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A sheaf-theoretic approach to tropical homology

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We introduce a sheaf-theoretic approach to tropical homology, especially for tropical homology with potentially non-compact supports. Our setup is suited to study the functorial properties of tropical homology, and we show that it behaves analogously to classical Borel-Moore homology in the sense that there are proper push-forwards, cross products, and cup products with tropical cohomology classes, and that it satisfies identities like the projection formula and the Künneth theorem. Our framework allows for a natural definition of the tropical cycle class map, which we show to be a natural transformation. Finally, we characterize the rational polyhedral spaces that satisfy Poincaré-Verdier duality as those that are smooth.

Contents

1. Introduction . . . 578
2. Rational polyhedral spaces and tropical forms . . . 583
3. Tropical cycles on rational polyhedral spaces . . . 589
4. Tropical (co)homology and its functorial properties . . . 595
5. The tropical cycle class map . . . 612
6. Poincaré-Verdier duality . . . 626

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Data availability 630
Appendix A. Complexes of singular chains on CS sets 630
References 639

1. Introduction

1.1. Background

Tropical (co)homology theory is a new tool to associate algebraic invariants to the spaces appearing in tropical geometry. They were introduced in [20] where it was shown that the tropical cohomology groups of tropical manifolds have a Hodge-theoretic interpretation in algebraic geometry in case the tropical manifold arises as the tropicalization of a smooth projective variety. As one would expect by analogy to the algebro-geometric picture, tropical homology is also closely related to tropical intersection theory. In [32], Mikhalkin and Zharkov introduced the tropical cycle class map on rational polyhedral spaces equipped with a global face structure that assigns a class in tropical homology to every tropical cycle. This map has been further studied in [37] in the case of tropical surfaces, and in [22] with a special emphasis on methods that work in the locally finite setting. An excellent introduction to the subject can be found in [7].

1.2. Our contributions

We introduce a sheaf-theoretic viewpoint on tropical homology, by expressing tropical homology groups directly in terms of the sheaves Ω^p_X of tropical p-forms and the dualizing complex $\mathbb{D}_X$. This will allow us to avoid the need to work with any stratification or global face structure of the rational polyhedral space X. We also avoid any reference to locally finite chains in a very similar way as this is avoided in the classical development of Borel-Moore homology [8]. The basis for this is the following theorem:

Theorem A *(= Theorem 4.20). Let X be a rational polyhedral space. Then there exists a natural isomorphism*

$$H^{lf}_{p,q}(X) \cong \mathbb{H}^{-q} R\mathscr{H}om^{\bullet}(\Omega^p_X, \mathbb{D}_X)\ ,$$

where $H^{lf}_{p,q}(X)$ denotes the tropical homology groups defined in [22] via locally finite tropical chains over $\mathbb{Z}$ and $\mathbb{D}_X$ denotes the dualizing complex on X.

Motivated by this, and in analogy to the classical theory, we denote

$$H^{BM}_{p,q}(X) = \mathbb{H}^{-q} R\mathscr{H}om^{\bullet}(\Omega^p_X, \mathbb{D}_X)$$

and call it the (p, q)-th tropical Borel-Moore homology group. Replacing hypercohomology by hypercohomology with supports, we also define compactly supported tropical homology groups $H_{p,q}(X)$ and tropical homology groups $H_{p,q}(Z, X)$ supported on a closed subset Z of X. These correspond to the most common support families, but we remark that our construction allows to work with an arbitrary support family.

Remark. In [22], what we denote by $H^{lf}_{p,q}(X)$ is denoted by $H^{BM}_{p,q}(X)$. By Theorem A, this will not lead to conflicts with the existing literature.

Note also that the sheaves Ω^p_X agree, up to torsion, with the sheaves $\mathscr{F}^p_X$ that are usually used in the literature to define tropical cohomology. See Remark 2.8 for an elaboration on the subtleties involved. We call the elements in Ω^p_X tropical p-forms because they are represented by wedges of tropical 1-forms (as introduced in [31]).

Our sheaf-theoretic formulation of tropical homology makes it evident that the functorial behavior of tropical homology is completely determined by the functorial behavior of tropical p-forms on the one hand, and dualizing complexes on the other. With this in mind, the constructions of proper push-forwards, cross products, cup products, and cap products in tropical homology are straightforward generalizations of the classical constructions, at least after we establish some general functorial properties of the sheaves of tropical forms. Our point of view also sheds light on the identities that these operations satisfy. For example, we obtain a tropical version of the Künneth Theorem:

Theorem B *(= Theorem 4.16). Let X and Y be compactifiable rational polyhedral spaces with torsion-free homology groups. Then we have*

$$H^{BM}_{p,q}(X \times Y) \cong \bigoplus_{\substack{i+j=p \\ k+l=q}} H^{BM}_{i,k}(X) \otimes_{\mathbb{Z}} H^{BM}_{j,l}(Y)$$

and

$$H_{p,q}(X \times Y) \cong \bigoplus_{\substack{i+j=p \\ k+l=q}} H_{i,k}(X) \otimes_{\mathbb{Z}} H^{BM}_{j,l}(Y)\ .$$

The compactifiability on X and Y is a mild condition; see Definition 4.15 for details. Note that our Theorem 4.16 also deals with the torsion; we have omitted this here to simplify the statement. Finally, note that Smacka proved a tropical Künneth Theorem with real coefficients using superforms in [39].

Another advantage of a sheaf-theoretic view on tropical geometry is that sheaves are very well-suited to pass from local to global considerations. We exploit this in our definition of the tropical cycle class map, where we use the definition of [32] locally and then utilize the sheaf property to glue. This has the advantage of avoiding the necessity of dealing with global face structures or triangulations of the space, as one needs to in the definitions in [32,22]. Once the tropical cycle class map is defined, we prove that *it is* a natural transformation in the sense that it satisfies the compatibility conditions summarized in the following theorem:

Theorem C *(= Corollary 5.8, Proposition 5.9, Proposition 5.12). The tropical cycle class map commutes with proper push-forwards, cross products, and intersections with tropical Cartier divisors.*

Finally, we study Poincaré-Verdier duality on purely n-dimensional rational polyhedral spaces that are regular at infinity (in the sense of [32]) and admit a fundamental cycle; we say that X admits a fundamental cycle if and only if assigning weight 1 everywhere on X defines a tropical n-cycle. We will see that a fundamental cycle induces morphisms

$$\delta_p^X : \Omega_X^{n-p}[n] \to \mathscr{D}(\Omega_X^p)$$

for every $p \in \mathbb{Z}$, where $\mathscr{D}(\Omega_X^p)$ denotes the Verdier dual of Ω_X^p. We say that X satisfies Poincaré-Verdier duality if δ_p^X is an isomorphisms for every $p \in \mathbb{Z}$.

Theorem D *(= Theorem 6.7, Corollary 6.9). Let X be an n-dimensional rational polyhedral space that is regular at infinity and admits a tropical fundamental class. Then X satisfies Poincaré-Verdier duality if and only if every point $x \in X$ has a neighborhood isomorphic to an open subset in $F \times \mathbb{T}^n$ for some $n \in \mathbb{N}$ and a fan F that is smooth in the sense of Amini–Piquerez and Aksnes. In particular, if X is smooth, $p, q \in \mathbb{N}$, and Φ is any family of supports, then there is a natural isomorphism*

$$H_\Phi^{p,q}(X) \cong H^\Phi_{n-p,n-q}(X) \ , \tag{1.1}$$

induced by the cap product with the fundamental class.

By the results of [22], tropical manifolds (in the sense of [35,32]) are smooth in the sense of Amini–Piquerez and Aksnes. In the case where the family Φ consists of either all compact or all closed subsets of X, the isomorphism (1.1) has already established in [22] (and in [23] for real coefficients) for tropical manifolds and in [4,1] for rational polyhedral spaces that are smooth in the sense of Amini–Piquerez and Aksnes. The inclusion of arbitrary support families is new. Note that Theorem D, together with the smoothness of tropical manifolds proved in [22], implies that tropical manifolds satisfy Poincaré duality with arbitrary systems of supports. In [39], Smacka proved, using superforms, that δ_p^X is an isomorphism on tropical manifolds if one considers real coefficients.

1.3. Sheaves and cosheaves on rational polyhedral spaces

Let us briefly discuss why we introduce a sheaf-theoretic approach to tropical homology theory to complement the approach using cosheaves used previously in the literature.

First, let us briefly recall some facts about tropical cohomology. The (p, q)-th tropical cohomology group of a rational polyhedral space X can be defined as the cohomology group $H^q(X, \Omega^p_X)$ (see [32]). This description clearly encapsulates the analogy to Dolbeault cohomology, but also makes the tropical cohomology groups accessible to standard sheaf-theoretic tools like Čech cohomology. Furthermore, the sheaves Ω^p_X are closely related to sheaves of superforms [25,10,17] in the sense that an appropriate complex of superforms defines a soft resolution of $\Omega^p_X \otimes_{\mathbb{Z}} \mathbb{R}$ (see [23]). In particular, it is shown in [23] that tropical cohomology groups with real coefficients can be interpreted as tropical Dolbeault cohomology groups.

We have several techniques at our disposal to study the sheaves Ω^p_X of tropical p-forms. Ideally, one could define tropical homology by taking homology groups with coefficients in the dual sheaf of Ω^p_X. But, unfortunately, this does not yield the correct notion. As a remedy, one takes a different approach to dualizing Ω^p_X and obtains a constructible cosheaf $\mathscr{F}^X_p$ on X. Using suitable singular chains with coefficients in $\mathscr{F}^X_p$ one obtains the tropical homology groups $H^{\text{sing}}_{p,q}(X)$ and $H^{lf}_{p,q}(X)$, where for the latter one considers locally finite chains. As shown in [32], there is an isomorphism $H^{\text{sing}}_{p,q}(X) \cong H_q(\mathscr{F}^X_p)$, where the latter group is the q-th cosheaf homology of $\mathscr{F}^X_p$.

It may seem like a natural principle that sheaves are used for cohomology while cosheaves are used for homology. This is, however, not necessarily the case. First of all, the appearance of sheaves, most notably local systems, as coefficients in homology is ubiquitous [8,9]. Secondly, the homology theory of cosheaves does not allow to account for different support families. For example, since $H^{\text{sing}}_{p,q}(X)$ is isomorphic to a homology group of $\mathscr{F}^X_p$, one might expect $H^{lf}_{p,q}(X)$ to be isomorphic to a homology group of $\mathscr{F}^X_p$ with compact supports, but homology groups with compact supports are undefined for cosheaves. Finally, the definitions of $H^{\text{sing}}_{p,q}(X)$ and $H^{lf}_{p,q}(X)$ make use of the fact that $\mathscr{F}^X_p$ is constructible and use a stratification of X. While the resulting homology groups are independent of the chosen stratification, using a stratification in the definition cannot be avoided. In particular, the definition of singular tropical homology groups does not generalize to a homology theory for arbitrary cosheaves, whereas there are singular homology groups with coefficients in any given sheaf [8,9].

As mentioned, our main objects of interest are the locally-finite tropical homology groups $H^{lf}_{p,q}(X)$, since those are the target of the tropical cycle class map (for $p = q$). As noted above, there is no notion of cosheaf homology that expresses $H^{lf}_{p,q}(X)$ as a cosheaf homology group. Therefore, one needs to work with locally-finite tropical chains. Again as noted above, tropical chains are defined with respect to a suitable stratification of the base space X. This makes it impossible to simply apply standard results from algebraic topology, but one needs to be careful to always respect the stratification, as for example in the development of singular intersection homology [15,13]. As the acyclic model theorem does not apply to tropical homology groups either, proving identities like the Künneth theorem for tropical homology gets quite involved, even for finite chains.

1.4. Other related work

We are hopeful that our sheaf-theoretic approach to tropical homology can be applied to spaces that are not necessarily rational polyhedral, but possibly have singularities in their affine structures. The resulting notion is also strongly related to the invariants of integral affine manifolds appearing in the context of mirror symmetry [16,34,33], as recently shown in [42]. We also hope that our point of view on the tropical cycle class map could provide a new perspective for the cycle class map on non-Archimedean spaces [26]. This is, in turn, closely related to the study of tropical and non-Archimedean analogues of the Hodge conjecture [44,30,2,3]

1.5. Structure of the paper

In sections 2 and 3 we recall the definitions of the objects and operations needed in the main part of the paper. We try to follow the literature [29,32,20,22,36,5,7] as closely as possible, but will provide a new perspective on some things. Most notably, we deviate from the literature in our definition of tropical p-forms in section 2, and our treatment of tropical cycles in section 3 has an emphasis on working locally and highlighting their functorial properties.

In section 4 we introduce tropical Borel-Moore homology and study its functorial behavior. We define proper push-forwards, cross products, cup products, and cap products, and will prove Theorem B. Finally, we compare our theory with the on obtained via locally finite tropical chains by proving Theorem A.

In section 5 we will define the tropical cycle class map and show that it is compatible with proper push-forwards, cross-products, and intersections with Cartier divisors, proving Theorem C. Furthermore, we show that our tropical cycle class map coincides with the one introduced in [32,22] if we are given a (global) face structure.

Section 6 is devoted to prove Theorem D.

The main ingredients of our proof of Theorem A are of an entirely topological nature, dealing mostly with certain sheaves of singular chains on conically stratified spaces. As these results are of a very different flavor, and potentially of independent interest, we decided to put them in Appendix A.

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Conventions. The natural numbers $\mathbb{N}$ include 0. *All* homology and cohomology groups in this paper, whether classical or tropical, will be considered with *integer coefficients*.

2. Rational polyhedral spaces and tropical forms

In this section we recall and further develop the notion of rational polyhedral spaces and tropical p-forms. Varying incarnations of rational polyhedral spaces (for example, the tropical varieties from [29]) have been fundamental to the development of tropical geometry since its inception. Our definition of rational polyhedral spaces follows the one recently given in [22]. Tropical p-forms have been introduced in the context of tropical (co)homology and our definition is a variation of the one given in [32].

2.1. Rational polyhedral spaces

We denote $\mathbb{T} \coloneqq \mathbb{R} \cup \{+\infty\}$ and consider it with the order topology. For $n \in \mathbb{N}$, the n-fold product $\mathbb{T}^n$ has a natural stratification $\mathbb{T}^n = \bigsqcup_{I \subseteq \{1,\ldots,n\}} \mathbb{T}^n_I$, where the stratum

$$\mathbb{T}^n_I = \{(x_i)_{1 \leq i \leq n} \mid x_i = \infty \text{ if and only iff } i \in I\}$$

is naturally identified with $\mathbb{R}^{n-|I|}$. Recall that a *(rational) polyhedron* in $\mathbb{R}^n$ is a finite intersection of half-spaces of the form $\{x \in \mathbb{R}^n \mid \langle m, x\rangle \leq a\}$ with $m \in (\mathbb{Z}^n)^*$ and $a \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ denotes the evaluation pairing. By a polyhedron in $\mathbb{T}^n$ we mean any set occurring as the closure of a polyhedron in some stratum $\mathbb{T}^n_I$. Note that for any polyhedron σ in $\mathbb{R}^n$ and subset $I \subset \{1, \ldots, n\}$, the intersection $\overline{\sigma} \cap \mathbb{T}^n_I$ is a polyhedron in $\mathbb{T}^n_I$. A *polyhedral set* in $\mathbb{T}^n$ is a finite union of polyhedra.

An *integral affine linear function* on a subset $X \subseteq \mathbb{T}^n$ is a continuous function $f \colon X \to \mathbb{R}$ that is of the form $x \mapsto \langle m, x\rangle + a$ for some $m \in (\mathbb{Z}^n)^*$ and $a \in \mathbb{R}$ locally around every point in X. Here, we use the convention that $0 \cdot (\infty) = 0$, and that $\langle m, x\rangle$ is only defined if for all i such that the coordinate m_i is nonzero, we have $x_i \neq \infty$. In particular, if f is integral affine linear on X and $f(x) = \langle m, x\rangle + a$ for $x \in X \cap \mathbb{T}^n_I$, then $m_i = 0$ for $i \in I$. For every subset $X \subseteq \mathbb{T}^n$, the integral affine linear functions on open subsets of X define a sheaf of abelian groups on X, denoted by Aff_X.

Example 2.1. Consider the polyhedral set

$$X = \partial \operatorname{conv}\{(0,0), (1,0), (0,1), (1,1)\}$$

in $\mathbb{R}^2$, which is the boundary of a square with sides of length one. Consider the function on X that is given by 0 on the top and right edge of the square, and by $-1+x_1+x_2$ on the bottom and left edge of the square. This function is continuous and locally the restriction of an integral affine linear function on $\mathbb{R}^2$. In fact, it coincides with the restriction of an integral affine linear function on the union of any two adjacent edges of the square. However, it is not equal to the restriction of a single integral affine linear function on $\mathbb{R}^2$ everywhere because it has different slopes on parallel edges.

Definition 2.2 *(see [22, Definition 2.1]).* A *rational polyhedral space* is a second-countable Hausdorff topological space X, together with a sheaf Aff_X of continuous functions such that for every $x \in X$ there exists an open neighborhood $U \subseteq X$, an open subset V of a polyhedral subset of $\mathbb{T}^n$ for some $n \in \mathbb{N}$, and a homeomorphism $\varphi\colon U \to V$ that induces an isomorphism $\varphi^{-1} \operatorname{Aff}_V \to \operatorname{Aff}_U$ via the pullback of functions. The data of U, V and φ is called a *chart*.

Definition 2.3. A morphism between two rational polyhedral spaces X and Y is a continuous map $f\colon X \to Y$ that induces a morphism $f^{-1} \operatorname{Aff}_Y \to \operatorname{Aff}_X$ via the pullback of functions. A morphism is *proper*, if it is proper as a continuous map of topological spaces, that is if the preimages of compact subsets of Y are compact.

Definition 2.4 *(see [22, Definition 2.2]).* Let X be a rational polyhedral space.

(a) A *polyhedron* in X is a closed subset $P \subseteq X$ such that there exists a chart $X \supseteq U \xrightarrow{\varphi} V \subseteq \mathbb{T}^n$ such that $P \subset U$ and $\varphi(P) \subseteq \mathbb{T}^n$ is a polyhedron. The *faces* of P are the preimages under φ of the (finite or infinite) faces of $\varphi(P)$. The *relative interior* $\operatorname{relint}(P)$ of P is the complement in P of the union of its proper faces.
(b) A *local face structure* at a point $x \in X$ is a finite set Σ of polyhedra in X that is closed under taking faces and intersections (that is if τ is a face of $\sigma \in \Sigma$, then $\tau \in \Sigma$, and $\sigma \cap \delta \in \Sigma$ for all $\sigma, \delta \in \Sigma$), such that x is contained in the (topological) interior of $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$, there exists a chart $X \supseteq U \to V \subseteq \mathbb{T}^n$ with $|\Sigma| \subseteq U$, and such that $x \in \sigma$ for all inclusion-maximal $\sigma \in \Sigma$.
(c) A (global) *face structure* on X is a set Σ of polyhedra in X that is closed under taking faces and intersections such that $X = \bigcup_{\sigma\in\Sigma} \sigma$, and for every $x \in X$ the set of all faces of polyhedra in Σ that contain x is a local face structure at x.
(d) We say that a closed subset $S \subseteq X$ is *locally polyhedral* if at every point $x \in X$ there is a local face structure Σ and a subset $\Sigma' \subseteq \Sigma$ such that $S \cap |\Sigma| = \bigcup_{\sigma\in\Sigma'} \sigma$.

2.2. Tangent spaces

As constant functions are integral affine linear, there is an inclusion $\mathbb{R}_X \hookrightarrow \operatorname{Aff}_X$, where $\mathbb{R}_X$ denotes the constant sheaf associated to $\mathbb{R}$. Following [31], we denote the quotient sheaf $\operatorname{Aff}_X / \mathbb{R}_X$ by Ω^1_X and call it the *cotangent sheaf*. The sections of Ω^1_X are called *tropical* 1*-forms*. The reason for this is that the cotangent space at a point should consist of linear approximations of functions, and linear functions are simply affine linear ones modulo constants. For $x \in X$ we denote by

$$T^{\mathbb{Z}}_x X := \operatorname{Hom}_{\mathbb{Z}}(\Omega^1_{X,x}, \mathbb{Z}) \text{ and}$$
$$T_x X := \operatorname{Hom}_{\mathbb{Z}}(\Omega^1_{X,x}, \mathbb{R})$$

the (integral) tangent space of X at x. It follows immediately from the definitions that a morphism $f\colon X \to Y$ of rational polyhedral spaces induces a morphism

$$f^\sharp\colon f^{-1}\Omega^1_Y \to \Omega^1_X \ ,$$

and hence morphisms of stalks $\Omega^1_{Y,f(x)} \to \Omega^1_{X,x}$ for all $x \in X$. These dualize to a morphisms

$$d_x f\colon T^{\mathbb{Z}}_x X \to T^{\mathbb{Z}}_{f(x)} Y$$

between the integral tangent spaces. If $Y = \mathbb{R}$, that is if f is an affine function on X, the germ at x of the image of f in $\Gamma(X, \Omega^1_X)$ under the quotient morphism $\mathrm{Aff}_X \to \Omega^1_X$ defines a morphism $T^{\mathbb{Z}}_x X \to \mathbb{Z}$ which coincides with $d_x f$ modulo the natural identification $T^{\mathbb{Z}}_{f(x)}\mathbb{R} \cong \mathbb{Z}$. For this reason, we use the notation *df* for the image of f in $\Gamma(X, \Omega^1_X)$.

Unfortunately, there is no known interpretation of $T^{\mathbb{Z}}_x X$ or $T_x X$ as the set of equivalence classes of "smooth" paths through x as in differential geometry. There is, however, an interpretation of a subset $T^{\mathbb{Z}}_x X$ as germs of functions $(\mathbb{R}_{\geq 0}, 0) \to (X, x)$. Recall that such a germ is a morphism $[0, \varepsilon) \to X$ for some $\varepsilon > 0$ that sends 0 to x, up the equivalence relation that allows to shrink the interval, i.e. restricting to $[0, \varepsilon')$ for $\varepsilon' < \varepsilon$ does not change the germ. To every germ $\gamma\colon (\mathbb{R}_{\geq 0}, 0) \to (X, x)$ we can associate the tangent vector $d_x\gamma(1) \in T^{\mathbb{Z}}_x X$, where we identify $T^{\mathbb{Z}}_0(\mathbb{R}_{\geq 0})$ with $\mathbb{Z}$ in the natural way. In fact, since affine linear functions on $\mathbb{R}_{\geq 0}$ are completely determined by the value and slope at 0, the germ γ is uniquely determined by $d_x\gamma(1)$. We define the *local cone* of X at x as the subset of $T_x X$ given by

$$\mathrm{LC}_x X \coloneqq \{\lambda \cdot d_0\gamma(1) \mid \lambda \in \mathbb{R}_{\geq 0},\ \gamma\colon (\mathbb{R}_{\geq 0}, 0) \to (X, x) \text{ a germ}\}$$

Proposition 2.5. *Let X be a rational polyhedral space, and let $x \in X$. Then $\mathrm{LC}_x X$ is a rational polyhedral subspace of $T_x X$ with tangent space*

$$T_0(\mathrm{LC}_x X) = T_x X$$

at the origin. Furthermore, there exists a unique morphism of germs

$$(\mathrm{LC}_x X, 0) \to (X, x)$$

such that the induced map $T_x X = T_0(\mathrm{LC}_x X) \to T_x X$ is the identity.

