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ON TORSOR STRUCTURES ON SPANNING TREES*

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Abstract. We study two actions of the (degree 0) Picard group on the set of the spanning trees of a finite ribbon graph. It is known that these two actions, denoted β_q and ρ_q, respectively, are independent of the base vertex q if and only if the ribbon graph is planar. Baker and Wang [*Int. Math. Res. Not. IMRN*, 16 (2018), pp. 5120–5147] conjectured that in a nonplanar ribbon graph without multiple edges, there always exists a vertex q for which $\rho_q \neq \beta_q$. We prove the conjecture and extend it to a class of ribbon graphs with multiple edges. We also give explicit examples exploring the relationship between the two torsor structures in the nonplanar case.

Key words. graphs and ribbon graphs, spanning trees, divisors, surfaces, torsors, critical group

MSC codes. 05C10, 05C25

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1. Introduction.

The number of spanning trees of a finite loopless graph G is equal to the determinant of the reduced combinatorial Laplacian of G; this is the celebrated Kirchhoff matrix-tree theorem. In addition to counting spanning trees, this determinant also counts the number of elements of a certain group, known as the (degree 0) *Picard group,* critical group, or sandpile group of the graph. This group, denoted $\mathrm{Pic}^0(G)$, is defined as the additive group of chip-firing equivalence classes on the collection of divisors on the graph.

In addition to equality in cardinality of the set $\mathcal{T}(G)$ of spanning trees of G and the Picard group $\mathrm{Pic}^0(G)$, it is known that $\mathcal{T}(G)$ admits at least two distinct torsor structures for the Picard group; the Picard group of a graph acts simply and transitively on the set $\mathcal{T}(G)$. Two such actions, namely, the Bernardi action β_q and the rotor-routing action ρ_q, have been the subject of study in recent years [5, 3] (see also [10, 13, 8]). To define these torsors, we work with graphs endowed with a *ribbon structure,* a cyclic ordering of the edges incident to each vertex. By giving a ribbon structure to a finite graph, we determine an orientable surface in which the graph can be considered as embedded; for this reason, ribbon structures were originally termed "combinatorial embeddings" of graphs [4].

After fixing a ribbon structure on a graph G, the Bernardi torsor β_q and the rotor-routing torsor ρ_q rely on the choice of a base vertex vertex q of the graph. Chan, Church, and Grochow prove that the torsor ρ_q is independent of the initial data q if and only if the ribbon graph is planar [5]. Separately, Baker and Wang prove that the Bernardi torsor β_q is also independent of the initial data q if and only if the ribbon graph is planar. Moreover, Baker and Wang prove in [3] that in the case of a planar ribbon graph, the two torsor structures coincide; the Bernardi and rotor-routing processes produce the same simply transitive group action. Baker and Wang include the following conjecture:

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CONJECTURE 1.1 (Baker and Wang [3]). *Given a nonplanar ribbon graph G with no multiple edges and no loops, there exists a vertex q for which $\rho_q \neq \beta_q$.*

We resolve this conjecture and extend the result to a class of graphs having multiple edges but no loops. In particular, it is proven here that any nonplanar graph with or without multiple edges admits a vertex q for which $\rho_q \neq \beta_q$ when endowed with any ribbon structure. The criterion for the presence of such a vertex q in a nonplanar ribbon graph with multiple edges is the presence of what we call a *proper witness pair.* In Theorem 6.2, we prove that the presence of a proper witness pair in a nonplanar ribbon graph is sufficient to ensure the presence of a vertex q for which $\rho_q \neq \beta_q$. Further, we show in Proposition 3.3 that any nonplanar graph, when endowed with any ribbon structure, must contain a proper witness pair. In particular, any nonplanar graph endowed with any ribbon structure contains a vertex at which the torsor structures disagree. We prove Theorem 7.1, the Baker–Wang conjecture for loopless nonplanar ribbon graphs with no multiple edges admitting no proper witness pair. Finally, we consider two examples and some associated computations; in particular, we exhibit a nonplanar ribbon graph having distinct vertices p and q such that $\rho_p = \beta_p$ while $\rho_q \neq \beta_q$.

