] higher derivative corrections to the gravitational action (1.4) do not modify the bulk scalar mass/dimension of the dual operator relation. We consider $\Delta = \{2, 3\}$. The bulk viscosity in these models was not discussed in the literature before.

Models $\mathcal{D}_{4,\Delta}$ are interesting in that here the higher derivative corrections are associated with finite 't Hooft coupling corrections of the UV fixed point CFT, rather than with the difference between the central charges of the UV CFT, encoded by $\beta \cdot \alpha_3 \equiv \frac{c-a}{8c}$, as in models $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}_{2,\Delta}$.

In the models just introduced we compute the shear and the bulk viscosities using (1.5)–(1.8), and compare the results with direct computation of these quantities from the dispersion relation of the shear and the sound modes,

$$\begin{aligned} \text{shear}: \qquad & \mathfrak{w} = -i\,\frac{2\pi\eta}{s}\,\mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3)\,, \\ \text{sound}: \qquad & \mathfrak{w} = c_s \cdot \mathfrak{q} - i\,\frac{4\pi\eta}{3s}\left(1 + \frac{3\zeta}{4\eta}\right)\mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3)\,. \end{aligned} \tag{2.10}$$

These are the appropriate quasinormal modes of the dual black brane [38]. The first non-conformal gauge theory computations of the bulk viscosity from the dispersion relation were performed in [15]. In genuinely higher-derivative holographic models the shear and the sound mode dispersion relations where studied only in conformal $\mathcal{N} = 4$ SYM in [34]. In this paper, we generalize (and combine) the computation methods of [15] and [34]. Such analysis are much more involved and are extremely technical. We will not provide any details — in fact our motivation of developing the framework explained in sections A.2 and A.3 was precisely to avoid computation of the dispersion relations in the first place. As we already emphasized, here we use such dispersion computations in models $\mathcal{A} - \mathcal{D}$ as a check on our general framework.

Finally, we mention one additional test we performed. The speed of the sound waves c_s in (2.10) is related to the equation of state $P = P(\mathcal{E})$ of the holographic gauge theory plasma via

$$c_s^2 = \frac{\partial P}{\partial \mathcal{E}}\bigg|_{\lambda_\Delta = \text{const}}\,, \tag{2.11}$$

where in computing derivatives of the pressure P with respect to the energy density $\mathcal{E}$ one has to keep the non-normalizable coefficient λ_Δ of the bulk scalar (i.e., the coupling constant of the dual operator $\mathcal{O}_\Delta$ explicitly breaking the conformal invariance) constant.

As we explicitly show in this section, all the validations pass with excellence.

2.1 Shear viscosity in models $\mathcal{A} - \mathcal{D}$

Given (1.5), we find that in all models $\mathcal{A} - \mathcal{C}$ the shear viscosity to the entropy density is

$$\frac{\eta}{s}\bigg|_{\mathcal{A},\mathcal{B},\mathcal{C}} = \frac{1}{4\pi}\left(1 + \beta \cdot \delta_{\eta;\Delta}^{\mathcal{A},\mathcal{B},\mathcal{C}}\right)\,, \qquad \delta_{\eta;\Delta}^{\mathcal{A},\mathcal{B},\mathcal{C}} = -8 + \frac{\Delta(\Delta-4)}{3}\,(\phi_{0,0}^h)^2\,, \tag{2.12}$$

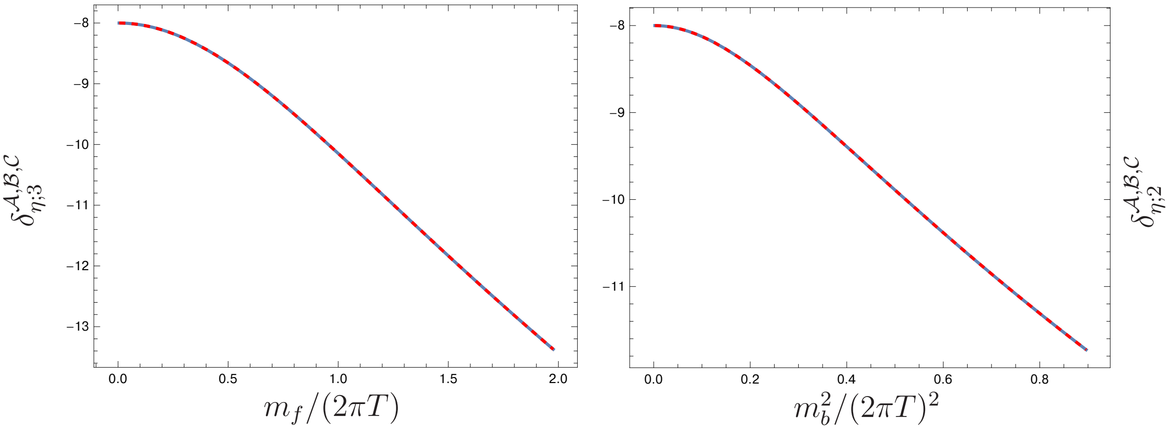

Figure 1. Corrections to the shear viscosity in models $\mathcal{A}-\mathcal{C}$ due to the UV fixed point central charges $c \neq a$ are universal, see (2.12). In the left panel we consider non-conformal gauge theories with a UV fixed point deformed by a dimension $\Delta = 3$ operator, $CFT \to CFT + m_f \mathcal{O}_3$, while in the right panel a UV fixed point is deformed by an operator of dimension $\Delta = 2$, $CFT \to CFT + m_b^2 \mathcal{O}_2$. Solid curves represent the corrections to the shear viscosity extracted from the shear channel quasinormal mode of the background black brane (2.10), while the red dashed curves are obtained applying (2.12).

while from (1.6) in models $\mathcal{D}$ the shear viscosity to the entropy density is

$$\frac{\eta}{s}\bigg|_{\mathcal{D}} = \frac{1}{4\pi}\left(1+\beta\cdot\delta_{\eta;\Delta}^{\mathcal{D}}\right), \qquad \delta_{\eta;\Delta}^{\mathcal{D}} = -\frac{1}{72}\left(\frac{\Delta(\Delta-4)(\phi_{0,0}^h)^2}{2}-12\right)\times \\ \times\left(5\left(\frac{1}{2}\Delta(\Delta-4)(\phi_{0,0}^h)^2-12\right)^2+3\Delta^2(\Delta-4)^2(\phi_{0,0}^h)^2\right), \tag{2.13}$$

where $\phi_{0,0}^h$ is the leading $\mathcal{O}(\beta^0)$ order horizon value of the bulk scalar. It is computed numerically solving the leading order $\mathcal{O}(\beta^0)$ background equations of motion (A.7)–(A.10), using the metric parameterization (A.23), subject to the boundary conditions:

$$\begin{aligned} r\to 0: &\qquad f\sim 1+\cdots\,, \qquad g\sim 1+\cdots\,, \qquad \phi\sim\begin{cases}\lambda_3\cdot r+\cdots\,, & \text{when } \Delta=3;\\ \lambda_2\cdot r^2\ln r+\cdots\,, & \text{when } \Delta=2,\end{cases} \\ r\to r_h: &\qquad f\sim 0+\cdots\,, \qquad g\sim g_{0,0}^h+\cdots\,, \qquad \phi\sim\phi_{0,0}^h\,. \end{aligned} \tag{2.14}$$

Without loss of generality we can fix $r_h = 1$, provided we present all results as dimensionless quantities. From (A.14) we find

$$2\pi T = \sqrt{g_{0,0}^h}\left(2+\frac{\Delta(4-\Delta)}{12}\,(\phi_{0,0}^h)^2\right). \tag{2.15}$$

Comparisons between the corrections to the shear viscosity (see (2.12) and (2.13)) for the holographic models $\mathcal{A}-\mathcal{D}$ extracted from the dispersion relation of the shear modes using (2.10) (the solid curves) and using (2.12) and (2.13) (dashed red curves) are presented in figures 1–2. In the left panels we consider models with $\Delta = 3$ with the non-normalizable

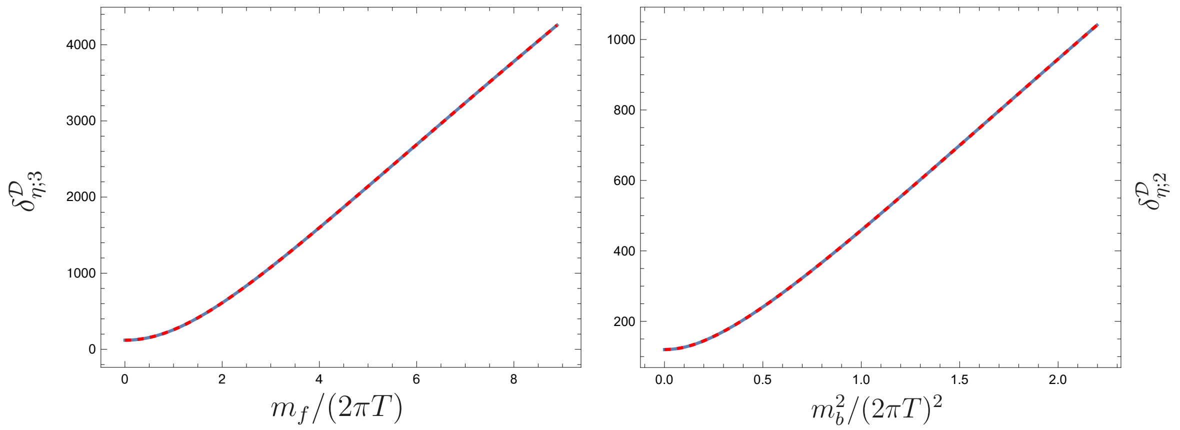

Figure 2. Corrections to the shear viscosity in models $\mathcal{D}$ due to finite 't Hooft coupling corrections, see (2.13). In the left panel we consider non-conformal gauge theories with a UV fixed point deformed by a dimension $\Delta = 3$ operator, $CFT \to CFT + m_f \mathcal{O}_3$, while in the right panel a UV fixed point is deformed by an operator of dimension $\Delta = 2$, $CFT \to CFT + m_b^2 \mathcal{O}_2$. Solid curves represent the corrections to the shear viscosity extracted from the shear channel quasinormal mode of the background black brane (2.10), while the red dashed curves are obtained applying (2.13).

gravitational bulk scalar coefficient λ_3 identified as a fermionic mass term $\lambda_3 \equiv m_f$ of the boundary gauge theory. In the right panels we consider models with $\Delta = 2$ with the non-normalizable gravitational bulk scalar coefficient λ_2 identified as a bosonic mass term $\lambda_2 \equiv m_b^2$ of the boundary gauge theory.[7] The difference between the solid and the dashed red curves is $\sim 10^{-8} \cdots 10^{-5}$ % over the ranges of $\lambda_\Delta / T^{4-\Delta}$ reported. When $m_f = 0$ and $m_b^2 = 0$ we recover the conformal gauge theory results (2.2):

$$\delta^{\mathcal{A},\mathcal{B},\mathcal{C}}_{\eta;\Delta}\bigg|_{m_f=m_b^2=0} = -8\,, \qquad \delta^{\mathcal{D}}_{\eta;\Delta}\bigg|_{m_f=m_b^2=0} = 120\,. \tag{2.16}$$

2.2 Bulk viscosity in models $\mathcal{A} - \mathcal{D}$

Unlike the shear viscosity, the bulk viscosity in models $\mathcal{A}$ and $\mathcal{B}, \mathcal{C}$ differs: using (1.7) we find

$$\begin{aligned}
\frac{\zeta}{\eta}\bigg|_{\mathcal{A}} &= \frac{4}{9}\, z_{0,0}^2 + \beta \cdot \delta^{\mathcal{A}}_{\zeta;\Delta}\,, \\
\delta^{\mathcal{A}}_{\zeta;\Delta} &= \frac{8}{9} z_{0,0} z_{0,1} + \frac{8}{27}\left(\frac{1}{2}\Delta(\Delta-4)(\phi^h_{0,0})^2 - 12\right) z_{0,0}^2 - \frac{16}{27}\, \frac{\Delta^2(\Delta-4)^2(\phi^h_{0,0})^2}{\frac{1}{2}\Delta(\Delta-4)(\phi^h_{0,0})^2 - 12}\, z_{0,0}^2 \\
&\quad - \frac{4}{9} z_{0,0}^2\, \delta^{\mathcal{A},\mathcal{B},\mathcal{C}}_{\eta;\Delta}\,,
\end{aligned} \tag{2.17}$$

$$\frac{\zeta}{\eta}\bigg|_{\mathcal{B},\mathcal{C}} = \frac{4}{9}\, z_{0,0}^2 + \beta \cdot \delta^{\mathcal{B},\mathcal{C}}_{\zeta;\Delta}\,, \qquad \delta^{\mathcal{B},\mathcal{C}}_{\zeta;\Delta} = \frac{8}{9} z_{0,0} z_{0,1} - \frac{4}{9} z_{0,0}^2\, \delta^{\mathcal{A},\mathcal{B},\mathcal{C}}_{\eta;\Delta}\,, \tag{2.18}$$

[7] We adopted m_f and m_b^2 labels from [39].

JHEP04(2024)032

and

$$\frac{\zeta}{\eta}\bigg|_{\mathcal{D}} = \frac{4}{9}\ z_{0,0}^2 + \beta\cdot\delta^{\mathcal{D}}_{\zeta;\Delta}\,,$$
$$\delta^{\mathcal{D}}_{\zeta;\Delta} = \frac{4}{9}z_{0,0}^2\left[\frac{5}{144}\left(\frac{1}{2}\Delta(\Delta-4)(\phi^h_{0,0})^2-12\right)^3 - \frac{5}{24}\left(\frac{1}{2}\Delta(\Delta-4)(\phi^h_{0,0})^2-12\right)\Delta^2(\Delta-4)^2(\phi^h_{0,0})^2\right]+\frac{8}{9}z_{0,0}z_{0,1}-\frac{4}{9}z_{0,0}^2\delta^{\mathcal{D}}_{\eta;\Delta}\,, \tag{2.19}$$

where $\phi^h_{0,0}$ is the leading $\mathcal{O}(\beta^0)$ order horizon value of the bulk scalar. In (2.17), (2.18) and (2.19) we denoted

$$z_{i,0}\equiv z_{0,0}+\beta\cdot z_{0,1}\,, \tag{2.20}$$

since our toy models have a single bulk scalar. To proceed further we need to evaluate the gauge invariant bulk scalar fluctuations z_0 at the horizon to order $\mathcal{O}(\beta)$ as emphasized in (2.20).

We present details for the model $\mathcal{A}_{2,\Delta=3}$, and only the final results for the other models.

