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Negative moments of the Riemann zeta-function

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Abstract. Assuming the Riemann Hypothesis, we study negative moments of the Riemann zeta-function and obtain asymptotic formulas in certain ranges of the shift in $\zeta(s)$. For example, integrating $|\zeta(\frac{1}{2}+\alpha+it)|^{-2k}$ with respect to t from T to $2T$, we obtain an asymptotic formula when the shift α is roughly bigger than $\frac{1}{\log T}$ and $k<\frac{1}{2}$. We also obtain non-trivial upper bounds for much smaller shifts, as long as $\log\frac{1}{\alpha}\ll\log\log T$. This provides partial progress towards a conjecture of Gonek on negative moments of the Riemann zeta-function, and settles the conjecture in certain ranges. As an application, we also obtain an upper bound for the average of the generalized Möbius function.

1. Introduction

For $k>0$, the $2k$th moment of the Riemann zeta-function is given by

$$I_k(T)=\int_0^T|\zeta(\tfrac{1}{2}+it)|^{2k}\,dt.$$

Hardy and Littlewood [16] computed the second moment, and Ingham [22] computed the fourth moment. It is conjectured that

$$I_k(T)\sim c_kT(\log T)^{k^2} \tag{1.1}$$

for an explicit constant c_k, whose value was predicted by Keating and Snaith [23] using analogies with random matrix theory. Their conjecture was later refined by Conrey, Farmer, Keating, Rubinstein and Snaith [8] to include lower order powers of $\log T$, for integer k. Under the Riemann Hypothesis (RH), Soundararajan [35] established almost sharp upper bounds for all the positive moments. This result was later improved by Harper [17], who obtained upper bounds of the conjectural magnitude as in (1.1). There is a wealth of literature on obtaining

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lower and upper bounds for positive moments of $\zeta(s)$; for (an incomplete) list of results, we refer the reader to [3, 18–20, 30, 31, 33, 34].

In this paper, we are interested in studying *negative* moments of the Riemann zeta-function. For $k > 0$ and $\alpha > 0$, let

$$I_{-k}(\alpha, T) = \frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\alpha+it)|^{-2k}\,dt.$$

Studying negative moments of L-functions or, more generally, moments of ratios of L-functions, has applications to many challenging problems in number theory. For a thorough list of applications of moments of ratios of L-functions, we refer the reader to Conrey and Snaith's paper [9]. Here, we will only mention a few. The Ratios Conjecture, due to Conrey, Farmer and Zirnbauer [7], predicts precise asymptotic formulas for averages of ratios of L-functions in families, and generalizes the study of the negative moments. Farmer observed in [11] that obtaining an asymptotic formula for the average of the ratio of two over two zeta-functions (a precursor of the Ratios Conjecture) implies Montgomery's Pair Correlation Conjecture. Focusing on the family of quadratic Dirichlet L-functions, Conrey and Snaith [9] showed that the Ratios Conjecture for this family implies that 100 % of $L(\frac{1}{2}, \chi_d) \neq 0$, where χ_d varies over real primitive characters. (We note that a conjecture due to Chowla [6] states that $L(\frac{1}{2}, \chi_d)$ never vanishes, and the Ratios Conjecture provides the strongest result in this direction.) More recently, the study of negative moments of L-functions has seen unexpected applications in other problems unrelated at first sight to L-functions. For example, assuming various conjectures about negative moments in families, Wang [39] showed that almost all integers (without any "local obstructions") can be written as the sum of three cubes.

A conjecture due to Gonek [15] states the following.

Conjecture 1.1 (Gonek). Let $k > 0$. Uniformly for $\frac{1}{\log T} \le \alpha \le 1$,

$$I_{-k}(\alpha, T) \asymp \left(\frac{1}{\alpha}\right)^{k^2},$$

and uniformly for $0 < \alpha \le \frac{1}{\log T}$,

$$I_{-k}(\alpha, T) \asymp \begin{cases} (\log T)^{k^2} & \text{if } k < \frac{1}{2}, \\ \left(\log \frac{e}{\alpha \log T}\right)(\log T)^{k^2} & \text{if } k = \frac{1}{2}. \end{cases}$$

Gonek's original conjecture predicted formulas for $k > \frac{1}{2}$ and $\alpha \le \frac{1}{\log T}$ as well, which seem to be contradicted however by more recent evidence (see the next paragraph). Under RH, Gonek [15] also proved lower bounds of the conjectured order of magnitude for all $k > 0$ and $\frac{1}{\log T} \le \alpha \le 1$, and for $k < \frac{1}{2}$ and $0 < \alpha \le \frac{1}{\log T}$.

No other progress has been made towards Gonek's conjecture so far, but more recent random matrix theory computations due to Berry and Keating [2], Fyodorov and Keating [14] and Forrester and Keating [13] suggest that certain corrections to the above conjecture are due in some ranges. Namely, when $\alpha \le \frac{1}{\log T}$, random matrix theory computations seem to contradict Gonek's prediction for the negative moments when $k \ge \frac{3}{2}$. In particular, the work in [2, 13] suggests certain transition regimes when $k = \frac{2n+1}{2}$, for n a positive integer (Gonek's conjecture already captures the first transition at $k = \frac{1}{2}$ featuring a logarithmic correction, and in this case the two conjectures do agree).

Reinterpreting the random matrix theory computations in [2], one would expect that for a shift $0 < \alpha \le \frac{1}{\log T}$ and j a natural number such that $\frac{2j-1}{2} < k < \frac{2j+1}{2}$,

$$I_{-k}(\alpha, T) \asymp (\log T)^{k^2} (\alpha \log T)^{-j(2k-j)}, \tag{1.2}$$

while for $k = \frac{2j-1}{2}$ and j a natural number, one would expect

$$I_{-k}(\alpha, T) \asymp \log\Big(\frac{e}{\alpha \log T}\Big) (\log T)^{k^2} (\alpha \log T)^{-j(2k-j)}.$$

We note that the above prediction indeed agrees with Conjecture 1.1 for $k = \frac{1}{2}$ and $\alpha \le \frac{1}{\log T}$. We remark that one could also predict (1.2) for integer k using heuristic ideas similar as in [8].

In this paper, we study the negative moments of the Riemann zeta-function. While obtaining lower bounds for the negative moments is a more tractable problem (see the comment after Conjecture 1.1), no progress has been made so far on obtaining asymptotic formulas or non-trivial upper bounds. When the shift α is "big enough", we obtain upper bounds which are almost sharp according to Conjecture 1.1, up to some logarithmic factors. We also obtain the first non-trivial upper bounds for the negative moments for a wide range of much smaller shifts α (roughly $\alpha \gg (\log T)^{-O(1)}$); however, the bounds in these cases are far from sharp.

More precisely, we prove the following.

Theorem 1.2. *Assume RH. Let $k \ge \frac{1}{2}, \alpha > 0$ and $\varepsilon, \delta > 0$, such that $u = \frac{\log \frac{1}{\alpha}}{\log\log T} \ll 1$. Then*

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} |\zeta(\tfrac12 + \alpha + it)|^{-2k}\, dt \\
&\ll (\log\log T)^k (\log T)^{k^2} && \text{if } \alpha \gg \frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}}, && (1.3)\\
&\ll \exp\Big(\frac{(4+\varepsilon)\log T \log\log\log T}{\log\log T}\Big) && \text{if } \frac{1}{(\log T)^{\frac{1}{2k}}} \ll \alpha = o\Big(\frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}}\Big), && (1.4)\\
&\ll T^{(1+\delta)(ku - \frac12 + k\varepsilon)} && \text{if } \alpha \ll \frac{1}{(\log T)^{\frac{1}{2k}}}. && (1.5)
\end{aligned}$$

We also have the following bounds for $k < \frac{1}{2}$.

Theorem 1.3. *Assume RH. Let $k < \frac{1}{2}, \alpha > 0$ and $\varepsilon, \delta > 0$, such that $u = \frac{\log \frac{1}{\alpha}}{\log\log T} \ll 1$. Then*

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} |\zeta(\tfrac12 + \alpha + it)|^{-2k}\, dt \\
&\ll (\log\log T)^k \big(\tfrac{\log(\alpha \log T)}{\alpha}\big)^{k^2} && \text{if } \alpha \gg \frac{\log\log T}{\log T}, && (1.6)\\
&\ll \exp\Big(C_1 (\log\log T)\big(\log \tfrac{\log\log T}{\alpha \log T}\big)^2\Big) && \text{if } \frac{1}{\log T} \ll \alpha = o\big(\tfrac{\log\log T}{\log T}\big), && (1.7)\\
&\ll \exp\Big(C_2 \big(\tfrac{1}{\alpha \log T}\big)^{\frac{2k}{1-2k-k\varepsilon}}\Big) && \text{if } \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}} \ll \alpha = o\big(\tfrac{1}{\log T}\big), && (1.8)\\
&\ll T^{(1+\delta)(ku - \frac12 + k\varepsilon)} \exp\Big(O\big(\tfrac{\log T \log\log\log T}{\log\log T}\big)\Big) && \text{if } \alpha = o\big(\tfrac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}}\big), && (1.9)
\end{aligned}$$

for some constants $C_1, C_2 > 0$.

Remarks. (1) We remark that results about negative moments of quadratic Dirichlet L-functions were recently proven in the function field setting, first in [4] for shifts big enough, and later improved in [12] to include small shifts.

(2) We note that the bound (1.3) in Theorem 1.2 and the bound (1.6) in Theorem 1.3 are "almost sharp". We remind the reader that in these cases, according to Gonek's Conjecture 1.1, we expect $I_{-k}(\alpha, T) \asymp (\frac{1}{\alpha})^{k^2}$. Hence, the former bound is off by a power of $\log T$ which depends on α, while the latter is off by a $\log\log T$ factor (namely, $(\log\log T)^k(\log(\alpha\log T))^{k^2}$).

We also further use the upper bounds to obtain an asymptotic formula for the negative moments in the following ranges.

Theorem 1.4. *Assume RH. Let $k > 0$, $C, \varepsilon > 0$ and*

$$\alpha \geq \max\left\{C\,\frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}},\ \frac{(1+\varepsilon)\log\log T}{2\log T}\right\}.$$

Then we have

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+\alpha+it)|^{-2k}\,dt$$
$$= (1+o(1))\zeta(1+2\alpha)^{k^2}\prod_p\left(1-\frac{1}{p^{1+2\alpha}}\right)^{k^2}\left(1+\sum_{j=1}^{\infty}\frac{\mu_k(p^j)^2}{p^{(1+2\alpha)j}}\right),$$

where $\mu_k(n)$ denotes the n^{th} Dirichlet coefficient of $\zeta(s)^{-k}$.

As an application, we study averages of the generalized Möbius function. Studying averages of the Möbius function has a long history due to its connections to RH. For example, RH is a consequence of the Mertens conjecture which states that, if we denote by

$$M(x) = \sum_{n\leq x}\mu(n),$$

then

$$|M(x)| \leq \sqrt{x},$$

for $x \geq 1$. The Mertens conjecture was disproven by Odlyzko and te Riele [29], who showed that

$$\liminf_{x\to\infty}\frac{M(x)}{\sqrt{x}} < -1.009 \quad \text{and} \quad \limsup_{x\to\infty}\frac{M(x)}{\sqrt{x}} > 1.06.$$

The true order of $M(x)$ is somewhat mysterious, and Odlyzko and te Riele [29] noted that "No good conjectures about the rate of growth of $M(x)$ are known".

Under RH, Littlewood [25] proved that $M(x) \ll x^{\frac{1}{2}+\varepsilon}$. This was improved in several works [24, 26, 38], and Soundararajan [36] showed that

$$M(x) \ll \sqrt{x}\exp\big(\sqrt{\log x}(\log\log x)^{14}\big)$$

on RH. A note of Balazard and de Roton [1] on Soundararajan's paper [36] improves the power of $\log\log x$ in the bound above, and it was stated in [1] that the power $(\log\log x)^{\frac{5}{2}+\varepsilon}$ is the limitation of the ideas in [1, 36].

Assuming RH and the conjectured order of magnitude of the negative second moment of $\zeta'(\rho)$ by Gonek [15] and Hejhal [21], Ng [28] also showed that

$$M(x) \ll \sqrt{x}(\log x)^{\frac{3}{2}}.$$

Using Theorem 1.2, we improve the result in [36]. As before, let $\mu_k(n)$ denote the nth Dirichlet coefficient of $\zeta(s)^{-k}$. Let

$$Z_{-k}(s) = \frac{1}{s((s-1)\zeta(s))^k}.$$

The function $Z_{-k}(s)$ is holomorphic in the disc $|s-1| < 1$ and has a Taylor series,

$$Z_{-k}(s) = \sum_{j=0}^{\infty} \gamma_{-k,j}(s-1)^j.$$

We prove the following.

Theorem 1.5. *Assume RH. For $k > 0$, we have*

$$\sum_{n \le x} \mu_k(n) = -\frac{\sin(k\pi)}{\pi} \frac{x}{(\log x)^{k+1}} \left(\sum_{j=0}^{N} \frac{(-1)^j \Gamma(k+j+1)\gamma_{-k,j}}{(\log x)^j} + O\left((N+1)^{k+\frac{1}{2}} \left(\frac{N+1}{\log x} \right)^{N+1} \right) \right) + E_k,$$

for any $N \ge 0$ with

$$E_k \ll \begin{cases} \sqrt{x}\exp(\varepsilon\sqrt{\log x}) & \text{if } k < 1, \\ \sqrt{x}\exp\big((\log x)^{\frac{k}{k+1}}(\log\log x)^{\frac{7}{k+1}+\varepsilon}\big) & \text{if } k \ge 1. \end{cases}$$

Also, if we assume Conjecture 1.1*, then*

$$E_k \ll \begin{cases} \sqrt{x}(\log x)^{\frac{k^2}{4}} & \text{if } k \le 2, \\ \sqrt{x}\exp\big((\log x)^{\frac{k-2}{k}}(\log\log x)^{-1+\varepsilon}\big) & \text{if } k > 2. \end{cases}$$

Remark 1. Notice that the main term above vanishes if $k \in \mathbb{Z}$.

Proving Theorem 1.5 requires good bounds for the negative moments of $\zeta(s)$ when roughly

$$\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}}}.$$

We remark that obtaining the bound (1.3) in Theorem 1.2 is the most delicate part of the proof and requires a recursive argument which allows us to obtain improved estimates at each step.

The ideas in the proof of Theorems 1.2 and 1.3 are sieve-theoretic inspired ideas, as in the work of Soundararajan [35] and Harper [17]. However, unlike in the work in [35] and [17], the contributions of zeros of $\zeta(s)$ play an important part, and one needs to be careful about the choice of parameters in the sieve-theoretic argument, in order to account for the big contributions coming from the zeros.

The paper is organized as follows. In Section 2, we prove some lemmas providing a lower bound for the logarithm of $\zeta(s)$, as well as a pointwise lower bound for $|\zeta(s)|$. In Section 3, we introduce the setup of the problem, and prove three main propositions. In Section 4, we consider the case of a "big" shift α, and do the first steps towards proving Theorems 1.2 and 1.3 in this case. In Section 5, we obtain the bound (1.3) in the smaller region

$$\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}},$$

and prove the bounds (1.6), (1.7) and (1.8) from Theorem 1.3 as well. In Section 6, we obtain the bound (1.3) in the wider region stated in the theorem by using a recursive argument. We consider the cases of "small" shifts in Section 7 and prove the bounds (1.5) and (1.9). We then prove Theorem 1.4 in Section 8, and Theorem 1.5 in Section 9.

Throughout the paper ε denotes an arbitrarily small positive number whose value may change from one line to the next.

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2. Preliminary lemmas

The first two lemmas concern the lower bounds for $|\zeta(s)|$.

