QUANTUM HYPERGRAPH HOMOMORPHISMS AND NON-LOCAL GAMES

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Abstract. Using the simulation paradigm in information theory, we define notions of quantum hypergraph homomorphisms and quantum hypergraph isomorphisms, and show that they constitute partial orders and equivalence relations, respectively. Specialising to the case where the underlying hypergraphs arise from non-local games, we define notions of quantum non-local game homomorphisms and quantum non-local game isomorphisms, and show that games, isomorphic with respect to a given correlation type, have equal values and asymptotic values relative to this type. We examine a new class of no-signalling correlations, which witness the existence of non-local game homomorphisms, and characterise them in terms of states on tensor products of canonical operator systems. We define jointly synchronous correlations and show that they correspond to traces on the tensor product of the canonical C*-algebras associated with the game parties.

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1. Introduction

The connections between operator algebra theory and quantum information theory are undergoing at present a phase of intensive development. One of the chief catalysts for this trend was the equivalence between the Connes Embedding Problem in von Neumann algebra theory and the Tsirelson Problem in quantum physics, established in [26, 29, 42]. Its culminations are arguably the refutal of the weak Tsirelson Problem in [54], the demonstration

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of the non-closedness of the set of quantum correlations presented in [53] (see also [22]) and, more recently, the announced in [29] resolution of the aforementioned Connes Embedding Problem.

Behind the developments in [54], [22] and [29] one discerns the role of non-local games and their optimal winning probabilities. These objects were first studied from the perspective of quantum information theory (see e.g. [14, 15, 41]), where they can effectively witness entanglement, leading to proofs of Bell's Theorem [6]. Important combinatorial ramifications were established in [12] and later in [40, 4], where quantum graph homomorphisms and quantum graph isomorphisms were defined and studied. A wealth of mathematical developments in non-local game theory took place within the past decade [22, 37, 38, 39, 44, 46, 51], relying on operator system theory, quantum group theory, quantum information theory and combinatorics, among others.

A non-local game is a cooperative game, played by two players, Alice and Bob, against a verifier. In each round of the game, the verifier selects an input pair (x, y) from the cartesian product of two finite sets X and Y, following a probability distribution π on $X \times Y$, and sends input x to Alice and input y to Bob. Alice produces an output a lying in a specified set A, and Bob – an output b lying in a specified set B; the combination (x, y, a, b) yields a win for the players if it satisfies a previously fixed predicate $\lambda : X \times Y \times A \times B \to \{0, 1\}$, representing the rules of the game.

During the course of the game, the players are not allowed to communicate; mathematically this is expressed by saying that the probabilistic strategies $p = \{(p(a, b|x, y))_{(a,b)\in A\times B} : (x, y) \in X \times Y\}$ that they are allowed to use are no-signalling correlations, that is, correlations $p(a, b|x, y)$ with well-defined conditional marginals $p(a|x)$ and $p(b|y)$. Several types of strategies are usually used: local (corresponding to classical resources), quantum (corresponding to finite dimensional entanglement), quantum approximate (corresponding to liminal entanglement) and quantum commuting (arising from the commuting model of quantum mechanics). Each correlation type gives rise to a corresponding game value and asymptotic game value: these are, respectively, the optimal winning probability in one round, and the optimal winning probability in the limit when independent rounds, forming an infinite sequence, are conducted.

One of the main motivations behind the present work is to identify conditions, upon which seemingly different games may have the same value with respect to a given strategy class. We propose notions of game homomorphisms and game isomorphisms, associated with a fixed correlation type. The existence of a homomorphism from a game G_1 into a game G_2 of type t leads to an inequality between the two t-values, while the existence of an isomorphism of type t – to an equality of these values.

In order to define game homomorphisms (resp. game isomorphisms) of a given type, we embed the two games into a larger game; we can think of the new game as a non-local "super-game", played by the verifiers of the original games and controlled by a "super-verifier". More specifically, suppose that game G_i has inputs from $X_i \times Y_i$ and outputs from $A_i \times B_i$, $i = 1, 2$. The super-verifier sends a pair x_2y_2 from $X_2 \times Y_2$ to the verifier of G_1, and a pair a_1b_1 from $A_1 \times B_1$ to the verifier of G_2. The two verifiers return pairs x_1y_1 from $X_1 \times Y_1$ and a_2b_2 from $A_2 \times B_2$, respectively. The rules of the super-game are appropriately determined by the rules of the individual games, requiring that $(x_2y_2, a_1b_1, x_1y_1, a_2b_2)$ yields a win for the super-game if (x_1, y_1, a_1, b_1) and (x_2, y_2, a_2, b_2) either simultaneously yield a win or simultaneously yield a lose for the games G_1 and G_2. Heuristically, in this setup, the goal of the two verifiers is to convince the super-verifier that the games G_1 and G_2 are equivalent.

Non-local game homomorphisms of a given type t are defined using no-signalling correlations of the same type, satisfying certain stronger conditions, which allow us to transport the perfect strategies of type t for the game G_1 to perfect strategies of the same type for the game G_2; the strategy transport thus achieved allows the comparison between the values of G_1 and G_2. In order to define non-local game isomorphisms of a given type, we employ a special kind of no-signalling correlations, defined in [10] and called therein bicorrelations, that allow reversibility. A game isomorphism of local type amounts to reshuffling the question-answer sets that transforms the rule predicates into each other. Much as in the case of quantum graph isomorphisms [4], the quantum identification of non-local games is a weaker equivalence relation, designed to take into account the possible presence of entanglement between the participating verifiers.

The construction described in the previous paragraphs is obtained as a special case of a more general setting hosting hypergraph homomorphisms (resp. isomorphisms); this is the second main motivation behind the present paper. The latter is achieved by employing the simulation paradigm in information theory [16], according to which, starting with a classical information channel from an alphabet X_1 to an alphabet A_1, using assistance with no-signalling resources over a quadruple (X_2, A_1, X_1, A_2), one can simulate an information channel from alphabet X_2 to alphabet A_2. Placing extra restrictions on the support of the input and the output channels, we define hypergraph homomorphism (resp. isomorphism) games. The perfect local strategies of the latter class of games correspond to a type of classical homomorphisms (resp. classical isomorphisms) of the underlying hypergraphs. Allowing non-classical resources, this leads to notions of quantum hypergraph homomorphisms (resp. isomorphisms).

We now describe the content of the paper in more detail. After collecting some general notation at the end of the present section, in Section 2 we recall the definition of the main no-signalling correlation types and introduce the simulation setup, showing that simulators can be composed with preservation of their types. In Section 3 we define an intermediate game, which we call the hypergraph quasi-homomorphism game, and show that quasi-homomorphism of a fixed type is a partial quasi-order on the set of all hypergraphs. We exhibit an example of hypergraphs that are quantum quasi-homomorphic but not locally quasi-homomorphic.

The hypergraph quasi-homomorphism game provides the base for defining, in Section 4, the hypergraph homomorphism game, which allows us to specify, for every correlation type t, a notion of t-homomorphic hypergraphs. In order to define t-isomorphic hypergraphs, we employ the notion of a classical bicorrelation of type t. The latter concept was introduced in [10] by specialising the notion of a quantum bicorrelation studied therein and using the correlation types defined in [55] based, in their own turn, on the setup of quantum no-signalling correlations of Duan and Winter [19]. Using the examples of the separation between quantum isomorphic and locally isomorphic graphs [4], we provide examples that separate quantum isomorphic hypergraphs from locally isomorphic ones. We further establish characterisations of hypergraph isomorphisms of quantum approximate, quantum commuting and no-signalling type in terms of states on operator system tensor products. The latter charaterisations are obtained as consequences of the characterisations of correlation types in [37] and [10].

Section 5 of the paper contains the definitions of the different types for a class of no-signalling correlations over a quadruple of the form $(X_2 \times Y_2) \times (A_1 \times B_1) \times (X_1 \times Y_1) \times (A_2 \times B_2)$, which we call strongly no-signalling, and collects some of their properties. In Section 6, we use the strongly no-signalling correlations as strategies for the homomorphism game between two given non-local games. This is achieved by applying the results of Section 4 in the case of hypergraphs that arise from non-local games. Theorem 6.1 and Theorem 6.4, in particular, establish the strategy transport for correlations and bicorrelations, respectively, allowing the comparisons of the values and the asymptotic values established in Theorem 6.12. We also observe that homomorphism (resp. isomorphism) between non-local games, of a given type, is a partial quasi-order (resp. an equivalence relation).

Section 7 is dedicated to the operator system representation of strongly no-signalling correlations. En route, we develop some basic multivariate tensor product theory in the operator system category, extending part of the work on bivariate operator systems tensor products in [34]. Our results can be seen as a continuation of the characterisations of general correlation types in [45, 37].

In Section 8, we restrict our attention to synchronous games, a class of non-local games first studied in [45] and having gained prominence through a number of recent developments (see e.g. [4, 20, 21, 22, 28, 29, 35, 44]). The usual synchronicity condition for a correlation [45] needs to be adapted for the case of the super-game under consideration; this leads to the definition of jointly synchronous correlations. The main result in this section is Theorem 8.3, which contains a tracial representation of jointly synchronous correlations, continuing the tracial characterisation thread from [45] and [35]. The strategy transport from Section 6 specialises in the synchronous case to transport of traces, leading to a necessary condition on the tracial state spaces of the game algebras (see [28]) whose games are quasi-homomorphic.

We point out that the work [5] establishes a simulation scheme between contextuality scenarios in the sense of [1], using the simulation paradigm in an identical way to the one utilised here. Noticing that contextuality scenarios have as their base ingredients some underlying hypergraphs, we see that [5, Definition 17] coincides with our definition of local hypergraph quasi-homomorphism. While the authors of [5] adopt a general categorical perspective and are mainly interested in the consequences for generalised probabilistic theories, our emphasis is placed on the hierarchy of the different concepts one obtains by varying the available correlation resources, and their operator theoretic characterisation. As a result, we make use of, and augment as necessary, the existing operator algebraic techniques in the area.

We use various concepts and results from operator space theory, and refer the reader to the monographs [43] and [48] for the general background.

In the remainder of this section we set notation, to be used throughout the paper. For a finite set X, we let $\mathbb{C}^X = \oplus_{x\in X}\mathbb{C}$ and write $(e_x)_{x\in X}$ for the canonical orthonormal basis of $\mathbb{C}^X$. We denote by M_X the algebra of all complex matrices over $X\times X$, and by $\mathcal{D}_X$ its subalgebra of all diagonal matrices. We write $\epsilon_{x,x'}$, $x,x'\in X$, for the canonical matrix units in M_X, denote by Tr the trace functional on M_X, and set $\langle S,T\rangle = \mathrm{Tr}(ST)$. We denote by $\mathcal{V}\otimes\mathcal{W}$ the algebraic tensor product of vector spaces $\mathcal{V}$ and $\mathcal{W}$, except when $\mathcal{V}$ and $\mathcal{W}$ are Hilbert spaces, in which case the notation is used for their Hilbertian tensor product. Given sets X_i, $i=1,\dots,n$, we abbreviate $X_1\cdots X_n = X_1\times\cdots\times X_n$, and write $M_{X_1\cdots X_n} = \otimes_{i=1}^n M_{X_i}$ and $\mathcal{D}_{X_1\cdots X_n} = \otimes_{i=1}^n \mathcal{D}_{X_i}$. We let $L_\omega : M_{X_1X_2}\to M_{X_1}$ be the slice map with respect to a given element $\omega\in M_{X_2}\equiv\mathcal{L}(\mathbb{C}^X)^*$; thus,

$$L_\omega(T_1\otimes T_2) = \langle\omega,T_2\rangle T_1,\quad T_i\in M_{X_i}, i=1,2.$$

The partial trace $\mathrm{Tr}_{X_2} : M_{X_1X_2}\to M_{X_1}$ is the slice map with respect to the identity operator I_{X_2} of M_{X_2}. Finally, for a Hilbert space H, we write $\mathcal{B}(H)$ for the C*-algebra of all bounded linear operators on H, and denote by I_H the identity operator on H.

2. General setup

A *hypergraph* is a subset $E\subseteq V\times W$, where V and W are finite sets. For $w\in W$, let $E(w)=\{v\in V : (v,w)\in E\}$. We refer to V as the set of *vertices* of E, and to $\{E(w) : w\in W\}$ as the set of its *edges*. The hypergraph E will be called *full* if for every $v\in V$ there exists $w\in W$ such that $(v,w)\in E$. The *dual* of the hypergraph $E\subseteq V\times W$ is the hypergraph

$$E^* := \{(w,v) : (v,w)\in E\}.$$

Let V and W be finite sets. A *(classical) information channel* from V to W is a positive trace preserving linear map $\mathcal{E} : \mathcal{D}_V\to\mathcal{D}_W$. We write

$V \overset{\mathcal{E}}{\mapsto} W$ and set

$$\mathcal{E}(w|v) = \langle \mathcal{E}(\epsilon_{v,v}), \epsilon_{w,w} \rangle, \quad v \in V, w \in W.$$

A channel $V \overset{\mathcal{E}}{\mapsto} W$ defines a hypergraph

$$E_{\mathcal{E}} = \{(v, w) \in V \times W : \mathcal{E}(w|v) > 0\}.$$

Given a hypergraph $E \subseteq V \times W$, we set

$$\mathcal{C}(E) = \left\{ V \overset{\mathcal{E}}{\mapsto} W \ : \ \text{a channel with } E_{\mathcal{E}} \subseteq E \right\};$$

if $\mathcal{E} \in \mathcal{C}(E)$, we say that $\mathcal{E}$ *fits* E. We note that if $\mathcal{C}(E) \neq \emptyset$ then E is full. The set $\mathcal{C}(V \times W)$ coincides with the set of all channels $V \overset{\mathcal{E}}{\mapsto} W$ and, if $E \subseteq V \times W$, then $\mathcal{C}(E)$ is a convex (with respect to the usual linear structure on the space of all linear maps from $\mathcal{D}_V$ to $\mathcal{D}_W$) subset of $\mathcal{C}(V \times W)$.

A channel $\mathcal{E} : \mathcal{D}_V \to \mathcal{D}_W$ is called *unital* if $\mathcal{E}(I_V) = I_W$. If $\mathcal{E}$ is unital then

$$|W| = \mathrm{Tr}(\mathcal{E}(I_V)) = \mathrm{Tr}(I_V) = |V|;$$

in this case it will be natural to assume that $V = W$. If $\mathcal{E} : \mathcal{D}_V \to \mathcal{D}_W$ is a unital channel then the map $\mathcal{E}^* : \mathcal{D}_W \to \mathcal{D}_V$, given by $\mathcal{E}^*(v|w) := \mathcal{E}(w|v)$ (or, equivalently,

$$\langle \mathcal{E}(S), T \rangle = \langle S, \mathcal{E}^*(T) \rangle, \quad S \in \mathcal{D}_V, T \in \mathcal{D}_W)$$

is also a channel.

Let V_i and W_i be finite sets, $i = 1, 2$. A *no-signalling (NS) correlation* on the quadruple (V_2, W_1, V_1, W_2) is an information channel $\Gamma : \mathcal{D}_{V_2 W_1} \to \mathcal{D}_{V_1 W_2}$ for which the *marginal channels*

$$\Gamma_{V_2 \to V_1} : \mathcal{D}_{V_2} \to \mathcal{D}_{V_1}, \quad \Gamma_{V_2 \to V_1}(v_1|v_2) := \sum_{w_2 \in W_2} \Gamma(v_1, w_2|v_2, w_1')$$

and

$$\Gamma^{W_1 \to W_2} : \mathcal{D}_{W_1} \to \mathcal{D}_{W_2}, \quad \Gamma^{W_1 \to W_2}(w_2|w_1) := \sum_{v_1 \in V_1} \Gamma(v_1, w_2|v_2', w_1)$$

are well-defined (independently of the choice of w_1' and v_2'). In the sequel, when there is no risk of confusion, the indicating sub/superscripts in the notation for the marginal channels will be dropped. We denote by $\mathcal{C}_{\rm ns}$ the collection of all NS correlations (the quadruple (V_2, W_1, V_1, W_2) will usually be understood from the context).

A *positive operator-valued measure (POVM)* is a (finite) family $(E_i)_{i=1}^k$ of positive operators acting on a Hilbert space H such that $\sum_{i=1}^k E_i = I$. An NS correlation $\Gamma : \mathcal{D}_{V_2 W_1} \to \mathcal{D}_{V_1 W_2}$ is called *quantum commuting* if there exists a Hilbert space H, a unit vector $\xi \in H$ and POVM's $(E_{v_2,v_1})_{v_1 \in V_1}$, $v_2 \in V_2$, and $(F_{w_1,w_2})_{w_2 \in W_2}$, $w_1 \in W_1$, such that

$$\Gamma(v_1, w_2|v_2, w_1) = \langle E_{v_2,v_1} F_{w_1,w_2} \xi, \xi \rangle, \quad v_i \in V_i, w_i \in W_i, i = 1, 2.$$

We call Γ *quantum* if it is quantum commuting and H can be chosen of the form $H = H_V \otimes H_W$ in such a way that H_V, H_W are finite dimensional, $E_{v_2,v_1} = E'_{v_2,v_1} \otimes I_W$ and $F_{w_1,w_2} = I_V \otimes F'_{w_1,w_2}$, $v_i \in V_i$, $w_i \in W_i$, $i = 1, 2$. The correlation Γ is *approximately quantum* if it is the limit of quantum correlations, and *local* if it is a convex combination of correlations of the form $\Gamma_V \otimes \Gamma_W$, where $\Gamma_V : \mathcal{D}_{V_2} \to \mathcal{D}_{V_1}$ and $\Gamma_W : \mathcal{D}_{W_1} \to \mathcal{D}_{W_2}$ are information channels. We refer the reader to [37, 45] for further details, and denote the subclasses of local, quantum, approximately quantum and quantum commuting NS correlations by $\mathcal{C}_{\rm loc}$, $\mathcal{C}_{\rm q}$, $\mathcal{C}_{\rm qa}$ and $\mathcal{C}_{\rm qc}$, respectively. We point out the inclusions

$$\mathcal{C}_{\rm loc} \subseteq \mathcal{C}_{\rm q} \subseteq \mathcal{C}_{\rm qa} \subseteq \mathcal{C}_{\rm qc} \subseteq \mathcal{C}_{\rm ns}, \tag{1}$$

all of them strict: $\mathcal{C}_{\rm loc} \neq \mathcal{C}_{\rm q}$ is the Bell Theorem [6], $\mathcal{C}_{\rm q} \neq \mathcal{C}_{\rm qa}$ is a negative answer to the weak Tsirelson Problem [53] (see also [22, 54]), and $\mathcal{C}_{\rm qa} \neq \mathcal{C}_{\rm qc}$ – in view of [26, 30, 42], a negative answer to the announced solution of the Connes Embedding Problem [29].

A *non-local game* on the quadruple (V_2, W_1, V_1, W_2) is a hypergraph $\Lambda \subseteq V_2W_1 \times V_1W_2$. For a non-local game Λ and ${\rm t} \in \{{\rm loc}, {\rm q}, {\rm qa}, {\rm qc}, {\rm ns}\}$, we write $\mathcal{C}_{\rm t}(\Lambda) = \mathcal{C}_{\rm t} \cap \mathcal{C}(\Lambda)$. The elements of $\mathcal{C}_{\rm t}(\Lambda)$ will be referred to as *perfect* t-*strategies* of the game Λ, and the set V_2W_1 (resp. V_1W_2) as the *question* (resp. *answer*) set for the two *players* of the game Λ. Note that non-local games are usually defined as a tuple $(V_2, W_1, V_1, W_2, \lambda)$, where $\lambda : V_2 \times W_1 \times V_1 \times W_2 \to \{0, 1\}$ is a function (referred to as a *rule function*); in the above definition, we have identified Λ with the support of λ.

We recall the *simulation paradigm* in information theory [16]. Given an NS correlation Γ on the quadruple (V_2, W_1, V_1, W_2) and a channel $V_1 \overset{\mathcal{E}}{\mapsto} W_1$, let $\Gamma[\mathcal{E}] : \mathcal{D}_{V_2} \to \mathcal{D}_{W_2}$ be the linear map, given by

$$\Gamma[\mathcal{E}](w_2|v_2) = \sum_{v_1 \in V_1} \sum_{w_1 \in W_1} \Gamma(v_1, w_2|v_2, w_1)\mathcal{E}(w_1|v_1). \tag{2}$$

It is straightforward to check that $\Gamma[\mathcal{E}]$ is a channel (see [16]) and that the map $\mathcal{E} \to \Gamma[\mathcal{E}]$ is an affine map from $\mathcal{C}(V_1 \times W_1)$ into $\mathcal{C}(V_2 \times W_2)$. We say that a channel $\mathcal{F} : \mathcal{D}_{V_2} \to \mathcal{D}_{W_2}$ is *simulated by* $\mathcal{E}$ *with the assistance of* Γ [16] if $\mathcal{F} = \Gamma[\mathcal{E}]$, we call Γ a *simulator*, and we write $(V_1 \mapsto W_1) \overset{\Gamma}{\to} (V_2 \mapsto W_2)$. The simulation procedure is illustrated by the diagram:

$$\begin{tikzcd} V_1 \arrow[r, "\mathcal{E}"] \arrow[d, leftarrow, dashed] & W_1 \arrow[d, dashed] \\ V_2 \arrow[r, "\Gamma[\mathcal{E}]"] & W_2. \end{tikzcd}$$

Suppose that $(V_1 \mapsto W_1) \overset{\Gamma_1}{\to} (V_2 \mapsto W_2)$ and $(V_2 \mapsto W_2) \overset{\Gamma_2}{\to} (V_3 \mapsto W_3)$, and define the linear map $\Gamma_2 * \Gamma_1 : \mathcal{D}_{V_3W_1} \to \mathcal{D}_{V_1W_3}$ by letting

$$(\Gamma_2 * \Gamma_1)(v_1, w_3|v_3, w_1) = \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_1(v_1, w_2|v_2, w_1)\Gamma_2(v_2, w_3|v_3, w_2).$$

Theorem 2.1. *If* $(V_1 \mapsto W_1) \stackrel{\Gamma_1}{\to} (V_2 \mapsto W_2)$ *and* $(V_2 \mapsto W_2) \stackrel{\Gamma_2}{\to} (V_3 \mapsto W_3)$ *then*

$$(V_1 \mapsto W_1) \stackrel{\Gamma_2 * \Gamma_1}{\longrightarrow} (V_3 \mapsto W_3).$$

Moreover,

(i) *if* $\mathrm{t} \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$ *and* $\Gamma_i \in \mathcal{C}_{\mathrm{t}}$, $i = 1, 2$, *then* $\Gamma_2 * \Gamma_1 \in \mathcal{C}_{\mathrm{t}}$*;*
(ii) *if* $\mathcal{E} \in \mathcal{C}(V_1 \times W_1)$ *then* $(\Gamma_2 * \Gamma_1)[\mathcal{E}] = \Gamma_2[\Gamma_1[\mathcal{E}]]$.

Proof. It is clear that the map $\Gamma_2 * \Gamma_1 : \mathcal{D}_{V_3 W_1} \to \mathcal{D}_{V_1 W_3}$ is positive. Fix $w_1 \in W_1$. Then

$$\begin{aligned} & \sum_{w_3 \in W_3} (\Gamma_2 * \Gamma_1)(v_1, w_3 | v_3, w_1) \\ = & \sum_{w_3 \in W_3} \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_1(v_1, w_2 | v_2, w_1) \Gamma_2(v_2, w_3 | v_3, w_2) \\ = & \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_1(v_1, w_2 | v_2, w_1) \Gamma_2(v_2 | v_3) = \sum_{v_2 \in V_2} \Gamma_1(v_1 | v_2) \Gamma_2(v_2 | v_3). \end{aligned}$$

Therefore, if $v_3 \in V_3$ then

$$\begin{aligned} \sum_{v_1 \in V_1} \sum_{w_3 \in W_3} (\Gamma_2 * \Gamma_1)(v_1, w_3 | v_3, w_1) & = \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} \Gamma_1(v_1 | v_2) \Gamma_2(v_2 | v_3) \\ & = \sum_{v_2 \in V_2} \Gamma_2(v_2 | v_3) = 1. \end{aligned}$$

It follows that $\Gamma_2 * \Gamma_1$ is trace preserving and the marginal channel $(\Gamma_2 * \Gamma_1)_{V_3 \to V_1}$ is well-defined. By symmetry, so is the marginal channel $(\Gamma_2 * \Gamma_1)^{W_1 \to W_3}$.

(i) Assume that $\Gamma_i \in \mathcal{C}_{\mathrm{qc}}$, $i = 1, 2$. Let H (resp. H') be a Hilbert space, $\xi \in H$ (resp. $\xi' \in H'$) a unit vector, and $(E_{v_2, v_1})_{v_1 \in V_1}$ and $(F_{w_1, w_2})_{w_2 \in W_2}$ (resp. $(E'_{v_3, v_2})_{v_2 \in V_2}$ and $(F'_{w_2, w_3})_{w_3 \in W_3}$) be families of POVM's such that $E_{v_2, v_1} F_{w_1, w_2} = F_{w_1, w_2} E_{v_2, v_1}$ (resp. $E'_{v_3, v_2} F'_{w_2, w_3} = F'_{w_2, w_3} E'_{v_3, v_2}$) for all $v_i \in V_i$, $w_i \in W_i$, $i = 1, 2, 3$, and

$$\Gamma_1(v_1, w_2 | v_2, w_1) = \langle E_{v_2, v_1} F_{w_1, w_2} \xi, \xi \rangle, \ \Gamma_2(v_2, w_3 | v_3, w_2) = \langle E'_{v_3, v_2} F'_{w_2, w_3} \xi', \xi' \rangle,$$

for all $v_i \in V_i, w_i \in W_i$, $i = 1, 2, 3$. Let $H'' = H \otimes H', \xi'' = \xi \otimes \xi'$, and

$$E''_{v_3, v_1} = \sum_{v_2 \in V_2} E_{v_2, v_1} \otimes E'_{v_3, v_2} \quad \text{and} \quad F''_{w_1, w_3} = \sum_{w_2 \in W_2} F_{w_1, w_2} \otimes F'_{w_2, w_3}.$$

It is clear that $E''_{v_3, v_1} F''_{w_1, w_3} = F''_{w_1, w_3} E''_{v_3, v_1}$ for all $v_i \in V_i$, $w_i \in W_i$, $i = 1, 3$. In addition, if $v_3 \in V_3$ then

$$\sum_{v_1 \in V_1} E''_{v_3, v_1} = \sum_{v_1 \in V_1} \sum_{v_2 \in V_2} E_{v_2, v_1} \otimes E'_{v_3, v_2} = \sum_{v_2 \in V_2} I \otimes E'_{v_3, v_2} = I.$$

Thus, $(E''_{v_3,v_1})_{v_1}$ is a POVM for every $v_3 \in V_3$. Similarly, $(F''_{w_1,w_3})_{w_3}$ is a POVM for every $w_1 \in W_1$. Finally, we see

$$\begin{aligned}
& \langle E''_{v_3,v_1} F''_{w_1,w_3} \xi'', \xi'' \rangle \\
= & \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \langle (E_{v_2,v_1} \otimes E'_{v_3,v_2})(F_{w_1,w_2} \otimes F'_{w_2,w_3})(\xi \otimes \xi'), (\xi \otimes \xi') \rangle \\
= & \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \langle E_{v_2,v_1} F_{w_1,w_2} \xi, \xi \rangle \langle E'_{v_3,v_2} F'_{w_2,w_3} \xi', \xi' \rangle \\
= & \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_1(v_1, w_2 | v_2, w_1) \Gamma_2(v_2, w_3 | v_3, w_2) = (\Gamma_2 * \Gamma_1)(v_1, w_3 | v_3, w_1).
\end{aligned}$$

This implies that $\Gamma_2 * \Gamma_1 \in \mathcal{C}_{\mathrm{qc}}$. The cases where $\mathrm{t} = \mathrm{q}$ and $\mathrm{t} = \mathrm{loc}$ are similar. Finally, the case $\mathrm{t} = \mathrm{qa}$ follows from the case $\mathrm{t} = \mathrm{q}$ and the continuity of the operation $(\Gamma_1, \Gamma_2) \to \Gamma_2 * \Gamma_1$.