2.2.1 Model $\mathcal{A}_{2,\Delta=3}$

Since we will need the gauge invariant bulk scalar fluctuations z_0 at the horizon to order $\mathcal{O}(\beta)$, we need the background geometry to order $\mathcal{O}(\beta)$. It is convenient to use the metric warp-factor parameterization as in (A.23). Explicitly,

$$c_1=\frac{f^{1/2}\sqrt{g+\beta\cdot g_{1,1}}}{r}\,,\qquad c_2=\frac{1}{r}\,,\qquad c_3=\frac{1-\frac{1}{3}\beta}{rf^{1/2}(1+\beta\cdot g_{2,1})}\,,\qquad \phi=\phi_0+\beta\cdot\phi_1\,. \tag{2.21}$$

From (A.7)–(A.10) we find

$$0=f'-\frac{r}{6}(\phi_0')^2f+\frac{\phi_0^2}{2r}-\frac{4f}{r}+\frac{4}{r}\,, \tag{2.22}$$

$$0=g'+\frac{g}{3}(\phi_0')^2r\,, \tag{2.23}$$

$$0=\phi_0''+\left(\frac{1}{r}-\frac{\phi_0^2}{2rf}-\frac{4}{rf}\right)\phi_0'+\frac{3\phi_0}{r^2f}\,, \tag{2.24}$$

at order $\mathcal{O}(\beta^0)$, and

$$0=g_{2,1}'-\left(\frac{\phi_0^2}{2rf}+\frac{4}{rf}\right)g_{2,1}-\frac{r}{6}\phi_0'\phi_1'-\frac{fr^5}{72}(\phi_0')^6-\frac{r^3}{24}(2\phi_0^2+f+16)(\phi_0')^4 +\frac{4}{9}\phi_0r^2(\phi_0')^3+\left(\frac{11}{3}rf+2r-\frac{8r}{f}-\frac{r}{8f}\phi_0^4-\left(\frac{2r}{f}-\frac{r}{12}\right)\phi_0^2\right)(\phi_0')^2 +\left(\frac{4\phi_0^3}{3f}+\frac{4(8-4f)\phi_0}{3f}\right)\phi_0'-\frac{\phi_0^4}{24rf}+\left(\frac{1}{r}-\frac{14}{3rf}\right)\phi_0^2+\frac{\phi_0\phi_1}{2rf}-\frac{4(f-1)^2}{rf}\,, \tag{2.25}$$

$$0=g_{1,1}'+\frac{r}{3}(\phi_0')^2g_{1,1}+\frac{gr}{3}\phi_0'\phi_1'-\frac{g(\phi_0^2+8)}{rf}g_{2,1}+\frac{g\phi_0}{rf}\phi_1+\frac{fr^5g}{108}(\phi_0')^6+\frac{r^3g}{36}(2\phi_0^2 -11f+16)(\phi_0')^4-\frac{2}{9}g\phi_0r^2(\phi_0')^3++\frac{rg}{12f}\left(\phi_0^4-2\phi_0^2(3f-8)+72f^2-48f+64\right)(\phi_0')^2 -\frac{2g\phi_0(\phi_0^2+8f+8)}{3f}\phi_0'-\frac{g}{12rf}\left(\phi_0^4-8\phi_0^2(3f-2)+96(f-1)^2\right)\,, \tag{2.26}$$

$$0=\phi_1''+\left(\frac{1}{r}-\frac{\phi_0^2}{2rf}-\frac{4}{rf}\right)\phi_1'-\frac{r\phi_0\phi_0'-3}{r^2f}\phi_1-g_{2,1}\frac{6\phi_0-r(\phi_0^2+8)\phi_0'}{r^2f}$$
$$+\frac{fr^5}{108}(\phi_0')^7+\frac{r^3}{36}(2\phi_0^2+7f+16)(\phi_0')^5-\frac{r^2\phi_0}{3}(\phi_0')^4+\frac{r}{12f}(\phi_0^4+2\phi_0^2(f+8)-80f^2$$
$$+64)(\phi_0')^3-\frac{\phi_0(\phi_0^2-8f+8)}{f}(\phi_0')^2+\frac{\phi_0'}{12rf}(\phi_0^4-8\phi_0^2(3f-8)+96(f-1)^2)\,, \tag{2.27}$$

at order $\mathcal{O}(\beta)$. The background equations (2.22)–(2.27) are solved subject to the following asymptotics:

- near the AdS boundary, i.e., as $r\to 0$,

$$\begin{aligned}
&\phi_0=\lambda_3 r+r^3\left(\phi_{0;3}+\frac{1}{6}\lambda_3^3\ln r\right)+\mathcal{O}(r^5\ln r)\,,\qquad g=1-\frac{1}{6}r^2\lambda_3^2+\mathcal{O}(r^4\ln r)\,,\\
&f=1+\frac{1}{6}r^2\lambda_3^2+r^4\left(f_4+\frac{1}{12}\lambda_3^4\ln r\right)+\mathcal{O}(r^6\ln^2 r)\,,\\
&\phi_1=r^3\left(\phi_{1;3}-\frac{2}{3}\lambda_3^3\ln r\right)+\mathcal{O}(r^5\ln r)\,,\\
&g_{1,1}=r^4\left(-\frac{1}{3}\lambda_3^4-\frac{2}{3}\lambda_3\phi_{0;3}-\frac{1}{2}\lambda_3\phi_{1;3}+2g_{2,1;4}\right)+\mathcal{O}(r^6)\,,\\
&g_{2,1}=-\frac{1}{3}r^2\lambda_3^2+r^4\left(g_{2,1;4}-\frac{1}{9}\lambda_3^4\ln r\right)+\mathcal{O}(r^6\ln^2 r)\,,
\end{aligned}\tag{2.28}$$

where $\lambda_3\equiv m_f$ is the non-normalizable coefficient of the bulk scalar, and the coefficients $\{\phi_{0;3},f_4,\phi_{1;3},g_{2,1;4}\}$ are related to the thermal expectation values of various boundary gauge theory operators;

- in the vicinity of the black brane horizon, i.e., as $y\equiv(1-r)\to 0$,

$$\begin{aligned}
&\phi_0=\phi_{0,0}^h+\mathcal{O}(y)\,,\qquad g=g_{0,0}^h+\mathcal{O}(y)\,,\qquad f=\left(4+\frac{1}{2}(\phi_{0,0}^h)^2\right)y+\mathcal{O}(y^2)\,,\\
&\phi_1=\phi_{1,0}^h+\mathcal{O}(y)\,,\qquad g_{1,1}=g_{1,1;0}^h+\mathcal{O}(y)\,,\\
&g_{2,1}=-\frac{(\phi_{0,0}^h)^4+28(\phi_{0,0}^h)^2-12\phi_{0,0}^h\phi_{1,0}^h+96}{12((\phi_{0,0}^h)^2+8)}+\mathcal{O}(y)\,,
\end{aligned}\tag{2.29}$$

specified by the set of coefficients $\{\phi_{0,0}^h,g_{0,0}^h,\phi_{1,0}^h,g_{1,1;0}^h\}$.

Using (2.29), from (A.14) we compute

$$\begin{aligned}
&2\pi T\equiv s_0+\beta\cdot s_1\,,\qquad s_0=\frac{(g_{0,0}^h)^{1/2}}{4}\left((\phi_{0,0}^h)^2+8\right),\\
&s_1=\frac{(g_{0,0}^h)^{-1/2}}{48}\left[12g_{0,0}^h\phi_{0,0}^h\phi_{1,0}^h+6g_{1,1;0}^h\left((\phi_{0,0}^h)^2+8\right)-g_{0,0}^h\left((\phi_{0,0}^h)^4+24(\phi_{0,0}^h)^2+64\right)\right].
\end{aligned}\tag{2.30}$$

It is important that $s_1\neq 0$, since as we will show it affects the representation of $\delta^{\mathcal{A}}_{\zeta;\Delta=3}$ in the plots.

We continue with the equations of motion for the gauge invariant fluctuations $z_{0,0}$ (at leading order in β) and $z_{0,1}$ at order $\mathcal{O}(\beta)$, see (A.66), (A.69) and (2.20),

$$0=z''_{0,0}-\frac{\phi_0^2-2f+8}{2rf}z'_{0,0}+\frac{z_{0,0}}{6r^2f}\Big((\phi_0')^2r^2(\phi_0^2+8)-12r\phi_0\phi_0'+18\Big)\,, \tag{2.31}$$

$$\begin{aligned}0=\,&z''_{0,1}-\frac{\phi_0^2-2f+8}{2rf}z'_{0,1}+\frac{z_{0,1}}{6r^2f}\Big((\phi_0')^2r^2(\phi_0^2+8)-12r\phi_0\phi_0'+18\Big)+\Bigg[\frac{7fr^5}{108}(\phi_0')^6\\&+\frac{5r^3}{36}(4\phi_0^2+7f+32)(\phi_0')^4-3\phi_0r^2(\phi_0')^3+\frac{r}{4f}\Big(3\phi_0^4-6\phi_0^2(f-8)\\&-16(f+2)(5f-6)\Big)(\phi_0')^2+\frac{1}{108rf}\Big(-684\phi_0^3r+144r\phi_0(21f-38)\Big)\phi_0'+\frac{1}{12rf}\Big(\phi_0^4\\&+4\phi_0^2(3g_{2,1}-6f+40)-12\phi_1\phi_0+96g_{2,1}+96(f-1)^2\Big)\Bigg]z'_{0,0}+\Bigg[\frac{fr^6}{324}(\phi_0')^8+\frac{r^4}{108}(4\phi_0^2\\&+7f+32)(\phi_0')^6-\frac{\phi_0r^3}{9}(\phi_0')^5+\frac{r^2}{36f}\Big(\phi_0^4-22\phi_0^2f+16\phi_0^2-80f^2-188f+64\Big)(\phi_0')^4\\&+\frac{\phi_0r}{2f}(\phi_0^2+8f+8)(\phi_0')^3-\frac{1}{36f}\Big(7\phi_0^4+4\phi_0^2(3g_{2,1}-6f+88)-12\phi_1\phi_0-96f^2\\&+96g_{2,1}-480f+1152\Big)(\phi_0')^2+\frac{r(\phi_0^2+8)\phi_0'-6\phi_0}{3rf}\phi_1'+\frac{1}{6f^2r^2}\Big(\phi_0^5r+4r(f+4)\phi_0^3\\&+8r(3g_{2,1}f-6f^2+16f+8)\phi_0-12f\phi_1r\Big)\phi_0'-\frac{\phi_0^4+6g_{2,1}f-4\phi_0^2(f-2)}{f^2r^2}\Bigg]z_{0,0}\,.\end{aligned} \tag{2.32}$$

The fluctuations $z_{0,0}$ and $z_{0,1}$ are then solved subject to the following asymptotics:

- near the AdS boundary, i.e., as $r\to 0$,

$$z_{0,0}=\frac{1}{2}\lambda_3r+r^3\Big(z_{0,0;3}+\frac{1}{4}\lambda_3^3\ln r\Big)+\mathcal{O}(r^5\ln r)\,,\qquad z_{0,1}=r^3\Big(z_{0,1;3}-\lambda_3^3\ln r\Big)+\mathcal{O}(r^5\ln r)\,, \tag{2.33}$$

where we note that the non-normalizable coefficients of $\{z_{0,0},z_{0,1}\}$, i.e., $\{\frac{1}{2}\lambda_3,0\}$ are precisely as required by (A.72);

- in the vicinity of the black brane horizon, i.e., as $y\equiv(1-r)\to 0$,

$$z_{0,0}=z^h_{0,0;0}+\mathcal{O}(y)\,,\qquad z_{0,1}=z^h_{0,1;0}+\mathcal{O}(y)\,. \tag{2.34}$$

Even though we will not present the equations for $z_{i,1}\equiv z_1\equiv z_{1,0}+\beta\cdot z_{1,1}$, we solved them as well, to validate the conservation of the imaginary part of the current (A.77) along the radial flow. Specifically, see (A.79),

$$\lim_{r\to 0}F\equiv F^{(b)}=\frac{1}{2}\lambda_3z_{1,0;3}\,,\qquad \lim_{r\to r_h=1}F\equiv F^{(h)}=\frac{(g^h_{0,0})^{1/2}}{8}\Big((\phi^h_{0,0})^2+8\Big)(z^h_{0,0;0})^2\,, \tag{2.35}$$

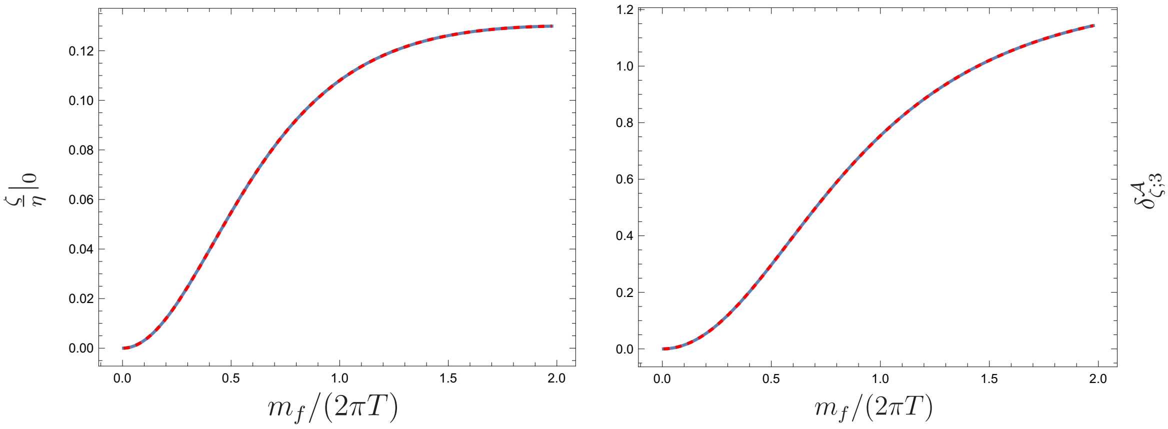

Figure 3. Bulk viscosity in model $(\mathcal{A}_{2,\Delta=3})$, (2.3). The solid curves represent the leading $\mathcal{O}(\beta^0)$ order bulk viscosity (the left panel) and its $\mathcal{O}(\beta)$ correction (the right panel) extracted from the sound wave channel quasinormal mode of the background black brane (2.10). The red dashed curves are obtained from (2.17).

where $z_{1,0;3}$ is the normalizable coefficient of $z_{1,0}$ (similar to $z_{0,0;3}$ of $z_{0,0}$ in (2.33)), and

$$\begin{split}\lim_{r\to 0}\delta F \equiv \delta F^{(b)} &= \left(\frac{1}{6}z_{1,0;3}+\frac{1}{2}z_{1,1;3}\right)\lambda_3\,,\\ \lim_{r\to r_h=1}\delta F \equiv \delta F^{(h)} &= (z^h_{0,0;0})^2\left(\frac{(g^h_{0,0})^{1/2}}{96}\left(-(\phi^h_{0,0})^4+72(\phi^h_{0,0})^2-64\right)+\frac{g^h_{1,1;0}}{16(g^h_{0,0})^{1/2}}\left((\phi^h_{0,0})^2+8\right)\right.\\ &\quad\left.+\frac{1}{8}(g^h_{0,0})^{1/2}(\phi^h_{0,0})\phi^h_{1,0}\right)+\frac{1}{4}z^h_{0,0;0}(g^h_{0,0})^{1/2}z^h_{0,1;0}\left((\phi^h_{0,0})^2+8\right),\end{split}\tag{2.36}$$

where $z_{1,1;3}$ is the normalizable coefficient of $z_{1,1}$ (similar to $z_{0,0;3}$ of $z_{0,0}$ in (2.33)). Conservation of the imaginary part of current (A.77) in particular requires that

$$\frac{F^{(b)}}{F^{(h)}}-1=0\,,\qquad \frac{\delta F^{(b)}}{\delta F^{(h)}}-1=0\,,\tag{2.37}$$

and provides a stringent test on our numerics.

In practice we solve numerically (2.22)–(2.27), (2.31)–(2.32), along with the equations for $z_{1,0}$ and $z_{1,1}$, parameterized by $\lambda_3\equiv m_f$. We use (2.17) to extract the bulk viscosity, and compare the results with the quasinormal modes computations[8] (2.10). It is important to present the physical results as dimensionless quantities, as we fixed the overall scale on the gravitational side of the computations setting $r_h=1$. From (2.30), the dimensionless quantity m_f/T is $\mathcal{O}(\beta)$ corrected,

$$\frac{m_f}{2\pi T}\equiv x(\lambda_3)+\beta\cdot\delta x(\lambda_3)\,.\tag{2.38}$$

Assume that we have a dimensionless quantity $\mathcal{K}$ that is $\mathcal{O}(\beta)$ corrected, and that is extracted from the numerics as a functions of λ_3, but we need to present it as a function of $m_f/(2\pi T)$. Then,

$$\mathcal{K}=\mathcal{K}_0(x+\beta\cdot\delta x)+\beta\cdot\mathcal{K}_1(x+\beta\cdot\delta x)=\mathcal{K}_0(x)+\beta\cdot\left(\delta x\cdot\frac{d\mathcal{K}_0(x)}{dx}+\mathcal{K}_1(x)\right),\tag{2.39}$$

[8]As we already mentioned, these computations are too technical to report in details here.