Lemma 2.1. *Assume RH. Let $\alpha > 0$. Then*

$$\begin{aligned}\log|\zeta(\tfrac12+\alpha+it)| \geq \frac{\log\frac{t}{2\pi}}{2\pi\Delta}\log\big(1-e^{-2\pi\Delta\alpha}\big) + \Re\bigg(\sum_{p\leq e^{2\pi\Delta}} \frac{(\log p)a_\alpha(p;\Delta)}{p^{\frac12+\alpha+it}}\bigg) \\ - \frac{\log\log\log T}{2} - \Big(\log\frac{1}{\Delta\alpha}\Big)^{\gamma(\Delta)} \\ + O\bigg(\frac{\Delta^2 e^{\pi\Delta}}{1+\Delta t} + \frac{\Delta\log(1+\Delta\sqrt{t})}{\sqrt{t}} + 1\bigg),\end{aligned}$$

where

$$a_\alpha(n;\Delta) = \sum_{j=0}^{\infty}\bigg(\frac{(j+1)}{\log n + 2\pi j\Delta}e^{-2\pi j\Delta\alpha} - \frac{(j+1)n^{2\alpha}}{2\pi(j+2)\Delta - \log n}e^{-2\pi(j+2)\Delta\alpha}\bigg) \tag{2.1}$$

and

$$\gamma(\Delta) = \begin{cases} 1 & \text{if } \Delta\alpha = o(1), \\ 0 & \text{if } \Delta\alpha \gg 1. \end{cases}$$

Proof. We use the work in [5]. Let

$$f_\alpha(x) = \log\bigg(\frac{4+x^2}{\alpha^2+x^2}\bigg),$$

and let $m_\Delta(x)$ be an extremal majorant for f_α, whose Fourier transform $\widehat{m}_\Delta$ is supported in $[-\Delta, \Delta]$, satisfying the properties from [5, Lemma 8]. By [5, Equation 3.1],

$$\log|\zeta(\tfrac12+\alpha+it)| \geq \left(1-\frac{\alpha}{2}\right)\log\frac{t}{2} - \frac12\sum_\gamma m_\Delta(t-\gamma) + O(1), \tag{2.2}$$

where the sum over γ is over the ordinates of the zeros of $\zeta(s)$ on the critical line. Using the explicit formula, [5, Equation 3.2] leads to

$$\begin{aligned}\sum_\gamma m_\Delta(t-\gamma) &= m_\Delta\left(t-\frac{1}{2i}\right) + m_\Delta\left(t+\frac{1}{2i}\right) - \frac{\log\pi}{2\pi}\widehat{m}_\Delta(0)\\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta(x)\Re\left(\frac{\Gamma'}{\Gamma}\left(\frac14+\frac{i(t-x)}{2}\right)\right)dx\\ &\quad - \frac{1}{\pi}\Re\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\frac12+it}}\widehat{m}_\Delta\left(\frac{\log n}{2\pi}\right)\right),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\widehat{m}_\Delta(\xi) = \sum_{j=0}^{\infty}\bigg(&\frac{j+1}{\xi+j\Delta}\left(e^{-2\pi(\xi+j\Delta)\alpha} - e^{-4\pi(\xi+j\Delta)}\right)\\ &- \frac{j+1}{(j+2)\Delta-\xi}\left(e^{-2\pi((j+2)\Delta-\xi)\alpha} - e^{-4\pi((j+2)\Delta-\xi)}\right)\bigg).\end{aligned}$$

For $n \leq e^{2\pi\Delta}$ we note that

$$\widehat{m}_\Delta\left(\frac{\log n}{2\pi}\right) = \frac{2\pi a_\alpha(n;\Delta)}{n^\alpha} + O\left(\frac{1}{n^2\log n}\right).$$

Combining that with (2.2), (2.3) and [5, Equations (3.3), (3.4), (3.5)], we obtain

$$\begin{aligned}\log|\zeta(\tfrac12+\alpha+it)| &\geq \frac{\log\frac{t}{2\pi}}{2\pi\Delta}\log(1-e^{-2\pi\Delta\alpha}) + \Re\left(\sum_{n\leq e^{2\pi\Delta}}\frac{\Lambda(n)a_\alpha(n;\Delta)}{n^{\frac12+\alpha+it}}\right)\\ &\quad + O\left(\frac{\Delta^2e^{\pi\Delta}}{1+\Delta t} + \frac{\Delta\log(1+\Delta\sqrt{t})}{\sqrt{t}} + 1\right).\end{aligned} \tag{2.4}$$

It is easy to see that $a_\alpha(n;\Delta) \geq 0$. Also, with $\log n \leq 2\pi\Delta$ we have

$$\begin{aligned}a_\alpha(n;\Delta) &< \frac{1}{\log n} + \sum_{j=1}^{\infty}(j+1)e^{-2\pi j\Delta\alpha}\left(\frac{1}{\log n+2\pi j\Delta} - \frac{1}{2\pi(j+2)\Delta-\log n}\right)\\ &\quad + \sum_{j=1}^{\infty}\frac{(j+1)e^{-2\pi j\Delta\alpha}}{2\pi(j+2)\Delta-\log n}(1-n^{2\alpha}e^{-4\pi\Delta\alpha})\\ &\leq \frac{1}{\log n} + 2(2\pi\Delta-\log n)\sum_{j=1}^{\infty}\frac{(j+1)e^{-2\pi j\Delta\alpha}}{(\log n+2\pi j\Delta)(2\pi(j+2)\Delta-\log n)} \qquad (2.5)\\ &\quad + \frac{1-n^{2\alpha}e^{-4\pi\Delta\alpha}}{2\pi\Delta}\sum_{j=1}^{\infty}e^{-2\pi j\Delta\alpha}\end{aligned}$$

$$< \frac{1}{\log n} + \frac{2(2\pi\Delta - \log n)}{(2\pi\Delta)^2} \sum_{j=1}^{\infty} \frac{e^{-2\pi j\Delta\alpha}}{j} + \frac{1 - n^{2\alpha}e^{-4\pi\Delta\alpha}}{2\pi\Delta(1 - e^{-2\pi\Delta\alpha})}$$

$$= \frac{1}{\log n} - \frac{2(2\pi\Delta - \log n)}{(2\pi\Delta)^2} \log(1 - e^{-2\pi\Delta\alpha}) + O\Big(\frac{1}{\Delta}\Big). \tag{2.6}$$

Note that if $\Delta\alpha \gg 1$, then the second term above is $O(\Delta^{-1})$, and if $\Delta\alpha = o(1)$, then it is

$$\leq \frac{1}{\pi\Delta} \log \frac{1}{\Delta\alpha}. \tag{2.7}$$

Now we use the expression (2.6) for $a_\alpha(n;\Delta)$ and evaluate the contribution from the prime squares in (2.4). When dealing with the main term of size $\frac{1}{\log n}$ above, we proceed similarly as in [17] or [36, Lemma 2]. For the second term, we trivially bound the sum over primes and use the bound (2.7), and it follows that the contribution from prime squares is

$$\geq -\frac{\log\log\log T}{2} - \Big(\log \frac{1}{\Delta\alpha}\Big)^{\gamma(\Delta)} - O(1).$$

Dealing similarly with the sum over prime cubes and higher powers and combining the equation above and (2.4) finishes the proof. □

Lemma 2.2. *Assume RH. If $0 < \alpha = o(\frac{1}{\log\log t})$, then*

$$\log|\zeta(\tfrac12 + \alpha + it)| \geq \frac{\log t}{2\log\log t} \log\big(1 - (\log t)^{-2\alpha}\big) + O\Big(\frac{\log t}{\log\log t}\Big) + O\Big(\frac{\log t}{(\log\log t)^2} \log \frac{1}{1 - (\log t)^{-2\alpha}}\Big), \tag{2.8}$$

and if $\frac12 - \alpha \ll \frac{1}{\log\log t}$, then

$$\log|\zeta(\tfrac12 + \alpha + it)| \geq -\log\log\log t + O(1),$$

otherwise

$$\log|\zeta(\tfrac12 + \alpha + it)| \geq -\Big(\frac12 + \frac{8\alpha}{1 - 4\alpha^2}\Big) \frac{(\log t)^{1-2\alpha}}{\log\log t} - \log\log\log t + O\Big(\frac{(\log t)^{1-2\alpha}}{(1 - 2\alpha)^2(\log\log t)^2}\Big).$$

Proof. Carneiro and Chandee established the last two bounds in [5, Theorem 2] and in the case $0 < \alpha = o(\frac{1}{\log\log t})$ they obtained

$$\log|\zeta(\tfrac12 + \alpha + it)| \geq \frac{\log t}{2\log\log t} \log\big(1 - (\log t)^{-2\alpha}\big) + O\Big(\frac{(\log t)^{1-2\alpha}}{(\log\log t)^2(1 - (\log t)^{-2\alpha})}\Big).$$

We will now prove the improved bound (2.8).

Let $\Delta = \frac{\log\log t}{\pi}$. From (2.4) and (2.6) we have

$$\log|\zeta(\tfrac12 + \alpha + it)| \geq \frac{\log \frac{t}{2\pi}}{2\log\log t} \log\big(1 - (\log t)^{-2\alpha}\big) - \sum_{n \leq (\log t)^2} \frac{\Lambda(n)}{n^{\frac12 + \alpha}} \Big(\frac{1}{\log n} - \frac{2(2\log\log t - \log n)}{(2\log\log t)^2} \log\big(1 - (\log t)^{-2\alpha}\big) + O\Big(\frac{1}{\log\log t}\Big)\Big) + O(1).$$

We now consider the sum over n above, which expands into three terms. The contribution of the first and the last term is

$$\ll \frac{\log t}{\log\log t},$$

while the second term is

$$\begin{aligned}
&= -\frac{\log(1-(\log t)^{-2\alpha})}{\log\log t}\Big(\frac{2(\log t)^{1-2\alpha}}{1-2\alpha} + O\big((\log\log t)^3\big)\Big)\\
&\qquad + \frac{\log(1-(\log t)^{-2\alpha})}{2(\log\log t)^2}\Big(\frac{4(\log t)^{1-2\alpha}\log\log t}{1-2\alpha} + O(\log t)\Big)\\
&= O\Big(\frac{\log t}{(\log\log t)^2}\log\frac{1}{1-(\log t)^{-2\alpha}}\Big),
\end{aligned}$$

by the prime number theorem and partial summation. This establishes (2.8). □

The next lemma is the classical mean value theorem (see, for instance, [27, Theorem 6.1]).

Lemma 2.3. *For any complex numbers $a(n)$ we have*

$$\int_T^{2T}\Big|\sum_{n\le N}\frac{a(n)}{n^{\frac12+it}}\Big|^2 dt = \big(T+O(N)\big)\sum_{n\le N}\frac{|a(n)|^2}{n}.$$

Now let ℓ be an integer, and for $z\in\mathbb{C}$, let

$$E_\ell(z) = \sum_{s\le\ell}\frac{z^s}{s!}.$$

If $z\in\mathbb{R}$, then for ℓ even, $E_\ell(z)>0$.

We have the following elementary inequality.

Lemma 2.4. *Let ℓ be an even integer and let z be a complex number such that $|z|\le\frac{\ell}{e^2}$. Then*

$$e^{\Re(z)} \le \max\Big\{1, |E_\ell(z)|\Big(1+\frac{1}{15e^\ell}\Big)\Big\}.$$

Proof. We have

$$e^{\Re(z)} = |e^z| \le |e^z - E_\ell(z)| + |E_\ell(z)|.$$

Now we have

$$|e^z - E_\ell(z)| \le \sum_{j=\ell+1}^{\infty}\frac{|z|^j}{j!},$$

and we can proceed as in [32, Lemma 1] to get that

$$|e^z - E_\ell(z)| \le \frac{1}{16e^\ell}.$$

Hence

$$e^{\Re(z)} \le |E_\ell(z)| + \frac{1}{16e^\ell}.$$

If $|E_\ell(z)| \le \frac{15}{16}$, then we have $e^{\Re(z)} \le 1$. Otherwise, if $|E_\ell(z)| > \frac{15}{16}$, we get that

$$e^{\Re(z)} \le |E_\ell(z)|\Big(1 + \frac{1}{15e^\ell}\Big),$$

and this completes the proof. □

Let $\nu(n)$ be the multiplicative function given by

$$\nu(p^j) = \frac{1}{j!},$$

for p a prime. We will frequently use the following fact. For any interval I, $s \in \mathbb{N}$ and $a(n)$ a completely multiplicative function, we have

$$\Big(\sum_{p\in I} a(p)\Big)^s = s! \sum_{\substack{p\,|\,n \Rightarrow p\in I\\ \Omega(n)=s}} a(n)\nu(n), \tag{2.9}$$

where $\Omega(n)$ denotes the number of prime factors of n, counting multiplicity.

3. Setup of the proof and main propositions

We will first introduce some notation and state some key lemmas, before proceeding to the proof of Theorems 1.2 and 1.3.

Let

$$I_0 = (1, T^{\beta_0}],\ I_1 = (T^{\beta_0}, T^{\beta_1}],\ \ldots,\ I_K = (T^{\beta_{K-1}}, T^{\beta_K}]$$

for a sequence $\beta_0, \ldots, \beta_K$ to be chosen later such that $\beta_{j+1} = r\beta_j$, for some $r > 1$.

Also let ℓ_j be even parameters which we will also choose later on. Let s_j be integers. For now, we can think of $s_j\beta_j \asymp 1$, and $\sum_{h=0}^K \ell_h\beta_h \ll 1$.

We let $T^{\beta_j} = e^{2\pi\Delta_j}$ for every $0 \le j \le K$ and let

$$P_{u,v}(t) = \sum_{p\in I_u} \frac{b_\alpha(p;\Delta_v)}{p^{\frac12+\alpha+it}},$$

where $b_\alpha(n;\Delta)$ is a completely multiplicative function in the first variable and

$$b_\alpha(p;\Delta) = -a_\alpha(p;\Delta)\log p.$$

For $p \le e^{2\pi\Delta}$, if $\Delta\alpha \gg 1$, we note from equation (2.5) that

$$\begin{aligned}|b_\alpha(p;\Delta)| &\le 1 + \frac{2\pi\Delta - \log p}{2\pi\Delta}\sum_{j=1}^\infty \frac{2\log p}{\log p + 2\pi j\Delta}e^{-2\pi j\Delta\alpha} + \frac{\log p}{2\pi\Delta}\sum_{j=1}^\infty e^{-2\pi j\Delta\alpha}\\ &\le 1 + \sum_{j=1}^\infty e^{-2\pi j\Delta\alpha} = \frac{1}{1-e^{-2\pi\Delta\alpha}}.\end{aligned}$$

On the other hand, if $\Delta\alpha = o(1)$, then from (2.6) we get that

$$|b_\alpha(p;\Delta)| \le \frac{2(2\pi\Delta - \log p)\log p}{(2\pi\Delta)^2}\sum_{j=1}^\infty \frac{e^{-2\pi j\Delta\alpha}}{j} + O(1) \le \frac12\log\frac{1}{\Delta\alpha} + O(1).$$

Thus we can rewrite the bound for $b_\alpha(p;\Delta)$ as a single bound,

$$|b_\alpha(p;\Delta)| \le b(\Delta)\bigg(\log\frac{1}{\Delta\alpha}\bigg)^{\gamma(\Delta)}, \tag{3.1}$$

where $\gamma(\Delta)=0$ if $\Delta\alpha \gg 1$ and $\gamma(\Delta)=1$ if $\Delta\alpha=o(1)$, and where

$$b(\Delta)=\begin{cases}\frac{1}{1-e^{-2\pi\Delta\alpha}} & \text{if } \Delta\alpha\gg 1,\\ \frac12+\varepsilon & \text{if } \Delta\alpha=o(1).\end{cases} \tag{3.2}$$

Let

$$\mathcal{T}_u=\bigg\{T\le t\le 2T : \max_{u\le v\le K}|P_{u,v}(t)|\le \frac{\ell_u}{ke^2}\bigg\}.$$

Denote the set of t such that $t\in\mathcal{T}_u$ for all $u\le K$ by $\mathcal{T}'$. For $0\le j\le K-1$, let $\mathcal{S}_j$ denote the subset of $t\in[T,2T]$ such that $t\in\mathcal{T}_h$ for all $h\le j$, but $t\notin\mathcal{T}_{j+1}$.

We will prove the following lemma.

Lemma 3.1. *For $t\in[T,2T]$, we either have*

$$\max_{0\le v\le K}|P_{0,v}(t)|>\frac{\ell_0}{ke^2},$$

or

$$|\zeta(\tfrac12+\alpha+it)|^{-2k}\le S_1(t)+S_2(t),$$

where

$$\begin{aligned}S_1(t)=(\log\log T)^k&\bigg(\frac{1}{1-T^{-\beta_K\alpha}}\bigg)^{\frac{2k}{\beta_K}}\exp\bigg(2k\bigg(\log\frac{1}{\Delta_K\alpha}\bigg)^{\gamma(\Delta_K)}\bigg)\\&\times\prod_{h=0}^{K}\max\bigg\{1,|E_{\ell_h}(kP_{h,K}(t))|^2\bigg(1+\frac{1}{15e^{\ell_h}}\bigg)^2\bigg\}\\&\times\exp\bigg(O\bigg(\frac{\Delta_K^2e^{\pi\Delta_K}}{1+\Delta_Kt}+\frac{\Delta_K\log(1+\Delta_K\sqrt{t})}{\sqrt{t}}+1\bigg)\bigg)\end{aligned}$$

and

$$\begin{aligned}S_2(t)=(\log\log T)^k&\sum_{j=0}^{K-1}\sum_{v=j+1}^{K}\bigg(\frac{1}{1-T^{-\beta_j\alpha}}\bigg)^{\frac{2k}{\beta_j}}\exp\bigg(2k\bigg(\log\frac{1}{\Delta_j\alpha}\bigg)^{\gamma(\Delta_j)}\bigg)\\&\times\prod_{h=0}^{j}\max\bigg\{1,|E_{\ell_h}(kP_{h,j}(t))|^2\bigg(1+\frac{1}{15e^{\ell_h}}\bigg)^2\bigg\}\bigg(\frac{ke^2}{\ell_{j+1}}|P_{j+1,v}(t)|\bigg)^{2s_{j+1}}\\&\times\exp\bigg(O\bigg(\frac{\Delta_j^2e^{\pi\Delta_j}}{1+\Delta_jt}+\frac{\Delta_j\log(1+\Delta_j\sqrt{t})}{\sqrt{t}}+1\bigg)\bigg).\end{aligned}$$

Proof. For $T\le t\le 2T$, we have the following possibilities:

(1) $t\notin\mathcal{T}_0$,

(2) $t\in\mathcal{T}'$,

(3) $t\in\mathcal{S}_j$ for some $0\le j\le K-1$.

