

PRODUCTS OF SYNCHRONOUS GAMES

L. MANČINSKA, V. I. PAULSEN, I. G. TODOROV, AND A. WINTER

ABSTRACT. We show that the $*$ -algebra of the product of two synchronous games is the tensor product of the corresponding $*$ -algebras. We prove that the product game has a perfect C^* -strategy if and only if each of the individual games does, and that in this case the C^* -algebra of the product game is $*$ -isomorphic to the maximal C^* -tensor product of the individual C^* -algebras. We provide examples of synchronous games whose synchronous values are strictly supermultiplicative.

1. INTRODUCTION

Non-local games have been extensively studied in Mathematics, Quantum Physics and Computer Science. Among their numerous applications is the explicit demonstration of the presence of entanglement. More specifically, there exist games that have no perfect classical strategies, but can nevertheless be demonstrated to possess perfect entanglement-assisted strategies. Some of the most notable such examples are certain types of linear binary constraint system (LBCS) games (see [4]).

A prominent family of non-local games is that of *synchronous games*, introduced in [11] which, by virtue of [8], include all LBCS games. Synchronous games play an important role in the recent (negative) resolution of Connes' Embedding Problem and an equivalent conjecture of Tsirelson [6]. In particular, the authors of [6] construct a synchronous game that has a perfect strategy in one mathematical model for entanglement description, but no perfect strategy in an alternate model, thus demonstrating that these two entanglement models are genuinely distinct. We note that an important reformulation of the two problems was obtained by Kirchberg (see the monograph [12]); in fact, Kirchberg's approach underlies many of the connections bridging operator algebras and non-local game theory (see [10]).

One of the reasons that synchronous games play a significant role in the resolution of the aforementioned problems, as well as in operator algebra theory in general, is that each synchronous game has a canonically associated $*$ -algebra, and the representation theory of this algebra captures the

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existence of perfect strategies for each of the mathematical models used to describe entanglement [2].

This is the starting point of the present paper, in which we pose, and answer, the question of the behaviour of the game algebras under parallel repetition. As one of our main results, we show that the $*$ -algebra affiliated with a product of synchronous games is the tensor product of the corresponding $*$ -algebras. Thus, the existence/non-existence of perfect strategies for the product game is related to the behaviour of the various types of representations, after the tensor products of the individual $*$ -algebras is taken. We show, in particular, that the product game has a perfect C^* -strategy [2] if and only if each of the individual game does so and that, in this case, the C^* -algebra of the product game is canonically C^* -isomorphic to the maximal C^* -tensor product of the two individual C^* -algebras.

Some of the central concepts in non-local game theory are the various notions of a *value* of a game. Given a probabilistic strategy for a non-local game and a prior probability distribution on its inputs, one may compute the expected winning probability for the game. The different notions of a value of the game arise by considering the optimal expected winning probability over different, fixed, sets of strategies. Thus, in addition to their classical values [13], non-local games have values, defined by using any preferred mathematical model of entanglement. For synchronous games, it is natural to study the optimal winning probabilities over families of *synchronous* strategies instead of arbitrary ones. In the present paper, we introduce *synchronous values* of a synchronous game, and make some first advances towards their understanding, showing that strict supermultiplicativity may occur in this new setting, as it does in the usual, non-synchronous, context.

Note on related work. When the first draft of the present paper was completed, it became clear that the paper [3] also introduces and studies the notion of the synchronous value of a synchronous game. More precisely, the authors of [3] examine the behaviour of synchronous values under parallel repetition and show that they can be strictly supermultiplicative, recovering some of our results in Section 4. The examples presented here are however different from the ones given in [3]. In addition, the emphasis in the present paper is placed on the behaviour of the game algebra with respect to the formation of products.

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2. PRELIMINARIES

Let X, Y, A and B be finite sets, and recall that a POVM on a Hilbert space H is a family $(E_i)_{i=1}^k$ of positive operators such that $\sum_{i=1}^k E_i = I$. A *correlation* is a family $p = \{(p(a, b|x, y))_{a \in A, b \in B} : x \in X, y \in Y\}$, where $(p(a, b|x, y))_{a \in A, b \in B}$ is a probability distribution for every $(x, y) \in X \times Y$. A correlation p is called

- (i) *local* if it is the convex combination of correlations of the form $\{(p_1(a|x)p_2(b|y))_{a \in A, b \in B} : x \in X, y \in Y\}$ (notation: $\mathcal{C}_{\text{loc}}$);
- (ii) *quantum* if there exist a finite dimensional Hilbert space H (resp. K), a unit vector $\xi \in H \otimes K$ and POVM's $(E_{x,a})_{a \in A}, x \in X$ on H (resp. $(F_{y,b})_{b \in B}, y \in Y$ on K), such that

$$p(a, b|x, y) = \langle (E_{x,a} \otimes F_{y,b})\xi, \xi \rangle$$

for all $x \in X, y \in Y, a \in A, b \in B$ (notation: $\mathcal{C}_q$);

- (iii) *approximately quantum* if it belongs to the closure $\overline{\mathcal{C}_q}$ of the set $\mathcal{C}_q$ (notation: $\mathcal{C}_{\text{qa}}$);

- (iv) *quantum commuting* if there exist a Hilbert space H , a unit vector $\xi \in H$ and POVM's $(E_{x,a})_{a \in A}, x \in X$ (resp. $(F_{y,b})_{b \in B}, y \in Y$) on H , such that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all x, y, a, b , and

$$p(a, b|x, y) = \langle E_{x,a}F_{y,b}\xi, \xi \rangle, \quad x \in X, y \in Y, a \in A, b \in B$$

for all $x \in X, y \in Y, a \in A, b \in B$ (notation: $\mathcal{C}_{\text{qc}}$).

We note that a correlation of any of the types just defined is *non-signalling* in the sense that

$$(2.1) \quad \sum_{b \in B} p(a, b|x, y) = \sum_{b \in B} p(a, b|x, y'), \quad x \in X, y, y' \in Y, a \in A,$$

and

$$(2.2) \quad \sum_{a \in A} p(a, b|x, y) = \sum_{a \in A} p(a, b|x', y), \quad x, x' \in X, y \in Y, b \in B.$$

We denote the (convex) set of all non-signalling correlations by $\mathcal{C}_{\text{ns}}$, and note the inclusions

$$\mathcal{C}_{\text{loc}} \subseteq \mathcal{C}_{\text{q}} \subseteq \mathcal{C}_{\text{qa}} \subseteq \mathcal{C}_{\text{qc}} \subseteq \mathcal{C}_{\text{ns}}.$$

A *non-local game* is a tuple $\mathbb{G} = (X, Y, A, B, \lambda)$, where X, Y, A, B are finite sets and $\lambda : X \times Y \times A \times B \rightarrow \{0, 1\}$ is a map, called the *rule function* of the game. We interpret X (resp. Y) as the set of questions posed by a Verifier to a player Alice (resp. Bob), and A (resp. B) as the set of her (resp. his) possible answers. Alice and Bob play cooperatively against the Verifier; if a pair $(a, b) \in A \times B$ of answers is received given the pair $(x, y) \in X \times Y$ of questions, the players win (resp. lose) the round of the game if $\lambda(x, y, a, b) = 1$ (resp. $\lambda(x, y, a, b) = 0$).