Proof. Since the definition of the local cone is local, we may assume that X is a polyhedral subset of $\mathbb{T}^n$ for some $n \in \mathbb{N}$, and $x = (x_1, \ldots, x_n)$. After a change of coordinates, we may further assume that there exists a $0 \leq k \leq n$ such that $x \in \{\infty\}^k \times \mathbb{R}^{n-k}$. For every connected open subset Y of a polyhedral set in $\mathbb{R}^n$, every morphism $Y \to \mathbb{T}^n$

whose image contains x has to map entirely into $\{\infty\}^k \times \mathbb{R}^{n-k}$. This applies in particular to open neighborhoods of 0 in $\mathbb{R}_{\geq 0}$ or $\mathrm{LC}_x X$, so both the local cone and the set of germs of morphisms $(\mathrm{LC}_x X, 0) \to (X, x)$ only depend on $X \cap (\{\infty\}^k \times \mathbb{R}^{n-k})$. The affine functions defined on a neighborhood of x are, after potentially shrinking the neighborhood, precisely those that are pullbacks of affine functions on $\mathbb{T}^{n-k}$ under the projection $X \to \mathbb{T}^{n-k}$ onto the last $n-k$ coordinates. Therefore, the tangent space of X at x only depends on $X \cap (\{\infty\}^k \times \mathbb{R}^{n-k})$ as well. After replacing X by $X \cap (\{\infty\}^k \times \mathbb{R}^{n-k})$, we may thus assume that $x \in \mathbb{R}^{n-k}$. In this case, the local cone at x is easily seen to be equal to the set

$$\{v \in \mathbb{R}^{n-k} \mid x + [0, \varepsilon)v \subseteq X \text{ for some } \varepsilon > 0\} ,$$

which is well-known to be a finite union of polyhedral cones, and in particular a polyhedral set. In this case, it is equally well known that x has a neighborhood in X that is isomorphic to a neighborhood of 0 in $\mathrm{LC}_x X$, so for the last part of the proof we may assume that $x = 0$ and $X = \mathrm{LC}_x X$. It follows immediately that $T_x X = T_0(\mathrm{LC}_x X)$ and that the identity map defines a germ of maps $(\mathrm{LC}_x X, 0) \to (X, x)$ inducing the identity on tangent spaces. Since such a germ is determined by the associated map on tangent spaces, this finishes the proof. □

Corollary 2.6. *Let X be a rational polyhedral space, and let $x \in X$. Then the local cone $\mathrm{LC}_x X$ spans the tangent space $T_x X$.*

Proof. Because $\mathrm{LC}_x X$ is invariant under scaling, a linear function on $T_x X$ vanishes on a neighborhood of 0 if and only if it vanishes on all of $\mathrm{LC}_x X$, which is true if and only if it vanishes on the span of $\mathrm{LC}_x X$. But by definition of the tangent space, a linear function on $T_x X$ vanishes on a neighborhood of 0 in $\mathrm{LC}_X X$ if and only if it vanishes on $T_0(\mathrm{LC}_x X)$. By Proposition 2.5, this shows that $\mathrm{LC}_x X$ spans $T_x X = T_0(\mathrm{LC}_x X)$. □

To define sheaves of tropical p-forms, one would like to take the p-th exterior power of Ω^1_X. Unfortunately, even for very well-behaved rational polyhedral spaces, $\bigwedge^p \Omega^1_X$ might very well be nontrivial for some $p > \dim(X)$. This is remedied with the following definition.

Definition 2.7. For a rational polyhedral space X we denote by $X^{\max}$ the set of points in X that has a neighborhood isomorphic to an open set in $\mathbb{R}^n$. By definition, $X^{\max}$ is an open subset of X. Let $\iota \colon X^{\max} \hookrightarrow X$ denote the inclusion. Then one defines the sheaf of graded rings Ω^*_X as the image of $\bigwedge^* \Omega^1_X$ in $\iota_* \left(\bigwedge^* \Omega^1_X|_{X^{\max}}\right)$. Sections of Ω^p_X are called *tropical p-forms.*

Note that $\Omega^1_X \to \iota_* \Omega^1_X|_{X^{\max}}$ is a monomorphism, so that the definition of Ω^1_X is unambiguous. Also note that for $p \in X^{\max}$ the rank of $\Omega^1_{X,p}$ equals the local dimension at p. Therefore, $\Omega^p_X = 0$ for $p > \dim X$.

Remark 2.8. The sheaf Ω^p_X of tropical p-forms is closely related to the sheaf $\mathscr{F}^p_X$ considered in [32]: given a point $x \in X$ and a local face structure Σ at x, the stalk $\mathscr{F}^p_{X,x}$ is the dual of the sublattice $\mathscr{F}_p^{X,x} = \sum \bigwedge^p T^{\mathbb{Z}}(\sigma)$ of $\bigwedge^p T^{\mathbb{Z}}_x X$, where $T^{\mathbb{Z}}(\sigma)$ denotes the integral tangent space of σ at any point in its relative interior, considered as a sublattice of $T^{\mathbb{Z}}_x X$. On the other hand, $\Omega^p_{X,x}$ is the sublattice of $\bigwedge^p \Omega^1_{X,x} = \bigwedge^p \mathrm{Hom}(T^{\mathbb{Z}}_x X, \mathbb{Z})$ consisting of all p-forms vanishing on $\mathscr{F}_p^{X,x}$. One concludes that $\Omega^p_{X,x}$ is the dual of the *saturation* of $\mathscr{F}_p^{X,x}$ in $\bigwedge^p T^{\mathbb{Z}}_x X$ and thus that $\Omega^p_{X,x}$ has finite index in $\mathscr{F}^p_{X,x}$. In particular, one has $\mathscr{F}^p_{X,x} = \Omega^p_{X,x}$ if $\mathscr{F}_p^{X,x}$ is saturated in $\bigwedge^p T^{\mathbb{Z}}_x X$ which happens, for example, if X is the tropical linear space associated to a loopless matroid (in this case, $\mathscr{F}^p_{X,x}$ is equal to the projective Orlik-Solomon algebra of a matroid [43]). In general, however, $\mathscr{F}^p_{X,x}$ and $\Omega^p_{X,x}$ do not coincide, not even for $p = 1$. A simple example where this happens is given by

$$X = \mathbb{R}\begin{pmatrix}1\\1\end{pmatrix} \cup \mathbb{R}\begin{pmatrix}1\\-1\end{pmatrix} ,$$

where $\Omega^1_{X,0}$ has index 2 in $\mathscr{F}^1_{X,0}$ because

$$\mathscr{F}_1^{X,0} = \mathbb{Z}\begin{pmatrix}1\\1\end{pmatrix} + \mathbb{Z}\begin{pmatrix}1\\-1\end{pmatrix}$$

has index 2 in $T^{\mathbb{Z}}_0 X = \mathbb{Z}^2$.

We use the sheaves Ω^p_X rather than the sheaves $\mathscr{F}^p_{\mathbb{Z}}$ because it is Ω^1_X rather than $\mathscr{F}^1_{\mathbb{Z}}$ that appears in the *tropical exponential sequence*

$$0 \to \mathbb{R}_X \to \mathrm{Aff}_X \to \Omega^1_X \to 0 .$$

We chose the notation Ω^p_X because it is clearly distinguishable from $\mathscr{F}^p_X$ and also because it stresses the analogy to algebraic geometry. It should be noted that our sheaves Ω^p_X do coincide with the sheaves $\mathscr{F}^p_{\mathbb{Z}}$ defined in the post-published version of [22].

Example 2.9. Let

$$X = \mathbb{R}_{\geq 0}(1,0) \cup \mathbb{R}_{\geq 0}(0,1) \cup \mathbb{R}_{\geq 0}(-1,-1) \subseteq \mathbb{R}^2$$

be the standard tropical line, depicted in Fig. 1. Then the stalk $\Omega^1_{X,0}$ of the sheaf of tropical 1-forms at the origin is isomorphic to the space of integer linear functions on $\mathbb{R}^2$, which we can identify with $\mathbb{Z}^2$. For every $x \in X$ not equal to 0, the stalk $\Omega^1_{X,x}$ is isomorphic to $\mathbb{Z}$. The set $X^{\max}$ is the complement of the origin, and the restriction $\Omega^1_X|_{X^{\max}}$ is locally free of rank 1. In particular, $\bigwedge^n \left(\Omega^1_X|_{X^{\max}}\right) = 0$ for all $n > 1$, and hence $\Omega^n_X = 0$ for $n > 1$. On the other hand, we have

$$\left(\bigwedge^2 \Omega^1_X\right)_0 \cong \bigwedge^2 \Omega^1_{X,0} \cong \bigwedge^2 \mathbb{Z}^2 \neq 0 .$$

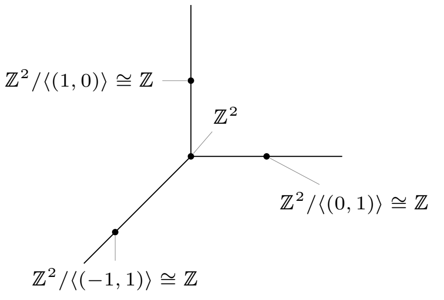

Fig. 1. The standard tropical line and the stalks of its sheaf of tropical 1-forms.

Finally, note that an integral linear function on $\mathbb{R}^2$ is completely determined by its slopes in the directions $(1,0)$ and $(0,1)$, and hence the natural morphism $\Omega^1_{X,0} \to (\iota_* \Omega^1_X|_{X^{\max}})_0$, where $\iota\colon X^{\max} \to X$ is the inclusion, is an embedding.

Example 2.10. Let N be the lattice generated by elements $e_0, \dots, e_3$ subject to the relation $\sum e_i = 0$, and let X be the union of the three half-planes in $N \otimes_{\mathbb{Z}} \mathbb{R}$ given by $H_i = \mathbb{R}e_0 + \mathbb{R}_{\geq 0} e_i$, where $i \in \{1,2,3\}$. Let $e_0^*, e_1^*, e_2^* \in \operatorname{Hom}(N, \mathbb{Z})$ be the dual basis to e_0, e_1, e_2. Then $\bigwedge^2 \Omega^1_{X,0}$ is freely generated by $e_0^* \wedge e_1^*$, $e_0^* \wedge e_2^*$, and $e_1^* \wedge e_2^*$. As both e_1^* and e_2^* vanish on e_0, the restrictions of $e_1^* \wedge e_2^*$ to the interiors of all three half planes H_i vanish. On the other hand, no linear combination of $e_0^* \wedge e_1^*$ and $e_0^* \wedge e_2^*$ vanishes on the interiors of all three half planes. Since $X^{\max} = \bigcup \mathring{H}_i$, we conclude that $\Omega^2_{X,0}$ is the quotient of $\bigwedge^2 \Omega^1_{X,0}$ by $\mathbb{Z}(e_1^* \wedge e_2^*)$.

Tropical p-forms can be pulled back along morphisms, as shown in the following proposition.

Proposition 2.11. *Let $f\colon X \to Y$ be a morphism of rational polyhedral spaces. Then the pullback*

$$f^\sharp\colon f^{-1}\Omega^1_Y \to \Omega^1_X$$

induces a pull-back

$$f^{-1}\Omega^*_Y \to \Omega^*_X \ ,$$

which we again denote by $f^\sharp$.

Proof. It is immediate that $f^\sharp$ induces a morphism

$$\bigwedge f^\sharp\colon f^{-1} \bigwedge \Omega^1_Y \to \bigwedge \Omega^1_X \ .$$

To see that this induces a morphism on the quotients $f^{-1}\Omega^*_Y \to \Omega^1_X$, we need to show that if $U \subset Y$ is open and we are given a section $\omega \in \Gamma(U, \bigwedge^* \Omega^1_Y)$ that restricts to zero on $U \cap Y^{\max}$, then $\bigwedge f^\sharp(\omega)$ vanishes on $f^{-1}U \cap X^{\max}$. Let $x \in f^{-1}U \cap X^{\max}$, and let Σ and Δ be local face structures around x and $f(x)$, respectively, such that $f(\sigma) \in \Delta$ for all $\sigma \in \Sigma$. Because $\bigwedge f^\sharp(\omega)$ is constant on a neighborhood of x, it suffices to show that it vanishes on a maximal cell $\sigma \in \Sigma$. We may thus replace X by σ, in which case f factors through the rational polyhedral space $f(\sigma)$. Since $f(\sigma) \in \Delta$, there exists a maximal cell $\delta \in \Delta$ containing $f(\sigma)$. As f factors through δ by construction, it then suffices to show that the pullback of ω to $\bigwedge^* \Omega^1_\delta$ vanishes. But as we assumed that the restriction of ω to the interior $\mathring{\delta}$ of δ vanishes, this follows from the fact that the sheaf $\bigwedge^* \Omega^1_\delta$ on δ is constant. □

3. Tropical cycles on rational polyhedral spaces

In this section we recall several well-known constructions from tropical intersection theory [29,5,14]. Since we will be working with the defining formulae for each construction, we will review them in some detail, both to establish notation and for the readers convenience.

3.1. Tropical cycles

To define tropical cycles on rational polyhedral spaces, we first need to recall their definition on affine space.

Definition 3.1. Let N be a lattice. A *tropical fan k-cycle* on $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ is an integer valued function $A\colon N_\mathbb{R} \to \mathbb{Z}$ such that

(1) For every $\lambda \in \mathbb{R}_{>0}$ and $x \in N_\mathbb{R}$ we have $A(\lambda x) = A(x)$,
(2) The *support* $|A| = \overline{\{x \in N_\mathbb{R} \mid A(x) \neq 0\}}$ of A is the support of a rational polyhedral fan in $N_\mathbb{R}$ of pure dimension k,
(3) A is locally constant on the open subset $|A|^{\max}$ of $|A|$ and 0 on $|A| \setminus |A|^{\max}$,
(4) A satisfies the so-called balancing condition: if Σ is a face structure on $|A|$ such that every $\sigma \in \Sigma$ is a cone, then A is constant on the relative interiors of the inclusion-maximal cells of Σ. Therefore, A and Σ define a weighted fan in the sense of Allermann and Rau [5]. We ask that this weighted fan satisfies the so-called balancing condition (see Remark 3.3), that is that it is a tropical fan in the sense of [5]. By [5, Lemma 2.11] this is independent of the choice of Σ.

Remark 3.2. It is immediate from the definition that every tropical fan cycle in the sense above defines a tropical fan cycle in the sense of [5]. If, conversely, $A = [(\Sigma, \omega)]$ is a tropical fan cycle in the sense of [5], where we use their notation here, then ω defines a locally constant integer-valued function on the subset of $|A|^{\max}$ consisting of the union of the relative interiors of all inclusion-maximal cones of Σ. This function can be extended uniquely to a locally constant function on all of $|A|^{\max}$ that is independent of the representative (Σ, ω) and, if extended by 0 to all of $N_\mathbb{R}$, is a tropical fan cycle in the sense of Definition 3.1.

Remark 3.3. We will rarely use the balancing-condition, but let us briefly recall its definition for the sake of being self-contained. If σ is a cone in $N_\mathbb{R}$ and τ is a codimension-1 face of σ, then a *lattice normal vector* for σ with respect to τ is an element $n \in \sigma \cap N$ such that the morphism

$$\bigwedge^{\dim(\tau)} T^\mathbb{Z}(\tau) \to \bigwedge^{\dim(\sigma)} T^\mathbb{Z}(\sigma), \quad \eta \mapsto n \wedge \eta$$

is an isomorphism, where the tangent spaces $T^\mathbb{Z}(\sigma)$ and $T^\mathbb{Z}(\tau)$ are taken at any point of the respective cones, and we consider them as sublattices of N. If Σ is a purely k-dimensional rational polyhedral fan in $N_\mathbb{R}$, and $\omega\colon \Sigma(k) \to \mathbb{Z}$ gives integer weights to its maximal cones, then (Σ, ω) satisfies the balancing condition if for every $\tau \in \Sigma(k-1)$ we have

$$\sum_{\sigma\colon\, \tau \subseteq \sigma \in \Sigma(k)} \omega(\sigma) n_{\sigma/\tau} \in T^\mathbb{Z}(\tau)$$

for any, and hence every, choice of lattice normal vectors $n_{\sigma/\tau}$ of σ with respect to τ.

Tropical fan k-cycles on $N_\mathbb{R}$ form an Abelian group. We remark that the sum of two such tropical cycles c and d is *not* simply the sum as integer-valued functions. This does hold, however, if we consider integer-valued functions modulo those functions whose support is a polyhedral set of dimension at most $k-1$.

Definition 3.4. Let X be a rational polyhedral space. We say that a function $A\colon X \to \mathbb{Z}$ is *locally constructible* if for every $x \in X$ there exists a local face structure Σ at x such that the restrictions $A|_{\mathrm{relint}(\sigma)}$ are constant for all $\sigma \in \Sigma$.

Every integer-valued function $A\colon X \to \mathbb{Z}$ on a rational polyhedral space X induces, at every $x \in X$, a function germ at the origin of the local cone $\mathrm{LC}_x X \subseteq T_x X$ via Proposition 2.5. If A is locally constructible, then for every $v \in \mathrm{LC}_x X$ the value $A(\varepsilon v)$ is independent of $\varepsilon > 0$, if chosen sufficiently small. This can be used to extend the germ to an $\mathbb{R}_{>0}$-invariant function $\mathrm{LC}_x X \to \mathbb{Z}$, which we extend by 0 to a function

$$\mathrm{LC}_x(A)\colon T_x X \to \mathbb{Z}\ .$$

Definition 3.5. Let X be a rational polyhedral space. A *tropical k-cycle* on X is a locally constructible function $A\colon X \to \mathbb{Z}$ such that $\mathrm{LC}_x(A)$ is a tropical fan k-cycle on $T_x X$ for all $x \in X$. The *support* of a tropical cycle A on X is the set $|A| = \overline{\{x \in X \mid A(x) \neq 0\}}$.

The addition for tropical fan k-cycles induces an addition for tropical k-cycles on a rational polyhedral space X so that there is an abelian group $Z_k(X)$ of tropical k-cycles.

Similarly as for tropical fan cycles, the sum $c + d$ of two tropical k-cycles agrees with the sum of A and B as integer-valued functions up to an integer-valued function whose support is a locally polyhedral subset of X of dimension at most $k - 1$. As both the definition of tropical k-cycles and the definition of the addition are local, the assignment $U \to Z_k(U)$ on open sets of X, with the obvious restriction morphisms, defines a sheaf $\mathscr{Z}_k^X$ of tropical k-cycles on X.

3.2. Proper push-forward of tropical cycles

Definition 3.6. Let A be a tropical k-cycle on a k-dimensional rational polyhedral space X, and let $f\colon X \to Y$ be a proper and surjective morphism of rational polyhedral spaces. Then for ever $y \in Y$ which is not an element of the at most $(k-1)$-dimensional locally polyhedral subset

$$f\left(X \setminus X^{\max}\right) \cup \left(Y \setminus Y^{\max}\right)$$

of Y, we define the *push-forward* of A along f as

$$f_*A(y) = \sum_{x \in f^{-1}\{y\}} [T_y^{\mathbb{Z}} X \colon d_x f(T_x^{\mathbb{Z}} X)] A(x) \ ,$$

where we consider the lattice index as 0 if it is not finite. We extend this function by 0 to a function on Y. Note that the sum over $f^{-1}\{y\}$ is in fact finite, since we can only get a nonzero contribution for isolated points of $f^{-1}\{y\}$, of which there can only be finitely many because f is proper.

In the general case, where X is not necessarily k-dimensional and f is not necessarily surjective (but still proper), we consider the (co)restriction $\widetilde{f}\colon |A| \to f|A|$ of f and define f_*A as the extension to Y by 0 of $\widetilde{f}_*A$.

If $f\colon X \to Y$ is a proper morphism of rational polyhedral spaces, and $A \in Z_k(X)$, then it follows immediately from the construction that f_*A is locally constructible. It is usually not a tropical cycle in the sense of Definition 3.5, but there is a unique tropical k-cycle B on Y such that f_*A and B coincide away from a locally polyhedral subset of dimension at most $(k-1)$. The uniqueness is clear, and for the existence part one only needs to show balancing, which can be proven locally and thus follows exactly as in [14, Proposition 2.25]. From this it is clear that the push-forward induces a morphism

$$Z_k(X) \to Z_k(Y)$$

of groups of tropical k-cycles, which, by abuse of notation, we denote by f_* as well.

3.3. Cross products of tropical cycles

Given two tropical cycles on two rational polyhedral spaces, one gets a tropical cycle on the product space by taking their cross-product [5].

Definition 3.7. Let X and Y be rational polyhedral spaces, and let $A \in Z_k(X)$ and $B \in Z_l(X)$. Then we define the *cross-product* of A and B as the function

$$A \times B \colon X \times Y \to \mathbb{Z},\ (x, y) \mapsto A(x) \cdot B(y) \ .$$

It is straightforward to check that this is a tropical $(k+l)$-cycle on $X \times Y$, and it is evident from the definition that the cross-product defines a bilinear map

$$Z_k(X) \times Z_l(Y) \to Z_{k+l}(X \times Y) \ .$$

3.4. Tropical Cartier divisors

Definition 3.8 *(see [22, Definition 4.1]).* Let X be a rational polyhedral space. A continuous function $\varphi \colon X \to \mathbb{R}$ is *tropically rational* if at every x in X there exists a local face structure Σ such that $\varphi|_\sigma \in \Gamma(\sigma, \mathrm{Aff}_\sigma)$ for all $\sigma \in \Sigma$. Sums of tropically rational functions are rational. We denote the group of tropically rational functions on X by $\mathscr{M}(X)$.

Remark 3.9. Tropically rational functions on a rational polyhedral space X are precisely the piecewise linear function on X with integral slopes. The terminology "rational" comes from the analogy with algebraic varieties, where the tropically rational functions play a similar role in the definition of divisors.

The condition on a function on a rational polyhedral space X to be tropically rational is a local condition. Therefore, the presheaf $U \mapsto \mathscr{M}(U)$ on X is in fact a sheaf, which we denote by $\mathscr{M}_X$. Every affine linear function on X is rational, so there is an inclusion $\mathrm{Aff}_X \hookrightarrow \mathscr{M}_X$. Its quotient is the sheaf $\mathscr{D}iv_X$ of Cartier divisors, that is $\mathscr{D}iv_X$ is defined as the unique sheaf fitting into a short exact sequence

$$0 \to \mathrm{Aff}_X \to \mathscr{M}_X \to \mathscr{D}iv_X \to 0 \ .$$

Definition 3.10. Let X be a rational polyhedral space. The group

$$\mathrm{Div}(X) := \Gamma(X, \mathscr{D}iv_X)$$

is the group of *Cartier divisors* on X. The *support* $|D|$ of $D \in \mathrm{Div}(X)$ is defined in the sheaf-theoretic sense as the support of D considered as a global section of $\mathscr{D}iv_X$.

If $f\colon X \to Y$ is a morphism of rational polyhedral spaces, then it is straightforward to check that for every tropically rational function φ on Y, the pull-back $f^*\varphi = \varphi \circ f$ is a tropically rational function on X. Since the pull-back of tropically rational functions is compatible with the pull-back of affine linear functions, we obtain a pull-back morphism

$$f^*\colon \operatorname{Div}(Y) \to \operatorname{Div}(X)$$

for Cartier divisors.

There is an intersection pairing

$$\operatorname{Div}(X) \times Z_k(X) \to Z_{k-1}(X)$$

on every rational polyhedral space X due to Allermann and Rau [5]. Let us briefly recall its construction. To define the product $D \cdot A$ of a divisor D with a tropical k-cycle A, we first pull back D to $|A|$, after which we can assume that $X = |A|$. We can then work locally around a point $x \in X$ and replace X by its local cone $\operatorname{LC}_x X$. This allows us to assume that $X = |\Sigma|$ for some rational polyhedral cone Σ in $\mathbb{R}^n$, that A is represented by a balanced weight function on the k-dimensional cones of Σ, and that D is represented by a piecewise linear function φ whose restrictions to the cones of Σ are linear. The intersection $D \cdot A$ is then represented by the weight $\Sigma(k-1) \to \mathbb{Z}$ that assigns to $\tau \in \Sigma(k-1)$ the integer

$$\sum_{\sigma\colon \tau \subseteq \sigma \in \Sigma(k)} \langle \varphi - l_\tau, n_{\sigma/\tau} \rangle A(\sigma) \ ,$$

which is independent of the choice of lattice normal vectors $n_{\sigma/\tau}$ (see Remark 3.3) and an integer linear function l_τ on $\mathbb{R}^n$ with $l_\tau|_\tau = \varphi|_\tau$.

Remark 3.11. Note that tropically rational functions on compact rational polyhedral spaces are bounded by continuity. If one allows tropically rational functions to obtain the value ∞, one obtains a less restrictive notion of tropical Cartier divisors, for which one should still be able to define the intersection pairing with tropical cycles, at least under some mild assumptions on the underlying rational polyhedral space. In the prototypical example of tropical toric varieties this has been done in [28].

3.5. Tropical line bundles

Following [31], we work with the following definition of tropical line bundles:

Definition 3.12. A *tropical line bundle* on a rational polyhedral space X is a morphism $Y \to X$ of rational polyhedral spaces such that locally on X there are identifications $Y \cong \mathbb{T} \times X$ of spaces over X. Two tropical line bundles are isomorphic if they are isomorphic as rational polyhedral spaces over X.

Note that the only automorphisms of $\mathbb{T}$ are the ones of the form $x \mapsto \lambda + x$ for some $\lambda \in \mathbb{R}$. Therefore, the automorphism group of $\mathbb{T} \times U$ is naturally isomorphic to $\Gamma(U, \mathrm{Aff}_U)$ for any rational polyhedral space U. Using standard arguments involving Čech cohomology, this leads to the following description of the set of isomorphism classes of tropical line bundles on a rational polyhedral space:

Proposition 3.13 *(cf. [31]). Let X be a rational polyhedral space. Then there is a natural bijection between the set of all isomorphism classes of tropical line bundles on X and the cohomology group $H^1(X, \mathrm{Aff}_X)$. In particular, the set of isomorphism classes of tropical line bundles on X is a group.*

If X is a rational polyhedral space, then the first boundary map in the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathrm{Aff}_X \to \mathscr{M}_X \to \mathscr{D}\!\mathit{iv}_X \to 0$$

associates to every Cartier divisor $D \in H^0(X, \mathscr{D}\!\mathit{iv}_X)$ a tropical line bundle

$$\mathscr{L}(D) \in H^1(X, \mathrm{Aff}_X) \ .$$

Remark 3.14. It follows from [22, Lemma 4.5] that every tropical line bundle is of this form if X admits a face structure. We expect this to remain true even in the absence of face structures, but will neither prove nor use this fact in the remainder of this paper.