If $t \notin \mathcal{T}_0$, then the first condition is automatically satisfied. Now suppose that $t \in \mathcal{T}_0$.

Suppose that $t \in \mathcal{T}'$. We use Lemma 2.1 with $\Delta = \Delta_K$ and the inequality in Lemma 2.4 with $z = k\Re P_{h,K}(t)$.

Now if $t \in \mathcal{S}_j$, then we use Lemma 2.1 with $\Delta = \Delta_j$, the inequality in Lemma 2.4 with $z = k\Re P_{h,j}(t)$ and the fact that there exists some $v \geq j+1$ such that

$$\frac{ke^2}{\ell_{j+1}}|P_{j+1,v}(t)| > 1. \qquad \square$$

We will also need the following propositions.

Proposition 3.2. *For $0 \leq v \leq K$ and $\beta_0 s_0 \leq 1$, we have*

$$\int_T^{2T} |P_{0,v}(t)|^{2s_0}\,dt \ll Ts_0!b(\Delta_0)^{2s_0}\Big(\log\frac{1}{\Delta_0\alpha}\Big)^{2s_0\gamma(\Delta_0)}(\log\log T^{\beta_0})^{s_0}.$$

Proof. Using (2.9), we have

$$P_{0,v}(t)^{s_0} = s_0! \sum_{\substack{p\,|\,n \Rightarrow p \leq T^{\beta_0}\\ \Omega(n)=s_0}} \frac{b_\alpha(n;\Delta_v)\nu(n)}{n^{\frac12+\alpha+it}}.$$

It then follows from Lemma 2.3 that

$$\begin{aligned}\int_T^{2T} |P_{0,v}(t)|^{2s_0}\,dt &= (T+O(T^{\beta_0 s_0}))(s_0!)^2 \sum_{\substack{p\,|\,n \Rightarrow p \leq T^{\beta_0}\\ \Omega(n)=s_0}} \frac{b_\alpha(n;\Delta_v)^2\nu(n)^2}{n^{1+2\alpha}}\\ &\ll Ts_0!\Big(\sum_{p\leq T^{\beta_0}}\frac{b_\alpha(p;\Delta_v)^2}{p^{1+2\alpha}}\Big)^{s_0}\\ &\leq Ts_0!b(\Delta_v)^{2s_0}\Big(\log\frac{1}{\Delta_v\alpha}\Big)^{2s_0\gamma(\Delta_v)}(\log\log T^{\beta_0})^{s_0},\end{aligned}$$

where we have used the fact that $\nu(n)^2 \leq \nu(n)$ and the bound (3.1). Now since

$$b(\Delta_v)\Big(\log\frac{1}{\Delta_v\alpha}\Big)^{2s_0\gamma(\Delta_v)} \leq b(\Delta_0)\Big(\log\frac{1}{\Delta_0\alpha}\Big)^{2s_0\gamma(\Delta_0)},$$

the conclusion follows. $\square$

Proposition 3.3. *Let $0 \leq j \leq K-1$. Then, for $\sum_{h=0}^{j}\ell_h\beta_h + s_{j+1}\beta_{j+1} \leq 1$ and for $j+1 \leq v \leq K$, we have*

$$\begin{aligned}&\int_T^{2T}\prod_{h=0}^{j}|E_{\ell_h}(kP_{h,j}(t))|^2|P_{j+1,v}(t)|^{2s_{j+1}}\,dt\\ &\quad\ll Ts_{j+1}!(\log T^{\beta_j})^{k^2b(\Delta_j)^2(\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}}\\ &\qquad\times b(\Delta_{j+1})^{2s_{j+1}}\Big(\log\frac{1}{\Delta_{j+1}\alpha}\Big)^{2s_{j+1}\gamma(\Delta_{j+1})}(\log r)^{s_{j+1}}.\end{aligned}$$

Proof. Using (2.9), we have

$$\int_T^{2T} \prod_{h=0}^{j} |E_{\ell_h}(kP_{h,j}(t))|^2 |P_{j+1,v}(t)|^{2s_{j+1}}\,dt \tag{3.3}$$
$$= (s_{j+1}!)^2 \int_T^{2T} \bigg| \sum_{n \le T^{\sum_{h=0}^{j} \ell_h\beta_h + s_{j+1}\beta_{j+1}}} \frac{c(n)\nu(n)}{n^{\frac12+\alpha+it}} \bigg|^2 dt,$$

where

$$c(n) = \sum_{\substack{n=n_1\dots n_{j+1}\\ p\,|\,n_h \Rightarrow p\in I_h\\ \Omega(n_h)\le \ell_h, h=0,\dots,j\\ \Omega(n_{j+1})=s_{j+1}}} \bigg(\prod_{h=0}^{j} b_\alpha(n_h;\Delta_j)k^{\Omega(n_h)}\bigg) b_\alpha(n_{j+1};\Delta_v).$$

Using Lemma 2.3 in (3.3) and the fact that $\nu(m)^2 \le \nu(m)$ for any m, we obtain that

$$\begin{aligned}(3.3) &\ll \big(T + T^{\sum_{h=0}^{j} \ell_h\beta_h + s_{j+1}\beta_{j+1}}\big)(s_{j+1}!)^2\\ &\quad\times \bigg(\prod_{h=0}^{j} \sum_{\substack{p\,|\,n_h \Rightarrow p\in I_h\\ \Omega(n_h)\le \ell_h}} \frac{|b_\alpha(n_h;\Delta_j)|^2 k^{2\Omega(n_h)}\nu(n_h)}{n_h^{1+2\alpha}}\bigg)\\ &\quad\times \sum_{\substack{p\,|\,n_{j+1} \Rightarrow p\in I_{j+1}\\ \Omega(n_{j+1})=s_{j+1}}} \frac{|b_\alpha(n_{j+1};\Delta_v)|^2\nu(n_{j+1})}{n_{j+1}^{1+2\alpha}}.\end{aligned}$$

Now we use the assumption that $\sum_{h=0}^{j} \ell_h\beta_h + s_{j+1}\beta_{j+1} \le 1$. Bounding $\nu(n_h) \le 1$, removing the condition on the number of primes of n_h, and using (3.1), we get that

$$\begin{aligned}(3.3) &\ll T s_{j+1}! \prod_{p\le T^{\beta_j}} \bigg(1 - \frac{k^2|b_\alpha(p;\Delta_j)|^2}{p^{1+2\alpha}}\bigg)^{-1} \bigg(\sum_{p\in I_{j+1}} \frac{|b_\alpha(p;\Delta_v|^2}{p^{1+2\alpha}}\bigg)^{s_{j+1}}\\ &\ll T s_{j+1}! (\log T^{\beta_j})^{k^2 b(\Delta_j)^2 (\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}} b(\Delta_v)^{2s_{j+1}}\\ &\quad\times \bigg(\log\frac{1}{\Delta_v\alpha}\bigg)^{2s_{j+1}\gamma(\Delta_v)} \bigg(\log\frac{\beta_{j+1}}{\beta_j}\bigg)^{s_{j+1}}\\ &\ll T s_{j+1}! (\log T^{\beta_j})^{k^2 b(\Delta_j)^2 (\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}} b(\Delta_{j+1})^{2s_{j+1}}\\ &\quad\times \bigg(\log\frac{1}{\Delta_{j+1}\alpha}\bigg)^{2s_{j+1}\gamma(\Delta_{j+1})} (\log r)^{s_{j+1}},\end{aligned}$$

which finishes the proof of the proposition. □

A minor modification of the proposition above (where we do not have the contribution from the $j+1$ interval) yields the following.

Proposition 3.4. *For $\sum_{h=0}^{K} \ell_h\beta_h \le 1$, we have*

$$\int_T^{2T} \prod_{h=0}^{K} |E_{\ell_h}(kP_{h,K}(t))|^2\,dt \ll T(\log T^{\beta_K})^{k^2 b(\Delta_K)^2 (\log\frac{1}{\Delta_K\alpha})^{2\gamma(\Delta_K)}}.$$

4. The case $\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}}$, first steps

Here, we will consider the case

$$\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}},$$

which will be a starting point in the proof of Theorems 1.2 and 1.3.

We choose

$$\beta_0 = \frac{a(2d-1)(\log\log T)^2}{(1+2\varepsilon)k(\log T)\big(\log \frac{1}{1-(\log T)^{-2\alpha}}\big)}, \quad s_0 = \left[\frac{a}{\beta_0}\right], \quad \ell_0 = 2\left\lceil\frac{s_0^d}{2}\right\rceil \tag{4.1}$$

and

$$\beta_j = r^j\beta_0, \quad s_j = \left[\frac{a}{\beta_j}\right], \quad \ell_j = 2\left\lceil\frac{s_j^d}{2}\right\rceil, \quad 1 \le j \le K, \tag{4.2}$$

where we can pick, for example,

$$a = \frac{4-3k\varepsilon}{2(2-k\varepsilon)}, \quad r = \frac{2}{2-k\varepsilon}, \quad d = \frac{8-7k\varepsilon}{2(4-3k\varepsilon)},$$

so that

$$\frac{a(2d-1)}{r} = 1 - k\varepsilon. \tag{4.3}$$

Here, K is chosen such that it is the maximal integer for which

$$\beta_K \le \begin{cases} \frac{\log(\alpha\log T)}{\alpha\log T} & \text{if } \alpha\log T \to \infty, \\ c & \text{if } \alpha \ll \frac{1}{\log T}, \end{cases} \tag{4.4}$$

where $c > 0$ is a small constant such that

$$c^{1-d}\left(\frac{a^d r^{1-d}}{r^{1-d}-1} + \frac{2r}{r-1}\right) \le 1 - a. \tag{4.5}$$

Note that the above ensures that the conditions in Propositions 3.3 and 3.4 are satisfied.

If $t \notin \mathcal{T}_0$, then there exists $0 \le v \le K$ such that $\frac{ke^2}{\ell_0}|P_{0,v}(t)| > 1$. Then we have

$$\begin{aligned}\int_{[T,2T]\setminus\mathcal{T}_0} &|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \\ &\le \int_T^{2T}\left(\frac{ke^2}{\ell_0}|P_{0,v}(t)|\right)^{2s_0}|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \\ &\le \left(\frac{ke^2}{\ell_0}\right)^{2s_0}\left(\frac{1}{1-(\log T)^{-2\alpha}}\right)^{\frac{(1+\varepsilon)k\log T}{\log\log T}}\int_T^{2T}|P_{0,v}(t)|^{2s_0}\,dt,\end{aligned}$$

by using the pointwise bound in Lemma 2.2,

$$|\zeta(\tfrac12+\alpha+it)|^{-1} \ll \left(\frac{1}{1-(\log T)^{-2\alpha}}\right)^{\frac{(1+\varepsilon)\log T}{2\log\log T}}. \tag{4.6}$$

By Proposition 3.2 and Stirling's formula, we get that

$$
\begin{aligned}
(4.7)\qquad &\int_{[T,2T]\backslash\mathcal{T}_0} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\
&\ll T\bigg(\frac{1}{1-(\log T)^{-2\alpha}}\bigg)^{\frac{(1+\varepsilon)k\log T}{\log\log T}}\sqrt{s_0}\exp\bigg(-(2d-1)s_0\log s_0\\
&\qquad\qquad + 2s_0\log\bigg(ke^{\frac32}b(\Delta_0)\bigg(\log\frac{1}{\Delta_0\alpha}\bigg)^{\gamma(\Delta_0)}\sqrt{\log\log T^{\beta_0}}\bigg)\bigg)\\
&= o(T),
\end{aligned}
$$

using the choice of s_0 in (4.1).

Now assume that $t\in\mathcal{T}_0$. Using Lemma 3.1, we have

$$\int_{\mathcal{T}_0}|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \le \int_T^{2T} S_1(t)\,dt + \int_T^{2T} S_2(t)\,dt.$$

With the choice (4.4) of β_K, we have that

$$\bigg(\frac{1}{1-T^{-\beta_K\alpha}}\bigg)^{\frac{2k}{\beta_K}} \ll \begin{cases} 1 & \text{if } \alpha \gg \frac{1}{\log T},\\ (\log T)^{O(1)} & \text{if } \alpha = o\big(\frac{1}{\log T}\big)\end{cases}$$

and

$$\exp\bigg(O\bigg(\frac{\Delta_K^2 e^{\pi\Delta_K}}{1+\Delta_K t} + \frac{\Delta_K\log(1+\Delta_K\sqrt{t})}{\sqrt{t}} + 1\bigg)\bigg) = O(1),$$

so

$$
\begin{aligned}
\int_T^{2T} S_1(t)\,dt &\ll (\log\log T)^k\exp\bigg(2k\bigg(\log\frac{1}{\Delta_K\alpha}\bigg)^{\gamma(\Delta_K)}\bigg)(\log T)^{c_1\gamma(\Delta_K)}\\
&\qquad\times\int_T^{2T}\prod_{h=0}^{K}\max\bigg\{1,|E_{\ell_h}(kP_{h,K}(t))|^2\bigg(1+\frac{1}{15e^{\ell_h}}\bigg)^2\bigg\}\,dt.
\end{aligned}
$$

Note that in the inequality above, we can assume without loss of generality that

$$\max\bigg\{1,|E_{\ell_h}(kP_{h,K}(t))|^2\bigg(1+\frac{1}{15e^{\ell_h}}\bigg)^2\bigg\} = |E_{\ell_h}(kP_{h,K}(t))|^2\bigg(1+\frac{1}{15e^{\ell_h}}\bigg)^2.$$

Using Proposition 3.4 and the observation that $\gamma(\Delta_K)=0$ in the first two cases, and $\gamma(\Delta_K)=1$ in the third case, we get that

$$
\begin{aligned}
(4.8)\qquad &\int_T^{2T} S_1(t)\,dt\\
&\ll \begin{cases} T(\log\log T)^k\big(\frac{\log(\alpha\log T)}{\alpha}\big)^{k^2} & \text{if } \alpha\log T\to\infty,\\ T(\log\log T)^k(\log T)^{k^2(\frac{1}{1-T^{-c\alpha}})^2} & \text{if } \alpha\asymp\frac{1}{\log T},\\ T(\log\log T)^k\exp\Big(k^2\big(\frac14+\varepsilon\big)(\log\log T)\big(\log\frac{1}{\alpha\log T}\big)^2\Big) & \text{if } \alpha = o\big(\frac{1}{\log T}\big).\end{cases}
\end{aligned}
$$

Now we will need to bound the contribution from $S_2(t)$. Using Proposition 3.3 and Stirling's formula, we have

$$\begin{aligned}\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k \sum_{j=0}^{K-1}(K-j)\sqrt{s_{j+1}}\exp\bigg(&\frac{2k}{\beta_j}\log\frac{1}{1-T^{-\beta_j\alpha}}\\ &-(2d-1)s_{j+1}\log s_{j+1}\\ &+2s_{j+1}\log\bigg(ke^{\frac32}b(\Delta_{j+1})\Big(\log\frac{1}{\Delta_{j+1}\alpha}\Big)^{\gamma(\Delta_{j+1})}\bigg)\\ &+2k\Big(\log\frac{1}{\Delta_j\alpha}\Big)^{\gamma(\Delta_j)}\bigg)\\ \times(\log T^{\beta_j})^{k^2b(\Delta_j)^2(\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}}&.\end{aligned}$$

In the equation above, note that if $\beta_j\alpha\log T\geq\varepsilon$, then

$$\frac{2k}{\beta_j}\log\frac{1}{1-T^{-\beta_j\alpha}}\leq\frac{2k}{\beta_j}\log\frac{1}{1-e^{-\varepsilon}}<(2d-1)s_{j+1}\log s_{j+1},$$

so the contribution in this case will be $o(T(\log\log T)^k)$. Now if $\beta_j\alpha\log T<\varepsilon$, then

$$\log\frac{1}{1-T^{-\beta_j\alpha}}<\log\frac{1}{\alpha}-\log(\beta_j\log T)+\log\frac{1}{1-\frac{\varepsilon}{2}},$$

so we obtain that

$$\begin{aligned}&\int_T^{2T} S_2(t)\,dt \qquad (4.9)\\ &\ll T(\log\log T)^k\sum_{j=0}^{K-1}(K-j)\sqrt{\frac{1}{\beta_{j+1}}}\\ &\quad\times\exp\bigg(\frac{\log\log T}{\beta_j}\Big(2k\frac{\log\frac{1}{\alpha}}{\log\log T}-\frac{a(2d-1)}{r}\Big)\\ &\qquad+\frac{\log(\beta_j\log T)}{\beta_j}\Big(\frac{a(2d-1)}{r}-2k\Big)\\ &\qquad+\frac{2a}{r\beta_j}\log\bigg(ke^{\frac32}b(\Delta_{j+1})\Big(\log\frac{1}{\Delta_{j+1}\alpha}\Big)^{\gamma(\Delta_{j+1})}\bigg)\\ &\qquad+\frac{2k}{\beta_j}\log\frac{1}{1-\frac{\varepsilon}{2}}+\frac{a(2d-1)}{r\beta_j}\log\frac{r}{a}+2k\Big(\log\frac{1}{\Delta_j\alpha}\Big)^{\gamma(\Delta_j)}\bigg)\\ &\quad\times(\log T^{\beta_j})^{k^2b(\Delta_j)^2(\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}}.\end{aligned}$$

5. Proof of Theorems 1.2 and 1.3 for "big" shifts α

Here, we will prove the bound (1.3) in Theorem 1.2 when

$$\alpha\gg\frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}},$$

and the bounds (1.6), (1.7), (1.8) in Theorem 1.3. Recall that

$$u = \frac{\log\frac{1}{\alpha}}{\log\log T}.$$

The contribution from $t \notin \mathcal{T}_0$ and from $S_1(t)$ has already been bounded in Section 4 (see equations (4.7) and (4.8)). Now we focus on bounding the contribution from $S_2(t)$.