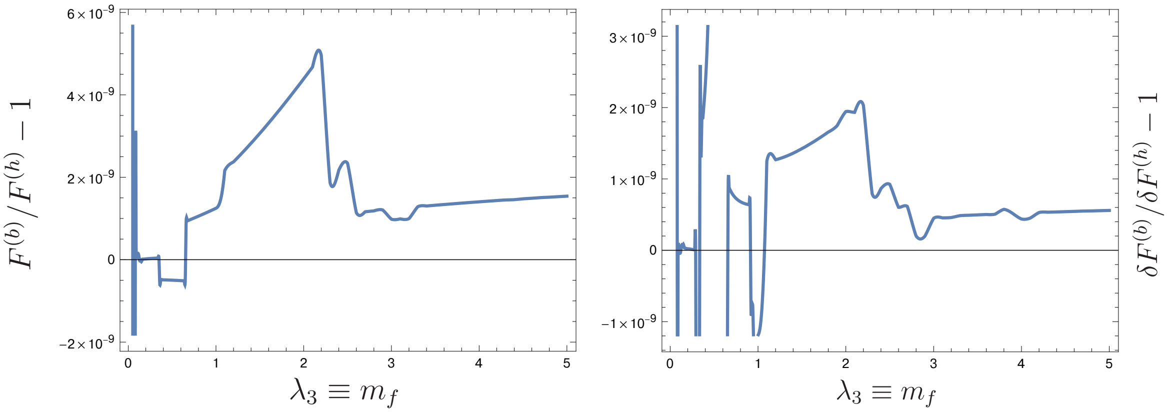

Figure 4. Numerical test of the conservation of the imaginary part of current (A.77) in model $(\mathcal{A}_{2,\Delta=3})$ to leading order $\mathcal{O}(\beta^0)$ (the left panel), and to subleading order $\mathcal{O}(\beta)$ (the right panel). See (2.37) for more details.

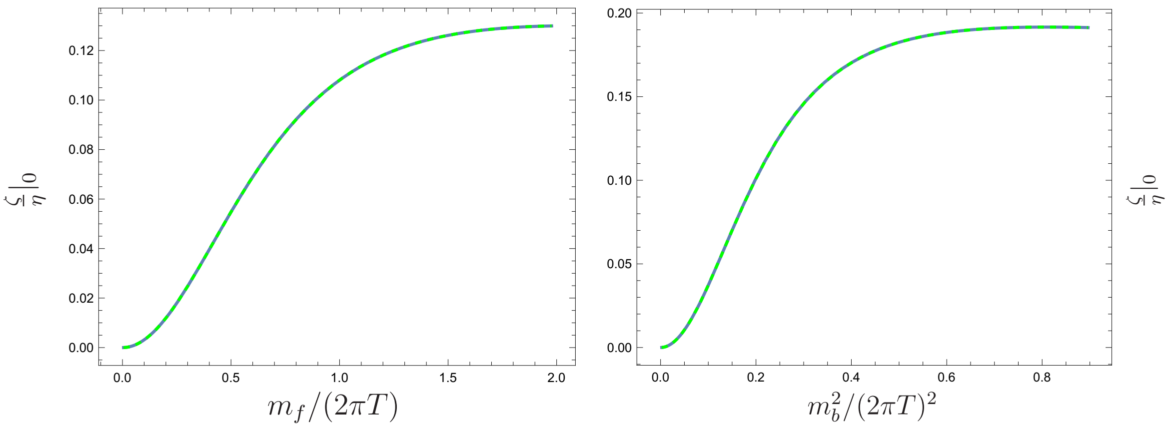

Figure 5. To order $\mathcal{O}(\beta^0)$ there is a perfect agreement between the ratio of bulk viscosity to the shear viscosity evaluated using our novel formula (2.41), shown in the solid curves, and the Eling-Oz expression (2.40), shown in the dashed green curves.

i.e., the $\mathcal{O}(\beta)$ correction of the quantity $\mathcal{K}$ receives an extra derivative in the $\mathcal{K}_0$ term. This was not an issue in our discussion of the shear viscosity to the entropy density ratio in section A.2, since there the appropriate quantity $\mathcal{K}_0 \equiv \frac{1}{4\pi}$ is a constant.

In figure 3 we compare the leading $\mathcal{O}(\beta^0)$ (the left panel) and the subleading $\mathcal{O}(\beta)$ correction (the right panel) of the ratio of the bulk viscosity to shear viscosity using the formalism of section A.3 (the red dashed curves), and the same quantities obtained from the computation of the sound channel quasinormal mode of the background black brane (2.10). The difference between the solid and the dashed red curves is $\sim 10^{-6} \cdots 10^{-4}$ %.

In figure 4 we numerically validate the conservation of the imaginary part of current (A.77).

2.2.2 Models $\mathcal{A} - \mathcal{D}$

Analysis of the remaining models proceeds as detailed for the model $(\mathcal{A}_{2,\Delta=3})$ in section 2.2.1. As for model $(\mathcal{A}_{2,\Delta=3})$, there is an excellent agreement between the general computation

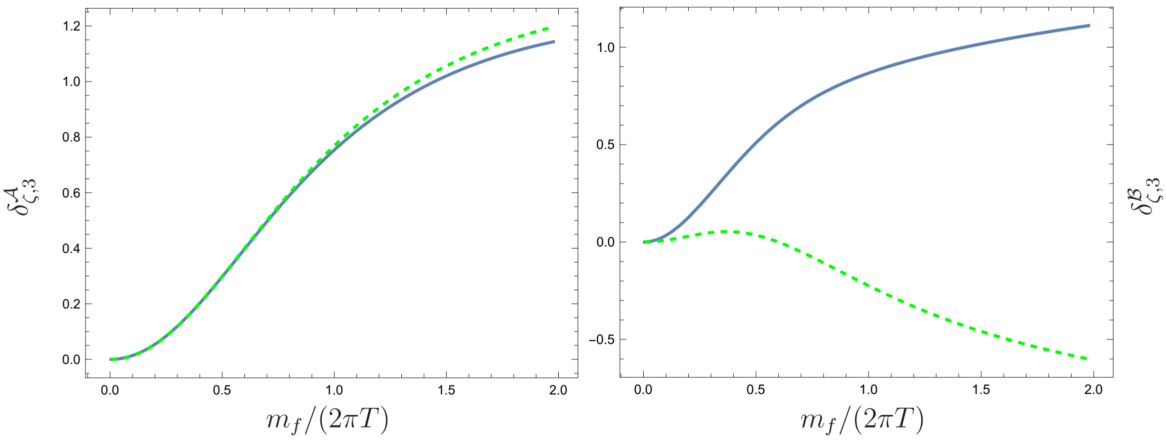

Figure 6. At order $\mathcal{O}(\beta)$ there is a disagreement between the ratio of bulk viscosity to the shear viscosity evaluated using (2.17) (the left panel) and (2.18) (the right panel), shown in the solid curves, and the extension of the Eling-Oz formula (2.40) to order $\mathcal{O}(\beta)$, shown in the dashed green curves.

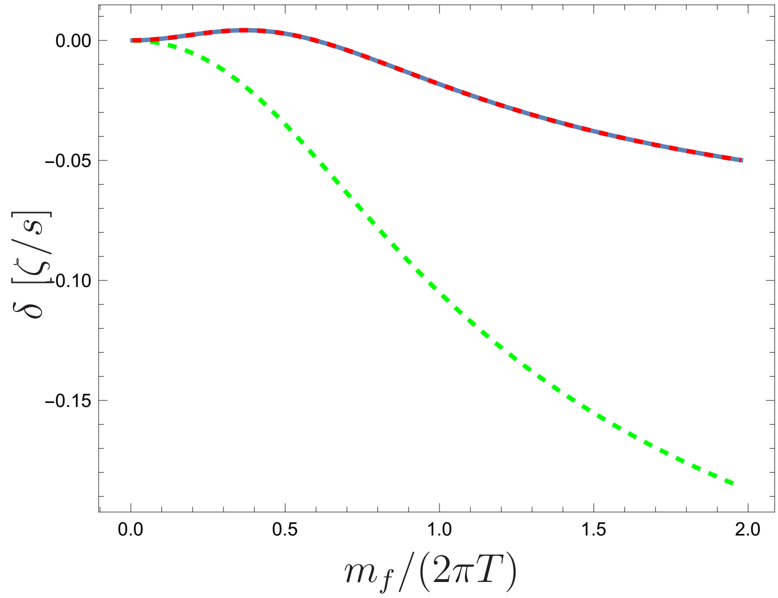

Figure 7. We compare $\mathcal{O}(\beta)$ correction to $\frac{\zeta}{s}$ in the holographic model $\mathcal{B}_{2,3}$: solid curve is obtained from the quasinormal mode (2.10) analysis, the red dashed curve is obtained from (2.42), and the green dashed curve is the application of the EO formula (2.43).

framework of section A.3 and the alternative extraction of the bulk viscosity from the quasinormal modes (2.10).

A simple formula to compute the bulk viscosity in two-derivative holographic models was proposed in [22] (EO):

$$\frac{\zeta}{\eta} = c_s^4 \sum_i \left(\frac{\partial \phi_i}{\partial \ln T}\right)^2 \bigg|_{\lambda_\Delta=\text{const}}, \tag{2.40}$$

where c_s is the speed of the sound waves in the holographic plasma (2.11), and the bulk scalar derivatives are evaluated at the black brane horizon, keeping the non-normalizable

JHEP04(2024)032

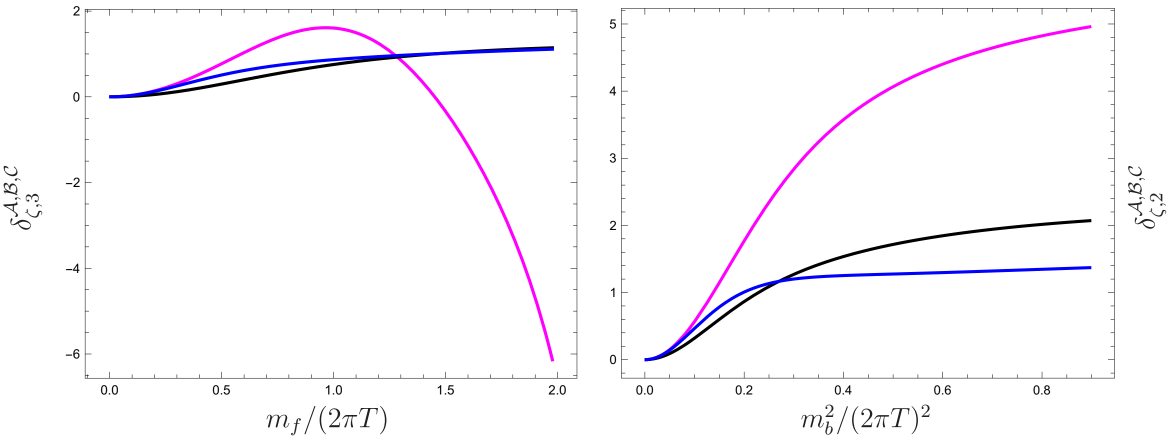

Figure 8. Although models $\mathcal{A} - \mathcal{C}$ have the same value of the higher derivative coupling $\alpha_3 = 1$, and the coupling constants α_1 and α_2 can be removed by a metric redefinition, such a redefinition modifies the scalar sector of the model resulting in distinct bulk viscosity corrections. Black curves represent $\mathcal{A}_{2,\Delta}$ models, blue curves represent $\mathcal{B}_{2,\Delta}$ models, and magenta curves represent $\mathcal{C}_{2,\Delta}$ models.

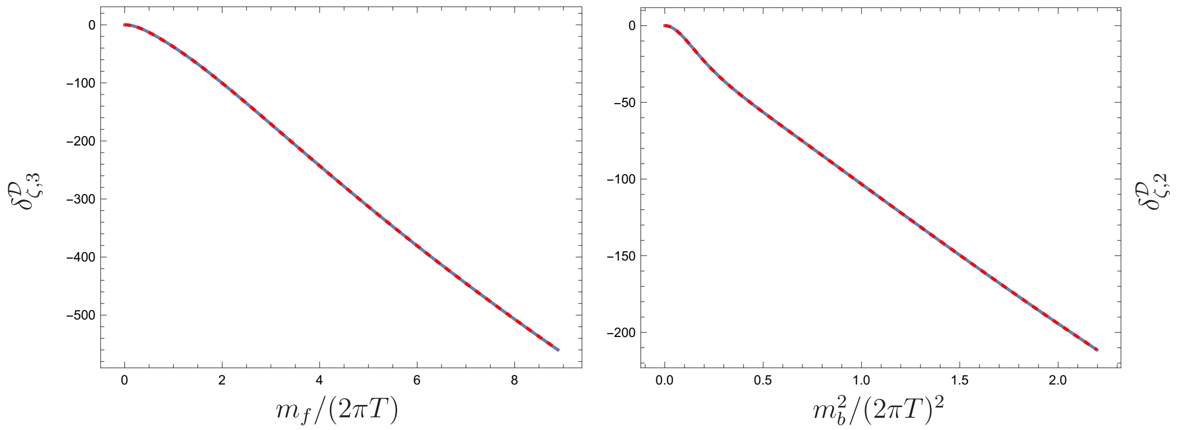

Figure 9. Bulk viscosity corrections in the $\mathcal{D}_{4,3}$ model (the left panel) and in the $\mathcal{D}_{4,2}$ model (the right panel). Solid curves represent corrections extracted from the sound wave channel quasinormal mode of the background black brane (2.10). The red curves are obtained from (2.19).

coefficients of these scalars — the mass terms of the boundary gauge theory — fixed [20, 23]. The EO was extensively tested in many models, and it is verified to leading order $\mathcal{O}(\beta^0)$ in the models discussed here. Specifically, in figure 5 we compare $\frac{\zeta}{\eta}\Big|_0$ for $\mathcal{A} - \mathcal{D}$ models with $\Delta = 3$ and $\Delta = 2$ (the green gashed curves) with the predictions (1.7) of the framework discussed in section A.3 (the solid curves):

$$\frac{\zeta}{s}\bigg|_0 = \frac{1}{9\pi}\sum_i z_{i,0}^2 \qquad \Longrightarrow \qquad \frac{\zeta}{\eta}\bigg|_0 = \frac{4}{9}\sum_i z_{i,0}^2 \,. \tag{2.41}$$

We stress that (2.41) is a simple, novel expression for the bulk viscosity in two-derivative holographic models: unlike [18], there is no restriction to a single bulk scalar, and there is no need to compute scalar derivatives at the horizon — $z_{i,0}$ are values of the gauge invariant scalar

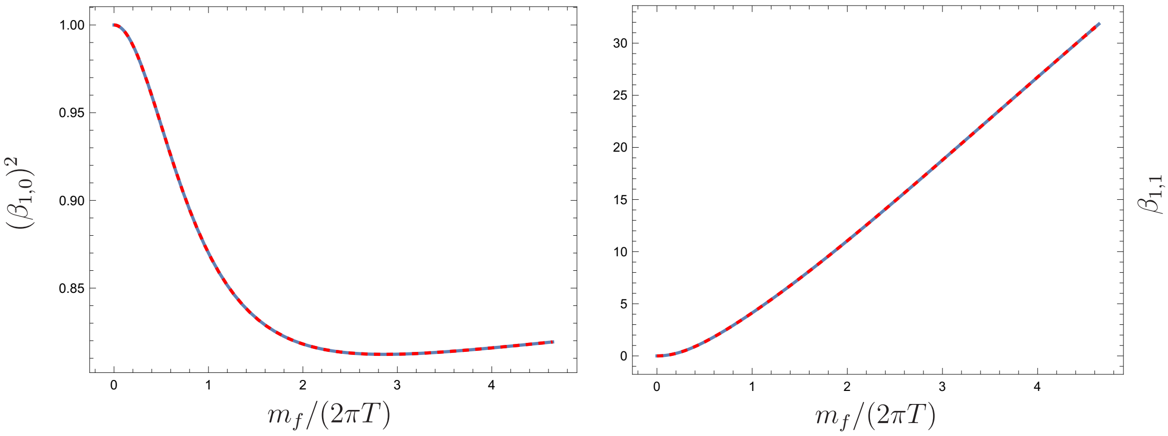

Figure 10. We parameterize the speed of the sound waves in the gauge theory plasma as in (2.44). The solid curves represent $(\beta_{1,0})^2$ and $\beta_{1,1}$ in the holographic model $\mathcal{D}_{4,3}$ obtained from the sound wave channel quasinormal mode (2.10); the dashed red curves represent the same data obtained applying the equation of state (2.11).

fluctuations[9] at zero frequency evaluated at the horizon. Furthermore, the first expression in (2.41) is true even in higher-derivative holographic models, provided the dual gravitational physics is effectively two-derivative (as in the Gauss-Bonnet models $\mathcal{B}_{(2,\Delta)}$), or, at least, effectively two-derivative at the horizon[10] (as in the class of models $\mathcal{C}_{(2,\Delta)}$), see (1.7),

$$\frac{\zeta}{s}\bigg|_{\mathcal{B},\mathcal{C}} = \frac{1}{9\pi}\sum_i z_{i,0}^2\,, \tag{2.42}$$

where $z_{i,0}$ has to be evaluated at the horizon including $\mathcal{O}(\beta)$ corrections.