If $f\colon X \to Y$ is a morphism between rational polyhedral spaces, then applying H^1 to the pull-back morphism $f^\sharp\colon f^{-1}\,\mathrm{Aff}_Y \to \mathrm{Aff}_X$ induces a pull-back morphism

$$f^*\colon H^1(Y, \mathrm{Aff}_Y) \to H^1(X, \mathrm{Aff}_X)$$

for tropical line bundles.

Proposition 3.15. *Let $f\colon X \to Y$ be a morphism between rational polyhedral spaces, and let $D \in \mathrm{Div}(Y)$ be a Cartier divisor on Y. Then we have*

$$f^*\mathscr{L}(D) = \mathscr{L}(f^*D) \ .$$

Proof. This follows immediately from the commutativity of the diagram

$$\begin{array}{ccc}
H^0(Y, \mathscr{D}\!\mathit{iv}_Y) & \longrightarrow & H^1(Y, \mathrm{Aff}_Y) \\
\big\downarrow f^{-1} & & \big\downarrow f^{-1} \\
H^0(X, f^{-1}\,\mathscr{D}\!\mathit{iv}_Y) & \longrightarrow & H^1(X, f^{-1}\,\mathrm{Aff}_Y) \\
\big\downarrow & & \big\downarrow H^1(f^\sharp) \\
H^0(X, \mathscr{D}\!\mathit{iv}_X) & \longrightarrow & H^1(X, \mathrm{Aff}_X)
\end{array} ,$$

where the horizontal morphisms are the first boundary maps in the long exact cohomology sequences associated to the short exact sequences

$$\begin{array}{ll} 0 \longrightarrow \mathrm{Aff}_Y \longrightarrow \mathscr{M}_Y \longrightarrow \mathscr{Div}_Y \longrightarrow 0 & , \\ 0 \to f^{-1}\,\mathrm{Aff}_Y \longrightarrow f^{-1}\,\mathscr{M}_Y \longrightarrow f^{-1}\,\mathscr{Div}_Y \to 0 & , \text{ and} \\ 0 \longrightarrow \mathrm{Aff}_X \longrightarrow \mathscr{M}_X \longrightarrow \mathscr{Div}_X \longrightarrow 0 & , \end{array}$$

respectively. □

4. Tropical (co)homology and its functorial properties

This section is devoted to giving a sheaf-theoretic definition of the tropical homology groups of [20,32,22] and using our understanding of sheaves to study the functorial behavior of tropical homology. In Theorem 4.20 we show that our definition of tropical homology agrees with that of [22].

Notation and general references. We will denote the constant sheaf associated to an abelian group A on a topological space X by A_X. If $\mathscr{F}$ is any sheaf on X and $S \subseteq X$ is a locally closed subset, we will denote $\mathscr{F}_S = \iota_!\iota^{-1}\mathscr{F}$, where $\iota\colon S \to X$ denotes the inclusion. For an abelian group A, we will sometimes denote $(A_X)_S$ by A_S if X is clear from the context. We will denote the group of morphisms between two $\mathbb{Z}_X$-modules (sheaves of abelian groups on X, that is) $\mathscr{F}$ and $\mathscr{G}$ by $\mathrm{Hom}_{\mathbb{Z}_X}(\mathscr{F}, \mathscr{G})$, where we will omit the subscript $\mathbb{Z}_X$ if X is clear from the context. The bounded derived category of $\mathbb{Z}_X$-modules will be denoted by $D(\mathbb{Z}_X)$, and we will omit the subscript X if X is a point, that is $D(\mathbb{Z})$ denotes the bounded derived category of abelian groups. If $\mathscr{C}^\bullet$ and $\mathscr{D}^\bullet$ are two cochain complexes of $\mathbb{Z}_X$-modules, then $\mathrm{Hom}_{D(\mathbb{Z}_X)}(\mathscr{C}^\bullet, \mathscr{D}^\bullet)$ denotes the group of morphisms between them in $D(\mathbb{Z}_X)$. As usual $\mathrm{Hom}^\bullet(\mathscr{C}^\bullet, \mathscr{D}^\bullet)$ denotes the Hom-complex and $R\,\mathrm{Hom}^\bullet(\mathscr{C}^\bullet, \mathscr{D}^\bullet)$ the derived Hom-complex, and similarly for the internal hom $\mathscr{Hom}$. The i-th cohomology sheaf of $\mathscr{C}^\bullet$ will be denoted by $H^i(\mathscr{C}^\bullet)$, whereas the i-th hypercohomology will be denoted by $\mathbb{H}^i(\mathscr{C}^\bullet)$.

A family of supports on a topological space X is a set Φ of closed subsets of X that is closed under taking closed subsets and finite unions.

For background on sheaf theory and Verdier duality we refer the reader to [6,9,21,24]

4.1. Tropical cohomology

Tropical cohomology groups are defined in analogy to Dolbeault cohomology groups in algebraic geometry.

Definition 4.1. Let X be a rational polyhedral space, let $p, q \in \mathbb{Z}$, and let Φ be a family of supports. Then the (p, q)-*th tropical cohomology group with supports in* Φ is defined as

$$H^{p,q}_{\Phi}(X) = H^q_{\Phi}(X, \Omega^p_X) \ .$$

If Φ consists of all closed subsets of X we usually omit it. If Φ consists of all compact subsets of X we also denote the cohomology group by $H^{p,q}_c(X)$, and if Φ consists of all closed subsets of a given closed subset Z of X we also denote the cohomology group by $H^{p,q}_Z(X)$.

Remark 4.2. We will frequently use the isomorphism

$$H^{p,q}_{\Phi}(X) \cong \varinjlim_{Z \in \Phi} \mathrm{Hom}_{D(\mathbb{Z}_X)}(\mathbb{Z}_Z, \Omega^p_X[q])$$

and thus represent a tropical (p, q)-cohomology class $\alpha \in H^{p,q}_{\Phi}(X)$ by an arrow $\mathbb{Z}_Z \xrightarrow{\alpha} \Omega^p_X[q]$ in $D(\mathbb{Z}_X)$ for some $Z \in \Phi$.

4.2. Tropical Borel-Moore homology

Similarly to the definition of the classical (i.e. non-tropical) Borel-Moore homology, our definition of tropical Borel-Moore homology will utilize the dualizing complex.

The *dualizing complex* $\mathbb{D}_X$ of a rational polyhedral space X is an element of $D(\mathbb{Z}_X)$ representing the functor

$$D(\mathbb{Z}_X) \to D(\mathbb{Z})\colon A \mapsto \mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma_c A, \mathbb{Z}) \ ,$$

where $R\Gamma_c$ is the (total) right derived functor of taking global sections with compact support, and $\mathbb{Z}$ is considered as a complex concentrated in degree 0. The universal element of the representation, that is the image of $\mathrm{id}_{\mathbb{D}_X}$ under the isomorphism

$$\mathrm{Hom}_{D(\mathbb{Z}_X)}(\mathbb{D}_X, \mathbb{D}_X) \xrightarrow{\cong} \mathrm{Hom}_{D(\mathbb{Z})}(R\Gamma_c \mathbb{D}_X, \mathbb{Z}) \ ,$$

is called the *trace map* and we will denote it by $\int_X$.

Definition 4.3. Let X be a rational polyhedral space, let $p, q \in \mathbb{Z}$, and let Φ be a family of supports. We define the *(p, q)-th (integral) tropical homology with supports in* Φ as

$$H^{\Phi}_{p,q}(X) := \mathbb{H}^0_{\Phi} R\,\mathscr{H}\!om^{\bullet}(\Omega^p_X[q], \mathbb{D}_X) \ .$$

If Φ contains all closed subsets of X we also denote the homology group by $H^{BM}_{p,q}(X)$, where the superscript stands for Borel-Moore. Moreover, if Φ consists of all compact subsets of X we usually omit Φ from the notation of the homology group, and if Φ consists of all closed subsets of a given closed subset Z of X we denote the homology group by $H^Z_{p,q}(X)$.

Remark 4.4. We will often represent Borel-Moore homology classes by morphisms in the derived category. To do so, we use the identification

$$\mathbb{H}^0_Z R\mathscr{H}om^\bullet(\Omega^p_X[q], \mathbb{D}_X) \cong H^0 R\operatorname{Hom}^\bullet((\Omega^p_X)_Z[q], \mathbb{D}_X) \cong \operatorname{Hom}_{D(\mathbb{Z}_X)}((\Omega^p_X)_Z[q], \mathbb{D}_X)$$

that allows us to represent an element $\alpha \in H^\Phi_{p,q}(X) = \varinjlim_{Z\in\Phi} H^Z_{p,q}(X)$ by a morphism $(\Omega^p_X)_Z[q] \xrightarrow{\alpha} \mathbb{D}_X$ for some $Z \in \Phi$.

Remark 4.5. Using Verdier duality, one obtains an identification

$$H^{BM}_{p,q}(X) = \mathbb{H}^0 R\mathscr{H}om^\bullet(\Omega^p_X[q], \mathbb{D}_X) \cong$$
$$\cong H^0 R\operatorname{Hom}^\bullet(\Omega^p_X[q], \mathbb{D}_X) \cong H^{-q} R\operatorname{Hom}^\bullet(R\Gamma_c\Omega^p_X, \mathbb{Z}) \ .$$

The dualizing complex $\mathbb{D}_X$ on a rational polyhedral space can be described explicitly in terms of sheaves of singular chains.

Definition 4.6 *(see [40, §VI], [9]).* Let X be a rational polyhedral space.

(a) For $i \in \mathbb{N}$, let Δ^i_X denote the sheafification of the presheaf

$$U \mapsto C_{-i}(X, X \setminus U) \ ,$$

where $C_j(A, B)$ denotes the group of relative singular j-chains with $\mathbb{Z}$-coefficients of the pair $A \supseteq B$. With the usual (co)boundary maps we obtain a cochain complex $\Delta^\bullet_X$.

(b) The i-th homology sheaf $\mathscr{H}^i_X := H^{-i}(\Delta^\bullet_X)$ is the sheafification of the presheaf

$$U \mapsto H_i(X, X \setminus U) \ ,$$

whose stalk at $x \in X$ is canonically identified with $H_i(X, X \setminus \{x\})$.

The complex $\Delta^\bullet_X$ is homotopically fine [40, VI, Proposition 7], which implies that the natural morphism

$$\Gamma_c\Delta^\bullet_X \to R\Gamma_c\Delta^\bullet$$

is a quasi-isomorphism. The global sections with compact support of Δ^{-i}_X are naturally identified with $C_i(X)$, so the natural augmentation $C_\bullet(X) \to \mathbb{Z}$ defined by taking degrees of 0-chains defines a morphism $\Delta^\bullet_X \to \mathbb{D}_X$ in the derived category (using the universal property of $\mathbb{D}_X$), which is well-known to be an isomorphism. In particular, we have $\mathscr{H}^i_X \cong H^{-i}(\mathbb{D}_X)$. This is one way of seeing that $H^i(\mathbb{D}_X) = 0$ for $i < -\dim(X)$.

Remark 4.7. If X is a rational polyhedral space with a face structure Σ, the dualizing complex has a completely combinatorial description due to Shepard [38]. More precisely, $\mathbb{D}_X$ is quasi-isomorphic to the complex which is given by $\bigoplus_{\sigma\in\Sigma(k)} \mathbb{Z}_\sigma$ in degree $-k$, where $\Sigma(k)$ denotes the set of all k-dimensional polyhedra in Σ. To define the differentials, one needs to pick orientations on all $\sigma \in \Sigma$. Having done this, the differential is given by the homomorphism

$$\bigoplus_{\sigma\in\Sigma(k)} \mathbb{Z}_\sigma \to \bigoplus_{\tau\in\Sigma(k-1)} \mathbb{Z}_\tau$$

that is given by 0 between the components $\mathbb{Z}_\sigma$ and $\mathbb{Z}_\tau$ if $\tau \nsubseteq \sigma$, and otherwise by multiplication by $\varepsilon_{\sigma/\tau}$, where

$$\varepsilon_{\sigma/\tau} = \begin{cases} 1 & \text{, if the orientations on } \sigma \text{ and } \tau \text{ agree} \\ -1 & \text{, else.} \end{cases}$$

Lemma 4.8.

(a) *The classical q-th (non-tropical) Borel-Moore homology group of X is isomorphic to $H^{BM}_{0,q}(X)$.*
(b) *Let Z be a locally polyhedral subset of dimension d, and let $\iota\colon Z \to X$ denote the inclusion. Then for all $p \in \mathbb{N}$ we have*

$$H^Z_{p,d}(X) = \operatorname{Hom}_{\mathbb{Z}_X}(\Omega^p_X, \iota_*\mathscr{H}^d_Z)\ .$$

In particular, the presheaf $U \mapsto H^{U\cap Z}_{p,d}(U)$ on X is a sheaf on X.

Proof. For (a) we use the fact that $\Omega^0_X = \mathbb{Z}_X$ and the natural isomorphisms

$$H^{BM}_{0,q}(X) = H^0 R\operatorname{Hom}^\bullet(\mathbb{Z}_X[q], \mathbb{D}_X) \cong H^{-q}R\Gamma\mathbb{D}_X = \mathbb{H}^{-q}\mathbb{D}_X\ .$$

By definition, $\mathbb{H}^{-q}\mathbb{D}_X$ is the q-th classical Borel-Moore homology group of X (as introduced in [8]).

For part (b) we can use Remark 4.4 and the universal property of the dualizing complex, to obtain the isomorphism

$$H^Z_{p,d}(X) \cong \operatorname{Hom}_{D(\mathbb{Z}_X)}((\Omega^p_X)_Z[d], \mathbb{D}_X) \cong \operatorname{Hom}_{D(\mathbb{Z}_Z)}(\Omega^p_X|_Z[d], \mathbb{D}_Z)\ .$$

Since $d = \dim(Z)$, the cohomology groups $H^i(\mathbb{D}_Z)$ vanish for $i < -d$. Therefore, $\mathbb{D}_Z$ is quasi-isomorphic to a complex of injectives that is 0 in degrees $< -d$. It follows that

$$\operatorname{Hom}_{D(\mathbb{Z}_Z)}(\Omega^p_X|_Z[d], \mathbb{D}_Z) \cong \operatorname{Hom}_{\mathbb{Z}_Z}(\Omega^p_X|_Z, H^{-d}(\mathbb{D}_Z))\ ,$$

which equals $\mathrm{Hom}_{\mathbb{Z}_Z}(\Omega^p_X|_Z, \mathscr{H}^d_Z)$ by definition of $\mathscr{H}^d_Z$. This in turn is isomorphic to $\mathrm{Hom}_{\mathbb{Z}_X}(\Omega^p_X, \iota_*\mathscr{H}^d_Z)$. For the "in particular"-statement we note that for every open subset $U \subseteq X$ we have $\mathbb{D}_U \cong \mathbb{D}_X|_U$. Therefore, the presheaf $U \mapsto H^{U\cap Z}_{p,d}(U)$ is isomorphic to the presheaf $U \mapsto \mathrm{Hom}_{\mathbb{Z}_U}(\Omega^p_X|_U, \iota_*\mathscr{H}^d_Z|_U)$, which equals the sheaf $\mathscr{H}om_{\mathbb{Z}_X}(\Omega^p_X, \iota_*\mathscr{H}^d_Z)$. □

4.3. Pull-backs

Let $f\colon X \to Y$ be a morphism of rational polyhedral spaces. Recall from Proposition 2.11 that pulling back tropical forms defines a morphism of graded sheaves of rings

$$f^\sharp\colon f^{-1}\Omega^*_Y \to \Omega^*_X \ .$$

Let $(\mathbb{Z}_Y \xrightarrow{c} \Omega^p_Y[q]) \in H^{p,q}(Y)$ be a tropical (p,q)-cohomology class. As the pull-back f^{-1} of sheaves of abelian groups defines an exact functor, it induces a functor $f^{-1}\colon D(\mathbb{Z}_Y) \to D(\mathbb{Z}_X)$. Applying this functor to c and composing the resulting arrow with $f^\sharp$ defines the *pull-back*

$$f^*c \in H^{p,q}(X) \ .$$

In other words, f^*c is represented by the composite

$$\mathbb{Z}_X \cong f^{-1}\mathbb{Z}_Y \xrightarrow{f^{-1}c} f^{-1}\Omega^p_Y[q] \xrightarrow{f^\sharp[q]} \Omega^p_X[q] \ .$$

The map $f^*\colon H^{p,q}(Y) \to H^{p,q}(X)$ is a morphism of abelian groups. If V is a closed subset of Y, then $f^{-1}\mathbb{Z}_W = \mathbb{Z}_{f^{-1}W}$. So the pull-back morphism can be refined to a morphism $H^{p,q}_W(Y) \to H^{p,q}_{f^{-1}W}(X)$. More generally, if Ψ is a family of supports on Y and we denote by $f^{-1}(\Psi)$ the family of supports on X consisting of all closed subsets of sets of the form $f^{-1}W$ for some $W \in \Psi$, then there is a pull-back morphism

$$f^*\colon H^{p,q}_\Psi(Y) \to H^{p,q}_{f^{-1}\Psi}(X) \ .$$

4.4. Proper push-forwards

If $f\colon X \to Y$ is a proper morphism of rational polyhedral spaces, then precomposing the trace $\int_X\colon R\Gamma_c\mathbb{D}_X \to \mathbb{Z}$ with the natural isomorphism $R\Gamma_c \circ Rf_*(\mathbb{D}_X) \xrightarrow{\cong} R\Gamma_c\mathbb{D}_X$ defines a morphism $R\Gamma_c(Rf_*\mathbb{D}_X) \to \mathbb{Z}$. By the universal property of the dualizing complex $\mathbb{D}_Y$, this corresponds to a morphism

$$Rf_*\mathbb{D}_X \to \mathbb{D}_Y \ . \tag{4.1}$$

Together with the composite

$$\Omega^p_Y \to f_*\Omega^p_X \to Rf_*\Omega^p_X \tag{4.2}$$

obtained by pulling back tropical p-forms, this defines a push-forward on tropical Borel-Moore homology:

Definition 4.9. Let $f\colon X \to Y$ be a proper morphism of rational polyhedral spaces, and let $p, q \in \mathbb{N}$. The *pushforward map*

$$f_*\colon H^{BM}_{p,q}(X) \to H^{BM}_{p,q}(Y)$$

associated to f is the composite of the morphism

$$\mathrm{Hom}_{D(\mathbb{Z}_X)}(\Omega^p_X[q], \mathbb{D}_X) \to \mathrm{Hom}_{D(\mathbb{Z}_Y)}(Rf_*\Omega^p_X[q], Rf_*\mathbb{D}_X)$$

obtained by taking the derived push-forward and the morphism

$$\mathrm{Hom}_{D(\mathbb{Z}_Y)}(Rf_*\Omega^p_X[q], Rf_*\mathbb{D}_X) \to \mathrm{Hom}_{D(\mathbb{Z}_Y)}(\Omega^p_Y[q], \mathbb{D}_Y)$$

defined via composition with the natural morphisms $\Omega^p_Y \to Rf_*\Omega^p_X$ in (4.2) and $Rf_*\mathbb{D}_X \to \mathbb{D}_Y$ in (4.1).

For every closed subset Z of X, the morphism $\Omega^p_Y \to f_*\Omega^p_X$ induces a morphism $(\Omega^p_Y)_{f(Z)} \to f_*((\Omega^p_X)_Z)$. Therefore, the push-forward map can be refined to respect supports; there is a push-forward morphism $H^Z_{p,q}(X) \to H^{f(X)}_{p,q}(Y)$. More generally, if Φ and Ψ are families of supports on X and Y, respectively, such that $\Phi \subseteq f^{-1}\Psi$, then there is a push-forward morphism

$$f_*\colon H^\Phi_{p,q}(X) \to H^\Psi_{p,q}(Y) .$$

It follows immediately from the functoriality of the derived push-forward and the pull-back of tropical forms that the push-forward on tropical Borel-Moore homology is functorial, that is $(f \circ g)_* = f_* \circ g_*$ whenever f and g are composable proper morphisms of rational polyhedral spaces.

For a better understanding of the push-forward we will need the following lemma:

Lemma 4.10. *Let $f\colon X \to Y$ be a proper morphisms of rational polyhedral spaces, and let $n = \dim X$. Then the morphism*

$$f_*\mathscr{H}^n_X \to \mathscr{H}^n_Y$$

induced by the natural morphism $Rf_*\mathbb{D}_X \to \mathbb{D}_Y$ in (4.1) is induced by the push-forwards $H_n(X, X \setminus f^{-1}U) \to H_n(Y, U)$ of relative singular cycles for open subsets $U \subseteq Y$.

Proof. We will describe the morphism $Rf_*\Delta^\bullet_X \to \Delta^\bullet_Y$ induced by the natural morphism $Rf_*\mathbb{D}_X \to \mathbb{D}_Y$ explicitly. To do so, we will first need to describe $Rf_*\Delta^\bullet_X$. Note that the barycentric subdivision defines an endomorphism on $\Delta_X^\bullet$. Let $\mathscr{S}_X^\bullet = \varinjlim_{\mathbb{N}} \Delta_X^\bullet$ denote the direct limit obtained by allowing repeated barycentric subdivision, and define $\mathscr{S}_Y^\bullet$ similarly. The natural morphism $\Delta_X^\bullet \to \mathscr{S}_X^\bullet$ is a quasi-isomorphisms since taking homology commutes with direct limits and the barycentric subdivision is a quasi-isomorphism. In particular, $Rf_*\Delta_X^\bullet = Rf_*\mathscr{S}_X^\bullet$. By [9, Prop. V-1.8 and Thm. V-12.14], $\mathscr{S}_X^\bullet$ is a complex of soft sheaves, so the natural morphisms $f_*\mathscr{S}_X^\bullet \to Rf_*\mathscr{S}_X^\bullet$ and $\Gamma_c\mathscr{S}_X^\bullet \to R\Gamma_c\mathscr{S}_X^\bullet$ are quasi-isomorphisms.

Next, we will show that the push-forward of relative singular cycles induces a morphism $f_*\mathscr{S}_X^\bullet \to \mathscr{S}_Y^\bullet$. If $\mathscr{S}_X^{pre,i}$ denotes the presheaf $U \mapsto \varinjlim_{\mathbb{N}} C_{-i}(X, X \setminus U)$ on X, and $\mathscr{S}^{pre,i}$ the analogous presheaf on Y, the push-forwards of relative chains $C_i(X, X \setminus f^{-1}U) \to C_i(Y, Y \setminus U)$ for $U \subseteq Y$ open and $i \in \mathbb{Z}$ induce a morphism $f_*\mathscr{S}_X^{pre,\bullet} \to \mathscr{S}_Y^{pre,\bullet}$ by the functoriality of the barycentric subdivision. Note that this does *not* automatically induce a morphism of complexes of sheaves $f_*\mathscr{S}_X^\bullet \to \mathscr{S}_Y^\bullet$ because push-forward does a priori not commute with sheafification. However using [9, V-Lemma 1.7] one sees that the presheaves $\mathscr{S}_X^{pre,i}$ have the property that for every compact set $K \subseteq X$ there is an equality

$$\varinjlim_{K \subseteq U} \mathscr{S}_X^{pre,i}(U) = \varinjlim_{K \subseteq U} \mathscr{S}_X^i(U) = \mathscr{S}_X^i(K) \ ,$$

where the direct limits are taken over all open subsets containing K (note that the second equality holds for every sheaf). Applying this to the fibers of f we obtain isomorphisms of stalks $(f_*\mathscr{S}_X^{pre,i})_y \cong \mathscr{S}_X^i(f^{-1}\{y\}) \cong (f_*\mathscr{S}_X^i)_y$ for all y, and hence sheafification commutes with push-forwards for the presheaves $\mathscr{S}_X^{pre,i}$. Consequently, the push-forward of relative singular cycles does in fact induce a morphism $f_*\mathscr{S}_X^\bullet \to \mathscr{S}_Y^\bullet$. To show that this coincides with the natural morphism $Rf_*\mathbb{D}_X \to \mathbb{D}_Y$ we apply $R\Gamma_c$ and obtain a morphism

$$\Gamma_c\mathscr{S}_X^\bullet = \Gamma_c f_*\mathscr{S}_X^\bullet \to \Gamma_c\mathscr{S}_Y^\bullet \ ,$$

which is the one induced by pushing forward singular chains. In particular, in degree 0 it is the morphism

$$C_0(X) \to C_0(Y)$$

that pushes forward points along f. This commutes with the degree morphisms to $\mathbb{Z}$, which are the traces defining the isomorphisms $\mathscr{S}_X^\bullet \cong \mathbb{D}_X$ and $\mathscr{S}_Y^\bullet \cong \mathbb{D}Y$. So by the definition of the natural morphism $Rf_*\mathbb{D}_X \to \mathbb{D}_Y$ it must agree with the morphism $f_*\mathscr{S}_X^\bullet \to \mathscr{S}_Y^\bullet$ obtained by pushing forward relative chains. From this description it is clear that the morphism

$$f_*\mathscr{H}_X^n \to \mathscr{H}_Y^n$$

is induced by the push-forwards of relative singular cycles as well. □

4.5. Cross products and the Künneth theorem

In this section we study the tropical homology group on a product $X \times Y$ of two rational polyhedral spaces X and Y. Let $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ denote the projections. In what follows we will use the notation

$$\mathscr{F} \boxtimes \mathscr{G} = p_X^{-1} \mathscr{F} \otimes_{\mathbb{Z}_{X\times Y}} p_Y^{-1} \mathscr{G}$$

for sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y, and

$$\mathscr{C}^\bullet \boxtimes^L \mathscr{D}^\bullet = p_X^{-1} \mathscr{C}^\bullet \otimes^L_{\mathbb{Z}_{X\times Y}} p_Y^{-1} \mathscr{D}^\bullet$$

for complexes of sheaves $\mathscr{C}^\bullet$ on X and $\mathscr{D}^\bullet$ on Y, where $\otimes^L$ denotes the derived tensor product.