First assume that

$$\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}} \quad \text{and} \quad k \geq \frac{1}{2}.$$

We rewrite (4.9) as

$$\begin{aligned}\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k \sum_{j=0}^{K-1}(K-j)\sqrt{\frac{1}{\beta_{j+1}}}\exp\bigg(&\frac{\log\log T}{\beta_j}\Big(2ku - \frac{a(2d-1)}{r}\Big)\\ &+ \frac{\log(\beta_j \log T)}{\beta_j}\Big(\frac{a(2d-1)}{r} - 2k\Big) + 2k\Big(\log\frac{1}{\Delta_j\alpha}\Big)^{\gamma(\Delta_j)}\\ &+ O\Big(\frac{\log\log\log T}{\beta_j}\Big)\bigg)(\log T^{\beta_j})^{k^2 b(\Delta_j)^2(\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}}.\end{aligned}$$

Note that

$$\frac{a(2d-1)}{r} - 2k < 0,$$

and using (4.3) and the fact that $\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}}$, it follows that

$$2ku - \frac{a(2d-1)}{r} \leq -k\varepsilon.$$

Hence we get

$$\begin{aligned}\text{(5.1)}\qquad \int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k &\sum_{j=0}^{K-1}(K-j)\sqrt{\frac{1}{\beta_{j+1}}}\\ &\times\exp\bigg(-\frac{k\varepsilon\log\log T}{\beta_j} + 2k\Big(\log\frac{1}{\Delta_j\alpha}\Big)^{\gamma(\Delta_j)}\\ &+ O\Big(\frac{\log\log\log T}{\beta_j}\Big)\bigg)(\log T^{\beta_j})^{k^2 b(\Delta_j)^2(\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}}.\end{aligned}$$

We first consider the contribution from those j for which $\gamma(\Delta_j) = 1$. Let R_1 denote this contribution. Using the fact that $K \ll \log\log T$ and after a relabeling of the ε, we have that

$$\begin{aligned}R_1 \ll T(\log\log T)^k \sum_j \exp\bigg(&-\frac{k\varepsilon\log\log T}{\beta_j} + 2k\log\frac{1}{\beta_j\alpha\log T} + O\Big(\frac{\log\log\log T}{\beta_j}\Big)\\ &+ \Big(\frac{1}{4}+\varepsilon\Big)k^2\Big(\log\frac{1}{\beta_j\alpha\log T}\Big)^2\log(\beta_j\log T)\bigg),\end{aligned}$$

where the sum over j is such that $\gamma(\Delta_j) = 1$. Since $\alpha \gg \frac{1}{\log T}$, we have

$$\Big(\log\frac{1}{\beta_j\alpha\log T}\Big)^2\log(\beta_j\log T) \ll \Big(\log\frac{1}{\beta_j}\Big)^2\log\log T.$$

As $\gamma(\Delta_j)=1$, we have $\beta_j\to 0$ in the sum in R_1 above, and then it follows that

$$R_1 = o(T(\log\log T)^k). \tag{5.2}$$

Now we consider the contribution in (5.1) from those j with $\gamma(\Delta_j)=0$. Let R_2 denote that term. We have that

$$\begin{aligned}R_2 \ll T(\log\log T)^k \sum_j \exp\bigg(&-\frac{k\varepsilon\log\log T}{\beta_j}+O\Big(\frac{\log\log\log T}{\beta_j}\Big)\\ &+k^2 b(\Delta_j)^2\log(\beta_j\log T)\bigg),\end{aligned}$$

where the sum is over j such that $\gamma(\Delta_j)=0$. Keeping in mind the choices for β_j (equations (4.2) and (4.4)), it then follows that

$$R_2 = \begin{cases} o(T(\log\log T)^k) & \text{if } \alpha\log T\to\infty,\\ o\big(T(\log\log T)^k(\log T)^{k^2(\frac{1}{1-T^{-c\alpha}})^2}\big) & \text{if } \alpha\asymp\frac{1}{\log T}.\end{cases} \tag{5.3}$$

Combining the bounds (4.7), (4.8), (5.2) and (5.3), the bound (1.3) follows when

$$\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}} \quad\text{and}\quad k\geq\frac{1}{2}.$$

Now assume that

$$k<\frac{1}{2}\quad\text{and}\quad \alpha\gg\frac{1}{\log T}.$$

If $\frac{a(2d-1)}{r}-2k\leq 0$, then the same argument as before works. Hence we assume we have

$$\frac{a(2d-1)}{r}-2k>0.$$

We rewrite the bound (4.9) for $S_2(t)$ as

$$\begin{aligned}\int_T^{2T} & S_2(t)\,dt \\ &\ll T(\log\log T)^k\sum_{j=0}^{K-1}(K-j)\sqrt{\frac{1}{\beta_{j+1}}}\\ &\quad\times\exp\bigg(-\frac{2k\log(\alpha\log T)}{\beta_j}+\frac{\log\beta_j}{\beta_j}\Big(\frac{a(2d-1)}{r}-2k\Big)\\ &\qquad+\frac{2}{\beta_j}\Big(\log\log\frac{1}{\Delta_{j+1}\alpha}\Big)^{\gamma(\Delta_{j+1})}+2k\Big(\log\frac{1}{\Delta_j\alpha}\Big)^{\gamma(\Delta_j)}\\ &\qquad+k^2b(\Delta_j)^2\Big(\log\frac{1}{\Delta_j\alpha}\Big)^{2\gamma(\Delta_j)}\log(\beta_j\log T)+O\Big(\frac{1}{\beta_j}\Big)\bigg).\end{aligned} \tag{5.4}$$

In (5.4), we first consider the contribution from those j for which $\gamma(\Delta_j)=1$, i.e., those j for which $\beta_j=o(\frac{1}{\alpha\log T})$. As before, we denote this contribution by R_1. We let

$$\begin{aligned}f(x) = &-\frac{2k\log(\alpha\log T)}{x}+\frac{\log x}{x}\Big(\frac{a(2d-1)}{r}-2k\Big)+\frac{2}{x}\log\log\frac{1}{x\alpha\log T}\\ &+2k\log\frac{1}{x\alpha\log T}+k^2\Big(\frac{1}{4}+\varepsilon\Big)\Big(\log\frac{1}{x\alpha\log T}\Big)^2\log(x\log T).\end{aligned}$$

By taking the derivative, we see that when $\frac{1}{\log T} \ll \alpha = o\big(\frac{\log\log T}{\log T}\big)$, the maximum of $f(x)$ (when $x = o(\frac{1}{\alpha \log T})$) is attained at some

$$x_0 \asymp \frac{1}{\log\log T},$$

while if $\alpha \gg \frac{\log\log T}{\log T}$, the function $f(x)$ is increasing on the interval under consideration. Hence, we get that

$$R_1 = \begin{cases} o(T(\log\log T)^k) & \text{if } \alpha \gg \frac{\log\log T}{\log T}, \\ O\Big(T\exp\Big(C_1(\log\log T)\big(\log\frac{\log\log T}{\alpha\log T}\big)^2\Big)\Big) & \text{if } \frac{1}{\log T} \ll \alpha = o\big(\frac{\log\log T}{\log T}\big), \end{cases} \tag{5.5}$$

for some $C_1 > 0$.

Now we bound the contribution in (5.4) from those j for which $\gamma(\Delta_j) = 0$. It is easy to see that in this case, the function in (5.4) is decreasing in j, so

$$R_2 = o(T(\log\log T)^k).$$

Combining the above with (4.7), (4.8) and (5.5), the bounds (1.6) and (1.7) follow.

Now we assume that

$$\frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}} \ll \alpha = o\Big(\frac{1}{\log T}\Big).$$

We rewrite the bound (4.9) for $S_2(t)$ as

$$\begin{aligned}\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k \sum_{j=0}^{K-1}(K-j)\sqrt{\frac{1}{\beta_{j+1}}}\exp\Big(&-\frac{2k\log(\alpha\log T)}{\beta_j} \\ &+ \frac{\log\beta_j}{\beta_j}\Big(\frac{a(2d-1)}{r} - 2k\Big) + \frac{2}{\beta_j}\log\log\frac{1}{\Delta_{j+1}\alpha} \\ &+ 2k\log\frac{1}{\Delta_j\alpha} + O\Big((\log\log T)^3 + \frac{1}{\beta_j}\Big)\Big).\end{aligned}$$

In the sum over j above, the maximum is attained at j_0 such that $\beta_{j_0} \asymp (\alpha\log T)^{\frac{2k}{\frac{a(2d-1)}{r}-2k}}$. It then follows that

$$\int_T^{2T} S_2(t)\,dt \ll T\exp\bigg(C_2\Big(\frac{1}{\alpha\log T}\Big)^{\frac{2k}{1-2k-k\varepsilon}}\bigg),$$

for some $C_2 > 0$ (and after a relabeling of the ε). Combining the above and (4.7), (4.8), the bound (1.8) follows when $\frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}} \ll \alpha = o(\frac{1}{\log T})$.

6. Proof of Theorem 1.2, bounds (1.3) and (1.4); some recursive estimates

Here, we will prove the bounds (1.3) and (1.4). To do that, we will use an inductive argument, which will be performed in Section 6.2. The first step of the argument in carried out in the next subsection.

6.1. The range $\alpha \gg (\log\log T)^{\frac{4}{k}+\varepsilon}/(\log T)^{\frac{1}{2k}}$, $k \geq \frac{1}{2}$, the first step. We previously obtained the bound (1.3) in the region

$$\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}} \quad \text{and} \quad k \geq \frac{1}{2}.$$

Hence, we will assume that

$$\frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}} \ll \alpha = o\bigg(\frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}}\bigg), \tag{6.1}$$

when $k \geq \frac{1}{2}$.

Let

$$\alpha = \frac{(\log\log T)^b}{(\log T)^{\frac{1}{2k}}}, \tag{6.2}$$

where $b \geq \frac{4}{k} + \varepsilon$. From (6.1), we have that $b = o(\frac{\log\log T}{\log\log\log T})$. We will show that for any $\delta > 0$, we have

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+\alpha+it)|^{-2k}\,dt \ll T\exp\Bigg(\frac{(\log T)^{\frac{3(1+\delta)}{kb-1}}}{\exp\big(\frac{2\log\log T\,\log\log\log\log T}{(kb-1)\log\log\log T}\big)}\Bigg).$$

We choose β_0, ℓ_0, s_0 as in (4.1) and β_j, ℓ_j, s_j are chosen as in (4.2). We choose a, d, r such that

$$\frac{a(2d-1)}{r} = 1 - \frac{n\log\log\log T}{\log\log T}, \tag{6.3}$$

where

$$n = (2kb-2)\bigg(1 - \frac{10\delta}{12(1+\delta)}\bigg). \tag{6.4}$$

For simplicity of notation, let $x = \frac{\log\log\log T}{\log\log T}$. We can take

$$a = \frac{1-n(1-\frac{\delta}{24})x}{1-n(1-\frac{\delta}{12})x}, \quad d = 1 - \frac{n}{2}\bigg(1-\frac{\delta}{12}\bigg)x, \quad r = \frac{1-n(1-\frac{\delta}{24})x}{1-nx}. \tag{6.5}$$

We choose β_K such that

$$\beta_K^{1-d}\bigg(\frac{a^d r^{1-d}}{r^{1-d}-1} + \frac{2r}{r-1}\bigg) \leq 1 - a.$$

Again, the above inequality ensures that the conditions in Propositions 3.3 and 3.4 are satisfied. Note that the condition above can be re-expressed as

$$\beta_K^{1-d} \leq c_1(1-a)(r-1)(1-d),$$

for some constant $c_1 > 0$. We then choose K such that β_K is the largest of the form in (4.2) such that

$$\beta_K \leq \frac{\exp\big(\frac{6\log\log T\,\log\log\log\log T}{n\log\log\log T}\big)}{(\log T)^{\frac{6}{n(1-\frac{\delta}{12})}}}. \tag{6.6}$$

If $t \notin \mathcal{T}_0$, then we proceed as in Section 5 and similarly to equation (4.7), we get that

$$\begin{aligned}(6.7)\qquad &\int_{[T,2T]\setminus\mathcal{T}_0} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\ll T\Big(\frac{1}{1-(\log T)^{-2\alpha}}\Big)^{\frac{(1+\varepsilon)k\log T}{\log\log T}}\sqrt{s_0}\exp\bigg(-(2d-1)s_0\log s_0\\ &\qquad + 2s_0\log\bigg(ke^{\frac32}b(\Delta_0)\Big(\log\frac{1}{\Delta_0\alpha}\Big)^{\gamma(\Delta_0)}\sqrt{\log\log T^{\beta_0}}\bigg)\bigg)\\ &= o(T).\end{aligned}$$

Now we suppose that $t \in \mathcal{T}_0$. Similarly as in Section 5, using Proposition 3.4 we get that

$$\begin{aligned}(6.8)\qquad &\int_T^{2T} S_1(t)\,dt\\ &\ll T(\log\log T)^k\exp\Bigg(c_1\frac{(\log T)^{\frac{6}{n(1-\frac{\delta}{12})}}\log\log T}{\exp\big(\frac{6\log\log T\log\log\log\log T}{n(1-\frac{\delta}{12})\log\log\log T}\big)}\Bigg)\exp\big(k^2(\log\log T)^3\big)\\ &\ll T\exp\Bigg(\frac{(\log T)^{\frac{6}{n(1-\frac{\delta}{12})}}}{\exp\big(\frac{4\log\log T\log\log\log\log T}{n\log\log\log T}\big)}\Bigg),\end{aligned}$$

for some $c_1 > 0$ (note that the constant c_1 can change from line to line).