(ii) We have

$$\begin{aligned}
& (\Gamma_2 * \Gamma_1)[\mathcal{E}](w_3 | v_3) \\
= & \sum_{v_1 \in V_1} \sum_{w_1 \in W_1} (\Gamma_2 * \Gamma_1)(v_1, w_3 | v_3, w_1) \mathcal{E}(w_1 | v_1) \\
= & \sum_{v_1 \in V_1} \sum_{w_1 \in W_1} \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_1(v_1, w_2 | v_2, w_1) \Gamma_2(v_2, w_3 | v_3, w_2) \mathcal{E}(w_1 | v_1) \\
= & \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_2(v_2, w_3 | v_3, w_2) \Gamma_1[\mathcal{E}](w_2 | v_2) = \Gamma_2[\Gamma_1[\mathcal{E}]](w_3 | v_3).
\end{aligned}$$

$\square$

3. Quasi-homomorphism games

In this section, we study an auxiliary notion, which will be specialised to the main cases of interest in Section 4. Let $E_1 \subseteq V_1 \times W_1$ and $E_2 \subseteq V_2 \times W_2$. We write

$$E_1 \rightsquigarrow E_2 \;=\; \{(v_2, w_1, v_1, w_2) : (v_1, w_1) \in E_1 \Rightarrow (v_2, w_2) \in E_2\};$$

thus,

$$E_1 \rightsquigarrow E_2 = \{(v_2, w_1, v_1, w_2) : (v_1, w_1) \in E_1^c \text{ or } (v_1, w_1, v_2, w_2) \in E_1 \times E_2\}.$$

We consider $E_1 \rightsquigarrow E_2$ as a hypergraph in $V_2 W_1 \times V_1 W_2$, and hence as a game with question and answer sets $V_2 \times W_1$ and $V_1 \times W_2$, respectively.

Definition 3.1. *Let $E_i \subseteq V_i \times W_i$ be a hypergraph, $i = 1, 2$, and $\mathrm{t} \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$. We say that E_1 is* t-quasi-homomorphic *to E_2 (denoted $E_1 \rightsquigarrow_{\mathrm{t}} E_2$) if $\mathcal{C}_{\mathrm{t}}(E_1 \rightsquigarrow E_2) \neq \emptyset$.*

If $\Gamma \in \mathcal{C}_{\mathrm{t}}(E_1 \rightsquigarrow E_2)$, we say that $E_1 \rightsquigarrow_{\mathrm{t}} E_2$ *via* Γ.

Proposition 3.2. *Let $E_i \subseteq V_i \times W_i$ be a full hypergraph, $i = 1, 2$, and Γ be an NS correlation over (V_2, W_1, V_1, W_2). Then $E_1 \rightsquigarrow_{\mathrm{ns}} E_2$ via Γ if and only if $\mathcal{E} \to \Gamma[\mathcal{E}]$ restricts to a well-defined affine map from $\mathcal{C}(E_1)$ into $\mathcal{C}(E_2)$.*

Proof. Assume that $E_1 \rightsquigarrow_{\rm ns} E_2$ via Γ and let $\mathcal{E} \in \mathcal{C}(E_1)$. Suppose that $(v_2, w_2) \notin E_2$. If $\mathcal{E}(w_1|v_1) \neq 0$ then $(v_1, w_1) \in E_1$ and hence, since Γ fits $E_1 \rightsquigarrow E_2$, we have that $\Gamma(v_1, w_2|v_2, w_1) = 0$. It follows that $\Gamma[\mathcal{E}](w_2|v_2) = 0$; thus, $E_{\Gamma[\mathcal{E}]} \subseteq E_2$.

Conversely, assume that $\mathcal{E} \to \Gamma[\mathcal{E}]$ restricts to a well-defined map from $\mathcal{C}(E_1)$ into $\mathcal{C}(E_2)$. Suppose that $(v_1, w_1) \in E_1$ and $(v_2, w_2) \notin E_2$. Let $\mathcal{E} : \mathcal{D}_{V_1} \to \mathcal{D}_{W_1}$ be any channel that fits E_1 such that $\mathcal{E}(w_1|v_1) = 1$ (note that the existence of such a channel is guaranteed by the fact that E_1 is full). We have that

$$\Gamma[\mathcal{E}](w_2|v_2) = \Gamma(v_1, w_2|v_2, w_1) + \sum_{(v_1', w_1') \neq (v_1, w_1)} \Gamma(v_1', w_2|v_2, w_1')\mathcal{E}(w_1'|v_1'). \tag{3}$$

Since $(v_2, w_2) \notin E_2$, we have that $\Gamma[\mathcal{E}](w_2|v_2) = 0$. Now (3) implies that $\Gamma(v_1, w_2|v_2, w_1) = 0$; thus, Γ fits $E_1 \rightsquigarrow E_2$, that is, $E_1 \rightsquigarrow_{\rm ns} E_2$ via Γ. $\square$

Theorem 3.3. *Let* $\mathrm{t} \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$. *The relation* $\rightsquigarrow_{\rm t}$ *is a quasi-order on the set of all hypergraphs.*

Proof. Fix $\mathrm{t} \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$. Reflexivity follows from the fact that the identity channel is a local correlation. Suppose that E_1, E_2 and E_3 are hypergraphs such that $E_1 \rightsquigarrow_{\rm t} E_2$ via Γ_1 and $E_2 \rightsquigarrow_{\rm t} E_3$ via Γ_2. By Theorem 2.1, $\Gamma_2 * \Gamma_1 \in \mathcal{C}_{\rm t}$. It suffices to show that $\Gamma_2 * \Gamma_1$ fits $E_1 \to E_3$. Suppose that $(v_1, w_1) \in E_1$ and $(\Gamma_2 * \Gamma_1)(v_1, w_3|v_3, w_1) \neq 0$. Then there exists $(v_2, w_2) \in V_2 \times W_2$ such that

$$\Gamma_1(v_1, w_2|v_2, w_1) \neq 0 \quad \text{and} \quad \Gamma_2(v_2, w_3|v_3, w_2) \neq 0.$$

Since Γ_1 fits $E_1 \rightsquigarrow E_2$, we have that $(v_2, w_2) \in E_2$, and since Γ_2 fits $E_2 \rightsquigarrow E_3$, we conclude that $(v_3, w_3) \in E_3$. The proof is complete. $\square$

Let $E_1 \subseteq V_1 \times W_1$ and $E_2 \subseteq V_2 \times W_2$ be hypergraphs. A map $f : V_2 \to V_1$ will be called a *quasi-homomorphism* from E_1 into E_2 if $f^{-1}(\alpha)$ is contained in an edge of E_2 for every edge α of E_1. A quasi-homomorphism $f : V_2 \to V_1$ gives rise to an accompanying map $g : W_1 \to W_2$ such that

$$f^{-1}(E_1(w_1)) \subseteq E_2(g(w_1)) \quad \text{for every } w_1 \in W_1; \tag{4}$$

conversely, if the maps $f : V_2 \to V_1$ and $g : W_1 \to W_2$ satisfy (4), then f is a quasi-homomorphism. If there exist a quasi-homomorphism from E_1 into E_2, we say that E_1 is *quasi-homomorphic* to E_2.

Proposition 3.4. *Let* $E_1 \subseteq V_1 \times W_1$ *and* $E_2 \subseteq V_2 \times W_2$ *be hypergraphs. Then* $E_1 \rightsquigarrow_{\rm loc} E_2$ *if and only* E_1 *is quasi-homomorphic to* E_2.

Proof. Assume that Γ is a local correlation that fits $E_1 \rightsquigarrow E_2$. We may assume that Γ is an extreme point in $\mathcal{C}_{\rm loc}$ and hence, by no-signalling, there exist functions $f : V_2 \to V_1$ and $g : W_1 \to W_2$ such that

$$\{(v_2, w_1, f(v_2), g(w_1)) : w_1 \in W_1, v_2 \in V_2\} \subseteq E_1 \rightsquigarrow E_2.$$

This implies

$$f(v_2) \in E_1(w_1) \implies v_2 \in E_2(g(w_1)) \quad \text{for all } w_1 \in W_1, \tag{5}$$

which means that (f, g) determines a quasi-homomorphism from E_1 to E_2.

Conversely, suppose that (5) is satisfied. Let $\Phi : \mathcal{D}_{V_2} \to \mathcal{D}_{V_1}$ (resp. $\Psi : \mathcal{D}_{W_1} \to \mathcal{D}_{W_2}$) be the channel given by $\Phi(v_1|v_2) = \delta_{v_1,f(v_2)}$ (resp. $\Psi(w_2|w_1) = \delta_{w_2,g(w_1)}$) and $\Gamma = \Phi \otimes \Psi$; then Γ fits $E_1 \rightsquigarrow E_2$. Indeed, assume that $(v_1, w_1) \in E_1$ and $(v_2, w_2) \notin E_2$, but

$$\delta_{v_1,f(v_2)}\delta_{w_2,g(w_1)} = \Gamma(v_1, w_2|v_2, w_1) \neq 0.$$

This means that $f(v_2) = v_1$ and hence $f(v_2) \in E_1(w_1)$, implying $v_2 \in E_2(g(w_1))$. Since $g(w_1) = w_2$, we have $(v_2, w_2) \in E_2$, a contradiction. □

It is clear that, if t and t' are correlation types such that $\mathcal{C}_{\mathrm{t}} \subseteq \mathcal{C}_{\mathrm{t}'}$, then

$$E_1 \rightsquigarrow_{\mathrm{t}} E_2 \implies E_1 \rightsquigarrow_{\mathrm{t}'} E_2.$$

We next show the irreversibility of the latter implication for some of the hypergraph quasi-homomorphism types. If G is a simple graph with vertex set X, we write $x \sim_G x'$ if $\{x, x'\}$ is an edge of G, and $x \simeq_G x'$ if $x \sim x'$ or $x = x'$ (if G is understood from the context, we write $x \sim x'$ and $x \simeq x'$, respectively). We let $\alpha(G)$ be the *independence number* of G, defined as the maximum cardinality $|S|$ of an independent set S of vertices (that is, a subset $S \subseteq X$ such that $x, x' \in S \Rightarrow x \not\sim x'$). We fix a hypergraph $E \subseteq X \times Y$, and write G_E for the corresponding *confusability graph*: the vertex set of G_E is X and the adjacency is given by letting

$$x \sim x' \text{ if } x \neq x' \text{ and } \exists\, y \in Y \text{ s.t. } (x, y) \in E \text{ and } (x', y) \in E.$$

If $X \overset{\mathcal{E}}{\mapsto} Y$, the confusability graph $G_{\mathcal{E}}$ of $\mathcal{E}$ [52] is defined by letting $G_{\mathcal{E}} = G_{E_{\mathcal{E}}}$. For a given set Z, let

$$\Delta_Z = \{(z, z) : z \in Z\}$$

be the diagonal over Z, considered as a hypergraph in $Z \times Z$.

Lemma 3.5. *Let $E \subseteq X \times Y$ be a full hypergraph. We have that $E \rightsquigarrow_{\mathrm{loc}} \Delta_Z$ if and only if $\alpha(G_E) \geq |Z|$.*

Proof. Suppose that $E \rightsquigarrow_{\mathrm{loc}} \Delta_Z$ and, using Proposition 3.4, let $f : Z \to X$ and $g : Y \to Z$ be maps realising a quasi-homomorphism from E into Δ_Z. This means

$$(f(z), y) \in E \implies z = g(y). \tag{6}$$

Let $z, z' \in Z$ with $f(z) \simeq f(z')$ in G_E. Since E is full, there exists $y \in Y$ such that $(f(z), y) \in E$ and $(f(z'), y) \in E$. By (6), $g(y) = z = z'$. Thus,

$$z \neq z' \Rightarrow f(z) \not\simeq_{G_E} f(z'); \tag{7}$$

it follows that $f(Z)$ is an independent set in G_E and hence, since f is injective by (7), $\alpha(G_E) \geq |Z|$.

Conversely, let $f : Z \to X$ be an injective map such that $f(Z)$ is an independent set in G_E. Fix $z_0 \in Z$. Let $y \in Y$; then $E(y)$ is a clique in G_E; thus, $|f(Z) \cap E(y)| \leq 1$. If $f(Z) \cap E(y) = \{f(z)\}$ for some $z \in Z$, define $g(y) = z$; if $f(Z) \cap E(y) = \emptyset$, define $g(y) = z_0$. It is straightforward that the pair (f, g) of maps realises a quasi-homomorphism from E into Δ_Z. $\square$

Proposition 3.6. *The implications*

$$E \rightsquigarrow_{\rm loc} F \;\Rightarrow\; E \rightsquigarrow_{\rm q} F \quad \text{and} \quad E \rightsquigarrow_{\rm q} F \;\Rightarrow\; E \rightsquigarrow_{\rm ns} F$$

are not reversible.

Proof. Recall [16, 18] that the quantum independence number $\alpha_{\rm q}(G)$ of a graph G with vertex set X is defined by letting

$$\alpha_{\rm q}(G) = \max\{k \in \mathbb{N} : \exists\ X \overset{\mathcal{E}}{\mapsto} Y \text{ with } G_{\mathcal{E}} = G \text{ and } \Gamma \in \mathcal{C}_{\rm q} \text{ s.t. } \Gamma[\mathcal{E}] = {\rm id}_{\mathcal{D}_k}\}.$$

By [16, Theorem 13], there exists a graph G such that $\alpha(G) < \alpha_{\rm q}(G)$. Let Y be a (finite) set and $\mathcal{E} : \mathcal{D}_X \to \mathcal{D}_Y$ be a channel that achieves the maximum in the definition of $\alpha_{\rm q}(G)$. Let $E = E_{\mathcal{E}}$; since E is the hypergraph of a channel, E is full. Letting Z be a set with $|Z| = \alpha_{\rm q}(G)$, we have that $E \rightsquigarrow_{\rm q} \Delta_Z$. On the other hand, by Lemma 3.5, $E \not\rightsquigarrow_{\rm loc} \Delta_Z$.

To show that the second implication fails in general, use [16, Theorem 7], according to which, if $E \subseteq X \times Y$ is a hypergraph then

$$\max\{Z : E \rightsquigarrow_{\rm ns} \Delta_Z\} = \lfloor \alpha^*(E) \rfloor,$$

where $\alpha^*(E)$ is the fractional packing number of E (see [16, Definition 5]). By the first paragraph, it suffices to exhibit an example of a hypergraph E such that $\lfloor \alpha^*(E) \rfloor > \alpha_{\rm q}(G_E)$. Let $\vartheta(G)$ (resp. $\bar{\chi}_{\rm f}(G)$) be the the Lovász number [36] of a graph G (resp. the fractional chromatic number of the complement of G). By [18] and the discussion surrounding [16, Proposition 6], we have the inequalities

$$\alpha_{\rm q}(G_E) \leq \vartheta(G_E) \leq \bar{\chi}_{\rm f}(G_E) \leq \alpha^*(E).$$

It hence suffices to exhibit an example of a graph G with $\lceil \vartheta(G) \rceil < \lfloor \bar{\chi}_{\rm f}(G) \rfloor$. Let $n, r \in \mathbb{N}$ with $r \leq n$ and let $K(n, r)$ be the graph whose vertices are the subsets of $[n]$ of cardinality r, with two such subsets S and T being adjacent if $S \cap T = \emptyset$. By (the proof of) [36, Theorem 13], $\vartheta(K(n, r)) = \binom{n-1}{r-1}$ while, as stated after [36, Corollary 7], $\bar{\chi}_{\rm f}(K(n, r)) = \binom{n}{r} / \lfloor \frac{n}{r} \rfloor$. Thus, an example is furnished by letting, e.g., $n = 5$ and $r = 3$. $\square$

Remark. We do not have counterexamples that show the irreversibility of the implications $E_1 \rightsquigarrow_{\rm q} E_2 \;\Rightarrow\; E_1 \rightsquigarrow_{\rm qa} E_2$ or $E_1 \rightsquigarrow_{\rm qa} E_2 \;\Rightarrow\; E_1 \rightsquigarrow_{\rm qc} E_2$. Such counterexamples would provide an alternative way to observe the inequalities $\mathcal{C}_{\rm q} \neq \mathcal{C}_{\rm qa}$ and $\mathcal{C}_{\rm qa} \neq \mathcal{C}_{\rm qc}$, respectively, and would thus be of substantial interest.

In the rest of the section, we link hypergraph quasi-homomorphisms to tensor products of operator systems. Recall that an *operator system* is a selfadjoint subspace of $\mathcal{B}(H)$, for some Hilbert space H, containing I_H. If

$\mathcal{S}$ is an operator system, we write $M_n(\mathcal{S})^+$ for the cone of positive elements in the space $M_n(\mathcal{S})$ of all n by n matrices with entries in $\mathcal{S}$. If $\mathcal{S}$ and $\mathcal{T}$ are operator systems and $\phi : \mathcal{S} \to \mathcal{T}$ is a linear map, we let $\phi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ be the map, given by $\phi^{(n)}((x_{i,j})_{i,j}) = (\phi(x_{i,j}))_{i,j}$. The map ϕ is called *positive* if $\phi(\mathcal{S}^+) \subseteq \mathcal{T}^+$, *completely positive* if $\phi^{(n)}$ is positive for every $n \in \mathbb{N}$, and *unital* if $\phi(1) = 1$. We call $\mathcal{S}$ and $\mathcal{T}$ *completely order isomorphic*, and write $\mathcal{S} \cong_{\text{c.o.i.}} \mathcal{T}$, if there exists a unital completely positive bijection $\phi : \mathcal{S} \to \mathcal{T}$ with completely positive inverse. We write $\mathcal{S} \subseteq_{\text{c.o.i.}} \mathcal{T}$ if $\mathcal{S} \subseteq \mathcal{T}$ and the inclusion map $\mathcal{S} \to \mathcal{T}$ is a complete order isomorphism onto its range. We note that if $\mathcal{S}$ is a finite dimensional operator system, the Banach space dual $\mathcal{S}^{\rm d}$ can be viewed, via Choi-Effros Theorem [43, Theorem 13.1], as an operator system [13, Corollary 4.5].

We refer to [43] for further details about operator systems, and recall here the three types operator system tensor products [34] of operator systems $\mathcal{S}$ and $\mathcal{T}$ that will be used in the sequel:

(i) the *minimal* operator system tensor product $\mathcal{S} \otimes_{\min} \mathcal{T}$ arises from viewing $\mathcal{S} \otimes \mathcal{T}$ as a subspace of $\mathcal{B}(H \otimes K)$, where $\mathcal{S}$ and $\mathcal{T}$ are realised as operator systems in $\mathcal{B}(H)$ and $\mathcal{B}(K)$, respectively (and H and K are Hilbert spaces);
(ii) the *commuting* tensor product $\mathcal{S} \otimes_{\rm c} \mathcal{T}$ has the smallest family of matricial cones that makes the maps $\phi \cdot \psi$, where $\phi : \mathcal{S} \to \mathcal{B}(H)$ and $\psi : \mathcal{T} \to \mathcal{B}(H)$ and completely positive maps with commuting ranges, completely positive; here, $\phi \cdot \psi$ is the linear map, given by $(\phi \cdot \psi)(x \otimes y) = \phi(x)\psi(y)$, $x \in \mathcal{S}$, $y \in \mathcal{T}$;
(iii) the *maximal* tensor product $\mathcal{S} \otimes_{\max} \mathcal{T}$ has matricial cones generated by the elementary tensors of the form $S \otimes T$, where $S \in M_n(\mathcal{S})^+$ and $T \in M_m(\mathcal{T})^+$, $n, m \in \mathbb{N}$.

For finite sets X and A, let $\mathcal{A}_{X,A} = \underbrace{\mathcal{D}_A *_1 \cdots *_1 \mathcal{D}_A}_{|X| \text{ times}}$, a C*-algebra free product, amalgamated over the units. Let $(\tilde{e}_{x,a})_{a \in A}$ be the standard basis of the x-th copy of $\mathcal{D}_A$ in $\mathcal{A}_{X,A}$, and

$$\mathcal{S}_{X,A} = \text{span}\{\tilde{e}_{x,a} : x \in X, a \in A\},$$

viewed as an operator subsystem of $\mathcal{A}_{X,A}$. As is readily seen, the operator system $\mathcal{S}_{X,A}$ satisfies the following universal property: for every family $\{(E_{x,a})_{a \in A} : x \in X\}$ of POVM's, acting on a Hilbert space H, there exists a unital completely positive map $\phi : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ such that $\phi(\tilde{e}_{x,a}) = E_{x,a}$, $x \in X$, $a \in A$; conversely, if $\phi : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ is a unital completely positive map for some Hilbert space H, then $(\phi(\tilde{e}_{x,a}))_{a \in A}$ is a POVM, $x \in X$.

Set

$$\mathcal{R}_{X,A} = \left\{ (\lambda_{x,a})_{x \in X, a \in A} : \lambda_{x,a} \in \mathbb{C}, \ \sum_{a \in A} \lambda_{x,a} = \sum_{a \in A} \lambda_{x',a}, \ x, x' \in X \right\},$$

viewed as an operator subsystem of $\mathcal{D}_{XA}$. By [24, Theorem 5.9],

$$\mathcal{S}^{\mathrm{d}}_{X,A} \cong_{\text{c.o.i.}} \mathcal{R}_{X,A}. \tag{8}$$

Remark 3.7. By [37, Theorem 3.1], Γ is a no-signalling correlation over the quadruple (V_2, W_1, V_1, W_2) if and only if there exists a state $s : \mathcal{S}_{V_2,V_1} \otimes_{\max} \mathcal{S}_{W_1,W_2} \to \mathbb{C}$ such that

$$\Gamma(v_1, w_2|v_2, w_1) = s\left(\tilde{e}_{v_2,v_1} \otimes \tilde{e}_{w_1,w_2}\right), \quad v_i \in V_i, w_i \in W_i,\ i = 1, 2.$$

By (8) and [25, Proposition 1.9],

$$(\mathcal{S}_{V_2,V_1} \otimes_{\max} \mathcal{S}_{W_1,W_2})^{\mathrm{d}} \cong_{\text{c.o.i.}} \mathcal{R}_{V_2,V_1} \otimes_{\min} \mathcal{R}_{W_1,W_2};$$

thus, the simulators $(V_1 \mapsto W_1) \xrightarrow{\Gamma} (V_2 \mapsto W_2)$ correspond canonically to the elements of the subset

$$\left\{\Lambda \in (\mathcal{R}_{V_2,V_1} \otimes_{\min} \mathcal{R}_{W_1,W_2})^+ : (\mathrm{Tr}_{V_1} \otimes \mathrm{Tr}_{W_2})(\Lambda) = 1\right\}.$$

Hence we have the following are equivalent for hypergraphs $E_1 \subseteq V_1 \times W_1$ and $E_2 \subseteq V_2 \times W_2$:

(i) the relation $E_1 \rightsquigarrow_{\mathrm{ns}} E_2$ holds true;
(ii) there exists a matrix $\Lambda \in (\mathcal{R}_{V_2,V_1} \otimes_{\min} \mathcal{R}_{W_1,W_2})^+$ supported on the set $E_1 \rightsquigarrow E_2$.

Remark 3.8. In Proposition 3.2, we saw that a simulator Γ that fits $E_1 \rightsquigarrow E_2$ induces an affine map from $\mathcal{C}(E_1)$ to $\mathcal{C}(E_2)$. We point out that not all such affine maps arise via simulation. We identify the set $\mathcal{C}(V \times W)$ of all information channels $V \mapsto W$ with the subset

$$\{\Lambda \in \mathcal{R}_{V,W} : \mathrm{Tr}_W(\Lambda) = 1\} \tag{9}$$

of the operator system $\mathcal{R}_{V,W}$. Let $\Phi : \mathcal{C}(V_1 \times W_1) \to \mathcal{C}(V_2 \times W_2)$ be an affine map, and extend it linearly to a map (denoted in the same way) $\Phi : \mathcal{R}_{V_1,W_1} \to \mathcal{R}_{V_2,W_2}$. By [43, Theorem 3.9], Φ is completely positive. By [13, Lemma 4.6], Φ corresponds in a canonical fashion to an element $\varphi \in \mathcal{R}_{V_1,W_1} \otimes_{\min} \mathcal{R}^{\mathrm{d}}_{V_2,W_2}$; by (8), we can view φ as an element of $\mathcal{R}_{V_1,W_1} \otimes_{\min} \mathcal{S}_{V_2,W_2}$. Reversing these steps, we see that every element $\varphi \in \mathcal{R}_{V_1,W_1} \otimes_{\min} \mathcal{S}_{V_2,W_2}$ gives rise in a canonical fashion to an affine map $\Phi : \mathcal{C}(V_1 \times W_1) \to \mathcal{C}(V_2 \times W_2)$.

On the other hand, suppose that the map $\Phi : \mathcal{C}(V_1 \times W_1) \to \mathcal{C}(V_2 \times W_2)$ has the form $\Phi(\cdot) = \Gamma[\cdot]$ for some simulator Γ. According to Remark 3.7, Γ can be canonically identified with an element γ of $\mathcal{R}_{V_2,V_1} \otimes_{\min} \mathcal{R}_{W_1,W_2}$. Matrix multiplication

$$\mathfrak{m} : \mathcal{R}_{V_2,V_1} \otimes_{\min} \mathcal{R}_{V_1,W_1} \otimes_{\min} \mathcal{R}_{W_1,W_2} \to \mathcal{R}_{V_2,W_2}$$

is completely positive and can hence be viewed, via [13, Lemma 4.6], as a positive element

$$\tilde{\mathfrak{m}} \in \mathcal{R}_{V_2,V_1} \otimes_{\min} \mathcal{R}_{V_1,W_1} \otimes_{\min} \mathcal{R}_{W_1,W_2} \otimes_{\min} \mathcal{R}^{\mathrm{d}}_{V_2,W_2}$$

which, taking into account (8), induces a completely positive map

$$\hat{\mathfrak{m}} : \mathcal{R}_{V_2,V_1} \otimes_{\min} \mathcal{R}_{W_1,W_2} \to \mathcal{S}_{V_1,W_1} \otimes_{\max} \mathcal{R}_{V_2,W_2}.$$

The simulator Γ can thus be identified with the element $\hat{\mathfrak{m}}(\gamma)$. The difference between all affine maps $\mathcal{C}(V_1 \times W_1) \to \mathcal{C}(V_2 \times W_2)$ and the simulators can now be visualised as the difference between the operator systems $\mathcal{R}_{V_1,W_1} \otimes_{\min} \mathcal{S}_{V_2,W_2}$ and $\mathcal{S}_{V_1,W_1} \otimes_{\max} \mathcal{R}_{V_2,W_2}$.

To be more specific, fix $w_1^0 \in W_1$ and let $\mathcal{E} \in \mathcal{C}(V_1 \times W_1)$ be the channel, given by

$$\mathcal{E}(w_1|v_1) = \delta_{w_1,w_1^0}, \quad v_1 \in V_1, w_1 \in W_1.$$

If $(V_1 \mapsto W_1) \xrightarrow{\Gamma} (V_2 \mapsto W_2)$ then

$$\Gamma[\mathcal{E}](w_2|v_2) = \sum_{v_1 \in V_1} \Gamma(v_1, w_2|v_2, w_1^0) = \Gamma(w_2|w_1^0).$$

We see that the probability distribution $\Gamma[E](\cdot|v_2)$ is independent of the variable v_2 and of the choice of w_1^0, a property not enjoyed by arbitrary affine maps from $\mathcal{C}(V_1 \times W_1)$ to $\mathcal{C}(V_2 \times W_2)$.

4. Homomorphism games

In this section, we adapt the set-up from Section 3 to define quantum versions of hypergraph homomorphisms and hypergraph isomorphisms.

4.1. **Bicorrelations.** We recall the definitions of a bicorrelation and of the various bicorrelation types introduced in [10], which will be needed in the sequel. Suppose that $V_1 = V_2 =: V$ and $W_1 = W_2 =: W$. We call an element $\Gamma \in \mathcal{C}_{\rm ns}$ a *no-signalling (NS) bicorrelation* [10] if Γ is a unital channel and its dual Γ^* is a no-signalling correlation.