To define the cross-product in tropical homology, we first need to relate the dualizing complex of $X \times Y$ the dualizing complexes of the factors X and Y. The trace maps $\int_X \colon R\Gamma_c \mathbb{D}_X \to \mathbb{Z}$ and $\int_Y \colon R\Gamma_c \mathbb{D}_Y \to \mathbb{Z}$ induce a morphism

$$R\Gamma_c \mathbb{D}_X \otimes^L_{\mathbb{Z}} R\Gamma_c \mathbb{D}_Y \to \mathbb{Z} \otimes^L_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z} \ .$$

By the Künneth formula [21, VII-2.7], $R\Gamma_c \mathbb{D}_X \otimes^L R\Gamma_c \mathbb{D}_Y$ is naturally isomorphic to $R\Gamma_c(\mathbb{D}_X \boxtimes^L \mathbb{D}_Y)$, so by the universal property of $\mathbb{D}_{X\times Y}$ there is an induced morphism $\mathbb{D}_X \boxtimes^L \mathbb{D}_Y \to \mathbb{D}_{X\times Y}$. Indeed, this is an isomorphism by [6, V, 10.26].

It will be convenient for us later to have an explicit description of this isomorphism in terms of sheaves of singular chains.

Lemma 4.11. *Let X and Y be rational polyhedral spaces. Then for all $i, j \in \mathbb{N}$, the morphism*

$$\mathscr{H}^i_X \boxtimes \mathscr{H}^j_Y \to \mathscr{H}^{i+j}_{X\times Y}$$

defined via the natural isomorphism $\mathbb{D}_X \boxtimes^L \mathbb{D}_Y \xrightarrow{\cong} \mathbb{D}_{X\times Y}$ is the one induced by the relative cross products

$$H_i(X, X \setminus U) \otimes_{\mathbb{Z}} H_j(Y, Y \setminus V) \to H_{i+j}(X \times Y, (X \times Y) \setminus (U \times V))$$

for open subsets $U \subseteq X$ and $V \subseteq Y$.

Proof. The Eilenberg-Zilber map defines a morphism

$$C_\bullet(X) \otimes_Z C_\bullet(Y) \to C_\bullet(X \times Y)$$

that induces morphisms

$$C_\bullet(X, X\setminus U)\otimes_{\mathbb{Z}} C_\bullet(Y, Y\setminus V)\to C_\bullet(X\times Y, (X\times Y)\setminus(U\times V))$$

for all pairs of open subsets $U\subseteq X$ and $V\subseteq Y$. Since the products $U\times V$ for $U\subseteq X$ and $V\subseteq Y$ open form a basis for $X\times Y$, we obtain a morphism

$$\Delta_X^\bullet\boxtimes\Delta_Y^\bullet\to\Delta_{X\times Y}^\bullet$$

after sheafifying. By construction, the morphisms $\mathscr{H}_X^i\boxtimes\mathscr{H}_Y^j\to\mathscr{H}_{X\times Y}^{i+j}$ induced by this is defined by relative cross products. It thus suffices to show that this morphism describes the natural morphism $\mathbb{D}_X\boxtimes^L\mathbb{D}_Y\to\mathbb{D}_{X\times Y}$.

Since $\Delta_X^\bullet$ and $\Delta_Y^\bullet$ are complexes of flat sheaves, the natural isomorphisms $\Delta_X^\bullet\xrightarrow{\cong}\mathbb{D}_X$ and $\Delta_Y^\bullet\xrightarrow{\cong}\mathbb{D}_Y$ define an isomorphism

$$\Delta_X^\bullet\boxtimes\Delta_Y^\bullet\xrightarrow{\cong}\mathbb{D}_X\boxtimes^L\mathbb{D}_Y\ .$$

To finish the proof, we must show that the diagram

$$\begin{array}{ccc}\Delta_X^\bullet\boxtimes\Delta_Y^\bullet & \longrightarrow & \Delta_{X\times Y}^\bullet\\ \downarrow{\cong} & & \downarrow{\cong}\\ \mathbb{D}_X\boxtimes^L\mathbb{D}_Y & \longrightarrow & \mathbb{D}_{X\times Y}\end{array}$$

is commutative. By the universal property of $\mathbb{D}_{X\times Y}$, we can apply $R\Gamma_c$ and need to show that the two morphisms $C_{-\bullet}(X)\otimes_{\mathbb{Z}} C_{-\bullet}(Y)\to\mathbb{Z}$ in the diagram

$$\begin{array}{ccccccc}C_{-\bullet}(X)\otimes_{\mathbb{Z}} C_{-\bullet}(Y) & \xrightarrow{\cong} & R\Gamma_c(\Delta_X^\bullet\boxtimes\Delta_Y^\bullet) & \longrightarrow & C_{-\bullet}(X\times Y) & & \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\ R\Gamma_c\mathbb{D}_X\otimes_{\mathbb{Z}}^L R\Gamma_c\mathbb{D}_Y & \xrightarrow{\cong} & R\Gamma_c(\mathbb{D}_X\boxtimes^L\mathbb{D}_Y) & \longrightarrow & R\Gamma_c(\mathbb{D}_{X\times Y}) & \xrightarrow{\int_{X\times Y}} & \mathbb{Z}\ ,\end{array}$$

where the leftmost horizontal isomorphisms use the Künneth formula [21, VII-2.7]. Note that the left square of the diagram is commutative by the functoriality of the Künneth formula, coincide. By construction, the composite of the two morphisms in the top row is the Eilenberg-Zilber map. In particular the morphism $C_{-\bullet}(X)\otimes_{\mathbb{Z}} C_{-\bullet}(Y)\to\mathbb{Z}$ obtained by moving clockwise through the diagram assigns 1 to a pure tensor $[x]\otimes[y]\in C_0(X)\otimes_{\mathbb{Z}} C_0(Y)$ of 0-simplices (i.e. a point in $X\times Y$). On the other hand, by the definition of the natural morphism $\mathbb{D}_X\boxtimes^L\mathbb{D}_Y\to\mathbb{D}_{X\times Y}$, the composite of the three morphisms in the lower row is the tensor product of the traces on X and Y. By the definition of the morphisms $\Delta_X^\bullet\to\mathbb{D}_X$ and $\Delta_Y^\bullet\to\mathbb{D}_Y$ it follows that the morphism $C_{-\bullet}(X)\otimes_{\mathbb{Z}} C_{-\bullet}(Y)\to\mathbb{Z}$ obtained by moving counterclockwise through the diagram is the tensor product of the two augmentations $C_{-\bullet}(X)\to\mathbb{Z}$ and $C_{-\bullet}(Y)\to\mathbb{Z}$ defined by the degree of 0-cycles. This product also assigns to 1 to any pure tensor $[x]\otimes[y]\in C_0(X)\otimes C_0(Y)$. This finishes the proof $\square$

Remark 4.12. In the proof of Lemma 4.11, we did not use the fact that the Eilenberg-Zilber map $C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$ is a chain homotopy equivalence. If one incorporates this into the proof of the lemma carefully, then it also shows that the natural morphism $\mathbb{D}_X \boxtimes^L \mathbb{D}_Y \to \mathbb{D}_{X\times Y}$ is an isomorphism.

To construct the tropical cross product we also need to relate the sheaves of tropical forms on $X \times Y$ with the sheaves of tropical forms on the factors. The projections p_X and p_Y induce morphisms

$$p_X^\sharp : p_X^{-1}\Omega_X^* \to \Omega_{X\times Y}^* \quad , \quad p_Y^\sharp : p_Y^{-1}\Omega_Y^* \to \Omega_{X\times Y}^*$$

of sheaves of skew-commutative graded rings. These morphisms induce a morphism

$$p_X^\sharp \otimes p_Y^\sharp : \Omega_X^* \boxtimes \Omega_Y^* \to \Omega_{X\times Y}^*$$

of sheaves of skew-commutative graded rings, where we view $\Omega_X^* \boxtimes \Omega_Y^*$ as the skew tensor product (i.e. the usual tensor product of $\mathbb{Z}$-algebras with a slightly modified multiplication to make it skew-symmetric (cf. [11, p. 571])) of $p_X^{-1}\Omega_X^*$ and $p_Y^{-1}\Omega_Y^*$.

Lemma 4.13. *The morphism*

$$p_X^\sharp \otimes p_Y^\sharp : \Omega_X^* \boxtimes \Omega_Y^* \to \Omega_{X\times Y}^*$$

is an isomorphism.

Proof. This is obvious in degree 1: working in charts this comes down to the facts that the linear span of a product is the product of the linear spans and that the dual of a direct sum of lattices is the direct sum of the duals. Because the exterior product of a sum is the skew tensor product of the exterior products of the summands, and everything commutes with pullbacks, we obtain an isomorphism

$$\bigwedge^* \Omega_X^1 \boxtimes \bigwedge^* \Omega_Y^1 \xrightarrow{\cong} \bigwedge^* \Omega_{X\times Y}^1$$

induced by p_X and p_Y. What is left to show is that if $\alpha \boxtimes \beta$ vanishes on $(X\times Y)^{\max}$, then either α vanishes on $X^{\max}$ or β vanishes on $Y^{\max}$. Assume the opposite. Then there exists a point $x \in X^{\max}$ at which α is nonzero and a point $y \in Y^{\max}$ at which β is nonzero. But since the stalks of $\bigwedge^* \Omega_X^1$ are all free, this implies that $\alpha \boxtimes \beta$ is nonzero an (x, y), which is a point in $(X \times Y)^{\max}$, a contradiction. □

Since $p_X^\sharp \otimes p_Y^\sharp$ is an isomorphism, we obtain a canonical splitting of the inclusion $\Omega_X^p \boxtimes \Omega_Y^{p'} \to \Omega_{X\times Y}^{p+p'}$ for every $p, p' \in \mathbb{N}$. Having established this we are ready to define the cross-product in tropical Borel-Moore homology.

Definition 4.14. Let X and Y be rational polyhedral spaces, and let $\alpha \in H^{BM}_{p,q}(X)$ and $\beta \in H^{BM}_{p',q'}(Y)$. Then the composite

$$\Omega^{p+p'}_{X\times Y}[q+q'] \to \Omega^p_X[q] \boxtimes \Omega^{p'}_Y[q'] \xrightarrow{\alpha \boxtimes^L \beta} \mathbb{D}_X \boxtimes^L \mathbb{D}_Y \cong \mathbb{D}_{X\times Y} ,$$

where the leftmost morphism is the natural splitting of $\Omega^p_X \boxtimes \Omega^{p'}_Y \to \Omega^{p+p'}_{X\times Y}$, defines an element $\alpha \times \beta \in H^{BM}_{p+p',q+q'}(X \times Y)$, the *cross product* of α and β. This defines a graded bilinear morphism

$$\times\colon H^{BM}_{*,*}(X) \otimes H^{BM}_{*,*}(Y) \to H^{BM}_{*,*}(X \times Y) .$$

By construction, the cross-product can be refined to respect supports; if Φ and Ψ are families of supports on X and Y, respectively, and $\Phi \times \Psi$ denotes the family of supports on $X \times Y$ consisting of closed subsets of sets of the form $V \times W$, where $V \in \Phi$ and $W \in \Psi$, there exists a bilinear map

$$\times\colon H^{\Phi}_{*,*}(X) \otimes H^{\Psi}_{*,*}(Y) \to H^{\Phi\times\Psi}_{*,*}(X \times Y)$$

As both the identification $\mathbb{D}_X \boxtimes^L \mathbb{D}_Y \cong \mathbb{D}_{X\times Y}$ and the pull-back of tropical forms is functorial, the same is true for cross-products. In other words, if $f\colon X \to X'$ and $g\colon Y \to Y'$ are proper morphisms of rational polyhedral spaces, then

$$f_*(\alpha) \times g_*(\beta) = (f \times g)_*(\alpha \times \beta)$$

for all $\alpha \in H^{BM}_{*,*}(X)$ and $\beta \in H^{BM}_{*,*}(Y)$.

We now turn our attention to a tropical Künneth formula. To prove it, we rely on results from [6] and need to make the mild assumption of the spaces involved being compactifiable. To state this assumption we note that every rational polyhedral space X has a natural filtration

$$\mathscr{X} : X = X_{\dim(X)} \supseteq X_{\dim(X)-1} \supset \ldots \supset X_0 ,$$

where X_i is obtained from X_{i+1} by removing the $(i+1)$-dimensional components of $(X_{i+1})^{\max}$. With this stratification X is an unrestricted pseudomanifold in the sense of [6].

Definition 4.15. A family of supports Φ on a rational polyhedral space X is *compactifiable* if for every $V \in \Phi$ there exists $W \in \Phi$ containing V and a refinement $\mathscr{X}'$ of the natural stratification on X such that $(X, \mathscr{X}')$ is a pseudomanifold, the set W is a union of strata of $(X, \mathscr{X}')$, and $(X, \mathscr{X}')$ is compactifiable in the sense of [6]: there exists a compact pseudomanifold $(Y, \mathscr{Y})$ containing $(X, \mathscr{X}')$ as a dense open sub-pseudomanifold. We say that X is compactifiable if the family of closed supports on X is compactifiable.

Theorem 4.16 *(Tropical Künneth theorem). Let X and Y be rational polyhedral spaces and let Φ and Ψ be compactifiable families of supports on X and Y, respectively. Then for every $p, q \in \mathbb{N}$, there is a natural decomposition $H^{\Phi}_{p,q}(X \times Y) = \bigoplus_{i+j=p} A_{i,j,q}$, and for each $i, j \in \mathbb{N}$ there is a short exact sequence*

$$0 \to \bigoplus_{k+l=q} H^{\Phi}_{i,k}(X) \otimes_{\mathbb{Z}} H^{\Psi}_{j,l}(Y) \to A_{i,j,q} \to \bigoplus_{k+l=q-1} \operatorname{Tor}_1^{\mathbb{Z}}(H^{\Phi}_{i,k}(X), H^{\Psi}_{j,l}(Y)) \to 0 \ .$$

Proof. By Lemma 4.13, there is a decomposition

$$R\mathscr{H}\!om^\bullet(\Omega^p_{X\times Y}, \mathbb{D}_{X\times Y}) \cong \bigoplus_{i+j=p} R\mathscr{H}\!om^\bullet(\Omega^i_X \boxtimes \Omega^j_Y, \mathbb{D}_{X\times Y}) \ .$$

We denote

$$A_{i,j,q} = \mathbb{H}^{-q}_{\Phi\times\Psi}(R\mathscr{H}\!om^\bullet(\Omega^i_X \boxtimes \Omega^j_Y, \mathbb{D}_{X\times Y}))$$

and for $V \in \Phi$ and $W \in \Psi$ we denote

$$A^{V,W}_{i,j,q} = \mathbb{H}^{-q}_{V\times W}(R\mathscr{H}\!om^\bullet(\Omega^i_X \boxtimes \Omega^j_Y, \mathbb{D}_{X\times Y})) \ .$$

Then we obtain a splitting

$$H^{\Phi\times\Psi}_{p,q}(X \times Y) = \bigoplus_{i+j=p} A_{i,j,q}$$

and we can write the summands as direct limits via

$$A_{i,j,q} = \varinjlim_{V,W} A^{V,W}_{i,j,q} \ .$$

Moreover, the sheaves Ω^i_X (resp. Ω^j_Y) are $\mathscr{X}$-cc (resp. $\mathscr{X}$-cc) in the sense of [6]. In particular, we may apply the results of [27][1] to X and Y and obtain that the natural morphism

$$R\mathscr{H}\!om^\bullet(\Omega^i_X, \mathbb{D}_X) \boxtimes R\mathscr{H}\!om^\bullet(\Omega^j_Y, \mathbb{D}_Y) \to R\mathscr{H}\!om^\bullet(\Omega^i_X \boxtimes \Omega^j_Y, \mathbb{D}_{X\times Y})$$

is in fact an isomorphism [27, Corollary 2.8]. Furthermore, since we may assume that $\mathbb{Z}_V$ is $\mathscr{X}'$-cc for some refinement $\mathscr{X}'$ of the canonical stratification $\mathscr{X}$ with $(X, \mathscr{X}')$ compactifiable, and similarly for $\mathbb{Z}_W$, for any pair of $\mathscr{X}$-cc (resp. $\mathscr{Y}$-cc) complexes $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ on X and Y the natural morphism

[1] The results in [27] are stated over $\mathbb{C}$, but their proofs only use identities from [6] that also hold over $\mathbb{Z}$.

$$R\Gamma_V \mathscr{F}^\bullet \otimes^L_{\mathbb{Z}} R\Gamma_W \mathscr{G}^\bullet = R\operatorname{Hom}^\bullet(\mathbb{Z}_V, \mathscr{F}^\bullet) \otimes^L_{\mathbb{Z}} R\operatorname{Hom}^\bullet(\mathbb{Z}_W, \mathscr{G}^\bullet) \to$$
$$\to R\operatorname{Hom}^\bullet(\mathbb{Z}_V \boxtimes \mathbb{Z}_W, \mathscr{F}^\bullet \boxtimes^L \mathscr{G}^\bullet) = R\Gamma_{V\times W}(\mathscr{F}^\bullet \boxtimes^L \mathscr{G}^\bullet)$$

is an isomorphism by [27, Corollary 4.2] (this is where the compactifiable condition is needed). Applying this to $\mathscr{F}^\bullet = R\mathscr{H}om^\bullet(\Omega^i_X, \mathbb{D}_X)$ and $\mathscr{G}^\bullet = R\mathscr{H}om^\bullet(\Omega^j_Y, \mathbb{D}_Y)$, we obtain

$$A^{V,W}_{i,j,q} = \mathbb{H}^{-q}_{V\times W}(R\mathscr{H}om^\bullet(\Omega^i_X, \mathbb{D}_X) \boxtimes^L R\mathscr{H}om^\bullet(\Omega^j_Y, \mathbb{D}_Y)) =$$
$$= \mathbb{H}^{-q}(R\Gamma_V\, \mathscr{H}om^\bullet(\Omega^i_X, \mathbb{D}_X) \boxtimes R\Gamma_W\, \mathscr{H}om^\bullet(\Omega^j_X, \mathbb{D}_Y))\ .$$

Now the Künneth theorem for complexes [41, Theorem 3.6.3] yields a short exact sequence

$$0 \to \bigoplus_{k+l=q} H^V_{i,k}(X) \otimes_{\mathbb{Z}} H^W_{j,l}(Y) \to A^{V,W}_{i,j,q} \to \bigoplus_{k+l=q-1} \operatorname{Tor}^{\mathbb{Z}}_1(H^V_{i,k}(X), H^W_{j,l}(Y)) \to 0\ ,$$

and taking the direct limit over all pairs (V, W) completes the proof. □

Corollary 4.17. *Let X and Y be compactifiable rational polyhedral spaces such that either $H^{BM}_{*,*}(X)$ or $H^{BM}_{*,*}(Y)$ is torsion free. Then the cross-product*

$$H^{BM}_{*,*}(X) \otimes_{\mathbb{Z}} H^{BM}_{*,*}(Y) \to H^{BM}_{*,*}(X \times Y)$$

is an isomorphism.

Proof. This follows immediately from Theorem 4.16 and the fact that the groups $\operatorname{Tor}^{\mathbb{Z}}_1(H^{BM}_{i,k}(X), H^{BM}_{j,l}(Y))$ all vanish. □

4.6. Cup and cap products

Let X be a rational polyhedral space. Since Ω^*_X is a sheaf of rings, its cohomology group has a ring structure again, the multiplication being the so-called *cup product*. If

$$(\mathbb{Z}_V \xrightarrow{c} \Omega^p_X[q]) \in H^{p,q}_V(X)\ \text{, and}$$
$$(\mathbb{Z}_W \xrightarrow{d} \Omega^{p'}_X[q']) \in H^{p',q'}_W(X)\ ,$$

where p, p', q, q' are integers and V and W are locally polyhedral subsets of X, then their cup product

$$c \smile d \in H^{p+p',q+q'}_{V\cap W}(X)$$

is represented by the composite

$$\mathbb{Z}_{V\cap W} \xrightarrow{\cong} \mathbb{Z}_V \otimes_{\mathbb{Z}_X} \mathbb{Z}_W \xrightarrow{c\otimes d} \Omega_X^p \otimes_{\mathbb{Z}_X} \Omega_X^{p'}[q+q'] \to \Omega_X^{p+p'}[q+q'] \ .$$

Here, the last morphism is the product on Ω_X^*, and the morphism in the middle can be obtained using the fact that $\Omega_X^p \otimes_{\mathbb{Z}_X} \Omega_X^{p'} \cong \Omega_X^p \otimes_{\mathbb{Z}_X}^L \Omega_X^{p'}$ because Ω_X^* is flat and the functoriality of the derived tensor product.

The cup product is clearly compatible with supports, so if Φ and Ψ are two families of supports, then the cup product defines a bilinear map

$$H_\Phi^{p,q}(X) \times H_\Psi^{p',q'}(X) \to H_{\Phi\cap\Psi}^{p+p',q+q'}(X) \ .$$

It follows directly from the associativity property of the sheaf of rings Ω_X^* that the cup product on $H^{*,*}(X)$ is associative. It is also unital, the unity being represented by the identity map on $\mathbb{Z}_X \to \mathbb{Z}_X = \Omega_X^0$. It is clear from the construction that the restriction of the cup product to $H^{0,*}(X)$ is the classical cup product on the cohomology of X (cf. Lemma 4.8 (a) and [21][II.9.9]).

Proposition 4.18. *Let $f\colon X \to Y$ be a morphism of rational polyhedral spaces. Then the pull-back (defined in §4.3)*

$$f^*\colon H^{*,*}(Y) \to H^{*,*}(X)$$

is a ring homomorphism.

Proof. Examining the definitions of cup products and pull-backs, we see that this directly follows from the fact that the pull-back $f^\sharp\colon f^{-1}\Omega_Y^* \to \Omega_X^*$ of tropical forms is a morphism of sheaves of graded rings. □

Similarly as the cup product, the *cap product* also generalizes from the classical to the tropical setting. To define it, let

$$(\mathbb{Z}_V \xrightarrow{c} \Omega_X^i[j]) \in H_V^{i,j}(X) \quad \text{, and}$$
$$((\Omega_X^p)_W[q] \xrightarrow{\alpha} \mathbb{D}_X) \in H_{p,q}^W(X) \ ,$$

where p, q, i, j are integers and V and W are closed subsets of X. Then the *cap product*

$$\alpha \frown c \in H_{p-i,q-j}^{V\cap W}(X)$$

is represented by the composite

$$(\Omega_X^{p-i})_{V\cap W}[q-j] \xrightarrow{\cong} \mathbb{Z}_V \otimes_{\mathbb{Z}_X} (\Omega_X^{p-i})_W[q-j] \xrightarrow{c\otimes \mathrm{id}} \Omega_X^i \otimes_{\mathbb{Z}_X} (\Omega_X^{p-i})_W[q] \to$$
$$\to (\Omega_X^p)_W[q] \xrightarrow{\alpha} \mathbb{D}_X \ ,$$

where the second arrow can be defined using the functoriality of the derived tensor product, and the third morphism is the product on Ω_X^*. This construction clearly is compatible with supports, so whenever Ψ and Φ are families of supports, we obtain a bilinear map

$$H_{p,q}^{\Psi}(X) \times H_{\Phi}^{i,j}(X) \to H_{p-i,q-j}^{\Phi\cap\Psi}(X) \ .$$

By the associativity property of the sheaf of rings Ω_X^*, the cap product makes $H_{*,*}^{BM}(X)$ a right $H^{*,*}(X)$-module. It is clear from the construction that the restriction

$$H_{0,*}^{BM}(X) \times H^{0,*}(X) \to H_{0,*}^{BM}(X)$$

of the cap product is the cap product in classical Borel-Moore homology (cf. Lemma 4.8 (a) and [21, IX.3]).