To bound the contribution from $S_2(t)$, we proceed as in equation (4.9) and obtain that

$$\begin{aligned}(6.9)\qquad &\int_T^{2T} S_2(t)\,dt\\ &\ll T(\log\log T)^k\sum_{j=0}^{K-1}(K-j)\sqrt{\frac{1}{\beta_{j+1}}}\exp\bigg(\frac{\log\log T}{\beta_j}\Big(2ku-\frac{a(2d-1)}{r}\Big)\\ &\quad+\frac{\log(\beta_j\log T)}{\beta_j}\Big(\frac{a(2d-1)}{r}-2k\Big)\\ &\quad+\frac{2a}{r\beta_j}\log\bigg(ke^{\frac32}b(\Delta_{j+1})\Big(\log\frac{1}{\Delta_{j+1}\alpha}\Big)^{\gamma(\Delta_{j+1})}\bigg)\\ &\quad+\frac{2k}{\beta_j}\log\frac{1}{1-\frac{\varepsilon}{2}}+\frac{a(2d-1)}{r\beta_j}\log\frac{r}{a}\\ &\quad+2k\Big(\log\frac{1}{\Delta_j\alpha}\Big)^{\gamma(\Delta_j)}\bigg)(\log T^{\beta_j})^{k^2b(\Delta_j)^2(\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}}.\end{aligned}$$

Since

$$\alpha = \frac{(\log\log T)^b}{(\log T)^{\frac{1}{2k}}}$$

and given (6.3), we get that

$$\begin{aligned}\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k\sum_{j=0}^{K-1}\exp\bigg(&\frac{\log\log\log T}{\beta_j}(n-2kb+2)\\ &+(\log\log T)^3+O\Big(\frac{1}{\beta_j}\Big)\bigg),\end{aligned}$$

where we used the fact that $K \ll \log\log T$ and the fact that

$$\log\log\frac{1}{\beta_j\alpha\log T} \le \log\log\frac{1}{\beta_0\alpha\log T} \le \frac{\log\log\log T}{2k}.$$

With the choice (6.4), we have that $n - 2kb + 2 \le -\delta$, and then

$$\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k \sum_{j=0}^{K-1} \exp\bigg(-\frac{\delta\log\log\log T}{\beta_j} + (\log\log T)^3 + O\Big(\frac{1}{\beta_j}\Big)\bigg).$$

In the sum over j above, the maximum is attained at $j = K-1$, and given the choice (6.6) for β_K, it follows that

$$\int_T^{2T} S_2(t)\,dt = o(T). \tag{6.10}$$

Combining equations (6.7), (6.8) and (6.10), it follows that

$$\int_T^{2T} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \ll T\exp\Bigg(\frac{(\log T)^{\frac{3(1+\delta)}{kb-1}}}{\exp\big(\frac{2\log\log T\log\log\log\log T}{(kb-1)\log\log\log T}\big)}\Bigg). \tag{6.11}$$

6.2. The range $\alpha \gg (\log\log T)^{\frac{4}{k}+\varepsilon}/(\log T)^{\frac{1}{2k}}$, $k \ge \frac12$, a recursive bound. Here, we have the same setup as in the previous subsection. Namely, we assume (6.1) and (6.2).

We will perform the same argument as before, but with a different choice of parameters. We suppose that at step $m-1$, for any $\delta > 0$, we have the bound

$$\begin{aligned}\int_T^{2T} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \ll T\exp\Bigg(&\frac{(\log T)^{(1+\delta)(\frac{3}{kb-1})^{m-1}}\log\log T}{\exp\big(\frac{2\cdot 3^{m-2}\log\log T\log\log\log\log T}{(kb-1)^{m-1}\log\log\log T}\big)}\Bigg)\\ &\times\exp\big((1+\delta)k^2(\log\log T)^3\big).\end{aligned} \tag{6.12}$$

Note that we proved the first step of the induction in Section 6.1 (see equation (6.11)). Using (6.12), we will show that

$$\begin{aligned}\int_T^{2T} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \ll T\exp\Bigg(&\frac{(\log T)^{(1+\delta)(\frac{3}{kb-1})^{m}}\log\log T}{\exp\big(\frac{2\cdot 3^{m-1}\log\log T\log\log\log\log T}{(kb-1)^{m}\log\log\log T}\big)}\Bigg)\\ &\times\exp\big((1+\delta)k^2(\log\log T)^3\big).\end{aligned} \tag{6.13}$$

Let

$$\varepsilon' = \frac{\delta(kb-1)\log\log T}{4kb(m-1)\big(\log\log T - 2kb\log\log\log T + \frac{\delta(kb-1)\log\log\log T}{2(m-1)}\big)}, \tag{6.14}$$

and

$$p = \frac{\log\log T}{\log\log T - 2kb\varepsilon'\log\log\log T}, \quad q = \frac{\log\log T}{2kb\varepsilon'\log\log\log T}, \tag{6.15}$$

so that $\frac1p + \frac1q = 1$. Let

$$f = b(1-\varepsilon'). \tag{6.16}$$

We will perform the inductive argument as long as

$$\left(\frac{kpf-1}{3}\right)^{m-1}<\frac{\delta\log\log T}{9\log\log\log T}. \tag{6.17}$$

We choose

$$\beta_0=\frac{3(2d-1)\big(1+\frac{\delta}{3}\big)\big(\frac{3}{kpf-1}\big)^{m-1}\exp\big(\frac{2\cdot 3^{m-2}\log\log T\log\log\log\log T}{(kpf-1)^{m-1}\log\log\log T}\big)}{4q(\log T)^{(1+\frac{\delta}{3})(\frac{3}{kpf-1})^{m-1}}}, \tag{6.18}$$

$$s_0=\left[\frac{1}{q\beta_0}\right],$$

$$\ell_0=2\left\lceil\frac{s_0^d}{2}\right\rceil,$$

where

$$a=\frac{1-n(1-\frac{\delta}{12})x}{1-n(1-\frac{\delta}{6})x},\quad d=1-\frac{n}{2}\left(1-\frac{\delta}{6}\right)x,\quad r=\frac{1-n\big(1-\frac{\delta}{12}\big)x}{1-nx}.$$

As before, we have

$$\frac{a(2d-1)}{r}=1-nx.$$

Recall that $x=\frac{\log\log\log T}{\log\log T}$ and we choose

$$n=\frac{2(kb-1)(kpf-1)^{m-1}\big(1-\frac{\delta}{24}\big)}{3^{m-1}\big(1+\frac{\delta}{3}\big)}. \tag{6.19}$$

We also choose β_K such that K is the maximal integer for which

$$\beta_K\le\frac{\exp\big(\frac{6\log\log T\log\log\log\log T}{n\log\log\log T}\big)}{(\log T)^{\frac{6}{n(1-\frac{\delta}{24})}}}. \tag{6.20}$$

If $t\notin\mathcal{T}_0$, there exists $0\le v\le K$ such that $\frac{ke^2}{\ell_0}|P_{0,v}(t)|>1$. Using Hölder's inequality, we have

$$\begin{aligned}&\int_{[T,2T]\setminus\mathcal{T}_0}|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\quad\le\int_T^{2T}\left(\frac{ke^2}{\ell_0}|P_{0,v}(t)|\right)^{2s_0}|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\quad\le\left(\frac{ke^2}{\ell_0}\right)^{2s_0}\left(\int_T^{2T}|\zeta(\tfrac12+\alpha+it)|^{-2kp}\,dt\right)^{\frac1p}\left(\int_T^{2T}|P_{0,v}(t)|^{2s_0q}\,dt\right)^{\frac1q}.\end{aligned}$$

For the first integral above, we will use the bound (6.12) with $\delta\mapsto\frac{\delta}{3}$. Note that

$$\alpha=\frac{(\log\log T)^f}{(\log T)^{\frac{1}{2kp}}}.$$

Then using the recursive bound (6.12), we obtain that

$$\begin{aligned}\int_T^{2T}|\zeta(\tfrac12+\alpha+it)|^{-2kp}\ll T\exp\left(\frac{(\log T)^{(1+\frac{\delta}{3})(\frac{3}{kpf-1})^{m-1}}\log\log T}{\exp\big(\frac{2\cdot 3^{m-2}\log\log T\log\log\log\log T}{(kpf-1)^{m-1}\log\log\log T}\big)}\right)\\ \times\exp\left(\left(1+\frac{\delta}{3}\right)p^2k^2(\log\log T)^3\right).\end{aligned} \tag{6.21}$$

Now using Proposition (3.2), we have that

$$\int_T^{2T} |P_{0,v}(t)|^{2s_0q}\,dt \ll T(s_0q)!b(\Delta_0)^{2s_0q}\Big(\log\frac{1}{\Delta_0\alpha}\Big)^{2s_0q\gamma(\Delta_0)}(\log\log T^{\beta_0})^{s_0q}.$$

Combining the bound above and (6.21) and using Stirling's formula, we get that

$$\begin{aligned}(6.22)\qquad &\int_{[T,2T]\setminus\mathcal{T}_0} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\ll T\exp\Bigg(\frac{(\log T)^{(1+\frac{\delta}{3})(\frac{3}{kpf-1})^{m-1}}\log\log T}{\exp(\frac{2\cdot 3^{m-2}\log\log T\log\log\log\log T}{(kpf-1)^{m-1}\log\log\log T})}\Bigg)\\ &\quad\times\exp\Big(\Big(1+\frac{\delta}{3}\Big)pk^2(\log\log T)^3\Big)s_0^{\frac{1}{2q}}\exp\Big(-(2d-1)s_0\log s_0\\ &\quad+2s_0\log\Big(ke^{\frac32}\sqrt{q}b(\Delta_0)\Big(\log\frac{1}{\Delta_0\alpha}\Big)^{\gamma(\Delta_0)}\sqrt{\log\log T^{\beta_0}}\Big)\Big).\end{aligned}$$

Recall the choice (6.18) for s_0. Note that we have

$$\log\log\frac{1}{\Delta_0\alpha}\le\log\log\log T,\quad \log\log(\beta_0\log T)<\log\log\log T$$

and

$$\log q\le\log\log\log T.$$

Using the three bounds above in (6.2), it follows that

$$\begin{aligned}&\int_{[T,2T]\setminus\mathcal{T}_0} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\ll T\exp\Bigg(\frac{(\log T)^{(1+\frac{\delta}{3})(\frac{3}{kpf-1})^{m-1}}\log\log T}{\exp(\frac{2\cdot 3^{m-2}\log\log T\log\log\log\log T}{(kpf-1)^{m-1}\log\log\log T})}\Bigg)\exp\Big(\Big(1+\frac{\delta}{3}\Big)pk^2(\log\log T)^3\Big)\\ &\quad\times\exp\big(-(2d-1)s_0\log s_0+4s_0\log\log\log T+O(s_0)\big).\end{aligned}$$

By (6.17) we get

$$\begin{aligned}(6.23)\qquad &\int_{[T,2T]\setminus\mathcal{T}_0} |\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\ll T\exp\Bigg(-\frac{(\log T)^{(1+\frac{\delta}{3})(\frac{3}{kpf-1})^{m-1}}\log\log T}{4\exp(\frac{2\cdot 3^{m-2}\log\log T\log\log\log\log T}{(kpf-1)^{m-1}\log\log\log T})}\Bigg)\exp\big((1+\delta)k^2(\log\log T)^3\big)\\ &\ll T\exp\big((1+\delta)k^2(\log\log T)^3\big).\end{aligned}$$

Now suppose that $t\in\mathcal{T}_0$. Using Proposition 3.4 and proceeding as before, we get that

$$\begin{aligned}(6.24)\qquad \int_T^{2T} S_1(t)\,dt &\ll (\log\log T)^k\exp\Big(c_1(\log T)^{\frac{3^m(1+\frac{\delta}{3})}{(kb-1)(kpf-1)^{m-1}(1-\frac{\delta}{24})^2}}\log\log T\\ &\quad\times\exp\Big(-\frac{3^m\log\log T\log\log\log\log T}{(kb-1)(kpf-1)^{m-1}\log\log\log T}\Big)\Big)\\ &\quad\times\exp\big((1+\delta)k^2(\log\log T)^3\big),\end{aligned}$$

for some $c_1 > 0$, and where we trivially bounded $\gamma(\Delta_K) \leq 1$. Now given the choice of parameters in (6.14), (6.15), (6.16), we have

$$kpf - 1 = (kb-1)\Big(1 - \frac{\delta}{4(m-1)}\Big) + O\Big(\frac{\log\log\log T}{\log\log T}\Big) > (kb-1)\Big(1 - \frac{7\delta}{24(m-1)}\Big).$$

Using this in (6.24) leads to

$$\int_T^{2T} S_1(t)\,dt \ll T\exp\bigg(\frac{(\log T)^{(1+\delta)(\frac{3}{kb-1})^m}\log\log T}{\exp\big(\frac{2\cdot 3^{m-1}\log\log T\log\log\log\log T}{(kb-1)^m\log\log\log T}\big)}\bigg) \times \exp\big((1+\delta)k^2(\log\log T)^3\big). \tag{6.25}$$

To bound the contribution from $S_2(t)$, we proceed as in (6.9). We rewrite

$$\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k\sum_{j=0}^{K-1}\exp\bigg(\frac{\log\log\log T}{\beta_j}(n-2kb+2) - \frac{n\log\log\log T\log(\beta_j\log T)}{\beta_j\log\log T} + \log\log T + k^2(\log\log T)^3 + O\Big(\frac{1}{\beta_j}\Big)\bigg), \tag{6.26}$$

where we used the fact that $k \geq \frac{1}{2}$. Note that the maximum in the sum over j is attained either at $j = 0$ or $j = K-1$. Now given the choices (6.18) and (6.19), the contribution from $j = 0$ is

$$\ll \exp\bigg(\frac{\log\log\log T}{\beta_0}\Big(-\frac{\delta(kb-1)}{4} + \frac{n\log\big(\frac{kpf-1}{3}\big)^{m-1}}{\log\log T}\Big) + O\Big(\frac{1}{\beta_0} + (\log\log T)^3\Big)\bigg),$$

and again using the choices (6.19) and (6.17), it follows that the contribution from $j = 0$ is negligible.

For the contribution from $j = K-1$, proceeding similarly as in the bound for $S_1(t)$, it follows that

$$\int_T^{2T} S_2(t)\,dt \ll T\exp\bigg(\frac{(\log T)^{(1+\delta)(\frac{3}{kb-1})^m}\log\log T}{\exp\big(\frac{2\cdot 3^{m-1}\log\log T\log\log\log\log T}{(kb-1)^m\log\log\log T}\big)}\bigg)\exp\big((1+\delta)k^2(\log\log T)^3\big).$$

Combining the above, (6.23) and (6.25), the induction conclusion (6.13) follows.

Now taking m maximal as in (6.17), we get that

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+\alpha+it)|^{-2k}\,dt \ll T\exp\big((1+\delta)k^2(\log\log T)^3\big). \tag{6.27}$$

6.3. The range $\alpha \gg (\log\log T)^{\frac{4}{k}+\varepsilon}/(\log T)^{\frac{1}{2k}}$, $k \geq \frac{1}{2}$, once more. Here, we use the same setup as in Section 6.2. Once again, we assume (6.1) and (6.2).

We will improve the bound (6.27). The proof is similar to the proof in the previous cases, so we will skip some of the details.

As before, let

$$p = \frac{\log\log T}{\log\log T - 2kb\delta\log\log\log T}$$

and

$$q=\frac{\log\log T}{2kb\delta\log\log\log T}, \tag{6.28}$$

so that $\frac{1}{p}+\frac{1}{q}=1$. We choose

$$\beta_0=\frac{(6d-5)\log\log\log T}{(1+2\delta)pqk^2(\log\log T)^3},\quad s_0=\left[\frac{1}{q\beta_0}\right],\quad \ell_0=2\left\lceil\frac{s_0^d}{2}\right\rceil. \tag{6.29}$$

We choose β_j,s_j,ℓ_j as in (4.2), and a,d,r are such that

$$\frac{a(2d-1)}{r}=1-\delta.$$

We also choose K maximal such that

$$\beta_K\le c, \tag{6.30}$$

where c is a small constant as in (4.5). Note that the conditions in Propositions 3.3 and 3.4 are satisfied.