Recall [17] that, if H is a Hilbert space, a *quantum magic square* over V on H is a block operator matrix $(E_{v_1,v_2})_{v_1,v_2 \in V}$ whose entries are positive operators, and

$$\sum_{v_2' \in V} E_{v_1,v_2'} = \sum_{v_1' \in V} E_{v_1',v_2} = I, \quad v_1, v_2 \in V.$$

An NS bicorrelation Γ is called *quantum commuting* if there exists a Hilbert space H, a unit vector $\xi \in H$ and quantum magic squares $(E_{v_2,v_1})_{v_2,v_1 \in V}$ and $(F_{w_1,w_2})_{w_1,w_2 \in W}$ with commuting entries, such that

$$\Gamma(v_1, w_2|v_2, w_1) = \langle E_{v_2,v_1} F_{w_1,w_2} \xi, \xi \rangle, \quad v_i \in V, w_i \in W, i = 1, 2. \tag{10}$$

The bicorrelation Γ is called *quantum* if the expression (10) is achieved for $H = H_A \otimes H_B$, where H_A and H_B are finite dimensional Hilbert spaces, $E_{v_2,v_1} = E'_{v_2,v_1} \otimes I_{H_B}$ and $F_{w_1,w_2} = I_{H_A} \otimes F'_{w_1,w_2}$, $v_i \in V$, $w_i \in W$, $i = 1, 2$. The *quantum approximate* bicorrelations are the limits of quantum bicorrelations. Finally, the *local bicorrelations* are the convex combinations of correlations of the form $p^{(1)}(v_1|v_2)p^{(2)}(w_2|w_1)$, where $(p^{(1)}(v_1|v_2))_{v_1,v_2}$ and $(p^{(2)}(w_2|w_1))_{w_2,w_1}$ are (scalar) bistochastic matrices.

We use the notation $\mathcal{C}_{\rm t}^{\rm bi}$ for the (convex) set of all bicorrelations of type t. For a subset $\Lambda \subseteq V_2W_1 \times V_1W_2$, we set $\mathcal{C}_{\rm t}^{\rm bi}(\Lambda) = \mathcal{C}_{\rm t}^{\rm bi} \cap \mathcal{C}_{\rm ns}(\Lambda)$.

Remark 4.1. Let t be a correlation type. If $\Gamma \in \mathcal{C}_{\rm t}^{\rm bi}$ then $\Gamma^* \in \mathcal{C}_{\rm t}^{\rm bi}$. For $\rm t \neq ns$, this is a consequence of the fact that the transpose of a quantum magic square is again a quantum magic square, while for $\rm t = ns$ this is part of the definition.

4.2. **Game definitions and properties.** We fix finite sets V_i and W_i, and let $E_i \subseteq V_i \times W_i$ be a hypergraph, $i = 1, 2$. Set

$$E_1 \leftrightarrow E_2 \;=\; \{(v_2, w_1, v_1, w_2) : (v_1, w_1) \in E_1 \Leftrightarrow (v_2, w_2) \in E_2\};$$

thus, $E_1 \leftrightarrow E_2$ consist of the quadruples (v_2, w_1, v_1, w_2) for which

$$(v_1, w_1, v_2, w_2) \in E_1 \times E_2 \;\text{ or }\; (v_1, w_1, v_2, w_2) \in E_1^c \times E_2^c.$$

We consider $E_1 \leftrightarrow E_2$ as a non-local game with question and answer sets V_2W_1 and V_1W_2, respectively.

Definition 4.2. *Let $E_i \subseteq V_i \times W_i$ be a hypergraph, $i = 1, 2$, and* $\rm t \in \{loc, q, qa, qc, ns\}$. *We say that*

(i) *E_1 is* t-homomorphic *to E_2 (denoted $E_1 \to_{\rm t} E_2$) if $\mathcal{C}_{\rm t}(E_1 \leftrightarrow E_2) \neq \emptyset$;*
(ii) *E_1 is* t-isomorphic *to E_2 (denoted $E_1 \simeq_{\rm t} E_2$) if $V_1 = V_2$, $W_1 = W_2$ and $\mathcal{C}_{\rm t}^{\rm bi}(E_1 \leftrightarrow E_2) \neq \emptyset$.*

An element Γ of $\mathcal{C}_{\rm t}(E_1 \leftrightarrow E_2)$ (resp. $\mathcal{C}_{\rm t}^{\rm bi}(E_1 \leftrightarrow E_2)$) will be referred to as a t-homomorphism (resp. t-isomorphism) from E_1 to E_2. It is clear that

$$E_1 \simeq_{\rm t} E_2 \implies E_1 \to_{\rm t} E_2 \implies E_1 \rightsquigarrow_{\rm t} E_2. \tag{11}$$

Remark 4.3. Let $\Gamma \in \mathcal{C}_{\rm ns}^{\rm bi}$. The following are equivalent:

(i) $E_1 \simeq_{\rm ns} E_2$ via Γ;
(ii) the map $\mathcal{E} \to \Gamma[\mathcal{E}]$ is well-defined from $\mathcal{C}(E_1)$ into $\mathcal{C}(E_2)$, and the map $\mathcal{F} \to \Gamma^*[\mathcal{F}]$ is well-defined from $\mathcal{C}(E_2)$ into $\mathcal{C}(E_1)$.

Indeed, taking into account (11), condition (i) implies that $E_1 \rightsquigarrow_{\rm t} E_2$ via Γ and $E_2 \rightsquigarrow_{\rm t} E_1$ via Γ^*, and (ii) follows from Proposition 3.2. Conversely, assuming (ii), Proposition 3.2 implies that Γ fits $E_1 \to E_2$ while Γ^* fits $E_2 \to E_1$. This means that Γ fits $E_1 \leftrightarrow E_2$.

Theorem 4.4. *For* $\rm t \in \{loc, q, qa, qc, ns\}$, *the relation $\to_{\rm t}$ (resp. $\simeq_{\rm t}$) is a quasi-order (resp. an equivalence relation).*

Proof. Similarly to the proof of Theorem 3.3, one can verify that if Γ_1 and Γ_2 are correlations such that Γ_1 fits $E_1 \leftrightarrow E_2$, while Γ_2 fits $E_2 \leftrightarrow E_3$, then the correlation $\Gamma_2 * \Gamma_1$ fits $E_1 \leftrightarrow E_3$. The claim about $\to_{\rm t}$ now follows from Theorem 2.1 (i).

To see the claim about the relation $\simeq_{\rm t}$, it suffices to establish its transitivity. It is therefore enough to show that whenever $\Gamma_1, \Gamma_2 \in \mathcal{C}_{\rm t}^{\rm bi}$, we also have $\Gamma_2 * \Gamma_1 \in \mathcal{C}_{\rm t}^{\rm bi}$. In the case $\rm t = ns$, the claim is a consequence of Remark 4.1, Theorem 2.1 and the fact that $(\Gamma_2 * \Gamma_1)^* = \Gamma_1^* * \Gamma_2^*$, which we verify:

$$\begin{aligned}
(\Gamma_2 * \Gamma_1)^*(v_3, w_1 | v_1, w_3) &= (\Gamma_2 * \Gamma_1)(v_1, w_3 | v_3, w_1) \\
&= \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_1(v_1, w_2 | v_2, w_1) \Gamma_2(v_2, w_3 | v_3, w_2) \\
&= \sum_{v_2 \in V_2} \sum_{w_2 \in W_2} \Gamma_2^*(v_3, w_2 | v_2, w_3) \Gamma_1^*(v_2, w_1 | v_1, w_2) \\
&= (\Gamma_1^* * \Gamma_2^*)(v_3, w_1 | v_1, w_3).
\end{aligned}$$

In the case $\rm t = qc$, the claim follows from the proof of Theorem 2.1 and the fact that the transpose $(E_{v_2,v_1})_{v_1,v_2}$ of a quantum magic square $(E_{v_1,v_2})_{v_1,v_2}$ is also a quantum magic square. The claim in the case of the remaining types, $\rm t = qa, q, loc$, follow similarly. □

Let $E_i \subseteq V_i \times W_i$ be a hypergraph, $i = 1, 2$. A map $f : V_2 \to V_1$ is called a *homomorphism* from E_1 to E_2 if $f^{-1}(\alpha)$ is an edge of E_2 whenever α is an edge of E_1; equivalently, $f : V_2 \to V_1$ is a homomorphism precisely when there exists a map $g : W_1 \to W_2$ such that

$$f^{-1}(E_1(w_1)) = E_2(g(w_1)) \quad \text{for every } w_1 \in W_1. \tag{12}$$

If $V_1 = V_2$ and $W_1 = W_2$, an *isomorphism* from E_1 to E_2 is a bijective homomorphism f, for which the map g in (12) can be chosen to be a bijection.

Proposition 4.5. *Let $E_1 \subseteq V_1 \times W_1$ and $E_2 \subseteq V_2 \times W_2$ be hypergraphs. Then*

(i) *$E_1 \to_{\rm loc} E_2$ if and only if there exists a homomorphism from E_1 to E_2;*
(ii) *if $V_1 = V_2$ and $W_1 = W_2$, then $E_1 \simeq_{\rm loc} E_2$ if and only if the hypergraphs E_1 and E_2 are isomorphic.*

Proof. (i) As in the proof of Proposition 3.4, the existence of a perfect local strategy for the homomorphism game $E_1 \to E_2$ implies the existence of maps $f : V_2 \to V_1$ and $g : W_1 \to W_2$ such that

$$(f(v_2), w_1) \in E_1 \iff (v_2, g(w_1)) \in E_2,$$

which is equivalent to (12).

Conversely, assuming (12) and adopting the notation from the proof of Proposition 3.4, we have that

$$(\Phi \otimes \Psi)(v_1, w_2 | v_2, w_1) = \begin{cases} 1 & \text{if } v_1 = f(v_2) \text{ and } w_2 = g(w_1) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, assuming that $(\Phi \otimes \Psi)(v_1, w_2 | v_2, w_1) = 1$, we have that

$$(v_1, w_1) \in E_1 \ \Leftrightarrow \ (f(v_2), w_1) \in E_1 \ \Leftrightarrow \ (v_2, g(w_1)) \in E_2 \ \Leftrightarrow \ (v_2, w_2) \in E_2,$$

which shows that $\Phi \otimes \Psi$ fits $E_1 \leftrightarrow E_2$.

(ii) Assume that the bicorrelation Γ is a perfect local strategy for the hypergraph isomorphism game $E_1 \leftrightarrow E_2$. By definition, $\Gamma = \sum_{i=1}^k \lambda_i \Phi_i \otimes \Psi_i$ as a convex combination, where $\Phi_i : \mathcal{D}_V \to \mathcal{D}_V$ and $\Psi_i : \mathcal{D}_W \to \mathcal{D}_W$ arise from (scalar) bistochastic matrices. Using Birkhhoff's Theorem, we decompose these bistochastic matrices as convex combinations of permutation matrices; this allows us to assume that Φ_i and Ψ_i each arise from permutation matrices. By positivity, $\Phi_i \otimes \Psi_i$ is a perfect strategy for the game $E_1 \leftrightarrow E_2$. We may thus assume that $\Gamma = \Phi \otimes \Psi$, where $\Phi(v_1|v_2) = \delta_{f(v_2),v_1}$ and $\Psi(w_2|w_1) = \delta_{g(w_1),w_2}$ for some bijections $f : V_2 \to V_1$ and $g : W_1 \to W_2$. Identifying Φ and Ψ with the corresponding conditional probability distributions, we have that

$$\mathrm{supp}(\Phi \otimes \Psi) = \{(v_2, w_1, f(v_2), g(w_1)) : v_2, w_1 \in V\}.$$

It follows that E_1 and E_2 are isomorphic via the pair (f, g).

Conversely, assuming that f and g are bijections that fulfill (12), the channel $\Gamma = \Phi \otimes \Psi$, defined in the previous paragraph, is a bicorrelation that is a perfect strategy for the game $E_1 \leftrightarrow E_2$. $\square$

4.3. **Values of probabilistic hypergraphs.** A *probabilistic hypergraph* is a hypergraph $E \subseteq V \times W$, equipped with a probability distribution $\pi : V \to [0,1]$ on its vertex set. Given a convex subset $\mathfrak{E} \subseteq \mathcal{C}(V \times W)$ of channels from V to W, we let

$$\omega_{\mathfrak{E}}(E, \pi) = \sup_{\mathcal{E} \in \mathfrak{E}} \sum_{(v,w) \in E} \pi(v) \mathcal{E}(w|v) \tag{13}$$

be the $\mathfrak{E}$-*value* of (E, π). Suppose that Γ is a perfect no-signalling strategy for the homomorphism game $E_1 \to E_2$. Given a probability distribution π_2 on V_2, let $\pi_1 = \Gamma_{V_2 \to V_1}(\pi_2)$; thus, π_1 is the probability distribution on V_1 given by

$$\pi_1(v_1) = \sum_{v_2 \in V_2} \Gamma_{V_2 \to V_1}(v_1|v_2) \pi_2(v_2), \quad v_1 \in V_1.$$

Similarly, for a probability distribution π_1 on V_1, let $\pi_2 = \Gamma^*_{V_1 \to V_2}(\pi_1)$. If $V_1 = V_2 =: V$, the probability distribution π on V will be called Γ-*stationary* if $V_1 = V_2$ and $\pi = \pi_1 = \pi_2$.

Proposition 4.6. *Let $E_i \subseteq V_i \times W_i$ be a hypergraph, $\mathfrak{E}_i \subseteq \mathcal{C}(V_i \times W_i)$, $i = 1, 2$, π_2 be a probability distribution on V_2, and $\pi_1 = \Gamma_{V_2 \to V_1}(\pi_2)$.*

(i) *Suppose that $E_1 \to_{\mathrm{ns}} E_2$ via a correlation Γ such that $\Gamma[\mathfrak{E}_1] \subseteq \mathfrak{E}_2$. Then*

$$\omega_{\mathfrak{E}_1}(E_1, \pi_1) \leq \omega_{\mathfrak{E}_2}(E_2, \pi_2).$$

(ii) *Suppose that $V_1 = V_2 =: V$, that π is a Γ-stationary probability distribution on V and that $E_1 \simeq_{\mathrm{ns}} E_2$ via a bicorrelation Γ such that $\Gamma[\mathfrak{E}_1] \subseteq \mathfrak{E}_2$ and $\Gamma^*[\mathfrak{E}_2] \subseteq \mathfrak{E}_1$. Then $\omega_{\mathfrak{E}_1}(E_1, \pi) = \omega_{\mathfrak{E}_2}(E_2, \pi)$.*

Proof. (i) Let Γ be a perfect no-signalling strategy for the homomorphism game $E_1 \to E_2$. Let $\mathcal{E} \in \mathfrak{E}_1$ and $\mathcal{F} = \Gamma[\mathcal{E}]$. We have that

$$\begin{aligned}
& \sum_{(v_2,w_2)\in E_2} \mathcal{F}(w_2|v_2)\pi_2(v_2) \\
= & \sum_{(v_2,w_2)\in E_2} \sum_{v_1\in V_1} \sum_{w_1\in W_1} \Gamma(v_1,w_2|v_2,w_1)\mathcal{E}(w_1|v_1)\pi_2(v_2) \\
\geq & \sum_{(v_2,w_2)\in E_2} \sum_{(v_1,w_1)\in E_1} \Gamma(v_1,w_2|v_2,w_1)\mathcal{E}(w_1|v_1)\pi_2(v_2) \\
= & \sum_{(v_1,w_1)\in E_1} \sum_{v_2\in V_2} \sum_{w_2\in W_2} \Gamma(v_1,w_2|v_2,w_1)\mathcal{E}(w_1|v_1)\pi_2(v_2) \\
= & \sum_{(v_1,w_1)\in E_1} \sum_{v_2\in V_2} \Gamma(v_1|v_2)\mathcal{E}(w_1|v_1)\pi_2(v_2) = \sum_{(v_1,w_1)\in E_1} \mathcal{E}(w_1|v_1)\pi_1(v_1).
\end{aligned}$$

Taking the supremum over all $\mathcal{E} \in \mathfrak{E}_1$ yields the desired inequality.

(ii) follows by symmetry from (i). $\square$

If $E \subseteq V \times W$ and $F \subseteq X \times Y$ are hypergraphs, their *product* is the hypergraph $E \otimes F \subseteq (VX) \times (WY)$, given by

$$E \otimes F = \{(v,x,w,y) : (v,w) \in E, (x,y) \in F\}.$$

We write $E^{\otimes n} = \otimes_{i=1}^{n} E$. Given a probability distribution π on V, let $\pi^n = \otimes_{i=1}^{n} \pi$ be the n-fold product distribution of π on the vertex set V^n of $E^{\otimes n}$. We fix subsets $\mathfrak{E}_n \subseteq \mathcal{C}(V^n \times W^n)$ with the property that

$$\mathcal{E} \in \mathfrak{E}_n,\ \mathcal{F} \in \mathfrak{E}_m \implies \mathcal{E} \otimes \mathcal{F} \in \mathfrak{E}_{n+m}, \quad n, m \in \mathbb{N}, \tag{14}$$

and write $\bar{\mathfrak{E}} = (\mathfrak{E}_n)_{n\in\mathbb{N}}$. The *asymptotic* $\bar{\mathfrak{E}}$*-value* of (E,π) is the quantity

$$\bar{\omega}_{\bar{\mathfrak{E}}}(E,\pi) = \limsup_{n\in\mathbb{N}} \omega_{\mathfrak{E}_n}(E^{\otimes n}, \pi^n)^{\frac{1}{n}}. \tag{15}$$

Remark 4.7. Let $\bar{\mathfrak{E}} = (\mathfrak{E}_n)_{n\in\mathbb{N}}$ be a sequence of families of channels, where $\mathfrak{E}_n \subseteq \mathcal{C}(V^n \times W^n)$, $n \in \mathbb{N}$, satisfying (14). By the definition of the value (13), we then have

$$\omega_{\mathfrak{E}_n}(E^{\otimes n}, \pi^n)\omega_{\mathfrak{E}_m}(E^{\otimes m}, \pi^m) \leq \omega_{\mathfrak{E}_{n+m}}(E^{\otimes (n+m)}, \pi^{n+m})$$

for all $n, m \in \mathbb{N}$. Thus, by Fekete's Lemma (see [23]), the limsup in the definition (15) can be replaced by a limit.

Corollary 4.8. *Let $E_i \subseteq V_i \times W_i$ be a hypergraph and $\bar{\mathfrak{E}}_i = (\mathfrak{E}_n^{(i)})_{n\in\mathbb{N}}$ be sequences of families of channels, where $\mathfrak{E}_n^{(i)} \subseteq \mathcal{C}(V_i^n \times W_i^n)$, satisfying (14), $i = 1, 2$.*

(i) *Suppose that $E_1 \to_{\rm ns} E_2$ via a correlation Γ such that $\Gamma^{\otimes n}[\mathfrak{E}_n^{(1)}] \subseteq \mathfrak{E}_n^{(2)}$, $n \in \mathbb{N}$, and let π_2 be a probability distribution on V_2. Set $\pi_1 = \Gamma_{V_2 \to V_1}(\pi_2)$. Then*

$$\bar{\omega}_{\bar{\mathfrak{E}}_1}(E_1,\pi_1) \leq \bar{\omega}_{\bar{\mathfrak{E}}_2}(E_2,\pi_2).$$

(ii) *Suppose that $V_1 = V_2 =: V$, that π is a Γ-stationary probability distribution on V and that $E_1 \simeq_{\rm ns} E_2$ via a bicorrelation Γ such that $\Gamma^{\otimes n}[\mathfrak{E}_n^{(1)}] \subseteq \mathfrak{E}_n^{(2)}$ and $\Gamma^{*\otimes n}[\mathfrak{E}_n^{(2)}] \subseteq \mathfrak{E}_n^{(1)}$. Then $\bar{\omega}_{\bar{\mathfrak{E}}_1}(E_1,\pi) = \bar{\omega}_{\bar{\mathfrak{E}}_2}(E_2,\pi)$.*

Proof. Proposition 4.6 (i) implies that

$$\bar{\omega}_{\bar{\mathfrak{E}}_1}(E_1^{\otimes n},\pi_1^n) \leq \bar{\omega}_{\bar{\mathfrak{E}}_2}(E_2^{\otimes n},\pi_2^n), \quad n \in \mathbb{N},$$

implying part (i). Part (ii) follows by symmetry, applying Proposition 4.6 (ii). □

4.4. **An operator system approach.** We recall the universal operator system for bicorrelations, introduced in [10]. A *ternary ring of operators (TRO)* is a subspace $\mathcal{V} \subseteq \mathcal{B}(H,K)$, for some Hilbert spaces H and K, such that $ST^*R \in \mathcal{V}$ whenever $S,T,R \in \mathcal{V}$ (see e.g. [7, 32]). Let $\mathcal{V}_V$ be the universal TRO generated by the entries $u_{v,v'}$ of a *bi-isometry*, that is, a block operator matrix $U = (u_{v,v'})_{v,v' \in V}$ such that both U and its transpose $U^{\rm t} := (u_{v',v})_{v,v' \in V}$ are isometries. Thus, $\mathcal{V}_V$ is the universal TRO with generators $u_{v,v'}$, $v,v' \in V$, and relations

$$\sum_{a \in V}[u_{a'',x''},u_{a,x},u_{a,x'}] = \delta_{x,x'}u_{a'',x''} \text{ and } \sum_{x \in V}[u_{a'',x''},u_{a,x},u_{a',x}] = \delta_{a,a'}u_{a'',x''},$$

for all $x,x',x'',a,a',a'' \in V$. Let $\mathcal{C}_V$ be the right C*-algebra of $\mathcal{V}_V$, when the latter is viewed as an imprimitivity bimodule [50]; thus, up to a *-isomorphism, we have that $\mathcal{C}_V \simeq \overline{\rm span}(\theta(\mathcal{V}_V)^*\theta(\mathcal{V}_V))$, for any faithful ternary representation $\theta : \mathcal{V}_V \to \mathcal{B}(H,K)$ (H and K being Hilbert spaces). We write

$$e_{v_1,v_1',v_2,v_2'} := u_{v_2,v_1}^* u_{v_2',v_1'}, \quad v_1,v_2,v_1',v_2' \in V;$$

note that $\mathcal{C}_V$ is generated, as a C*-algebra, by the elements $e_{v_1,v_1',v_2,v_2'}$, v_1, v_1', v_2, $v_2' \in V$. Set $e_{v_1,v_2} := e_{v_1,v_1,v_2,v_2}$, $v_1,v_2 \in V$, and let

$$\mathcal{S}_V = {\rm span}\{e_{v_1,v_2} : v_1,v_2 \in V\} \text{ and } \mathcal{T}_V = {\rm span}\{e_{v_1,v_1',v_2,v_2'} : v_1,v_1',v_2,v_2' \in V\},$$

viewed as operator subsystems of $\mathcal{C}_V$.

The following was shown in [10]:

Theorem 4.9 ([10])**.** *Let H be a Hilbert space. If $\phi : \mathcal{S}_V \to \mathcal{B}(H)$ is a unital completely positive map then $(\phi(e_{v_1,v_2}))_{v_1,v_2 \in V}$ is a quantum magic square. Conversely, if $(E_{v_1,v_2})_{v_1,v_2 \in V}$ is a quantum magic square on H then there exists a (unique) unital completely positive map $\phi : \mathcal{S}_V \to \mathcal{B}(H)$ such that $E_{v_1,v_2} = \phi(e_{v_1,v_2})$, $v_1,v_2 \in V$.*

Lemma 4.10. *The flip map $\mathfrak{f} : e_{v_1,v_2} \to e_{v_2,v_1}$ extends to a unital complete order automorphism of $\mathcal{S}_V$.*

Proof. Let $\phi : \mathcal{S}_V \to \mathcal{B}(H)$ be a unital complete order embedding and set $E_{v_1,v_2} = \phi(e_{v_1,v_2})$, $v_1, v_2 \in V$. By Theorem 4.9, $(E_{v_1,v_2})_{v_1,v_2\in V}$ is a quantum magic square. Therefore, $(E_{v_2,v_1})_{v_1,v_2\in V}$ is a quantum magic square and Theorem 4.9 gives rise to a unital completely positive map $\psi : \mathcal{S}_V \to \mathcal{B}(H)$ with the property

$$\psi(e_{v_1,v_2}) = E_{v_2,v_1}, \quad v_1, v_2 \in V.$$

Note that $\psi = \phi \circ \mathfrak{f}$; hence $\mathfrak{f}$ is completely positive. By symmetry, $\mathfrak{f}$ is a complete order isomorphism. □

We now assume that $V_1 = V_2 =: V$ and $W_1 = W_2 =: W$. Let

$$E_1 \Leftrightarrow E_2 \ := \ (E_1 \times E_2) \cup (E_1^c \times E_2^c);$$

consider $E_1 \Leftrightarrow E_2$ as a non-local game with question and answer sets V_1W_1 and V_2W_2, respectively, and refer to it as an *equivalence game*. Note that, if

$$\mathfrak{r} : V_2 \times W_1 \times V_1 \times W_2 \to V_1 \times W_1 \times V_2 \times W_2$$

is the shuffle map, given by $\mathfrak{r}(v_2, w_1, v_1, w_2) = (v_1, w_1, v_2, w_2)$, then $E_1 \Leftrightarrow E_2 = \mathfrak{r}(E_1 \leftrightarrow E_2)$.

For clarity, we denote the canonical generators of the operator system $\mathcal{S}_W$ by f_{w_1,w_2}, $w_1, w_2 \in W$. Given a linear functional $s : \mathcal{S}_V \otimes \mathcal{S}_W \to \mathbb{C}$, we let $\Gamma_s : \mathcal{D}_{VW} \to \mathcal{D}_{VW}$ be the linear map given by

$$\Gamma_s(v_1, w_2|v_2, w_1) = s(e_{v_1,v_2} \otimes f_{w_1,w_2}).$$

For an NS correlation Γ over (V_2, W_1, V_1, W_2), let $\mathfrak{F}(\Gamma) : \mathcal{D}_{V_1W_1} \to \mathcal{D}_{V_2W_2}$ be the linear map, defined by letting

$$\mathfrak{F}(\Gamma)(v_2, w_2|v_1, w_1) := \Gamma(v_1, w_2|v_2, w_1).$$

Proposition 4.11. *Let* $\mathrm{t} \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$. *The map* $\Gamma \to \mathfrak{F}(\Gamma)$ *is an affine isomorphism between* $\mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}(E_1 \leftrightarrow E_2)$ *and* $\mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}(E_1 \Leftrightarrow E_2)$.

Proof. Let $\mathrm{t} = \mathrm{qc}$. and $\Gamma \in \mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}(E_1 \leftrightarrow E_2)$. By [10], there exists a state $s : \mathcal{S}_V \otimes_{\mathrm{c}} \mathcal{S}_W \to \mathbb{C}$ such that $\Gamma = \Gamma_s$. Let $\tilde{s} := s \circ (\mathfrak{f} \otimes \mathrm{id})$; by Lemma 4.10 and the functoriality of the commuting tensor product, $\tilde{s}$ is a state on the operator system $\mathcal{S}_V \otimes_{\mathrm{c}} \mathcal{S}_W$. Since $\mathfrak{F}(\Gamma) = \Gamma_{\tilde{s}}$, we have that $\mathfrak{F}(\Gamma) \in \mathcal{C}_{\mathrm{qc}}^{\mathrm{bi}}$. The fact that $\mathrm{supp}(\mathfrak{F}(\Gamma)) = \mathfrak{r}(\mathrm{supp}(\Gamma))$ is straightforward. We finally note that $\mathfrak{F}^2 = \mathrm{id}$, showing that $\mathfrak{F}$ is an isomorphism.

The cases $\mathrm{t} = \mathrm{qa}$ and $\mathrm{t} = \mathrm{ns}$ are analogous, using the minimal (resp. maximal) tensor product instead of the commuting one.