In an independent work, Changxin Ding [7] also resolves this conjecture. As in the current paper, Ding considers multiple cases, but the approach taken there utilizes decompositions of graphs to facilitate study of the difference between the two torsors. In the present document, we work instead with classes of graphs depending on the presence or absence of certain subgraphs, referred to here as *witness pairs.*

2. Background and terminology. By a *graph*, we mean a finite connected multigraph with no loops; for a graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its collection of edges, given as unordered pairs of vertices. Occasionally, we use V and E to denote the vertices and edges of a graph G when the graph is unambiguous. Unless otherwise specified, multiple parallel edges are allowed. A *ribbon graph* is a finite graph together with a cyclic ordering of the edges around each vertex, referred to as a *ribbon structure.* Note that a ribbon structure on a graph induces a natural ribbon structure on any graph minor. Given two edges e_0, e_1 incident to a given vertex v, we will write $e_0 \prec e_1$ if e_1 immediately succeeds e_0 in the order around v. If a vertex v has two edges e_0 and e_1, the *interval* $[e_0, e_1]$ between e_0 and e_1 is the collection of edges $\{e_0, f_1, \dots, f_k, e_1\}$, where

$$e_0 \prec f_1 \prec \cdots \prec f_k \prec e_1.$$

Analogously, we define the open interval $(e_0, e_1) = \{f_2, \dots, f_k\}$. The *length* of an interval is its cardinality as a set.

A finite ribbon graph is equivalent to an embedding of the underlying graph into a closed oriented surface such that the ribbon structure agrees with the orientation of the surface (see, e.g., [15, Theorem 3.7]). In all figures in this paper, the ribbon structure is assumed to be given by the counterclockwise orientation of the underlying surface.

A *path* P in a ribbon graph G is the image of a mapping $P_k \to G$ of the length-k path into G that is injective on edges but not necessarily on vertices. A *cycle* in G is the image of a mapping $C_k \to G$ of the length-k cycle into G that is injective on both edges and vertices. We sometimes refer to paths and cycles as sets of edges rather than as subgraphs. Let C be a cycle, and endow C with an orientation. A vertex v of C is incident to precisely two edges of C, e_{in} and e_{out}, where the labeling of these edges agrees with the orientation of C. An edge e incident to v is said to be to the *left* of C if $e \in (e_{\text{out}}, e_{\text{in}})$ and to the *right* of C if $e \in (e_{\text{in}}, e_{\text{out}})$.

Given a group Γ, we consider a group action of Γ on a set X as a function

$$\Gamma \times X \longrightarrow X\,.$$

We say that a group action is *regular* if for every $\gamma \in \Gamma$ and every $x \in X$, the restricted mappings $\{\gamma\} \times X \to X$ and $\Gamma \times \{x\} \to X$ are isomorphisms of sets. Stated differently, an action of Γ on X is regular if Γ acts simply and transitively on X. If $X = \{1, \ldots, N\}$, we can consider a group action as a homomorphism from Γ to the symmetric group on N symbols $\Gamma \to S_N$. We therefore sometimes work with the image of such a mapping, representing group elements as permutations. All multiplication of permutations will be from right to left, so that, for example, we have $(123)(34) = (1234)$.

3. Cycles and witnesses. Here we are concerned in particular with nonplanar graphs. Following [5], we discuss planarity and nonplanarity of ribbon graphs in terms of the absence or presence of nonseparating cycles.

Definition 3.1. *A cycle C is nonseparating if for any orientation of C, there exists a path P that intersects C only in the endpoints of P such that the first edge of P is on the left of C and the last edge of P is on the right of C. We will call P a witness for the nonseparating cycle C and refer to the ordered pair (C, P) as a witness pair.*

Definition 3.2. *Given a nonseparating cycle C, we will say that a witness P for C is a proper witness if the endpoints of P are distinct vertices of C. In this case, we call the pair (C, P) a proper witness pair (see Figure 3.1).*

We say that a connected ribbon graph G is *nonplanar* if and only if it contains a nonseparating cycle. Note that this definition of nonplanarity is not equivalent to the underlying graph G being nonplanar in the typical sense, but all nonplanar graphs yield nonplanar ribbon graphs when endowed with any ribbon structure. A planar graph G can be endowed with a ribbon structure such that the resulting ribbon graph is nonplanar.