A *probabilistic strategy* for the game $\mathbb{G}$ is a family $p = \{p(a, b|x, y)\}_{a \in A, b \in B} : x \in X, y \in Y\}$ of probability distributions; $p(a, b|x, y)$ is interpreted as the probability that the players respond with the pair (a, b) of answers when they are asked the pair (x, y) of questions. A probabilistic strategy p for $\mathbb{G}$ is called *non-signalling* if p is a non-signalling correlation; such a strategy being used by the players expresses the fact that they do not communicate after the start of the game. A non-signalling strategy p for $\mathbb{G}$ is called *perfect* if

$$\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0, \quad x \in X, y \in Y, a \in A, b \in B.$$

For $x \in \{\text{loc}, \text{q}, \text{qa}, \text{qc}, \text{ns}\}$, let $\mathcal{C}_x(\lambda)$ be the set of all elements of $\mathcal{C}_x$ that are perfect strategies for the game $\mathbb{G}$ with rule function λ .

The game $\mathbb{G}$ is called *synchronous* if $X = Y$, $A = B$, and $\lambda(x, x, a, b) = 0$ if $a \neq b$; in this case we write $\mathbb{G} = (X, A, \lambda)$. The *synchronicity game* has rule function $\lambda : X \times X \times A \times A \rightarrow \{0, 1\}$ given by $\lambda(x, y, a, b) = 1 - (1 - \delta_{a,b})\delta_{x,y}$. For synchronous games, there are two further types of winning strategies which will be of interest to us. Namely, a synchronous game $\mathbb{G} = (X, A, \lambda)$ is said to have a *perfect algebraic strategy* (see [2]) if there exists a unital complex $*$ -algebra with a generating set $\{e_{x,a} : x \in X, a \in A\}$ satisfying the following relations:

$$\begin{aligned} e_{x,a} &= e_{x,a}^* = e_{x,a}^2, \quad x \in X, a \in A; \\ (2.3) \quad \sum_{a \in A} e_{x,a} &= 1, \quad x \in X, \text{ and} \\ \lambda(x, y, a, b) &= 0 \implies e_{x,a}e_{y,b} = 0. \end{aligned}$$

If $\mathbb{G}$ has a perfect algebraic strategy then there exists a universal $*$ -algebra $\mathcal{A}(\mathbb{G})$ satisfying the conditions (2.3) (in the sense that any unital $*$ -algebra satisfying (2.3) is a canonical quotient of $\mathcal{A}(\mathbb{G})$) and that $\mathcal{A}(\mathbb{G})$ is unique up

to a unital $*$ -isomorphism. We will reserve the notation $e_{x,a}$, $x \in X$, $a \in A$, for a family of generators of $\mathcal{A}(\mathbb{G})$ satisfying (2.3).

The game $\mathbb{G}$ is said to have a *perfect C^* -strategy* if there exists a (universal) unital C^* -algebra $C^*(\mathbb{G})$ with generating set $\{e_{x,a} : x \in X, a \in A\}$ satisfying the conditions (2.3). It is clear that if $\mathbb{G}$ has a perfect C^* -strategy then it has a perfect algebraic strategy. The following observation is also clear.

Remark 2.1. For $i \in \{1, 2\}$, let $\mathbb{G}_i$ be a synchronous game with rule function $\lambda_i : X_i \times X_i \times A_i \times A_i \rightarrow \{0, 1\}$, $i = 1, 2$. Suppose that $\lambda_1 \leq \lambda_2$. If $\mathbb{G}_1$ has a perfect algebraic strategy (resp. perfect C^* -strategy) then $\mathbb{G}_2$ has a perfect algebraic strategy (resp. perfect C^* -strategy).

The synchronicity game with a question set X and an answer set A has a perfect C^* -strategy, and its C^* -algebra, which will be denoted by $\mathcal{A}_{X,A}$, is $*$ -isomorphic, via Fourier transform, to the group C^* -algebra $C^*(\mathbb{F}_{X,A})$, where

$$\mathbb{F}_{X,A} = \underbrace{\mathbb{Z}_{|A|} * \cdots * \mathbb{Z}_{|A|}}_{|X| \text{ times}}.$$

The following facts were established in [10, 8, 11].

Theorem 2.2. *Let $p \in \mathcal{C}_{\text{ns}}$ be a synchronous correlation over the pair (X, A) of finite sets.*

(i) $p \in \mathcal{C}_{\text{qc}}$ if and only if there exists a trace $\tau : \mathcal{A}_{X,A} \rightarrow \mathbb{C}$ such that

$$(2.4) \quad p(x, y, a, b) = \tau(e_{x,a}e_{y,b}), \quad x, y \in X, a, b \in A;$$

(ii) $p \in \mathcal{C}_{\text{qa}}$ if and only if there exists an amenable trace $\tau : \mathcal{A}_{X,A} \rightarrow \mathbb{C}$ such that (2.4) is satisfied;

(iii) $p \in \mathcal{C}_{\text{q}}$ if and only if there exist a finite dimensional Hilbert space H , a $*$ -representation $\pi : \mathcal{A}_{X,A} \rightarrow \mathcal{B}(H)$ and a trace $\tau_{\text{fin}} : \mathcal{B}(H) \rightarrow \mathbb{C}$ such that

$$(2.5) \quad p(x, y, a, b) = (\tau_{\text{fin}} \circ \pi)(e_{x,a}e_{y,b}), \quad x, y \in X, a, b \in A;$$

(iv) $p \in \mathcal{C}_{\text{loc}}$ if and only if there exists a finite dimensional Hilbert space H , a $*$ -representation $\pi : \mathcal{A}_{X,A} \rightarrow \mathcal{B}(H)$ with an abelian image, and a trace $\tau_{\text{fin}} : \mathcal{B}(H) \rightarrow \mathbb{C}$ such that (2.5) is satisfied.

3. ALGEBRAIC STRATEGIES OF PRODUCT GAMES

Let $\mathbb{G}_1 = (X_1, A_1, \lambda_1)$ and $\mathbb{G}_2 = (X_2, A_2, \lambda_2)$ be synchronous games. Their *product* $\mathbb{G}_1 \times \mathbb{G}_2$ is the game with pair of input sets $(X_1 \times X_2, X_1 \times X_2)$,

pair of output sets $(A_1 \times A_2, A_1 \times A_2)$, and a rule function

$$\lambda_1 \times \lambda_2 : X_1 \times X_2 \times X_1 \times X_2 \times A_1 \times A_2 \times A_1 \times A_2 \rightarrow \{0, 1\}$$

given by

$$(\lambda_1 \times \lambda_2)((x_1, x_2), (y_1, y_2), (a_1, a_2), (b_1, b_2)) = \lambda_1(x_1, y_1, a_1, b_1) \lambda_2(x_2, y_2, a_2, b_2).$$

Remarks. (i) If $\mathbb{G}_1$ and $\mathbb{G}_2$ are synchronous then so is $\mathbb{G}_1 \times \mathbb{G}_2$; we thus write $\mathbb{G}_1 \times \mathbb{G}_2 = (X_1 \times X_2, A_1 \times A_2, \lambda_1 \times \lambda_2)$. A similar product can be defined for any (not necessarily synchronous) pair of non-local games, but in the sequel we will only be interested in the synchronous case.