For the case $\mathrm{t} = \mathrm{q}$, assuming that $\Gamma \in \mathcal{C}_{\mathrm{t}}^{\mathrm{bi}}(E_1 \leftrightarrow E_2)$, there exist quantum magic squares $(E_{v_1,v_2})_{v_1,v_2\in V}$ (resp. $(F_{w_1,w_2})_{w_1,w_2\in W}$) acting on Hilbert spaces H_V (resp. H_W) and unit vectors $\xi \in H_V, \eta \in H_W$ so that

$$\Gamma(v_1, w_2|v_2, w_1) = \langle (E_{v_2,v_1} \otimes F_{w_1,w_2})(\xi \otimes \eta), \xi \otimes \eta \rangle.$$

Let $\tilde{E} = (\tilde{E}_{v_2,v_1})_{v_1,v_2\in V}$, where $\tilde{E}_{v_2,v_1} := E_{v_1,v_2}$; it is clear that $\tilde{E}$ is a quantum magic square and that

$$\mathfrak{F}(\Gamma)(v_2, w_2|v_1, w_1) = \langle (\tilde{E}_{v_1,v_2} \otimes F_{w_1,w_2})(\xi \otimes \eta), \xi \otimes \eta \rangle.$$

Thus, $\mathfrak{F}$ induces an isomorphism from $\mathcal{C}_{\rm q}^{\rm bi}(E_1 \leftrightarrow E_2)$ onto $\mathcal{C}_{\rm q}^{\rm bi}(E_1 \Leftrightarrow E_2)$.

Assume that $\Gamma \in \mathcal{C}_{\rm loc}^{\rm bi}(E_1 \leftrightarrow E_2)$; thus, $\Gamma = \sum_{i=1}^k \lambda_i p_i^{(1)} \otimes p_i^{(2)}$, where $p_i^{(1)} = \{(p_i^{(1)}(v_1|v_2))_{v_1} : v_2 \in V\}$ (resp. $p_i^{(2)} = \{(p_i^{(2)}(w_2|w_1))_{w_2} : w_1 \in W\}$) is a channel in $\mathcal{C}(V_2 \times V_1)$ (resp. $\mathcal{C}(W_1 \times W_2)$), with the property that the matrices $(p_i^{(1)}(v_1|v_2))_{v_2,v_1}$ and $(p_i^{(2)}(w_2|w_1))_{w_1,w_2}$ are bistochastic. It follows that the matrix $\tilde{p}_i^{(1)} := (p_i^{(1)}(v_2|v_1))_{v_1,v_2}$, $i = 1, \dots, k$, is bistochastic. In addition,

$$\mathfrak{F}(\Gamma) = \sum_{i=1}^k \lambda_i \tilde{p}_i^{(1)} \otimes p_i^{(2)}.$$

As in the first paragraph, $\mathfrak{F}$ induces an isomorphism from $\mathcal{C}_{\rm loc}^{\rm bi}(E_1 \leftrightarrow E_2)$ onto $\mathcal{C}_{\rm loc}^{\rm bi}(E_1 \Leftrightarrow E_2)$. $\square$

Let $E_1 \subseteq V_1 \times W_1$ and $E_2 \subseteq V_2 \times W_2$ be hypergraphs, and let

$$\mathcal{J} = \text{span}\{e_{v_2,v_1} \otimes f_{w_1,w_2} : (v_2, w_1, v_1, w_2) \notin E_1 \leftrightarrow E_2\}.$$

Corollary 4.12. *The map $s \to \Gamma_s$ is an affine surjective correspondence between*

(i) *the states of $\mathcal{S}_V \otimes_{\max} \mathcal{S}_W$ that annihilate $\mathcal{J}$ and the perfect* ns-*strategies of $E_1 \leftrightarrow E_2$;*
(ii) *the states of $\mathcal{S}_V \otimes_{\rm c} \mathcal{S}_W$ that annihilate $\mathcal{J}$ and the perfect* qc-*strategies of $E_1 \leftrightarrow E_2$;*
(iii) *the states of $\mathcal{S}_V \otimes_{\min} \mathcal{S}_W$ that annihilate $\mathcal{J}$ and the perfect* qa-*strategies of $E_1 \leftrightarrow E_2$.*

4.5. **Faithful isomorphisms.** In this subsection, we assume that $V_1 = W_1 = V_2 = W_2 =: V$.

Definition 4.13. *Let $E_1 \subseteq V_1 \times W_1$ and $E_2 \subseteq V_2 \times W_2$. A bicorrelation $\Gamma \in \mathcal{C}_{\rm t}^{\rm bi}$ over (V_2, W_1, V_1, W_2) is called* faithful *if*

$$\Gamma(v_1, w_2|v_2, w_1) = 0 \text{ whenever } (v_1 = w_1 \ \& \ v_2 \neq w_2) \text{ or } (v_1 \neq w_1 \ \& \ v_2 = w_2).$$

A faithful t-isomorphism *between E_1 and E_2 is a faithful bicorrelation $\Gamma \in \mathcal{C}_{\rm t}^{\rm bi}(E_1 \leftrightarrow E_2)$.*

A faithful isomorphism Γ between the hypergraphs E_1 and E_2 can be thought of as a means of mutually simulating the noiseless channels id : $V_2 \to W_2$ and id : $V_1 \to W_1$ by each other: every time the original channel $\mathcal{E} : V_1 \to W_1$ transmits faithfully a certain symbol $v \in V_1$, the simulated channel $\Gamma[\mathcal{E}]$ does so too, and vice versa.

We note that a correlation Γ over (V_2, W_1, V_1, W_2) is faithful if and only if the correlation $\mathfrak{F}(\Gamma)$ over (V_1, W_1, V_2, W_2) is bisynchronous in the sense of [44, Definition 1.2]. This enables us to use the works [38] and [44] in the sequel. Recall that a *quantum permutation* acting on Hilbert space H is a unitary matrix $(P_{v,v'})_{v,v' \in V}$, whose entries $P_{v,v'}$ are projections in

$\mathcal{B}(H)$. (We note that every quantum permutation is automatically a quantum magic square.) The quantum permutation group over V is the universal C*-algebra $\mathfrak{A}_V$ generated by the entries of a quantum permutation [38, Section 2.2]. We write $p_{v,w}$, $v,w \in V$, for a fixed family of generators of $\mathfrak{A}_V$ (so that $(p_{v,w})_{v,w\in V}$ is a universal quantum permutation). We call a quantum permutation $(P_{v,v'})_{v,v'\in V}$ a *quantum q-permutation* (resp. a *quantum qc-permutation*) if its entries act on a finite dimensional Hilbert space (resp. there exists a C*-algebra $\mathcal{A}$ with a trace containing its entries).

Given a hypergraph $E \subseteq V \times V$, let

$$\mathsf{A}_E = \sum_{(v,v')\in E} \epsilon_{v,v'}$$

be the incidence matrix of E.

Theorem 4.14. *Let $t \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qc}\}$. The following are equivalent:*

(i) *E_1 is faithfully* t*-isomorphic to E_2;*

(ii) *there exists a quantum* t*-permutation $P = (P_{v,v'})_{v,v'\in V}$ such that*

$$P(\mathsf{A}_{E_1} \otimes I_H) = (\mathsf{A}_{E_2} \otimes I_H)P. \tag{16}$$

Proof. The proof relies on the ideas from the proof of [4, Lemma 5.8]. We only consider the case $\mathrm{t} = \mathrm{qc}$.

(ii)⇒(i) Suppose that $\mathcal{A}$ is a unital C*-algebra, equipped with a trace τ (which can be assumed to be faithful), and that $(P_{v,v'})_{v,v'\in V}$ is a quantum permutation with entries in $\mathcal{A}$, satisfying (16). If $v, w' \in V$ then, denoting by $(\mathsf{A})_{v,w'}$ the (v,w')-entry of a matrix A over $V \times V$, we have

$$\sum_{v'\in E_2^*(v)} P_{v',w'} = ((\mathsf{A}_{E_2} \otimes I_{\mathcal{H}})P)_{v,w'} = (P(\mathsf{A}_{E_1} \otimes I_{\mathcal{H}}))_{v,w'} = \sum_{w\in E_1(w')} P_{v,w}. \tag{17}$$

Since the columns of P are PVM's,

$$\left(\sum_{v'\in E_2^*(v)} P_{v',w'}\right)^2 = \sum_{v'\in E_2^*(v)} P_{v',w'}.$$

Pairing this with (17), we see

$$\begin{aligned}
\sum_{v'\in E_2^*(v)} P_{v',w'} \sum_{w\in E_1(w')} P_{v,w} &= \left(\sum_{v'\in E_2^*(v)} P_{v',w'}\right)^2 \\
&= \sum_{v'\in E_2^*(v)} P_{v',w'} = \sum_{v'\in E_2^*(v)} P_{v',w'} \sum_{w\in V} P_{v,w}.
\end{aligned}$$

This implies

$$\sum_{v'\in E_2^*(v)} P_{v',w'} \sum_{w\notin E_1(w')} P_{v,w} = 0,$$

hence

$$\sum_{v'\in E_2^*(v)} \sum_{w\notin E_1(w')} \tau(P_{v',w'}P_{v,w}) = 0,$$

forcing $\tau(P_{v',w'}P_{v,w}) = 0$ whenever $(v,v') \in E_2$ while $(w,w') \notin E_1$. By symmetry, $\tau(P_{v',w'}P_{v,w}) = 0$ whenever $(w,w') \in E_1$ while $(v,v') \notin E_2$. As τ is faithful, this implies

$$P_{v',w'}P_{v,w} = P_{v,w}P_{v',w'} = 0 \quad \text{whenever } (v,v') \in E_2 \not\Leftrightarrow (w,w') \notin E_1. \tag{18}$$

Define the linear map $\Gamma : \mathcal{D}_{V_2W_1} \to \mathcal{D}_{V_1W_2}$ by letting

$$\Gamma(v_1, w_2 | v_2, w_1) := \tau(P_{v_2,v_1}P_{w_1,w_2}).$$

We claim $\Gamma \in \mathcal{C}_{\text{qc}}^{\text{bi}}(E_1 \leftrightarrow E_2)$, and that it is faithful. It is clear that Γ is a quantum commuting correlation. The unitality of Γ is straightforward, while the fact that Γ^* is quantum commuting follows from Remark 4.1. The faithfulness of Γ is an immediate consequence of the fact that the rows and columns of a quantum permutation are PVM's. Finally, since $\mathfrak{F}(\Gamma)$ is a perfect strategy for the equivalence game $E_1 \Leftrightarrow E_2$, Proposition 4.11 implies that Γ is a perfect (quantum commuting) strategy for the hypergraph isomorphism game $E_1 \leftrightarrow E_2$.

(i)$\Rightarrow$(ii) Assume that E_1 is faithfully qc-isomorphic to E_2 via $\Gamma \in \mathcal{C}_{\text{qc}}^{\text{bi}}(E_1 \leftrightarrow E_2)$. By Proposition 4.11, $\mathfrak{F}(\Gamma)$ is a perfect quantum commuting and bisynchronous strategy of the equivalence game $E_1 \Leftrightarrow E_2$. By [44, Theorem 2.2] and [4, Lemma 5.13], there exists a faithful tracial state τ on a C*-algebra $\mathcal{A}$, and a *-representation $\pi : \mathfrak{A}_V \to \mathcal{A}$ such that, if $P_{v,w} = \pi(p_{v,w})$ for $v, w \in V$, then

$$\Gamma(v_1, w_2 | v_2, w_1) = \tau(P_{v_1,v_2}P_{w_1,w_2}).$$

Since Γ fits $E_1 \leftrightarrow E_2$, we have that if $(v_1, w_1) \in E_1$ and $(v_2, w_2) \notin E_2$, then $\tau(P_{v_1,v_2}P_{w_1,w_2}) = 0$, whenever $(v_1, w_1) \in E_1 \not\Leftrightarrow (v_2, w_2) \in E_2$. It follows that

$$P_{v_1,v_2}P_{w_1,w_2} = 0 \;\text{ whenever } (v_1, w_1) \in E_1 \not\Leftrightarrow (v_2, w_2) \in E_2.$$

Let $P = (P_{v,w})_{v,w\in V}$; it is clear that P is a quantum permutation. For any $v, w \in V$, we have

$$\begin{aligned}
((\mathsf{A}_{E_2} \otimes I_{\mathcal{H}})P)_{v,w} &= \sum_{v' \in E_2^*(v)} P_{v',w} = \sum_{w' \in V} P_{v,w'} \sum_{v' \in E_2^*(v)} P_{v',w} \\
&= \sum_{v' \in E_2^*(v)} \sum_{w' \in V} P_{v,w'}P_{v',w} = \sum_{v' \in E_2^*(v)} \sum_{w' \in E_1(w)} P_{v,w'}P_{v',w} \\
&= \sum_{v' \in V} \sum_{w' \in E_1(w)} P_{v,w'}P_{v',w} = \sum_{w' \in E_1(w)} P_{v,w'} \sum_{v' \in V} P_{v',w} \\
&= \sum_{w' \in E_1(w)} P_{v,w'} = (P(\mathsf{A}_{E_1} \otimes I_{\mathcal{H}}))_{v,w}.
\end{aligned}$$

This shows the validity of (16). $\square$

Remark. Let G_1 and G_2 be graphs on a vertex set X. The graph isomorphism game $G_1 \cong G_2$ is defined in [4] and, according to [4, Theorems 5.9 and 5.14], G_1 and G_2 are quantum (resp. quantum commuting) isomorphic (denoted $G_1 \cong_{\rm q} G_2$ (resp. $G_1 \cong_{\rm qc} G_2$)) if and only if there exists a quantum q-permutation (resp. quantum qc-permutation) P acting on a Hilbert space H, such that $P(\mathsf{A}_{G_1} \otimes I_H) = (\mathsf{A}_{G_2} \otimes I_H)P$, where A_{G_1} and A_{G_2} are the adjacency matrices of the graphs G_1 and G_2, respectively. Thus, Theorem 4.14 generalises [4, Theorems 5.9 and 5.14].

Lemma 4.15. *Let $E_i \subseteq V_i \times W_i$ be a hypergraph, $i = 1, 2$. If the pair (f, g) of functions determines an isomorphism from E_1 to E_2, then g is an isomorphism from $G_{E_1^*}$ to $G_{E_2^*}$.*

Proof. We set $G_i = G_{E_i^*}$, $i = 1, 2$. Suppose that $x \sim x'$ in G_1, and let $v \in V$ be such that $v \in E_1(x) \cap E_1(x')$. Let $v' \in V$ be the unique element such that $f(v') = v$. We have that $v' \in f^{-1}(E_1(x)) \cap f^{-1}(E_1(x'))$. As $f^{-1}(E_1(x)) = E_2(g(x))$ and $f^{-1}(E_1(x')) = E_2(g(x'))$, we have $v' \in E_2(g(x)) \cap E_2(g(x'))$, that is, $g(x) \sim g(x')$ in G_2.

Now suppose $g(x) \sim g(x')$ in G_2, and let $v' \in V$ be such that $v' \in E_2(g(x)) \cap E_2(g(x'))$; then $f(v') \in f(E_2(g(x))) \cap f(E_2(g(x')))$. As $f^{-1}(E_1(x)) = E_2(g(x))$, this implies $E_1(x) = f(E_2(g(x)))$. Thus, $f(v') \in E_1(x) \cap E_1(x')$, meaning that $x \sim x'$ in G_1. This shows that g is an isomorphism. □

Let G be a graph with vertex set X. Recall [27, Section 1.7] that the *line graph* $L(G)$ of G has as a vertex set the set L of all edges of G and its adjacency relation is given by

$$l \sim_{L(G)} l' \text{ if there exist } x, y, z \in X \text{ s.t. } x \neq z, l = \{x, y\} \text{ and } l' = \{y, z\}$$

(in other words, $l \sim_{L(G)} l'$ precisely when l and l' are distinct edges that share a common vertex). Let

$$E_G = \{(x, x') \in X \times X : x \sim_G x'\},$$

considered as a hypergraph in $X \times X$, and

$$F_G = \{((x, y), y) : x \sim_G y\},$$

considered as a hypergraph in $XX \times X$.

Theorem 4.16. *Let G_1 and G_2 be graphs with vertex set X such that $G_1 \cong_{\rm q} G_2$ but $G_1 \not\cong G_2$. Then*

(i) *$E_{G_1} \simeq_{\rm q} E_{G_2}$ but $E_{G_1} \not\simeq_{\rm loc} E_{G_2}$;*
(ii) *$F_{G_1} \simeq_{\rm q} F_{G_2}$ but $F_{G_1} \not\simeq_{\rm loc} F_{G_2}$.*

Proof. (i) Set $E_i = E_{G_i}$, $i = 1, 2$. By [4, Theorem 5.8], there exist $d \in \mathbb{N}$ and a quantum permutation $P \in M_X \otimes M_d$ such that

$$(\mathsf{A}_{E_1} \otimes I_d)P = P(\mathsf{A}_{E_2} \otimes I_d).$$

Theorem 4.14 now implies that $E_1 \simeq_{\rm q} E_2$.

By Proposition 4.5, it now suffices to show that the hypergraphs E_1 and E_2 are not isomorphic. Assume, towards a contradiction, that there exists a pair (f, g) of bijections, where $f : X \to X$ and $g : X \to X$, such that

$$f^{-1}(E_1(x)) = E_2(g(x)), \quad x \in X.$$

By Lemma 4.15, g is an isomorphism from $G_{E_1^*}$ to $G_{E_2^*}$. Note that $G_{E_i^*}$ is either isomorphic to $L(G_i)$ or contains $L(G_i)$ as a connected component, potentially with additional isolated vertices. It follows that $L(G_1) \cong L(G_2)$. On the other hand, since quantum and classical isomorphism differ for the graphs G_1 and G_2, we have that the cardinality of the vertex sets of G_1 and G_2 exceeds four [51, Section 3]. Thus, $L(G_i) \ncong K_3, K_{1,3}$ for $i = 1, 2$ (where $K_{1,3}$ stands for the bipartite graph on four vertices with three vertices in one disjoint set, and the remaining in the other). Whitney's Isomorphism Theorem [56] now implies that $G_1 \cong G_2$, a contradiction.

(ii) As in [4, Section 2], let $\mathrm{rel}(x, y)$ denote the relation between vertices of the graph G, of either being adjacent ($\mathrm{rel}(x, y) = 1$), equal ($\mathrm{rel}(x, y) = 0$) or non-adjacent ($\mathrm{rel}(x, y) = -1$). By [4, Theorem 5.9], there exists a quantum permutation $P = (P_{x,y})_{x,y}$ over $X \times X$, acting on a (finite dimensional) Hilbert space, such that

$$P_{x,x'} P_{y,y'} = 0 \quad \text{if } \mathrm{rel}(x, y) \neq \mathrm{rel}(x', y'). \tag{19}$$

Abbreviating the notation (x, y) to xy, given pairs $xy, ab \in XX$, let $Q_{xy,ab} = P_{y,b} P_{x,a} P_{y,b}$. Note that

$$\sum_{ab \in XX} Q_{xy,ab} = \sum_{b \in X} P_{y,b} \left(\sum_{a \in X} P_{x,a} \right) P_{y,b} = \sum_{b \in X} P_{y,b} = I;$$

thus, the family $(Q_{xy,ab})_{ab \in XX}$ is a POVM, for every $xy \in XX$.

Suppose that $(xy, y) \in F_{G_1}$ but $(ab, c) \notin F_{G_2}$. Let $a \not\sim_{G_2} b$. Since $x \sim_{G_1} y$, by (19) we have

$$Q_{xy,ab} P_{y,c} = P_{y,b} (P_{x,a} P_{y,b}) P_{y,c} = 0.$$

On the other hand, if $a \sim_{G_2} b$ then $c \neq b$ and hence, again,

$$Q_{xy,ab} P_{y,c} = P_{y,b} P_{x,a} (P_{y,b} P_{y,c}) = 0.$$

Similarly, if $(xy, y) \notin F_{G_1}$ but $(ab, c) \in F_{G_2}$, we obtain $Q_{xy,ab} P_{y,c} = 0$. Let ξ be a maximally entangled vector in $H \otimes H$; thus,

$$\langle (S \otimes T)\xi, \xi \rangle = \mathrm{Tr}(ST^{\mathrm{t}}), \quad S, T \in \mathcal{B}(H).$$

A perfect quantum strategy p for the isomorphism game $F_{G_1} \cong F_{G_2}$ is then given by letting

$$p(ab, c|xy, z) = \left\langle (Q_{xy,ab} \otimes P_{y,c}^{\mathrm{t}})\xi, \xi \right\rangle, \quad x, y, z, a, b, c \in X.$$

By Proposition 4.11, p gives rise to a perfect quantum strategy for the hypergraph isomorphism game $F_{G_1} \simeq F_{G_2}$.

Suppose that $F_{G_1} \simeq_{\mathrm{loc}} F_{G_2}$. By Proposition 4.5, there exist bijections $f : XX \to XX$ and $g : X \to X$ such that

$$(xy, z) \in F_{G_1} \iff (f(xy), g(z)) \in F_{G_2}.$$

We check that f is an isomorphism from $L(G_1)$ onto $L(G_2)$, thus arriving at a contradiction as in (i). Suppose that $xy \sim_{L(G_1)} zy$, where $x, y, z \in X$, $x \neq z$, $x \sim_{G_1} y \sim_{G_1} z$. Write $f(xy) = ab$ and $f(zy) = cd$. Then $g(y) = b$

and hence $d = b$. Since $a \sim_{G_2} c \sim_{G_2} b$, we have that $ab \sim_{L(G_2)} cd$, that is, $f(xy) \sim_{L(G_2)} f(zy)$. By symmetry,

$$f(xy) \sim_{L(G_2)} f(uv) \iff xy \sim_{L(G_1)} uv.$$

□

Corollary 4.17. *There exist hypergraphs E_1 and E_2 such that $E_1 \simeq_{\mathrm{q}} E_2$ but $E_1 \not\simeq_{\mathrm{loc}} E_2$.*

Proof. By [4, Theorem 6.4], there exist graphs G_1 and G_2 that are quantum isomorphic, but not isomorphic. The statement now is a consequence of Theorem 4.16. □

5. Strongly no-signalling correlations

In the rest of the paper, we restrict the setup of Sections 2-4 to the special case where the underlying hypergraphs are non-local games. We start by introducing, in this section, the types of correlations that will serve as suitable strategies.

Let X, Y, A and B be finite sets, and H be a Hilbert space. In the sequel, to simplify notation, if there is no risk of confusion, we will abbreviate an ordered pair (x, y) in $X \times Y$ to xy. A positive operator $P = (P_{xy,ab})_{xy,ab} \in \mathcal{D}_{XYAB} \otimes \mathcal{B}(H)$ will be called a *no-signalling (NS) operator matrix* if the *marginal operators*

$$P_{x,a} := \sum_{b \in B} P_{xy,ab} \quad \text{and} \quad P^{y,b} := \sum_{a \in A} P_{xy,ab}$$

are well-defined, and $(P_{x,a})_{a \in A}$ (and hence $(P^{y,b})_{b \in B}$) is a POVM for every $x \in X$ (and every $y \in Y$). This notion formed the base for the concept of a *nonsignalling operator system* in [2, Definition 5.2], although it was not defined there explicitly. An NS operator matrix $P = (P_{xy,ab})_{xy,ab}$ is called *dilatable* if there exist a Hilbert space K, an isometry $V : H \to K$ and POVM's $(E_{x,a})_{a \in A}$ and $(F_{y,b})_{b \in B}$ on K, $x \in X$, $y \in Y$, such that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ and

$$P_{xy,ab} = V^* E_{x,a} F_{y,b} V, \quad x \in X, y \in Y, a \in A, b \in B. \tag{20}$$

Remark. If the entries $P_{xy,ab}$ of an NS operator matrix are projections then

$$P_{x,a}P^{y,b} = \sum_{a' \in A} \sum_{b' \in B} P_{xy,ab'} P_{xy,a'b} = P_{xy,ab}$$

for all x, y, a, b.

We recall the operator system $\mathcal{S}_{X,A}$ and the C*-algebra $\mathcal{A}_{X,A}$, introduced before Remark 3.7, whose canonical generators are the the elements $\tilde{e}_{x,a}$ of universal PVM's $\{\tilde{e}_{x,a}\}_{a \in A}$, $x \in X$. For clarity, we will denote the canonical generators of the operator system $\mathcal{S}_{Y,B}$ by $\tilde{f}_{y,b}$, $y \in Y$, $b \in B$.

Proposition 5.1. *If $P = (P_{xy,ab})_{xy,ab}$ is a dilatable NS operator matrix acting on the Hilbert space H then there exists a unital completely positive map $\gamma : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathcal{B}(H)$, such that $\gamma(\tilde{e}_{x,a} \otimes \tilde{f}_{y,b}) = P_{xy,ab}$. Conversely, if $\gamma : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathcal{B}(H)$ is a unital completely positive map then $\left(\gamma(\tilde{e}_{x,a} \otimes \tilde{f}_{y,b})\right)_{xy,ab}$ is a dilatable NS operator matrix.*

Proof. Let K be a Hilbert space, $V : H \to K$ be an isometry, and $(E_{x,a})_{a\in A}$ and $(F_{y,b})_{b\in B}$ be mutually commuting POVM's on K satisfying (20). The linear map $\phi : \mathcal{S}_{X,A} \to \mathcal{B}(K)$ (resp. $\psi : \mathcal{S}_{Y,B} \to \mathcal{B}(K)$), given by $\phi(\tilde{e}_{x,a}) = E_{x,a}$ (resp. $\psi(\tilde{f}_{x,a}) = F_{x,a}$) is (unital and) completely positive. By the definition of the commuting tensor product, the map $\phi \cdot \psi : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathcal{B}(K)$, given by $(\phi\cdot\psi)(u\otimes v) = \phi(u)\psi(v)$, is (unital and) completely positive. Set

$$\gamma(w) = V^*(\phi \cdot \psi)(w)V, \quad w \in \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B};$$

we have that γ is unital and completely positive, and $\gamma(\tilde{e}_{x,a} \otimes \tilde{f}_{y,b}) = P_{xy,ab}$, $x \in X$, $y \in Y$, $a \in A$, $b \in B$.

Conversely, suppose that $\gamma : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathcal{B}(H)$ is a unital completely positive map. By [46, Lemma 2.8], $\mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \subseteq_{\rm c.o.i} \mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}$. Using Arveson Extension Theorem, let $\tilde{\gamma} : \mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B} \to \mathcal{B}(H)$ be a completely positive extension of γ. Applying Stinespring's Theorem, write

$$\tilde{\gamma}(w) = V^*\pi(w)V, \quad w \in \mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B},$$

for some *-representation π of $\mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}$ on a Hilbert space K and an isometry $V : H \to K$. Letting $E_{x,a} = \pi(\tilde{e}_{x,a} \otimes 1)$ and $F_{y,b} = \pi(1 \otimes \tilde{f}_{y,b})$, we obtain a representation (20) for the matrix $\left(\gamma(\tilde{e}_{x,a} \otimes \tilde{f}_{y,b})\right)_{xy,ab}$. $\square$

The following fact is implicit in the proof of Proposition 5.1:

Corollary 5.2. *If $P = (P_{xy,ab})_{xy,ab}$ is a dilatable NS operator matrix then $(E_{x,a})_{a\in A}$, $x \in X$, and $(F_{y,b})_{b\in B}$, $y \in Y$, in (20) can be chosen to be PVM's.*

Remark 5.3. There exist non-dilatable NS operator matrices whenever the cardinalities of X, Y, A are at least 2. Indeed, let $\phi : \mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B} \to \mathcal{B}(H)$ be a unital complete order embedding, and set $P_{xy,ab} = \phi(\tilde{e}_{x,a} \otimes \tilde{f}_{y,b})$, $x \in X$, $y \in Y$, $a \in A$, $b \in B$. It is clear that $(P_{xy,ab})_{xy,ab}$ is an NS operator matrix. Suppose that it is dilatable; by Proposition 5.1, there exists a unital completely positive map $\psi : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathcal{B}(H)$ such that $\psi(\tilde{e}_{x,a} \otimes \tilde{f}_{y,b}) = P_{xy,ab}$, $x \in X$, $y \in Y$, $a \in A$, $b \in B$. The map $\phi^{-1} \circ \psi : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$ is completely positive, and by the extremal property of the maximal operator system tensor product (see [34, Theorem 5.5]), it is a (unital) complete order isomorphism. By virtue of [37, Theorem 3.1], this contradicts the fact that $\mathcal{C}_{\rm ns} \neq \mathcal{C}_{\rm qc}$ (see e.g. [24, Corollary 7.12]).

