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Convergence of a Ramshaw-Mesina iteration

Aytekin Çıbık [a,*], William Layton [b]

[a] Department of Mathematics, Gazi University, Ankara, 06560, Turkey
[b] Department of Mathematics, University of Pittsburgh, Pittsburgh, 15260, PA, USA

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ABSTRACT

In 1991 Ramshaw and Mesina introduced a clever synthesis of penalty methods and artificial compression methods. Its form makes it an interesting option to replace the pressure update in the Uzawa iteration. The result, for the Stokes problem, is

$$\begin{cases} Step1: & -\triangle u^{n+1} + \nabla p^n = f(x), \text{in}\Omega,\ u^{n+1}|_{\partial\Omega} = 0, \\ Step2: & p^{n+1} - p^n + \beta\nabla\cdot(u^{n+1} - u^n) + \alpha^2\nabla\cdot u^{n+1} = 0. \end{cases} \tag{1}$$

For saddle point problems, including Stokes, this iteration converges under a condition similar to the one required for Uzawa iteration.

1. Introduction

In 1991 Ramshaw and Mesina [1] gave a clever synthesis of regularization of incompressibility ($\nabla\cdot u = 0$) by artificial compression ($p_t + \alpha^2\nabla\cdot u = 0$) and a reformulation of penalization ($\frac{\partial}{\partial t}(p + \beta\nabla\cdot u = 0)$). For α, β suitably balanced, their regularization

$$p_t + \beta\nabla\cdot u_t + \alpha^2\nabla\cdot u = 0 \tag{2}$$

sped up satisfaction of incompressibility by several orders of magnitude. Since (2) reduces to $\nabla\cdot u = 0$ at steady state ($p_t = 0, u_t = 0$), it is natural to consider replacing the pressure update step in the Uzawa iteration by a discrete form of (2). For the Stokes problem, this yields: given u^n, p^n

$$\begin{cases} Step\ 1: & -\triangle u^{n+1} + \nabla p^n = f(x),\ \text{in}\ \Omega,\ u^{n+1}|_{\partial\Omega} = 0, \\ Step\ 2: & p^{n+1} - p^n + \beta\nabla\cdot(u^{n+1} - u^n) + \alpha^2\nabla\cdot u^{n+1} = 0. \end{cases} \tag{3}$$

If $\beta = 0$ this reduces to the standard Uzawa iteration. Section 2 proves convergence for (3) under the condition $\beta + \alpha^2 < M^{-1}$, where M is the maximum of the Rayleigh quotient for the Schur complement. Section 3 provides a numerical test.

Some related work. The utility of (2) to timestep to steady state developed in directions complemented herein by Ramshaw and Mousseau [2] and McHugh and Ramshaw [3]. The (short) proof herein builds on the analysis of the Uzawa iteration in Bacuta [4]. The regularization (2) was explored in different directions in [2,3,5,6]. At this point, experience with (3) is insufficient to classify its advantages, disadvantages, complexity differences. There are many possible further developments paralleling those of the Uzawa iteration in e.g. [7,8], (among hundreds of papers).

* Corresponding author.
E-mail address: abayram@gazi.edu.tr (A. Çıbık).

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2. Proof of convergence

We follow the framework of Bacuta [4] which we now summarize. Let $\mathbf{V}, P$ denote two Hilbert spaces. Denote linear operators $A : \mathbf{V} \to \mathbf{V}^*$ and $B : V \to P$, so that $B^* : P \to \mathbf{V}^*$. In this framework, we consider the reformulation of (3) as

$$\begin{cases} Au^{n+1} + B^* p^n = f, \\ p^{n+1} - p^n - \beta B(u^{n+1} - u^n) - \alpha^2 Bu^{n+1} = 0. \end{cases} \tag{4}$$

For the Stokes problem this corresponds to $A = -\triangle$, $B^* = \nabla$, $B = -\nabla\cdot$, $\mathbf{V} = (H_0^1(\Omega))^d$, $P = L_0^2(\Omega)$, with $f \in \mathbf{V}^*$. The Schur complement operator $\mathcal{A}$ is assumed invertible and that

$$\mathcal{A} := BA^{-1}B^* : P \to P$$

is a self-adjoint, bounded, positive definite operator. These properties are proven under mild conditions in Bacuta [4], Lemma 2.1 p.2635. In particular, Bacuta [4] shows that there are positive constants m, M such that, for $(\cdot,\cdot), |\cdot|$ the P-inner product and norm,

$$0 < m|q|^2 \leq (\mathcal{A}q, q) \leq M|q|^2 \text{ for all } 0 \neq q \in P. \tag{5}$$

Under these conditions, the $\mathcal{A}$-inner product and norm, $(p,q)_{\mathcal{A}} := (\mathcal{A}p, q), |p|_{\mathcal{A}} := (p,p)_{\mathcal{A}}^{1/2}$, are well-defined. We prove the following.