We now proceed as before. If $t\notin\mathcal{T}_0$, then as in Section 6.2, we have that

$$\begin{aligned}\int_{[T,2T]\backslash\mathcal{T}_0}&|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\le\Big(\frac{ke^2}{\ell_0}\Big)^{2s_0}\Big(\int_T^{2T}|\zeta(\tfrac12+\alpha+it)|^{-2kp}\,dt\Big)^{\frac1p}\Big(\int_T^{2T}|P_{0,v}(t)|^{2s_0q}\,dt\Big)^{\frac1q}\\ &\ll T\exp\big((1+\delta)pk^2(\log\log T)^3\big)\exp\Bigg(-(2d-1)s_0\log s_0\\ &\qquad+2s_0\log\Bigg(ke^{\frac32}\sqrt{q}b(\Delta_0)\Big(\log\frac{1}{\Delta_0\alpha}\Big)^{\gamma(\Delta_0)}\sqrt{\log\log T^{\beta_0}}\Bigg)\Bigg).\end{aligned}$$

Note that $\gamma(\Delta_0)=0$ with the choice of parameters (6.29). We deduce that

$$\begin{aligned}\int_{[T,2T]\backslash\mathcal{T}_0}&|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\ &\ll T\exp\big((1+\delta)pk^2(\log\log T)^3\big)\\ &\qquad\times\exp\big(-(2d-1)s_0\log s_0+2s_0\log\log\log T+O(s_0)\big)\\ &=o(T),\end{aligned} \tag{6.31}$$

by (6.28) and the choice of parameters in (6.29). Using the choice of β_K in (6.30), we also get that

$$\int_T^{2T}S_1(t)\,dt\ll T(\log\log T)^k(\log T)^{k^2}. \tag{6.32}$$

To bound the contribution from $S_2(t)$, we rewrite equation (6.9) using the fact that $\gamma(\Delta_j)=0$ for all j and that $K\ll\log\log T$. We have

$$\begin{aligned}\int_T^{2T}S_2(t)\,dt\ll T(\log\log T)^{k+1}\sum_{j=0}^{K-1}\exp\Bigg(&\frac{\log\log T}{\beta_j}\Big(1-2k-\frac{2kb\log\log\log T}{\log\log T}\Big)\\ &+\frac{4\log\log\log T}{\beta_j}\Big(2k-\frac{a(2d-1)}{r}\Big)+k^2\log\log T+O\Big(\frac{1}{\beta_j}\Big)\Bigg).\end{aligned}$$

In the sum over j above, note that if $k = \frac{1}{2}$, then the exponential term is bounded by

$$\exp\Big(-\frac{c_1 \log\log\log T}{\beta_j} + k^2 \log\log T\Big),$$

for some $c_1 > 0$. If $k > \frac{1}{2}$, then the exponential term is bounded by

$$\exp\Big(-\frac{c_1 \log\log T}{\beta_j} + k^2 \log\log T\Big),$$

for some $c_1 > 0$. In both cases, we obtain that

$$\int_T^{2T} S_2(t)\,dt = o\big(T(\log\log T)^k(\log T)^{k^2}\big).$$

Combining the above, (6.32) and (6.31), the bound (1.3) follows.

6.4. The range $1/(\log T)^{\frac{1}{2k}} \ll \alpha = o((\log\log T)^{\frac{4}{k}+\varepsilon}/(\log T)^{\frac{1}{2k}})$. Here, we prove the bound (1.4). We will only sketch the proof, since it is similar to the proof in the previous cases. We choose the parameters β_j, ℓ_j, s_j as in (4.1), (4.2), and a, d, r as in (6.3) and (6.5), where

$$n = \frac{6}{1-\delta}. \tag{6.33}$$

We also choose β_K as in (6.6). We now proceed as in Section 6.1.

As in (6.7), we have

$$\int_{[T,2T]\setminus\mathcal{T}_0} |\zeta(\tfrac{1}{2}+\alpha+it)|^{-2k}\,dt = o(T). \tag{6.34}$$

Also, as in (6.8), and keeping in mind the choice (6.33) for n, we get that

$$\int_T^{2T} S_1(t)\,dt \ll T\exp\bigg(\frac{\log T \log\log T}{\exp(\frac{2\log\log T\log\log\log\log T}{3\log\log\log T})}\bigg). \tag{6.35}$$

To bound the contribution from $S_2(t)$, we proceed as in equation (6.9). Since

$$\alpha \gg \frac{1}{(\log T)^{\frac{1}{2k}}},$$

we have

$$\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k \sum_{j=0}^{K-1} \exp\bigg(\frac{\log\log\log T}{\beta_j}(n+2) + (\log\log T)^3 + O\Big(\frac{1}{\beta_j}\Big)\bigg).$$

The maximum in the sum above is attained when $j = 0$. So, keeping in mind the choice of β_0 in (4.1), we obtain that

$$\int_T^{2T} S_2(t)\,dt \ll T\exp\bigg(\frac{(4+\delta)\log T\log\log\log T}{\log\log T}\bigg), \tag{6.36}$$

after a relabeling of the δ. Combining (6.34), (6.35) and (6.36), the conclusion follows in this case.

7. Proof of Theorems 1.2 and 1.3 for small shifts α

In this section, we will prove the bounds (1.5) and (1.9). The proof will be similar to the one in the previous subsection, but we choose the parameters differently. Recall that

$$u = \frac{\log \frac{1}{\alpha}}{\log\log T}.$$

Let $\delta > 0$. We choose

$$\beta_0 = \frac{(2ku + 2d - 1 - \frac{a(2d-1)}{r}) \log\log T}{(1+\delta)ku \log T}, \quad s_0 = \left[\frac{1}{\beta_0}\right], \quad \ell_0 = 2\left\lceil \frac{s_0^d}{2} \right\rceil, \tag{7.1}$$

where we pick

$$a = \frac{1-3k\varepsilon}{1-2k\varepsilon}, \quad r = \frac{1}{1-2k\varepsilon}, \quad d = \frac{2-7k\varepsilon}{2(1-3k\varepsilon)},$$

so that

$$\frac{a(2d-1)}{r} = 1 - 4k\varepsilon. \tag{7.2}$$

We further pick β_j, ℓ_j, s_j as in (4.2). We choose K to be maximal such that

$$\beta_K \le c, \tag{7.3}$$

for c a small constant such that

$$c^{1-d}\left(\frac{ra^d}{r^{1-d}-1} + \frac{2r}{r-1}\right) \le 1 - a - \beta_0^{1-d}. \tag{7.4}$$

Note that the above ensures that the conditions in Propositions 3.2, 3.3 and 3.4 are satisfied.

If $t \notin \mathcal{T}_0$, then we proceed as before, and similarly to equation (4.7) we get that

$$\begin{aligned}\int_{[T,2T]\setminus\mathcal{T}_0} &|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \\ &\ll T^{1+(1+\delta)ku} \exp\Bigg(-(2d-1)s_0\log s_0 \\ &\quad + 2s_0 \log\bigg(ke^{\frac32} b(\Delta_0)\Big(\log\frac{1}{\Delta_0\alpha}\Big)^{\gamma(\Delta_0)} \sqrt{\log\log T^{\beta_0}}\bigg)\Bigg).\end{aligned}$$

Keeping in mind the choice of parameters (7.1), we obtain that

$$\begin{aligned}\int_{[T,2T]\setminus\mathcal{T}_0} &|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt \\ &\ll T^{1+(1+\delta)ku\frac{2ku-\frac{a(2d-1)}{r}}{2ku-\frac{a(2d-1)}{r}+2d-1}} \exp\bigg(O\Big(\frac{\log T\log\log\log T}{\log\log T}\Big)\bigg).\end{aligned} \tag{7.5}$$

Now if $t \in \mathcal{T}_0$, then since $\gamma(\Delta_K) = 1$, we use Proposition 3.4 and the expression (7.3) for β_K as before to get that

$$\int_T^{2T} S_1(t)\,dt \ll T(\log T)^{O(1)} \exp\bigg(\Big(\frac14+\varepsilon\Big)k^2\Big(\log\frac{1}{\alpha}\Big)^2 \log\log T\bigg). \tag{7.6}$$

Similarly as before (see equation (4.9)), we have

$$\int_T^{2T} S_2(t)\,dt \ll T(\log\log T)^k \sum_{j=0}^{K-1}(K-j)\sqrt{\frac{1}{\beta_{j+1}}}\exp\bigg(\frac{\log\log T}{\beta_j}\Big(2ku-\frac{a(2d-1)}{r}\Big)$$
$$+\frac{\log(\beta_j\log T)}{\beta_j}\Big(\frac{a(2d-1)}{r}-2k\Big)$$
$$++\frac{2a}{r\beta_j}\log\bigg(e^{\frac{3}{2}}b(\Delta_{j+1})\Big(\log\frac{1}{\Delta_{j+1}\alpha}\Big)^{\gamma(\Delta_{j+1})}\bigg)$$
$$+\frac{2k}{\beta_j}\log\frac{1}{1-\frac{\varepsilon}{2}}\bigg)\big(\log T^{\beta_j}\big)^{k^2b(\Delta_j)^2(\log\frac{1}{\Delta_j\alpha})^{2\gamma(\Delta_j)}}.$$

As $K\ll\log\log T$, we further write the above as

$$\int_T^{2T} S_2(t)\,dt \ll T\sum_{j=0}^{K-1}\exp\bigg(\frac{\log\log T}{\beta_j}\Big(2ku-\frac{a(2d-1)}{r}\Big)+O\Big(\frac{\log T\log\log\log T}{\log\log T}\Big)\bigg).$$

Since

$$\alpha=o\bigg(\frac{1}{(\log T)^{\frac{1}{2k}-\varepsilon}}\bigg),$$

we have

$$2ku-\frac{a(2d-1)}{r}\geq 2k\varepsilon.$$

Hence the sum over j above achieves its maximum when $j=0$. Using (7.1), we obtain that

$$\int_T^{2T} S_2(t)\,dt \ll T^{1+(1+\delta)ku\frac{2ku-\frac{a(2d-1)}{r}}{2ku-\frac{a(2d-1)}{r}+2d-1}}\exp\bigg(O\Big(\frac{\log T\log\log\log T}{\log\log T}\Big)\bigg). \tag{7.7}$$

Combining equations (7.2), (7.5), (7.6) and (7.7) and after a relabeling of the ε, the bounds (1.5) and (1.9) follow.

8. The asymptotic formula

In this section, we shall prove Theorem 1.4. We have

$$\begin{aligned}\frac{1}{2\pi i}\int_{1-\frac{iT}{2}}^{1+\frac{iT}{2}} e^{\frac{z^2}{2}}X^z\frac{1}{\zeta(\frac12+\alpha+it+z)^k\zeta(\frac12+\alpha-it+z)^k}\frac{dz}{z}\\ =\sum_{m,n=1}^{\infty}\frac{\mu_k(m)\mu_k(n)}{(mn)^{\frac12+\alpha}}\Big(\frac{m}{n}\Big)^{-it}W\Big(\frac{mn}{X}\Big),\end{aligned} \tag{8.1}$$

where

$$W(x)=\frac{1}{2\pi i}\int_{1-\frac{iT}{2}}^{1+\frac{iT}{2}} e^{\frac{z^2}{2}}x^{-z}\frac{dz}{z}, \tag{8.2}$$

by writing the zeta-functions in (8.1) as Dirichlet series and integrating term-by-term. On the other hand, by deforming the line of integration, the left hand side of (8.1) is equal to

$$\begin{aligned}&|\zeta(\tfrac12+\alpha+it)|^{-2k}\\&\quad+\frac{1}{2\pi i}\int_{-(1-\varepsilon)\alpha-\frac{iT}{2}}^{-(1-\varepsilon)\alpha+\frac{iT}{2}} e^{\frac{z^2}{2}}X^z\frac{1}{\zeta(\frac12+\alpha+it+z)^k\zeta(\frac12+\alpha-it+z)^k}\frac{dz}{z}\\&\quad+O\Big(\frac{e^{-\frac{T^2}{8}}X}{T}\max_{\varepsilon\alpha\le\sigma\le1+\alpha}\frac{1}{|\zeta(\frac12+\sigma+i(t\pm\frac{T}{2}))|^k|\zeta(\frac12+\sigma-i(t\mp\frac{T}{2}))|^k}\Big).\end{aligned}$$

For $t\in[T,2T]$, the O-term is

$$\ll\frac{e^{-\frac{T^2}{8}}X}{T}T^{\frac{k\log\frac{1}{\varepsilon\alpha}}{\log\log T}}\ll e^{-\frac{T^2}{8}}XT^{O_k(1)},$$

by Lemma 2.2. Hence, integrating over $t\in[T,2T]$ we obtain that

$$\begin{aligned}&\int_T^{2T}|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\&=\sum_{m,n=1}^{\infty}\frac{\mu_k(m)\mu_k(n)}{(mn)^{\frac12+\alpha}}W\Big(\frac{mn}{X}\Big)\int_T^{2T}\Big(\frac{m}{n}\Big)^{-it}\,dt\\&\quad-\frac{1}{2\pi i}\int_{-(1-\varepsilon)\alpha-\frac{iT}{2}}^{-(1-\varepsilon)\alpha+\frac{iT}{2}} e^{\frac{z^2}{2}}X^z\int_T^{2T}\frac{1}{\zeta(\frac12+\alpha+it+z)^k\zeta(\frac12+\alpha-it+z)^k}\,dt\frac{dz}{z}\\&\quad+O\big(e^{-\frac{T^2}{8}}XT^{O_k(1)}\big).\end{aligned}$$

Furthermore, by the Cauchy–Schwarz inequality the second term above is

$$\begin{aligned}&\ll_\varepsilon\frac{X^{-(1-\varepsilon)\alpha}}{\alpha}\int_{-\frac{T}{2}}^{\frac{T}{2}}e^{-\frac{z^2}{2}}\Big(\int_T^{2T}|\zeta(\tfrac12+\varepsilon\alpha+i(t+z))|^{-2k}\,dt\Big)^{\frac12}\\&\qquad\times\Big(\int_T^{2T}|\zeta(\tfrac12+\varepsilon\alpha-i(t-z))|^{-2k}\,dt\Big)^{\frac12}dz\\&\ll_\varepsilon\frac{X^{-(1-\varepsilon)\alpha}}{\alpha}T(\log\log T)^k(\log T)^{k^2}\\&\ll_\varepsilon TX^{-(1-\varepsilon)\alpha}(\log T)^{k^2+1},\end{aligned}$$

in view of (1.3) and (1.6) in Theorems 1.2 and 1.3, and the fact that

$$\frac{1}{\alpha}\ll\min\Big\{\frac{(\log T)^{\frac{1}{2k}}}{(\log\log T)^{\frac4k+\varepsilon}},\frac{\log T}{\log\log T}\Big\}.$$

Thus,

$$\begin{aligned}\int_T^{2T}|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt&=\sum_{m,n=1}^{\infty}\frac{\mu_k(m)\mu_k(n)}{(mn)^{\frac12+\alpha}}W\Big(\frac{mn}{X}\Big)\int_T^{2T}\Big(\frac{m}{n}\Big)^{-it}\,dt\\&\quad+O\big(TX^{-(1-\varepsilon)\alpha}(\log T)^{k^2+1}\big)\\&\quad+O\big(e^{-\frac{T^2}{8}}XT^{O_k(1)}\big).\end{aligned}\tag{8.3}$$

We next consider the contribution of off-diagonal terms $m \neq n$ on the right-hand side of (8.3), which is

$$\ll \sum_{m\neq n} \frac{d_k(m)d_k(n)}{(mn)^{\frac{1}{2}+\alpha}\big|\log \frac{m}{n}\big|}\Big|W\Big(\frac{mn}{X}\Big)\Big|.$$

We note from (8.2) that $W(x) \ll x^{-1}$ trivially and

$$W(x) = 1 + O(x) + O\Big(\frac{e^{-\frac{T^2}{8}}}{xT}\Big)$$

if $x \leq 1$, by moving the contour to the -1-line, and so this contribution is bounded by

$$\begin{aligned}
&\ll \sum_{\substack{m\neq n\\ mn\leq X}} \frac{d_k(m)d_k(n)}{(mn)^{\frac{1}{2}+\alpha}\big|\log \frac{m}{n}\big|} + \frac{e^{-\frac{T^2}{8}}X}{T} \sum_{\substack{m\neq n\\ mn\leq X}} \frac{d_k(m)d_k(n)}{(mn)^{\frac{3}{2}+\alpha}\big|\log \frac{m}{n}\big|}\\
&\qquad + X \sum_{\substack{m\neq n\\ mn> X}} \frac{d_k(m)d_k(n)}{(mn)^{\frac{3}{2}+\alpha}\big|\log \frac{m}{n}\big|}\\
&= E_1 + E_2,
\end{aligned}$$

say, where E_1 and E_2 denote the sums with $\frac{m}{n} \notin [\frac{1}{2}, 2]$ and $\frac{m}{n} \in [\frac{1}{2}, 2]$, respectively. With E_1, $\big|\log \frac{m}{n}\big| \gg 1$ and we get

$$\begin{aligned}
E_1 &\ll \sum_{mn\leq X} \frac{d_k(m)d_k(n)}{(mn)^{\frac{1}{2}+\alpha}} + \frac{e^{-\frac{T^2}{8}}X}{T} \sum_{mn\leq X} \frac{d_k(m)d_k(n)}{(mn)^{\frac{3}{2}+\alpha}}\\
&\qquad + X \sum_{mn> X} \frac{d_k(m)d_k(n)}{(mn)^{\frac{3}{2}+\alpha}}\\
&\ll X^{\frac{1}{2}-\alpha}(\log X)^{2k-1} + \frac{e^{-\frac{T^2}{8}}X}{T}.
\end{aligned} \tag{8.4}$$