We have the following structural property of nonplanar graphs endowed with a ribbon structure.

Proposition 3.3. *A connected nonplanar graph G endowed with any ribbon structure admits a proper witness pair.*

Proof. Starting from an arbitrary ribbon graph G, first note that the operations of edge deletion and edge contraction cannot create a proper witness in G. Thus, by

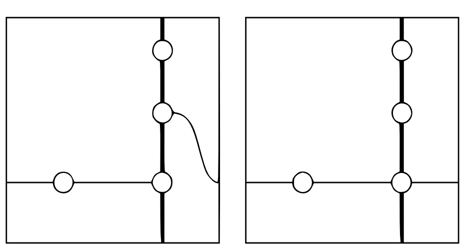


Fig. 3.1. *Nonseparating cycles (thickened) in two nonplanar ribbon graphs on the torus. The left image portrays a proper witness, while the right image portrays an improper witness.*

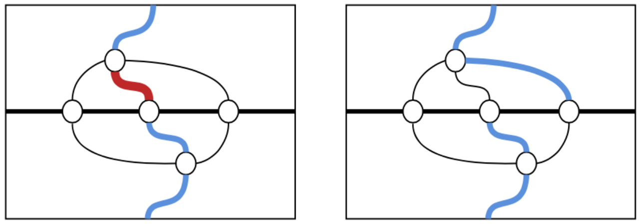


FIG. 3.2. *The construction of a proper witness in K_5 as described in the proof of Proposition 3.3. At the left, we see a thickened cycle of length 3 with an improper witness with edges in red and blue. After replacing the red edge (extra thickened), we have a proper witness in blue, depicted at the right.*

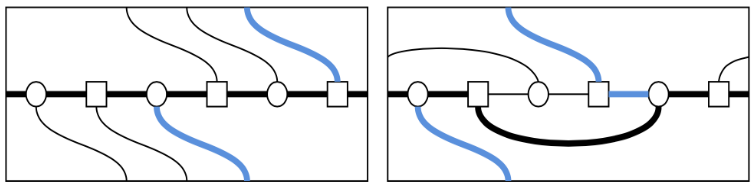


FIG. 3.3. *Two torus embeddings of $K_{3,3}$, each admitting proper witness pairs. In each, the two partition classes of vertices are represented by circles and squares. At the left, the thickened nonseparating cycle contains all 6 vertices; on the right, the thickened cycle contains only 4. Again, in each case, the embedding admits a proper witness pair.*

Kuratowski's theorem, it is enough to illustrate that K_5 and $K_{3,3}$ necessarily admit a proper witness pair when endowed with any ribbon structure.

Given any ribbon structure on K_5, there necessarily exists a nonseparating cycle C; if some witness path P is of length 1, then the fact that K_5 is loopless ensures that P must be proper since the endpoints of P in C are distinct. Further, if P is of length 2, then the fact that K_5 has no multiple edges ensures that P must be proper. Now, C either includes all vertices or excludes at least one vertex. In the case that C includes all vertices, any witness path must be of length 1 and so must be proper.

Suppose then that C excludes at least one vertex and that all witness paths have length greater than 1. Then, since C is necessarily of length 3 or 4, any witness path has length either 2 or 3. Let P be some such witness path. If P is proper, we are done. If P is improper, then P has length 3, and we can write $P = \{\{a,b\},\{b,c\},\{c,a\}\}$ and $C = \{\{a,d\},\{d,e\},\{e,a\}\}$. Suppose without loss of generality that $\{a,b\}$ is on the left of C. Then either the edges $\{d,b\},\{e,b\},\{d,c\}$, and $\{e,c\}$ are on the left of C or one is on the right. In the former case, we have a proper witness $P' = \{\{d,b\},\{b,c\},\{c,a\}\}$, and in the latter, we have some proper witness P' by replacing either $\{c,a\}$ or $\{b,c\},\{c,a\}$ with the edge on the right of C (see Figure 3.2).