Proposition 4.19 *(Projection formula). Let $f\colon X \to Y$ be a proper morphism of rational polyhedral spaces, let $V \subseteq X$ and $W \subseteq Y$ be closed subsets let $\alpha \in H_{p,q}^V(X)$ and $c \in H_W^{i,j}(Y)$. Then we have the equality*

$$f_*(\alpha \frown f^*c) = f_*\alpha \frown c$$

in $H_{p-i,q-j}^{f(V)\cap W}(Y)$.

Proof. Both sides of the equation correspond to a morphism

$$(\Omega_Y^{p-i})_{f(V)\cap W}[q-j] \to \mathbb{D}_Y \ .$$

More precisely, the left side of the equation corresponds to the morphism obtained by moving counterclockwise along the square in the diagram below, whereas the right side of the equation corresponds to the morphism obtained by moving clockwise along the square.

$$\begin{array}{ccccc}
(\Omega_Y^{p-i})_{f(V)\cap W}[q-j] & \xrightarrow{c\otimes \mathrm{id}} & \Omega_Y^i \otimes_{\mathbb{Z}_Y} (\Omega_Y^{p-i})_{f(V)}[q] & \longrightarrow & (\Omega_Y^p)_{f(V)}[q] \\
\downarrow & & & & \\
Rf_*(\Omega_X^{p-i})_{V\cap f^{-1}W}[q-j] & & & & \\
\downarrow{\scriptstyle Rf_*(f^{-1}c\otimes \mathrm{id})} & & & & \downarrow \\
Rf_*(f^{-1}\Omega_Y^i \otimes_{\mathbb{Z}_X} (\Omega_X^{p-i})_V)[q] & \longrightarrow & Rf_*(\Omega_X^i \otimes_{\mathbb{Z}_X} (\Omega_X^{p-i})_V)[q] & \longrightarrow & Rf_*(\Omega_X^p)_V[q] \\
 & & & \swarrow{\scriptstyle Rf_*\alpha} & \\
 & & Rf_*\mathbb{D}_X & \longrightarrow & \mathbb{D}_Y
\end{array}$$

It thus suffices to show that the square commutes. Using the fact that f^{-1} and Rf_* are adjoints, this boils down to the fact that $f^\sharp\colon f^{-1}\Omega_Y^* \to \Omega_X^*$ is a morphism of sheaves of rings. □

4.7. Comparison with singular tropical homology

In this section we show that our sheaf-theoretically defined tropical Borel-Moore homology groups $H_{p,q}^{BM}(X)$ are naturally isomorphic to the locally finite tropical homology groups $H_{p,q}^{lf}(X)$ of X that are used in [22]. The analogous statement for tropical cohomology groups has been proven in [23], where the authors show that $H^q(X, \Omega_X^p)$ is isomorphic to the singular cohomology group $H_{\text{sing}}^{p,q}(X)$. It should be noted that if the tropical cohomology groups of X are finitely generated, the isomorphism $H_{\text{sing}}^{p,q}(X) \cong H^q(X, \Omega_X^p)$ from [23], combined with the universal coefficient theorem, implies that the tropical homology group with compact support $H_{p,q}(X)$ is isomorphic to the singular tropical homology group $H_{p,q}^{\text{sing}}(X)$. However, the isomorphism obtained this way is, a priori, not compatible with the cycle class map. More importantly, the isomorphism of tropical cohomology groups does *not* imply that $H_{p,q}^{BM}(X)$ is isomorphic to $H_{p,q}^{lf}(X)$.

Similar to the cohomological case treated in [23], the idea to prove $H_{p,q}^{BM}(X) \cong H_{p,q}^{lf}(X)$ is to follow closely the classical argument (see [9]) in the case $p = 0$. However, there is a problem with that: the classical argument uses that the complex $\Delta_X^\bullet$ is homotopically fine and therefore its cohomology (i.e. $H^*(\Delta_X) = H^* \operatorname{Hom}^\bullet(\mathbb{Z}_X, \Delta_X^\bullet)$) agrees with its hypercohomology (i.e. $\mathbb{H}^*(\Delta_X^\bullet) = \operatorname{Ext}^*(\mathbb{Z}_X, \Delta_X^\bullet)$). It does *not* follow directly from the homotopical fineness that for $p > 0$ the analogous statement holds, that is that $H^* \operatorname{Hom}^\bullet(\Omega_X^p, \Delta_X^\bullet) = \operatorname{Ext}^*(\Omega_X^p, \Delta_X^\bullet)$. To solve this we prove a more general statement (Proposition A.9) on conically stratified spaces, building on work of Friedman [13,12]

Let X be a rational polyhedral space, and let Σ be a face structure on X. We say that a singular simplex $\sigma\colon \Delta^q \to X$ (where Δ^q denotes the standard q-simplex) respects the face structure Σ if for every face $\Theta \subseteq \Delta^q$ there exists a polyhedron $P \in \Sigma$ such that $\sigma(\operatorname{relint}(\Theta)) \subseteq P$. A *tropical* (p, q)*-simplex (with respect to* Σ*)*, is a pair (σ, s), where $\sigma\colon \Delta^q \to X$ is a singular q-simplex respecting the face structure Σ and $s \in \operatorname{Hom}(\Omega_X^p|_{\sigma(\Delta^q)}, \mathbb{Z}_{\sigma(\Delta^q)})$. We denote by $C_{p,q}(X; \Sigma)$ the free abelian group generated by tropical (p, q)-simplices (w.r.t. Σ). If (σ, s) is a tropical (p, q)-simplex, then pulling back (σ, s) along the i-th face morphism $\delta^{q,i}\colon \Delta^{q-1} \to \Delta^q$ yields a tropical $(p, q-1)$-simplex $\partial_{p,q,i}(\sigma, s) = (\sigma \circ \delta^{q,i}, s|_{\sigma(\delta^{q,i}(\Delta^{q-1}))})$. Extending $\partial_{p,q,i}$ by linearity and taking alternating sums, one defines the differentials

$$\partial_{p,q} = \sum_{i=0}^{q} (-1)^i \partial_{p,q,i}$$

and obtains a chain complexes $C_{p,\bullet}(X; \Sigma)$. We call their homology groups

$$H^{\text{sing}}_{p,q}(X;\Sigma) \coloneqq H_q(C_{p,\bullet}(X;\Sigma))$$

the *singular tropical homology groups of X (with respect to* Σ*)*. They agree with the tropical homology groups introduced in [20]. It is well-known that they do not depend on Σ, which will also follow from Theorem 4.20.

Allowing locally finite chains, that is infinite sums of tropical simplices such that every point has a neighborhood intersecting only finitely many of them, instead of only finite chains we obtain chain complexes $C^{lf}_{p,\bullet}(X;\Sigma)$ whose homology groups

$$H^{lf}_{p,q}(X;\Sigma) = H_q(C^{lf}_{p,\bullet}(X;\Sigma))$$

are the *locally finite tropical homology groups of X (with respect to* Σ). They agree with tropical homology groups studied in [22]. Again, it will follow from Theorem 4.20 that they are independent of the face structure Σ.

Theorem 4.20. *Let X be a rational polyhedral space equipped with a face structure* Σ*. Then there are natural isomorphisms*

$$H^{lf}_{p,q}(X;\Sigma) \cong H^{BM}_{p,q}(X) \quad , \textit{ and}$$
$$H^{\text{sing}}_{p,q}(X;\Sigma) \cong H_{p,q}(X) \ .$$

Proof. The face structure Σ defines an admissible stratification on the space X in the sense of Definition A.3. By Proposition A.5, the subcomplex $\Delta^{\Sigma,\bullet}_X$ of $\Delta^\bullet_X$ consisting of chains respecting Σ (we refer to Appendix A for a precise definition of $\Delta^{\Sigma,\bullet}_X$) is quasi-isomorphic to $\Delta^\bullet_X$. Furthermore, by Proposition A.9, there is a natural quasi-isomorphism

$$\mathscr{H}om^\bullet(\Omega^p_X, \Delta^{\Sigma,\bullet}_X) \xrightarrow{\cong} R\,\mathscr{H}om^\bullet(\Omega^p_X, \mathbb{D}_X) \ .$$

Taking hypercohomology with closed/compact supports we obtain natural isomorphisms

$$\mathbb{H}^{-q}\,\mathscr{H}om^\bullet(\Omega^p_X, \Delta^\bullet_{\mathscr{S}}) \xrightarrow{\cong} \mathbb{H}^{-q} R\,\mathscr{H}om^\bullet(\Omega^p_X, \mathbb{D}_X) = H^{BM}_{p,q}(X) \ , \text{ and}$$
$$\mathbb{H}^{-q}_c\,\mathscr{H}om^\bullet(\Omega^p_X, \Delta^\bullet_{\mathscr{S}}) \xrightarrow{\cong} \mathbb{H}^{-q}_c R\,\mathscr{H}om^\bullet(\Omega^p_X, \mathbb{D}_X) = H_{p,q}(X) \ .$$

Since $\Delta^{\Sigma,\bullet}_X$ is homotopically fine, the same is true for $\mathscr{H}om^\bullet(\Omega^p_X, \Delta^{\Sigma,\bullet}_X)$. It follows that the natural morphisms

$$H^{-q}\,\mathscr{H}om^\bullet(\Omega^p_X, \Delta^{\Sigma,\bullet}_X) \to \mathbb{H}^{-q}\,\mathscr{H}om^\bullet(\Omega^p_X, \Delta^{\Sigma,\bullet}_X) \ \text{ and}$$
$$H^{-q}_c\,\mathscr{H}om^\bullet(\Omega^p_X, \Delta^{\Sigma,\bullet}_X) \to \mathbb{H}^{-q}_c\,\mathscr{H}om^\bullet(\Omega^p_X, \Delta^{\Sigma,\bullet}_X)$$

are isomorphisms. Note that for each $i \in \mathbb{N}$ there is a morphism

$$\bigoplus_{\sigma} \mathbb{Z}_{\sigma(\Delta^i)} \to \Delta_X^{\Sigma,-i} ,$$

where the sum is taken over all singular i-simplices $\sigma\colon \Delta^i \to X$ respecting Σ, that sends the generator of $\mathbb{Z}_{\sigma(\Delta^i)}$ to the global section represented by $\sigma \in C_i(X;\Sigma) = C_i(X, X \setminus X;\Sigma)$ (see (A.1)). This morphism is in fact an isomorphism: the stalk of both $\bigoplus_\sigma \mathbb{Z}_{\sigma(\Delta^i)}$ and $\Delta_X^{\Sigma,-i}$ at $x \in X$ is isomorphic to the free abelian group on singular i-simplices that respect Σ and contain x in their image. Therefore, if we write $\iota_\sigma\colon \sigma(\Delta^i) \to X$ for the inclusion, we have

$$\mathscr{H}om\left(\Omega_X^p, \Delta_X^{\Sigma,-i}\right) \cong \mathscr{H}om\left(\Omega_X^p, \bigoplus_{\sigma} \mathbb{Z}_{\sigma(\Delta^i)}\right) \cong$$
$$\cong \bigoplus_{\sigma} \mathscr{H}om\left(\Omega_X^p, \mathbb{Z}_{\sigma(\Delta^i)}\right) \cong \bigoplus_{\sigma} (\iota_\sigma)_* \mathscr{H}om\left(\Omega_X^p|_{\sigma(\Delta^i)}, \mathbb{Z}_{\sigma(\Delta^i)}\right),$$

where the direct sum commutes with $\mathscr{H}om$ because Ω_X^p is constructible. The group of global sections of this last sheaf is precisely $C_{p,i}^{lf}(X;\Sigma)$, and the group of its global sections with compact support is precisely $C_{p,i}(X;\Sigma)$. Leaving the straightforward check that this identification commutes with the differentials to the reader, we obtain an isomorphism

$$\Gamma\left(\mathscr{H}om(\Omega_X^p, \Delta_X^{\Sigma,\bullet})\right) \cong C_{p,-\bullet}^{lf}(X;\Sigma) \quad \text{and}$$
$$\Gamma_c\left(\mathscr{H}om(\Omega_X^p, \Delta_X^{\Sigma,\bullet})\right) \cong C_{p,-\bullet}(X;\Sigma)$$

of cochain complexes of abelian groups. Taking their $(-q)$-th cohomology finishes the proof. □

5. The tropical cycle class map

In this section we define the cycle class maps

$$\text{cyc}\colon Z_k(X) \to H_{k,k}^{BM}(X)$$

on a rational polyhedral space X in our framework. These have been defined before in [32] in the case of compact supports and in [22] for closed supports. Our framework naturally allows to include families of supports, that is we show that the cycle class $A \in Z_k(X)$ actually lives in $H_{k,k}^{|A|}(X)$. Moreover, our construction is local in the sense that we do not make any assumption on the existence of global face structures. If a global face structure does exist, we show in Theorem 5.13 that our definition of the cycle class map agrees with that in [22].

5.1. Conventions for orientations

To define the tropical cycle class map one needs to make a choice regarding orientations. There are two ways of defining an orientation for $\mathbb{R}^n$ that are relevant for us, one being the choice of a generator for $\bigwedge^n T_0^{\mathbb{Z}} \mathbb{R}^n \cong \bigwedge^n \mathbb{Z}^n$, the other being the choice of a generator for $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. For the construction of the tropical cycle class map we need to choose, once and for all, an isomorphism

$$\bigwedge^n T_0^{\mathbb{Z}} \mathbb{R}^n \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

that allows us to compare these two notions of orientation, and our choice will be the one that sends $e_1 \wedge \ldots \wedge e_n$ to class of the (linearly embedded) singular simplex $[e_n, e_{n-1}, \ldots, e_1, e_0]$, where $e_1, \ldots, e_n$ is any basis for $\mathbb{Z}^n$ and $e_0 = -\sum_{i=1}^n e_i$.

Suppose that σ is a k-dimensional polyhedron in $\mathbb{R}^n$ and assume we have chosen an orientation $\eta_\sigma \in \bigwedge^k T^{\mathbb{Z}}(\sigma)$. Choosing an embedding $f \colon \sigma \to \mathbb{R}^k$, the orientation η_σ induces an element in $\bigwedge^k \mathbb{R}^k$ and hence a class $[\sigma]$ in

$$H(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) = H(f(\sigma), f(\sigma) \setminus \{f(x)\}) = H^{lf}(\sigma, \partial\sigma)$$

for every choice of a point $x \in \sigma \setminus \partial\sigma$ of which $[\sigma]$ is independent. The class $[\sigma]$ does not depend on f either; if the orientation of f is flipped, then the signs in both the identifications $\bigwedge^k T^{\mathbb{Z}}(\sigma) = \bigwedge^k \mathbb{R}^k$ and $H^{lf}(\sigma, \partial\sigma) = H(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\})$ are flipped, that is a change of orientation of f flips the sign of the class we are defining exactly twice. Now let $\tau \subseteq \sigma$ be a face of σ of codimension 1 and assume we have also chosen an orientation $\eta_\tau \in \bigwedge^{k-1} T^{\mathbb{Z}}(\tau)$. Then with our convention in place we also obtain a class $[\tau] \in H_{k-1}^{lf}(\tau, \partial\tau)$. If ∂_τ denotes the composite

$$H_k^{lf}(\sigma, \partial\sigma) \to H_{k-1}^{lf}(\partial\sigma) \to H_{k-1}^{lf}(\partial\sigma, \partial\sigma \setminus \operatorname{relint}(\tau)) \cong H_{k-1}^{lf}(\tau, \partial\tau),$$

and $n_{\sigma/\tau} \in T^{\mathbb{Z}}(\sigma)$ is any lattice normal vector of σ with respect to τ, the equalities $\partial_\tau[\sigma] = [\tau]$ and $\eta_\sigma = \eta_\tau \wedge n_{\sigma/\tau}$ are equivalent. In this case, we say that the chosen orientations on σ and τ are compatible.

5.2. Tropical cycles as sheaf hom

The following crucial observation, together with Lemma 4.8 (b), will allow us define the tropical cycle class map.

Proposition 5.1. *Let X be an n-dimensional rational polyhedral space. Then there is a natural isomorphism of sheaves*

$$\mathscr{Z}_n^X \cong \mathscr{H}\!om(\Omega_X^n, \mathscr{H}_X^n)\ .$$

Proof. We will first define the isomorphism locally around a point $x \in X$. Let Σ be a local face structure at x. After potentially shrinking Σ, we may assume that $|\Sigma|$ is compact, in which case Σ gives a CW-complex structure to the neighborhood $|\Sigma|$ of x. After choosing orientations $\eta_\sigma \in \bigwedge^{\dim(\sigma)} T^{\mathbb{Z}}(\sigma)$ for all $\sigma \in \Sigma$, cellular homology provides a description of $\mathscr{H}^n_{X,x} = H_n(X, X \setminus \{x\})$ as a subgroup of the free group on $\Sigma(n)$, where $\Sigma(i)$ denotes the set of i-dimensional cells of Σ. Namely, it is the group of all weights $w\colon \Sigma(n) \to \mathbb{Z}$ such that for every $(n-1)$-dimensional cell $\tau \in \Sigma$ containing x we have

$$\sum_{\sigma\colon\ \tau \subseteq \sigma \in \Sigma(n)} \varepsilon_{\sigma/\tau} w(\sigma) = 0\ , \tag{5.1}$$

where $\varepsilon_{\sigma/\tau}$ is either 1 or -1, depending on whether the chosen orientations on σ and τ agree or not. If y is a point in the interior of $|\Sigma|$, then the set

$$\Sigma_y = \{\sigma \in \Sigma \mid \text{ there exists } \tau \in \Sigma \text{ so that } \sigma \subseteq \tau \text{ and } y \in \tau\}$$

is a local face structure at y, so with the same reasoning we conclude that $\mathscr{H}^n_{X,y}$ is the group of weights $w\colon \Sigma_y(n) \to \mathbb{Z}$ satisfying the condition displayed in (5.1) for all $(n-1)$-dimensional cells $\tau \in \Sigma$ containing y. This description is continuous in the sense that we obtain an exact sequence

$$0 \to \mathscr{H}^n_X \to \bigoplus_{\sigma \in \Sigma(n)} \mathbb{Z}_\sigma \to \bigoplus_{\tau : x \in \tau \in \Sigma(n-1)} \mathbb{Z}_\tau$$

on the interior of $|\Sigma|$, where the map to the right is determined by the condition (5.1). Note that this is precisely the description of $\mathscr{H}^n_X$ one obtains from Shepard's combinatorial description of the dualizing complex mentioned in Remark 4.7. Applying $\mathscr{H}\!om(\Omega^n_X, -)$ to the sequence, we obtain an exact sequence

$$0 \to \mathscr{H}\!om(\Omega^n_X, \mathscr{H}^n_X) \to \bigoplus_{\sigma \in \Sigma(n)} \iota_{\sigma*} \mathscr{H}\!om(\Omega^n_X|_\sigma, \mathbb{Z}_\sigma) \to \bigoplus_{\tau : x \in \tau \in \Sigma(n-1)} \iota_{\tau*} \mathscr{H}\!om(\Omega^n_X|_\tau, \mathbb{Z}_\tau)\ ,$$

where $\iota_\delta\colon \delta \to X$ denotes the inclusion for every $\delta \in \Sigma$. By Lemma A.8, for every $\delta \in \Sigma$ we have an isomorphism

$$\mathscr{H}\!om(\Omega^n_X|_\delta, \mathbb{Z}_\delta) \xrightarrow{\cong} (\kappa_\delta)_* \mathscr{H}\!om(\Omega^n_X|_{\mathrm{relint}(\delta)}, \mathbb{Z}_{\mathrm{relint}(\delta)})\ ,$$

where $\kappa_\delta\colon\ \mathrm{relint}(\delta) \to \delta$ denotes the inclusion, and the latter sheaf is in turn naturally isomorphic to the constant sheaf

$$\Big(\bigwedge\nolimits^n T^{\mathbb{Z}}_\delta X\Big)_\delta$$

on δ. Here, we denote $T^{\mathbb{Z}}_\delta X := \Gamma(\mathrm{relint}(\delta), \Omega^1_X|_{\mathrm{relint}(\delta)})^*$, which is naturally isomorphic to $T^{\mathbb{Z}}_x X$ for any $x \in \mathrm{relint}(\delta)$. If $\sigma \in \Sigma(n)$, then $T^{\mathbb{Z}}_\sigma X$ is isomorphic to $\mathbb{Z}_\sigma$ and is generated by η_σ. We thus obtain an exact sequence

$$0 \to \mathscr{H}om(\Omega_X^n, \mathscr{H}_X^n) \to \bigoplus_{\sigma \in \Sigma(n)} \mathbb{Z}_\sigma \to \bigoplus_{\tau : x \in \tau \in \Sigma(n-1)} \left(\bigwedge^n T_\tau^{\mathbb{Z}} X \right)_\tau ,$$

where the component of the rightmost morphism going from $\mathbb{Z}_\sigma$ to $\left(\bigwedge^n T^{\mathbb{Z}}(\tau)\right)_\tau$ is 0 if τ is a face of σ at infinity, and sends the generator 1 of $\mathbb{Z}_\sigma$ to $\eta_\tau \wedge n_{\sigma/\tau}$ else, where $n_{\sigma/\tau} \in T^{\mathbb{Z}}(\sigma)$ is any lattice normal vector of σ relative to τ. This effectively yields a presentation of $\mathscr{H}om(\Omega_X^n, \mathscr{H}_X^n)$ on a neighborhood of x as the sheaf of locally constant functions A on $X^{\max}$ such that for every codimension-1 face $\tau \in \Sigma(n-1)$ we have

$$\eta_\tau \wedge \left(\sum_{\sigma \in \Sigma_\tau(n)} A(\sigma) n_{\sigma/\tau} = 0 \right) ,$$

where $\Sigma_\tau(n)$ denotes the subset of $\Sigma(n)$ consisting of cells that have τ as a finite face. This equality holds if and only if

$$\sum_{\sigma \in \Sigma_\tau(n)} A(\sigma) n_{\sigma/\tau} \in T^{\mathbb{Z}}(\tau) ,$$

which is precisely the balancing condition (see Remark 3.3). In other words, we obtain an isomorphism $\mathscr{Z}_n^X \cong \mathscr{H}om(\Omega_X^n, \mathscr{H}_X^n)$ in a neighborhood of x.

To show that these local isomorphisms glue to a global isomorphism, we essentially need to show that the local isomorphisms are independent of all choices. The choices we made were the local face structure and the orientations on them. We used the same orientations to pick generators for $\bigwedge^n T^{\mathbb{Z}}(\sigma)$ and $H_n(\sigma, \partial\sigma)$, so if we picked the opposite orientation on one of the cells σ, we would change signs twice and hence obtain the same isomorphism. It remains to show that a different local face structure would also provide the same isomorphism. But to compare two different choices of local face structures one can always pass to a common refinement, and it is clear that the construction of the isomorphism is compatible with refinements. □

Remark 5.2. Going through the proof of Proposition 5.1 we obtain the following description of the isomorphism $\mathscr{Z}_n^X \cong \mathscr{H}om(\Omega_X^n, \mathscr{H}_X^n)$ locally around a point x using a local face structure Σ at x and orientations $\eta_\sigma \in \bigwedge^n T^{\mathbb{Z}}(\sigma)$ on each of its maximal cells $\sigma \in \Sigma$. The description of the morphisms $\Omega_X^n \to \mathscr{H}_X^n$ corresponding to $A \in Z_n(X)$ uses three ingredients:

(1) Using the orientations and cellular homology, we can choose for each facet $\sigma \in \Sigma(n)$ of dimension n a chain $[\sigma]$ supported on σ such that $\mathscr{H}_X^n$ is, locally around x, isomorphic to the subsheaf of Δ_X^{-n} generated by the $[\sigma]$, $\sigma \in \Sigma$.

(2) Whenever $\omega \in \Gamma(U, \Omega_X^n)$ is a tropical n-form defined on an open subset of the interior of $|\Sigma|$, we can pair it with η_σ to obtain a locally constant, integer-valued function $\langle \omega, \eta_\sigma \rangle$ on $U \cap \sigma$.

(3) If A is a tropical n-cycle defined on the interior of $|\Sigma|$, then it defines a multiplicity $A(\sigma)$ on every $\sigma \in \Sigma(n)$.