In the next proposition, we identify the NS operator matrices that give rise to local NS correlations. Call a NS operator matrix $(P_{xy,ab})_{x,y,a,b}$ *locally* *dilatable* if it admits a dilation of the form (20), where the family $\{E_{x,a}, F_{y,b} : x \in X, y \in Y, a \in A, b \in B\}$ is commutative.

Proposition 5.4. *An NS correlation $p = \{(p(a,b|x,y))_{a,b} : (x,y) \in X \times Y\}$ over (X,Y,A,B) is local if and only if there exists a Hilbert space H, a locally dilatable NS operator matrix $(P_{xy,ab})_{xy,ab}$ and a unit vector $\xi \in H$ such that*

$$p(a,b|x,y) = \langle P_{xy,ab}\xi, \xi\rangle, \quad x \in X, y \in Y, a \in A, b \in B. \tag{21}$$

Proof. Assume that $p \in \mathcal{C}_{\rm loc}$, namely,

$$p = \sum_{i=1}^{k} \lambda_i p_i^{(1)} \otimes p_i^{(2)} \tag{22}$$

as a convex combination, where $p_i^{(1)} = (p_i^{(1)}(a|x))$ (resp. $p_i^{(2)} = (p_i^{(2)}(b|y))$) are conditional probability distributions. Set $P_{xy,ab} = (p_i^{(1)}(a|x)p_i^{(2)}(b|y))_{i=1}^k$, considered as a matrix in $\mathcal{D}_k$. The representation (21) is obtained by letting $\xi = (\sqrt{\lambda_i})_{i=1}^k \in \mathbb{C}^k$ and $E_{x,a}$ (resp. $F_{y,b}$) be the diagonal matrix with diagonal $(p_i^{(1)}(a|x))_{i=1}^k$ (resp. $(p_i^{(2)}(b|y))_{i=1}^k$).

Conversely, suppose that $(P_{xy,ab})_{xy,ab}$ is a locally dilatable NS operator matrix satisfying (21). By replacing the Hilbert space H with the Hilbert space K arising from the dilation (20) of $(P_{xy,ab})_{xy,ab}$, and the vector ξ with the vector $V\xi$, we may assume that $P_{xy,ab} = E_{x,a}F_{y,b}$, $x \in X$, $y \in Y$, $a \in A$, $b \in B$, where the family $\{E_{x,a}, F_{y,b} : x \in X, y \in Y, a \in A, b \in B\}$ is commutative.

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the C*-algebra, generated by $\{E_{x,a} : x \in X, a \in A\}$ (resp. $\{F_{y,b} : y \in Y, b \in B\}$), and let $s : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathbb{C}$ be the state, given by $s(S \otimes T) = \langle ST\xi, \xi\rangle$. Using the nuclearity of abelian C*-algebras, we view s as a state on $\mathcal{A} \otimes_{\min} \mathcal{B}$. Identify $\mathcal{A} = C(\Omega_1)$ and $\mathcal{B} = C(\Omega_2)$, for some compact Hausdorff spaces Ω_1 and Ω_2, and the state s with a Borel probability measure μ on the product topological space $\Omega_1 \times \Omega_2$. We thus have

$$p(a,b|x,y) = \int_{\Omega_1 \times \Omega_2} E_{x,a}(\omega_1)F_{y,b}(\omega_2)d\mu(\omega_1,\omega_2),$$

$x \in X$, $y \in Y$, $a \in A$, $b \in B$. Approximating μ with convex combinations of product measures $\mu_1 \times \mu_2$, we see that p can be approximated by convex combinations of the form (22). By the Carathéodory Theorem, the number of terms in the sum in each of the approximants of the form (22) can be chosen to be at most $|X||Y||A||B| + 1$. Using a standard compactness argument, we conclude that p is itself of the form (22). $\square$

Remark 5.5. Call an NS operator matrix $P = (P_{xy,ab})_{xy,ab}$, acting on a Hilbert space H, *quantum dilatable* if there exist families $(E_{x,a})_{a \in A}$ and $(F_{y,b})_{b \in B}$ of POVM's, acting on finite dimensional Hilbert spaces H_A and H_B, respectively, and an isometry $V : H \to H_A \otimes H_B$, such that

$$P_{xy,ab} = V^*(E_{x,a} \otimes F_{y,b})V, \quad x \in X, y \in Y, a \in A, b \in B.$$

It is clear that a NS correlation p over (X,Y,A,B) is quantum if and only if there exists a quantum dilatable NS operator matrix $P = (P_{xy,ab})_{xy,ab}$ acting on a Hilbert space H and a unit vector $\xi \in H$ such that

$$p(a,b|x,y) = \langle P_{xy,ab}\xi,\xi\rangle, \quad x\in X, y\in Y, a\in A, b\in B.$$

It follows from [3, Theorem 7.4] that an NS operator matrix is quantum dilatable precisely when its entries generate a quantum k-AOU space for some $k\in\mathbb{N}$ in the sense of [3] (we refer the reader to [3, Section 7] for the definition and the properties of the latter type of Archimedean ordered spaces).

We now introduce the suitable correlation types for hypergraph homomorphism games, provided the hypergraphs are non-local games in their own right. Thus, in the notation of Section 3, we assume that $V_i = X_iY_i$ and $W_i = A_iB_i$ for some finite sets X_i, Y_i, A_i and B_i, and let $E_i \subseteq X_iY_i \times A_iB_i$, $i=1,2$. We note that X_iY_i (resp. A_iB_i) is interpreted as the question (resp. answer) set for the game E_i, $i=1,2$. A channel

$$\Gamma : \mathcal{D}_{X_2Y_2\times A_1B_1} \to \mathcal{D}_{X_1Y_1\times A_2B_2}$$

will be called a *strongly no-signalling (SNS) correlation* if

$$\sum_{b_2\in B_2}\Gamma(x_1y_1,a_2b_2|x_2y_2,a_1b_1) = \sum_{b_2\in B_2}\Gamma(x_1y_1,a_2b_2|x_2y_2,a_1b_1'), \quad b_1,b_1'\in B_1,$$

$$\sum_{a_2\in A_2}\Gamma(x_1y_1,a_2b_2|x_2y_2,a_1b_1) = \sum_{a_2\in A_2}\Gamma(x_1y_1,a_2b_2|x_2y_2,a_1'b_1), \quad a_1,a_1'\in A_1,$$

$$\sum_{y_1\in Y_1}\Gamma(x_1y_1,a_2b_2|x_2y_2,a_1b_1) = \sum_{y_1\in Y_1}\Gamma(x_1y_1,a_2b_2|x_2y_2',a_1b_1), \quad y_2,y_2'\in Y_2,$$

and

$$\sum_{x_1\in X_1}\Gamma(x_1y_1,a_2b_2|x_2y_2,a_1b_1) = \sum_{x_1\in X_1}\Gamma(x_1y_1,a_2b_2|x_2'y_2,a_1b_1), \quad x_2,x_2'\in X_2.$$

We denote by $\mathcal{C}_{\rm sns}$ the (convex) set of all SNS correlations (the specific question-answer sets will be understood from the context), and note that $\mathcal{C}_{\rm sns}\subseteq\mathcal{C}_{\rm ns}$. For a subset $\Lambda\subseteq V_2\times W_1\times V_1\times W_2$, we write $\mathcal{C}_{\rm sns}(\Lambda)$ for the set of all SNS correlations with support contained in Λ. If $\Gamma\in\mathcal{C}_{\rm sns}$, we write

$$\Gamma(x_1y_1,a_2|x_2y_2,a_1), \quad \Gamma(x_1y_1,b_2|x_2y_2,b_1),$$

$$\Gamma(x_1,a_2b_2|x_2,a_1b_1) \ \text{ and } \ \Gamma(y_1,a_2b_2|y_2,a_1b_1)$$

for the corresponding marginal conditional probability distributions, which are well-defined by the definition of strong no-signalling. The SNS conditions imply that the further conditional probability distributions

$$\Gamma(x_1,a_2|x_2,a_1) = \sum_{y_1\in Y_1}\Gamma(x_1y_1,a_2|x_2y_2,a_1),$$

$$\Gamma(y_1,b_2|y_2,b_1) = \sum_{x_1\in X_1}\Gamma(x_1y_1,b_2|x_2y_2,b_1)$$

$$\Gamma(y_1, a_2|y_2, a_1) = \sum_{x_1 \in X_1} \Gamma(x_1y_1, a_2|x_2y_2, a_1)$$

and

$$\Gamma(x_1, b_2|x_2, b_1) = \sum_{y_1 \in Y_1} \Gamma(x_1y_1, b_2|x_2y_2, b_1)$$

are well-defined and are no-signalling correlations in their own right; for example, for $x_1 \in X_1$, $x_2 \in X_2$, $a_1, a_1' \in A_1$ and $a_2 \in A_2$ we have

$$\begin{aligned}
\sum_{a_2 \in A_2} \Gamma(x_1, a_2|x_2, a_1) &= \sum_{a_2 \in A_2} \sum_{y_1 \in Y_1} \Gamma(x_1y_1, a_2|x_2y_2, a_1) \\
&= \sum_{a_2 \in A_2} \sum_{y_1 \in Y_1} \sum_{b_2 \in B_2} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1') \\
&= \sum_{y_1 \in Y_1} \sum_{b_2 \in B_2} \sum_{a_2 \in A_2} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1') \\
&= \sum_{y_1 \in Y_1} \sum_{b_2 \in B_2} \sum_{a_2 \in A_2} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1'b_1') \\
&= \sum_{a_2 \in A_2} \Gamma(x_1, a_2|x_2, a_1').
\end{aligned}$$

Definition 5.6. *An SNS correlation Γ over the quadruple $(X_2Y_2, A_1B_1, X_1Y_1, A_2B_2)$ is called*

(i) quantum commuting *if there exist a Hilbert space H, dilatable NS operator matrices $P = (P_{x_2y_2,x_1y_1})_{x_2y_2,x_1y_1}$ and $Q = (Q_{a_1b_1,a_2b_2})_{a_1b_1,a_2b_2}$ on H with mutually commuting entries, and a unit vector $\xi \in H$, such that*

$$\Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \langle P_{x_2y_2,x_1y_1} Q_{a_1b_1,a_2b_2} \xi, \xi \rangle \tag{23}$$

for all $x_i \in X_i$, $y_i \in Y_i$, $a_i \in A_i$ and $b_i \in B_i$, $i = 1, 2$;

(ii) quantum *if there exist finite dimensional Hilbert spaces H and K, quantum dilatable NS operator matrices*

$$M = (M_{x_2a_1,x_1a_2})_{x_2,a_1,x_1,a_2} \text{ on } H, \text{ and } N = (N_{y_2b_1,y_1b_2})_{y_2,b_1,y_1,b_2} \text{ on } K,$$

and a unit vector $\xi \in H \otimes K$, such that

$$\Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \langle (M_{x_2a_1,x_1a_2} \otimes N_{y_2b_1,y_1b_2}) \xi, \xi \rangle \tag{24}$$

for all $x_i \in X_i$, $y_i \in Y_i$, $a_i \in A_i$ and $b_i \in B_i$, $i = 1, 2$;

(iii) approximately quantum *if it is a limit of quantum SNS correlations;*

(iv) local *if it is quantum, and the matrices M and N from (ii) can be chosen to be locally dilatable.*

We denote by $\mathcal{C}_{\rm sqc}$ (resp. $\mathcal{C}_{\rm sqa}$, $\mathcal{C}_{\rm sq}$ and $\mathcal{C}_{\rm sloc}$) the classes of quantum commuting (resp. approximately quantum, quantum and local) SNS correlations.

Remark 5.7. Let Γ be a local SNS correlation Γ over the quadruple $(X_2Y_2, A_1B_1, X_1Y_1, A_2B_2)$. By choosing dilations of the matrices M and N in (24) with mutually commuting entries, we can write Γ in the form

$$\Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \left\langle E_{x_2,x_1}E^{a_1,a_2}F_{y_2,y_1}F^{b_1,b_2}\xi, \xi\right\rangle,$$

where the POVM's $(E_{x_2,x_1})_{x_1}$, $(E^{a_1,a_2})_{a_2}$, $(F_{y_2,y_1})_{y_1}$ and $(F^{b_1,b_2})_{b_2}$ have mutually commuting entries. An argument, similar to the one in the proof of Proposition 5.4 now shows that $\Gamma = \sum_{i=1}^k \lambda_i \Phi_i \otimes \Psi_i$ as a convex combination for some local NS correlations $\Phi_i : \mathcal{D}_{X_2Y_2} \to \mathcal{D}_{X_1Y_1}$ and $\Psi_i : \mathcal{D}_{A_1B_1} \to \mathcal{D}_{A_2B_2}$, $i = 1, \dots, k$.

The following lemma will be used in Sections 6 and 7.

Lemma 5.8. *The SNS correlation Γ belongs to $\mathcal{C}_{\rm sqc}$ if and only if there exists a Hilbert space K, PVM's $(P_{x_2,x_1})_{x_1\in X_1}$, $(P^{y_2,y_1})_{y_1\in Y_1}$, $(Q_{a_1,a_2})_{a_2\in A_2}$ and $(Q^{b_1,b_2})_{b_2\in B_2}$ on K with mutually commuting entries, and a unit vector $\eta \in K$, such that*

$$\Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \left\langle P_{x_2,x_1}P^{y_2,y_1}Q_{a_1,a_2}Q^{b_1,b_2}\eta, \eta\right\rangle \tag{25}$$

for all $x_i \in X_i$, $y_i \in Y_i$, $a_i \in A_i$ and $b_i \in B_i$, $i = 1, 2$.

Proof. Let $\Gamma \in \mathcal{C}_{\rm sqc}$. Suppose that H is a Hilbert space,

$$P = (P_{x_2y_2,x_1y_1})_{x_2y_2,x_1y_1} \text{ and } Q = (Q_{a_1b_1,a_2b_2})_{a_1b_1,a_2b_2}$$

are dilatable NS operator matrices acting on H with mutually commuting entries, and $\xi \in H$ is a unit vector, for which (23) holds. By Proposition 5.1, [46, Lemma 2.8] and the Stinespring Theorem, there exist a Hilbert space K, a unital *-representation $\pi : \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{Y_2,Y_1} \to \mathcal{B}(K)$, and an isometry $V : H \to K$, such that

$$P_{x_2y_2,x_1y_1} = V^*\pi(\tilde{e}_{x_2,x_1} \otimes \tilde{e}_{y_2,y_1})V, \quad x_i \in X_i, y_i \in Y_i, i = 1, 2.$$

After replacing K with the closure of the span of $\pi(\mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{Y_2,Y_1})VH$, we may assume that the latter span is dense in K. Let $\mathcal{N}$ be the C*-algebra, generated by the family $\{Q_{a_1b_1,a_2b_2} : a_i \in A_i, b_i \in B_i, i = 1, 2\}$. By Arveson's Commutant Lifting Theorem [43, Theorem 12.7], there exists a *-representation

$$\rho : \mathcal{N} \to \pi(\mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{Y_2,Y_1})',$$

which is unital by the uniqueness clause of the theorem, such that

$$Vu = \rho(u)V, \quad u \in \mathcal{N}.$$

Set $\tilde{Q}_{a_1b_1,a_2b_2} = \rho(Q_{a_1b_1,a_2b_2})$, $a_i \in A_i$, $b_i \in B_i$, $i = 1, 2$. Since the map $\rho : \mathcal{S}_{A_1,A_2} \otimes_{\rm c} \mathcal{S}_{B_1,B_2} \to \mathcal{B}(K)$ is unital and completely positive, Proposition 5.1 implies that the NS operator matrix $(\tilde{Q}_{a_1b_1,a_2b_2})_{a_1,a_2,b_1,b_2}$ is dilatable. Let $\tilde{H}$ be a Hilbert space, $\tilde{\rho} : \mathcal{A}_{A_1,A_2} \otimes_{\max} \mathcal{A}_{B_1,B_2} \to \mathcal{B}(\tilde{H})$ be a unital *-representation, and $W : K \to \tilde{H}$ be an isometry, such that

$$\tilde{Q}_{a_1b_1,a_2b_2} = W^*\tilde{\rho}(\tilde{e}_{a_1,a_2} \otimes \tilde{e}_{b_1,b_2})W, \quad a_i \in A_i, b_i \in B_i, i = 1, 2;$$

assume, without loss of generality, that $\tilde{\rho}(\mathcal{A}_{A_1,A_2} \otimes_{\max} \mathcal{A}_{B_1,B_2})WK$ has dense span in $\tilde{H}$. Applying Arveson's Commutant Lifting Theorem again, we obtain a *-homomorphism

$$\tilde{\pi} : \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{Y_2,Y_1} \to \tilde{\rho}(\mathcal{A}_{A_1,A_2} \otimes_{\max} \mathcal{A}_{B_1,B_2})'$$

such that

$$W\pi(u) = \tilde{\pi}(u)W, \quad u \in \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{Y_2,Y_1}.$$

Set

$$\begin{aligned} P_{x_2,x_1} &= \tilde{\pi}(\tilde{e}_{x_2,x_1} \otimes 1), \quad & P^{y_2,y_1} &= \tilde{\pi}(1 \otimes \tilde{e}_{y_2,y_1}), \\ Q_{a_1,a_2} &= \tilde{\rho}(\tilde{e}_{a_1,a_2} \otimes 1), \quad & Q^{b_1,b_2} &= \tilde{\rho}(1 \otimes \tilde{e}_{b_1,b_2}), \end{aligned}$$

and $\eta = WV\xi$ to obtain the representation (25).

Conversely, assuming (25), we have that the NS operator matrices with entries $P_{x_2y_2,x_1y_1} := P_{x_2,x_1}P^{y_2,y_1}$ and $Q_{a_1b_1,a_2b_2} := Q_{a_1,a_2}Q^{b_1,b_2}$ are (trivially dilatable and) commuting, showing that Γ is a quantum commuting SNS correlation. $\square$

6. Homomorphisms between non-local games

In this section, we demonstrate how hypergraph homomorphisms, defined in Sections 3 and 4, give rise to homomorphisms between non-local games. In the next subsection we restrict the simulation paradigm to the case of NS correlations.

6.1. **Strategy transport.** Let X_i, Y_i, A_i and B_i be finite sets, $i = 1, 2$.

Theorem 6.1. *Let Γ be an SNS correlation over the quadruple $(X_2Y_2, A_1B_1, X_1Y_1, A_2B_2)$ and $\mathcal{E}$ be an NS correlation over the quadruple (X_1, Y_1, A_1, B_1). The following hold:*

(i) $\Gamma[\mathcal{E}] \in \mathcal{C}_{\rm ns}$;
(ii) *if* $\Gamma \in \mathcal{C}_{\rm sqc}$ *and* $\mathcal{E} \in \mathcal{C}_{\rm qc}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}_{\rm qc}$;
(iii) *if* $\Gamma \in \mathcal{C}_{\rm sqa}$ *and* $\mathcal{E} \in \mathcal{C}_{\rm qa}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}_{\rm qa}$;
(iv) *if* $\Gamma \in \mathcal{C}_{\rm sq}$ *and* $\mathcal{E} \in \mathcal{C}_{\rm q}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}_{\rm q}$;
(v) *if* $\Gamma \in \mathcal{C}_{\rm sloc}$ *and* $\mathcal{E} \in \mathcal{C}_{\rm loc}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}_{\rm loc}$.

Proof. (i) Set $\mathcal{F} = \Gamma[\mathcal{E}]$, and fix $x_2 \in X_2$, $y_2 \in Y_2$ and $a_2 \in A_2$. We have

$$\begin{aligned} & \sum_{b_2 \in B_2} \mathcal{F}(a_2, b_2|x_2, y_2) \\ = \quad & \sum_{b_2 \in B_2} \sum_{x_1y_1 \in X_1Y_1} \sum_{a_1b_1 \in A_1B_1} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1)\mathcal{E}(a_1, b_1|x_1, y_1) \\ = \quad & \sum_{x_1y_1 \in X_1Y_1} \sum_{a_1b_1 \in A_1B_1} \Gamma(x_1y_1, a_2|x_2y_2, a_1)\mathcal{E}(a_1, b_1|x_1, y_1) \\ = \quad & \sum_{x_1y_1 \in X_1Y_1} \sum_{a_1 \in A_1} \Gamma(x_1y_1, a_2|x_2y_2, a_1)\mathcal{E}(a_1|x_1) \\ = \quad & \sum_{x_1 \in X_1} \sum_{a_1 \in A_1} \Gamma(x_1, a_2|x_2, a_1)\mathcal{E}(a_1|x_1), \end{aligned}$$

and hence the marginal $\mathcal{F}(a_2|x_2)$ is well-defined. Similarly, the marginal $\mathcal{F}(b_2|y_2)$ is well-defined.

(ii) Appealing to Lemma 5.8, let $(P_{x_2,x_1})_{x_1\in X_1}$, $(P^{y_2,y_1})_{y_1\in Y_1}$, $(Q_{a_1,a_2})_{a_2\in A_2}$ and $(Q^{b_1,b_2})_{b_2\in B_2}$ be mutually commuting POVM's on a Hilbert space H and $\xi \in H$ be a unit vector, such that

$$\Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \left\langle P_{x_2,x_1}P^{y_2,y_1}Q_{a_1,a_2}Q^{b_1,b_2}\xi, \xi\right\rangle$$

for all x_i, y_i, a_i, b_i, $i = 1, 2$. Let $(E_{x_1,a_1})_{a_1\in A_1}$ and $(F_{y_1,b_1})_{b_1\in B_1}$ be mutually commuting families of POVM's on a Hilbert space K and $\eta \in H$ be a unit vector such that

$$\mathcal{E}(a_1, b_1|x_1, y_1) = \langle E_{x_1,a_1}F_{y_1,b_1}\eta, \eta\rangle, \quad x_1 \in X_1, y_1 \in Y_1, a_1 \in A_1, b_1 \in B_1.$$

Set

$$\tilde{E}_{x_2,a_2} = \sum_{x_1\in X_1}\sum_{a_1\in A_1} P_{x_2,x_1}Q_{a_1,a_2} \otimes E_{x_1,a_1} \tag{26}$$

and

$$\tilde{F}_{y_2,b_2} = \sum_{y_1\in Y_1}\sum_{b_1\in B_1} P^{y_2,y_1}Q^{b_1,b_2} \otimes F_{y_1,b_1}. \tag{27}$$

We have that $(\tilde{E}_{x_2,a_2})_{a_2\in A_2}$ (resp. $(\tilde{F}_{y_2,b_2})_{b_2\in B_2}$) is a POVM, $x_2 \in X_2$ (resp. $y_2 \in Y_2$), acting on the Hilbert space $H \otimes K$. In addition,

$$\tilde{E}_{x_2,a_2}\tilde{F}_{y_2,b_2} = \tilde{F}_{y_2,b_2}\tilde{E}_{x_2,a_2}, \quad x_2 \in X_2, y_2 \in Y_2, a_2 \in A_2, b_2 \in B_2,$$

and

$$\Gamma[\mathcal{E}](a_2, b_2|x_2, y_2) = \left\langle \tilde{E}_{x_2,a_2}\tilde{F}_{y_2,b_2}(\xi \otimes \eta), \xi \otimes \eta\right\rangle$$

for all x_2, y_2, a_2, b_2, showing that $\Gamma[\mathcal{E}]$ is quantum commuting.

(iv) Let $M = (M_{x_2a_1,x_1a_2})_{x_2,a_1,x_1,a_2}$ and $N = (N_{y_2b_1,y_1b_2})_{y_2,b_1,y_1,b_2}$ be quantum dilatable NS operator matrices, acting on finite dimensional Hilbert spaces H and K, respectively, and $\xi \in H \otimes K$ be a unit vector, for which Γ admits a representation of the form (24). Write

$$\mathcal{E}(a_1, b_1|x_1, y_1) = \langle (E_{x_1,a_1} \otimes F_{y_1,b_1})\eta, \eta\rangle, \; x_1 \in X_1, y_1 \in Y_1, a_1 \in A_1, b_1 \in B_1,$$

where $(E_{x_1,a_1})_{a_1\in A_1}$ and $(F_{y_1,b_1})_{b_1\in B_1}$ are finite dimensionally acting. Define

$$\tilde{E}_{x_2,a_2} = \sum_{x_1\in X_1}\sum_{a_1\in A_1} M_{x_2a_1,x_1a_2} \otimes E_{x_1,a_1}$$

and

$$\tilde{F}_{y_2,b_2} = \sum_{y_1\in Y_1}\sum_{b_1\in B_1} N_{y_2b_1,y_1b_2} \otimes F_{y_1,b_1};$$

it is straightforward to see that $(\tilde{E}_{x_2,a_2})_{a_2\in A_2}$ (resp. $(\tilde{F}_{y_2,b_2})_{b_2\in B_2}$) is a finite dimensionally acting POVM. The proof in (ii) can now continue without further modification.

(iii) is a direct consequence of (iv).

(v) follows from Remark 5.7 and the fact that, if $\Phi_X : \mathcal{D}_{X_2} \to \mathcal{D}_{X_1}$, $\Phi_Y : \mathcal{D}_{Y_2} \to \mathcal{D}_{Y_1}$, $\Psi_A : \mathcal{D}_{A_1} \to \mathcal{D}_{A_2}$ and $\Psi_B : \mathcal{D}_{B_1} \to \mathcal{D}_{B_2}$ are channels then

$$(\Phi_X \otimes \Phi_Y \otimes \Psi_A \otimes \Psi_B)[\mathcal{E} \otimes \mathcal{F}] = (\Phi_X \otimes \Psi_A)[\mathcal{E}] \otimes (\Phi_Y \otimes \Psi_B)[\mathcal{F}]$$

for all channels $\mathcal{E} : \mathcal{D}_{X_1} \to \mathcal{D}_{A_1}$ and $\mathcal{F} : \mathcal{D}_{Y_1} \to \mathcal{D}_{B_1}$. $\square$

Remark 6.2. The proof of Theorem 6.1 (i) shows that, in its notation, letting $\Gamma_{X_2A_1 \to X_1A_2}$ be the NS correlation determined by the conditional probability distributions $\Gamma(x_1, a_2|x_2, a_1)$, we have that

$$\Gamma_{X_2A_1 \to X_1A_2}[\mathcal{E}_{X_1 \to A_1}] = \Gamma[\mathcal{E}]_{X_2 \to A_2}.$$

We next define the suitable version of the notion of a bicorrelation, defined in [10], in the strongly no-signalling context. Assume that $X_1 = X_2 =: X$, $Y_1 = Y_2 =: Y$, $A_1 = A_2 =: A$ and $B_1 = B_2 =: B$. We further assume that $X = A$ and $Y = B$. A positive operator $P = (P_{xy,ab})_{xy,ab} \in \mathcal{D}_{XYAB} \otimes \mathcal{B}(H)$ will be called a *magic bisquare* if it is an NS operator matrix, and the matrices $(P_{x,a})_{x,a}$ and $(P^{y,b})_{y,b}$ are quantum magic squares. A magic bisquare $P = (P_{x,y,a,b})_{xy,ab}$ is called *dilatable* if there exist a Hilbert space K, an isometry $V : H \to K$ and quantum magic squares $(E_{x,a})_{x,a\in X}$ and $(F_{y,b})_{y,b\in Y}$ on K, $x \in X$, $y \in Y$, such that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ and relations (20) hold for all $x, a \in X$ and all $y, b \in Y$. *Quantum dilatable* and *locally dilatable* magic bisquares are described similarly to quantum and local NS operator matrices, using quantum magic squares in the place of families of POVM's.

An SNS correlation Γ will be called an *SNS bicorrelation* if Γ is unital and Γ^* is also an SNS correlation. An SNS bicorrelation Γ over the quadruple (XY, XY, XY, XY) is called *quantum commuting* if there exist a Hilbert space H, dilatable magic bisquares $P = (P_{x_2y_2,x_1y_1})_{x_2y_2,x_1y_1}$ and $Q = (Q_{a_1b_1,a_2b_2})_{a_1b_1,a_2b_2}$ on H with mutually commuting entries, and a unit vector $\xi \in H$, such that equation (23) holds. The classes of quantum SNS bicorrelations (denoted $\mathcal{C}^{\rm bi}_{\rm sq}$), approximately quantum SNS bicorrelations (denoted $\mathcal{C}^{\rm bi}_{\rm sqa}$), and local SNS bicorrelations (denoted $\mathcal{C}^{\rm bi}_{\rm sloc}$) are described similarly to the their correlation counterparts, using magic bisquares of the appropriate type in the place of NS operator matrices of that type. The following remark is straightforward from the definitions:

Remark 6.3. *For a correlation type* $\mathrm{t} \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}, \mathrm{ns}\}$*, if* $\Gamma \in \mathcal{C}^{\rm bi}_{\rm st}$ *then* $\Gamma^* \in \mathcal{C}^{\rm bi}_{\rm st}$.