Consider now $K_{3,3}$. Since $K_{3,3}$ is nonplanar, any ribbon structure admits a nonseparating cycle C. In the case that C contains all vertices, the fact that $K_{3,3}$ contains no loops ensures that any witness path P is a proper witness. Suppose C excludes at least one vertex. Since all cycles in bipartite graphs are of even length, any witness path P must have length 2. But since there are no multiple edges in $K_{3,3}$, the endpoints of P in C must be distinct; hence, P is proper (see Figure 3.3). ∎

We next prove a similar result describing ribbon graphs admitting K_4 as a minor in such a way that the inherited ribbon structure is nonplanar.

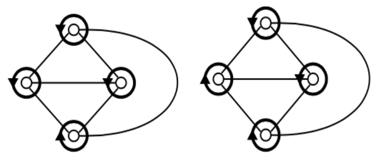


FIG. 3.4. *Two ribbon structures on K_4 yielding a nonplanar ribbon graph.*

PROPOSITION 3.4. *Suppose that G is a graph containing the complete graph K_4 as a minor. If G is endowed with a ribbon structure making the K_4 minor with the induced ribbon structure nonplanar, then G admits a proper witness pair.*

Proof. As in the proof of Proposition 3.3, we begin with the observation that the presence of a nonseparating cycle with proper witness in the graph K_4 is sufficient to guarantee the presence of such a cycle with proper witness in any graph G containing K_4 as a minor. It is therefore sufficient to show that each nonplanar ribbon structure on K_4 admits a proper witness pair.

Since each vertex in K_4 is of degree 3, there are only two choices of cyclic ordering of the edges around each vertex (see Figure 3.4). In particular, we can fix a planar embedding of K_4 and enumerate all ribbon structures by choosing either a clockwise or a counterclockwise orientation of edges about each vertex.

Moreover, by symmetry, we only need to consider the number of vertices oriented in either direction rather than their relative positions. We refer to the ribbon structure in which i vertices are oriented clockwise and j are oriented counterclockwise as being of type (i, j). By reversing the orientation of the plane, it is clear that the presence of a proper witness pair in the ribbon structure of type (i, j) ensures the presence of such a pair in the structure of type (j, i). Now, since the ribbon structures of type $(4, 0)$ and $(0, 4)$ give rise to planar ribbon graph structures, we have only two choices of ribbon structures yielding a nonplanar ribbon graph K_4, arising, respectively, from the pairs in types $(3, 1)$ and $(2, 2)$. For each of these two ribbon structures, one can directly find the associated surface of minimal genus and verify that there exists a proper witness (see Figure 3.5). ∎

In the case of a nonplanar ribbon graph which does not admit a proper witness pair, there is another kind of witness pair which will be of use to us.

DEFINITION 3.5. *Let G be a nonplanar ribbon graph that does not admit a proper witness pair. Suppose that G admits a witness pair (C, P) in G such that C and P share a unique vertex z. We say that (C, P) is a tight witness pair if the edges e and f incident to z in C are such that the interval $[e, f]$ in the ribbon structure about z has minimal length among all cycles C passing through z that have P as a witness path.*

The tightness of a witness pair (C, P) ensures that there are relatively few edges incident to z on one side of C. More precisely, this property means that for an oriented cycle C, the number of edges containing z on one side of C is as small possible. This definition is used in particular in the proof of Theorem 7.

Every finite nonplanar ribbon graph having no proper witness pair has at least one tight witness pair. This follows from the fact that G is a finite graph, and so we are taking a minimum over a finite collection of witness pairs.

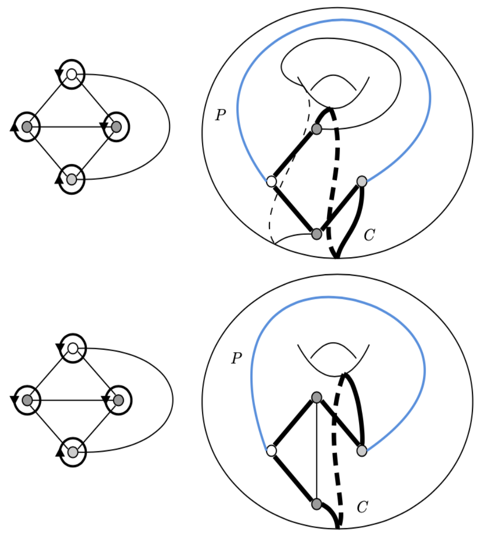


FIG. 3.5. *Proper witness pairs (C, P) in each of the nonplanar ribbon structures on K_4. In each case, the cycle C and its proper witness P are given.*