Figure 6 demonstrates that the naive application of the EO formula (2.40) does not work in higher-derivative models. By "naive" application we mean the evaluation of the speed of the sound waves and the background scalar derivatives at the horizon in (2.40) to order $\mathcal{O}(\beta)$. Again, the solid curves represent the corrections[11] to the bulk viscosity from (2.17) (the left panel) and (2.18) (the right panel), and the dashed green curves indicate corrections from the EO formula (2.40). Interestingly, there is a disagreement even for models belonging to class $\mathcal{B}$ (the right panel), which are effectively two-derivative in the bulk. One might wonder whether the comparison of the ratio $\frac{\zeta}{s}$, which is somewhat more universal as it is partly applicable to higher-derivative theories (see (2.42)), would fare better. From [22],

$$\frac{\zeta}{s} = \frac{c_s^4}{4\pi}\sum_i\left(\frac{\partial\phi_i}{\partial\ln T}\right)^2\bigg|_{\lambda_\Delta=\text{const}}\,. \tag{2.43}$$

As we show in figure 7, this is not the case for the effective two-derivative in the bulk model $\mathcal{B}_{2,3}$: the solid curve represents the $\mathcal{O}(\beta)$ correction $\delta\frac{\zeta}{s}$ extracted from the quasinormal

[9]From the quasinormal mode perspective, the z_i's are spatially SO(3) invariant background scalar fluctuations of the sound channel which decouple from the metric fluctuations as $\mathfrak{q}\to 0$.

[10]Recall that we refer to the physics as being effective two-derivative at the horizon if there is no distinction between the Bekenstein and the Wald entropy densities, see (A.28).

[11]The quasinormal mode analysis (2.10) validates these results.

mode (2.10) analysis, the red dashed curve is obtained from (2.42), and the green dashed curve is the application of the EO formula (2.43). It should probably not come as a surprise that the EO formula for the bulk viscosity fails in higher derivative as well as in Gauss-Bonnet holographic models: the naive application of the EO formalism does not capture the Gauss-Bonnet coupling correction to the shear viscosity either [9].

In the holographic models $\mathcal{A}-\mathcal{C}$ (1.3) doing a metric field redefinition removes the bulk higher derivative coupling constants α_1 and α_2 [9]. However, in the presence of the scalar sector, as in (1.2), such a redefinition generates new higher derivative coupling constants in the scalar sector of the form $R\cdot(\partial\phi)^2$ and $R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Because computing the shear viscosity involves only the metric fluctuations, see (A.30), such a redefinition does not affect it, and the final correction is universal for all these models, see (2.12). On the contrary, computing the bulk viscosity necessitates turning on the bulk scalar fluctuations, see (A.62). As a result, the bulk viscosity is different in models $\mathcal{A}_{2,\Delta}$, $\mathcal{B}_{2,\Delta}$ and $\mathcal{C}_{2,\Delta}$ even though all these models have the same value of $\alpha_3 = 1$. This is shown explicitly in figure 8. Black curves represent $\mathcal{A}_{2,\Delta}$ models, blue curves represent $\mathcal{B}_{2,\Delta}$ models, and magenta curves represent $\mathcal{C}_{2,\Delta}$ models.

In figure 9 we present bulk viscosity corrections in the models $\mathcal{D}_{4,3}$ (the left panel) and $\mathcal{D}_{4,2}$ (the right panel). The solid curves show the bulk viscosity corrections extracted from the dispersion relation (2.10), and the dashed red curves represent (2.19). This is an excellent validation of our computational framework in holographic models with Weyl^4 higher derivative corrections.

We conclude this section mentioning one of the numerous consistency tests we performed. The speed of the sound waves c_s can be extracted from the dispersion relation of the sound channel quasinormal modes of the background black brane (2.10), or from the background black brane equations of state (2.11). We parameterize the speed of the sound waves as

$$c_s = \frac{\beta_{1,0}}{\sqrt{3}}\left(1+\beta\cdot\beta_{1,1}\right). \tag{2.44}$$

In figure 10 we compare results for $(\beta_{1,0})^2$ (the left panel) and $\beta_{1,1}$ (the right panel). Solid curves indicate data from the dispersion relation (2.10), and the dashed red curves are the corresponding results obtained from the equation of state (2.11) in the holographic model $\mathcal{D}_{4,3}$.

3 Conclusion

In this paper we developed a novel framework for computing transport coefficients in holographic model with higher derivative corrections. This allowed us to produce compact expressions (1.5)–(1.8) for the shear and bulk viscosities in large classes of non-conformal holographic models with higher derivative corrections. We expect that these formulas would be useful in exploring conditions under which the shear viscosity [6] or the bulk viscosity [17] bounds are violated. The explicit expressions for the Wald entropy density (A.28) would be useful in searches of stable holographic conformal order [40–44]. Moreover, since holographic models with scalar fields have been used, depending on the choice of scalar potential, to generate a wide spectrum of temperature dependence for η, in addition to a non-zero ζ, our analysis is also useful for direct comparison to the physics of the QGP. In particular, our simple expressions for the shear and bulk viscosities in the presence of arbitrary scalars can facilitate holographic model building and guide the efforts to describe the behavior of the QGP near the deconfinement transition.

We demonstrated that there are particularly simple and universal expressions for the ratio $\frac{\zeta}{s}$, see (2.42), valid even in models with higher derivatives in the bulk, but effectively two-derivative physics at the horizon, specifically when there is no distinction between the Bekenstein and the Wald entropies of the gauge theory thermal state dual black brane horizon. We also explored the applicability of the Eling-Oz formula for the bulk viscosity [22], and demonstrated that its naive application fails even in effectively two-derivative holographic Gauss-Bonnet models. At this stage it is not clear to us how to extend [22] to capture theories with higher derivatives and whether that construction can be generalized in a simple way.

Specific models discussed in section 2 can be of interest to phenomenological applications in heavy ion collisions. To facilitate these applications we recall the relations of some of the parameters used on the gravitational side of the holographic correspondence to the gauge theory observables:

- If c and a are the two central charges of a gauge theory UV fixed point,

$$\beta \cdot \alpha_3 = \frac{c-a}{8c}\,. \tag{3.1}$$

  The holographic coupling α_3 appears in models $\mathcal{A} - \mathcal{C}$. We are not aware of the simple relation for the other two coupling constants, α_2 and α_3, in the models $\mathcal{B}_{2,\Delta}$ and $\mathcal{C}_{2,\Delta}$.

- Assume for simplicity[12] that the UV fixed point is $\mathcal{N} = 4$ SU(N_c) supersymmetric Yang-Mills theory with a gauge coupling $g^2_{\rm YM}$. Then, in models $\mathcal{D}_{4,\Delta}$,

$$\beta \equiv \frac{1}{8}\zeta(3)\ \left(g^2_{\rm YM} N_c\right)^{-3/2}\,. \tag{3.2}$$

In this paper we focused on the first-order transport coefficients, i.e., η and ζ, of the hydrodynamics theory derivative approximation. Stability and causality of the Landau-frame hydrodynamics can be ensured including higher-order transport coefficients. Holographic computations of the second-order transport coefficients of conformal gauge theories were first done in [46, 47],[13] and finite 't Hooft coupling corrections were discussed in [49–52]. It would be interesting, albeit challenging, to extend the computational framework proposed here to the analysis of these coefficients in holographic models with higher derivative corrections. We expect that such results will be sensitive to the holographic renormalization of the models, as well as to the details of the proper formulation of the variational principle, i.e., the precise expressions for the Gibbons-Hawking terms.

In the future, it would also be interesting to extend the results reported here to holographic models with conserved charges, and to capture the effects of a chemical potential. It is natural to wonder whether, in the presence of generic higher derivative terms, the conductivity can also be extracted from a radially conserved current, and thus entirely from the horizon of the geometry. Finally, it would be useful to extend our framework to magnetohydrodynamics, again in the presence of higher derivatives. We leave these questions to future work.

[12]See [45] for more examples.

[13]See also [48] for some extensions to nonconformal models.

Acknowledgments

Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. A.B.'s work was further supported by NSERC through the Discovery Grants program. The work of S.C. and L.E. is supported in part by the NSF grant PHY-2210271. This work was initiated during the program "The Many Faces of Relativistic Fluid Dynamics" at the Kavli Institute for Theoretical Physics at UC Santa Barbara. AB and SC thank KITP for their hospitality. This work was supported in part by the National Science Foundation under Grant No. NSF PHY-1748958.

A Transport coefficients from Kubo formulas

A.1 Black brane geometry dual to thermal states of the boundary theory

The background geometry dual to a thermal equilibrium state of a boundary gauge theory takes form

$$ds_5^2 = -c_1^2\, dt^2 + c_2^2\, d\boldsymbol{x}^2 + c_3^2\, dr^2\,, \tag{A.1}$$

where $c_i = c_i(r)$, and additionally $\phi_i = \phi_i(r)$. The radial coordinate is $r \in [0, r_h]$, with r_h being the location of the regular black brane horizon,

$$\lim_{r\to r_h} c_1 = 0\,. \tag{A.2}$$

Notice that at this stage we do not fix the residual diffeomorphism associated with the reparametrization of the radial coordinate.

One can efficiently compute the background equations of motion from the effective one dimensional action,

$$S_1 = \frac{1}{16\pi G_N}\int_0^{r_h} dr\left[\mathcal{I} + \beta\cdot\delta\mathcal{I}\right], \tag{A.3}$$

obtained from the evaluation of (1.2) on the ansatz (A.1). Here ($' \equiv \frac{d}{dr}$),

$$\mathcal{I} = c_1c_2^3c_3\left(12 - \frac{2c_1''}{c_1c_3^2} - \frac{6c_2''}{c_2c_3^2} - \frac{6(c_2')^2}{c_2^2c_3^2} + \frac{6c_3'c_2'}{c_2c_3^3} - \frac{6c_1'c_2'}{c_2c_1c_3^2} + \frac{2c_1'c_3'}{c_1c_3^3} - \frac{1}{2c_3^2}\sum_i(\phi_i')^2 - V\right), \tag{A.4}$$

with the higher derivative contributions in model (1.3) given by

$$\begin{aligned}
\delta\mathcal{I}_2 = c_1c_2^3c_3\Bigg(4\alpha_1\left[\frac{3c_2''}{c_2c_3^2} + \frac{c_1''}{c_1c_3^2} - \frac{c_1'c_3'}{c_1c_3^3} + \frac{3c_1'c_2'}{c_2c_1c_3^2} - \frac{3c_3'c_2'}{c_2c_3^3} + \frac{3(c_2')^2}{c_2^2c_3^2}\right]^2 + \alpha_2\Bigg[\frac{6c_1'c_2'c_2''}{c_2^2c_1c_3^4} \\
- \frac{6c_1'c_3'c_2''}{c_2c_1c_3^5} - \frac{6c_2'c_3'c_1''}{c_2c_1c_3^5} - \frac{6c_1'(c_2,')^2c_3'}{c_2^2c_1c_3^5} - \frac{24c_2'c_3'c_2''}{c_2^2c_3^5} - \frac{12(c_2')^3c_3'}{c_2^3c_3^5} + \frac{6c_1'c_2'(c_3')^2}{c_2c_1c_3^6} + \frac{6c_1'c_2'c_1''}{c_2c_1^2c_3^4} \\
+ \frac{6c_1''c_2''}{c_2c_1c_3^4} + \frac{12c_1'(c_2')^3}{c_2^3c_1c_3^4} + \frac{12(c_2')^2c_2''}{c_2^3c_3^4} - \frac{4c_1'c_3'c_1''}{c_1^2c_3^5} - \frac{6(c_1')^2c_2'c_3'}{c_2c_1^2c_3^5} + \frac{2(c_1'')^2}{c_1^2c_3^4} + \frac{12(c_2')^4}{c_2^4c_3^4} \\
+ \frac{2(c_1')^2(c_3')^2}{c_1^2c_3^6} + \frac{12(c_1')^2(c_2')^2}{c_2^2c_1^2c_3^4} + \frac{12(c_2'')^2}{c_2^2c_3^4} + \frac{12(c_2')^2(c_3')^2}{c_2^2c_3^6}\Bigg] + \alpha_3\Bigg[\frac{4(c_1'')^2}{c_1^2c_3^4} + \frac{12(c_1')^2(c_2')^2}{c_2^2c_1^2c_3^4} \\
+ \frac{12(c_2'')^2}{c_2^2c_3^4} + \frac{12(c_2')^4}{c_2^4c_3^4} - \frac{8c_1'c_3'c_1''}{c_1^2c_3^5} - \frac{24c_2'c_3'c_2''}{c_2^2c_3^5} + \frac{4(c_1')^2(c_3')^2}{c_1^2c_3^6} + \frac{12(c_2')^2(c_3')^2}{c_2^2c_3^6}\Bigg],
\end{aligned} \tag{A.5}$$

JHEP04(2024)032

and in model (1.4) by

$$\delta\mathcal{I}_4 = \frac{5}{36} c_1 c_2^3 c_3 \left(\frac{c_1' c_2'}{c_2 c_1 c_3^2} + \frac{c_1' c_3'}{c_1 c_3^3} - \frac{(c_2')^2}{c_2^2 c_3^2} - \frac{c_3' c_2'}{c_2 c_3^3} - \frac{c_1''}{c_1 c_3^2} + \frac{c_2''}{c_2 c_3^2} \right)^4 . \tag{A.6}$$

From (A.3) we obtain the following equations of motion:[14]

$$0 = c_1'' - \frac{c_1 (c_2')^2}{c_2^2} + \frac{2 c_2' c_1'}{c_2} - \frac{c_3' c_1'}{c_3} + \frac{1}{12} c_1 \sum_i (\phi_i')^2 + \frac{1}{6} c_3^2 c_1 (V - 12) + \beta \cdot \Big[\cdots \Big] , \tag{A.7}$$

$$0 = c_2'' - \frac{c_2' c_3'}{c_3} + \frac{(c_2')^2}{c_2} + \frac{1}{12} c_2 \sum_i (\phi_i')^2 + \frac{1}{6} c_2 c_3^2 (V - 12) + \beta \cdot \Big[\cdots \Big] , \tag{A.8}$$

$$0 = \sum_i (\phi_i')^2 - \frac{12 (c_2')^2}{c_2^2} - \frac{12 c_2' c_1'}{c_2 c_1} - 2 c_3^2 (V - 12) + \beta \cdot \Big[\cdots \Big] , \tag{A.9}$$

$$0 = \phi_i'' - \frac{\phi_i' c_3'}{c_3} + \frac{3 \phi_i' c_2'}{c_2} + \frac{c_1' \phi_i'}{c_1} - c_3^2 \, \partial_i V . \tag{A.10}$$

We verified that the constraint (A.9) is consistent with the remaining equations to order $\mathcal{O}(\beta)$ inclusive.