(ii) The product game $\mathbb{G}_1 \times \mathbb{G}_2$ can be thought of as a game where Alice and Bob play the games $\mathbb{G}_1$ and $\mathbb{G}_2$ in parallel, receiving a question for each of them. In order to win the combined game, it is necessary and sufficient to win both $\mathbb{G}_1$ and $\mathbb{G}_2$. Thus, $\mathbb{G}_1 \times \mathbb{G}_2$ can be thought of as a conjunctive product of $\mathbb{G}_1$ and $\mathbb{G}_2$. We note that there are also other natural ways to combine the games $\mathbb{G}_1$ and $\mathbb{G}_2$ (notably, by forming a disjunctive product), but they will not be considered in this paper.

We will denote by $\mathcal{A}_1 \otimes \mathcal{A}_2$ the algebraic tensor product of $*$ -algebras $\mathcal{A}_i$, $i = 1, 2$. Recall that $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a $*$ -algebra with an underlying vector space is the algebraic tensor product of the underlying spaces, multiplication given by

$$(u_1 \otimes u_2)(v_1 \otimes v_2) := (u_1 v_1) \otimes (u_2 v_2),$$

and involution given by

$$(u_1 \otimes u_2)^* = u_1^* \otimes u_2^*.$$

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are moreover C^* -algebras, we write as customary $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$ for their *maximal C^* -tensor product*; thus, $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$ is the unique C^* -algebra containing $\mathcal{A}_1$ and $\mathcal{A}_2$ as unital C^* -subalgebras, generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ as a C^* -algebra, and satisfying the following universal property: if H is a Hilbert space and $\pi_i : \mathcal{A}_i \rightarrow \mathcal{B}(H)$ is a $*$ -representation, $i = 1, 2$, such that $\pi_1(u_1)\pi_2(u_2) = \pi_2(u_2)\pi_1(u_1)$ for all $u_1 \in \mathcal{A}_1$, $u_2 \in \mathcal{A}_2$, then there exists a unique $*$ -representation $\pi_1 \otimes_{\max} \pi_2 : \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \rightarrow \mathcal{B}(H)$ such that $(\pi_1 \otimes_{\max} \pi_2)(u_i) = \pi_i(u_i)$, if $u_i \in \mathcal{A}_i$, $i = 1, 2$.

Theorem 3.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be synchronous games.*

- (i) *The games $\mathbb{G}_1$ and $\mathbb{G}_2$ have perfect algebraic strategies if and only if $\mathbb{G}_1 \times \mathbb{G}_2$ does. In this case, $\mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2) \cong \mathcal{A}(\mathbb{G}_1) \otimes \mathcal{A}(\mathbb{G}_2)$.*
- (ii) *The games $\mathbb{G}_1$ and $\mathbb{G}_2$ have perfect C^* -strategies if and only if $\mathbb{G}_1 \times \mathbb{G}_2$ does. In this case, $C^*(\mathbb{G}_1 \times \mathbb{G}_2) \cong C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2)$.*

Proof. Write $\mathbb{G}_1 = (X_1, A_1, \lambda_1)$, $\mathbb{G}_2 = (X_2, A_2, \lambda_2)$; set $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2$ and $\lambda = \lambda_1 \times \lambda_2$.

(i) Assume that $\mathbb{G}_1$ and $\mathbb{G}_2$ have perfect algebraic strategies. For $(x_1, x_2) \in X_1 \times X_2$ and $(a_1, a_2) \in A_1 \times A_2$, set $g_{(x_1, x_2), (a_1, a_2)} = e_{x_1, a_1} \otimes e_{x_2, a_2}$; then conditions (2.3) are clearly satisfied for the family $\{g_{(x_1, x_2), (a_1, a_2)} : x_i \in X_i, a_i \in A_i, i = 1, 2\}$. It follows that $\mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2)$ exists and that there exists a surjective $*$ -homomorphism $\pi : \mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2) \rightarrow \mathcal{A}(\mathbb{G}_1) \otimes \mathcal{A}(\mathbb{G}_2)$ such that

$$\pi(e_{(x_1, x_2), (a_1, a_2)}) = e_{x_1, a_1} \otimes e_{x_2, a_2}, \quad x_1 \in X_1, x_2 \in X_2, a_1 \in A_1, a_2 \in A_2.$$

Conversely, suppose that $\mathbb{G}_1 \times \mathbb{G}_2$ has a perfect algebraic strategy. For $x_1 \in X_1$, $x_2 \in X_2$ and $a_1 \in A_1$, let

$$f_{x_1, x_2, a_1} = \sum_{b \in A_2} e_{(x_1, x_2), (a_1, b)}.$$

We have

$$(3.2) \quad \sum_{a \in A_1} f_{x_1, x_2, a} = 1.$$

Let $x_1, x_2, x'_2 \in X$, $a_1 \in A_1$, and suppose that $x_2 \neq x'_2$. Since $\mathbb{G}_1$ is synchronous,

$$(3.3) \quad e_{(x_1, x_2), (a, b)} e_{(x_1, x'_2), (a_1, c)} = 0, \quad \text{for all } a, a_1 \in A_1, b, c \in A_2, a \neq a_1.$$

By (3.2) and (3.3),

$$\begin{aligned} f_{x_1, x_2, a_1} f_{x_1, x'_2, a_1} &= \left(1 - \sum_{a \in A_1, a \neq a_1} f_{x_1, x_2, a}\right) f_{x_1, x'_2, a_1} \\ &= f_{x_1, x'_2, a_1} - \sum_{a \in A_1, a \neq a_1} \sum_{b, c \in A_2} e_{(x_1, x_2), (a, b)} e_{(x_1, x'_2), (a_1, c)} \\ &= f_{x_1, x'_2, a_1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f_{x_1, x_2, a_1} f_{x_1, x'_2, a_1} &= f_{x_1, x_2, a_1} \left(1 - \sum_{a \in A_1, a \neq a_1} f_{x_1, x'_2, a}\right) \\ (3.4) \quad &= f_{x_1, x_2, a_1} - \sum_{a \in A_1, a \neq a_1} \sum_{b, c \in A_2} e_{(x_1, x_2), (a_1, c)} e_{(x_1, x'_2), (a, b)} \\ &= f_{x_1, x_2, a_1}. \end{aligned}$$

Thus, $f_{x_1, x_2, a_1} = f_{x_1, x'_2, a_1}$; we set $f_{x_1, a_1} = f_{x_1, x_2, a_1}$ for any $x_2 \in X$. From the calculation (3.4) it now follows that $f_{x_1, a_1}^2 = f_{x_1, a_1}$. Clearly, we also have that $f_{x_1, a_1}^* = f_{x_1, a_1}$. Finally, if $\lambda_1(x_1, x'_1, a_1, a'_1) = 0$ then

$$\lambda(x_1, x_2, x'_1, x'_2, a_1, a_2, a'_1, a'_2) = 0, \quad x_2, x'_2 \in X_2, a_2, a'_2 \in A_2,$$

and hence, fixing some $x_2 \in X_2$, we have

$$f_{x_1, a_1} f_{x'_1, a'_1} = \sum_{b, c \in A_2} e_{(x_1, x_2), (a_1, b)} e_{(x'_1, x_2), (a'_1, c)} = 0.$$

It follows that the subalgebra $\mathcal{A}_1$ of $\mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2)$, generated by the family $\{f_{x_1, a_1} : x_1 \in X_1, a_1 \in A_1\}$, satisfies the conditions of a perfect algebraic strategy for $\mathbb{G}_1$.