4. Divisors on graphs. In this section, we review the basic notions of the divisor theory of graphs, providing a setting in which to discuss the Picard group of a graph as the group of chip-firing equivalence classes. Alternative presentations of this material can be found in Chapter 4 of [12] or in Chapter 2 of [6]. Given a graph G, let $\mathrm{Div}(G)$ denote the free abelian group with formal generators corresponding to the vertices of G. Elements of $\mathrm{Div}(G)$ are called *divisors* on G; a given divisor D has the form

$$D = \sum_{v \in V} a_v (v)\,,$$

where $a_v \in \mathbb{Z}$ for all $v \in V$ and (v) is the formal generator corresponding to v. It is often convenient to think of a divisor as an arrangement of *chips* placed at the vertices of the graph G. The *degree* of a divisor D is the sum $\sum_{v \in V} a_v$. The collection of degree d divisors is denoted by $\mathrm{Div}^d(G)$. Note that, given any vertex q of G, the group $\mathrm{Div}^0(G)$ is generated by the divisors $\{(v) - (q) : v \in V\}$.

Denote by $\mathcal{M}(G)$ the collection of functions $f : V \to \mathbb{Z}$. The *combinatorial Laplacian* on G is the mapping $\Delta : \mathcal{M}(G) \to \mathrm{Div}^0(G)$ given by

$$\Delta f := \sum_{v \in V(G)} \Bigg(\sum_{e = \{v,w\}} [f(v) - f(w)] \Bigg) (v)\,.$$

The image of Δ is the set of *principal* divisors on G; such divisors form a subgroup $\mathrm{Prin}(G)$ of $\mathrm{Div}^0(G)$. We say that two divisors D and D' are *linearly equivalent* if they differ by a principal divisor, so that $D' = D + \Delta f$ for some $f \in \mathcal{M}(G)$. The (degree 0) *Picard group* of G is the quotient group

$$\mathrm{Pic}^0(G) = \frac{\mathrm{Div}^0(G)}{\mathrm{Prin}(G)}\,.$$

In general, we write $\mathrm{Pic}^d(G) = \mathrm{Div}^d(G)/\,\mathrm{Prin}(G)$; this is the collection of linear equivalence classes of degree-d divisors. $\mathrm{Pic}^0(G)$ admits a regular action on $\mathrm{Pic}^d(G)$ for any d given by addition of divisor classes. It is known that the cardinality of the Picard group of G is equal to the number of spanning trees of the graph G (see [2] and references therein).

For a graph G, the *genus* of G is the first Betti number $g = |E(G)| - |V(G)| + 1$; any spanning tree of G has exactly $|E(G)| - g$ edges. Given a spanning tree $T \in \mathcal{T}(G)$ excluding edges $e_1, \ldots e_g$, a divisor $B \in \mathrm{Div}^g(G)$ is referred to as a *break divisor* for T if B is of the form $B = \sum_{i=1}^{g} s(e_i)$, where $s(e_i)$ is one vertex of the edge e_i. Note that there are many possible distinct break divisors associated to a given tree. We have the following property of break divisors.

PROPOSITION 4.1 (Theorem 1.1 of [1]). *Each divisor class of* $\mathrm{Pic}^g(G)$ *contains a unique break divisor.*

Consequently, we can use break divisors as canonical representatives of the equivalence classes in $\mathrm{Pic}^g(G)$.

5. Bernardi and rotor-routing torsors. In addition to having the same cardinality as $\mathcal{T}(G)$, the Picard group $\mathrm{Pic}^0(G)$ also admits at least two regular group actions, or *torsor structures,* on $\mathcal{T}(G)$. One is derived from a process due to Bernardi, and the other arises from a process referred to as rotor-routing [4], [10]. We deal with torsor structures that depend on a choice of base vertex in the graph G. We remark that Kálmán, Lee, and Tóthmérész have recently shown in [11] that, using the machinery of planar trinities, it is possible to give a canonical definition of the Bernardi action on planar graphs without an initial choice of a base vertex. Moreover, this work extends these results to the situation of balanced plane digraphs.