On-shell, i.e., evaluated when (A.7)–(A.10) hold, the effective action (A.3) is a total derivative. Specifically, we find

$$\mathcal{I} + \beta \cdot \delta\mathcal{I} = - \frac{6 c_2^2 c_1}{c_3} \cdot \text{eq.(A.8)} + \frac{d}{dr} \left\{ - \frac{2 c_2^3 c_1'}{c_3} + \beta \cdot \delta\mathcal{B} \right\} , \tag{A.11}$$

with the higher derivative terms $\delta\mathcal{B}$ given by

$$\begin{aligned}
\delta\mathcal{B}_2 = & - 4(2\alpha_1 + \alpha_2 + 2\alpha_3) \frac{c_2^3 c_1'''}{c_3^3} - 6(4\alpha_1 + \alpha_2) \frac{c_1 c_2^2 c_2'''}{c_3^3} + 12(4\alpha_1 + \alpha_2) \frac{c_1 (c_2')^3}{c_3^3} \\
& + \frac{2 c_2 (c_1')^2}{c_3^5 c_1} \Big(24 c_2' c_2 c_3^2 \alpha_1 + 6 c_2' c_2 c_3^2 \alpha_2 - 8 c_3' c_2^2 c_3 \alpha_1 - 4 c_3' c_2^2 c_3 \alpha_2 - 8 c_3' c_2^2 c_3 \alpha_3 \Big) + \frac{4 c_2 c_1'}{c_3^5} \Big(\\
& 2 c_3'' c_2^2 c_3 \alpha_1 + c_3'' c_2^2 c_3 \alpha_2 + 2 c_3'' c_2^2 c_3 \alpha_3 + 12 (c_2')^2 c_3^2 \alpha_1 + 3 (c_2')^2 c_3^2 \alpha_2 + 6 (c_2')^2 c_3^2 \alpha_3 \\
& + 6 c_2' c_2 c_3 \alpha_1 c_3' + 3 c_2' c_2 c_3 \alpha_2 c_3' + 6 c_3' c_2 c_3 \alpha_3 c_2' - 6 (c_3')^2 c_2^2 \alpha_1 - 3 (c_3')^2 c_2^2 \alpha_2 - 6 (c_3')^2 c_2^2 \alpha_3 \Big) \\
& - 4(2\alpha_1 + \alpha_2 + 2\alpha_3) \frac{c_2^2 c_1''}{c_3^4 c_1} \Big(3 c_2' c_3 c_1 - 3 c_3' c_2 c_1 - 2 c_2 c_3 c_1' \Big) + 6(4\alpha_1 + \alpha_2) \frac{c_2 c_1 c_3' (c_2')^2}{c_3^4} \\
& + 6(4\alpha_1 + \alpha_2) \frac{c_2^2 c_1 c_2'}{c_3^5} \Big(c_3'' c_3 - 3 (c_3')^2 \Big) + 6(4\alpha_1 + \alpha_2) \frac{c_2''}{c_3^6} \Big(- c_2' c_2 c_3^3 c_1 + 3 c_3' c_2^2 c_3^2 c_1 \Big) ,
\end{aligned} \tag{A.12}$$

[14] The $\cdots$ represent the $\mathcal{O}(\beta)$ terms that we omit for readability. Of course, these terms must be taken into account to obtain the correct results.

JHEP04(2024)032

and

$$\delta\mathcal{B}_4=\frac{5}{9}\left(\frac{(c_2')^2}{c_2^2}+\frac{c_2'c_3'}{c_3c_2}-\frac{c_2''}{c_2}-\frac{c_2'c_1'}{c_2c_1}-\frac{c_3'c_1'}{c_3c_1}+\frac{c_1''}{c_1}\right)^2\cdot\left(\frac{3c_3''c_1'c_2^3}{c_3^8}-\frac{3c_1c_3''c_2'c_2^2}{c_3^8}-\frac{3c_1'''c_2^3}{c_3^7}\right.$$
$$+\frac{3c_1c_2'''c_2^2}{c_3^7}-\frac{4(c_1')^2c_2'c_2^2}{c_3^7c_1}-\frac{4(c_1')^2c_2^3c_3'}{c_3^8c_1}+\frac{2c_1'(c_2')^2c_2}{c_3^7}-\frac{c_1'c_2'c_2^2c_3'}{c_3^8}+\frac{4c_1'c_2^3c_1''}{c_3^7c_1}+\frac{2c_1'c_2^2c_2''}{c_3^7}$$
$$-\frac{9c_1'c_2^3(c_3')^2}{c_3^9}+\frac{2c_1(c_2')^3}{c_3^7}+\frac{5c_1(c_2')^2c_2c_3'}{c_3^8}-\frac{c_2'c_2^2c_1''}{c_3^7}-\frac{5c_1c_2'c_2c_2''}{c_3^7}+\frac{9c_1c_2'c_2^2(c_3')^2}{c_3^9}+\frac{9c_2^3c_3'c_1''}{c_3^8}$$
$$\left.-\frac{9c_1c_2^2c_3'c_2''}{c_3^8}\right). \tag{A.13}$$

In what follows we will need the entropy density s and the temperature T of the boundary thermal state. The temperature is determined by requiring the vanishing of the conical deficit angle of the analytical continuation of the geometry (A.1),

$$2\pi T=\lim_{r\to r_h}\left[-\frac{c_2}{c_3}\left(\frac{c_1}{c_2}\right)'\right]=\lim_{r\to r_h}\left[-\frac{c_1'}{c_3}+\frac{c_1c_2'}{c_2c_3}\right]=\lim_{r\to r_h}\left[-\frac{c_1'}{c_3}\right], \tag{A.14}$$

where to obtain the last equality we used (A.2). The thermal entropy density of the boundary gauge theory is identified with the entropy density of the dual black brane [53]. Since our holographic model contains higher-derivative terms, the Bekenstein entropy s_B,

$$s_B=\lim_{r\to r_h}\frac{c_2^3}{4G_N}, \tag{A.15}$$

must be replaced with the Wald entropy s_W [54],

$$s_W=-\frac{1}{8\pi G_N}\lim_{r\to r_h}\left[c_2^3\ \epsilon_{\mu\nu}\epsilon_{\rho\lambda}\frac{\delta L_5}{\delta R_{\mu\nu\rho\lambda}}\right], \tag{A.16}$$

i.e., $s=s_W$. The simplest way to compute the Wald entropy density is instead to use the boundary thermodynamics:

- According to the holographic correspondence [55, 56], the on-shell gravitational action S_1, properly renormalized [57], has to be identified with the boundary gauge theory free energy density $\mathcal{F}$ as follows,

$$-\mathcal{F}=S_1\Big|_{\text{on-shell}}=\frac{1}{16\pi G_N}\int_0^{r_h}dr\ \frac{d}{dr}\left\{-\frac{2c_2^3c_1'}{c_3}+\beta\cdot\delta\mathcal{B}\right\}+\lim_{r\to0}\left[S_{\text{GH}}+S_{ct}\right], \tag{A.17}$$

where we used (A.11). S_{GH} is a generalized Gibbons-Hawking term [8], necessary to have a well-defined variational principle, and S_{ct} is the counter-term action — we will not need the explicit form of either of these corrections.

- eq. (A.17) can be rearranged to explicitly implement the basic thermodynamic relation $-\mathcal{F}=sT-\mathcal{E}$ between the free energy density $\mathcal{F}$, the energy density $\mathcal{E}$ and the entropy density s [58]:

$$-\mathcal{F}=\frac{1}{16\pi G_N}\lim_{r\to r_h}\left[-\frac{2c_2^3c_1'}{c_3}+\beta\cdot\delta\mathcal{B}\right]-\lim_{r\to0}\left[\frac{1}{16\pi G_N}\left(-\frac{2c_2^3c_1'}{c_3}+\beta\cdot\delta\mathcal{B}\right)+S_{\text{GH}}+S_{ct}\right]. \tag{A.18}$$

- Finally, we identify[15]

$$sT \equiv s_W T = \frac{1}{16\pi G_N} \lim_{r\to r_h}\left[-\frac{2c_2^3 c_1'}{c_3} + \beta\cdot\delta\mathcal{B}\right]. \tag{A.19}$$

Notice that to leading order $\mathcal{O}(\beta^0)$, using (A.14), we recover from (A.19) the Bekenstein entropy (A.15); the order $\mathcal{O}(\beta)$ term implements the correction to get the Wald entropy density. In what follows we will need the ratio of the Wald and the Bekenstein entropy densities of the black brane, evaluated at the same temperature:

$$\frac{s_W}{s_B} \equiv 1 + \beta\cdot\lim_{r\to r_h}\kappa(r)\,, \qquad \kappa \equiv -\frac{c_3}{2c_2^3 c_1'}\cdot\delta\mathcal{B}\,. \tag{A.20}$$

We proceed to evaluate κ for our two holographic models (1.3) and (1.4). Since we work to order $\mathcal{O}(\beta)$ in (A.20), we can evaluate $\delta\mathcal{B}$ to leading order in β, i.e., we can use the leading order equations of motion (A.7)–(A.10), and algebraically eliminate $c_1''', c_2''', c_1'', c_2''$ from $\delta\mathcal{B}_2$ and $\delta\mathcal{B}_4$:

$$\begin{aligned}\delta\mathcal{B}_2 = &(16\alpha_1+5\alpha_2+4\alpha_3)\frac{2c_2^3c_1}{3c_3}\sum_i \partial_i V\cdot\phi_i' - c_2^3\bigg((28\alpha_1+11\alpha_2+16\alpha_3)\frac{2c_1'}{3c_3}\\ &+(4\alpha_1+\alpha_2)\frac{6c_1c_2'}{c_3c_2}\bigg)V - \frac{36c_1(c_2')^3}{c_3^3}(4\alpha_1+\alpha_2) - \frac{24c_1'c_2(c_2')^2}{c_3^3}(9\alpha_1+3\alpha_2+2\alpha_3)\\ &-\frac{36c_2^2c_2'(c_1')^2}{c_3^3c_1}(2\alpha_1+\alpha_2+2\alpha_3) + \frac{8c_2^3c_1'}{c_3}(28\alpha_1+11\alpha_2+16\alpha_3) + \frac{72c_2^2c_2'c_1}{c_3}(4\alpha_1+\alpha_2)\,,\end{aligned} \tag{A.21}$$

$$\begin{aligned}\delta\mathcal{B}_4 = &-\frac{15(c_2')^2}{c_2^2}\left(\frac{c_1'}{c_1}-\frac{c_2'}{c_2}\right)^2\bigg(\frac{(c_1'c_2-c_2'c_1)c_2^3}{c_3^5}V - \frac{12c_1'c_2^3}{c_3^5} + \frac{12c_1c_2'c_2^2}{c_3^5} + \frac{7(c_1')^2c_2'c_2^2}{c_3^7c_1}\\ &+\frac{7c_1'(c_2')^2c_2}{c_3^7} - \frac{14c_1(c_2')^3}{c_3^7}\bigg)\,.\end{aligned} \tag{A.22}$$

We need to evaluate (A.21) and (A.22) at the horizon, i.e., as $r\to r_h$. It is convenient to fix the residual diffeomorphism in (A.1) as[16]

$$c_1 = \frac{f^{1/2}(g+\mathcal{O}(\beta))^{1/2}}{r}\,, \qquad c_2 = \frac{1}{r}\,, \qquad c_3 = \frac{1}{rf^{1/2}(1+\mathcal{O}(\beta))}\,, \tag{A.23}$$

where $\{f,g\} = \{f,g\}(r)$. Given (A.7)–(A.9), to leading order in β,

$$f' = \frac{f}{6r}\sum_i(\phi_i')^2 + \frac{4f}{r} + \frac{V}{3r} - \frac{4}{r}\,, \qquad g' = -\frac{gr}{3}\sum_i(\phi_i')^2\,. \tag{A.24}$$

From (A.2), the horizon is located at r_h, such that

$$\lim_{r\to r_h} f = 0\,. \tag{A.25}$$

[15] Strictly speaking, (A.19) is correct up to an arbitrary constant. But this constant must be set to zero from the comparison with thermal AdS, in which case the black brane geometry is dual to a thermal state of a boundary CFT with vanishing entropy in the limit $T\to 0$.

[16] Of course, final results are independent of this choice.

Regularity of ϕ_i at the horizon then implies from (A.10) that[17]

$$\phi_i' \sim \frac{3\partial_i V}{r(V-12)}\,, \qquad \text{as} \qquad r \to r_h\,. \tag{A.26}$$

Using (A.24) and (A.26) we find from (A.21), (A.22) and (A.20)

$$\kappa_2 \sim \frac{2}{3}(V-12)(5\alpha_1+\alpha_2-\alpha_3) \qquad \text{and} \qquad \kappa_4 \sim -\frac{5}{144}(V-12)^3 \quad \text{as} \quad r \to r_h\,, \tag{A.27}$$

i.e.,

$$\frac{s_W}{s_B} = 1 + \beta \cdot \begin{cases} \frac{2}{3}(V-12)(5\alpha_1+\alpha_2-\alpha_3)\,, & \text{when } \delta\mathcal{L} = \delta\mathcal{L}_2\,; \\ -\frac{5}{144}(V-12)^3\,, & \text{when } \delta\mathcal{L} = \delta\mathcal{L}_4\,, \end{cases} \tag{A.28}$$

where all quantities in (A.28) are to be evaluated at the black brane horizon.

Finally, note that in the class of models $\delta\mathcal{L}_2$ there are two special cases for which the Wald and Bekenstein entropies coincide to order $\mathcal{O}(\beta)$:

- $\alpha_1 = 1\,,\ \alpha_2 = -4\,,\ \alpha_3 = 1$ — the combination in (1.3) assembles into a Gauss-Bonnet term, which renders the full gravitational action (1.2) two-derivative;

- $\alpha_1 = 0\,,\ \alpha_2 = \alpha_3 = 1$ — the gravitational action (1.2) is higher-derivative in the bulk, but is effectively two-derivative in the vicinity of the black brane horizon.

This will be relevant to our discussion later on, when we examine the applicability of the Eling-Oz construction [22].

A.2 $\frac{\eta}{s}$ with higher derivative corrections

Following [2, 8], we use the Kubo formula to compute the shear viscosity from the two-point retarded correlation function of the boundary stress-energy tensor with indices along the spatial directions $_{12}$,

$$\eta = -\lim_{w\to 0}\frac{1}{w}\mathrm{Im}\,G_R(w)\,, \qquad G_R(w) = -i\int dt d\boldsymbol{x} e^{iwt}\theta(t)\langle[T_{12}(t,\boldsymbol{x}), T_{12}(0,\boldsymbol{0})]\rangle\,. \tag{A.29}$$

To compute the retarded thermal two-point function of the components of the stress-energy tensor entering (A.29) we add the bulk metric perturbation

$$ds_5^2 \to ds_5^2 + 2h_{12}(t,r)\ dx_1 dx_2\,. \tag{A.30}$$

Simple symmetry arguments [59] show that all the remaining metric and bulk scalars fluctuations can be consistently set to zero; additionally, we can restrict to SO(3) invariant metric perturbations.

It is convenient to use the idea of the complexified effective action for the fluctuations introduced in [18]. This complexified action is a functional of $h_{12,w}(r)$ and $h^*_{12,w}(r) \equiv h_{12,-w}(r)$

$$S^{(2)} = \frac{1}{16\pi G_N}\int_0^{r_h} dr\ \mathcal{L}_{\mathbb{C}}\{h_{12,w}, h^*_{12,w}\}\,, \tag{A.31}$$

[17] We use $l.h.s. \sim r.h.s.$ to denote that $\lim_{r\to r_h}\frac{l.h.s.}{r.h.s.} = 1$.

and is constructed in such a way that the $h_{12,w}$ equation of motion obtained from $\mathcal{L}_{\mathbb{C}}$,

$$0 = \frac{\delta S^{(2)}}{\delta h^*_{12,w}}\,, \tag{A.32}$$

is identical to the one obtained from the effective action (1.2), assuming the harmonic dependence for the fluctuations $h_{12}(t,r) = e^{-iwt}h_{12,w}(r)$. In practice, it is straightforward to construct $\mathcal{L}_{\mathbb{C}}$:

- Evaluate (1.2) to quadratic order in the fluctuations (A.30).
- Complexify every term of the resulting quadratic action as follows: e.g., replace

$$\begin{aligned} h_{12}\,\partial^2_{rr}h_{12} \ &\longrightarrow\ \frac{1}{2}\left(h_{12}\,\partial^2_{rr}h^*_{12} + h^*_{12}\,\partial^2_{rr}h_{12}\right)\,, \\ &\text{or} \\ \partial_t h_{12}\,\partial^2_{tr}h_{12} \ &\longrightarrow\ \frac{1}{2}\left(\partial_t h_{12}\,\partial^2_{tr}h^*_{12} + \partial_t h^*_{12}\,\partial^2_{tr}h_{12}\right)\,. \end{aligned} \tag{A.33}$$

- Introduce the harmonic dependence as

$$h_{12} = e^{-iwt}h_{12,w}(r)\,, \qquad h^*_{12} = e^{iwt}h_{12,-w}(r)\,. \tag{A.34}$$

- The resulting action gives $\mathcal{L}_{\mathbb{C}}$.