Theorem 2.1. *Under the above assumptions, in particular (5), the iteration (3) converges if $\beta \geq 0$ and $\beta + \alpha^2 < \frac{1}{M}$.*

Proof. Step 1 of (4) is used to eliminate the velocity in a standard way using $u^{n+1} = A^{-1}(f - B^* p^n)$ which reduces Step 2 to

$$p^{n+1} - p^n + \beta\mathcal{A}(p^n - p^{n-1}) + \alpha^2\mathcal{A}p^n = \alpha^2 BA^{-1}f.$$

The true p satisfies this exactly. Thus the error $e^n = p - p^n$ satisfies, by subtraction and rearrangement,

$$e^{n+1} - \left[I - (\beta + \alpha^2)\mathcal{A}\right] e^n - \beta\mathcal{A}e^{n-1} = 0.$$

Take the inner-product with e^{n+1}. This gives:

$$|e^{n+1}|^2 - (e^n, e^{n+1}) + (\beta + \alpha^2)(e^n, e^{n+1})_{\mathcal{A}} - \beta(e^{n-1}, e^{n+1})_{\mathcal{A}} = 0.$$

Using the polarization identity on $(e^n, e^{n+1}), (e^n, e^{n+1})_{\mathcal{A}}$ and $(e^{n-1}, e^{n+1})_{\mathcal{A}}$ gives

$$\begin{aligned} &|e^{n+1}|^2 - \left[\frac{1}{2}|e^n|^2 + \frac{1}{2}|e^{n+1}|^2 - \frac{1}{2}|e^{n+1} - e^n|^2\right] + \\ &(\beta + \alpha^2)\left[\frac{1}{2}|e^n|_{\mathcal{A}}^2 + \frac{1}{2}|e^{n+1}|_{\mathcal{A}}^2 - \frac{1}{2}|e^{n+1} - e^n|_{\mathcal{A}}^2\right] \\ &-\beta\left[\frac{1}{2}|e^{n-1}|_{\mathcal{A}}^2 + \frac{1}{2}|e^{n+1}|_{\mathcal{A}}^2 - \frac{1}{2}|e^{n+1} - e^{n-1}|_{\mathcal{A}}^2\right] = 0. \end{aligned}$$

Multiply by 2 and regroup terms by

$$\begin{aligned} &\left[|e^{n+1}|^2 + \beta|e^n|_{\mathcal{A}}^2 + \alpha^2|e^{n+1}|_{\mathcal{A}}^2 + \left\{|e^{n+1} - e^n|^2 - (\beta + \alpha^2)|e^{n+1} - e^n|_{\mathcal{A}}^2\right\}\right] \\ &- \left[|e^n|^2 + \beta|e^{n-1}|_{\mathcal{A}}^2 + \alpha^2|e^n|_{\mathcal{A}}^2 + \left\{|e^n - e^{n-1}|^2 - (\beta + \alpha^2)|e^n - e^{n-1}|_{\mathcal{A}}^2\right\}\right] \\ &+ \left\{|e^n - e^{n-1}|^2 - (\beta + \alpha^2)|e^n - e^{n-1}|_{\mathcal{A}}^2\right\} \\ &+ 2\alpha^2|e^n|_{\mathcal{A}}^2 + \beta|e^{n+1} - e^{n-1}|_{\mathcal{A}}^2 = 0. \end{aligned} \tag{6}$$

Notice that in three places (in braces, $\{\cdot\}$) there occurs $|q|^2 - (\beta + \alpha^2)|q|_{\mathcal{A}}^2$ with $q = e^{n+1} - e^n$ and $q = e^n - e^{n-1}$. Using $0 < m|q|^2 \leq (\mathcal{A}q, q) \leq M|q|^2$ we bound this term as

$$\left[1 - M(\beta + \alpha^2)\right]|q|^2 \leq |q|^2 - (\beta + \alpha^2)|q|_{\mathcal{A}}^2 \leq \left[1 - m(\beta + \alpha^2)\right]|q|^2.$$

Thus, this term is positive if $\beta + \alpha^2 < 1/M$. Thus, (6) takes the abstract form $\left[E^{n+1}\right] - \left[E^n\right] + P^n = 0$ where both $E \& P$ are non-negative. Summing we have E^N and $\sum_{n=1}^N P^n$, a series with non-negative terms, are uniformly bounded. Hence $\sum_{n=1}^\infty P^n$ is convergent and the nth term $P^n \to 0$ as $n \to \infty$ where

$$P^n = \left\{|e^n - e^{n-1}|^2 - (\beta + \alpha^2)|e^n - e^{n-1}|_{\mathcal{A}}^2\right\} + 2\alpha^2|e^n|_{\mathcal{A}}^2 + \beta|e^{n+1} - e^{n-1}|_{\mathcal{A}}^2.$$

This implies $e^n \to 0$, concluding the proof. □

3. Numerical test

We compare (1) to the standard Uzawa algorithm for the lid-driven cavity. The domain is a unit square with the top lid sliding in the positive x-direction according to $u(x, 1) = (g(x), 0)^T$, $g(x) = 4x(1 - x)$. The boundary conditions are $u = 0$ on the other walls. FreeFem++ is used for computations and the inf-sup stable Taylor-Hood finite element pair is used for the test.