5.1. Bernardi torsor. The first torsor structure considered in this work arises from a family of combinatorial maps, collectively referred to as the Bernardi process. As mentioned above, $\mathrm{Pic}^0(G)$ is in bijective correspondence with the set of spanning trees $\mathcal{T}(G)$. The works of Bernardi and Baker and Wang establish a family of bijections $\beta_{(q,e)} : \mathcal{T}(G) \to \mathrm{Pic}^g(G)$ witnessing this correspondence by mapping spanning trees to break divisors [4], [3]. These maps are parametrized by pairs (q, e) consisting of a vertex q and an edge e incident to q.

The Bernardi process uses the data (q, e) and a spanning tree T to perform a *tour* $\tau_{(q,e)}(T)$ of the graph G. This tour can be thought of as a special walk in G beginning and ending at q, traversing each edge of T twice, and excluding each edge outside T. More specifically, the tour $\tau_{(q,e)}(T)$ takes the form $\tau_{(q,e)}(T) = (v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0 = v_k = q$ and $e = e_1 = \{v_0, v_1\}$.

Given e_i and v_i, the edge e_{i+1} is chosen to be the first edge of T succeeding e_i in the ribbon structure about v_i; the process can be thought of as originating with the edge e_0 immediately preceding e in the ribbon structure about q. We think of this process as starting with the initial data, traversing each edge of T to the other endpoint, and cycling through the ribbon structure until finding another edge of T, ignoring each edge f not included in T. In such a tour, we ignore g many edges $f_1, \ldots, f_g \notin T$, each being passed over two distinct times, once from each endpoint.

In performing such a tour of G, we construct a divisor on G by placing a chip at our feet the first time we ignore a given edge $f \notin T$. In this case, we say that f is *cut* at the first endpoint, from which we ignore it. The divisor obtained in this fashion is necessarily a break divisor for the tree T. The mapping $\beta_{(q,e)} : \mathcal{T}(G) \to B(G)$ is given by mapping a spanning tree T to the resulting break divisor.

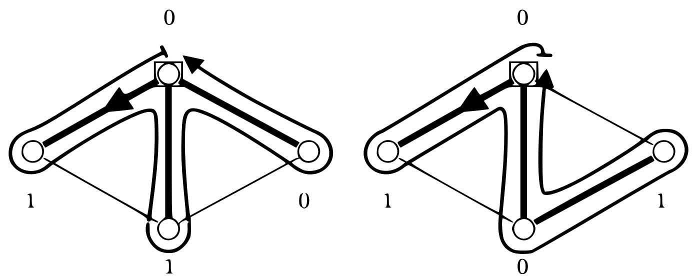


FIG. 5.1. *Bernardi tours and break divisors associated to two spanning trees of the given graph. Spanning trees are denoted by thickened edges. In each case, the initial data (q,e) are represented by a boxed vertex and an arrow along the initial edge.*

The process of deriving the break divisor $\beta_{(q,e)}(T)$ from T is illustrated graphically in Figure 5.1. Since, by Proposition 4.1, each divisor class of $\mathrm{Pic}^g(G)$ contains a unique break divisor, it follows that the maps $\beta_{(q,e)}$ induce explicit combinatorial bijections between spanning trees and divisor classes of degree g.

From these bijections, Baker and Wang define a group action $\beta_q : \mathrm{Pic}^0(G) \times \mathcal{T}(G) \to \mathcal{T}(G)$ using the natural action of $\mathrm{Pic}^0(G)$ on $\mathrm{Pic}^g(G)$ [3]. Fix initial data (q,e). Given any divisor class $[D] \in \mathrm{Pic}^0(G)$, define

$$\beta_q([D],T) := \beta^{-1}_{(q,e)}([D] + [\beta_{(q,e)}(T)])\,. \tag{5.1}$$

A central result of the paper of Baker and Wang states that the action (5.1) is in fact independent of the chosen edge e used in defining the bijection $\beta_{(q,e)}$ (see [3, Theorem 4.1]). As a result of this, one can define a group action of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$ after fixing only a vertex of G, the *base vertex* of the action.