Theorem 6.4. *Let* Γ *be an SNS bicorrelation over the quadruple* (XY, AB, XY, AB) *and* $\mathcal{E}$ *be an NS correlation over the quadruple* (X, Y, A, B)*. The following hold:*

(i) $\Gamma[\mathcal{E}] \in \mathcal{C}^{\rm bi}_{\rm ns}$;
(ii) *if* $\Gamma \in \mathcal{C}^{\rm bi}_{\rm sqc}$ *and* $\mathcal{E} \in \mathcal{C}^{\rm bi}_{\rm qc}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}^{\rm bi}_{\rm qc}$;
(iii) *if* $\Gamma \in \mathcal{C}^{\rm bi}_{\rm sqa}$ *and* $\mathcal{E} \in \mathcal{C}^{\rm bi}_{\rm qa}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}^{\rm bi}_{\rm qa}$;
(iv) *if* $\Gamma \in \mathcal{C}^{\rm bi}_{\rm sq}$ *and* $\mathcal{E} \in \mathcal{C}^{\rm bi}_{\rm q}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}^{\rm bi}_{\rm q}$;

(v) *if* $\Gamma \in \mathcal{C}_{\rm sloc}^{\rm bi}$ *and* $\mathcal{E} \in \mathcal{C}_{\rm loc}^{\rm bi}$ *then* $\Gamma[\mathcal{E}] \in \mathcal{C}_{\rm loc}^{\rm bi}$.

Proof. (i) The claim follows as in Theorem 6.1, using the fact that in our case we have, in addition, that $\Gamma[\mathcal{E}]^* = \Gamma^*[\mathcal{E}^*]$. We verify that latter identity:

$$\begin{aligned}
& \Gamma[\mathcal{E}]^*(x_2, y_2|a_2, b_2) \\
= & \sum_{x_1y_1 \in X_1Y_1} \sum_{a_1b_1 \in A_1B_1} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1)\mathcal{E}(a_1, b_1|x_1, y_1) \\
= & \sum_{a_1b_1 \in A_1B_1} \sum_{x_1y_1 \in X_1Y_1} \Gamma^*(x_2y_2, a_1b_1|x_1y_1, a_2b_2)\mathcal{E}^*(x_1, y_1|a_1, b_1) \\
= & \Gamma^*[\mathcal{E}^*](x_2, y_2|a_2, b_2),
\end{aligned}$$

for all $x_2 \in X, y_2 \in Y, a_2 \in A, b_2 \in B$.

(ii) follows as Theorem 6.1 (ii), after noting that the POVM's defined in (26) and (27) are quantum magic squares. The proofs of (iii), (iv) and (v) is now similar to those of Theorem 6.1 (iii), (iv) and (v). $\square$

Definition 6.5. *Let* $E_i \subseteq X_iY_i \times A_iB_i$ *be a non-local game,* $i = 1, 2$, *and* $\rm t \in \{loc, q, qa, qc, ns\}$. *We say that*

(i) E_1 is t-quasi-homomorphic to E_2 *(and write* $E_1 \rightsquigarrow_{\rm st} E_2$*) if the hypergraph quasi-homomorphism game* $E_1 \rightsquigarrow E_2$ *has a perfect strategy* $\Gamma \in \mathcal{C}_{\rm st}$*;*
(ii) E_1 is t-homomorphic to E_2 *(and write* $E_1 \to_{\rm st} E_2$*) if the hypergraph homomorphism game* $E_1 \to E_2$ *has a perfect strategy* $\Gamma \in \mathcal{C}_{\rm st}^{\rm bi}$*;*
(iii) E_1 is t-isomorphic to E_2 *(and write* $E_1 \simeq_{\rm st} E_2$*) if the hypergraph isomorphism game* $E_1 \leftrightarrow E_2$ *has a perfect strategy* $\Gamma \in \mathcal{C}_{\rm st}^{\rm bi}$.

Corollary 6.6. *Let* $E_i \subseteq X_iY_i \times A_iB_i$ *be a non-local game,* $i = 1, 2$, *and* $\rm t \in \{loc, q, qa, qc, ns\}$. *If* $E_1 \to_{\rm st} E_2$ *and* E_1 *has a perfect* t*-strategy then* E_2 *has a perfect* t*-strategy.*

Proof. The statement follows after an application of Theorem 6.1 and Proposition 3.2. $\square$

Theorem 6.7. *For* $\rm t \in \{loc, q, qa, qc, ns\}$, *the relations* $\to_{\rm st}$ *and* $\rightsquigarrow_{\rm st}$ *(resp. the relation* $\simeq_{\rm st}$*) are quasi-orders (resp. is an equivalence relation) on the set of non-local games.*

Proof. Let Γ_1 and Γ_2 be elements of $\mathcal{C}_{\rm sns}$. The range of the variables in the summations below being understood from the context, we have

$$\begin{aligned}
&\sum_{b_3}(\Gamma_2 * \Gamma_1)(x_1y_1, a_3b_3|x_3y_3, a_1b_1)\\
= \;&\sum_{b_3}\sum_{x_2,y_2}\sum_{a_2,b_2}\Gamma_1(x_1y_1, a_2b_2|x_2y_2, a_1b_1)\Gamma_2(x_2y_2, a_3b_3|x_3y_3, a_2b_2)\\
= \;&\sum_{x_2,y_2}\sum_{a_2,b_2}\Gamma_1(x_1y_1, a_2b_2|x_2y_2, a_1b_1)\Gamma_2(x_2y_2, a_3|x_3y_3, a_2)\\
= \;&\sum_{x_2,y_2}\sum_{a_2}\Gamma_1(x_1y_1, a_2|x_2y_2, a_1)\Gamma_2(x_2y_2, a_3|x_3y_3, a_2)\\
= \;&\sum_{b_3}(\Gamma_2 * \Gamma_1)(x_1y_1, a_3b_3|x_3y_3, a_1b_1')
\end{aligned}$$

for all b_1, b_1'. One verifies similarly the remaining three relations required in the definition of an SNS correlations; thus, $\Gamma_2 * \Gamma_1 \in \mathcal{C}_{\rm sns}$. Using Remark 6.3, we now also see that, if Γ_1 and Γ_2 be elements of $\mathcal{C}_{\rm sns}^{\rm bi}$ then $\Gamma_2 * \Gamma_1 \in \mathcal{C}_{\rm sns}^{\rm bi}$. The cases where $\mathrm{t} = \mathrm{ns}$ now follows taking into account Theorem 3.3.

Among the remaining correlation types, we only consider the case $\mathrm{t} = \mathrm{qc}$. Suppose that $\Gamma_1, \Gamma_2 \in \mathcal{C}_{\rm sqc}$. Suppose that $P = (P_{x_2y_2,x_1y_1})_{x_2y_2,x_1y_1}$ and $Q = (Q_{a_1b_1,a_2b_2})_{a_1b_1,a_2b_2}$ are dilatable NS operator matrices on a Hilbert space H with mutually commuting entries, and $\xi \in H$ is a unit vector, such that

$$\Gamma_1(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \langle P_{x_2y_2,x_1y_1}Q_{a_1b_1,a_2b_2}\xi, \xi\rangle$$

for all $x_i \in X_i$, $y_i \in Y_i$, $a_i \in A_i$ and $b_i \in B_i$, $i = 1, 2$. Similarly, write

$$\Gamma_2(x_2y_2, a_3b_3|x_3y_3, a_2b_2) = \left\langle P'_{x_3y_3,x_2y_2}Q'_{a_2b_2,a_3b_3}\xi', \xi'\right\rangle,$$

for all $x_i \in X_i$, $y_i \in Y_i$, $a_i \in A_i$ and $b_i \in B_i$, $i = 2, 3$, in a Hilbert space H'. Similarly to the proof of Theorem 2.1, set

$$P''_{x_3y_3,x_1y_1} = \sum_{x_2\in X_2}\sum_{y_2\in Y_2} P_{x_2y_2,x_1y_1} \otimes P'_{x_3y_3,x_2y_2},$$

$$Q''_{a_1b_1,a_3b_3} = \sum_{a_2\in A_2}\sum_{b_2\in B_2} Q_{a_1b_1,a_2b_2} \otimes Q'_{a_2b_2,a_3b_3},$$

and $\xi'' = \xi \otimes \xi'$, in the Hilbert space $H'' := H \otimes H'$. It is straighforward that $(P''_{x_3y_3,x_1y_1})$ and $(Q''_{a_1b_1,a_3b_3})$ are dilatable NS operator matrices with commuting entries, which give rise to the correlation $\Gamma_2 * \Gamma_1$ as in (23).

The case where Γ_1 and Γ_2 are (quantum commuting) SNS bicorrelations follows from the previous paragraph and the fact that the matrices $(P''_{x_3y_3,x_1y_1})$ and $(Q''_{a_1b_1,a_3b_3})$ are magic bisquares, provided $(P_{x_2y_2,x_1y_1})$, $(Q_{a_1b_1,a_2b_2})$, $(P'_{x_3y_3,x_2y_2})$ and $(Q'_{a_2b_2,a_3b_3})$ are such. $\square$

Remark 6.8. The statements in Theorem 6.1 are not reversible: e.g. not every affine map $\mathcal{C}_{\rm ns}(E_1) \mapsto \mathcal{C}_{\rm ns}(E_2)$ has the form $\mathcal{E} \to \Gamma[\mathcal{E}]$ for some $\Gamma \in \mathcal{C}_{\rm sqc}$

that fits $E_1 \leftrightarrow E_2$. Indeed, let $E_i = [1] \times Y_i \times [1] \times B_i$, $i = 1, 2$. In this case, $\mathcal{C}_{\rm ns}(E_i) = \mathcal{C}(Y_i \times B_i)$, $i = 1, 2$, and the statement follows from Remark 3.8.

Remark 6.9. Let $E_i \subseteq X_iY_i \times A_iB_i$ be a non-local game, $i = 1, 2$. It follows from Remark 5.7 and Proposition 4.5 (i) that $E_1 \to_{\rm loc} E_2$ if and only if there exist maps $f_X : X_2 \to X_1$, $f_Y : Y_2 \to Y_1$, $g_A : A_1 \to A_2$ and $g_B : B_1 \to B_2$ such that

$$(28)\quad (f_X \times f_Y)^{-1}(E_1((a_1, b_1))) = E_2((g_A(a_1), g_B(b_1))), \quad (a_1, b_1) \in A_1 \times B_1.$$

Similarly, if $X_1 = X_2$, $Y_1 = Y_2$, $A_1 = A_2$ and $B_1 = B_2$ then $E_1 \simeq_{\rm loc} E_2$ if and only if the maps f_X, f_Y, g_A and g_B can be chosen bijective, that is, there exist permutations on the corresponding question and answer sets of the four players that transform the rules functions of the two games into one another. Finally, $E_1 \rightsquigarrow_{\rm loc} E_2$ amounts to having inclusions in (28) instead of equalities.

Remark 6.10. By Proposition 3.6, there exists hypergraphs $E_i \subseteq X_i \times A_i$, $i = 1, 2$, such that $E_1 \rightsquigarrow_{\rm q} E_2$ via a quantum correlation Γ, but $E_1 \not\rightsquigarrow_{\rm loc} E_2$. Let

$$\tilde{E}_i = \{((1, x), (1, a)) : (x, a) \in E_i\},$$

considered as a non-local game over $([1], X_i, [1], A_i)$, $i = 1, 2$. It is straightforward that the correlation $\tilde{\Gamma}$, given by

$$\tilde{\Gamma}(1x_1, 1a_2|1x_2, 1a_1) := \Gamma(x_1, a_2|x_2, a_1),$$

is a quantum SNS correlation that realises a homomorphism $\tilde{E}_1 \rightsquigarrow_{\rm q} \tilde{E}_2$. On the other hand, $\tilde{E}_1 \not\rightsquigarrow_{\rm loc} \tilde{E}_2$ as the relation $\tilde{E}_1 \rightsquigarrow_{\rm loc} \tilde{E}_2$ would force $E_1 \rightsquigarrow_{\rm loc} E_2$.

6.2. **Optimal winning probabilities.** The results in this subsection can be viewed as a strengthening of Corollary 6.6. We first recall the notion of a *product game* from non-local game theory (see e.g. [39]). Given non-local games $\Lambda_k \subseteq X_2^{(k)}Y_1^{(k)} \times X_1^{(k)}Y_2^{(k)}$, $k = 1, 2$, their *product* $\Lambda_1 \otimes \Lambda_2$ is the non-local game on $X_2^{(1)}X_2^{(2)}Y_1^{(1)}Y_1^{(2)} \times X_1^{(1)}X_1^{(2)}Y_2^{(1)}Y_2^{(2)}$ arising from the product set $\Lambda_1 \times \Lambda_2$ after the natural reshuffling (see [39, Section 3]). Here, $(X_2^{(1)}X_2^{(2)}, Y_1^{(1)}Y_1^{(2)})$ (resp. $(X_1^{(1)}X_1^{(2)}, Y_2^{(1)}Y_2^{(2)})$) is the corresponding question (resp. answer) set.

Lemma 6.11. *Let $X_i^{(k)}$, $Y_i^{(k)}$, $A_i^{(k)}$ and $B_i^{(k)}$ be finite sets, $i, k = 1, 2$, and*

$$\Gamma_k : \mathcal{D}_{X_2^{(k)}Y_2^{(k)} \times A_1^{(k)}B_1^{(k)}} \to \mathcal{D}_{X_1^{(k)}Y_1^{(k)} \times A_2^{(k)}B_2^{(k)}}$$

be an SNS correlation, $k = 1, 2$.

(i) *If $\Gamma_k \in \mathcal{C}_{\rm st}$, $k = 1, 2$, then $\Gamma_1 \otimes \Gamma_2 \in \mathcal{C}_{\rm st}$;*
(ii) *If $\Gamma_k \in \mathcal{C}_{\rm st}^{\rm bi}$, $k = 1, 2$, then $\Gamma_1 \otimes \Gamma_2 \in \mathcal{C}_{\rm st}^{\rm bi}$.*

Proof. We give details for the case $\rm t = qc$ only; the arguments for the rest of the correlation types follow along similar lines.

(i) Let $P^{(k)} = (P^{(k)}_{x_2y_2,x_1y_1})_{x_2y_2,x_1y_1}$ and $Q^{(k)} = (Q^{(k)}_{a_1b_1,a_2b_2})_{a_1b_1,a_2b_2}$ on H_k with mutually commuting entries, and a unit vector $\xi_k \in H$, such that

$$\Gamma^{(k)}(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \left\langle P^{(k)}_{x_2y_2,x_1y_1} Q^{(k)}_{a_1b_1,a_2b_2}\xi, \xi\right\rangle, \quad k = 1, 2,$$

for all $x_i \in X_i^{(k)}$, $y_i \in Y_i^{(k)}$, $a_i \in A_i^{(k)}$ and $b_i \in B_i^{(k)}$, $i = 1, 2$. Letting $H = H_1 \otimes H_2$, $\xi = \xi_1 \otimes \xi_2$,

$$P_{x_2y_2,x_1y_1} = P^{(1)}_{x_2y_2,x_1y_1} \otimes P^{(2)}_{x_2y_2,x_1y_1} \quad \text{and} \quad Q_{a_1b_1,a_2b_2} = Q^{(1)}_{a_1b_1,a_2b_2} \otimes Q^{(2)}_{a_1b_1,a_2b_2},$$

we see that the operator matrices $P = (P_{x_2y_2,x_1y_1})$ and $Q = (Q_{a_1b_1,a_2b_2})$ are dilatable and have mutually commuting entries. In addition, $\Gamma_1 \otimes \Gamma_2$ arises as in equation (23) from Definition 5.6 via the quadruple (H, ξ, P, Q).

(ii) is similar to (i); the case $\mathrm{t} = \mathrm{ns}$ uses the fact that $(\Gamma_1 \otimes \Gamma_2)^* = \Gamma_1^* \otimes \Gamma_2^*$, the case $\mathrm{t} = \mathrm{qc}$ – the fact that the NS operator matrices $(P_{x_2y_2,x_1y_1})_{x_2y_2,x_1y_1}$ and $(Q_{a_1b_1,a_2b_2})_{a_1b_1,a_2b_2}$ defined in (i) are magic bisquares, and the rest of the cases are analogous. □

Given a correlation type t, the t-*value* of a non-local game $E \subseteq V_2W_1 \times V_1W_2$, equipped with a probability distribution π on V_2W_1, is the parameter $\omega_{\mathrm{t}}(E, \pi) = \omega_{\mathcal{C}_{\mathrm{t}}}(E, \pi)$ (see (13) in Subsection 4.3). We set

$$\bar{\omega}_{\mathrm{t}}(E, \pi) = \limsup_{n\in\mathbb{N}} \omega_{\mathcal{C}_{\mathrm{t}}}(E^{\otimes n}, \pi^n)^{\frac{1}{n}};$$

the parameter $\bar{\omega}_{\mathrm{t}}(E, \pi)$ is the optimal t-value of the game E under *parallel repetition* [49].

Theorem 6.12. *Let $E_i \subseteq X_iY_i \times A_iB_i$ be a non-local game, π_2 be a probability distribution on X_2Y_2, and $\pi_1 = \Gamma_{X_2Y_2\to X_1Y_1}(\pi_2)$.*

(i) *If $E_1 \to_{\mathrm{st}} E_2$ via $\Gamma \in \mathcal{C}_{\mathrm{st}}$ then*

$$\omega_{\mathrm{t}}(E_1, \pi_1) \leq \omega_{\mathrm{t}}(E_2, \pi_2) \quad \textit{and} \quad \bar{\omega}_{\mathrm{t}}(E_1, \pi_1) \leq \bar{\omega}_{\mathrm{t}}(E_2, \pi_2);$$

(ii) *Suppose that $X_1 = X_2 =: X$, $Y_1 = Y_2 =: Y$ and that that π is a Γ-stationary probability distribution on XY If $E_1 \simeq_{\mathrm{st}} E_2$ via $\Gamma \in \mathcal{C}_{\mathrm{st}}^{\mathrm{bi}}$ then*

$$\omega_{\mathrm{t}}(E_1, \pi) = \omega_{\mathrm{t}}(E_2, \pi) \quad \textit{and} \quad \bar{\omega}_{\mathrm{t}}(E_1, \pi) = \bar{\omega}_{\mathrm{t}}(E_2, \pi).$$

Proof. The statements in (i) and (ii) regarding the t-values follow from Theorems 6.1 and 6.4, and Proposition 4.6. The statements regarding the parameters $\bar{\omega}_{\mathrm{t}}$ follow from Lemma 6.11 and Corollary 4.8. □

7. Representations of SNS correlations

Our goal in this section is to obtain representations of the quantum commuting and the approximately quantum correlation types in terms of operator system tensor products; this is achieved in Subsection 7.2. In the next subsection, we develop the required multivariate tensor product theory in the operator system category, which extends the bivariate theory developed in [34] and may be of interest in its own right. We will emphasise the differences with the bivariate theory, and will omit those arguments that can be easily adapted from [34].

7.1. **Multivariate operator system tensor products.** We fix throughout this subsection operator systems $\mathcal{S}_1, \dots, \mathcal{S}_k$. Following [34], we define a *tensor product* of $\mathcal{S}_1, \dots, \mathcal{S}_k$ (in this order) to be an operator system structure $\sigma = (\Sigma_n)_{n\in\mathbb{N}}$ on the algebraic tensor product $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_k$ satisfying the following properties:

(T1) $(\otimes_{j=1}^k \mathcal{S}_j, (\Sigma_n)_{n\in\mathbb{N}}, \otimes_{j=1}^k 1_{\mathcal{S}_j})$ is an operator system;
(T2) if $P_j \in M_{n_j}(\mathcal{S}_j)^+$, $j \in [k]$, then $\otimes_{j=1}^k P_j \in \Sigma_{n_1\cdots n_k}$, and
(T3) if $f_j : \mathcal{S}_j \to M_{l_j}$ is a unital completely positive map, $j \in [k]$, then $\otimes_{j=1}^k f_j$ is completely positive.

A *tensor k-product* in the operator system category is an assignment σ of a tensor product $\sigma\text{-}\otimes_{j=1}^k \mathcal{S}_j$ to each ordered k-tuple $(\mathcal{S}_1, \dots, \mathcal{S}_k)$ of operator systems. We call σ *functorial* if, whenever $\mathcal{T}_j$ are operator systems and $\psi_j : \mathcal{S}_j \to \mathcal{T}_j$ are unital completely positive maps, $j \in [k]$, we have that the tensor product map $\otimes_{j=1}^k \psi_j : \sigma\text{-}\otimes_{j=1}^k \mathcal{S}_j \to \sigma\text{-}\otimes_{j=1}^k \mathcal{T}_j$ is (unital and) completely positive.

We will see that, as in the bivariate case, the operator system category admits natural minimal, maximal and commuting tensor k-products. Equip the algebraic tensor product $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_k$ with the involution, given by

$$(u_1 \otimes \cdots \otimes u_k)^* := u_1^* \otimes \cdots \otimes u_k^*,$$

and extend it to an involution on $M_n(\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_k)$ by letting $(u_{i,j})_{i,j}^* := (u_{j,i}^*)_{i,j}$. Write $M_n(\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_k)_h$ for the real vector space of all hermitian elements of $M_n(\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_k)$.

7.1.1. *The maximal tensor product.* By [34, Theorem 5.5], the maximal tensor product between two operator systems is associative; we can thus unambiguously give a meaning to the multivariate maximal tensor product

$$\max\text{-}\otimes_{j=1}^k \mathcal{S}_j := \mathcal{S}_1 \otimes_{\max} \cdots \otimes_{\max} \mathcal{S}_k.$$

We will need an explicit description of its positive cones, in the spirit of the one given in [34, Section 5]. For notational simplicity, we restrict to the case $k = 3$. For each $n \in \mathbb{N}$, let

$$D_n^{\max} = \{\alpha(P_1 \otimes P_2 \otimes P_3)\alpha^* : \ P_i \in M_{n_i}(\mathcal{S}_i)^+, \ \alpha \in M_{n,n_1n_2n_3}, n_i \in \mathbb{N}, i \in [3]\}.$$

Remark 7.1. Let $\mathcal{S}_i, i = 1, 2, 3$ be operator systems and $(D_n)_{n=1}^\infty$ be a matrix ordering, with $D_n \subseteq M_n(\mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \mathcal{S}_3)$, satisfying property (T2) from the start of Subsection 7.1. Then the compatibility condition implies that $D_n^{\max} \subseteq D_n$ for all $n \in \mathbb{N}$.

It is straightforward to verify that $1_{\mathcal{S}_1} \otimes 1_{\mathcal{S}_2} \otimes 1_{\mathcal{S}_3}$ is a matrix order unit for the matrix ordering $(D_n^{\max})_{n=1}^\infty$; we let $(C_n^{\max})_{n=1}^\infty$ be its Archimedeanisation [47, Section 3.1].

Proposition 7.2. *Let $\mathcal{S}_i, i = 1, 2, 3,$ be operator systems. Then*

$$C_n^{\max} = M_n\left((\mathcal{S}_1 \otimes_{\max} \mathcal{S}_2) \otimes_{\max} \mathcal{S}_3\right)^+, \quad n \in \mathbb{N}.$$

Proof. Set $C_n = M_n\left((\mathcal{S}_1 \otimes_{\max} \mathcal{S}_2) \otimes_{\max} \mathcal{S}_3\right)^+$, $n \in \mathbb{N}$. By Remark 7.1, $C_n^{\max} \subseteq C_n$. Let $Q \in M_n((\mathcal{S}_1 \otimes_{\max} \mathcal{S}_2) \otimes_{\max} \mathcal{S}_3)^+$; without loss of generality, we may assume that $Q = \alpha(Q_1 \otimes R_1)\alpha^*$ for $Q_1 \in M_k(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_2)^+$, $R_1 \in M_m(\mathcal{S}_3)^+$ and $\alpha \in M_{n,km}$. As $Q_1 \in M_k(\mathcal{S}_1 \otimes_{\max} \mathcal{S}_2)^+$, again without loss of generality we may assume that $Q_1 = \beta(P_1 \otimes P_2)\beta^*$, where $P_1 \in M_\ell(\mathcal{S}_1)^+, P_2 \in M_p(\mathcal{S}_2)^+$, and $\beta \in M_{k,\ell p}$. Write $\beta = \sum_{i=1}^{r} C_i \otimes D_i$ for some $C_i \in M_{n_1,\ell}$ and $D_i \in M_{n_2,p}$ with $n_1 n_2 = k$. We see that

$$\beta(P_1 \otimes P_2)\beta^* = \sum_{i,j=1}^{r} C_i P_1 C_j^* \otimes D_i P_2 D_j^*.$$

Let $P_1' = (C_i P_1 C_j^*)_{i,j}^r$ and $P_2' = (D_i P_2 D_j^*)_{i,j}^r$; thus, $P_1' \in M_{rn_1}(\mathcal{S}_1)^+$ and $P_2' \in M_{rn_2}(\mathcal{S}_2)^+$, and we view $P_1' \otimes P_2'$ as a $r^2k \times r^2k$ matrix.

Let $\beta' = (\beta_1, \dots, \beta_{r^2})$, where $\beta_j \in M_k$ for $1 \leq j \leq r^2$ and

$$\beta_1 = \beta_{r+2} = \beta_{2r+3} = \cdots = \beta_{r^2} = I_k,$$

with $\beta_j = 0$ otherwise. We have that $\beta' \in M_{k,kr^2}$; thus, $\gamma := \beta' \otimes I_m \in M_{km,r^2km}$, and

$$\begin{aligned}\gamma(P_1' \otimes P_2' \otimes R_1)\gamma^* &= (\beta' \otimes I_m)((P_1' \otimes P_2') \otimes R_1)(\beta' \otimes I_m)^* \\ &= \left(\sum_{i,j=1}^{r} C_i P_1 C_j^* \otimes D_i P_2 D_j^*\right) \otimes R_1 = \beta(P_1 \otimes P_2)\beta^* \otimes R_1.\end{aligned}$$

Thus,

$$\begin{aligned}\alpha(\beta(P_1 \otimes P_2)\beta^* \otimes R_1)\alpha^* &= \alpha(\gamma(P_1' \otimes P_2' \otimes R_1)\gamma^*)\alpha^* \\ &= (\alpha\gamma)(P_1' \otimes P_2' \otimes R_1)(\alpha\gamma)^*,\end{aligned}$$

where $\alpha\gamma \in M_{n,r^3m}$. This shows $Q \in D_n^{\max}$, and so $C_n = C_n^{\max}$ for each $n \in \mathbb{N}$. $\square$

Corollary 7.3. *We have that $M_n(\max\text{-}\otimes_{j=1}^k \mathcal{S}_j)^+ = C_n^{\max}$, $n \in \mathbb{N}$.*

Proposition 7.4. *If $\mathcal{A}_j$, $j \in [k]$, are unital C^*-algebras then* $\max\text{-}\otimes_{j=1}^k \mathcal{A}_j$ *is completely order isomorphic to the image of $\otimes_{j=1}^k \mathcal{A}_j$ inside the maximal C^*-algebra tensor product* $\mathrm{C}^*\max\text{-}\otimes_{j=1}^k \mathcal{A}_j$.