With notation as in (1), (2), and (3), the morphism $\Omega_X^n \to \mathscr{H}_X^n$ that A is mapped to by the isomorphism can now be described by the rule

$$\omega \mapsto \sum_{\sigma \in \Sigma(n)} A(\sigma)\langle \omega, \eta_\sigma \rangle [\sigma] \ .$$

5.3. The tropical cycle class map

We can now define the tropical cycle class map. We do this in two steps, first for top-dimensional tropical cycles and then in general.

Definition 5.3. Let X be an n-dimensional rational polyhedral space. We define the *tropical cycle class map on n-dimensional tropical cycles*

$$\widetilde{\mathrm{cyc}}_X \colon Z_n(X) \to H_{n,n}^{BM}(X)$$

as the composite of the canonical map

$$Z_n(X) \to \mathrm{Hom}(\Omega_X^n, \mathscr{H}_X^n)$$

and the canonical identification

$$\mathrm{Hom}(\Omega_X^n, \mathscr{H}_X^n) = \mathrm{Hom}_{D(X)}(\Omega_X^n[n], \mathbb{D}_X) = H_{n,n}^{BM}(X)\,,$$

where the first equality holds since $H^{-j}(\mathbb{D}_X) = \mathscr{H}_X^j = 0$ for $j > n$.

To define the tropical cycle class map in the remaining dimensions we use push-forwards:

Definition 5.4. Let X be an n-dimensional rational polyhedral space. For a tropical cycle $A \in Z_i(X)$, $i \in \mathbb{N}$, we define its *tropical cycle class* by

$$\mathrm{cyc}_X(A) := \iota_*(\widetilde{\mathrm{cyc}}_{|A|}(A)) \in H_{i,i}^{BM}(X)\,,$$

where $\iota \colon |A| \hookrightarrow X$ is the inclusion map. Note that $\widetilde{\mathrm{cyc}}_{|A|}(A)$ is defined, since $i = \dim |A|$.

Remark 5.5. It is also possible to define a refined cycle class that respects supports. Namely, we can view the tropical cycle class of $A \in Z_i(X)$ as an element of $H_{i,i}^{|A|}(X)$

For top-dimensional tropical cycles, we now have two ways to take their tropical cycle classes which are a priori different: one by applying $\widetilde{\mathrm{cyc}}$ and one by applying cyc. As a byproduct of the compatibility with push-forwards we will see that they agree (see Corollary 5.7).

5.4. Compatibility with push-forwards

As we have seen, both tropical cycle groups and tropical Borel-Moore homology groups are functorial with respect to proper morphisms. We will now show that the tropical cycle class map respects push-forwards, that is that it defines a natural transformation between tropical cycle groups and tropical homology groups.

Proposition 5.6. *Let $f\colon X \to Y$ be a proper morphism of n-dimensional rational polyhedral spaces. Then the tropical cycle class map $\widetilde{\mathrm{cyc}}$ commutes with push-forwards. In other words, the diagram*

$$\begin{array}{ccc} Z_n(X) & \xrightarrow{f_*} & Z_n(Y) \\ \downarrow{\scriptstyle \widetilde{\mathrm{cyc}}_X} & & \downarrow{\scriptstyle \widetilde{\mathrm{cyc}}_Y} \\ H^{BM}_{n,n}(X) & \xrightarrow{f_*} & H^{BM}_{n,n}(Y) \end{array}$$

is commutative.

Proof. Inspecting the definitions of $\widetilde{\mathrm{cyc}}_X$, $\widetilde{\mathrm{cyc}}_Y$, and the push-forward in homology, we see that the statement boils down purely formally to proving the commutativity of the diagram

$$\begin{array}{ccc} Z_n(X) & \xrightarrow{f_*} & Z_n(Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Omega^n_X, \mathscr{H}^n_X) & \longrightarrow & \mathrm{Hom}(\Omega^n_Y, \mathscr{H}^n_Y) \end{array},$$

where the vertical maps are induced by the natural isomorphisms of Proposition 5.1, and the lower horizontal map sends a morphism $\Omega^n_X \to \mathscr{H}^n_X$ to the composite

$$\Omega^n_Y \to f_*\Omega^n_Y \to f_*\mathscr{H}^n_X \to \mathscr{H}^n_Y,$$

where the rightmost morphism is the $(-n)$-th cohomology of the natural morphism $Rf_*\mathbb{D}_X \to \mathbb{D}_Y$. By Lemma 4.10, this morphism is induced by the push-forward morphisms $H_n(X, X \setminus f^{-1}U) \to H_n(Y, U)$ between the singular relative homology groups. So let $A \in Z_n(X)$. We will compare the two morphisms in $\mathrm{Hom}(\Omega^n_Y, \mathscr{H}^n_Y)$ obtained from A. Since tropical n-cycles on Y are determined by their restriction to a dense open subset of $Y^{\max}$, it suffices to do this locally at a point $y \in Y^{\max} \setminus f(X \setminus X^{\max})$. Let $\varphi \in \Omega^n_{Y,y}$. To compute the image of φ under the morphism obtained from A by moving along the square counterclockwise, we first take its image $f^*\varphi$ in $(f_*\Omega^n_X)_y = \Gamma(f^{-1}\{y\}, \Omega^n_X)$. We note that the support of $f^*\varphi$ only consists of isolated points of $f^{-1}\{y\}$. Indeed, $f^*\varphi$ is nonzero at $x \in f^{-1}\{y\}$, if and only if $d_x f\colon T^{\mathbb{Z}}_x X \to T^{\mathbb{Z}}_y Y$ is injective, which is the case if and only if f is injective on a neighborhood of x. In particular, we see that $f^*\varphi$ is supported on finitely many points, call them $x_1, \dots, x_k$, because $f^{-1}\{y\}$ is compact. Now we take the image of $f^*\varphi$ under the morphism

$$(f_*\Omega^n_X)_x = \Gamma(f^{-1}\{y\}, \Omega^n_X) \to \Gamma(f^{-1}\{y\}, \mathscr{H}^n_X) = (f_*\mathscr{H}^n_X)_y$$

induced by A. To understand it we use the explicit description of the canonical morphism $Z_n(X) \to \mathrm{Hom}(\Omega^n_X, \mathscr{H}^n_X)$ given in Remark 5.2, which is particularly simple here because we are working on $X^{\max}$. We pick an orientation $\eta_y \in \bigwedge^n T^{\mathbb{Z}}_y Y$, which induces orientations $\eta_{x_i} \in \bigwedge^n T^{\mathbb{Z}}_{x_i} X$ for all $1 \le i \le k$. These orientations define generators $[\sigma_{x_i}] \in \mathscr{H}^n_{X,x_i}$ and $[\delta] \in \mathscr{H}^n_{Y,y}$. The image of $f^*\varphi$ in $\Gamma(f^{-1}\{y\}, \mathscr{H}^n_X)$ is then represented by

$$\sum_{i=1}^k A(x_i)\langle f^*\varphi, \eta_{x_i}\rangle[\sigma_{x_i}] \ .$$

We observe that

$$\langle f^*\varphi, \eta_{x_i}\rangle = [T^{\mathbb{Z}}_y Y \colon d_{x_i} f(T^{\mathbb{Z}}_{x_i} X)]\langle \varphi, \eta_y\rangle \ .$$

So when we finally apply the morphism

$$(f_*\mathscr{H}^n_X)_y = \Gamma(f^{-1}\{y\}, \mathscr{H}^n_X) \to \mathscr{H}^n_{Y,y},$$

we see that

$$\left(\sum_{i=1}^k [T^{\mathbb{Z}}_y Y \colon d_{x_i} f(T^{\mathbb{Z}}_{x_i} X)] A(x_i)\right) \langle \varphi, \eta_y\rangle[\delta]$$

is the image of φ. Looking back at Definition 3.6, we see that this is precisely the image of φ under the morphism $\Omega^n_{Y,y} \to \mathscr{H}^n_{Y,y}$ induced by f_*A. □

Corollary 5.7. *Let X be an n-dimensional rational polyhedral space. Then the two morphisms $\widetilde{\mathrm{cyc}}_X, \mathrm{cyc}_X \colon Z_n(X) \to H^{BM}_{n,n}(X)$ coincide.*

Proof. Let $A \in Z_n(X)$, and let $\iota\colon |A| \to X$ be the inclusion. The statement is trivially true for $A = 0$, so we may assume that $|A|$ is an n-dimensional rational polyhedral subspace of X. By definition of cyc_X, we have $\mathrm{cyc}_X(A) = \iota_*(\widetilde{\mathrm{cyc}}_{|A|}(A))$, which equals $\widetilde{\mathrm{cyc}}_X(\iota_* A)$ by Proposition 5.6. As $\iota_* A = A$, this finishes the proof. □

Corollary 5.8. *Let $f\colon X \to Y$ be a proper morphism of rational polyhedral spaces. Then the tropical cycle class map* cyc *commutes with push-forwards. In other words, the diagram*

$$\begin{array}{ccc} Z_i(X) & \xrightarrow{f_*} & Z_i(Y) \\ \big\downarrow{\scriptstyle \mathrm{cyc}_X} & & \big\downarrow{\scriptstyle \mathrm{cyc}_Y} \\ H^{BM}_{i,i}(X) & \xrightarrow{f_*} & H^{BM}_{i,i}(Y) \end{array}$$

is commutative for all $i \in \mathbb{N}$.

Proof. By definition of the tropical cycle class map, the assertion holds if X is a rational polyhedral subspace of Y and f is the inclusion. Now assume f is general and $A \in Z_i(X)$. Using the result for inclusions and the fact that push-forwards are functorial for both tropical homology classes and tropical cycles, we reduce to the case where $X = |A|$, $Y = f(|A|)$, and $\dim(X) = i$. If $\dim(Y) < \dim(X)$, then $H^{BM}_{i,i}(Y) = 0$ and the statement is trivial, so we may assume $i = \dim(Y) = \dim(X)$. In this case, the result follow from Proposition 5.1 and Corollary 5.7. □

5.5. Compatibility with cross products

Given tropical cycles A and B on locally polyhedral spaces X and Y, we have defined their cross-product (§3.3) and the cross-product of their tropical cycle classes (§4.5). We now show that the tropical cycle class of the former equals the latter.

Proposition 5.9. *Let X, Y be rational polyhedral spaces, and let $i, j \in \mathbb{N}$. Then the tropical cycle class map takes cross products to cross products. In other words, the diagram*

$$\begin{array}{ccc} Z_i(X) \otimes_{\mathbb{Z}} Z_j(Y) & \xrightarrow{\times} & Z_{i+j}(X \times Y) \\ \big\downarrow{\scriptstyle \mathrm{cyc}_X \otimes \mathrm{cyc}_Y} & & \big\downarrow{\scriptstyle \mathrm{cyc}_{X\times Y}} \\ H^{BM}_{i,i}(X) \otimes_{\mathbb{Z}} H^{BM}_{j,j}(Y) & \xrightarrow{\times} & H^{BM}_{i+j,i+j}(X \times Y) \end{array}$$

is commutative.

Proof. Let $A \in Z_i(X)$ and $B \in Z_j(Y)$. By the functoriality of the cross-product we may assume that $X = |A|$ is purely i-dimensional, and $Y = |B|$ is purely j-dimensional. In this case, both morphisms

$$\mathrm{cyc}(A) \times \mathrm{cyc}(B)\colon \Omega^{i+j}_{X\times Y}[i+j] \to \mathbb{D}_{X\times Y} \ , \text{ and}$$
$$\mathrm{cyc}(A \times B)\colon \Omega^{i+j}_{X\times Y}[i+j] \to \mathbb{D}_{X\times Y}$$

are completely determined by the morphisms $\Omega^{i+j}_{X\times Y} \to \mathscr{H}^{i+j}_X$ they induce by taking cohomology in degree $-(i+j)$. By Proposition 5.1, it suffices to compare these morphisms locally at a point $z = (x, y) \in (X \times Y)^{\max}$. Then $x \in X^{\max}$ and $y \in Y^{\max}$. Let η_x be a generator of $\bigwedge^i T^{\mathbb{Z}}_x X$ and η_y a generator of $\bigwedge^j T^{\mathbb{Z}}_y Y$, and let $[\sigma_x]$ and $[\sigma_y]$ be the corresponding generator of $\mathscr{H}^i_{X,x}$ and $\mathscr{H}^j_{Y,y}$. Then as explained in Remark 5.2, the morphism $\Omega^i_{X,x} \to \mathscr{H}^i_{X,x}$ defined by X is given by $\omega \mapsto \langle\omega, \eta_x\rangle A(x)[\sigma_x]$, and the morphism $\Omega^j_{Y,y} \to \mathscr{H}^i_{Y,y}$ defined by B is given by $\omega \mapsto \langle\omega, \eta_y\rangle B(y)[\sigma_y]$. By definition of the cross product and Lemma 4.11, the morphism $\Omega^{i+j}_{X\times Y,z} \to \mathscr{H}^{i+j}_{X\times Y,z}$ induced by $\operatorname{cyc}(A) \times \operatorname{cyc}(B)$ takes $\omega \otimes \omega' \in \Omega^i_{X,x} \otimes \Omega^j_{Y,y} \cong \Omega^{i+j}_{X\times Y,z}$ to

$$\langle\omega, \eta_x\rangle\langle\omega', \eta_y\rangle A(x)B(y)[\sigma_x] \times [\sigma_y].$$

Here $[\sigma_x] \times [\sigma_y]$ denotes the classical cross-product, which equals the generator $[\sigma_z]$ of $\mathscr{H}^{i+j}_{X\times Y}$ corresponding to the generator $\eta_z = \eta_x \otimes \eta_y$ of $\bigwedge^{i+j} T^{\mathbb{Z}}_z(X \times Y)$. Therefore, the expression above for the image of $\omega \otimes \omega'$ equals

$$\langle\omega \otimes \omega', \eta_x \otimes \eta_y\rangle(A \times B)(z)[\sigma_z]$$

which is precisely the image of $\omega \otimes \omega'$ under the morphism $\Omega^{i+j}_{X\times Y,z} \to \mathscr{H}^{i+j}_{X\times Y,z}$ induced by $\operatorname{cyc}_{X\times Y}(A \times B)$. □

5.6. The first Chern class of a divisor

Let X be a rational polyhedral space. As explained in §3.5, the set of isomorphism classes of tropical line bundles on X is an abelian group, naturally isomorphic to $H^1(X, \operatorname{Aff}_X)$.

Definition 5.10 *(see [31, Section 5] and [22, Definition 3.6]).* Let X be a rational polyhedral space, and let $d\colon \operatorname{Aff}_X \to \Omega^1_X$ be the quotient map. Then the *first Chern class* is defined as the morphism

$$c_1 \coloneqq H^1(d)\colon H^1(X, \operatorname{Aff}_X) \to H^1(X, \Omega^1_X) = H^{1,1}(X)\ .$$

If $\mathscr{L}$ is a tropical line bundle on X, corresponding to $\alpha \in H^1(X, \operatorname{Aff}_X)$, then the first Chern class of $\mathscr{L}$ is $c_1(\mathscr{L}) \coloneqq c_1(\alpha)$.

Proposition 5.11. *Let $f\colon X \to Y$ be a morphism of rational polyhedral spaces, and let $\mathscr{L}$ be a tropical line bundle on Y. Then*

$$c_1(f^*\mathscr{L}) = f^*(c_1(\mathscr{L}))\ .$$

Proof. This follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} H^1(Y, \mathrm{Aff}_Y) & \longrightarrow & H^1(Y, \Omega^1_Y) \\ \downarrow f^{-1} & & \downarrow f^{-1} \\ H^1(X, f^{-1}\,\mathrm{Aff}_Y) & \longrightarrow & H^1(X, f^{-1}\Omega^1_Y) \\ \downarrow H^1(f^\sharp) & & \downarrow H^1(f^\sharp) \\ H^1(X, \mathrm{Aff}_X) & \longrightarrow & H^1(X, \Omega^1_X) \end{array} ,$$

where the horizontal morphisms are induced by the quotient morphisms $\mathrm{Aff} \to \Omega^1$. □

Also recall from §3.4 that there is a sheaf of tropical Cartier divisors $\mathscr{Div}_X$ which fits into a short exact sequence

$$0 \to \mathrm{Aff}_X \to \mathscr{M}_X \to \mathscr{Div}_X \to 0 , \tag{5.2}$$

the first connecting homomorphism of whose associated long exact cohomology sequence is the map

$$\mathrm{Div}(X) = H^0(X, \mathscr{Div}_X) \to H^1(X, \mathrm{Aff}_X), \;\; D \mapsto \mathscr{L}(D) .$$

Composing this with the first Chern class defines a map

$$\mathrm{Div}(X) \to H^{1,1}(X), \;\; D \mapsto c_1(\mathscr{L}(D))$$

that assigns a $(1,1)$-cohomology class to every Cartier divisor. We will need to work with a factorization of this map. A Cartier divisor $D \in \mathrm{Div}(X)$ is, by definition, a global section of $\mathscr{Div}_X$ whose support is, again by definition, equal to $|D|$. Therefore, it defines a morphism $\mathbb{Z}_{|D|} \to \mathscr{Div}_X$. Composing this with the morphism $\mathscr{Div}_X \to \mathrm{Aff}_X[1]$ defined by (5.2) and the projection morphism $\mathrm{Aff}_X[1] \to \Omega^1_X[1]$ defines a morphism $\mathbb{Z}_{|D|} \to \Omega^1_X[1]$, and hence an element in $H^{1,1}_{|D|}(X)$ (see Remark 4.2), the image of which in $H^{1,1}(X)$ is $c_1(\mathscr{L}(D))$. We can thus view $c_1(\mathscr{L}(D))$ as an element in $H^{1,1}_{|D|}(X)$.

We will now show that taking the cap product with the first Chern class $c_1(\mathscr{L}(D))$ corresponds to intersecting with D. This is a generalization of [22, Theorem 4.15], where the statement is shown without supports. As our setup does not assume a global face structure to exist, the main difficulty is to reduce to a local computation where one can use a local face structure. In principle, this local computation should be a straightforward computation in a suitable exterior algebra. Yet obtaining the correct sign is surprisingly subtle. One only obtains the correct sign if the cap product and the intersection product are defined in a way compatible with the orientation convention chosen in §5.1.

Proposition 5.12. *Let X be a rational polyhedral space. Then for every tropical Cartier divisor $D \in \mathrm{Div}(X)$ and for every tropical cycle $A \in Z_i(X)$ we have*

$$\operatorname{cyc}_X(D \cdot A) = \operatorname{cyc}_X(A) \frown c_1(\mathscr{L}(D))$$

in $H^{|D| \cap |A|}_{i-1,i-1}(X)$.

Proof. First we reduce to the case $X = |A|$. Let $\iota\colon |A| \hookrightarrow X$ be the inclusion map. Then by definition of the intersection pairing, and by Corollary 5.8, we have

$$\operatorname{cyc}_X(D \cdot A) = \operatorname{cyc}_X(\iota_*(\iota^* D \cdot A)) = \iota_* \operatorname{cyc}_{|A|}(\iota^* D \cdot A)$$

in $H^{|D| \cap |A|}_{i-1,i-1}(X)$. On the other hand, by the projection formula (Proposition 4.19) we have

$$\operatorname{cyc}_X(A) \frown c_1(\mathscr{L}(D)) = \iota_* \operatorname{cyc}_{|A|}(A) c_1(\mathscr{L}(D)) = \iota_*(\operatorname{cyc}_{|A|}(A) \frown \iota^*(c_1(\mathscr{L}(D))))$$

in $H^{|D| \cap |A|}_{i-1,i-1}(X)$, which equals $\iota_*(\operatorname{cyc}_{|A|}(A) \frown c_1(\mathscr{L}(\iota^* D)))$ by Propositions 3.15 and 5.11. It thus suffices to show that

$$\operatorname{cyc}_{|A|}(\iota^* D \cdot A) = \operatorname{cyc}_{|A|}(A) \frown c_1(\mathscr{L}(\iota^* D))$$

in $H^{BM}_{i-1,i-1}(|\iota^* D|)$, which allows us to assume $X = |A|$. In this case, we will interpret $c_1(\mathscr{L}(D))$ as an element of $H^{1,1}_{|D|}(X)$ and show that the equality holds in $H^{|D|}_{i-1,i-1}(X)$. Since $|D|$ is at most $(i-1)$-dimensional, the presheaf

$$U \mapsto H^{U \cap |D|}_{i-1,i-1}(U)$$

on X is a sheaf by Lemma 4.8 (b). We can thus work locally around a point $x \in X$, where we can use a local face structure Σ. After potentially shrinking and refining Σ, we can assume that the divisor D is principal, say $D = \operatorname{div}(\varphi)$, and that $\varphi|_\sigma \in \Gamma(\sigma, \operatorname{Aff}_\sigma)$ for all $\sigma \in \Sigma$. As usual, we choose an orientation on each $\sigma \in \Sigma$ in form of a generator

$$\eta_\sigma \in \bigwedge^{\dim \sigma} T^{\mathbb{Z}}(\sigma),$$

and these orientations define classes $[\sigma] \in \Gamma(X, \Delta_X^{-\dim \sigma})$ supported on σ for all $\sigma \in \Sigma$.

The tropically rational function φ defines a morphism $\mathbb{Z}_{|D|} \to \mathscr{Div}_X$, and, by definition, the first Chern class $c_1(\mathscr{L}(D)) \in H^{1,1}_{|D|}(X)$ is the composite of this morphism with the composite $\mathscr{Div}_X \to \operatorname{Aff}_X[1] \to \Omega^1_X[1]$, where the first morphism comes from the exact sequence (5.2). These are morphisms in the derived category $D(\mathbb{Z}_X)$ that are not represented by morphisms between the complexes involved. To remedy this, we note that the exact sequence (5.2) yields an isomorphism between $\mathscr{Div}_X$ and the complex

$$\ldots \to 0 \to \operatorname{Aff}_X \to \mathscr{M}_X \to 0 \to \ldots\ ,$$

where $\mathscr{M}_X$ sits in degree 0, and that the short exact sequence

$$0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to \mathbb{Z}_{|D|} \to 0 \ ,$$

where $U = X \setminus |D|$, yields an isomorphism between $\mathbb{Z}_{|D|}$ and the complex

$$\ldots \to 0 \to \mathbb{Z}_U \to \mathbb{Z}_X \to 0 \to \ldots \ ,$$

where $\mathbb{Z}_X$ sits in degree 0. The first Chern class $c_1(\mathscr{L}(D))$ is then represented by the diagram

$$\begin{array}{ccccccc} \mathbb{Z}_X & \xrightarrow{\cdot\varphi} & \mathscr{M}_X & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbb{Z}_U & \xrightarrow{\cdot\varphi} & \mathrm{Aff}_X & \xrightarrow{-\,\mathrm{id}} & \mathrm{Aff}_X & \longrightarrow & \Omega^1_X \ , \end{array}$$

where each column represents an element in $D(\mathbb{Z}_X)$ and the minus sign in the middle morphism in the lower row is there by convention. Tensoring with $\Omega^{i-1}_X[i-1]$ from the right and composing with the multiplication morphism $\Omega^{i-1}_X \otimes_{\mathbb{Z}} \Omega^1_X \to \Omega^i_X$, and the morphism $\Omega^i_X[i] \to \Delta^\bullet_X$ representing $\mathrm{cyc}_X(A)$, we see that $\mathrm{cyc}(A) \frown c_1(\mathscr{L}(D))$ is represented by the morphism $((\Omega^{i-1}_X)_U \to \Omega^{i-1}_X)) \to \Delta^\bullet_X$, where the former complex sits in degrees $-i$ and $-(i-1)$, that is given by

$$(\Omega^{i-1}_X)_U \to \Delta^{-i}_X\colon \quad \omega \mapsto -\sum_{\sigma\in\Sigma(i)} \langle d(\varphi|_\sigma) \wedge \omega|_\sigma, \eta_\sigma\rangle A(\sigma)[\sigma] \ ,$$

in degree $-i$ and 0 in every other degree. The morphism in degree $-i$ extends to a morphism $\Omega^{i-1}_X \to \Delta^{-i}_X$, defining a chain homotopy between the morphism of cochain complexes just defined, and the morphism that is given by

$$\Omega^{i-1}_X \to \Delta^{-(i-1)}_X\colon \quad \omega \mapsto \partial\left(\sum_{\sigma\in\Sigma(i)} \langle d(\varphi|_\sigma) \wedge \omega, \eta_\sigma\rangle A(\sigma)[\sigma]\right)$$

in degree $-(i-1)$ and is 0 in all other degrees. To simplify the expression on degree $-(i-1)$ we pick for every finite codimension-1 face τ of a cone $\sigma \in \Sigma(i)$ a lattice normal vector $n_{\sigma/\tau}$, and we set $\varepsilon_{\sigma/\tau}$ equal to 1 if the chosen orientations on σ and τ are compatible, and to -1 otherwise. Recall that this means that

$$\varepsilon_{\sigma/\tau}\eta_\sigma = \eta_\tau \wedge n_{\sigma/\tau} \ .$$

We now compute that

$$\partial\left(\sum_{\sigma\in\Sigma(i)}\langle d(\varphi|_\sigma)\wedge\omega,\eta_\sigma\rangle A(\sigma)[\sigma]\right)=$$

$$=\sum_{\tau\in\Sigma(i-1)}\sum_{\sigma\in\Sigma_\tau(i)}\langle d(\varphi|_\sigma)\wedge\omega,\varepsilon_{\sigma/\tau}\eta_\tau\wedge n_{\sigma/\tau}\rangle A(\sigma)\varepsilon_{\sigma/\tau}[\tau]=$$

$$\sum_{\tau\in\Sigma(i-1)}\sum_{\sigma\in\Sigma_\tau(i)}\langle\omega\wedge d(\varphi|_\sigma),n_{\sigma/\tau}\wedge\eta_\tau\rangle A(\sigma)[\tau]\ ,$$

where $\Sigma_\tau(i)$ denotes the set of cells $\sigma\in\Sigma(i)$ that have τ as a finite face. Note that we only need to consider finite faces because if ω is defined at a point of an infinite face of σ, then $\langle d(\varphi|_\sigma)\wedge\omega,\eta_\sigma\rangle=0$. Let l_τ be an affine linear function on a neighborhood of $|\Sigma|$ such that $l_\tau|_\tau=\varphi|_\tau$. Then we can rewrite the coefficient of $[\tau]$ in the expression above as

$$\sum_{\sigma\in\Sigma_\tau(i)}\Big(\big\langle\omega\wedge(d(\varphi|_\sigma)-dl_\tau),n_{\sigma/\tau}\wedge\eta_\tau\big\rangle A(\sigma)+\big\langle\omega\wedge dl_\tau,n_{\sigma/\tau}\wedge\eta_\tau\big\rangle A(\sigma)\Big)=$$

$$=\sum_{\sigma\in\Sigma_\tau(i)}\big\langle\omega,\eta_\tau\big\rangle\cdot\big\langle(d(\varphi|_\sigma)-dl_\tau),n_{\sigma/\tau}\big\rangle A(\sigma)+$$

$$+\left\langle\omega\wedge dl_\tau,\left(\sum_{\sigma\in\Sigma_\tau(i)}A(\sigma)n_{\sigma/\tau}\right)\wedge\eta_\tau\right\rangle\ .$$