5.2. Rotor-routing torsor. Another regular action is derived from the *rotor-routing* process on G, introduced in [14] and studied further in [10] and [9]. The rotor-routing process is a discrete-time dynamical system derived from G. The state space consists of *rotor configurations* on G: pairs (σ, v), where v is a vertex of G, interpreted as the location of a chip, and $\sigma : V(G) \to E(G)$, a *rotor function* assigning to each vertex w of G an edge $\sigma(w)$ containing w. The edge $\sigma(w)$ is referred to as the *rotor at* w.

A single step of the rotor-routing process takes the rotor configuration (σ, v) to (σ', w), where σ' is the rotor configuration for which $\sigma'(u) = \sigma(u)$ for all $u \neq v$, $\sigma'(v)$ is the successor of $\sigma(v)$ in the ribbon structure about v, and w is the other vertex of $\sigma'(v)$. Informally, we move the rotor at v once according to the ribbon structure about v and move the chip along the new edge to its other vertex w.

For the purposes of defining the action of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$, we choose a vertex q of G as a sink of the system and ignore the rotor at q. Given a spanning tree T of G and a vertex q of G, we can obtain a rotor function σ_T such that $\sigma_T(v)$ is the first edge on the unique path in T from v to q. Conversely, given a rotor function σ, we can obtain a subgraph H_σ of G whose only edges are the rotors of σ.

Given any vertex q of G and any spanning tree T of G, iterating the rotor-routing process from any configuration (σ_T, v) will eventually yield a configuration (τ, q); the chip will necessarily reach the vertex q [9, Lemma 3.6]. Moreover, it can be shown that the graph H_τ derived from τ will be a spanning tree of G [9, Lemma 3.10].

Using this observation, we can construct a group action ρ_q of $\mathrm{Div}^0(G)$ on $\mathcal{T}(G)$ by defining the action of generators $\{(v) - (q) : v \in V(G)\}$ and extending linearly.

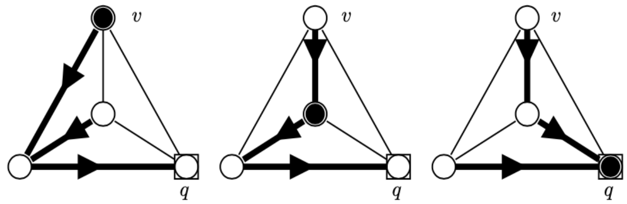


FIG. 5.2. *Rotor-routing process applied to a spanning tree of K_4. The original spanning tree T, indicated at the left by thickened edges, is mapped to the rightmost spanning tree $\rho_q([(v)-(q)])(T)$. At each stage, the location of the chip is indicated by a filled vertex.*

Beginning from the rotor configuration (σ_T, v), iterate the process until reaching some (τ, q), and let T' be the spanning tree given by the final rotor function. This action is trivial on any divisor in $\mathrm{Prin}(G)$ and so descends to an action by $\mathrm{Pic}^0(G)$. The rotor-routing action $\rho_q : \mathrm{Pic}^0(G) \times \mathcal{T}(G) \to \mathcal{T}(G)$ is then given by setting

$$\rho_q([(v)-(q)], T) = T'. \tag{5.2}$$

See Figure 5.2 for an illustration of the rotor-routing action. In the investigation of [5], the following definition is of crucial use to establish the main findings.

DEFINITION 5.1. *A rotor configuration (σ, v) is called a unicycle if the image of the rotor function σ contains a unique directed cycle and v is a vertex of this cycle or if v has some neighbor w such that $\sigma(v) = \sigma(w)$.*

This definition will be of particular use to us due to the interaction between the rotor-routing action and unicycles.

LEMMA 5.2 (Lemma 4.9 of [10]). *Let (σ, v) be a unicycle on a graph with m edges. Then, in iterating the rotor-routing process $2m$ times from (σ, v), the chip traverses each edge of G exactly once in each direction, each rotor makes exactly one full rotation, and the final state is again (σ, v).*

The following lemma is an easy consequence of Lemma 5.2.