Let the projections $f_{x_2, a_2} \in \mathcal{A}$, $x_2 \in X_2$, $a_2 \in A_2$, be defined similarly, and let $\mathcal{A}_2$ be the subalgebra of $\mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2)$ generated by them. Let $\rho_i : \mathcal{A}(\mathbb{G}_i) \rightarrow \mathcal{A}_i$ be the quotient map satisfying

$$\rho_i(e_{x_i, a_i}) = f_{x_i, a_i}, \quad x_i \in X_i, a_i \in A_i,$$

which exists by the virtue of the universality of $\mathcal{A}(\mathbb{G}_i)$, $i = 1, 2$. Then $\rho_1 \otimes \rho_2 : \mathcal{A}(\mathbb{G}_1) \otimes \mathcal{A}(\mathbb{G}_2) \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ is a *-homomorphism.

For $x_i \in X_i$ and $a_i \in A_i$, $i = 1, 2$, we have

$$\begin{aligned} f_{x_1, a_1} f_{x_2, a_2} &= \left(\sum_{b \in A_2} e_{(x_1, x_2), (a_1, b)} \right) \left(\sum_{a \in A_1} e_{(x_1, x_2), (a, a_2)} \right) \\ &= e_{(x_1, x_2), (a_1, a_2)} = f_{x_2, a_2} f_{x_1, a_1}. \end{aligned}$$

It follows that $s_1 s_2 = s_2 s_1$ for all $s_i \in \mathcal{A}_i$, $i = 1, 2$; thus, by the universal property of the tensor product, the mapping

$$m : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2), \quad m(s_1 \otimes s_2) = s_1 s_2,$$

is a *-homomorphism. It follows that the map $\rho := m \circ (\rho_1 \otimes \rho_2) : \mathcal{A}(\mathbb{G}_1) \otimes \mathcal{A}(\mathbb{G}_2) \rightarrow \mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2)$ is a *-homomorphism; moreover,

$$(3.5) \quad \rho(e_{x_1, a_1} \otimes e_{x_2, a_2}) = e_{(x_1, x_2), (a_1, a_2)}, \quad x_i \in X_i, a_i \in A_i, i = 1, 2.$$

Equations (3.1) and (3.5) show that $\pi \circ \rho$ agrees with the identity on the generators of $\mathcal{A}(\mathbb{G}_1) \otimes \mathcal{A}(\mathbb{G}_2)$, and hence on the whole algebra. It follows that π is a *-isomorphism from $\mathcal{A}(\mathbb{G}_1 \times \mathbb{G}_2)$ onto $\mathcal{A}(\mathbb{G}_1) \otimes \mathcal{A}(\mathbb{G}_2)$.

(ii) If $C^*(\mathbb{G}_1)$ and $C^*(\mathbb{G}_2)$ exist then, by setting $g_{(x_1, x_2), (a_1, a_2)} = e_{x_1, a_1} \otimes e_{x_2, a_2}$ inside the maximal tensor product $C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2)$, we obtain a family of operators that satisfy the relations defining the C*-algebra of $\mathbb{G}_1 \times \mathbb{G}_2$; thus, $C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2)$ implements a perfect C*-strategy for $\mathbb{G}_1 \times \mathbb{G}_2$. Conversely, if $\mathbb{G}_1 \times \mathbb{G}_2$ has a perfect C*-strategy then the closure $\tilde{\mathcal{A}}_1$ in $C^*(\mathbb{G}_1 \times \mathbb{G}_2)$ of the *-algebra $\mathcal{A}_1$ defined in the proof of (i) provides a perfect C*-strategy for $\mathbb{G}_1$. By symmetry, such exists for $\mathbb{G}_2$ as well.

Now suppose that $\mathbb{G}_1$ and $\mathbb{G}_2$ have perfect C*-strategies. Then, by the universal properties of $C^*(\mathbb{G}_1 \times \mathbb{G}_2)$ and $C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2)$, the maps π and ρ defined in (i) extend to *-homomorphisms $\tilde{\pi} : C^*(\mathbb{G}_1 \times \mathbb{G}_2) \rightarrow$

$C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2)$ and $\tilde{\rho} : C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2) \rightarrow C^*(\mathbb{G}_1 \times \mathbb{G}_2)$, which shows that $C^*(\mathbb{G}_1 \times \mathbb{G}_2) \cong C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2)$ canonically. $\square$

Remark. Let $\mathbb{G}_i = (X_i, A_i, \lambda_i)$, $i = 1, 2$, be synchronous games and $x \in \{\text{loc}, q, \text{qa}, \text{qc}, \text{ns}\}$. If $p_i \in \mathcal{C}_x(\lambda_i)$, $i = 1, 2$,

$$p_1 \otimes p_2 : A_1 \times A_2 \times A_1 \times A_2 \times X_1 \times X_2 \times X_1 \times X_2 \rightarrow [0, 1],$$

and

$$(p_1 \otimes p_2)(a_1, a_2, b_1, b_2 | x_1, x_2, y_1, y_2) = p_1(a_1, b_1 | x_1, y_1) p_2(a_2, b_2 | x_2, y_2),$$

then $p_1 \otimes p_2 \in \mathcal{C}_x(\lambda_1 \times \lambda_2)$. Indeed, it is straightforward that $p_1 \otimes p_2 \in \mathcal{C}_{\text{ns}}(\lambda_1 \times \lambda_2)$. Suppose that $p_i \in \mathcal{C}_{\text{qc}}$, $i = 1, 2$, and write

$$p_i(a_i, b_i | x_i, y_i) = \langle E_{x_i, a_i} F_{y_i, b_i} \xi_i, \xi_i \rangle, \quad i = 1, 2,$$

as in the definition of quantum commuting correlations. Then

$$\begin{aligned} & (p_1 \otimes p_2)(a_1, a_2, b_1, b_2 | x_1, x_2, y_1, y_2) \\ &= \langle ((E_{x_1, a_1} \otimes E_{x_2, a_2})(F_{y_1, b_1} \otimes F_{y_2, b_2})(\xi_1 \otimes \xi_2), (\xi_1 \otimes \xi_2)) \rangle, \end{aligned}$$

showing that $p_1 \times p_2 \in \mathcal{C}_{\text{qc}}$. The argument for $x = q$ and $x = \text{loc}$ is similar, while the statement about qa follows from the definition of $\mathcal{C}_{\text{qa}}$ as the closure of $\mathcal{C}_q$.

It is well-known, and easily seen that, in the converse direction, if $p \in \mathcal{C}_x(\lambda_1 \times \lambda_2)$ and $x_2, y_2 \in X_2$, then the correlation

$$p_{x_2, y_2} : A_1 \times A_1 \times X_1 \times X_1 \rightarrow [0, 1],$$

given by

$$p_{x_2, y_2}(a_1, b_1 | x_1, y_1) = \sum_{a_2, b_2 \in A_2} p(a_1, a_2, b_1, b_2 | x_1, x_2, y_1, y_2),$$

is a perfect strategy for $\mathbb{G}_1$. The argument from Theorem 3.1 shows that, if the games $\mathbb{G}_1$ and $\mathbb{G}_2$ are synchronous, the correlation p_{x_2, y_2} is independent of the choice of x_2 and y_2 .