By the balancing condition and the fact that $\bigwedge^i T^{\mathbb{Z}}(\tau)=0$, the second summand of the last expression vanishes and we see that the coefficient of $[\tau]$ is given by

$$\langle\omega,\eta_\tau\rangle\left(\sum_{\sigma\in\Sigma_\tau(i)}\big\langle(d(\varphi|_\sigma)-dl_\tau),n_{\sigma/\tau}\big\rangle A(\sigma)\right)\ .$$

But this is precisely the coefficient of $[\tau]$ one gets in $\mathrm{cyc}_X(D\cdot A)$ (compare the coefficient with the weights of $D\cdot A$ described in §3.4), finishing the proof. □

5.7. Compatibility with the singular cycle class map

Let X be a locally polyhedral space, equipped with a face structure Σ whose cells are compact, and let $A\in Z_k(X)$ be a tropical k-cycle on X. There exists a face structure Σ' on $|A|$ such that every cell of Σ' is contained in a cell of Σ. For each $\sigma\in\Sigma'(k)$ we choose an orientation $\eta_\sigma\in\bigwedge^k T^{\mathbb{Z}}(\sigma)$, which give rise to chains $[\sigma]\in\Gamma(X,\Delta_X^{-k})$ supported on σ via cellular homology. The *locally finite cycle class of* A, as considered in [22, Section 4] (also see [32, Section 4]) is defined as

$$\mathrm{cyc}^{lf}_X(A)=\sum_{\sigma\in\Sigma'(k)}A(\sigma)\cdot[\sigma]\otimes\eta_\sigma\ ,$$

where $[\sigma] \otimes \eta_\sigma$ is the locally finite tropical (k,k)-chain defined by $[\sigma]$ and η_σ, and $A(\sigma)$ is the constant value that A has on σ. Since any two face structures have a common refinement, this is independent of the choice of Σ' and defines a morphism

$$\mathrm{cyc}^{lf}_X \colon Z_k(X) \to H^{lf}_{k,k,}(X;\Sigma) \ .$$

Theorem 5.13. *Let X be a rational polyhedral space, equipped with a face structure Σ with compact cells. Then the natural isomorphism (Theorem 4.20)*

$$H^{lf}_{k,k}(X;\Sigma) \xrightarrow{\cong} H^{BM}_{k,k}(X)$$

is compatible with the two tropical cycle class maps. In other words, the diagram

$$\begin{array}{ccc} & Z_k(X) & \\ {\scriptstyle \mathrm{cyc}^{lf}_X} \swarrow & & \searrow {\scriptstyle \mathrm{cyc}_X} \\ H^{lf}_{k,k}(X;\Sigma) & \xrightarrow{\cong} & H^{BM}_{k,k}(X) \end{array}$$

is commutative.

Proof. Let $A \in Z_k(X)$, and let Σ' be a face structure on $|A|$ whose cells are contained in cells of Σ. Choose orientations $\eta_\sigma \in \bigwedge^k T^{\mathbb{Z}}(\sigma)$ for all $\sigma \in \Sigma'$, and let $[\sigma] \in \Gamma(X, \Delta_X^{-k})$ be the chain supported on σ obtained from η_σ via cellular homology. Using the face structure Σ', the morphism $\mathrm{cyc}_X(A) \colon \Omega^k_X[k] \to \mathbb{D}_X$ in $D(\mathbb{Z}_X)$ can actually be represented as a morphism of complexes $\Omega^k_X[k] \to \Delta_X^{\Sigma,\bullet}$, namely as the morphism whose component in degree $-k$ is given by

$$\Omega^n_X[k] \ni \omega \mapsto \sum_{\sigma \in \Sigma'(k)} A(\sigma) \cdot \langle \omega, \eta_\sigma \rangle [\sigma] \in \Delta_X^{\Sigma,-k} \ .$$

This corresponds to the locally finite tropical (k,k)-chain

$$\sum_{\sigma \in \Sigma'(k)} A(\sigma) \cdot [\sigma] \otimes \eta_\sigma \in C^{lf}_{k,k}(X;\Sigma)$$

under the isomorphism $\mathrm{Hom}(\Omega^k_X, \Delta_X^{\Sigma,-k}) \cong C^{lf}_{k,k}(X;\Sigma)$ used in the proof of Theorem 4.20. Noting that this tropical chain represents $\mathrm{cyc}_X(A)$ by definition finishes the proof. $\square$

6. Poincaré-Verdier duality

In this section we study when a rational polyhedral space satisfies Poincaré-Verdier duality. Our goal is to prove that this is the case if and only if they are smooth in the sense of [4,1]. Note that the most commonly used example of smoothness for rational polyhedral spaces in the literature is being locally matroidal. By the results of [22], locally matroidal rational polyhedral spaces are smooth in the sense of [4,1].

6.1. Smoothness

Let X be a rational polyhedral space of dimension n. We say that X *admits a fundamental class* if it is pure-dimensional and the constant function with value 1 defines an element $[X] \in Z_n(X)$. By the definition of balancing, if X admits a fundamental class, then the same is true for the local cones $\mathrm{LC}_x(X)$ for all $x \in X$.

We say that X is *regular at infinity* if every point of X has a neighborhood isomorphic to an open subset of $|F| \times \mathbb{T}^n$ for some fan F and some $n \in \mathbb{Z}$. This notion is equivalent to the one introduced in [32].

Following [22], we say that an n-dimensional rational polyhedral space X admitting a fundamental class *satisfies Poincaré duality* if for all $p, q \in \mathbb{Z}$ the morphism

$$H^{p,q}(X) \to H^{BM}_{n-p,n-q}(X), \;\; \alpha \mapsto [X] \frown \alpha$$

is an isomorphism.

Following [4] and [1] we make the following definition.

Definition 6.1. Let X be a rational polyhedral space. We say that X is *smooth* if it is regular at infinity, admits a fundamental class, and $\mathrm{LC}_x X$ satisfies Poincaré duality for all $x \in X$.

Remark 6.2.

a) If Σ is a fan, then a local cone $\mathrm{LC}_x |\Sigma|$ only depends on the unique cone σ of Σ containing x in its relative interior. It coincides with what is sometimes called the star of Σ at σ. Aksnes follows this convention on stars in [1] and it follows that Definition 6.1 agrees with the definition of what Aksnes calls "local Poincaré duality space" as that condition is precisely that all stars of a fan should satisfy Poincaré duality. Amini and Piquerez [4] use the same condition, namely that all stars should satisfy Poincaré duality, for what they call "smooth" fans, but they use the other common convention on stars: for them, the star of Σ at σ is the image of $\mathrm{LC}_x |\Sigma|$ in the quotient $\mathrm{Span}\, |\Sigma| / \mathrm{Span}\, \sigma$ (where we still assume $x \in \mathrm{relint}(\sigma)$). As Aksnes has pointed out [1, page 24], the Künneth formula (Theorem 4.16) implies that a fan, and hence any rational polyhedral space, is a "Poincaré duality space" in the sense of Aksnes if and only if it is "smooth" in the sense of Amini-Piquerez.

b) While we follow Amini-Piquerez in their usage of the adjective "smooth", tropically smooth spaces are analogous to classical smooth spaces only in that they locally satisfy Poincaré duality. Classically, there is the much larger class of orientable homology manifolds that also satisfy Poincaré duality locally but are not considered smooth. Prior to the work of Amini–Piquerez and Aksnes, the adjective "smooth" had been reserved for tropical manifolds, which are connected rational polyhedral spaces that are regular at infinity and whose local cones are Bergman fans of matroids [35,32]. It was shown in [22] that tropical manifolds are smooth in the sense of Definition 6.1.

Lemma 6.3. *Let X be a rational polyhedral space. If X is smooth, then so is $\mathrm{LC}_x X$ for all $x \in X$. In particular X is smooth if an only if it is pure-dimensional and every point of X is isomorphic to an open subset of $F \times \mathbb{T}^n$ for some smooth fan F and some nonnegative integer n.*

Proof. For every $y \in \mathrm{LC}_x X$ there exists $z \in X$ close to x with $\mathrm{LC}_z X = \mathrm{LC}_y(\mathrm{LC}_x X)$. Therefore, smoothness of X implies smoothness of $\mathrm{LC}_x X$.

The "in particular" statement follows from the fact that X is regular at infinity if and only if every $x \in X$ has a neighborhood isomorphic to an open subset of $\mathrm{LC}_x X \times \mathbb{T}^n$ for some nonnegative integer n. □

6.2. Poincaré-Verdier duality

We now study the Verdier dual of the sheaf of tropical p-forms on a smooth rational polyhedral space X. Recall that, for any complex $\mathscr{C}^\bullet \in D(\mathbb{Z}_X)$, its *Verdier dual* is defined as

$$\mathscr{D}(\mathscr{C}^\bullet) := R\,\mathscr{H}\!om^\bullet(\mathscr{C}^\bullet, \mathbb{D}_X)\ .$$

It is immediate from this definition that $\mathscr{D}(\mathbb{Z}_X) = \mathbb{D}_X$. What is less obvious is that for a constructible complex $\mathscr{C}^\bullet \in D(\mathbb{Z}_X)$ there is a natural isomorphism $\mathscr{D}(\mathscr{D}(\mathscr{C}^\bullet)) \cong \mathscr{C}^\bullet$, justifying the terminology "dual".

If X is purely n-dimensional and admits a fundamental class, then there is a *duality morphism*

$$\delta_p\colon \Omega_X^{n-p}[n] \to \mathscr{D}(\Omega_X^p)$$

defined as the composite

$$\Omega_X^{n-p}[n] \to \mathscr{H}\!om(\Omega_X^p, \Omega_X^n[n]) \to R\,\mathscr{H}\!om^\bullet(\Omega_X^p, \Omega_X^n[n]) \to R\,\mathscr{H}\!om^\bullet(\Omega_X^p, \mathbb{D}_X) = \mathscr{D}(\Omega_X^p)\ ,$$

where the last morphism is the composition with the morphism $\Omega_X^n[n] \xrightarrow{[X]} \mathbb{D}_X$ given by the fundamental class.

Definition 6.4. We say that a rational polyhedral space X admitting a fundamental class satisfies *Poincaré-Verdier duality* if for all $p \in \mathbb{N}$ the duality morphism δ_p is an isomorphism.

Note that δ_p being an isomorphism, and hence satisfying Poincaré-Verdier duality, is a local condition.

Lemma 6.5. *Let X be an n-dimensional rational polyhedral space admitting a fundamental class and satisfying Poincaré-Verdier duality. Furthermore, let Φ be a family of supports on an open subset $U \subseteq X$. Then for all $p, q \in \mathbb{Z}$, the cap product with the fundamental class $[U]$ of U induces an isomorphism*

$$H_\Phi^{n-p,n-q}(U) \xrightarrow{[U]\frown(-)} H^\Phi_{p,q}(U) \ .$$

Proof. Since satisfying Poincaré-Verdier duality is a local condition, U satisfies Poincaré-Verdier duality and we may assume $U = X$. Applying hypercohomolgy with supports is functorial, so X satisfying Poincaré-Verdier duality implies that δ_p induces isomorphisms on hypercohomology. Applying hypercohomolgy with supports in Φ in degree $-q$ to δ_p we thus obtain an isomorphism

$$H_\Phi^{n-p,n-q}(X) = \mathbb{H}_\Phi^{-q}(\Omega_X^{n-p}[n]) \xrightarrow{\mathbb{H}_\Phi^{-q}(\delta_p|_X)} \mathbb{H}_\Phi^{-q}(\mathscr{D}(\Omega_X^p)) = H^\Phi_{p,q}(X) \ .$$

By definition of δ_p, this is precisely the cap product with the fundamental class $[X]$. □

Proposition 6.6. *Let X be an n-dimensional rational polyhedral space admitting a fundamental class. Then the following are equivalent:*

a) X satisfies Poincaré-Verdier duality.
b) Every open subset of X satisfies Poincaré duality.
c) Every point of X has a neighborhood basis of open subsets satisfying Poincaré duality.

Proof. That a) implies b) is precisely the content of Lemma 6.5 when one takes as the family of supports all closes subsets. Clearly, b) implies c). Now assume c) holds, let $x \in X$, and let $\mathscr{U}$ be a neighborhood basis for x consisting of open subsets of X satisfying Poincaré duality. As observed above, the morphism

$$\mathbb{H}^{-q}(\Omega_X^{n-p}[n]|_U) \to \mathbb{H}^{-q}(\mathscr{D}(\Omega_X^p)|_U)$$

induced by δ_p is given by the cap product with the fundamental class, and therefore is an isomorphism for all $U \in \mathscr{U}$ by assumption. It follows that the morphism

$$\mathscr{H}^{-q}(\Omega_X^{n-p}[n])_x = \varinjlim_{U \in \mathscr{U}} \mathbb{H}^{-q}(\Omega_X^{n-p}[n]|_U) \to \varinjlim_{U \in \mathscr{U}} \mathbb{H}^{-q}(\mathscr{D}(\Omega_X^p)|_U) = \mathscr{H}^{-q}(\mathscr{D}(\Omega_X^p))_x$$

induced by δ_p is an isomorphism. Since x, p, and q were arbitrary, this implies that all duality morphisms are isomorphisms, implying a). $\square$

Theorem 6.7. *Let X be an n-dimensional rational polyhedral space that is regular at infinity and admits a fundamental class. Then X satisfies Poincaré-Verdier duality if and only if X is smooth.*

Proof. Let $x \in X$. By Proposition 6.6 it suffices to show that $\mathrm{LC}_x X$ satisfies Poincaré duality if any only if x has a neighborhood basis consisting of open sets satisfying Poincaré duality. We first prove this in the case where $X = L \times \mathbb{T}^k$ for some fan L and $k \in \mathbb{Z}_{\geq 0}$, and x is the unique point of $\{0\} \times \{\infty\}^k$. In this case, X satisfies Poincaré duality if and only if $L = \mathrm{LC}_x X$ satisfies Poincaré duality by [22, Lemma 5.8]. It is left to show that there exist sufficiently many neighborhoods of x in X that satisfy Poincaré duality. While x does not have enough neighborhoods that are isomorphic to X as rational polyhedral spaces, it is easy to construct neighborhoods V of x that are homeomorphic to X in a way respecting any fan structure of L, and this is enough to conclude that V satisfies Poincaré duality if and only if X does. We now provide the details of this argument: the point x has a neighborhood basis consisting of sets of the form

$$(U \cap L) \times (\alpha, \infty]^k \ ,$$

for sufficiently small balls U in $T_x X$ and sufficiently large $\alpha \in \mathbb{R}$. Let V be such a neighborhood. There exists a homeomorphism

$$f_1 \colon L \to U \cap L$$

such that $f_1(v)$ is a positive multiple of v for all $v \in L$ and a strictly increasing homeomorphism $f_2 \colon \mathbb{T} \to (\alpha, \infty]$. Let $f = f_1 \times f_2^k \colon X \to V$ be the induced homeomorphism. By construction, the homeomorphism f maps each stratum of X into itself. Therefore, for each $p \in \mathbb{Z}_{\geq 0}$ the sheaves $f^{-1}\Omega^p_V$ and Ω^p_X have isomorphic stalks. Since both sheaves are locally constant on the strata of X and these locally constant sheaves are glued identically, they are in fact isomorphic. As f is a homeomorphism we also have $f^{-1}\mathbb{D}_V = \mathbb{D}_X$ and f induces isomorphisms

$$f^{-1} \colon H^{p,q}(V) \xrightarrow{\cong} H^{p,q}(X) \ , \text{ and}$$

$$f^{-1} \colon H^{BM}_{p,q}(V) \xrightarrow{\cong} H^{BM}_{p,q}(X) \ .$$

If either V or X satisfies Poincaré duality, then $H^{BM}_{n,n}(V) = H^{BM}_{n,n}(X) = \mathbb{Z}$. Therefore, $f^{-1}[V] = \pm[X]$, which in turn implies that $f^{-1}\delta^V_p = \pm f^{-1}\delta^X_p$. We conclude that V satisfies Poincaré duality if and only if X satisfies Poincaré duality. As already explained, this happens if and only if $L = \mathrm{LC}_x X$ satisfies Poincaré duality.

If X is a general polyhedral space, then x has a neighborhood isomorphic to $\mathrm{LC}_x X \times \mathbb{T}^k$ for some $k \in \mathbb{Z}_{\geq 0}$ in a way that x corresponds to the unique point of $\{0\} \times \{\infty\}^k$. The assertion is thus reduced to the case treated above. □

Corollary 6.8. *Let X be a smooth n-dimensional rational polyhedral space. Then there is a natural isomorphism $\mathbb{D}_X \cong \Omega_X^n[n]$.*

Proof. The dualizing complex $\mathbb{D}_X$ is canonically isomorphic to $\mathscr{D}(\mathbb{Z}_X)$. Since $\mathbb{Z}_X = \Omega_X^0$, we obtain

$$\mathbb{D}_X = \mathscr{D}(\mathbb{Z}_X) = \mathscr{D}(\Omega_X^0) \cong \Omega_X^n[n]$$

by applying Theorem 6.7. □

Another immediate consequence of Theorem 6.7 is that smooth rational polyhedral spaces satisfy Poincaré duality with respect to arbitrary families of supports. As already mentioned in the introduction, for compact and closed supports this has also been proved in [4,1] building on the Mayer-Vietoris argument given for tropical manifolds in [22]. To apply the result in practice, one first needs to show that the rational polyhedral space X in question is smooth. For this one can use [22], where it is proved that tropical manifolds are smooth, or the results of [4] about shellable fans.

Corollary 6.9. *Let X be a smooth rational polyhedral space of dimension n. Then for every $p, q \in \mathbb{Z}$ and every family of supports Φ the morphism*

$$H_\Phi^{n-p,n-q}(X) \to H_{p,q}^\Phi(X), \quad \alpha \mapsto [X] \frown \alpha$$

is an isomorphism.

Proof. This follows directly from Lemma 6.5 and Theorem 6.7. □

Data availability

No data was used for the research described in the article.

Appendix A. Complexes of singular chains on CS sets

The goal of this appendix is to construct an explicit complex $\Delta_X^{\mathscr{S},\bullet}$ quasi-isomorphic to $\mathbb{D}_X$ on a rational polyhedral space X such that

$$\mathrm{Hom}^\bullet(\Omega_X^p, \Delta_X^{\mathscr{S},\bullet}) = R\,\mathrm{Hom}^\bullet(\Omega_X^p, \Delta_X^{\mathscr{S},\bullet})$$

for all $p \in \mathbb{N}$. To prove this result, we generalize it to a result about constructible sheaves on conically stratified spaces, which are well-studied in the context of intersection cohomology. We refer to [6] for an early definition of conically stratified spaces and the classical study of their intersection cohomology (which is sheaf theoretic), and to [13] for a treatment with a focus on singular intersection homology, which is particular relevant in this appendix.

Definition A.1 *(see [13, Definition 2.2.16]).* Let X be a topological space. A *stratification* of X is a collection $\mathscr{S}$ of disjoint locally closed subsets of X such that $X = \bigcup_{S\in\mathscr{S}} S$, each $S \in \mathscr{S}$ is a pure-dimensional topological manifold, and such that for every $S \in \mathscr{S}$ the closure $\overline{S}$ is a union of strata of dimension less than $\dim(S)$.

Next we recall the definition of conically stratified spaces. If L is a stratified space, we will use the notation $\mathring{c}(L)$ for the open cone over L, that for the space $(\mathbb{T} \times L)/(\{\infty\} \times L)$. The open cone $\mathring{c}(L)$ has an induced stratification, with the cone point being the unique 0-dimensional stratum, and all other strata being of the form $\mathbb{R} \times S$, where S is a stratum of L.

Definition A.2 *(see [13, Definition 2.3.1]).* Let X be a topological space, equipped with a stratification $\mathscr{S}$. We say that X is *conically stratified*, or a *CS set* for short, if for all $S \in \mathscr{S}$ and $x \in \mathscr{S}$ there exist a neighborhood U of x in S, a neighborhood V of x in X, and a compact stratified space L such that V is homeomorphic to $U \times \mathring{c}(L)$ in a way respecting the stratification.

Definition A.3. We say that a stratification $\mathscr{S}$ of a topological space X is *admissible*, if the stratified space $(X, \mathscr{S})$ is conically stratified and for every stratum $S \in \mathscr{S}$ the pair $(\overline{S}, S)$ is homeomorphic to a pair $(U, \mathring{D}^n)$, where $\mathring{D}^n$ is the open unit disc in $\mathbb{R}^n$ and U is an open subset of the closed unit disc D^n that contains $\mathring{D}^n$.

Example A.4. If X is a rational polyhedral space with a face structure Σ, then the relative interiors of the polyhedra in Σ stratify X, and this stratification is admissible.

Let X be a topological space equipped with a stratification $\mathscr{S}$. Exactly as for face structures (see §4.7), we say that a singular simplex $\sigma\colon \Delta^q \to X$ (where Δ^q denotes the standard q-simplex) respects the stratification $\mathscr{S}$ if the relative interior of any face of Δ^q is mapped into a stratum of $\mathscr{S}$. For every open set $U \subseteq X$ and $i \in \mathbb{Z}$ we denote by $C_i(U; \mathscr{S})$ the free abelian group on all singular i-simplices in U respecting the stratification $\mathscr{S}$. Since faces of simplices respecting the stratification respect the stratification again, we obtain a chain complex $C_\bullet(U, \mathscr{S})$, and a quotient

$$C_\bullet(X, U; \mathscr{S}) = C_\bullet(X; \mathscr{S})/C_\bullet(U; \mathscr{S}) \tag{A.1}$$

of relative chains that respect the stratification. We denote the i-th homology of these complexes by

$$H_i(U;\mathscr{S}) = H_i(C_\bullet(U;\mathscr{S})) \ , \text{ and}$$
$$H_i(X,U;\mathscr{S}) = H_i(C_\bullet(X,U;\mathscr{S})) \ .$$

For every k we denote by $\Delta_X^{\mathscr{S},-k}$ the sheafification of the presheaf $U \mapsto C_k(X, X\setminus\overline{U};\mathscr{S})$. The differentials on the complexes of relative chains that respect the stratification induce a differential that makes $\Delta_X^{\mathscr{S},\bullet}$ a cochain complex. By definition, $\Delta_X^{\mathscr{S},\bullet}$ is a subcomplex of $\Delta_X^\bullet$.

Proposition A.5. *Let X be a conically stratified space with stratification $\mathscr{S}$. Then the inclusion map*

$$\Delta_X^{\mathscr{S},\bullet} \to \Delta_X^\bullet$$

is a quasi-isomorphism.