Table 1
Iterations for different meshes for varying β and $\alpha^2 = 1.5$.

Mesh	$\beta = 0$	$\beta = 10^{-4}$	$\beta = 10^{-2}$	$\beta = 0.1$	$\beta = 0.2$
10 × 10	74	74	74	75	89
20 × 20	75	75	75	77	85
40 × 40	77	77	77	78	85

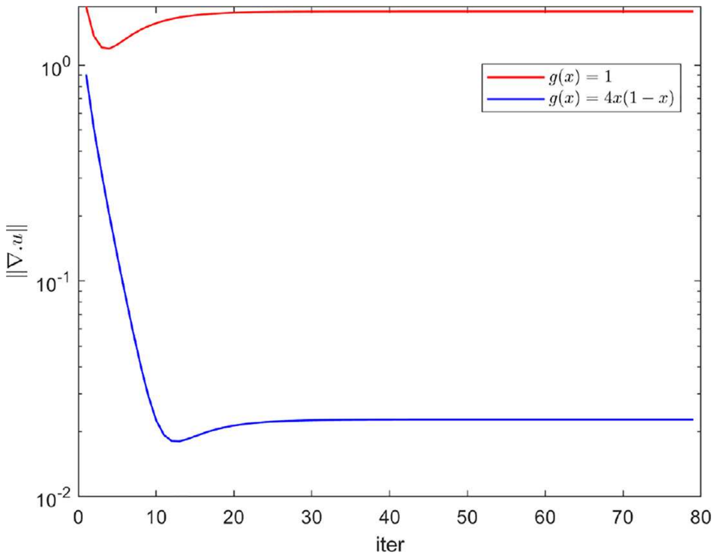

Fig. 1. Comparison of $\|\nabla.u\|$ for different selections of $g(x)$.

The stopping criteria for the iterations is:

$$\max\{\|u^{n+1} - u^n\|, \|p^{n+1} - p^n\|\} \leq 10^{-6}. \tag{7}$$

According to [9], the value of α^2 should be $\alpha^2 < 2$. Our numerical tests confirmed that for any $\alpha^2 \geq 2$ the method diverges and the optimal selection of α^2 is $\alpha^2 = 1.5$ independent of β.

Table 1 indicates that mesh refinement has a very slight effect on convergence and increasing β up to 0.1 has almost no effect on number of iterations. For $\beta > 0.2$, (1) diverged due to the violation of convergence criteria. Further tests for smaller α and larger β did not show a significant change in number of iterations. In this test, the method had almost the same convergence properties as standard Uzawa.

Next we also tested the effect of solution regularity by changing the lid to $u(x, 1) = (1, 0)^T$ with fixed α and β values ($\alpha^2 = 1.5$, $\beta = 0.05$)

Fig. 1 indicates that solution regularity has a notable effect on $\|\nabla.u\|$.

4. Conclusion

If the pressure of Uzawa is replaced as in (1), convergence seems to be similar to standard Uzawa. This suggests the dramatic improvement observed by Ramshaw and Mesina is linked to explicit time stepping. Replacing Step 1 in (1) by a first order Richardson step is therefore where the impact of Step 2 should be next explored.

Data availability

No data was used for the research described in the article.

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References

[1] J.D. Ramshaw, G.L. Mesina, A hybrid penalty-pseudocompressibility method for transient incompressible fluid flow, Comput. & Fluids 20 (2) (1991) 165–175.
[2] J.D. Ramshaw, V.A. Mousseau, Damped artificial compressibility method for steady-state low-speed flow calculations, Comput. & Fluids 20 (2) (1991) 177–186.
[3] P.R. McHugh, J.D. Ramshaw, Damped artificial compressibility iteration scheme for implicit calculations of unsteady incompressible flow, Int. J. Numer. Methods Fluids 21 (2) (1995) 141–153.
[4] C. Bacuta, A unified approach for Uzawa algorithms, SIAM J. Numer. Anal. 44 (6) (2006) 2633–2649.
[5] J.D. Ramshaw, G.L. Mesina, Simplified heuristic Fourier analysis of iteration convergence rates, Commun. Numer. Methods. Eng. 10 (6) (1994) 481–487.
[6] J.K. Dukowicz, Computational efficiency of the hybrid penalty-pseudocompressibility method for incompressible flow, Comput. & Fluids (ISSN: 0045-7930) 23 (2) (1994) 479–486.
[7] R. Bank, B. Welfert, H. Yserentant, A class of iterative methods for solving saddle point problems, Numer. Math. 56 (7) (1989) 645–666.
[8] H.C. Elman, G.H. Golub, Inexact and preconditioned Uzawa algorithms for saddle point problems, SIAM J. Numer. Anal. 31 (6) (1994) 1645–1661.
[9] E. Bänsch, P. Morin, R.H. Nochetto, An adaptive Uzawa FEM for the Stokes problem: Convergence without the inf-sup condition, SIAM J. Numer. Anal. 40 (4) (2003) 1207–1229.