Corollary 3.2. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be synchronous games and $t \in \{\text{loc}, q, \text{qc}\}$. The following are equivalent, for a no-signalling correlation p :*

- (i) *p is a perfect t -strategy for the product game $\mathbb{G}_1 \times \mathbb{G}_2$;*
- (ii) *p is the limit of convex combinations of strategies of the form $p_1 \otimes p_2$, where p_i is a perfect t -strategy for $\mathbb{G}_i$, $i = 1, 2$.*

Proof. (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii) Assume that p is a quantum commuting perfect strategy of $\mathbb{G}_1 \times \mathbb{G}_2$. By [11, Corollary 5.6], there exists a tracial state τ on $C^*(\mathbb{G}_1 \times \mathbb{G}_2)$ such that

$$(3.6) \quad p((a_1, a_2), (b_1, b_2) | (x_1, x_2), (y_1, y_2)) = \tau(e_{(x_1, x_2), (a_1, a_2)} e_{(y_1, y_2), (b_1, b_2)}),$$

for all $x_i \in X_i$, $y_i \in Y_i$, $a_i \in A_i$ and $b_i \in B_i$, $i = 1, 2$. By Theorem 3.1, τ can be viewed as a tracial state on $C^*(\mathbb{G}_1) \otimes_{\max} C^*(\mathbb{G}_2)$ in a canonical way. By [5, Corollary 2.2], τ is the limit of convex combinations of states of the form $\tau_1 \otimes \tau_2$, where τ_i is a tracial state on $C^*(\mathbb{G}_i)$, $i = 1, 2$. Using once again [11, Corollary 5.6], we see that (ii) holds in the case $t = qc$.

We next consider the case $t = q$. By [2, Theorem 3.2], there exist a finite dimensional Hilbert space H , a $*$ -representation $\pi : C^*(\mathbb{G}_1 \times \mathbb{G}_2) \rightarrow \mathcal{B}(H)$ and a tracial state $\tilde{\tau} : \pi(C^*(\mathbb{G}_1 \times \mathbb{G}_2)) \rightarrow \mathbb{C}$ such that (3.6) is fulfilled for $\tau = \tilde{\tau} \circ \pi$. Set $\mathcal{A} = \pi(C^*(\mathbb{G}_1 \times \mathbb{G}_2))$. Using Theorem 3.1, write $\mathcal{A}_1 = \pi(C^*(\mathbb{G}_1) \otimes 1)$ and $\mathcal{A}_2 = \pi(1 \otimes C^*(\mathbb{G}_2))$; note that $\mathcal{A}$ is generated by its finite dimensional and mutually commuting C^* -subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$. It follows that $\mathcal{A} = \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 = \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$; by [5, Corollary 2.2], $\tilde{\tau}$ is the limit of convex combinations of states of the form $\tilde{\tau}_1 \otimes \tilde{\tau}_2$, where $\tilde{\tau}_i$ is a tracial state on $\mathcal{A}_i$, $i = 1, 2$. The functional $\tau_i = \tilde{\tau}_i \circ \pi|_{C^*(\mathbb{G}_i)}$ is a tracial state on $C^*(\mathbb{G}_i)$, $i = 1, 2$. Another application of [2, Theorem 3.2] now completes the proof of the statement in this case.

The case $t = \text{loc}$ follows similar lines, again using [2, Theorem 3.2]; the details are omitted. $\square$

4. SYNCHRONOUS VALUES OF GAMES

A synchronous game $\mathbb{G} = (X, A, \lambda)$, equipped with a probability distribution π on the set $X \times X$ of questions, will be called a *synchronous game with density*. For each $x \in \{\text{loc}, \text{qa}, \text{qc}, \text{ns}\}$, we define the *synchronous x-value* $\omega_x^s(\mathbb{G}, \pi)$ of $(\mathbb{G}, \pi)$ by setting

$$(4.1) \quad \omega_x^s(\mathbb{G}, \pi) = \sup \left\{ \sum_{x \in X, y \in Y} \sum_{a \in A, b \in B} \pi(x, y) \lambda(x, y, a, b) p(a, b | x, y) : p \in \mathcal{C}_x^s \right\},$$

where $\mathcal{C}_x^s$ denotes the set of all synchronous correlations over (X, A) of class x .

Note that each term in the supremum is the expected probability of the players winning the game given that they use the conditional probability density p . So the synchronous x -value represents the optimal winning probability over the corresponding set of synchronous correlations.

- Remark 4.1.** (i) Let $x \in \{\text{loc}, \text{qa}, \text{qc}, \text{ns}\}$. Then $\omega_x^s(\mathbb{G}, \pi) \leq 1$ and, since $\mathcal{C}_x^s$ is closed, the supremum in the definition of $\omega_x(\mathbb{G}, \pi)$ is achieved. It follows that a synchronous game G has a perfect x -strategy if and only if $\omega_x^s(\mathbb{G}) = 1$.
- (ii) It was shown in [8] that $\mathcal{C}_{\text{qa}}^s = \overline{\mathcal{C}_{\text{q}}^s}$. Thus, if in the right hand side of (4.1) one employs correlations of the class $\mathcal{C}_{\text{q}}^s$, the corresponding supremum coincides with $\omega_{\text{qa}}^s(\mathbb{G}, \pi)$.

Given synchronous games with densities $(\mathbb{G}_i, \pi_i) = (X_i, A_i, \lambda_i, \pi_i)$, $i = 1, 2$, equip the question set $(X_1 \times X_2) \times (X_1 \times X_2)$ of the game $\mathbb{G}_1 \times \mathbb{G}_2$ with the probability distribution $\pi_1 \times \pi_2$, given by

$$(\pi_1 \times \pi_2)((x_1, x_2), (y_1, y_2)) = \pi_1(x_1, y_1)\pi_2(x_2, y_2).$$

We write $(\mathbb{G}_1 \times \mathbb{G}_2, \pi_1 \times \pi_2)$ for this synchronous game with density.

Proposition 4.2. *Let $(\mathbb{G}_i, \pi_i) = (X_i, A_i, \lambda_i, \pi_i)$, $i = 1, 2$, be synchronous games with densities and $x \in \{\text{loc}, \text{qa}, \text{qc}, \text{ns}\}$. Then*

$$\omega_x^s(\mathbb{G}_1, \pi_1)\omega_x^s(\mathbb{G}_2, \pi_2) \leq \omega_x^s(\mathbb{G}_1 \times \mathbb{G}_2, \pi_1 \times \pi_2).$$

Proof. If $p_i(a_i, b_i|x_i, y_i) \in \mathcal{C}_x$, $i = 1, 2$, then applying Theorem 2.2, it is easily checked that

$$p((a_1, a_2), (b_1, b_2)|(x_1, x_2), (y_1, y_2)) := p_1(a_1, b_1|x_1, y_1)p_2(a_2, b_2|x_2, y_2) \in \mathcal{C}_x.$$

Consequently,

$$\begin{aligned} & \omega_x^s(\mathbb{G}_1, \pi_1)\omega_x^s(\mathbb{G}_2, \pi_2) \\ &= \prod_{i=1}^2 \sup\left\{ \sum_{x_i, y_i, a_i, b_i} \pi_i(x_i, y_i)\lambda_i(x_i, y_i, a_i, b_i)p(a_i, b_i|x_i, y_i) : p_i \in \mathcal{C}_x^s \right\} \\ &\leq \sup\left\{ \sum_{x_i, y_i, a_i, b_i, i=1,2} \pi_1(x_1, y_1)\pi_2(x_2, y_2)\lambda(x_1, x_2, y_1, y_2, a_1, a_2, b_1, b_2) \right. \\ &\quad \left. p(a_1, a_2, b_1, b_2|x_1, x_2, y_1, y_2) : p \in \mathcal{C}_x^s \right\} \\ &= \omega_x^s(\mathbb{G}_1 \times \mathbb{G}_2, \pi_1 \times \pi_2). \end{aligned}$$

□

The inequality in Theorem 4.2 can be strict, even in the case of the classical value ω_{loc}^s . We provide two examples towards this end. The first is an example of a synchronous game with uniform distribution on the inputs for which the inequality is strict. The second example is of a *symmetric* synchronous game with uniform distribution on the inputs for which the inequality is strict.