For the model (1.3) we find $\mathcal{L}_{\mathbb{C}} \equiv \mathcal{L}_{\mathbb{C};2}$,

$$\begin{aligned} \mathcal{L}_{\mathbb{C};2} = &B_1\ h''_{12,-w}h''_{12,w} + \frac{B_2}{2}\ \left(h''_{12,w}h'_{12,-w} + h''_{12,-w}h'_{12,w}\right) \\ &-\frac{1}{2}(B_3w^2 - A_1 - B_4)\left(h''_{12,w}h_{12,-w} + h''_{12,-w}h_{12,w}\right) + (B_{10}w^2 + B_{11} + A_4)\ h'_{12,w}h'_{12,-w} \\ &-\frac{1}{2}((B_6 - B_9)w^2 - A_5 - B_{12})\ \left(h'_{12,w}h_{12,-w} + h'_{12,-w}h_{12,w}\right) \\ &+h_{12,w}h_{12,-w}\ \left(B_5w^4 - w^2(B_7 - B_8 + A_2 - A_3) + B_{13} + A_6\right)\,, \end{aligned} \tag{A.35}$$

where the connection coefficients $A_i(r) = \mathcal{O}(\beta^0)$ and $B_i(r) = \mathcal{O}(\beta)$ are functionals of the background geometry (A.1). Coefficients A_i are presented in appendix B; coefficients B_i are too long to be reported here. For the model (1.4) we find $\mathcal{L}_{\mathbb{C}} \equiv \mathcal{L}_{\mathbb{C};4}$ of the form as in (A.35), albeit with a distinct set of the connection coefficients B_i.

On-shell, the effective action (A.35) can be re-expressed as a total derivative,

$$\mathcal{L}_{\mathbb{C}} = 16\pi G_N \cdot h^*_{12,w}\cdot \frac{\delta S^{(2)}}{\delta h^*_{12,w}} + \partial_r J_w\,, \tag{A.36}$$

with a current

$$\begin{aligned} J_w = &\left[B_1h''_{12,w} + \frac{B_2}{2}h'_{12,w} - \frac{1}{2}(B_3w^2 - A_1 - B_4)h_{12,w}\right]h'_{12,-w} + \Big[-B_1h'''_{12,w} - B'_1h''_{12,w} \\ &+ \left(\left(B_{10} + \frac{1}{2}B_3\right)w^2 - \frac{1}{2}B'_2 + B_{11} + A_4 - \frac{1}{2}A_1 - \frac{1}{2}B_4\right)h'_{12,w} \\ &+ \left(\left(-\frac{1}{2}B_6 + \frac{1}{2}B_9 + \frac{1}{2}B'_3\right)w^2 - \frac{1}{2}A'_1 - \frac{1}{2}B'_4 + \frac{1}{2}A_5 + \frac{1}{2}B_{12}\right)h_{12,w}\Big]h_{12,-w}\,. \end{aligned} \tag{A.37}$$

JHEP04(2024)032

A crucial observation originally made in [18] was that an analog of J_w in two-derivative holographic models (no B_i coefficients in (A.37)) has a radially conserved imaginary part, on-shell. It is straightforward to verify that this property holds, even in the presence of higher derivatives of the effective action:

$$\partial_r(J_w - J_{-w})\Big|_{on-shell} = 2\,\partial_r \mathrm{Im}\,J_w\Big|_{on-shell} = 0\,. \tag{A.38}$$

The conservation law (A.38) is a direct consequence of the exact U(1) symmetry of $\mathcal{L}_{\mathbb{C}}$ (A.31) that rotates the phase of fluctuations, namely $h_{12,w} \to e^{i\theta} h_{12,w}$ and $h_{12,-w} \to e^{-i\theta} h_{12,-w}$. The conserved Noether charge associated with this symmetry is precisely $\mathrm{Im}\,J_w$. Indeed,

$$16\pi G_N\,\frac{\delta S^{(2)}}{\delta\theta} = -i\,\partial_r\theta\cdot(J_w - J_{-w})\,, \tag{A.39}$$

for infinitesimal θ-rotations. In [18] this conserved charge was interpreted as the radially conserved number flux of gravitons[18], [19].

Essentially following the discussion of [60], the retarded two point correlation function of the stress-energy tensor (A.29) has to be identified with the boundary limit (i.e., $r \to 0$) of the current J_w (A.37). It is easy to see that J_w diverges as one approaches the AdS boundary — it must be regularized and renormalized [57] by adding an appropriate counter-term J_{ct}. Additionally, one must add a generalized Gibbons-Hawking boundary term J_{GH} [8, 61] to have a well-defined variational principle for $S^{(2)}$. Thus,

$$G_R(w) = \frac{1}{8\pi G_N}\lim_{r\to 0}(J_w + J_{\mathrm{GH}} + J_{ct})\,. \tag{A.40}$$

Recall from (A.29) that, in order to extract the shear viscosity, we need only the imaginary part of $G_R(w)$. However, both J_{GH} and J_{ct} can not contribute to $\mathrm{Im}\,G_R$. For example, from the representation (A.36), the variation $\frac{\delta S^{(2)}}{\delta h^*_{12,w}}$ would produce a boundary term $-A_1 h_{12,w}\cdot\delta h'_{12,-w}$. Such a term must be cancelled with the appropriate term in the variation of J_{GH}:

$$\delta\Big(+A_1\left(h_{12,w}h'_{12,-w} + h_{12,-w}h'_{12,w}\right)\Big) = +A_1 h_{12,w}\cdot\delta h'_{12,-w}\,. \tag{A.41}$$

However, $\mathrm{Im}\,[A_1\left(h_{12,w}h'_{12,-w} + h_{12,-w}h'_{12,w}\right)] = 0$. Clearly, this will be the case for all terms in J_{GH} and J_{ct}: indeed, originally the Gibbons-Hawking term and the counterterms are real, and the complexification as in (A.33) will not change this fact. As a result,

$$\mathrm{Im}\,G_R(w) = \frac{1}{8\pi G_N}\lim_{r\to 0}\mathrm{Im}\,J_w = \frac{1}{8\pi G_N}\lim_{r\to r_h}\mathrm{Im}\,J_w\,, \tag{A.42}$$

where in the second equality we used the fact that $\mathrm{Im}\,J_w$ is conserved along the radial flow, and so *can be evaluated at the horizon.* Of course, this is the reason underlying the original claims of the universality of the shear viscosity in two-derivative holographic models [3, 62].

[18] See [18] for further discussion and related earlier work.

[19] It is an interesting open question as to why the quadratic action for the fluctuations has this peculiar property, (A.36).

Before we can evaluate J_w at the black brane horizon, we need to derive the equation of motion for $h_{12,w}$. From (A.32) we find

$$
\begin{aligned}
0 = &(A_1 - A_4)\ h''_{12,w} + (A'_1 - A'_4)\ h'_{12,w} + \left(\frac{1}{2}A''_1 - \frac{1}{2}A'_5 + A_6 + (A_3 - A_2)w^2\right)\ h_{12,w} \\
&+ \left\{B_1\ h''''_{12,w} + 2B'_1\ h''_{12,w} + \left(B''_1 + \frac{1}{2}B'_2 + B_4 - B_{11} - w^2(B_3 + B_{10})\right)\ h'_{12,w}\right. \\
&+ \left(\frac{1}{2}B''_2 - B'_{11} + B'_4 - (B'_3 + B'_{10})w^2\right)\ h'_{12,w} + \left(B_5 w^4 + \left(B_8 - B_7 + \frac{1}{2}B'_6 - \frac{1}{2}B'_9\right.\right. \\
&\left.\left.\left. - \frac{1}{2}B''_3\right)w^2 + B_{13} + \frac{1}{2}B''_4 - \frac{1}{2}B'_{12}\right)\ h_{12,w}\right\}. \qquad (A.43)
\end{aligned}
$$

All the terms in the bracket $\{\}$ above are $\mathcal{O}(\beta)$, and the equation (A.43) can reduced to a second order equation for $h_{12,w}$ eliminating $h''''_{12,w}$, $h'''_{12,w}$, $h''_{12,w}$ with B_i connection coefficients using the $\mathcal{O}(\beta^0)$ equation of motion. We will need to solve the equation (A.43) in the hydrodynamic approximation, specifically to order $\mathcal{O}(w)$. It is convenient to introduce

$$
h_{12,w} = c_2^2\ H_{12,w}\,, \qquad H_{12,w} = \left(\frac{c_1}{c_2}\right)^{-i\mathfrak{w}} (H_0 + i\mathfrak{w}\ H_1)\ , \qquad (A.44)
$$

where we set

$$
\mathfrak{w} = \frac{w}{2\pi T}\,. \qquad (A.45)
$$

With (A.44), the incoming wave boundary condition for the fluctuations, and the correct normalization at the boundary, result in

$$
\begin{aligned}
&\lim_{r\to 0} H_0(r) = 1\,, \qquad \lim_{r\to 0} H_1(r) = 0\,, \\
&\lim_{r\to r_h} H_0(r) = \text{finite}\,, \qquad \lim_{r\to r_h} H_1(r) = \text{finite}\,.
\end{aligned} \qquad (A.46)
$$

Using the black brane background equations of motion (A.7)–(A.10), the equation for the fluctuations (A.43), and (A.44), we can evaluate the $\mathcal{O}(\mathfrak{w})$ part of J_w from (A.37)

$$
J_w = J_0 - i\mathfrak{w}\ J_1\,, \qquad J_1 \equiv F + \beta\cdot\delta F\,, \qquad (A.47)
$$

$$
F = -\frac{c_2^2(c_2c'_1 - c'_2c_1)}{2c_3}\ H_0^2 + \frac{c_2^3 c_1}{2c_3}\ (H_0H'_1 - H'_0H_1)\,, \qquad (A.48)
$$

while for the model (1.3), $\delta F \equiv \delta F_2$,

$$
\begin{aligned}
\delta F_2 = &-\frac{2}{3}\left(H_0^2(c_2c'_1 - c'_2c_1) - c_2c_1(H_0H'_1 - H'_0H_1)\right)\left(\frac{2c_2^2}{c_3}(2\alpha_1 + \alpha_2 + 2\alpha_3)V\right. \\
&\left. + \left(\frac{c_2c'_1c'_2}{c_1c_3^3} + \frac{(c'_2)^2}{c_3^3}\right)(9\alpha_1 + 9\alpha_2 + 24\alpha_3) + \frac{3(c'_2)^2}{c_3^3}\alpha_3 - \frac{24c_2^2}{c_3}(2\alpha_1 + \alpha_2 + 2\alpha_3)\right),
\end{aligned} \qquad (A.49)
$$

and for the model (1.4), $\delta F \equiv \delta F_4$,

$$
\begin{aligned}
\delta F_4 = &-\frac{c'_2(c_1c'_2 - c_2c'_1)^2}{4}\left(H_0^2(c_2c'_1 - c'_2c_1) - c_2c_1(H_0H'_1 - H'_0H_1)\right)\left(\frac{24c'_2(c'_1)^2}{c_1^4c_2^2c_3^7} - \frac{48c'_1}{c_1^3c_2c_3^5}\right. \\
&\left. - \frac{96c'_2}{c_1^2c_2^2c_3^5} + \frac{117(c'_2)^2c'_1}{c_1^3c_2^3c_3^7} + \frac{111(c'_2)^3}{c_1^2c_2^4c_3^7} + \left(\frac{4c'_1}{c_1^3c_2c_3^5} + \frac{8c'_2}{c_1^2c_2^2c_3^5}\right)V\right). \qquad (A.50)
\end{aligned}
$$

Finally, from (A.29) and (A.42) we obtain

$$\eta = -\frac{1}{8\pi G_N}\lim_{w\to 0}\frac{1}{w}\mathrm{Im}\, J_w = -\frac{1}{8\pi G_N}\lim_{w\to 0}(-1)\cdot\frac{\mathfrak{w}}{w}J_1 = \frac{1}{8\pi G_N}\cdot\frac{1}{2\pi T}\cdot(F+\beta\cdot\delta F)\,. \tag{A.51}$$

Further simplifications occur when (A.48), (A.49) and (A.50) are evaluated at the horizon. Using (A.23)–(A.26), see also footnote 17,

$$\begin{aligned}
F &\sim (4G_N s_B)(2\pi T)\cdot\left(\frac{1}{2}H_0^2\right)\,,\\
\delta F_2 &\sim (4G_N s_B)(2\pi T)\cdot\left(\frac{H_0^2}{3}\cdot(V-12)(5\alpha_1+\alpha_2)\right)\,,\\
\delta F_4 &\sim (4G_N s_B)(2\pi T)\cdot\left(-\frac{H_0^2}{96}\cdot(V-12)\left[5(V-12)^2+2\sum_i(\partial_i V)^2\right]\right)\,,
\end{aligned} \tag{A.52}$$

where the Bekenstein entropy density is given by (A.15) and the temperature is determined by (A.14).

Notice that in (A.52) we only need to know the value of H_0 at the black brane horizon. The equation of motion for H_0 is determined from (A.43) setting $w=0$,

$$0 = H_0'' + H_0'\left[\ln\frac{c_1 c_2^3}{c_3}\right]' + \beta\cdot\delta eq\,, \tag{A.53}$$

where for the model (1.3), $\delta eq \equiv \delta eq_2$,

$$\begin{aligned}
\delta eq_2 = \frac{4H_0'}{3}\Bigg[&-\frac{54(c_2')^3}{c_2^3c_3^2}(\alpha_1+\alpha_2+3\alpha_3)-\frac{18c_1'(c_2')^2}{c_1c_2^2c_3^2}(4\alpha_1+4\alpha_2+11\alpha_3)\\
&+\left(\frac{12c_1'}{c_1}-\frac{6c_2'(c_1')^2}{c_2c_3^2c_1^2}\right)(3\alpha_1+3\alpha_2+8\alpha_3)+\frac{12c_2'}{c_2}(9\alpha_1+9\alpha_2+26\alpha_3)+2(2\alpha_1+\alpha_2\\
&+2\alpha_3)\sum_i\partial_i V\cdot\phi_i'-\left((9\alpha_1+9\alpha_2+26\alpha_3)\frac{c_2'}{c_2}+\frac{c_1'}{c_1}(3\alpha_1+3\alpha_2+8\alpha_3)\right)V\Bigg]\,,
\end{aligned} \tag{A.54}$$

and for the model (1.4), $\delta eq \equiv \delta eq_4$,

$$\begin{aligned}
\delta eq_4 = \frac{H_0'}{6}\left(\frac{c_1'}{c_1}-\frac{c_2'}{c_2}\right)^2\Bigg[&\frac{12c_2'(2c_1c_2'+c_2c_1')}{c_1c_3^4c_2^2}\sum_i\partial_i V\cdot\phi_i'-\left(\frac{20c_2'}{c_3^2c_2}+\frac{4c_1'}{c_3^2c_1}\right)V^2+\left(\frac{480c_2'}{c_3^2c_2}\right.\\
&\left.+\frac{96c_1'}{c_3^2c_1}-\frac{849(c_2')^3}{c_3^4c_2^3}-\frac{639c_1'(c_2')^2}{c_3^4c_2^2c_1}-\frac{96(c_1')^2c_2'}{c_3^4c_2c_1^2}\right)V-\frac{2880c_2'}{c_3^2c_2}-\frac{576c_1'}{c_3^2c_1}+\frac{10188(c_2')^3}{c_3^4c_2^3}\\
&+\frac{7668c_1'(c_2')^2}{c_3^4c_2^2c_1}+\frac{1152(c_1')^2c_2'}{c_3^4c_2c_1^2}-\frac{5994(c_2')^5}{c_2^5c_3^6}-\frac{8316c_1'(c_2')^4}{c_1c_2^4c_3^6}-\frac{3402(c_1')^2(c_2')^3}{c_1^2c_2^3c_3^6}\\
&-\frac{432(c_1')^3(c_2')^2}{c_1^3c_2^2c_3^6}\Bigg]\,.
\end{aligned} \tag{A.55}$$

We seek solution of (A.53) recursively in β,

$$H_0 = H_{0,0}+\beta\ H_{0,1}\,. \tag{A.56}$$

To leading order, we have

$$H_{0,0} = \mathcal{C}_{1,0} + \mathcal{C}_{2,0} \int dr \; \frac{c_3}{c_2^3 c_1} \,. \tag{A.57}$$

Regularity of $H_{0,0}$ at the horizon fixes $\mathcal{C}_{2,0} = 0$ (see (A.23) and (A.25)), while the normalization as $r \to 0$ sets $\mathcal{C}_{1,0} = 1$. With $H_{0,0} \equiv 1$, the general solution for $H_{0,1}$ once again takes the form (A.57), albeit now the boundary conditions (A.46) set $\mathcal{C}_{1,1} = \mathcal{C}_{2,1} = 0$. Thus, putting all this together, we have

$$H_0(r) \equiv 1 + \mathcal{O}(\beta^2) \,. \tag{A.58}$$

Finally, collecting (A.51), (A.52) and (A.58) we find

$$4\pi \frac{\eta}{s_B} = 1 + \beta \cdot \begin{cases} \frac{2}{3}(5\alpha_1 + \alpha_2)(V-12), & \text{when } \delta\mathcal{L} = \delta\mathcal{L}_2\,; \\ -\frac{1}{48}(V-12)\left[5(V-12)^2 + 2\sum_i (\partial_i V)^2\right], & \text{when } \delta\mathcal{L} = \delta\mathcal{L}_4\,. \end{cases} \tag{A.59}$$

Noting that

$$4\pi \frac{\eta}{s} = 4\pi \frac{\eta}{s_B} \cdot \frac{s_B}{s_W} \,, \tag{A.60}$$

and using (A.28) we arrive at the results reported in (1.5) and (1.6) for the final shear viscosity to entropy ratio in the presence of higher derivative corrections.