For E_2, we use the fact that

$$\frac{d_k(m)d_k(n)}{(mn)^{\sigma}} \ll \frac{d_k(m)^2}{m^{2\sigma}} + \frac{d_k(n)^2}{n^{2\sigma}}, \tag{8.5}$$

and we have

$$\begin{aligned}
E_2 &\ll \sum_{m\leq \sqrt{2X}} \frac{d_k(m)^2}{m^{1+2\alpha}} \sum_{\substack{n\neq m\\ \frac{m}{2}\leq n\leq 2m}} \frac{1}{\big|\log \frac{m}{n}\big|}\\
&\qquad + \frac{e^{-\frac{T^2}{8}}X}{T} \sum_{m\leq \sqrt{2X}} \frac{d_k(m)^2}{m^{3+2\alpha}} \sum_{\substack{n\neq m\\ \frac{m}{2}\leq n\leq 2m}} \frac{1}{\big|\log \frac{m}{n}\big|}\\
&\qquad + X \sum_{m> \sqrt{\frac{X}{2}}} \frac{d_k(m)^2}{m^{3+2\alpha}} \sum_{\substack{n\neq m\\ \frac{m}{2}\leq n\leq 2m}} \frac{1}{\big|\log \frac{m}{n}\big|}
\end{aligned} \tag{8.6}$$

$$
\begin{aligned}
&\ll \log X \sum_{m\leq\sqrt{2X}} \frac{d_k(m)^2}{m^{2\alpha}} + \frac{e^{-\frac{T^2}{8}} X\log X}{T} \sum_{m\leq\sqrt{2X}} \frac{d_k(m)^2}{m^{2+2\alpha}}\\
&\qquad + X \sum_{m>\sqrt{\frac{X}{2}}} \frac{d_k(m)^2\log m}{m^{2+2\alpha}}\\
&\ll X^{\frac12-\alpha}(\log X)^{k^2} + \frac{e^{-\frac{T^2}{8}} X\log X}{T}.
\end{aligned}
$$

We are left with the contribution of the diagonal terms $m = n$ on the right-hand side of (8.3). By (8.2) this is

$$
\begin{aligned}
T\sum_{n=1}^{\infty}\frac{\mu_k(n)^2}{n^{1+2\alpha}}W\Big(\frac{n^2}{X}\Big) &= \frac{T}{2\pi i}\int_{1-\frac{iT}{2}}^{1+\frac{iT}{2}} e^{\frac{z^2}{2}}X^z\sum_{n=1}^{\infty}\frac{\mu_k(n)^2}{n^{1+2\alpha+2z}}\frac{dz}{z}\\
&= \frac{T}{2\pi i}\int_{1-\frac{iT}{2}}^{1+\frac{iT}{2}} e^{\frac{z^2}{2}}X^z\zeta(1+2\alpha+2z)^{k^2}\\
&\qquad\times\prod_p\Big(1-\frac{1}{p^{1+2\alpha+2z}}\Big)^{k^2}\Big(1+\sum_{j=1}^{\infty}\frac{\mu_k(p^j)^2}{p^{(1+2\alpha+2z)j}}\Big)\frac{dz}{z}.
\end{aligned}
$$

We move the contour to the $-(1-\varepsilon)\alpha$-line, crossing a simple pole at $z = 0$. In doing so, we get that this is equal to

$$
\begin{aligned}
&T\zeta(1+2\alpha)^{k^2}\prod_p\Big(1-\frac{1}{p^{1+2\alpha}}\Big)^{k^2}\Big(1+\sum_{j=1}^{\infty}\frac{\mu_k(p^j)^2}{p^{(1+2\alpha)j}}\Big)\\
&\qquad + O\big(TX^{-(1-\varepsilon)\alpha}\alpha^{-(k^2+1)}\big) + O\big(e^{-\frac{T^2}{8}}XT^{O_k(1)}\big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_T^{2T}|\zeta(\tfrac12+\alpha+it)|^{-2k}\,dt\\
&= T\zeta(1+2\alpha)^{k^2}\prod_p\Big(1-\frac{1}{p^{1+2\alpha}}\Big)^{k^2}\Big(1+\sum_{j=1}^{\infty}\frac{\mu_k(p^j)^2}{p^{(1+2\alpha)j}}\Big)\\
&\qquad + O\big(TX^{-(1-\varepsilon)\alpha}(\log T)^{k^2+1}\big) + O\big(e^{-\frac{T^2}{8}}XT^{O_k(1)}\log X\big) + O\big(X^{\frac12-\alpha}(\log X)^{k^2}\big),
\end{aligned}
$$

by combining the above with (8.3), (8.4) and (8.6). We choose $X = T^2$, then the error terms above become $T^{1-2(1-\varepsilon)\alpha}(\log T)^{k^2+1}$, and the conclusion follows after a relabeling of the ε.

9. Proof of Theorem 1.5

9.1. Assuming RH: $k \geq 1$. Following the arguments in [10, Chapter 17], it is standard from Perron's formula that

$$\sum_{n\leq x}\mu_k(n) = \frac{1}{2\pi i}\int_{c-i[x]}^{c+i[x]}\frac{x^s}{\zeta(s)^k}\frac{ds}{s} + O\big((\log x)^k\big) \tag{9.1}$$

with $c = 1 + \frac{1}{\log x}$. We will now deform the contour. The choice of the new contour depends on whether k is an integer or not because of the singularity of $\zeta(s)^{-k}$ at $s = 1$ when $k \notin \mathbb{Z}$. We will use the Selberg–Delange method in this case (see, for instance, [37, Chapter 5]).

We first describe the contour deformation when $k \in \mathbb{Z}$. Let

$$x_0 = \exp\Big((\log x)^{\frac{k}{k+1}} (\log\log x)^{\frac{k+8}{k+1}}\Big), \quad \alpha_0 = \frac{(\log\log x_0)^{\frac{8}{k}+\varepsilon}}{(\log x_0)^{\frac{1}{k}}}$$

and

$$J = \left\lceil \log_2 \frac{[x]}{x_0} \right\rceil \ll \log x.$$

We replace the line segment $c + it$, $|t| \le [x]$, with a piecewise linear path comprising of a number of horizontal and vertical line segments,

$$\bigcup_{j=1}^{J} (V_j \cup H_j) \cup V_0,$$

where

$$\begin{aligned}
&V_0\colon \quad s = \frac{1}{2} + \alpha_0 + it, \quad |t| \le x_0,\\
&V_j\colon \quad s = \frac{1}{2} + \frac{(\log\log(2^{j-1}x_0))^{\frac{8}{k}+\varepsilon}}{(\log(2^{j-1}x_0))^{\frac{1}{k}}} + it, \quad 2^{j-1}x_0 \le |t| \le \min\{2^j x_0, [x]\},\\
&\qquad\ 1 \le j \le J,\\
&H_j\colon \quad s = \sigma \pm i2^j x_0, \quad \frac{1}{2} + \frac{(\log\log(2^j x_0))^{\frac{8}{k}+\varepsilon}}{(\log(2^j x_0))^{\frac{1}{k}}} \le \sigma \le \frac{1}{2} + \frac{(\log\log(2^{j-1}x_0))^{\frac{8}{k}+\varepsilon}}{(\log(2^{j-1}x_0))^{\frac{1}{k}}},\\
&\qquad\ 1 \le j \le J-1,\\
&H_J\colon \quad s = \sigma \pm i[x], \quad \frac{1}{2} + \frac{(\log\log(2^{J-1}x_0))^{\frac{8}{k}+\varepsilon}}{(\log(2^{J-1}x_0))^{\frac{1}{k}}} \le \sigma \le 1 + \frac{1}{\log x}.
\end{aligned}$$

If $k \notin \mathbb{Z}$, we use the contour deformation above, but replace V_0 by $V_0 = V_0' \cup \Gamma$, where Γ is the truncated Hankel-type contour made up of the anticlockwise circle $|s-1| = \frac{1}{\log x}$ excluding the point $1 - \frac{1}{\log x}$, and the linear paths joining $\frac{1}{2} + \alpha_0$ to $1 - \frac{1}{\log x}$ and then back to $\frac{1}{2} + \alpha_0$ with arguments $-\pi$ and $+\pi$, respectively.

Notice that in either case, we encounter no singularity in moving the contour.

We first consider the integral over Γ when $k \notin \mathbb{Z}$. We will see that the main contribution only arises from this. Recall that

$$Z_{-k}(s) = \frac{1}{s((s-1)\zeta(s))^k}$$

and $Z_{-k}(s)$ is holomorphic in the disc $|s-1| < 1$. By Cauchy's formula we have

$$Z_{-k}(s) = \sum_{j=0}^{\infty} \gamma_{-k,j}(s-1)^j = \sum_{j=0}^{N} \gamma_{-k,j}(s-1)^j + O\Big(\big((1+\varepsilon)|s-1|\big)^{N+1}\Big)$$

uniformly for $|s-1|<\frac{1}{2}$. Notice that Γ is contained in the disc $|s-1|<\frac{1}{2}$ so

$$\begin{aligned}\tag{9.2} \frac{1}{2\pi i}\int_\Gamma \frac{x^s}{\zeta(s)^k}\frac{ds}{s} &= \frac{1}{2\pi i}\int_\Gamma x^s(s-1)^k Z_{-k}(s)\,ds\\ &= \sum_{j=0}^N \gamma_{-k,j}\frac{1}{2\pi i}\int_\Gamma x^s(s-1)^{k+j}\,ds\\ &\qquad + O\bigg((1+\varepsilon)^{N+1}\int_\Gamma x^{\Re(s)}|s-1|^{k+N+1}|ds|\bigg).\end{aligned}$$

The O-term above is

$$\begin{aligned}&\ll (1+\varepsilon)^{N+1}\int_{\frac12}^{1-\frac{1}{\log x}} x^\sigma(1-\sigma)^{k+N+1}d\sigma + \frac{x}{(\log x)^{k+1}}\Big(\frac{1+\varepsilon}{\log x}\Big)^{N+1}\\ &\ll \frac{x}{(\log x)^{k+1}}\Big(\frac{1+\varepsilon}{\log x}\Big)^{N+1}\int_1^\infty e^{-t}t^{k+N+1}\,dt + \frac{x}{(\log x)^{k+1}}\Big(\frac{1+\varepsilon}{\log x}\Big)^{N+1}\\ &\ll \frac{x\Gamma(k+N+2)}{(\log x)^{k+1}}\Big(\frac{1+\varepsilon}{\log x}\Big)^{N+1}\\ &\ll \frac{x(N+1)^{k+\frac12}}{(\log x)^{k+1}}\Big(\frac{N+1}{\log x}\Big)^{N+1}.\end{aligned}$$

Furthermore, the first term in (9.2) is

$$\frac{x}{(\log x)^{k+1}}\sum_{j=0}^N \frac{\gamma_{-k,j}}{(\log x)^j}\frac{1}{2\pi i}\int_{\Gamma'} e^z z^{k+j}\,dz,$$

after a change of variables, where Γ' is made up of the anticlockwise circle $|z|=1$ excluding the point -1, and the linear paths joining $-\frac{\log x}{2}+\alpha_0\log x$ to -1 and then back to $-\frac{\log x}{2}+\alpha_0\log x$ with arguments $-\pi$ and $+\pi$, respectively. We extend the linear paths to from $-\infty$ to -1 and then back to ∞. The error term arising from doing so is

$$\begin{aligned}&\ll \frac{x}{(\log x)^{k+1}}\sum_{j=0}^N \frac{|\gamma_{-k,j}|}{(\log x)^j}\int_{-\infty}^{-\frac{\log x}{2}+\alpha_0\log x} e^z|z|^{k+j}\,dz\\ &\ll x\exp\Big(-\frac{\log x}{2}+\alpha_0\log x\Big)\\ &= \sqrt{x}\exp\bigg(\frac{\log x(\log\log x_0)^{\frac8k+\varepsilon}}{(\log x_0)^{\frac1k}}\bigg)\\ &\ll \sqrt{x}\exp\big((\log x)^{\frac{k}{k+1}}(\log\log x)^{\frac{7}{k+1}+\varepsilon}\big).\end{aligned}$$

For the main term, the contour is the Hankel contour, and hence by [37, Chapter 5, Theorem 2], it is equal to

$$\begin{aligned}&\frac{x}{(\log x)^{k+1}}\sum_{j=0}^N \frac{\gamma_{-k,j}}{(\log x)^j}\frac{1}{\Gamma(-(k+j))}\\ &\quad= -\frac{\sin(k\pi)}{\pi}\frac{x}{(\log x)^{k+1}}\sum_{j=0}^N \frac{(-1)^j\Gamma(k+j+1)\gamma_{-k,j}}{(\log x)^j}.\end{aligned}$$

Thus

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}=-\frac{\sin(k\pi)}{\pi}\frac{x}{(\log x)^{k+1}}\Bigg(\sum_{j=0}^{N}\frac{(-1)^j\Gamma(k+j+1)\gamma_{-k,j}}{(\log x)^j}$$
$$+O\bigg((N+1)^{k+\frac{1}{2}}\Big(\frac{N+1}{\log x}\Big)^{N+1}\bigg)\Bigg)$$
$$+O\Big(\sqrt{x}\exp\big((\log x)^{\frac{k}{k+1}}(\log\log x)^{\frac{7}{k+1}+\varepsilon}\big)\Big).$$

We next consider the integrals along V_0 and V_0'. Let

$$T_0=\exp\bigg(\frac{\log x_0}{(\log\log x_0)^8}\bigg).$$

For $T_0\leq T\leq x_0$ we have

$$\frac{1}{(\log T)^{\frac{1}{k}}}\ll\alpha_0\ll\frac{(\log\log T)^{\frac{8}{k}+\varepsilon}}{(\log T)^{\frac{1}{k}}}$$

and so

$$\frac{1}{2\pi i}\int_{\frac{1}{2}+\alpha_0+\frac{iT}{2}}^{\frac{1}{2}+\alpha_0+iT}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}\ll\frac{x^{\frac{1}{2}+\alpha_0}}{T}\int_{\frac{1}{2}+\alpha_0+\frac{iT}{2}}^{\frac{1}{2}+\alpha_0+iT}\bigg|\frac{ds}{\zeta(s)^k}\bigg|$$
$$\ll x^{\frac{1}{2}+\alpha_0}\exp\big(\log T(\log\log T)^{-1+\varepsilon}\big),$$

by (1.4). We hence get

$$\frac{1}{2\pi i}\int_{\frac{1}{2}+\alpha_0+iT_0}^{\frac{1}{2}+\alpha_0+ix_0}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}\tag{9.3}$$
$$\ll x^{\frac{1}{2}+\alpha_0}\exp\big(\log x_0(\log\log x_0)^{-1+\varepsilon}\big)$$
$$=\sqrt{x}\exp\bigg(\frac{\log x(\log\log x_0)^{\frac{8}{k}+\varepsilon}}{(\log x_0)^{\frac{1}{k}}}+\log x_0(\log\log x_0)^{-1+\varepsilon}\bigg)$$
$$\ll\sqrt{x}\exp\big((\log x)^{\frac{k}{k+1}}(\log\log x)^{\frac{7}{k+1}+\varepsilon}\big),$$

by diving the segment of integration into dyadic intervals. The same bound holds for the integral along

$$\int_{\frac{1}{2}+\alpha_0-ix_0}^{\frac{1}{2}+\alpha_0-iT_0}.$$

Furthermore, we note from Lemma 2.2 that

$$|\zeta(s)|^{-1}\ll(|t|+2)^{(\frac{1}{2k}+\varepsilon)\frac{\log\log x_0}{\log\log(|t|+2)}}\tag{9.4}$$

for $s\in V_0$. So

$$\frac{1}{2\pi i}\int_{\frac{1}{2}+\alpha_0-iT_0}^{\frac{1}{2}+\alpha_0+iT_0}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}\tag{9.5}$$
$$\ll\sqrt{x}\exp\bigg(\frac{\log x(\log\log x_0)^{\frac{8}{k}+\varepsilon}}{(\log x_0)^{\frac{1}{k}}}+\frac{\log\log x_0\log T_0}{\log\log T_0}\bigg)$$
$$\ll\sqrt{x}\exp\bigg((\log x)^{\frac{k}{k+1}}(\log\log x)^{\frac{7}{k+1}+\varepsilon}\bigg).$$

Combining (9.3) and (9.5) we obtain that

$$\frac{1}{2\pi i}\int_{V_0}\frac{x^s}{\zeta(s)^k}\frac{ds}{s} \ll \sqrt{x}\exp\big((\log x)^{\frac{k}{k+1}}(\log\log x)^{\frac{7}{k+1}+\varepsilon}\big).$$

The same bound holds for the integral over V_0'.