A rule function λ is called *symmetric* if $\lambda(x, y, a, b) = \lambda(y, x, b, a)$. Note that, since synchronous correlations p arise from traces, we have that

$$p(a, b|x, y) = \tau(e_{x,a}e_{y,b}) = \tau(e_{y,b}e_{x,a}) = p(b, a|y, x);$$

thus, they satisfy an extra transposition invariance. For this reason, arguably the most suitable context to study synchronous values is that of symmetric synchronous games with symmetric distributions π on their inputs, that is, ones satisfying $\pi(x, y) = \pi(y, x)$ for all x and y . Our examples illustrate that even in this more restrictive setting the synchronous value can be strictly supermultiplicative.

For a rule function $\lambda : X \times Y \times A \times B \rightarrow \{0, 1\}$ of a game over the quadruple (X, Y, A, B) , and $(x, y) \in X \times Y$, let

$$E_{x,y} = \{(a, b) \in A \times B : \lambda(x, y, a, b) = 1\}.$$

Clearly, specifying λ is equivalent to specifying the family $\{E_{x,y} : x, y\}$ of subsets of $A \times B$. In synchronous games, we have $X = Y$, $A = B$, and $E_{x,x} \subseteq \{(a, a) : a \in A\}$, for every $x \in X$. Note that the local synchronous strategies of a synchronous game arise from functions $f : X \rightarrow A$: a function $f : X \rightarrow A$ gives rise to the local no-signalling correlation $p_f = (p(a, b|x, y))$ given by $p_f(a, b|x, y) = 1$ if $a = f(x)$ and $b = f(y)$, and $p(a, b|x, y) = 0$ otherwise; conversely, every local synchronous strategy is the convex combination of strategies of the form p_f (this can be inferred, e.g., from [11, Corollary 5.6]).

For a synchronous non-local game $\mathbb{G}$ with rule function $\lambda : X \times X \times A \times A \rightarrow \{0, 1\}$ and probability distribution $\pi : X \times X \rightarrow [0, 1]$, we write

$$h_{\mathbb{G}} = \sum_{x,y \in X} \pi(x, y) \sum_{a,b \in A} \lambda(x, y, a, b) e_{x,a} e_{y,b},$$

viewed as an element of the C^* -algebra $\mathcal{A}_{X,A}$.

Example 4.3. Let $\mathbb{G}$ be the game with $X = Y = A = B = [2] := \{1, 2\}$ and rule function λ specified by setting

$$E_{1,1} = E_{2,2} = \{(1, 1), (2, 2)\}, E_{1,2} = \{(1, 1)\}, E_{2,1} = \{(1, 2)\}.$$

We claim that, if $[2] \times [2]$ is equipped with the uniform probability distribution π , then

$$(4.2) \quad \omega_{\text{loc}}^s(G, \pi) = \omega_{\text{q}}^s(G, \pi) = \omega_{\text{qc}}^s(G, \pi) = 3/4,$$

while

$$(4.3) \quad \omega_{\text{loc}}^s(\mathbb{G} \times \mathbb{G}, \pi \times \pi) = \omega_{\text{q}}^s(\mathbb{G} \times \mathbb{G}, \pi \times \pi) = \omega_{\text{qc}}^s(\mathbb{G} \times \mathbb{G}, \pi \times \pi) = 10/16.$$

Indeed, we have that

$$h_{\mathbb{G}} = \frac{1}{4}(2I + e_{1,1}e_{2,1} + e_{2,1}e_{1,2}).$$

The synchronous values of $\mathbb{G}$ are obtained by computing the supremum of the quantities $\tau(h_{\mathbb{G}})$ over traces τ of $\mathcal{A}(\mathbb{G})$ of particular type. Setting $p = e_{1,1}$ and $q = e_{2,1}$, we have

$$4h_{\mathbb{G}} = 2I + pq + q(I - p) = 2I + q,$$

and (4.2) follows.

Write $X_1 = X_2 = X$ and $A_1 = A_2 = A$ and let

$$\{E_{(x_1, x_2), (y_1, y_2)} : (x_1, y_1) \in X_1 \times X_1, (x_2, y_2) \in X_2 \times X_2\}$$

be the family of subsets of $(A_1 \times A_2) \times (A_1 \times A_2)$ that determine $\lambda \times \lambda$. Note that

$$E_{(x_1, x_2), (y_1, y_2)} = \{((a_1, a_2), (b_1, b_2)) : (a_1, b_1) \in E_{x_1, y_1} \text{ and } (a_2, b_2) \in E_{x_2, y_2}\}.$$

Consider the function $h : X_1 \times X_2 \rightarrow A_1 \times A_2$, given by

$$(1, 1) \rightarrow (1, 1), \quad (1, 2) \rightarrow (1, 1), \quad (2, 1) \rightarrow (1, 1), \quad (2, 2) \rightarrow (1, 1).$$

For each $((x_1, x_2), (y_1, y_2))$ listed below, we indicate whether the value $(h \times h)((x_1, x_2), (y_1, y_2))$ belongs to the set $E_{(x_1, x_2), (y_1, y_2)}$:

- $E_{(1,1), (1,2)} = \{((1, 1), (1, 1)), ((2, 1), (2, 1))\} \implies \text{Yes};$
- $E_{(1,1), (2,1)} = \{((1, 1), (1, 1)), ((1, 2), (1, 2))\} \implies \text{Yes};$
- $E_{(1,1), (2,2)} = \{((1, 1), (1, 1))\} \implies \text{Yes};$
- $E_{(1,2), (1,1)} = \{((1, 1), (1, 2)), ((2, 1), (2, 2))\} \implies \text{No};$
- $E_{(1,2), (2,1)} = \{((1, 1), (1, 2))\} \implies \text{No};$
- $E_{(1,2), (2,2)} = \{((1, 1), (1, 1)), ((1, 2), (1, 2))\} \implies \text{Yes};$
- $E_{(2,1), (1,2)} = \{((1, 1), (2, 1))\} \implies \text{No};$
- $E_{(1,2), (2,1)} = \{((1, 1), (1, 2))\} \implies \text{No};$
- $E_{(2,1), (2,2)} = \{((1, 1), (1, 1)), ((2, 1), (2, 1))\} \implies \text{Yes};$
- $E_{(2,2), (1,1)} = \{((1, 1), (2, 2))\} \implies \text{No};$
- $E_{(2,2), (1,2)} = \{((1, 1), (2, 1)), ((1, 2), (2, 2))\} \implies \text{No};$
- $E_{(2,2), (2,1)} = \{((1, 1), (1, 1)), ((2, 1), (2, 1))\} \implies \text{Yes}.$

The remaining pairs $((x_1, x_2), (y_1, y_2))$ are of the form $((x_1, x_2), (x_1, x_2))$; since the game λ is synchronous, we have that $(h \times h)((x_1, x_2), (x_1, x_2))$ meets $E_{(x_1, x_2), (x_1, x_2)}$. This shows that

$$\omega_{\text{loc}}(\mathbb{G} \times \mathbb{G}, \pi \times \pi) \geq \frac{10}{16}.$$