Proof. The proof relies on the ideas in the proof of [34, Theorem 5.12]; we only consider the case $k = 3$. Let $\mathcal{C} = \mathcal{A}_1 \otimes_{\mathrm{C}^*\max} \mathcal{A}_2 \otimes_{\mathrm{C}^*\max} \mathcal{A}_3$ denote the maximal C*-algebraic tensor product of $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$. The faithful inclusion of $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3 \subseteq \mathcal{C}$ endows $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ with an operator system structure; let $\tau\text{-}\otimes_{j=1}^3 \mathcal{A}_j$ denote this operator system, and let $C_n^{\max}$, $n \in \mathbb{N}$, be the matricial cones of $\max\text{-}\otimes_{j=1}^3 \mathcal{A}_j$. For $n \in \mathbb{N}$, let $D_n = M_n(\tau\text{-}\otimes_{j=1}^3 \mathcal{A}_j)^+$. By maximality, $C_n^{\max} \subseteq D_n$. For the converse inclusion, note that, if $X = \sum_{i=1}^{k} a_i^{(1)} \otimes a_i^{(2)} \otimes a_i^{(3)}$ where $a_i^{(1)} \in M_n(\mathcal{A}_1), a_i^{(2)} \in \mathcal{A}_2$ and $a_i^{(3)} \in \mathcal{A}_3$, then

$$XX^* = \sum_{i,j=1}^{k} a_i^{(1)} a_j^{(1)*} \otimes a_i^{(2)} a_j^{(2)*} \otimes a_i^{(3)} a_j^{(3)*}.$$

Let $A^{(1)} = (a_i^{(1)} a_j^{(1)*})_{i,j}$, $A^{(2)} = (a_i^{(2)} a_j^{(2)*})_{i,j}$ and $A^{(3)} = (a_i^{(3)} a_j^{(3)*})_{i,j}$; then $A^{(1)} \in M_k(M_n(\mathcal{A}_1))^+, A^{(2)} \in M_k(\mathcal{A}_2)^+$ and $A^{(3)} \in M_k(\mathcal{A}_3)^+$. Similarly to the proof of Proposition 7.2, there exists $\beta \in M_{n,nk^3}$ such that

$$\beta(A^{(1)} \otimes A^{(2)} \otimes A^{(3)})\beta^* = \sum_{i,j=1}^{k} a_i^{(1)} a_j^{(1)*} \otimes a_i^{(2)} a_j^{(2)*} \otimes a_i^{(3)} a_j^{(3)*} = XX^*,$$

and hence $XX^* \in C_n^{\max}$. Since D_n is generated, as a closed convex set, by elements of the form X^*X, we have that $D_n \subseteq C_n^{\max}$. The proof is complete. $\square$

7.1.2. *The minimal tensor product.* According to [34, Theorem 4.6], the bivariate minimal tensor product is associative and hence one can unambiguously define the operator system

$$\text{min-}\otimes_{j=1}^{k} \mathcal{S}_j := \mathcal{S}_1 \otimes_{\min} \cdots \otimes_{\min} \mathcal{S}_k.$$

Proposition 7.5. *Let $n \in \mathbb{N}$. Then*

$$M_n(\text{min-}\otimes_{j=1}^{k} \mathcal{S}_j)^+ = \{u \in M_n(\otimes_{j=1}^{k}\mathcal{S}_j) : (\otimes_{j=1}^{k} f_j)^{(n)}(u) \in M_{nn_1\dots n_k}^+$$
$$\textit{for all u.c.p. maps } f_j : \mathcal{S}_j \to M_{n_j}, n_j \in \mathbb{N}, j \in [k]\}.$$

Proof. We consider the case $k = 3$ only. For $n \in \mathbb{N}$, let

$$C_n = \{u \in M_n(\otimes_{j=1}^{3}\mathcal{S}_j) : \ (\otimes_{j=1}^{3} f_j)^{(n)}(u) \in M_{nn_1n_2n_3}^+$$
$$\text{for all u.c.p. maps } f_j : \mathcal{S}_j \to M_{n_j},\ n_j \in \mathbb{N},\ j = 1,2,3\}.$$

We will show that $C_n = M_n((\mathcal{S}_1 \otimes_{\min} \mathcal{S}_2) \otimes_{\min} \mathcal{S}_3)^+$. We assume, without loss of generality, that $\mathcal{S}_j \subseteq \mathcal{B}(\mathcal{H}_j)$, $j = 1,2,3$. Let $P \in C_n$ and suppose that $P = \sum_{i=1}^{\ell} X_i \otimes y_i \otimes z_i$, where $X_i \in M_n(\mathcal{S}_1), y_i \in \mathcal{S}_2$ and $z_i \in \mathcal{S}_3$ for $i \in [l]$. Furthermore, let $\psi = \sum_{s=1}^{k} \xi_s \otimes \eta_s \otimes \beta_s$ where $\xi_s \in \mathcal{H}_1^{(n)}, \eta_s \in \mathcal{H}_2$ and $\beta_s \in \mathcal{H}_3$, $s \in [k]$. Let $\Phi_1 : M_n(\mathcal{S}_1) \to M_k$, $\Phi_2 : \mathcal{S}_2 \to M_k$ and $\Phi_3 : \mathcal{S}_3 \to M_k$ be the maps given by

$$\Phi_1(X) = (\langle X\xi_s, \xi_t\rangle)_{s,t}, \quad \Phi_2(y) = (\langle y\eta_s, \eta_t\rangle)_{s,t}, \quad \Phi_3(z) = (\langle z\beta_s, \beta_t\rangle)_{s,t}.$$

It is clear that Φ_j is completely positive, $j = 1,2,3$. Note that $(f_1 \otimes f_2 \otimes f_3)^{(n)} = f_1^{(n)} \otimes f_2 \otimes f_3$ for all linear maps $f_j : \mathcal{S}_j \to M_m$, $j = 1,2,3$. Thus, $(f_1^{(n)} \otimes f_2 \otimes f_3)(P) \in M_{nk^3}^+$ if $f_j : \mathcal{S}_j \to M_k$ is unital and completely positive, $i = 1,2,3$. By [34, Lemma 4.2], $(\Phi_1 \otimes \Phi_2 \otimes \Phi_3)(P) \in M_{nk^3}^+$. Let $e = (e_1 \otimes e_1, \dots, e_k \otimes e_k) \in \mathbb{C}^{k^3}$ where $\{e_i\}_{i=1}^k$ is the standard orthonormal basis of $\mathbb{C}^k$. We have

$$\begin{aligned}
\langle P\psi, \psi\rangle &= \sum_{i=1}^{\ell}\sum_{s,t=1}^{k} \langle X_i\xi_s, \xi_t\rangle\langle y_i\eta_s, \eta_t\rangle\langle z_i\beta_s, \beta_t\rangle \\
&= \sum_{i=1}^{\ell}\langle(\Phi_1(X_i)\otimes\Phi_2(y_i)\otimes\Phi_3(z_i))e, e\rangle \\
&= \langle(\Phi_1\otimes\Phi_2\otimes\Phi_3)(P)e, e\rangle.
\end{aligned}$$

It follows that $P \in \mathcal{B}((\otimes_{j=1}^3\mathcal{H}_j)^n)^+$ and hence $C_n \subseteq D_n$. The reverse inclusion follows easily using the functoriality and injectivity of the minimal C*-algebraic tensor product; the details are omitted. $\square$

Remark 7.6. By their definition, and the fact that the bivariate minimal and maximal operator system tensor products are functorial [34, Theorems 4.6 and 5.5], the minimal and the maximal multivariate operator system tensor products are functorial. Similarly, the minimal multivariate operator system tensor product is injective (see [34, Section 3]).

7.1.3. *The commuting tensor product.* Let H be a Hilbert space and $\phi_j : \mathcal{S}_j \to \mathcal{B}(H)$ be a completely positive map, $j \in [k]$. We call the family $(\phi_j)_{j=1}^k$ *commuting* if ϕ_i and ϕ_j have mutually commuting ranges whenever $i \neq j$. Let $\Pi_{j=1}^k\phi_i : \otimes_{j=1}^k\mathcal{S}_j \to \mathcal{B}(H)$ be the linear map, given by

$$\left(\Pi_{j=1}^k\phi_j\right)(\otimes_{j=1}^k u_j) = \Pi_{j=1}^k\phi_j(u_j), \quad u_j \in \mathcal{S}_j, j \in [k].$$

For $n \in \mathbb{N}$, let

$$\begin{aligned}
C_n^{\mathrm{c}} = \{&u \in M_n(\otimes_{j=1}^k\mathcal{S}_j) : (\Pi_{j=1}^k\phi_j)^{(n)}(u) \in M_n(\mathcal{B}(H))^+, \\
&\text{for all commuting families } (\phi_j)_{j=1}^k \text{ and all Hilbert spaces } H\}.
\end{aligned}$$

Lemma 7.7. *Let $\mathcal{S}_1, \dots, \mathcal{S}_k$ be operator systems and $\mathcal{S} = \otimes_{j=1}^k\mathcal{S}_i$. Then C_n^{c} is a cone in $M_n(\mathcal{S})$ and $\mathcal{S}$, equipped with the family $(C_n^{\mathrm{c}})_{n\in\mathbb{N}}$ and the element $1 := 1_{\mathcal{S}_1}\otimes\cdots\otimes 1_{\mathcal{S}_k}$ as an Archimedean matrix order unit, is an operator system.*

Proof. It is straightforward to verify that C_n^{c} is a cone, $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$, $\alpha \in M_{n,m}$ and $u \in C_m^{\mathrm{c}}$. We have

$$\left(\Pi_{j=1}^k\phi_j\right)^{(n)}(\alpha u\alpha^*) = \alpha\left[\left(\Pi_{j=1}^k\phi_j\right)^{(m)}(u)\right]\alpha^* \in M_n(\mathcal{B}(H))^+;$$

thus, the family $(C_n^{\mathrm{c}})_{n=1}^\infty$ is compatible.

Let $\phi_j : \mathcal{S}_j \to M_{\ell_j}$ be a unital completely positive map, $j \in [k]$, and $\tilde{\phi}_j : \mathcal{S}_j \to M_{\ell_1\cdots\ell_k}$ be the map, given by

$$\tilde{\phi}_j(u) = 1_{\mathcal{S}_1}\otimes\cdots\otimes\phi_j(u)\otimes\cdots\otimes 1_{\mathcal{S}_k}, \quad u \in \mathcal{S}_j.$$

Then $\tilde{\phi}_j$ is completely positive for each $j \in [k]$, and $(\tilde{\phi}_j)_{j=1}^k$ is a commuting family. Therefore

$$(\phi_1 \otimes \cdots \otimes \phi_k)^{(n)}(u) = \left(\Pi_{j=1}^k \tilde{\phi}_j\right)^{(n)}(u) \geq 0, \quad u \in C_n^{\mathrm{c}}.$$

By Proposition 7.5, $C_n^{\mathrm{c}} \subseteq M_n(\text{min-}\otimes_{j=1}^k \mathcal{S}_j)^+$, $n \in \mathbb{N}$. In particular, $C_n^{\mathrm{c}} \cap (-C_n^{\mathrm{c}}) = \{0\}$ for each $n \in \mathbb{N}$. On the other hand, by Remark 7.1, $M_n(\text{max-}\otimes_{j=1}^k \mathcal{S}_j)^+ \subseteq C_n^{\mathrm{c}}$, $n \in \mathbb{N}$. In particular, $1_{\mathcal{S}_1} \otimes \cdots \otimes 1_{\mathcal{S}_k}$ a matrix order unit; it is straightforward to verify that it is Archimedean. $\square$

We denote the operator system $(\mathcal{S}, (C_n^{\mathrm{c}})_{n\in\mathbb{N}}, 1)$ from Theorem 7.7 by c-$\otimes_{j=1}^k \mathcal{S}_j$.

Theorem 7.8. *The map $(\mathcal{S}_1 \ldots, \mathcal{S}_k) \to$ c-$\otimes_{j=1}^k \mathcal{S}_j$ is a functorial operator system tensor product.*

Proof. Lemma 7.7 and its proof establish properties (T1) and (T3), while Remark 7.1 implies property (T2). To show functoriality, let $\mathcal{S}_j$ and $\mathcal{T}_j$ be operator systems, $\phi_j : \mathcal{S}_j \to \mathcal{T}_j$ be completely positive maps, $j \in [k]$, and $u \in M_n(\text{c-}\otimes_{j=1}^k \mathcal{S}_j)^+$. If $\psi_j : \mathcal{T}_j \to \mathcal{B}(H)$, $j \in [k]$, are completely positive maps such that the family $(\psi_j)_{j=1}^k$ is commuting then the family $(\psi_j \circ \phi_j)_{j=1}^k$ is commuting, and hence

$$(\Pi_{j=1}^k \psi_j)((\otimes_{j=1}^k \phi_j)(u)) = (\Pi_{j=1}^k (\psi_j \circ \phi_j))(u) \in M_n(\mathcal{B}(H))^+.$$

It follows that $\otimes_{j=1}^k \phi_j :$ c-$\otimes_{j=1}^k \mathcal{S}_j \to$ c-$\otimes_{j=1}^k \mathcal{T}_j$ is completely positive. $\square$

Proposition 7.9. *If $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are unital C^*-algebras, then $\mathcal{A}_1 \otimes_{\max} \cdots \otimes_{\max} \mathcal{A}_k = \mathcal{A}_1 \otimes_{\mathrm{c}} \cdots \otimes_{\mathrm{c}} \mathcal{A}_k$.*

Proof. By Remark 7.1, $M_n(\text{max-}\otimes_{j=1}^k \mathcal{A}_j)^+ \subseteq M_n(\text{c-}\otimes_{j=1}^k \mathcal{A}_j)^+$, $n \in \mathbb{N}$. For the reverse inclusion, fix $u \in M_n(\text{c-}\otimes_{j=1}^k \mathcal{A}_j)^+$. It suffices to show that $\phi^{(n)}(u) \geq 0$ for all unital completely positive maps $\phi : \mathcal{A}_1 \otimes_{\max} \cdots \otimes_{\max} \mathcal{A}_k \to \mathcal{B}(H)$. By Proposition 7.4, and an application of Stinespring's Theorem, we may further assume ϕ is a $*$-homomorphism. By associativity of the maximal tensor product of operator systems (see [34, Theorem 5.5]) and the universal property of the maximal tensor product of C*-algebras, we write $\phi = \Pi_{j=1}^k \phi_j$, where $\phi_j : \mathcal{A}_j \to \mathcal{B}(H)$ are $*$-homomorphisms with commuting ranges. By assumption, we have

$$\phi^{(n)}(u) = \left(\Pi_{j=1}^k \phi_j\right)^{(n)}(u) \geq 0,$$

and so $u \in M_n(\text{max-}\otimes_{j=1}^k \mathcal{A}_j)^+$. The proof is complete. $\square$

For the next theorem, recall that the *coproduct* $\mathcal{S}_1 \oplus_1 \mathcal{S}_2$ of two operator systems is the (unique, up to a unital complete order isomorphism) operator system, containing $\mathcal{S}_1$ and $\mathcal{S}_2$ as operator subsystems, satisfying the following universal property: for every operator system $\mathcal{R}$ and unital completely positive maps $\phi_i : \mathcal{S}_i \to \mathcal{R}$, $i = 1, 2$, there exists a unique unital completely positive map $\phi : \mathcal{S}_1 \oplus_1 \mathcal{S}_2 \to \mathcal{R}$ such that $\phi|_{\mathcal{S}_i} = \phi_i$, $i = 1, 2$ (see [24, 33]). We denote by $\mathcal{A} *_1 \mathcal{B}$ the C*-free product of the unital C*-algebras $\mathcal{A}$ and $\mathcal{B}$, amalgamated over their units. The next statement extends [46, Lemma 2.8] to the multivariate case.

Theorem 7.10. *Let $\mathcal{A}_i^{(j)}$ be a unital C*-algebra, $i \in [n_j]$, $j \in [k]$, and set $\mathcal{S}_j = \mathcal{A}_1^{(j)} \oplus_1 \cdots \oplus_1 \mathcal{A}_{n_j}^{(j)}$ and $\mathcal{A}_j = \mathcal{A}_1^{(j)} *_1 \cdots *_1 \mathcal{A}_{n_j}^{(j)}$, $j \in [k]$. Then*

$$\text{c-}\otimes_{j=1}^k \mathcal{S}_j \subseteq_{\text{c.o.i.}} \mathcal{A}_1 \otimes_{\max} \cdots \otimes_{\max} \mathcal{A}_k.$$

Proof. Set $\mathcal{S} = \text{c-}\otimes_{j=1}^k \mathcal{S}_j$ for brevity. By Theorem 7.8 and Proposition 7.9, the inclusion map $\iota : \text{c-}\otimes_{j=1}^k \mathcal{S}_i \to \text{max-}\otimes_{j=1}^k \mathcal{A}_i$ is completely positive. Let $u \in M_n(\mathcal{S}) \cap M_n(\text{max-}\otimes_{j=1}^k \mathcal{A}_j)^+$; we will show that $u \in M_n(\mathcal{S})^+$. For $j \in [k]$, let $\phi_j : \mathcal{S}_j \to \mathcal{B}(H)$ be a completely positive map, $j \in [k]$, such that the family $(\phi_j)_{j=1}^k$ is commuting. Assume first that ϕ_j is unital for each $j \in [k]$. Using [24, Theorem 5.2], identify $\mathcal{S}_j$ with the linear span of $\mathcal{A}_1^{(j)}, \dots, \mathcal{A}_{n_j}^{(j)}$ inside $\mathcal{A}_j$, $j \in [k]$. Each map ϕ_j is determined by a family $(\phi_\ell^j)_{\ell=1}^{n_j}$ of maps $\phi_\ell^j : \mathcal{A}_\ell^{(j)} \to \mathcal{B}(H)$, which are unital and completely positive, in that

$$\phi_j(a_1^j + \cdots + a_{n_j}^j) = \phi_1^j(a_1^j) + \cdots + \phi_{n_j}^j(a_{n_j}^j),$$

$a_\ell^j \in \mathcal{A}_\ell^{(j)}$, $\ell = 1, \dots, n_j$. As shown in [8], if $j \in [k]$, there exists a unital completely positive map $\tilde{\phi}_j : \mathcal{A}_j \to \mathcal{B}(H)$, given by

$$\tilde{\phi}_j(a_{\ell_1}^j \cdots a_{\ell_m}^j) = \phi_{\ell_1}^j(a_{\ell_1}^j) \cdots \phi_{\ell_m}^j(a_{\ell_m}^j),$$

where $a_{\ell_p}^j \in \mathcal{A}_{\ell_p}^j$ for each p, and $\ell_1 \neq \cdots \neq \ell_m$. Since $\text{ran}\phi_\ell^j \subseteq \text{ran}\phi_j$, $j \in [k]$, we have that $\phi_\ell^i(\mathcal{A}_\ell^{(i)})$ and $\phi_m^j(\mathcal{A}_m^{(j)})$ mutually commute whenever $i \neq j$. As $\mathcal{A}_j$ is generated, as a C*-algebra, by the operator subsystems $\mathcal{A}_i^{(j)}$, $i \in [n_j]$, we have that $\tilde{\phi}_i(\mathcal{A}_i)$ and $\tilde{\phi}_j(\mathcal{A}_j)$ commute whenever $i \neq j$. By Proposition 7.9, the map $\Pi_{j=1}^k \tilde{\phi}_j$ is completely positive on $\mathcal{A}_1 \otimes_{\max} \cdots \otimes_{\max} \mathcal{A}_k$, and thus

$$\left(\Pi_{j=1}^k \phi_j\right)^{(n)}(u) = (\tilde{\phi}_1 \cdots \tilde{\phi}_k)^{(n)}(u) \geq 0,$$

showing $u \in M_n(\mathcal{S})^+$.

Now relax the assumption on the unitality of the maps ϕ_j. Without loss of generality, assume that $\|\phi_j\| \leq 1$, so that the operator $T_j := \phi_j(1_{\mathcal{S}_j})$ is a positive contraction. Assume first that T_j is invertible for each $j \in [k]$. Let $\mathcal{M}_j$ be the von Neumann algebra generated by $\phi_j(\mathcal{S}_j)$; as $T_i \in \text{ran}(\phi_i)$, we have that $T_i \in \mathcal{M}_j'$ if $i \neq j$. We conclude that $T_i^{-1}, T_i^{1/2}$ and $T_i^{-1/2}$ are in $\mathcal{M}_j'$, whenever $i \neq j$. Define the maps $\hat{\phi}_j : \mathcal{S}_j \to \mathcal{B}(H)$ via

$$\hat{\phi}_j(u) = T_j^{-1/2} \phi_j(u) T_j^{-1/2}, \quad u \in \mathcal{S}_j, \, j \in [k].$$

As $T_j^{-1/2}$ is positive, $\hat{\phi}_j$ is a (unital) completely positive map, $j \in [k]$. We have that $T_i \in \mathcal{M}_j'$ if $i \neq j$; in particular, $T_iT_j = T_jT_i$ when $i \neq j$, and hence

$$T_i^{-1/2}T_j^{-1/2} = T_j^{-1/2}T_i^{-1/2} \quad \text{and} \quad T_i^{1/2}T_j^{1/2} = T_j^{1/2}T_i^{1/2}, \qquad i \neq j.$$

If $u \in \mathcal{S}_i$, $v \in \mathcal{S}_j$ and $i \neq j$, then

$$\begin{aligned}
\hat{\phi}_i(u)\hat{\phi}_j(v) &= T_i^{-1/2}\phi_i(u)T_i^{-1/2}T_j^{-1/2}\phi_j(v)T_j^{-1/2}\\
&= T_j^{-1/2}T_i^{-1/2}\phi_i(u)T_i^{-1/2}\phi_j(v)T_j^{-1/2}\\
&= T_j^{-1/2}\phi_j(v)T_i^{-1/2}\phi_i(u)T_i^{-1/2}T_j^{-1/2}\\
&= T_j^{-1/2}\phi_j(v)T_j^{-1/2}T_i^{-1/2}\phi_i(u)T_i^{-1/2} = \hat{\phi}_j(v)\hat{\phi}_i(u).
\end{aligned}$$

Thus, the family $(\hat{\phi}_j)_{j=1}^k$ is commuting. Using the previous paragraph, we see that $\Pi_{j=1}^k\hat{\phi}_j(u) \geq 0$. Let $T = T_1^{1/2} \cdots T_k^{1/2}$; then T is a positive operator and thus

$$\left(\Pi_{j=1}^k \phi_j\right)^{(n)}(u) = (T \otimes I_n)((\Pi_{j=1}^k\hat{\phi}_j)(u))(T \otimes I_n) \geq 0,$$

implying $u \in M_n(\mathcal{S})^+$.

Finally, relax the assumption on the invertibility of the operators T_j, $j \in [k]$. Let $\epsilon > 0$; then $T_j + \epsilon I$ (positive and) invertible, $j \in [k]$. Let $f_j : \mathcal{S}_j \to \mathbb{C}$ be a state, and define the map $\hat{\phi}_j : \mathcal{S}_j \to \mathcal{B}(H)$ by letting

$$\hat{\phi}_j(u) = \phi_j(u) + \epsilon f_j(u)I, \quad u \in \mathcal{S}_j,\ j \in [k].$$

We have that $(\hat{\phi}_j)_{j=1}^k$ is a commuting family of completely positive maps. Furthermore, $\hat{\phi}_j(1_{\mathcal{S}_j}) = T_j + \epsilon I$, and so $\hat{\phi}_j(1_{\mathcal{S}_j})$ is invertible, $j \in [k]$. By the previous paragraph, $(\Pi_{j=1}^k\hat{\phi}_j)^{(n)}(u) \in M_n(\mathcal{B}(H))^+$. Letting $\epsilon \to 0$, we obtain $(\Pi_{j=1}^k\phi_j)^{(n)}(u) \in M_n(\mathcal{B}(H))^+$; thus $u \in M_n(\mathcal{S})^+$ and the proof is complete. □

7.2. **Representations of correlations via operator systems.** In this subsection, we describe the correlations from the classes $\mathcal{C}_{\rm sns}$, $\mathcal{C}_{\rm sqc}$ and $\mathcal{C}_{\rm sqa}$ in terms of states on operator system tensor products. Recall that $\tilde{e}_{x,y}$ are the canonical generators of the operator system $\mathcal{S}_{X,Y}$ and, for a linear functional

$$s : \mathcal{S}_{X_2,X_1} \otimes \mathcal{S}_{Y_2,Y_1} \otimes \mathcal{S}_{A_1,A_2} \otimes \mathcal{S}_{B_1,B_2} \to \mathbb{C},$$

let $\Gamma : \mathcal{D}_{X_2Y_2} \otimes \mathcal{D}_{A_1B_1} \to \mathcal{D}_{X_1Y_1} \otimes \mathcal{D}_{A_2B_2}$ be the linear map, given by

$$\Gamma_s\left(x_1y_1, a_2b_2|x_2y_2, a_1b_1\right) := s\left(\tilde{e}_{x_2,x_1} \otimes \tilde{e}_{y_2,y_1} \otimes \tilde{e}_{a_1,b_1} \otimes \tilde{e}_{a_2,b_2}\right).$$

Theorem 7.11. *The map $s \to \Gamma_s$ is an affine isomorphism from*

(i) *the state space of $\mathcal{S}_{X_2,X_1} \otimes_{\max} \mathcal{S}_{Y_2,Y_1} \otimes_{\max} \mathcal{S}_{A_1,A_2} \otimes_{\max} \mathcal{S}_{B_1,B_2}$ onto $\mathcal{C}_{\rm sns}$;*

(ii) *the state space of $\mathcal{S}_{X_2,X_1} \otimes_{\rm c} \mathcal{S}_{Y_2,Y_1} \otimes_{\rm c} \mathcal{S}_{A_1,A_2} \otimes_{\rm c} \mathcal{S}_{B_1,B_2}$ onto $\mathcal{C}_{\rm sqc}$;*

(iii) *the state space of* $\mathcal{S}_{X_2,X_1} \otimes_{\min} \mathcal{S}_{Y_2,Y_1} \otimes_{\min} \mathcal{S}_{A_1,A_2} \otimes_{\min} \mathcal{S}_{B_1,B_2}$ *onto* $\mathcal{C}_{\rm sqa}$.

Proof. (i) Let $\Gamma \in \mathcal{C}_{\rm sns}$, and consider it as an element of $\mathcal{D}_{X_2X_1} \otimes_{\min} \mathcal{D}_{Y_2Y_1} \otimes_{\min} \mathcal{D}_{A_1A_2} \otimes_{\min} \mathcal{D}_{B_1B_2}$. For an element $\omega \in \mathcal{D}_{Y_2Y_1} \otimes_{\min} \mathcal{D}_{B_1B_2}$, let

$$L_\omega : \mathcal{D}_{X_2X_1} \otimes_{\min} \mathcal{D}_{Y_2Y_1} \otimes_{\min} \mathcal{D}_{A_1A_2} \otimes_{\min} \mathcal{D}_{B_1B_2} \to \mathcal{D}_{X_2X_1} \otimes_{\min} \mathcal{D}_{A_1A_2}$$

be the corresponding slice map. Based on the remarks preceeding Definition 5.6, the strongly no-signalling conditions imply that, if $\omega = \epsilon_{y_2,y_2} \otimes \epsilon_{y_1,y_1} \otimes \epsilon_{b_1,b_1} \otimes \epsilon_{b_2,b_2}$ then $L_\omega(\Gamma) \in \mathcal{C}_{\rm ns}$. By [37, Theorem 3.1],

$$L_\omega(\Gamma) \in \mathcal{R}_{X_2X_1} \otimes_{\min} \mathcal{R}_{A_1A_2}, \quad \omega \in \mathcal{D}_{Y_2Y_1} \otimes_{\min} \mathcal{D}_{B_1B_2}.$$

By symmetry,

$$L_{\omega'}(\Gamma) \in \mathcal{R}_{Y_2Y_1} \otimes_{\min} \mathcal{R}_{B_1B_2}, \quad \omega' \in \mathcal{D}_{X_2X_1} \otimes_{\min} \mathcal{D}_{A_1A_2}.$$

It follows that

$$\Gamma \in \mathcal{R}_{X_2,X_1} \otimes_{\min} \mathcal{R}_{Y_2,Y_1} \otimes_{\min} \mathcal{R}_{A_1,A_2} \otimes_{\min} \mathcal{R}_{B_1,B_2}.$$

On the other hand, relation (8) implies that

$$(\mathcal{S}_{X_2,X_1} \otimes_{\max} \mathcal{S}_{Y_2,Y_1} \otimes_{\max} \mathcal{S}_{A_1,A_2} \otimes_{\max} \mathcal{S}_{B_1,B_2})^{\rm d} \cong_{\rm c.o.i.} \mathcal{R}_{X_2,X_1} \otimes_{\min} \mathcal{R}_{Y_2,Y_1} \otimes_{\min} \mathcal{R}_{A_1,A_2} \otimes_{\min} \mathcal{R}_{B_1,B_2}.$$

The claim is proved.