A.3 $\frac{\zeta}{s}$ with higher derivative corrections

The computation of the bulk viscosity ζ parallels the discussion of section A.2. The starting point is the Kubo formula [18]

$$\zeta = -\frac{4}{9} \lim_{w\to 0} \frac{1}{w} \mathrm{Im}\, G_R(w) \,, \qquad G_R(w) = -i \int dt d\boldsymbol{x} e^{iwt} \theta(t) \langle [\tfrac{1}{2} T_i^i(t,\boldsymbol{x}), \tfrac{1}{2} T_j^j(0,\boldsymbol{0})] \rangle \,. \tag{A.61}$$

To compute the relevant retarded correlation function we consider the decoupled set of SO(3) invariant metric fluctuations and the bulk scalars

$$\begin{aligned} ds_5^2 &\to ds_5^2 + h_{tt}(t,r)\; dt^2 + h_{11}(t,r)\; d\boldsymbol{x}^2 + 2h_{tr}(t,r)\; dtdr + h_{rr}(t,r)\; dr^2 \,, \\ \phi_i &\to \phi_i + \psi_i(t,r) \,. \end{aligned} \tag{A.62}$$

For convenience, we fix the axial gauge as

$$h_{tr} = h_{rr} = 0 \,. \tag{A.63}$$

To study the equations of motion for the fluctuations, it is convenient to introduce new variables H_{00} , H_{11}

$$h_{tt}(t,r) = e^{-iwt}\; c_1^2\; H_{00}(r) \,, \qquad h_{11}(t,r) = e^{-iwt}\; c_2^2\; H_{11}(r) \,, \tag{A.64}$$

and the gauge invariant scalar fluctuations Z_i as in [15]

$$\psi_i(t,r) = e^{-iwt}\Big(Z_i(r) + \frac{\phi_i' c_2}{2c_2'}\; H_{11}(r) \Big) \,. \tag{A.65}$$

The equations of motion for Z_i completely decouple from the equations for the metric fluctuations,

$$\begin{aligned}0 = Z_i'' + \left(\ln \frac{c_1 c_2^3}{c_3}\right)' Z_i' + \frac{c_3^2 w^2}{c_1^2}\, Z_i - \phi_i' \cdot (V-12)\, \frac{c_2^2 c_3^2}{9(c_2')^2} \sum_j Z_j \cdot \phi_j' \\ - \frac{c_2 c_3^2}{3c_2'}\left(\partial_i V \sum_j Z_j \cdot \phi_j' + \phi_i' \sum_j \partial_j V \cdot Z_j\right) - c_3^2 \sum_j Z_j \cdot \partial_{ij}^2 V + \beta \cdot \Big[\cdots\Big].\end{aligned} \tag{A.66}$$

Furthermore, we have

$$H_{11} \equiv \frac{c_2'}{c_2 c_3}\, H\,, \qquad 0 = H' + \frac{c_3 c_2}{3c_2'} \sum_i Z_i \cdot \phi_i' + \beta \cdot \Big[\cdots\Big], \tag{A.67}$$

$$\begin{aligned}0 = H_{00}' + \frac{c_2}{3c_2'} \sum_i \phi_i' \cdot Z_i' - \frac{1}{3}\left(c_3(V-12) + \frac{3c_3 w^2}{c_1^2} + \frac{9c_1' c_2'}{c_3 c_2 c_1} + \frac{3(c_1')^2}{c_3 c_1^2}\right) H \\ - \frac{c_2}{9c_1(c_2')^2}\left(c_1 c_2 c_3^2 (V-12) + 3c_1' c_2'\right) \sum_i Z_i \cdot \phi_i' - \frac{c_2 c_3^2}{3c_2'} \sum_i \partial_i V \cdot Z_i + \beta \cdot \Big[\cdots\Big],\end{aligned} \tag{A.68}$$

where once again we omit the $\mathcal{O}(\beta)$ terms for readability. We will need to solve the equations (A.66)–(A.68) in the hydrodynamic approximation, specifically to order $\mathcal{O}(w)$ (see (A.45) for the definition of $\mathfrak{w}$),

$$Z_i = \left(\frac{c_1}{c_2}\right)^{-i\mathfrak{w}} (z_{i,0} + i\mathfrak{w}\, z_{i,1})\,, \qquad H = H_0 + i\mathfrak{w}\, H_1\,, \qquad H_{00} = H_{00,0} + i\mathfrak{w}\, H_{00,1}\,. \tag{A.69}$$

The set of the gauge invariant scalar equations is solved first. The incoming wave boundary condition implies that $z_{i,0}$ and $z_{i,1}$ must be regular at the black brane horizon. To correctly normalize the retarded correction function in (A.61) we must set

$$\lim_{r\to 0} H_{11}(r) = 1\,, \qquad \lim_{r\to 0} H_{00}(r) = 0\,. \tag{A.70}$$

Additionally, the coefficients of ψ_i that are non-normalizable near the AdS boundary must vanish [15]. From (A.65) this implies that if the non-normalizable coefficient λ_i of the background scalar ϕ_i dual to a gauge theory operator of dimension Δ_i is nonzero, i.e.,

$$\phi_i = \lambda_i \cdot r^{4-\Delta_i} + \cdots\,, \qquad \text{as } r \to 0\,, \tag{A.71}$$

the near-AdS boundary asymptotic of Z_i much be

$$\lim_{r\to 0} \frac{Z_i}{r^{4-\Delta_i}} = \frac{4-\Delta_i}{2} \cdot \lambda_i = \lim_{r\to 0} \frac{z_{i,0}}{r^{4-\Delta_i}}\,, \qquad \lim_{r\to 0} \frac{z_{i,1}}{r^{4-\Delta_i}} = 0\,. \tag{A.72}$$

As we show shortly, we will need only the values of the scalars $z_{i,0}$ near the horizon. These would have to be determined numerically.

Parallel to our discussion in section A.2, we compute the complexified action for the fluctuations. This action is a functional of $\{h_{00,w}\,, h_{11,w}\,, p_{i,w}\}$,

$$h_{tt}(t,r) = e^{-iwt}\, h_{00,w}\,, \qquad h_{11}(t,r) = e^{-iwt}\, h_{11,w}\,, \qquad \psi_i(t,r) = e^{-iwt}\, p_{i,w}\,, \tag{A.73}$$

JHEP04(2024)032

and their complex conjugates:

$$S^{(2)} = \frac{1}{16\pi G_N} \int_0^{r_h} dr\ \mathcal{L}_{\mathbb{C}}\{h_{00,w}\,,\ h_{11,w}\,,\ p_{i,w}, h^*_{00,w}\,,\ h^*_{11,w}\,,\ p^*_{i,w}\}\,, \tag{A.74}$$

with

$$\begin{aligned}
\mathcal{L}_{\mathbb{C}} = &\frac{1}{2}A_1\left(h''_{11,w}h_{00,-w} + h''_{11,-w}h_{00,w}\right) + \frac{1}{2}A_2\left(h''_{11,w}h_{11,-w} + h''_{11,-w}h_{11,w}\right)\\
&+\frac{1}{2}A_3\left(h''_{00,w}h_{11,-w} + h''_{00,-w}h_{11,w}\right) + \frac{1}{2}A_4\left(h''_{00,w}h_{00,-w} + h''_{00,-w}h_{00,w}\right)\\
&+\frac{1}{2}A_5\left(h'_{11,w}h'_{00,-w} + h'_{11,-w}h'_{00,w}\right) + \frac{1}{2}A_6\left(h'_{11,w}h_{00,-w} + h'_{11,-w}h_{00,w}\right)\\
&+\frac{1}{2}A_7\left(h'_{11,w}h_{11,-w} + h'_{11,-w}h_{11,w}\right) + \frac{1}{2}A_8\left(h_{11,-w}h'_{00,w} + h_{11,w}h'_{00,-w}\right)\\
&+\frac{1}{2}A_9\left(h_{00,-w}h'_{00,w} + h_{00,w}h'_{00,-w}\right) + A_{13}h'_{00,-w}h'_{00,w}\\
&+\frac{1}{2}\sum_i A_{17,i}\cdot\left(h_{00,-w}p'_{i,w} + h_{00,w}p'_{i,-w}\right) + \frac{1}{2}\sum_i A_{18,i}\cdot\left(h_{11,-w}p'_{i,w} + h_{11,w}p'_{i,-w}\right)\\
&+A_{19}\sum_i p'_{i,w}p'_{i,-w} + \frac{1}{2}(A_{16} + w^2(A_{12} - A_{10}))(h_{00,-w}h_{11,w} + h_{00,w}h_{11,-w})\\
&+A_{14}h_{00,w}h_{00,-w} + (A_{15} - w^2A_{11})h_{11,w}h_{11,-w} + \frac{1}{2}\sum_i A_{23,i}\cdot(p_{i,w}h_{00,-w} + p_{i,-w}h_{00,w})\\
&+\sum_i (A_{21,i} + w^2A_{20})\cdot p_{i,w}p_{i,-w} + \frac{1}{2}\sum_i A_{22,i}\cdot(h_{11,-w}p_{i,w} + h_{11,w}p_{i,-w})\\
&+\frac{1}{2}\sum_{i\neq j} A_{24,ij}(p_{i,w}p_{j,-w} + p_{i,-w}p_{j,w}) + \beta\Big[\cdots\Big]\,,
\end{aligned} \tag{A.75}$$

where for readability we suppressed $\mathcal{O}(\beta)$ terms. The $\mathcal{O}(\beta^0)$ connection coefficients A_i are collected in appendix C.

On-shell, the effective action (A.75) can be re-expressed as a total derivative,

$$\mathcal{L}_{\mathbb{C}} = 16\pi G_N\cdot\left(h^*_{00,w}\cdot\frac{\delta S^{(2)}}{\delta h^*_{00,w}} + h^*_{11,w}\cdot\frac{\delta S^{(2)}}{\delta h^*_{11,w}} + p^*_{i,w}\cdot\frac{\delta S^{(2)}}{\delta p^*_{i,w}}\right) + \partial_r J_w\,, \tag{A.76}$$

with a current given by

$$\begin{aligned}
J_w = &\left[\frac{A_4}{2}h_{00,w} + \frac{A_3}{2}h_{11,w}\right]h'_{00,-w} + \left[\frac{A_1}{2}h_{00,w} + \frac{A_2}{2}h_{11,w}\right]h'_{11,-w} + \left[\frac{1}{2}(A_5 - A_1)h'_{00,w}\right.\\
&\left.-\frac{1}{2}A_2h'_{11,w} + \frac{1}{2}(A_7 - A'_2)h_{11,w} + \frac{1}{2}(A_6 - A'_1)h_{00,w}\right]h_{11,-w} + \left[\frac{1}{2}(A_5 - A_3)h'_{11,w}\right.\\
&\left.+\left(A_{13} - \frac{1}{2}A_4\right)h'_{00,w} + \frac{1}{2}(A_9 - A'_4)h_{00,w} + \frac{1}{2}(A_8 - A'_3)h_{11,w}\right]h_{00,-w} + \left[\frac{A_{17,i}}{2}h_{00,w}\right.\\
&\left.+\frac{A_{18,i}}{2}h_{11,w} + A_{19}p_{i,w}\right]p_{i,-w} + \beta\Big\{\cdots\Big\}\,.
\end{aligned} \tag{A.77}$$

Parallel to the discussion of the shear channel fluctuations in section A.2, the imaginary part of J_w in (A.77) is radially conserved. The same arguments as in section A.2 lead to

$$\operatorname{Im} G_R(w) = \frac{1}{8\pi G_N}\lim_{r\to r_h}\operatorname{Im} J_w\,, \tag{A.78}$$

with (see (A.69))

$$J_w = J_0 - i\mathfrak{w}\ J_1\,, \qquad J_1 \equiv F + \beta\cdot\delta F\,, \tag{A.79}$$

$$F = -\frac{c_2^2(c_2c_1' - c_2'c_1)}{2c_3}\ \sum_i (z_{i,0})^2 + \frac{c_2^3c_1}{2c_3}\ \sum_i \left(z_{i,1}'z_{i,0} - z_{i,0}'z_{i,1}\right)\,, \tag{A.80}$$

where we used the equations of motion (A.67) and (A.68) to eliminate all derivatives of H_{11} and H_{00}. What is remarkable is that the final result (A.80) depends only on $z_{i,0}$ and $z_{i,1}$. While we will not present here the results for δF's, we note that they are functionals of $z_{i,0}$ and $z_{i,1}$ only as well.

As in section A.2, further simplification occurs when J_1 is evaluated at the horizon. We find:

$$F \sim (4G_Ns_B)(2\pi T)\cdot\left(\frac{1}{2}\sum_i (z_{i,0})^2\right)\,, \tag{A.81}$$

$$\delta F_2 \sim (4G_Ns_B)(2\pi T)\cdot\left(\frac{2(5\alpha_1+\alpha_2-\alpha_3)}{3(V-12)}\ \sum_i (z_{i,0}\cdot\partial_i V)^2\right)\,, \tag{A.82}$$

$$\delta F_4 \sim (4G_Ns_B)(2\pi T)\cdot\left(-\frac{5}{48}(V-12)\ \sum_i (z_{i,0}\cdot\partial_i V)^2\right)\,, \tag{A.83}$$

$$\tag{A.84}$$

where the Bekenstein entropy density is given by (A.15) and the temperature is determined by (A.14). As before, we used $\delta F \equiv \delta F_2$ to refer to the model (1.3), and $\delta F \equiv \delta F_4$ to refer to the model (1.4).

Finally, using (A.78) and (A.84), we obtain from (A.61) the following expressions for the bulk viscosity,

$$\zeta = \frac{s_B}{4\pi}\cdot\frac{4}{9}\left[\qquad \sum_i z_{i,0}^2 + \beta\cdot\begin{cases}\frac{4(5\alpha_1+\alpha_2-\alpha_3)}{3(V-12)}\ \sum_i (z_{i,0}\cdot\partial_i V)^2\,, & \text{when } \delta\mathcal{L} = \delta\mathcal{L}_2\,;\\ -\frac{5}{24}(V-12)\ \sum_i (z_{i,0}\cdot\partial_i V)^2\,, & \text{when } \delta\mathcal{L} = \delta\mathcal{L}_4\end{cases}\right]\,. \tag{A.85}$$

Noting that

$$9\pi\frac{\zeta}{s} = 9\pi\frac{\zeta}{s_B}\cdot\frac{s_B}{s_W}\,, \tag{A.86}$$

and using (A.28), we arrive at the results reported in (1.7) and (1.8).