For the contribution from the vertical segments $\bigcup_{j=1}^{J} V_j$, we deduce from (1.3) that it is bounded by

$$\begin{aligned}(9.6)\qquad &\ll \sqrt{x}(\log x)^{\frac{k^2}{4}+\varepsilon}\sum_{j=0}^{J-1}\exp\bigg(\frac{\log x(\log\log(2^j x_0))^{\frac{8}{k}+\varepsilon}}{(\log(2^j x_0))^{\frac{1}{k}}}\bigg)\\ &\ll \sqrt{x}\exp\bigg(\frac{\log x(\log\log x)^{\frac{8}{k}+\varepsilon}}{(\log x_0)^{\frac{1}{k}}}\bigg)\\ &\ll \sqrt{x}\exp\bigg((\log x)^{\frac{k}{k+1}}(\log\log x)^{\frac{7}{k+1}+\varepsilon}\bigg).\end{aligned}$$

For $s\in H_j$, $1\leq j\leq J-1$, again like (9.4) we have

$$|\zeta(s)|^{-1} \ll (2^j x_0)^{\frac{1}{2k}+\varepsilon}.$$

So the contribution to (9.1) from the horizontal segments $\bigcup_{j=1}^{J-1} H_j$ is

$$\begin{aligned}(9.7)\qquad &\ll \sqrt{x}\sum_{j=0}^{J-2}(2^j x_0)^{-\frac{1}{2}+\varepsilon}\exp\bigg(\frac{\log x(\log\log(2^j x_0))^{\frac{8}{k}+\varepsilon}}{(\log(2^j x_0))^{\frac{1}{k}}}\bigg)\\ &\ll \sqrt{x}x_0^{-\frac{1}{2}+\varepsilon}\exp\bigg(\frac{\log x(\log\log x)^{\frac{8}{k}+\varepsilon}}{(\log x_0)^{\frac{1}{k}}}\bigg)\\ &\ll \sqrt{x}.\end{aligned}$$

We are left with the integral along H_J, which is bounded by

$$(9.8)\qquad \max_{\frac{1}{2}+\frac{(\log\log x)^{8/k+\varepsilon}}{(\log x)^{1/k}}\leq\sigma\leq 1+\frac{1}{\log x}}\exp\big((\sigma-1)\log x - k\log|\zeta(\sigma\pm ix)|\big).$$

For

$$\frac{1}{2}+\frac{(\log\log x)^{\frac{8}{k}+\varepsilon}}{(\log x)^{\frac{1}{k}}}\leq\sigma\leq\frac{1}{2}+o\bigg(\frac{1}{\log\log x}\bigg),$$

Lemma 2.2 implies that this is

$$\begin{aligned}(9.9)\quad &\ll \exp\bigg((\sigma-1)\log x+\frac{k\log x}{2\log\log x}\log\frac{1}{1-(\log x)^{1-2\sigma}}\\ &\qquad + O_k\Big(\frac{\log x}{\log\log x}\Big)+O_k\Big(\frac{\log x}{(\log\log x)^2}\log\frac{1}{1-(\log x)^{1-2\sigma}}\Big)\bigg)\\ &\ll \exp\bigg((\sigma-1)\log x+\frac{k\log x}{2\log\log x}\log\frac{1}{(2\sigma-1)\log\log x}+O_k\big((2\sigma-1)\log x\big)\\ &\qquad + O_k\Big(\frac{\log x}{\log\log x}\Big)+O_k\Big(\frac{\log x}{(\log\log x)^2}\log\frac{1}{2\sigma-1}\Big)\bigg),\end{aligned}$$

which is decreasing with respect to σ, and, hence,

$$\ll \exp\bigg(-\frac{(8+k)\log x\log\log\log x}{2\log\log x}+O_k\Big(\frac{\log x}{\log\log x}\Big)\bigg)\ll 1. \tag{9.10}$$

For

$$1-\frac{1}{\log\log x}\le\sigma\le 1+\frac{1}{\log x},$$

the second estimate in Lemma 2.2 leads to a bound of size

$$\ll \exp\big((\sigma-1)\log x+k\log\log\log x+O_k(1)\big)\ll(\log\log x)^k. \tag{9.11}$$

Finally, for

$$\frac12+O\Big(\frac{1}{\log\log x}\Big)\le\sigma\le 1-\frac{1}{\log\log x},$$

we use the last estimate in Lemma 2.2 to get the bound

$$\begin{aligned}\ll \exp\bigg(&(\sigma-1)\log x+\frac{(\log x)^{2-2\sigma}}{(1-\sigma)\log\log x}+\varepsilon\log x\\&+O_k\Big(\frac{(\log x)^{2-2\sigma}}{(1-\sigma)^2(\log\log x)^2}\Big)\bigg)\\ \ll x^{\varepsilon}.&\end{aligned} \tag{9.12}$$

Combining the estimates we obtain the first part of the theorem for $k\ge 1$.

9.2. Assuming RH: $k<1$. The arguments are similar to the previous case. Let

$$x_1=\exp\big(\sqrt{\varepsilon\log x}\log\log x\big),\quad \alpha_1=\varepsilon\frac{\log\log x_1}{\log x_1}$$

and $J=\lceil\log_2\frac{[x]}{x_1}\rceil\ll\log x$. We replace the contour in (9.1) with

$$\bigcup_{j=1}^{J}(V_j\cup H_j)\cup V_0'\cup\Gamma,$$

where $V_0'\cup\Gamma$ is the contour joining the points $\frac12+\alpha_1-ix_1$ and $\frac12+\alpha_1+ix_1$, as described below, and

$$\begin{aligned}
&V_j:\quad s=\frac12+\varepsilon\frac{\log\log(2^{j-1}x_1)}{\log(2^{j-1}x_1)}+it,\quad 2^{j-1}x_1\le|t|\le\min\{2^jx_1,[x]\},\quad 1\le j\le J,\\
&H_j:\quad s=\sigma\pm i2^jx_1,\quad \frac12+\varepsilon\frac{\log\log(2^jx_1)}{\log(2^jx_1)}\le\sigma\le\frac12+\varepsilon\frac{\log\log(2^{j-1}x_1)}{\log(2^{j-1}x_1)},\quad 1\le j\le J-1,\\
&H_J:\quad s=\sigma\pm i[x],\quad \frac12+\varepsilon\frac{\log\log(2^{J-1}x_1)}{\log(2^{J-1}x_1)}\le\sigma\le 1+\frac{1}{\log x}.
\end{aligned}$$

Here Γ is the truncated Hankel-type contour made up of the anticlockwise circle $|s-1|=\frac{1}{\log x}$ excluding the point $1-\frac{1}{\log x}$, and the linear paths joining $\frac12+\alpha_1$ to $1-\frac{1}{\log x}$ and then back to $\frac{1}{2}+\alpha_1$ with arguments $-\pi$ and $+\pi$, respectively, and V_0' consists of the two vertical segments joining the Hankel contour and the point $\frac{1}{2}+\alpha_1-ix_1$ and $\frac{1}{2}+\alpha_1+ix_1$ respectively. Again we encounter no singularity in doing so.

For the integral over Γ, we have the main term as before. Also, the error term arising from extending the linear paths to from $-\infty$ to -1 and then back to ∞ is now

$$\begin{aligned}
&\ll \frac{x}{(\log x)^{k+1}}\sum_{j=0}^{N}\frac{|\gamma_{-k,j}|}{(\log x)^j}\int_{-\infty}^{-\frac{\log x}{2}+\alpha_1\log x} e^z|z|^{k+j}\,dz\\
&\ll x\exp\Big(-\frac{\log x}{2}+\alpha_1\log x\Big)\\
&= \sqrt{x}\exp\Big(\varepsilon\frac{\log x\log\log x_1}{\log x_1}\Big)\\
&\ll \sqrt{x}\exp\big(\varepsilon\sqrt{\log x}\big).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Gamma}\frac{x^s}{\zeta(s)^k}\frac{ds}{s} = -\frac{\sin(k\pi)}{\pi}\frac{x}{(\log x)^{k+1}}\Bigg(&\sum_{j=0}^{N}\frac{(-1)^j\Gamma(k+j+1)\gamma_{-k,j}}{(\log x)^j}\\
&+O\bigg((N+1)^{k+\frac{1}{2}}\Big(\frac{+1N}{\log x}\Big)^{N+1}\bigg)\Bigg)\\
&+O\Big(\sqrt{x}\exp\big(\varepsilon\sqrt{\log x}\big)\Big).
\end{aligned}$$

For the integral along V_0, let

$$T_1=\exp\Big(\frac{\log x_1}{\log\log x_1}\Big).$$

If $T_1\le T\le x_1$, then $\frac{1}{\log T}\ll\alpha_1\ll\frac{\log\log T}{\log T}$, and hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\frac{1}{2}+\alpha_1+\frac{iT}{2}}^{\frac{1}{2}+\alpha_1+iT}\frac{x^s}{\zeta(s)^k}\frac{ds}{s} &\ll \frac{x^{\frac{1}{2}+\alpha_1}}{T}\int_{\frac{1}{2}+\alpha_1+\frac{iT}{2}}^{\frac{1}{2}+\alpha_1+iT}\left|\frac{ds}{\zeta(s)^k}\right|\\
&\ll x^{\frac{1}{2}+\alpha_1}\exp\big((\log\log T)^{1+\varepsilon}\big),
\end{aligned}$$

by (1.7). It follows that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\frac{1}{2}+\alpha_1+iT_1}^{\frac{1}{2}+\alpha_1+ix_1}\frac{x^s}{\zeta(s)^k}\frac{ds}{s} &\ll x^{\frac{1}{2}+\alpha_1}\exp\big((\log\log x_1)^{1+\varepsilon}\big)\\
&= \sqrt{x}\exp\Big(\varepsilon\frac{\log x\log\log x_1}{\log x_1}+(\log\log x_1)^{1+\varepsilon}\Big)\\
&\ll \sqrt{x}\exp\big(\varepsilon\sqrt{\log x}\big),
\end{aligned}\tag{9.13}$$

by relabelling ε. The same bound holds for the integral along

$$\int_{\frac{1}{2}+\alpha_1-ix_1}^{\frac{1}{2}+\alpha_1-iT_1}.$$

Also, by (9.4) we have

$$\begin{aligned}\text{(9.14)}\qquad &\frac{1}{2\pi i}\int_{\frac12+\alpha_1-iT_1}^{\frac12+\alpha_1+iT_1}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}\\ &\ll\sqrt{x}\exp\Big(\varepsilon\frac{\log x\log\log x_1}{\log x_1}+\Big(\frac{k}{2}+\varepsilon\Big)\frac{\log\log x_1\log T_1}{\log\log T_1}\Big)\\ &\ll\sqrt{x}\exp\big(\varepsilon\sqrt{\log x}\big),\end{aligned}$$

by another relabelling of ε. Combining (9.13) and (9.14), we obtain that

$$\text{(9.15)}\qquad \frac{1}{2\pi i}\int_{V_0}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}\ll\sqrt{x}\exp\big(\varepsilon\sqrt{\log x}\big).$$

The same bound holds for the other integrals. Indeed, similar to (9.6) and (9.7) we have

$$\begin{aligned}\text{(9.16)}\qquad \sum_{j=1}^{J}\frac{1}{2\pi i}\int_{V_j}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}&\ll\sqrt{x}(\log x)^{\frac{k^2}{4}+\varepsilon}\sum_{j=0}^{J-1}\exp\Big(\varepsilon\frac{\log x\log\log(2^jx_1)}{\log(2^jx_1)}\Big)\\ &\ll\sqrt{x}(\log x)^{\frac{k^2}{4}+1+\varepsilon}\exp\Big(\varepsilon\frac{\log x\log\log x_1}{\log x_1}\Big)\\ &\ll\sqrt{x}\exp\big(\varepsilon\sqrt{\log x}\big)\end{aligned}$$

and

$$\begin{aligned}\text{(9.17)}\qquad \sum_{j=1}^{J-1}\frac{1}{2\pi i}\int_{H_j}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}&\ll\sqrt{x}\sum_{j=0}^{J-2}(2^jx_1)^{\frac{k}{2}-1+\varepsilon}\exp\Big(\varepsilon\frac{\log x\log\log(2^jx_1)}{\log(2^jx_1)}\Big)\\ &\ll\sqrt{x}x_1^{-\frac12+\varepsilon}(\log x)\exp\Big(\varepsilon\frac{\log x\log\log x_1}{\log x_1}\Big)\\ &\ll\sqrt{x}\exp\big(\varepsilon\sqrt{\log x}\big).\end{aligned}$$

Similar to (9.8), (9.9), (9.10), (9.11) and (9.12) we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{H_J}\frac{x^s}{\zeta(s)^k}\frac{ds}{s}\\ &\quad\ll\max_{\frac12+\varepsilon\frac{\log\log x}{\log x}\le\sigma\le1+\frac{1}{\log x}}\exp\big((\sigma-1)\log x-k\log|\zeta(\sigma\pm ix)|\big)\\ &\quad\ll\begin{cases}\exp\Big(\frac{(k-1)\log x}{2}-\frac{k\log x\log\log\log x}{\log\log x}\\ \qquad+O_k\big(\frac{\log x}{\log\log x}\big)\Big) & \text{if } \frac12+\varepsilon\frac{\log\log x}{\log x}\le\sigma\le\frac12+o(\frac{1}{\log\log x}),\\ (\log\log x)^k & \text{if } 1-\frac{1}{\log\log x}\le\sigma\le1+\frac{1}{\log x},\\ x^{\varepsilon} & \text{if } \frac12+O(\frac{1}{\log\log x})\le\sigma\le1-\frac{1}{\log\log x},\end{cases}\\ &\quad\ll x^{\varepsilon},\end{aligned}$$

by Lemma 2.2. This, together with (9.15), (9.16) and (9.17), establishes the first part of the theorem for $k<1$.

9.3. Assuming Conjecture 1.1. We will prove the theorem in the case $k \in \mathbb{Z}$. When $k \notin \mathbb{Z}$, we have the main term just like in previous subsections.

We replace the contour in (9.1) by $H_1 \cup V_0$, where

$$\begin{aligned} &V_0: \quad s = \frac{1}{2} + \alpha_2 + it, \quad |t| \le [x], \\ &H_1: \quad s = \sigma \pm i[x], \quad \frac{1}{2} + \alpha_2 \le \sigma \le 1 + \frac{1}{\log x} \end{aligned}$$

for some $\frac{1}{\log x} \le \alpha_2 \le 1$ to be chosen later. We encounter no pole in doing so.

On one hand, by Conjecture 1.1 we get

$$\frac{1}{2\pi i}\int_{V_0} \frac{x^s}{\zeta(s)^k}\frac{ds}{s} \ll \sqrt{x}x^{\alpha_2}\Big(\frac{1}{\alpha_2}\Big)^{\frac{k^2}{4}}. \tag{9.18}$$

On the other hand, we have

$$\frac{1}{2\pi i}\int_{H_1} \frac{x^s}{\zeta(s)^k}\frac{ds}{s} \ll \max_{\frac{1}{2}+\alpha_2\le\sigma\le 1+\frac{1}{\log x}} \exp\big((\sigma-1)\log x - k\log|\zeta(\sigma\pm i[x])|\big).$$

Like in (9.9) and (9.10), this is

$$\begin{aligned} &\ll \exp\bigg((\sigma-1)\log x + \frac{k\log x}{2\log\log x}\log\frac{1}{(2\sigma-1)\log\log x} \\ &\qquad + O_k\big((2\sigma-1)\log x\big) + O_k\Big(\frac{\log x}{\log\log x}\Big) \\ &\qquad + O_k\Big(\frac{\log x}{(\log\log x)^2}\log\frac{1}{2\sigma-1}\Big)\bigg) \\ &\ll \exp\bigg(-\frac{\log x}{2} + \frac{k\log x\log\frac{1}{\alpha_2}}{2\log\log x} - \frac{k\log x\log\log\log x}{2\log\log x} \\ &\qquad + O_k\big(\alpha_2\log x\big) + O_k\Big(\frac{\log x}{\log\log x}\Big) + O_k\Big(\frac{\log x\log\frac{1}{\alpha_2}}{(\log\log x)^2}\Big)\bigg) \end{aligned} \tag{9.19}$$

for $\frac{1}{2}+\alpha_2 \le \sigma \le \frac{1}{2} + o(\frac{1}{\log\log x})$, and as in (9.11) and (9.12), it is

$$\ll (\log\log x)^k + x^{\varepsilon}$$

for $\frac{1}{2} + O(\frac{1}{\log\log x}) \le \sigma \le 1 + \frac{1}{\log x}$. In view of (9.19), we choose

$$\alpha_2 = \begin{cases} \frac{1}{\log x} & \text{if } k \le 2, \\ \frac{1}{(\log x)^{\frac{2}{k}}(\log\log x)^{1-\varepsilon}} & \text{if } k > 2, \end{cases}$$

and then

$$\frac{1}{2\pi i}\int_{H_1} \frac{x^s}{\zeta(s)^k}\frac{ds}{s} \ll \sqrt{x}.$$

Combining with (9.18) we obtain the second part of the theorem.

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