(ii) Suppose that

$$s : \mathcal{S}_{X_2,X_1} \otimes_{\rm c} \mathcal{S}_{Y_2,Y_1} \otimes_{\rm c} \mathcal{S}_{A_1,A_2} \otimes_{\rm c} \mathcal{S}_{B_1,B_2} \to \mathbb{C}$$

is a state. By Theorem 7.10, we may assume that s is the restriction of a state

$$\tilde{s} : \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{Y_2,Y_1} \otimes_{\max} \mathcal{A}_{A_1,A_2} \otimes_{\max} \mathcal{A}_{B_1,B_2} \to \mathbb{C}.$$

The GNS construction applied to s produces a Hilbert space H, a unit vector $\xi \in H$ and mutually commuting projection-valued measures $(E_{x_2,x_1})_{x_1\in X_1}$, $(E^{y_2,y_1})_{y_1\in Y_1}$, $(F_{a_1,a_2})_{a_2\in A_2}$ and $(F^{b_1,b_2})_{b_2\in B_2}$ on H such that

$$\Gamma_s\left(x_1y_1, a_2b_2 | x_2y_2, a_1b_1\right) = \left\langle E_{x_2,x_1}E^{y_2,y_1}F_{a_1,a_2}F^{b_1,b_2}\xi, \xi\right\rangle,$$

for all x_i, y_i, a_i, b_i, $i = 1, 2$. Setting $P_{x_2y_2,x_1y_1} = E_{x_2,x_1}E^{y_2,y_1}$ and $Q_{a_1b_1,a_2b_2} = F_{a_1,a_2}F^{b_1,b_2}$, we have that the NS operator matrices $P = (P_{x_2y_2,x_1y_1})_{x_2,x_1,y_2,y_1}$ and $Q = (Q_{a_1a_2,b_1b_2})_{a_1,a_2,b_1,b_2}$ are dilatable and have mutually commuting entries; thus, $\Gamma_s \in \mathcal{C}_{\rm sqc}$.

Conversely, suppose that $\Gamma \in \mathcal{C}_{\rm sqc}$ and use Lemma 5.8 to write Γ in the form (25) for some mutually commuting PVM's $P_X = (P_{x_2,x_1})_{x_1\in X_1}$, $P_Y = (P^{y_2,y_1})_{y_1\in Y_1}$, $P_A = (P_{a_1,a_2})_{a_2\in A_2}$, $P_B = (P^{b_1,a_2})_{a_2\in A_2}$, $x_2 \in X_2$, $y_2 \in Y_2$, $a_1 \in A_1$, $b_1 \in B_1$. Let π_X, π_Y, π_A and π_B be the unital *-representations of $\mathcal{A}_{X_2,X_1}$, $\mathcal{A}_{Y_2,Y_1}$, $\mathcal{A}_{A_1,A_2}$ and $\mathcal{A}_{B_1,B_2}$, arising canonically from P_X, P_Y, P_A and P_B, respectively. Then $\pi := \pi_X \otimes \pi_Y \otimes \pi_A \otimes \pi_B$ is a unital *-representation of the C*-algebra $\mathcal{A} := \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{Y_2,Y_1} \otimes_{\max} \mathcal{A}_{A_1,A_2} \otimes_{\max} \mathcal{A}_{B_1,B_2}$. Using

Theorem 7.10, let s be the restriction to $\mathcal{S}_{X_2,X_1} \otimes_{\rm c} \mathcal{S}_{Y_2,Y_1} \otimes_{\rm c} \mathcal{S}_{A_1,A_2} \otimes_{\rm c} \mathcal{S}_{B_1,B_2}$ of the state on $\mathcal{A}$, given by $w \to \langle \pi(w)\eta, \eta\rangle$. We have that $\Gamma = \Gamma_s$.

(iii) Let Γ be a quantum SNS correlation; without loss of generality, we may thus assume that

$$\Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \left\langle \left(M_{x_2,x_1} \otimes M^{a_1,a_2} \otimes N_{y_2,y_1} \otimes N^{b_1,b_2}\right)\eta, \eta\right\rangle,$$

for all $x_i \in X_i$, $y_i \in Y_i$, $a_i \in A_i$ and $b_i \in B_i$, $i = 1,2$, where the families $M_X = (M_{x_2,x_1})_{x_1\in X_1}$, $M_A = (M^{a_1,a_2})_{a_2\in A_2}$, $N_Y = (N_{y_2,y_1})_{y_1\in Y_1}$ and $N_B = (N^{b_1,b_2})_{b_2\in B_2}$ are POVM's acting on finite dimensional Hilbert spaces H_X, H_A, H_Y and H_B, respectively, and $\eta \in H_X \otimes H_A \otimes H_Y \otimes H_B$ is a unit vector. After an application of Naimark's Theorem, we can further assume that M_X, M_A, N_Y and N_B are PVM's. Let $\pi_X : \mathcal{A}_{X_2,X_1} \to \mathcal{B}(H_X)$, $\pi_Y : \mathcal{A}_{Y_2,Y_1} \to \mathcal{B}(H_Y)$, $\pi_A : \mathcal{A}_{A_1,A_2} \to \mathcal{B}(H_A)$ and $\pi_B : \mathcal{A}_{B_1,B_2} \to \mathcal{B}(H_B)$ be the unital *-representations, canonically arising from M_X, M_A, N_Y and N_B, respectively. Letting $\pi = \pi_X \otimes \pi_Y \otimes \pi_A \otimes \pi_B$, $\tilde{\eta} \in H_X \otimes H_Y \otimes H_A \otimes H_B$ be the vector obtained from η after applying the canonical shuffle, $\tilde{s}$ be the state on $\mathcal{A}_{X_2,X_1} \otimes_{\min} \mathcal{A}_{Y_2,Y_1} \otimes_{\min} \mathcal{A}_{A_1,A_2} \otimes_{\min} \mathcal{A}_{B_1,B_2}$ given by $\tilde{s}(w) = \langle \pi(w)\tilde{\eta}, \tilde{\eta}\rangle$, and s be its restriction to $\mathcal{S}_{X_2,X_1} \otimes_{\min} \mathcal{S}_{Y_2,Y_1} \otimes_{\min} \mathcal{S}_{A_1,A_2} \otimes_{\min} \mathcal{S}_{B_1,B_2}$, we obtain that $\Gamma = \Gamma_s$.

Suppose that $\Gamma \in \mathcal{C}_{\rm sqa}$, and let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of quantum SNS correlations with $\Gamma_n \to_{n\to\infty} \Gamma$. Using the previous paragraph, choose a state s_n of $\mathcal{S}_{X_2,X_1} \otimes_{\min} \mathcal{S}_{Y_2,Y_1} \otimes_{\min} \mathcal{S}_{A_1,A_2} \otimes_{\min} \mathcal{S}_{B_1,B_2}$ such that $\Gamma_n = \Gamma_{s_n}$, $n \in \mathbb{N}$. Letting s be a cluster point of the sequence $(s_n)_{n\in\mathbb{N}}$ in the weak* topology, we have that $\Gamma = \Gamma_s$.

Conversely, let $s : \mathcal{S}_{X_2,X_1} \otimes_{\min} \mathcal{S}_{Y_2,Y_1} \otimes_{\min} \mathcal{S}_{A_1,A_2} \otimes_{\min} \mathcal{S}_{B_1,B_2} \to \mathbb{C}$ be a state, and, using the injectivity of the minimal operator system tensor product (pointed out in Remark 7.6), let $\tilde{s} : \mathcal{A}_{X_2,X_1} \otimes_{\min} \mathcal{A}_{Y_2,Y_1} \otimes_{\min} \mathcal{A}_{A_1,A_2} \otimes_{\min} \mathcal{A}_{B_1,B_2} \to \mathbb{C}$ be an extension of s. By [31, Corollary 4.3.10], s is in the weak* closure of the convex hull of vector functionals on $\pi_X(\mathcal{A}_{X_2,X_1}) \otimes_{\min} \pi_Y(\mathcal{A}_{Y_2,Y_1}) \otimes_{\min} \pi_A(\mathcal{A}_{A_1,A_2}) \otimes_{\min} \pi_B(\mathcal{A}_{B_1,B_2})$ for some unital *-representations π_X, π_Y, π_A and π_B. The argument given in the proof of [46, Theorem 2.10] can now be used to show that Γ is a limit of quantum SNS correlations. $\square$

8. Synchronous games

A *synchronous game* [45] over a quadruple (X, X, A, A) is a non-local game $E \subseteq XX \times AA$ such that

$$(x, x, a, b) \in E \implies a = b.$$

A perfect strategy for a synchronous game $E \subseteq XX \times AA$ is called a *synchronous correlation* over (X, X, A, A).

Assume that $Y_i = X_i$ and $B_i = A_i$, $i = 1,2$. In this section, we restrict our attention to the case where the games $E_1 \subseteq X_1X_1 \times A_1A_1$ and $E_2 \subseteq X_2X_2 \times A_2A_2$ participating in a quasi-homorphism game $E_1 \rightsquigarrow E_2$

are synchronous. We achieve tracial representations of SNS quantum commuting and approximately quantum correlations, which lead to necessary conditions for the existence of an isomorphism between two synchronous games in terms of the corresponding game algebras. As in the paragraph before Definition 5.6, we have that, if Γ is an SNS correlation over the quadruple $(X_2X_2, A_1A_1, X_1X_1, A_2A_2)$, then the linear maps $\Gamma_{X_2X_2\to X_1X_1} : \mathcal{D}_{X_2X_2} \to \mathcal{D}_{X_1X_1}$ and $\Gamma^{A_2A_2\to A_1A_1} : \mathcal{D}_{A_2A_2} \to \mathcal{D}_{A_1A_1}$, given by

$$\Gamma_{X_2X_2\to X_1X_1}(x_1, y_1|x_2, y_2) = \sum_{a_2,b_2\in A_2} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1) \tag{29}$$

and

$$\Gamma^{A_1A_1\to A_2A_2}(a_2, b_2|a_1, b_1) = \sum_{x_1,y_1\in X_1} \Gamma(x_1y_1, a_2b_2|x_2y_2, a_1b_1), \tag{30}$$

are NS correlations over (X_2, X_2, X_1, X_1) and (A_1, A_1, A_2, A_2), respectively.

Definition 8.1. *An SNS correlation Γ over $(X_2X_2, A_1A_1, X_1X_1, A_2A_2)$ is called* jointly synchronous *if $\Gamma_{X_2X_2\to X_1X_1}$ and $\Gamma^{A_1A_1\to A_2A_2}$ are synchronous correlations.*

Remark 8.2. Since the terms in (29) and (30) are non-negative, an element $\Gamma \in \mathcal{C}_{\rm sns}$ is jointly synchronous precisely when

$$\Gamma(x_1y_1, a_2b_2|x_2x_2, a_1b_1) \neq 0 \implies x_1 = y_1$$

for all $x_2 \in X_2$, $x_1, y_1 \in X_1$, $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, and

$$\Gamma(x_1y_1, a_2b_2|x_2y_2, a_1a_1) \neq 0 \implies a_2 = b_2.$$

for all $x_2, y_2 \in X_2$, $x_1, y_1 \in X_1$, $a_1 \in A_1$ and $a_2, b_2 \in A_2$.

For a linear functional T on $\mathcal{A}_{X_2,X_1} \otimes \mathcal{A}_{A_1,A_2}$, set

$$\Gamma_{\mathsf{T}}(x_1y_1, a_2b_2|x_2y_2, a_1b_1) = \mathsf{T}(e_{x_2,x_1}e_{y_2,y_1} \otimes e_{a_1,a_2}e_{b_1,b_2}),$$

where $x_i, y_i \in X_i, a_i, b_i \in A_i, i = 1, 2$. In the sequel, we will use the terms "trace" and "tracial state" interchangeangly.

Theorem 8.3. *The following hold:*

- (i) *If T is a trace on $\mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{A_1,A_2}$ then Γ_{T} is a jointly synchronous and quantum commuting correlation.*
- (ii) *If $\Gamma \in \mathcal{C}_{\rm sqc}$ is jointly synchronous then there exists a trace T on $\mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{A_1,A_2}$ such that $\Gamma = \Gamma_{\mathsf{T}}$.*
- (iii) *A jointly synchronous SNS correlation Γ is approximately quantum if and only if there exists an amenable trace T on $\mathcal{A}_{X_2,X_1} \otimes_{\min} \mathcal{A}_{A_1,A_2}$ such that $\Gamma = \Gamma_{\mathsf{T}}$ such that $\Gamma = \Gamma_{\mathsf{T}}$.*

Proof. (i) That Γ_{T} is strongly no-signalling is straightforward. The GNS construction, applied to T, yields a Hilbert space H, a unit vector $\xi \in H$ and a representation $\pi : \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{A_1,A_2} \to \mathcal{B}(H)$ such that $\mathsf{T}(w) = \langle \pi(w)\xi, \xi\rangle$, $w \in \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{A_1,A_2}$. Set

$$P_{x_2y_2,x_1y_1} = \pi(e_{x_2,x_1}e_{y_2,y_1} \otimes 1) \ \text{ and } \ Q_{a_1b_1,a_2b_2} = \pi(1 \otimes e_{a_1,a_2}e_{b_1,b_2}),$$

for all $x_i, y_i \in X_i$ and $a_i, b_i \in A_i$, $i = 1, 2$. Proposition 5.1, applied to the map $\phi : \mathcal{S}_{X_2,X_1} \otimes_{\rm c} \mathcal{S}_{X_2,X_1} \to \mathcal{B}(H)$, given by

$$\phi(e_{x_2,x_1} \otimes e_{y_2,y_1}) = P_{x_2y_2,x_1y_1}, \quad x_i, y_i \in X_i, i = 1, 2,$$

shows that $(P_{x_2y_2,x_1y_1})$ is a dilatable operator matrix; by symmetry, so is $(Q_{a_1b_1,a_2b_2})$. Thus, Γ_{T} is a quantum commuting SNS correlation. Finally, the strong synchronicity is immediate from Remark 8.2 and the fact that $(e_{x_2,x_1})_{x_1 \in X_1}$ and $(e_{a_1,a_2})_{a_2 \in A_2}$ are PVM's.

(ii) For brevity, write

$$\mathfrak{B} = \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{A_1,A_2} \otimes_{\max} \mathcal{A}_{A_1,A_2}$$

and

$$\mathfrak{A} = \mathcal{A}_{X_2,X_1} \otimes_{\max} \mathcal{A}_{A_1,A_2}.$$

Note that, up to a flip of the tensor terms, $\mathfrak{B} = \mathfrak{A} \otimes_{\max} \mathfrak{A}$; in the rest of the proof, the latter identification is used without explicit mention.

By the (proof of) Theorem 7.11, there exists a state $\tilde{s} : \mathfrak{B} \to \mathbb{C}$ such that $\Gamma = \Gamma_{\tilde{s}}$. If $w_1, w_2 \in \mathfrak{A} \otimes_{\max} \mathfrak{A}$, write $w_1 \sim w_2$ if $\tilde{s}(w_1 - w_2) = 0$. For $x_i \in X_i$ and $a_i, b_i \in A_i$, $i = 1, 2$, we have

$$\begin{aligned}
\tilde{s}(e_{x_2,x_1} \otimes 1 \otimes e_{a_1,a_2} \otimes e_{b_1,b_2}) &= \sum_{y_1 \in X_1} \tilde{s}(e_{x_2,x_1} \otimes e_{x_2,y_1} \otimes e_{a_1,a_2} \otimes e_{b_1,b_2}) \\
&= \tilde{s}(e_{x_2,x_1} \otimes e_{x_2,x_1} \otimes e_{a_1,a_2} \otimes e_{b_1,b_2}) \\
&= \Gamma(x_1x_1, a_2b_2 | x_2x_2, a_1b_1) \\
&= \sum_{y_1 \in X_1} \tilde{s}(e_{x_2,y_1} \otimes e_{x_2,x_1} \otimes e_{a_1,a_2} \otimes e_{b_1,b_2}) \\
&= \tilde{s}(1 \otimes e_{x_2,x_1} \otimes e_{a_1,a_2} \otimes e_{b_1,b_2});
\end{aligned}$$

thus,

$$\begin{aligned}
e_{x_2,x_1} \otimes 1 \otimes e_{a_1,a_2} \otimes e_{b_1,b_2} &\sim e_{x_2,x_1} \otimes e_{x_2,x_1} \otimes e_{a_1,a_2} \otimes e_{b_1,b_2} \\
&\sim 1 \otimes e_{x_2,x_1} \otimes e_{a_1,a_2} \otimes e_{b_1,b_2}.
\end{aligned}$$

It follows that

$$e_{x_2,x_1} \otimes 1 \otimes 1 \otimes 1 \sim e_{x_2,x_1} \otimes e_{x_2,x_1} \otimes 1 \otimes 1 \sim 1 \otimes e_{x_2,x_1} \otimes 1 \otimes 1,$$

for $x_i \in X_i, i = 1, 2$. Write

$$h = e_{x_2,x_1} \otimes 1 \otimes 1 \otimes 1 - 1 \otimes e_{x_2,x_1} \otimes 1 \otimes 1,$$

and note that

$$\begin{aligned}
h^2 &= e_{x_2,x_1} \otimes 1 \otimes 1 \otimes 1 - e_{x_2,x_1} \otimes e_{x_2,x_1} \otimes 1 \otimes 1 \\
&\quad - e_{x_2,x_1} \otimes e_{x_2,x_1} \otimes 1 \otimes 1 + 1 \otimes e_{x_2,x_1} \otimes 1 \otimes 1,
\end{aligned}$$

implying $h^2 \sim 0$. An application of the Cauchy-Schwarz inequality now shows that $wh \sim 0$ and $hw \sim 0$ whenever $w \in \mathfrak{B}$. In particular, for all $x_i \in X_i$, $i = 1, 2$, we have

$$ze_{x_2,x_1} \otimes 1 \otimes v \sim z \otimes e_{x_2,x_1} \otimes v \sim ze_{x_2,x_1} \otimes 1 \otimes v, \quad z \in \mathcal{A}_{X_2,X_1}, v \in \mathfrak{A}. \tag{31}$$

Similarly,

$$u\otimes z'e_{a_1,a_2}\otimes 1 \sim u\otimes z'\otimes e_{a_1,a_2} \sim u\otimes e_{a_1,a_2}z'\otimes 1, \quad z'\in\mathcal{A}_{A_1,A_2}, u\in\mathfrak{A}. \tag{32}$$

Equations (31) and (32) imply

$$ze_{x_2,x_1}\otimes 1\otimes z'e_{a_1,a_2}\otimes 1 \sim e_{x_2,x_1}\otimes z\otimes e_{a_1,a_2}\otimes z' \sim e_{x_2,x_1}z\otimes 1\otimes e_{a_1,a_2}z'\otimes 1 \tag{33}$$

for all $z\in\mathcal{A}_{X_2,X_1}$, all $z'\in\mathcal{A}_{A_1,A_2}$ and all $x_i\in X_i$, $a_i\in A_i$, $i=1,2$. An induction by the length of the words w on $\{e_{x_2,x_1} : x_i\in X_i, i=1,2\}$ and w' on $\{e_{a_1,a_2} : a_i\in A_i, i=1,2\}$, whose base step is provided by (33) shows that

$$zw\otimes 1\otimes z'w'\otimes 1 \sim wz\otimes 1\otimes w'z'\otimes 1, \quad z,z'\in\mathcal{A}_{X_2,X_1}, w,w'\in\mathcal{A}_{A_1,A_2}$$

(see the proof of [37, Theorem 6.1]). We conclude that the functional T on $\mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{A_1,A_2}$, given by

$$\mathsf{T}(u\otimes v) = \tilde{s}(u\otimes 1\otimes v\otimes 1), \quad u\in\mathcal{A}_{X_2,X_1}, v\in\mathcal{A}_{A_1,A_2},$$

is a tracial state. The fact that $\Gamma=\Gamma_{\mathsf{T}}$ follows from (33).

(iii) Suppose that Γ is an approximately quantum jointly synchronous SNS correlation. By Theorem 7.11, there exists a state s on $\mathcal{A}_{X_2,X_1}\otimes_{\min}\mathcal{A}_{X_2,X_1}\otimes_{\min}\mathcal{A}_{A_1,A_2}\otimes_{\min}\mathcal{A}_{A_1,A_2}$ such that $\Gamma=\Gamma_s$. Using (i), let $\mathsf{T} : \mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{A_1,A_2}\to\mathbb{C}$ be a tracial state such that $\Gamma=\Gamma_{\mathsf{T}}$. By [35, Lemma III.3], there exists a *-isomorphism $\partial : \mathcal{A}_{X_2,X_1}\to\mathcal{A}_{X_2,X_1}^{\rm op}$ such that $\partial(e_{x_2,x_1}) = e_{x_2,x_1}^{\rm op}$. Let $q : \mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{A_1,A_2}\to\mathcal{A}_{X_2,X_1}\otimes_{\min}\mathcal{A}_{A_1,A_2}$ be the quotient map,

$$\mathcal{F} : \mathcal{A}_{X_2,X_1}\otimes_{\min}\mathcal{A}_{A_1,A_2}\otimes_{\min}\mathcal{A}_{X_2,X_1}\otimes_{\min}\mathcal{A}_{A_1,A_2} \longrightarrow \mathcal{A}_{X_2,X_1}\otimes_{\min}\mathcal{A}_{X_2,X_1}\otimes_{\min}\mathcal{A}_{A_1,A_2}\otimes_{\min}\mathcal{A}_{A_1,A_2}$$

be the flip operation, and

$$\mu : (\mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{A_1,A_2})\otimes_{\min}(\mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{A_1,A_2})^{\rm op}\to\mathbb{C}$$

be the linear functional, defined by letting

$$\mu = s\circ\mathcal{F}\circ\left(q\otimes\left(q\circ(\partial^{-1}\otimes\partial^{-1})\right)\right).$$

It is then straightforward to check that

$$\mu(u\otimes v^{\rm op}) = \mathsf{T}(uv), \quad u,v\in\mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{A_1,A_2}.$$

By [11, Theorem 6.2.7], T is amenable. The converse direction follows by reversing these steps. □

Proposition 8.4. *If Γ is a jointly synchronous SNS correlation then $\Gamma[\mathcal{E}]$ is synchronous whenever $\mathcal{E}$ is such.*

Proof. Let $\mathcal{E}$ be a synchronous no-signalling correlation over (X_1, X_1, A_1, A_1). Suppose that $x_2 \in X_2$ and $a_2, b_2 \in A_2$. We have

$$\begin{aligned}\Gamma[\mathcal{E}](a_2, b_2|x_2, x_2) &= \sum_{x_1,y_1\in X_1}\sum_{a_1,b_1\in A_1}\Gamma(x_1y_1, a_2b_2|x_2x_2, a_1b_1)\mathcal{E}(a_1, b_1|x_1, y_1)\\ &= \sum_{x_1\in X_1}\sum_{a_1,b_1\in A_1}\Gamma(x_1x_1, a_2b_2|x_2x_2, a_1b_1)\mathcal{E}(a_1, b_1|x_1, x_1)\\ &= \sum_{x_1\in X_1}\sum_{a_1\in A_1}\Gamma(x_1x_1, a_2b_2|x_2x_2, a_1a_1)\mathcal{E}(a_1, a_1|x_1, x_1).\end{aligned}$$

Since Γ is jointly synchronous, $\Gamma(x_1x_1, a_2b_2|x_2x_2, a_1a_1) = 0$ whenever $a_2 \neq b_2$. Thus, $\Gamma[\mathcal{E}](a_2, b_2|x_2, x_2) = 0$ whenever $a_2 \neq b_2$, that is, $\Gamma[\mathcal{E}]$ is synchronous. $\square$

Remark 8.5. Let T be a tracial state on $\mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{A_1,A_2}$ and τ be a tracial state on $\mathcal{A}_{X_1,A_1}$. Let $\mathcal{E}_\tau : \mathcal{D}_{X_1X_1} \to \mathcal{D}_{A_1A_1}$ be the quantum commuting no-signalling correlation [45], given by

$$\mathcal{E}_\tau(a, b|x, y) = \tau(e_{x,a}e_{y,b}), \quad x, y \in X_1, a, b \in A_1. \tag{34}$$

By Theorem 6.1, $\Gamma_{\mathsf{T}}[\mathcal{E}_\tau]$ is a quantum commuting NS correlation while, by Proposition 8.4, it is synchronous. By [45, Theorem 5.5], there exists a tracial state $\mathsf{T}[\tau] : \mathcal{A}_{X_2,A_2} \to \mathbb{C}$ such that

$$\Gamma_{\mathsf{T}}[\mathcal{E}_\tau] = \mathcal{E}_{\mathsf{T}[\tau]}. \tag{35}$$

Let, on the other hand,

$$\mathsf{T}\odot\tau : \mathcal{A}_{X_2,X_1}\otimes_{\max}\mathcal{A}_{X_1,A_1}\otimes_{\max}\mathcal{A}_{A_1,A_2} \to \mathbb{C}$$

be the tracial state, given by

$$(\mathsf{T}\odot\tau)(u\otimes v\otimes w) = \mathsf{T}(u\otimes w)\tau(v), \quad u \in \mathcal{A}_{X_2,X_1}, v \in \mathcal{A}_{X_1,A_1}, w \in \mathcal{A}_{A_1,A_2}.$$

Equation (35) implies

$$\mathsf{T}[\tau](e_{x_2,a_2}e_{y_2,b_2}) = \sum_{x_1,y_1}\sum_{a_1,b_1}(\mathsf{T}\odot\tau)(e_{x_2,x_1}e_{y_2,y_1}\otimes e_{x_1,a_1}e_{y_1,b_1}\otimes e_{a_1,a_2}e_{b_1,b_2}).$$

We recall synchronous game $E \subseteq XX \times AA$ gives rise to the C*-algebra [28] $\mathcal{A}(E) = \mathcal{A}_{X,A}/\mathcal{J}_E$, where $\mathcal{J}_E$ is the closed ideal of $\mathcal{A}_{X,A}$, generated by the set $\{e_{x,a}e_{y,b} : (x, y, a, b) \notin E\}$ (the C*-algebra $\mathcal{A}(E)$ is known as the *game C*-algebra* of E). Write q_E for the quotient map from $\mathcal{A}_{X,A}$ onto $\mathcal{A}(E)$. We denote by $\mathrm{T}_2(\mathcal{A}(E))$ the convex set of all restrictions of tracial states on a C*-algebra $\mathcal{A}(E)$ to the subspace

$$\mathcal{A}_{(2)}(E) := \mathrm{span}\{q(e_{x,a}e_{y,b}) : x, y \in X, a, b \in A\}.$$

By [28, Theorem 3.2], the perfect quantum commuting strategies for E are in correspondence with the elements of $\mathrm{T}(\mathcal{A}(E))$, by associating with every $\tau \in \mathrm{T}(\mathcal{A}(E))$ the correlation defined in (34).

Corollary 8.6. *Let $E_i \subseteq X_iX_i \times A_iA_i$ be a synchronous game, $i = 1, 2$. If $E_1 \rightsquigarrow_{\mathrm{qc}} E_2$ then $\tau \to \Gamma[\tau]$ is an affine map from $\mathrm{T}_2(\mathcal{A}(E_1))$ into $\mathrm{T}_2(\mathcal{A}(E_2))$.*

Proof. For a tracial state τ on $\mathcal{A}(E)$, let $\tau_{(2)}$ be its restriction to the subspace $\mathcal{A}^{(2)}(E)$ of $\mathcal{A}(E)$. By Proposition 8.4 and Remark 8.5, it suffices to show that the map

$$\mathrm{T}_2(\mathcal{A}(E_1)) \to \mathrm{T}_2(\mathcal{A}(E_2)); \quad \tau_{(2)} \to \mathsf{T}[\tau]_{(2)},$$

is well-defined. This is immediate from Theorem 6.1 (ii) and the discussion in Remark 8.5. $\square$

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