On the other hand, let $h_{\mathbb{G} \times \mathbb{G}}$ be the element of $\mathcal{A}(\mathbb{G} \times \mathbb{G})$ with

$$(4.4) \quad 16h_{\mathbb{G} \times \mathbb{G}} = 4 \cdot I + e_{11,11}(e_{12,11} + e_{22,11} + e_{21,11}) + \\ e_{11,12}(e_{21,12} + e_{12,11}) + e_{11,21}e_{21,11} + \\ e_{11,22}(e_{22,11} + e_{21,12}) + \\ e_{12,21}(e_{21,12} + e_{22,11}) + e_{12,22}e_{22,12} + \\ e_{21,11}e_{22,11} + e_{21,21}e_{22,21} + e_{21,12}e_{22,11} + e_{21,22}e_{22,21},$$

and note that

$$\omega_{\text{qc}}^s(\mathbb{G} \times \mathbb{G}, \pi \times \pi) = \sup\{\tau(h_{\mathbb{G} \times \mathbb{G}}) : \tau \text{ a trace on } \mathcal{A}(\mathbb{G} \times \mathbb{G})\}.$$

For any trace τ , if $p, q \geq 0$ and $r \geq q$, then $0 \leq \tau(pq) \leq \tau(pr)$. We thus see that

$$16\tau(h_{\mathbb{G} \times \mathbb{G}}) \leq 4 + \tau(3e_{11,11}) + \tau(3e_{11,12}) + \tau(3e_{11,22}) + \tau(2e_{12,21}) \\ + \tau(e_{12,22}) + \tau(e_{21,11} + e_{21,21} + e_{21,12} + e_{21,22}) \\ \leq 4 + 3 + 2 + 1 = 10,$$

and (4.3) is proved.

The game, exhibited above, shows that strict inequality may take place in Proposition 4.2 even for the smallest non-trivial games. At the expense of increasing the number of inputs by one, the game can be made symmetric – this is achieved in the next example, which also exhibits a separation between the local and the quantum synchronous value.

Example 4.4. Let $X = [3]$, $A = [2]$, equip $X \times X$ with the uniform probability distribution, and let $\pi : X \times X \times A \times A \rightarrow \{0, 1\}$ be the rule function determined by the sets

- $E_{1,2} = E_{2,1} = E_{2,3} = E_{3,2} = \{(1, 2)\};$
- $E_{1,3} = E_{3,1} = \{(2, 1)\};$
- $E_{1,1} = E_{2,2} = E_{3,3} = \{(1, 1), (2, 2)\}.$

The eight possibilities for functions $g : X \rightarrow A$ are:

- $g_1 \times g_1 \times g_1 : (1, 2, 3) \rightarrow (1, 1, 1)$, yielding $R_{g_1} = 3;$
- $g_2 \times g_2 \times g_2 : (1, 2, 3) \rightarrow (1, 1, 2)$, yielding $R_{g_2} = 5;$
- $g_3 \times g_3 \times g_3 : (1, 2, 3) \rightarrow (1, 2, 1)$, yielding $R_{g_3} = 5;$
- $g_4 \times g_4 \times g_4 : (1, 2, 3) \rightarrow (1, 2, 2)$, yielding $R_{g_4} = 5;$
- $g_5 \times g_5 \times g_5 : (1, 2, 3) \rightarrow (2, 1, 1)$, yielding $R_{g_5} = 5;$
- $g_6 \times g_6 \times g_6 : (1, 2, 3) \rightarrow (2, 1, 2)$, yielding $R_{g_6} = 5;$
- $g_7 \times g_7 \times g_7 : (1, 2, 3) \rightarrow (2, 2, 1)$, yielding $R_{g_7} = 5;$
- $g_8 \times g_8 \times g_8 : (1, 2, 3) \rightarrow (2, 2, 2)$, yielding $R_{g_8} = 3.$

Thus, $\omega_{\text{loc}}(\mathbb{G}, \pi) = \frac{5}{9}$.

Letting as before $X_1 = X_2 = X$ and $A_1 = A_2 = A$, consider the function $\eta : X_1 \times X_2 \rightarrow A_1 \times A_2$, given by

$$\eta(x_1, x_2) = \begin{cases} (1, 1) & \text{if } (x_1, x_2) \in \{(1, 1), (1, 3), (3, 1), (3, 3)\} \\ (1, 2) & \text{if } (x_1, x_2) \in \{(1, 2), (3, 2)\} \\ (2, 1) & \text{if } (x_1, x_2) \in \{(2, 1), (2, 3)\} \\ (2, 2) & \text{if } (x_1, x_2) = (2, 2). \end{cases}$$

Among the inputs (x_1, x_2, y_1, y_2) of the product game $\mu \times \mu$ with $x_1 \leq x_2$, $y_1 \leq y_2$, and $(x_1, y_1) \neq (x_2, y_2)$, we single out the following

$$\begin{aligned} E_{(1,1),(1,2)} &= \{((1, 1), (1, 2)), ((2, 1), (2, 2))\}, \\ E_{(1,1),(2,1)} &= \{((1, 1), (2, 1)), ((1, 2), (2, 2))\}, \\ E_{(1,1),(2,2)} &= \{((1, 1), (2, 2))\}, \\ E_{(1,2),(2,2)} &= \{((1, 1), (2, 1)), ((1, 2), (2, 2))\}, \\ E_{(2,1),(2,2)} &= \{((1, 1), (1, 2)), ((2, 1), (2, 2))\}, \\ E_{(1,3),(2,3)} &= \{((1, 1), (2, 1)), ((1, 2), (2, 2))\}, \\ E_{(3,1),(3,2)} &= \{((1, 1), (1, 2)), ((2, 1), (2, 2))\}, \end{aligned}$$

which all meet the graph of the function $\eta \times \eta$. Among the inputs (x_1, x_2, y_1, y_2) of the product game $\mathbb{G} \times \mathbb{G}$ with $x_1 \leq x_2$, $y_1 \geq y_2$, and $(x_1, y_1) \neq (x_2, y_2)$, we single out

$$\begin{aligned} E_{(2,3),(2,2)} &= \{((1, 1), (1, 2)), ((2, 1), (2, 2))\}, \\ E_{(3,3),(3,2)} &= \{((1, 1), (2, 1)), ((2, 1), (2, 2))\}, \end{aligned}$$

which also meet the graph of $\eta \times \eta$. By symmetry, we thus have that

$$\omega_{\text{loc}}(\mathbb{G} \times \mathbb{G}, \pi \times \pi) \geq \frac{27}{81} > \frac{25}{81} = \omega_{\text{loc}}(\mathbb{G}, \pi)^2.$$

We finish this note by computing the quantum value of the game $\mathbb{G}$ from Example 4.4, implying that it is strictly greater than the local (classical) value. As a result, we can see that strict supermultiplicativity of the game value can take place also in the case the game demonstrates quantum advantage.