Proof. For the purpose of this proof we will denote

$$H_*^{\text{strat}}(X) = H_*(X;\mathscr{S})$$

for a conically stratified space X with stratification $\mathscr{S}$.

We need to show that $H^{-i}(\Delta_X^{\mathscr{S},\bullet}) \to H^{-i}(\Delta_X^\bullet)$ is an isomorphism of sheaves for all $i \in \mathbb{Z}$. At a point $x \in X$, the stalks of these sheaves are $H_i(X, X\setminus\{x\};\mathscr{S})$ and $H_i(X, X\setminus\{x\})$, respectively, so using the long exact sequence for relative homology and the five lemma, it suffices to show that the natural morphisms

$$H_*^{\text{strat}}(U) \to H_*(U)$$

are isomorphisms for all open subsets $U \subseteq X$. This follows from [13, Theorem 5.1.4] once we show that the four hypothesis of the theorem are satisfied.

(1) Since the barycentric subdivision restricts to an equivalence of complexes $C_\bullet(X;\mathscr{S}) \to C_\bullet(X;\mathscr{S})$, there are compatible Mayer-Vietoris sequences for H_*^{strat} and H_*.

(2) If $\{U_\alpha\}$ is an increasing collection of open subsets of a CS set X such that

$$H_*^{\text{strat}}(U_\alpha) \to H_*(U_\alpha)$$

is an isomorphism for each α, then

$$H_*^{\text{strat}}\left(\bigcup_\alpha U_\alpha\right) \to H_*\left(\bigcup_\alpha U_\alpha\right)$$

is also an isomorphism because

$$H_*^{\text{strat}}\left(\bigcup_\alpha U_\alpha\right) \cong \varinjlim_\alpha H_*^{\text{strat}}(U_\alpha) \text{ and}$$
$$H_*\left(\bigcup_\alpha U_\alpha\right) \cong \varinjlim_\alpha H_*(U_\alpha) \ .$$

(3) The statement is true if X is a point. It is also true if X is homeomorphic to $\mathbb{R}^n \times \mathring{c}(L)$ in a way respecting the stratification for some $n \in \mathbb{N}$ and some CS set L, because in this case X can be contracted to a point in a way that respects the stratification, reducing to the case where X is a point.

(4) If X only has a single stratum, then $C_\bullet^{\text{strat}}(X) = C_\bullet(X)$ and therefore the statement is true for X. □

The following Proposition is inspired by [12, Proposition 3.7].

Proposition A.6. *Let X be a conically stratified space with stratification $\mathscr{S}$, and let $U \subseteq X$ be an open subset. Then the inclusion*

$$\Delta_U^{U\cap\mathscr{S},\bullet} \to \Delta_X^{\mathscr{S},\bullet}|_U \ ,$$

where $U \cap \mathscr{S}$ is the induced stratification on U, is an equivalence of complexes

Proof. For every open subset $V \subseteq U$ with $\overline{V} \subseteq U$ the inclusion

$$C_\bullet(U, U \setminus \overline{V}; \mathscr{S}) \to C_\bullet(X, X \setminus \overline{V}; \mathscr{S})$$

is an equivalence by the excision theorem (see [19, Section 2.1]). To show that this stays an equivalence when sheafifying, we need to make sure that the homotopy inverses are compatible with restrictions. Let $S\colon C_\bullet(X;\mathscr{S}) \to C_\bullet(X;\mathscr{S})$ be the barycentric subdivision (see [19, Section 2.1]) and let $T\colon \text{ id} \Rightarrow S$ be a (functorial) chain homotopy between the identity and S. We use T to define a new chain homotopy T_U whose action on singular n-simplices is given by

$$T_U\sigma = \begin{cases} 0, & \text{if } \sigma(\Delta^n) \subseteq U, \\ T\sigma, & \text{else.} \end{cases}$$

By the functoriality of T, this induces morphisms

$$C_k(X, X \setminus \overline{V}; \mathscr{S}) \to C_{k+1}(X, X \setminus \overline{V}; \mathscr{S})$$

for all $k \in \mathbb{N}$ and open subsets $V \subseteq X$, which are compatible with restrictions. Therefore, they induce morphism $\Delta_X^{\mathscr{S},k} \to \Delta_X^{\mathscr{S},k-1}$ of sheaves for all $k \in \mathbb{Z}$, whose restrictions to U we denote by $\psi^k\colon \Delta_X^{\mathscr{S},k}|_U \to \Delta_X^{\mathscr{S},k-1}|_U$. Let $f = \text{id} - (d\psi + \psi d)$. We claim that the sequence $(f^i)_i$ of powers of f converges to a morphism f^∞ in the sense that for every section $s \in \Gamma(V, \Delta_X^{\mathscr{S},k})$, where $V \subseteq U$ is open, there exists a covering $V = \bigcup_\lambda V_\lambda$ such that the sequence $f^i(s)|_{V_\lambda}$ converges to $f^\infty(s)|_{V_\lambda}$ in the discrete topology on $\Gamma(V_\lambda, \Delta_X^{\mathscr{S},k})$ (that is the sequence is eventually constant). As the open subsets $V \subset U$ with $\overline{V} \subseteq U$ form a basis for the topology of U, it suffices to show that the sequence $(f^i(s))_i$ is eventually constant for every $s \in C_{-k}(X, X \setminus \overline{V}; \mathscr{S})$ for such an open subset V of U. By linearity, we may even assume that s is represented by a single singular $(-k)$-simplex σ. We have

$$f(\sigma) = \sigma - (\partial\psi + \psi\partial)\sigma = \sigma - (\partial T + T\partial)\sigma + (\partial(T - T_U) + (T - T_U)\partial)\sigma = \\ = S(\sigma) + (\partial(T - T_U) + (T - T_U)\partial)\sigma \ .$$

Note that for any singular j-simplex δ we have

$$(T - T_U)(\delta) = \begin{cases} T\delta, & \text{if } \delta(\Delta^j) \subseteq U \\ 0, & \text{else.} \end{cases}$$

In particular, $(T - T_U)c$ is a linear combination of simplices contained in U for every chain c. Therefore, $f(\sigma) - S\sigma$ is represented by a linear combination of singular simplices that are contained in U. As f is the identity on simplices that are contained in U, we see inductively that $f^i(\sigma) - S^i\sigma$ is represented by a linear combination of simplices that are contained in U for all $i \in \mathbb{N}$. Since we assumed that $\overline{V} \subset U$, the space X is the union of U and $X \setminus \overline{V}$. Thus, for i large enough the chain $S^i\sigma$ will be represented by a linear combination of simplices that are either contained in U or in $X \setminus \overline{V}$. The latter are 0 in $C_{-k}(X, X \setminus \overline{V}; \mathscr{S})$, so $f^i(\sigma)$ is represented by a linear combination of simplices that are contained in U. It follows immediately that the sequence $(f^i(\sigma))_i$ is eventually constant, and we conclude that f^∞ is well-defined. It also follows that $f^\infty(\sigma)$ is contained in $C_{-k}(U, U \setminus \overline{V}; \mathscr{S})$. This shows that f^∞ maps into the subcomplex $\Delta_U^{U\cap\mathscr{S},\bullet}$ of $\Delta_X^{\mathscr{S},\bullet}|_U$.

It is clear from the definitions that f^∞ respects the differentials, so we have constructed a morphism $\Delta_X^{\mathscr{S},\bullet}|_U \to \Delta_U^{U\cap\mathscr{S},\bullet}$. If $\iota\colon \Delta_U^{U\cap\mathscr{S},\bullet} \to \Delta_X^{\mathscr{S},\bullet}|_U$ denotes the inclusion, then $f^\infty \circ \iota = \mathrm{id}$ by construction. The construction of f^∞ also provides a chain homotopy $\mathrm{id} \Rightarrow \iota \circ f^\infty$, namely the limit

$$\sum_{i=0}^{\infty} \psi f^i \ .$$

This converges in the same sense as before, because once $f^i(s)$ in $\Delta_U^{U\cap\mathscr{S},\bullet}$ we have $\psi(f^i(s)) = 0$ by definition. We conclude that ι is an equivalence of complexes. □

Proposition A.7. *Let X be a conically stratified space with stratification $\mathscr{S}$, and let $A \subset X$ be a locally closed subset of X that is a union of strata. Then the morphism*

$$\mathrm{Hom}^\bullet(\mathbb{Z}_A, \Delta_X^{\mathscr{S},\bullet}) \to R\,\mathrm{Hom}^\bullet(\mathbb{Z}_A, \mathbb{D}_X)$$

that is induced by the inclusion $\Delta_X^{\mathscr{S},\bullet} \to \Delta_X^\bullet$ *and the natural identification* $\Delta_X^\bullet \cong \mathbb{D}_X$*, is an isomorphism in* $D(\mathbb{Z})$*.*

In particular, the natural morphism

$$\mathscr{H}om^\bullet(\mathbb{Z}_A, \Delta_X^{\mathscr{S},\bullet}) \to R\,\mathscr{H}om^\bullet(\mathbb{Z}_A, \mathbb{D}_X)$$

is an isomorphism in $D(\mathbb{Z}_X)$*.*

Proof. By Proposition A.5, we need to show that if $\Delta_X^{\mathscr{S},\bullet} \to \mathscr{I}^\bullet$ is an injective resolution, then the induced morphism

$$\mathrm{Hom}^\bullet(\mathbb{Z}_A, \Delta_X^{\mathscr{S},\bullet}) \to \mathrm{Hom}^\bullet(\mathbb{Z}_A, \mathscr{I}^\bullet)$$

is a quasi-isomorphism. Let $U \subseteq X$ be an open subset such that A is closed in U. Then the morphism above is a quasi-isomorphism if and only if the morphism

$$\mathrm{Hom}^\bullet(\mathbb{Z}_A, \Delta_X^{\mathscr{S},\bullet}|_U) \to \mathrm{Hom}^\bullet(\mathbb{Z}_A, \mathscr{I}^\bullet|_U)$$

is a quasi-isomorphism. Since the natural morphism $\Delta_U^{U\cap\mathscr{S},\bullet} \to \Delta_X^{\mathscr{S},\bullet}|_U$ is a chain equivalence by A.6, the induced morphism

$$\mathrm{Hom}^\bullet(\mathbb{Z}_A, \Delta_U^{U\cap\mathscr{S},\bullet}) \to \mathrm{Hom}^\bullet(\mathbb{Z}_A, \Delta_X^{\mathscr{S},\bullet}|_U)$$

is a quasi-isomorphism as well, so it suffices to show that the morphism

$$\mathrm{Hom}^\bullet(\mathbb{Z}_A, \Delta_U^{U\cap\mathscr{S},\bullet}) \to \mathrm{Hom}^\bullet(\mathbb{Z}_A, \mathscr{I}^\bullet|_U)$$

that is induced by the composite $\Delta_U^{U\cap\mathscr{S},\bullet} \to \Delta_X^{\mathscr{S},\bullet}|_U \to \mathscr{I}^\bullet|_U$ is a quasi-isomorphism. As this composite is an injective resolution of $\Delta_U^{U\cap\mathscr{S},\bullet}$ we may replace X by U and assume that A is closed in X.

Let $i\colon A \to X$ be the inclusion. Since $i^!$ is right-adjoint to i_*, it suffices to show that the morphism

$$\Gamma_A(X, \Delta_X^{\mathscr{S},\bullet}) = \mathrm{Hom}^\bullet\left(\mathbb{Z}_A, i^!\left(\Delta_X^{\mathscr{S},\bullet}\right)\right) \to \mathrm{Hom}^\bullet\left(\mathbb{Z}_A, i^!\mathscr{I}^\bullet\right) = \Gamma(A, i^!\mathscr{I}^\bullet)$$

is a quasi-isomorphism. The natural morphism $\Delta_A^{A\cap\mathscr{S},\bullet} \to i^!(\Delta_X^{\mathscr{S},\bullet})$ defined by push-forwards of chains along i defines an isomorphism

$$\Gamma(A, \Delta_A^{A\cap\mathscr{S},\bullet}) \cong \Gamma_A(X, \Delta_X^{\mathscr{S},\bullet})$$

on global sections because both sides are the chain complexes of locally finite chains in A that respect the stratification. It thus suffices to show that the morphism

$$\Gamma(A, \Delta_A^{A\cap\mathscr{S},\bullet}) \to \Gamma(A, i^! \mathscr{I}^\bullet) \tag{A.2}$$

induced by the composite $\Delta_A^{A\cap\mathscr{S},\bullet} \to i^!\Delta_X^{\mathscr{S},\bullet} \to i^!\mathscr{I}^\bullet$ is a quasi-isomorphism. By construction, there is a commutative diagram

$$\begin{array}{ccc} \Delta_A^{A\cap\mathscr{S},\bullet} & \longrightarrow & i^!\mathscr{I}^\bullet \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{D}_A & \longrightarrow & i^!\mathbb{D}_X \end{array}$$

in $D(\mathbb{Z}_A)$ whose vertical arrows are isomorphism. Since we defined the upper horizontal morphism via the push-forward of singular cycles, which is, of course, compatible with the trace morphisms, the lower horizontal morphism is the natural isomorphism $\mathbb{D}_A \cong i^!\mathbb{D}_X$. We conclude that the upper horizontal morphism is an isomorphism in $D(\mathbb{Z}_A)$ as well. Because $i^!$ is the right-adjoint of the exact functor i_*, the upper horizontal morphism in the diagram is in fact an injective resolution. That the morphism displayed in (A.2) is a quasi-isomorphism now follows from the fact that $\Delta_A^{A\cap\mathscr{S},\bullet}$ is homotopically fine and from [9, IV Theorem 2.2], finishing the proof of the main statement.

For the "in particular" statement we note that

$$\mathrm{Hom}^\bullet(\mathbb{Z}_{U\cap A}, \Delta_U^{U\cap\mathscr{S},\bullet}) \to R\,\mathrm{Hom}^\bullet(\mathbb{Z}_{U\cap A}, \mathbb{D}_U)$$

is a quasi-isomorphism for every open subset $U \subseteq X$ by the main statement. Together with Proposition A.6 we see that

$$\mathrm{Hom}^\bullet(\mathbb{Z}_A|_U, \Delta_X^{\mathscr{S},\bullet}|_U) \to R\,\mathrm{Hom}^\bullet(\mathbb{Z}_A|_U, \mathbb{D}_X|_U)$$

is a quasi-isomorphism for all open subsets U of X, which directly implies the claim. □

Lemma A.8. *Let Y be a subset of the closed unit disc $D^n \subseteq \mathbb{R}^n$ such that its intersection $Z = Y \cap \mathring{D}^n$ with the open unit disc $\mathring{D}^n$ is nonempty and connected. Furthermore, let $\mathscr{F}$ be sheaf of abelian groups on D^n such that the restriction $\mathscr{F}|_{\mathring{D}^n}$ is constant. Then the restriction maps*

$$\mathrm{Hom}(\mathscr{F}|_{\mathring{D}^n}, \mathbb{Z}_{\mathring{D}^n}) \to \mathrm{Hom}(\mathscr{F}|_Z, \mathbb{Z}_Z)\ , \ \textit{and}$$
$$\mathrm{Hom}(\mathscr{F}|_Y, \mathbb{Z}_Y) \to \mathrm{Hom}(\mathscr{F}|_Z, \mathbb{Z}_Z)$$

are isomorphisms.

Proof. The fact that

$$\mathrm{Hom}(\mathscr{F}|_{\mathring{D}^n}, \mathbb{Z}_{\mathring{D}^n}) \to \mathrm{Hom}(\mathscr{F}|_Z, \mathbb{Z}_Z)$$

is an isomorphism follows immediately from the fact that $\mathscr{F}|_{\mathring{D}^n}$ is constant and Z is connected. For the second map we consider the short exact sequence

$$0 \to \mathscr{F}_Z \to \mathscr{F}_Y \to \mathscr{F}_{Y\setminus Z} \to 0$$

of sheaves on Y. Applying $\mathrm{Hom}(-, \mathbb{Z}_Y)$ we obtain an exact sequence

$$0 \to \mathrm{Hom}(\mathscr{F}_{Y\setminus Z}, \mathbb{Z}_Y) \to \mathrm{Hom}(\mathscr{F}_Y, \mathbb{Z}_Y) \to \mathrm{Hom}(\mathscr{F}_Z, \mathbb{Z}_Y) \to \mathrm{Ext}^1(\mathscr{F}_{Y\setminus Z}, \mathbb{Z}_Y) \ .$$

As $\mathbb{Z}_Y$ does not have any sections supported on a proper closed subset of Y, we have

$$\mathrm{Hom}(\mathscr{F}_{Y\setminus Z}, \mathbb{Z}_Y) = 0 \ .$$

Furthermore, there is a natural isomorphism

$$\mathrm{Hom}(\mathscr{F}_Z, \mathbb{Z}_Y) \cong \mathrm{Hom}(\mathscr{F}|_Z, \mathbb{Z}_Z) \ .$$

We conclude that the restriction

$$\mathrm{Hom}(\mathscr{F}|_Y, \mathbb{Z}_Y) \to \mathrm{Hom}(\mathscr{F}|_Z, \mathbb{Z}_Z)$$

is injective. To show that it is surjective as well we consider the commutative square

$$\begin{array}{ccc} \mathrm{Hom}(\mathscr{F}, \mathbb{Z}_{D^n}) & \longrightarrow & \mathrm{Hom}(\mathscr{F}|_{\mathring{D}^n}, \mathbb{Z}_{\mathring{D}^n}) \\ \downarrow & & \downarrow \cong \\ \mathrm{Hom}(\mathscr{F}|_Y, \mathbb{Z}_Y) & \longrightarrow & \mathrm{Hom}(\mathscr{F}|_Z, \mathbb{Z}_Z) \ . \end{array}$$

From the discussion above we know that the vertical arrow on the right is an isomorphism, and that the horizontal arrows are injective (set $Y = D^n$ in the discussion above for the top arrow). So, to finish the proof, it suffices to show that the top horizontal arrow is surjective. In other words, it suffices to prove the result for $Y = D^n$, that is to show the surjectivity of

$$\mathrm{Hom}(\mathscr{F}, \mathbb{Z}_{D^n}) \to \mathrm{Hom}(\mathscr{F}|_{\mathring{D}^n}, \mathbb{Z}_{\mathring{D}^n}) \ .$$

In the exact sequence from above we can see that this map is surjective if and only if $\mathrm{Ext}^1(\mathscr{F}_{S^{n-1}}, \mathbb{Z}_{D^n}) = 0$, where $S^{n-1} = D^n \setminus \mathring{D}^n$. Let $i \colon S^{n-1} \to D^n$ be the inclusion. By Verdier duality for i we see that

$$\mathrm{Ext}^1(\mathscr{F}_{S^{n-1}}, \mathbb{Z}_{D^n}) = \mathrm{Ext}^1(\mathscr{F}|_{S^{n-1}}, i^!\mathbb{Z}_{D^n}) \ .$$

For $k \in \mathbb{N}$, the k-th cohomology sheaf $H^k(i^!\mathbb{Z}_{D^n})$ is the restriction to S^{n-1} of the sheaf associated to the presheaf

$$U \to H^k(U, U \cap \mathring{D}^n)$$

on D^n (cf. [18, pp. 14-15] for a closely related example). As the pair $(D^n, \mathring{D}^n)$ is locally homeomorphic to the pair given by an open half-space in $\mathbb{R}^n$ and its closure, these sheaves are all zero and hence $i^!\mathbb{Z}_{D^n} = 0$, finishing the proof. □

Proposition A.9. *Let X be a conically stratified space with an admissible stratification $\mathscr{S}$, and let $\mathscr{F}$ be a sheaf of abelian groups on X such that $\mathscr{F}|_S$ is locally free of finite rank for every stratum $S \in \mathscr{S}$. Then the natural morphism*

$$\mathscr{H}\!om^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S}}) \to R\,\mathscr{H}\!om^\bullet(\mathscr{F}, \mathbb{D}_X)$$

is an isomorphism in $D(\mathbb{Z}_X)$. In particular, the natural morphism

$$\mathrm{Hom}^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S}}) \to R\,\mathrm{Hom}^\bullet(\mathscr{F}, \mathbb{D}_X)$$

is an isomorphism in $D(\mathbb{Z})$.

Proof. The statement is local on X, so we may assume that the stratification $\mathscr{S}$ is finite. We do induction on the number of strata on which $\mathscr{F}$ is nontrivial. If this number is 0, then $\mathscr{F} = 0$ and the statement is trivial. So let us assume there is a stratum on which $\mathscr{F}$ is nontrivial, and let $S \in \mathscr{S}$ be maximal with that property. The stratum S is an open subset of the support $\mathrm{supp}(\mathscr{F})$, so if $A = \mathrm{supp}(\mathscr{F}) \setminus S$ we obtain an exact sequence

$$0 \to \mathscr{F}_S \to \mathscr{F} \to \mathscr{F}_A \to 0 \ .$$

We can use this to obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathscr{H}\!om^\bullet(\mathscr{F}_A, \Delta_X^{\mathscr{S}}) & \longrightarrow & \mathscr{H}\!om^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S}}) & \longrightarrow & \mathscr{H}\!om^\bullet(\mathscr{F}_S, \Delta_X^{\mathscr{S}}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
R\,\mathscr{H}\!om^\bullet(\mathscr{F}_A, \Delta_X^{\mathscr{S}}) & \longrightarrow & R\,\mathscr{H}\!om^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S}}) & \longrightarrow & R\,\mathscr{H}\!om^\bullet(\mathscr{F}_S, \Delta_X^{\mathscr{S}}) & \longrightarrow &
\end{array}$$

where the lower row is an exact triangle. Note that the vertical arrow on the left is an isomorphism by the induction hypothesis. Since $\mathscr{F}_S$ is locally free and S is simply connected, the sheaf $\mathscr{F}_S$ is isomorphic to a finite sum of several copies of $\mathbb{Z}_S$. So by Proposition A.7, the right arrow is an isomorphism as well. If we can show that the morphisms of complexes in the upper row of the diagram defines a short exact sequence in every degree, the statement follows from the five lemma. We recall from the proof of Theorem 4.20 that for every $i \in \mathbb{Z}$ there is an isomorphism

$$\Delta_X^{\mathscr{S}, -i} \cong \bigoplus_{\sigma} \mathbb{Z}_{\sigma(\Delta^i)} \ ,$$

where the direct sum is over all singular i-simplices $\sigma\colon \Delta^i \to X$ respecting the stratification. Consequentially, for every sheaf of Abelian groups $\mathscr{G}$ that is locally free of finite rank when restricted to any stratum in $\mathscr{S}$, we have

$$\mathscr{H}om(\mathscr{G}, \Delta_X^{\mathscr{S},-i}) = \mathscr{H}om\left(\mathscr{G}, \bigoplus_{\sigma} \mathbb{Z}_{\sigma(\Delta^i)}\right) = \bigoplus_{\sigma} \mathscr{H}om(\mathscr{G}, \mathbb{Z}_{\sigma(\Delta^i)}) \ ,$$

where the last equality holds because $\mathscr{G}$ is constructible. If for a simplex σ appearing in the direct sum we denote by $T_\sigma \in \mathscr{S}$ the unique stratum into which the relative interior of Δ^i maps, there is an isomorphism

$$\mathscr{H}om(\mathscr{G}, \mathbb{Z}_{\sigma(\Delta^i)}) \cong (\mathrm{Hom}(\mathscr{G}|_{T_\sigma}, \mathbb{Z}_{T_\sigma}))_{\sigma(\Delta^i)}$$

induced by restricting sections by Lemma A.8. So, to finish the proof, it suffices to show that for every stratum $T \in \mathscr{S}$ the sequence

$$0 \to \mathrm{Hom}((\mathscr{F}_A)|_T, \mathbb{Z}_T) \to \mathrm{Hom}(\mathscr{F}|_T, \mathbb{Z}_T) \to \mathrm{Hom}((\mathscr{F}_S)|_T, \mathbb{Z}_T) \to 0$$

is exact. If $T = S$, this is the case because the second morphism is an isomorphism and the first group is trivial, whereas if $T \neq S$ this is the case because the first morphism is an isomorphism and the last group is trivial.

For the "in particular" statement we apply $R\Gamma$ to the isomorphism

$$\mathscr{H}om^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S},\bullet}) \to R\,\mathscr{H}om^\bullet(\mathscr{F}, \mathbb{D}_X)$$

and note that the natural morphism

$$\mathrm{Hom}^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S},\bullet}) = \Gamma(\mathscr{H}om^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S},\bullet})) \to R\Gamma\,\mathscr{H}om^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S},\bullet})$$

is an isomorphism because $\Delta_X^{\mathscr{S},\bullet}$, and hence $\mathscr{H}om^\bullet(\mathscr{F}, \Delta_X^{\mathscr{S},\bullet})$, is homotopically fine [9, IV Theorem 2.2]. □

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