B Connection coefficients of (A.35)

$$A_1 = \frac{2c_1}{c_2c_3}\,, \tag{B.1}$$

$$A_2 = -\frac{2c_3}{c_1c_2}\,, \tag{B.2}$$

$$A_3 = -\frac{3c_3}{2c_1c_2}\,, \tag{B.3}$$

$$A_4 = \frac{3c_1}{2c_2c_3}\,, \tag{B.4}$$

$$A_5 = -\frac{2c_1c_3'}{c_2c_3^2} - \frac{6c_1c_2'}{c_2^2c_3} + \frac{2c_1'}{c_2c_3}\,, \tag{B.5}$$

$$\begin{aligned} A_6 = {}& \frac{c_3c_1}{2c_2}\,V + \frac{c_1}{4c_2c_3}\,\sum_i(\phi_i')^2 - \frac{6c_3c_1}{c_2} - \frac{c_2''c_1}{c_2^2c_3} + \frac{c_3'c_2'c_1}{c_2^2c_3^2} + \frac{5(c_2')^2c_1}{c_2^3c_3} + \frac{c_1''}{c_2c_3} \\ & - \frac{c_3'c_1'}{c_2c_3^2} - \frac{c_1'c_2'}{c_2^2c_3}\,. \end{aligned} \tag{B.6}$$

C Connection coefficients of (A.75)

$$A_1 = \frac{3c_2}{2c_3c_1}\,, \tag{C.1}$$

$$A_2 = -\,\frac{3c_1}{2c_2c_3}\,, \tag{C.2}$$

$$A_3 = \frac{3c_2}{2c_3c_1}\,, \tag{C.3}$$

$$A_4 = \frac{c_2^3}{2c_3c_1^3}\,, \tag{C.4}$$

$$A_5 = \frac{3c_2}{2c_3c_1}\,, \tag{C.5}$$

$$A_6 = -\,\frac{3c_2c_1'}{2c_1^2c_3} - \frac{3c_2c_3'}{2c_1c_3^2}\,, \tag{C.6}$$

$$A_7 = -\,\frac{3c_1'}{2c_2c_3} + \frac{3c_1c_3'}{2c_2c_3^2}\,, \tag{C.7}$$

$$A_8 = -\,\frac{3c_2c_1'}{c_1^2c_3} - \frac{3c_2c_3'}{2c_1c_3^2} + \frac{3c_2'}{2c_1c_3}\,, \tag{C.8}$$

$$A_9 = -\,\frac{3c_2^3c_1'}{c_1^4c_3} - \frac{c_2^3c_3'}{2c_1^3c_3^2} + \frac{3c_2^2c_2'}{2c_1^3c_3}\,, \tag{C.9}$$

$$A_{10} = \frac{3c_2c_3}{2c_1^3}\,, \tag{C.10}$$

$$A_{11} = \frac{3c_3}{2c_2c_1}\,, \tag{C.11}$$

$$A_{12} = \frac{3c_2c_3}{2c_1^3}\,, \tag{C.12}$$

$$A_{13} = \frac{c_2^3}{2c_3c_1^3}\,, \tag{C.13}$$

$$\begin{aligned} A_{14} = {}& \frac{c_2^3c_3}{8c_1^3}\,V + \frac{c_2^3}{16c_3c_1^3}\sum_i(\phi_i')^2 - \frac{3c_2^3c_3}{2c_1^3} + \frac{3c_2^2c_2''}{4c_3c_1^3} - \frac{3c_2^3c_1''}{4c_3c_1^4} + \frac{3c_2^3(c_1')^2}{c_3c_1^5} + \frac{3c_2^3c_1'c_3'}{4c_3^2c_1^4} \\ & - \frac{9c_2^2c_1'c_2'}{4c_3c_1^4} - \frac{3c_2^2c_2'c_3'}{4c_3^2c_1^3} + \frac{3c_2(c_2')^2}{4c_3c_1^3}\,, \end{aligned} \tag{C.14}$$

$$A_{15} = -\frac{3c_3c_1}{8c_2}\,V - \frac{3c_1}{16c_3c_2}\sum_i(\phi_i')^2 + \frac{9c_3c_1}{2c_2} + \frac{3c_2''c_1}{4c_3c_2^2} - \frac{3c_1''}{4c_3c_2} + \frac{3c_1'c_3'}{4c_3^2c_2} + \frac{3c_1'c_2'}{4c_3c_2^2}$$
$$- \frac{3c_1c_2'c_3'}{4c_3^2c_2^2} + \frac{3c_1(c_2')^2}{4c_3c_2^3}\,, \tag{C.15}$$

$$A_{16} = \frac{3c_2c_3}{4c_1}\,V + \frac{3c_2}{8c_1c_3}\sum_i(\phi_i')^2 - \frac{9c_2c_3}{c_1} + \frac{3c_2''}{2c_1c_3} - \frac{3c_2c_1''}{2c_1^2c_3} + \frac{3c_2(c_1')^2}{c_1^3c_3} + \frac{3c_2c_1'c_3'}{2c_1^2c_3^2}$$
$$- \frac{3c_1'c_2'}{2c_1^2c_3} - \frac{3c_2'c_3'}{2c_1c_3^2} + \frac{3(c_2')^2}{2c_2c_1c_3}\,, \tag{C.16}$$

$$A_{17,i} = \frac{c_2^3}{2c_3c_1}\,\phi_i'\,, \tag{C.17}$$

$$A_{18,i} = -\frac{3c_2c_1}{2c_3}\,\phi_i'\,, \tag{C.18}$$

$$A_{19} = -\frac{c_1c_2^3}{2c_3}\,, \tag{C.19}$$

$$A_{20} = \frac{c_2^3c_3}{2c_1}\,, \tag{C.20}$$

$$A_{21,i} = -\frac{c_2^3c_3c_1}{2}\,\partial_{ii}^2V\,, \tag{C.21}$$

$$A_{22,i} = -\frac{3c_2c_3c_1}{2}\,\partial_iV\,, \tag{C.22}$$

$$A_{23,i} = \frac{c_3c_2^3}{2c_1}\,\partial_iV\,, \tag{C.23}$$

$$A_{24,ij} = -c_2^3c_3c_1\,\partial_{ij}^2V\,. \tag{C.24}$$

References

[1] J. Casalderrey-Solana et al., *Gauge/String Duality, Hot QCD and Heavy Ion Collisions*, Cambridge University Press (2014) [DOI:10.1017/9781009403504] [INSPIRE].

[2] G. Policastro, D.T. Son and A.O. Starinets, *The Shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **87** (2001) 081601 [hep-th/0104066] [INSPIRE].

[3] A. Buchel and J.T. Liu, *Universality of the shear viscosity in supergravity*, *Phys. Rev. Lett.* **93** (2004) 090602 [hep-th/0311175] [INSPIRE].

[4] M. Baggioli et al., *Breaking rotations without violating the KSS viscosity bound*, *JHEP* **07** (2023) 016 [arXiv:2304.01807] [INSPIRE].

[5] P. Kovtun, D.T. Son and A.O. Starinets, *Holography and hydrodynamics: Diffusion on stretched horizons*, *JHEP* **10** (2003) 064 [hep-th/0309213] [INSPIRE].

JHEP04(2024)032

[6] P. Kovtun, D.T. Son and A.O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, Phys. Rev. Lett. **94** (2005) 111601 [hep-th/0405231] [INSPIRE].

[7] S. Cremonini, *The Shear Viscosity to Entropy Ratio: A Status Report*, Mod. Phys. Lett. B **25** (2011) 1867 [arXiv:1108.0677] [INSPIRE].

[8] A. Buchel, J.T. Liu and A.O. Starinets, *Coupling constant dependence of the shear viscosity in $N = 4$ supersymmetric Yang-Mills theory*, Nucl. Phys. B **707** (2005) 56 [hep-th/0406264] [INSPIRE].

[9] Y. Kats and P. Petrov, *Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory*, JHEP **01** (2009) 044 [arXiv:0712.0743] [INSPIRE].

[10] M. Brigante et al., *The Viscosity Bound and Causality Violation*, Phys. Rev. Lett. **100** (2008) 191601 [arXiv:0802.3318] [INSPIRE].

[11] A. Buchel and S. Cremonini, *Viscosity Bound and Causality in Superfluid Plasma*, JHEP **10** (2010) 026 [arXiv:1007.2963] [INSPIRE].

[12] S. Cremonini, U. Gürsoy and P. Szepietowski, *On the Temperature Dependence of the Shear Viscosity and Holography*, JHEP **08** (2012) 167 [arXiv:1206.3581] [INSPIRE].

[13] R. Rougemont et al., *Hot QCD phase diagram from holographic Einstein-Maxwell-Dilaton models*, Prog. Part. Nucl. Phys. **135** (2024) 104093 [arXiv:2307.03885] [INSPIRE].

[14] A. Parnachev and A. Starinets, *The silence of the little strings*, JHEP **10** (2005) 027 [hep-th/0506144] [INSPIRE].

[15] P. Benincasa, A. Buchel and A.O. Starinets, *Sound waves in strongly coupled non-conformal gauge theory plasma*, Nucl. Phys. B **733** (2006) 160 [hep-th/0507026] [INSPIRE].

[16] A. Buchel, *Transport properties of cascading gauge theories*, Phys. Rev. D **72** (2005) 106002 [hep-th/0509083] [INSPIRE].

[17] A. Buchel, *Bulk viscosity of gauge theory plasma at strong coupling*, Phys. Lett. B **663** (2008) 286 [arXiv:0708.3459] [INSPIRE].

[18] S.S. Gubser, S.S. Pufu and F.D. Rocha, *Bulk viscosity of strongly coupled plasmas with holographic duals*, JHEP **08** (2008) 085 [arXiv:0806.0407] [INSPIRE].

[19] A. Buchel and C. Pagnutti, *Bulk viscosity of $N = 2^*$ plasma*, Nucl. Phys. B **816** (2009) 62 [arXiv:0812.3623] [INSPIRE].

[20] A. Buchel, U. Gürsoy and E. Kiritsis, *Holographic bulk viscosity: GPR versus EO*, JHEP **09** (2011) 095 [arXiv:1104.2058] [INSPIRE].

[21] A. Buchel, *Violation of the holographic bulk viscosity bound*, Phys. Rev. D **85** (2012) 066004 [arXiv:1110.0063] [INSPIRE].

[22] C. Eling and Y. Oz, *A Novel Formula for Bulk Viscosity from the Null Horizon Focusing Equation*, JHEP **06** (2011) 007 [arXiv:1103.1657] [INSPIRE].

[23] A. Buchel, *On Eling-Oz formula for the holographic bulk viscosity*, JHEP **05** (2011) 065 [arXiv:1103.3733] [INSPIRE].

[24] U. Gürsoy and E. Kiritsis, *Exploring improved holographic theories for QCD: Part I*, JHEP **02** (2008) 032 [arXiv:0707.1324] [INSPIRE].

[25] U. Gürsoy, E. Kiritsis and F. Nitti, *Exploring improved holographic theories for QCD: Part II*, JHEP **02** (2008) 019 [arXiv:0707.1349] [INSPIRE].

JHEP04(2024)032

[26] A.O. Starinets, *Quasinormal spectrum and the black hole membrane paradigm*, Phys. Lett. B **670** (2009) 442 [arXiv:0806.3797] [INSPIRE].

[27] T. Damour, *Quelques proprietes mecaniques, electromagnet iques, thermodynamiques et quantiques des trous noir*, Ph.D. thesis, Laboratoire de Physique Nucléaire et de Hautes Energies (LPNHE) UPMC, F-75005 Paris, France (1979) [INSPIRE].

[28] K.S. Thorne, R.H. Price and D.A. Macdonald, *Black Holes: The Membrane Paradigm*, Yale University Press (1986) [INSPIRE].

[29] N. Iqbal and H. Liu, *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*, Phys. Rev. D **79** (2009) 025023 [arXiv:0809.3808] [INSPIRE].

[30] T. Demircik et al., *A Novel Method for Holographic Transport*, arXiv:2311.00042 [INSPIRE].

[31] A. Donos, P. Kailidis and C. Pantelidou, *Holographic dissipation from the symplectic current*, JHEP **10** (2022) 058 [arXiv:2208.05911] [INSPIRE].

[32] R.A. Davison, B. Goutéraux and E. Mefford, *Zero sound and higher-form symmetries in compressible holographic phases*, JHEP **12** (2023) 040 [arXiv:2210.14802] [INSPIRE].

[33] A. Donos and P. Kailidis, *Dissipative effects in finite density holographic superfluids*, JHEP **11** (2022) 053 [arXiv:2209.06893] [INSPIRE].

[34] P. Benincasa and A. Buchel, *Transport properties of $N = 4$ supersymmetric Yang-Mills theory at finite coupling*, JHEP **01** (2006) 103 [hep-th/0510041] [INSPIRE].

[35] A. Buchel, *Shear viscosity of boost invariant plasma at finite coupling*, Nucl. Phys. B **802** (2008) 281 [arXiv:0801.4421] [INSPIRE].

[36] A. Buchel, *Resolving disagreement for η/s in a CFT plasma at finite coupling*, Nucl. Phys. B **803** (2008) 166 [arXiv:0805.2683] [INSPIRE].

[37] A. Buchel, *Non-conformal holographic Gauss-Bonnet hydrodynamics*, JHEP **03** (2018) 037 [arXiv:1801.06165] [INSPIRE].

[38] P.K. Kovtun and A.O. Starinets, *Quasinormal modes and holography*, Phys. Rev. D **72** (2005) 086009 [hep-th/0506184] [INSPIRE].

[39] A. Buchel, S. Deakin, P. Kerner and J.T. Liu, *Thermodynamics of the $N = 2^*$ strongly coupled plasma*, Nucl. Phys. B **784** (2007) 72 [hep-th/0701142] [INSPIRE].

[40] N. Chai et al., *Thermal Order in Conformal Theories*, Phys. Rev. D **102** (2020) 065014 [arXiv:2005.03676] [INSPIRE].

[41] A. Buchel, *Thermal order in holographic CFTs and no-hair theorem violation in black branes*, Nucl. Phys. B **967** (2021) 115425 [arXiv:2005.07833] [INSPIRE].

[42] A. Buchel, *Holographic conformal order in supergravity*, Phys. Lett. B **814** (2021) 136111 [arXiv:2007.09420] [INSPIRE].

[43] A. Buchel, *Fate of the conformal order*, Phys. Rev. D **103** (2021) 026008 [arXiv:2011.11509] [INSPIRE].

[44] A. Buchel, *The quest for a conifold conformal order*, JHEP **08** (2022) 080 [arXiv:2205.00612] [INSPIRE].

[45] A. Buchel, R.C. Myers, M.F. Paulos and A. Sinha, *Universal holographic hydrodynamics at finite coupling*, Phys. Lett. B **669** (2008) 364 [arXiv:0808.1837] [INSPIRE].

[46] R. Baier et al., *Relativistic viscous hydrodynamics, conformal invariance, and holography*, JHEP **04** (2008) 100 [arXiv:0712.2451] [INSPIRE].

JHEP04(2024)032

[47] S. Bhattacharyya, V.E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, *JHEP* **02** (2008) 045 [arXiv:0712.2456] [INSPIRE].

[48] A. Buchel, *Relaxation time of non-conformal plasma*, *Phys. Lett. B* **681** (2009) 200 [arXiv:0908.0108] [INSPIRE].

[49] A. Buchel and M. Paulos, *Relaxation time of a CFT plasma at finite coupling*, *Nucl. Phys. B* **805** (2008) 59 [arXiv:0806.0788] [INSPIRE].

[50] A. Buchel and M. Paulos, *Second order hydrodynamics of a CFT plasma from boost invariant expansion*, *Nucl. Phys. B* **810** (2009) 40 [arXiv:0808.1601] [INSPIRE].

[51] O. Saremi and K.A. Sohrabi, *Causal three-point functions and nonlinear second-order hydrodynamic coefficients in AdS/CFT*, *JHEP* **11** (2011) 147 [arXiv:1105.4870] [INSPIRE].

[52] S. Grozdanov and A.O. Starinets, *Second-order transport, quasinormal modes and zero-viscosity limit in the Gauss-Bonnet holographic fluid*, *JHEP* **03** (2017) 166 [arXiv:1611.07053] [INSPIRE].

[53] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131] [INSPIRE].

[54] R.M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev. D* **48** (1993) R3427 [gr-qc/9307038] [INSPIRE].

[55] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200] [INSPIRE].

[56] O. Aharony et al., *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [INSPIRE].

[57] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [hep-th/0209067] [INSPIRE].

[58] A. Buchel, *$N = 2^*$ hydrodynamics*, *Nucl. Phys. B* **708** (2005) 451 [hep-th/0406200] [INSPIRE].

[59] G. Policastro, D.T. Son and A.O. Starinets, *From AdS/CFT correspondence to hydrodynamics*, *JHEP* **09** (2002) 043 [hep-th/0205052] [INSPIRE].

[60] D.T. Son and A.O. Starinets, *Minkowski space correlators in AdS/CFT correspondence: Recipe and applications*, *JHEP* **09** (2002) 042 [hep-th/0205051] [INSPIRE].

[61] S. Cremonini, J.T. Liu and P. Szepietowski, *Higher Derivative Corrections to R-charged Black Holes: Boundary Counterterms and the Mass-Charge Relation*, *JHEP* **03** (2010) 042 [arXiv:0910.5159] [INSPIRE].

[62] A. Buchel, *On universality of stress-energy tensor correlation functions in supergravity*, *Phys. Lett. B* **609** (2005) 392 [hep-th/0408095] [INSPIRE].