Theorem 4.5. *Let $(\mathbb{G}, \pi)$ be the non-local game from Example 4.4. Then*

$$\omega_{\text{qc}}^s(\mathbb{G}, \pi) = \omega_{\text{q}}^s(\mathbb{G}, \pi) = 7/12 > 5/9 = \omega_{\text{loc}}^s(\mathbb{G}, \pi).$$

Proof. By [14], the set $\mathcal{C}_{\text{q}}^s$ in our (three input, two output) case is closed. Thus, the supremum (4.1) used to compute $\omega_{\text{q}}(\mathbb{G}, \pi)$ is achieved. We write for brevity e_{xa} in the place of $e_{x,a}$. Let τ be a trace on a finite dimensional

representation of $\mathcal{A}_{X,A}$ for which the value $\omega_q^s(\mathbb{G}, \pi)$ is attained. By splitting the representation into irreducible blocks, we can assume that all projections e_{xa} are contained in M_n for some $n \in \mathbb{N}$. We note that

$$(4.5) \quad 9h_{\mathbb{G}} = 3I + e_{11}e_{22} + e_{21}e_{12} + e_{21}e_{32} + e_{31}e_{22} + e_{12}e_{31} + e_{32}e_{11}.$$

Setting $p = e_{11}$, $q = e_{21}$, and $r = e_{31}$, so that $e_{12} = I - p$, $e_{22} = I - q$, $e_{32} = I - r$ we have

$$9h_{\mathbb{G}} = 3I + p(1 - q) + q(1 - p) + q(1 - r) + r(1 - q) + (1 - p)r + (1 - r)p.$$

By [1], p commutes with $q + r$, q commutes with $p + r$ and r commutes with $p + q$. This implies that $p + q + r$ commutes with p , q and r . Thus,

$$9\tau(h_{\mathbb{G}}) = 3 + \tau(2p(1 - q - r) + 2r + 2q(1 - r)).$$

By [9], there exists $t \in \mathbb{R}^+$ such that

$$(4.6) \quad p + q + r = tI_n.$$

Write

$$p = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}, \text{ so that } q + r = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix},$$

and note that (4.6) yields

$$C = (t - 1)I_k \text{ and } D = tI_{n-k}.$$

Writing

$$q = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix},$$

we have

$$r = \begin{pmatrix} (t - 1)I_k - A & -X \\ -X^* & tI_{n-k} - B \end{pmatrix}.$$

The identities $q^2 = q$ and $r^2 = r$ yield

$$A^2 + XX^* = A, \quad ((t - 1)I_k - A)^2 + XX^* = (t - 1)I_k - A$$

and

$$B^2 + X^*X = B, \quad (tI_{n-k} - B)^2 + X^*X = tI_{n-k} - B.$$

Thus,

$$((t - 1)I_k - A)^2 + A - A^2 = (t - 1)I_k - A \implies A = \frac{t - 1}{2}I_k,$$

and

$$(tI_{n-k} - B)^2 + B - B^2 = tI_{n-k} - B \implies B = \frac{t}{2}I_{n-k}.$$

Therefore

$$q = \begin{pmatrix} \frac{t-1}{2}I_k & X \\ X^* & \frac{t}{2}I_{n-k} \end{pmatrix}$$

and using again the fact that q is an idempotent, we see that

$$XX^* = \left(\frac{t-1}{2} - \frac{t-1}{2} \right)^2 I_k = \frac{4t-t^2-3}{4} I_k$$

and

$$X^*X = \frac{2t-t^2}{4} I_{n-k}.$$

Since $\tau(X^*X) = \tau(XX^*)$, we have

$$k(4t-t^2-3) = (2t-t^2)(n-k) \implies k(6t-2t^2-3) = (2t-t^2)n.$$

This shows that

$$k/n = \frac{2t-t^2}{6t-2t^2-3}.$$

By [9], the following values of t are feasible:

$$t = 0, 1, 2, 3, 3/2.$$

This leads to

$$k/n = 0, 1, 0, 1, 1/2.$$

The case where $t = 3/2$ and $k/n = 1/2$ can be realised in M_2 and we have that

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix}, r = \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix}.$$

In this case

$$\tau(h_{\mathbb{G}}) = 1/9\tau(3I_2 + 2p(I - (tI - p) + 2r + 2q(1 - r))) = 7/12.$$

If $t = 0$, then q has a negative entry, so can be discarded.

If $t = 1$, then again q is not a projection.

If $t = 2$, then $p = 0$ and

$$q = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, r = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = 1 - q,$$

and in this case

$$\tau(h_G) = 1/9(\text{tr}(3I_2 + 2r + 2q^2)) = 5/9 < 7/12.$$

Finally, if $t = 3$, then $p = q = r = I$ and so

$$\tau(h_{\mathbb{G}}) = 1/9(3 + 2(-1) + 2 + 0) = 1/3.$$

So we get that the value occurs when $t = 3/2$; note that we have also shown that, in addition, it can be achieved by matrices in M_2 .

We turn to the quantum commuting value. Let $\tau : \mathcal{A}_{X,A} \rightarrow \mathbb{C}$ be a tracial state and $\pi_\tau : \mathcal{A}_{X,A} \rightarrow \mathcal{B}(H)$ be the GNS representation of $\mathcal{A}_{X,A}$, associated with τ . We let $\mathcal{A} = \pi_\tau(\mathcal{A}_{X,A})''$ and extend τ to a normal trace on $\mathcal{A}$ in the canonical fashion. Let $\mathcal{Z}$ be the centre of $\mathcal{A}$; up to a normal *-isomorphism, we have that $\mathcal{Z} = L^\infty(Z, \mu)$ for a suitable probability space (Z, μ) . In the

sequel, we use the terminology of [7, Chapter 14]. Using [7, Theorem 14.2.2], we write

$$H = \int_Z H_z d\mu(z) \quad \text{and} \quad \mathcal{A} = \int_Z \mathcal{A}(z) d\mu(z)$$

in their corresponding direct integral decompositions. Note that $\mathcal{A}(z) \subseteq \mathcal{B}(H_z)$ is a factor for μ -almost all $z \in Z$ (see [7, Theorem 14.2.2]). For an element $T \in \mathcal{A}$, we write $T = \int_Z T(z) d\mu(z)$ for its direct integral decomposition.

Suppressing notation, we write p, q and r for their images under π_τ . By the arguments from the first part of the proof, $p + q + r \in \mathcal{Z}$; thus, there exists a measurable function $t : Z \rightarrow \mathbb{R}_+$, such that

$$p(z) + q(z) + r(z) = t(z)I_{H_z}, \quad \text{for } \mu\text{-almost all } z \in Z.$$

We continue to use the notation from the first part of the proof. Thus, C and D are the (unique) operators in $\mathcal{A}$, determined by the requirement

$$q + r = pCp + p^\perp Dp^\perp,$$

while A, B and X are determined by

$$q = pAp + p^\perp Bp^\perp + pXp^\perp + p^\perp X^*p.$$

The previous arguments now imply

$$\tau(p(z))(6t(z) - 2t(z)^2 - 3) = (2t(z) - t(z)^2).$$

Since the function $z \rightarrow t(z)$ is measurable (and takes finitely many values), the arguments above now show that there is a partition $Z = Z_{3/2} \cup Z_2 \cup Z_3$, where

$$Z_\alpha = \{z \in Z : t(z) = \alpha\}.$$

Since μ is a probability measure, we have that $\tau(h_{\mathbb{G}}) \leq 7/12$. The proof is complete. $\square$

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QMATH, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN

Email address: mancinska@math.ku.dk

INSTITUTE FOR QUANTUM COMPUTING AND DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, CANADA

Email address: vpaulsen@uwaterloo.ca

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, 501 EWING HALL, NEWARK, DE 19716, USA

Email address: todorov@udel.edu

ICREA AND GRUP D'INFORMACIÓ QUÀNTICA, DEPARTAMENT DE FÍSICA, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA (BARCELONA), SPAIN

Email address: andreas.winter@uab.cat