LEMMA 5.3. *Suppose that G is a ribbon graph, (σ, w) a unicycle on G, and z a vertex satisfying $\sigma(z) = \{z, w\}$. Then, for all neighbors $v \neq w$ of z satisfying $\sigma(v) = \{v, z\}$, the rotor at z reaches $\{z, v\}$ before the rotor at v completes a full rotation.*

Proof. Let v be a neighbor of z such that $\sigma(v) = \{v, z\}$. Suppose for the sake of contradiction that in iterating the rotor-routing process from (σ, z), the rotor at v completes a full rotation before the rotor at z reaches $\{z, v\}$. Then, in order for the rotor at z to complete a full rotation, we pass through some rotor configuration (ρ, v) such that $\rho(z) = \{z, v\}$. Then, iterating the rotor-routing process one additional time, the rotor at v has completed more than one full turn before the rotor at z returns to its starting position, contradicting Lemma 5.2. ∎

5.3. Comparison of torsor structures. The rotor-routing and Bernardi processes described in the previous subsections give rise to regular group actions of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$, yielding two torsor structures on $\mathcal{T}(G)$ via (5.1) and (5.2). Each of the two structures is dependent on a choice of some vertex of G as base vertex for the action. These torsor structures have been the subject of much study in recent years [3, 5, 13, 11].

Fix some vertex $q \in V(G)$, and consider ρ_q and β_q. To see that the two homomorphisms ρ_q and β_q agree, we must have that

$$\rho_q([D],T) = \beta_q([D],T)$$

for each $T \in \mathcal{T}(G)$ for each divisor class $[D] \in \mathrm{Pic}^0(G)$. In particular, since divisors of the form $(v)-(q)$ generate $\mathrm{Div}^0(G)$, by (5.1) and (5.2), it is sufficient to verify that for each $v \in V(G)$, we have the equality

$$\beta_{(q,e)}(\rho_q([(v)-(q)],T)) = [(v)-(q)] + \beta_{(q,e)}(T)\,.$$

In contrast, to show that $\rho_q \neq \beta_q$ we need only find a single tree $T \in \mathcal{T}$ for which the above equality does not hold; doing so immediately yields that, as permutations of $\mathcal{T}(G)$,

$$\rho_q([(v)-(q)],\,\cdot\,) \neq \beta_q([(v)-(q)],\,\cdot\,),$$

and therefore ρ_q and β_q cannot be equal.

We have the following theorem about the rotor-routing torsor structure on $\mathcal{T}(G)$.

Theorem 5.4 (Theorem 2 of [5]). *Let G be a connected ribbon graph. The action ρ_q of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$ is independent of the base vertex q if and only if G is a planar ribbon graph.*

The work of Baker and Wang established an analogous result for the torsor structure derived from the Bernardi process.

Theorem 5.5 (Theorems 5.1 and 5.4 of [3]). *Let G be a connected ribbon graph. The action β_q of $\mathrm{Pic}^0(G)$ on $\mathcal{T}(G)$ is independent of the base vertex q if and only if G is a planar ribbon graph.*

In the case of a planar ribbon graph G, neither the rotor-routing action nor the Bernardi action is dependent on the choice of base vertex; both actions are in this sense canonical.

Theorem 5.6 (Theorem 7.1 of [3]). *For a planar ribbon graph G, the Bernardi and rotor-routing processes define the same $\mathrm{Pic}^0(G)$-torsor structure on $\mathcal{T}(G)$.*

However, in the case of a nonplanar G, the situation was not as clear. The work [3] concludes with the following conjecture.

Conjecture 5.7 (Conjecture 7.2 of [3]). *Let G be a nonplanar ribbon graph with no multiple edges and no loops. Then there exists a vertex q of G such that $\rho_q \neq \beta_q$.*

6. Baker–Wang conjecture and proper witnesses.

We now prove the Baker–Wang conjecture in the case that the graph G admits a proper witness pair. This will be done by first proving the conjecture in a more restricted situation.

Lemma 6.1. *Suppose that G is a ribbon graph with proper witness pair (C,P) such that P has endpoints x and z. If the two edges e_0 and e_1 of C incident to z satisfy $e_0 \prec e_1$, then there exists a vertex q of G such that $\rho_q \neq \beta_q$.*

Proof. Suppose that $e_1 = \{z,q\}$, so that q is the other vertex of e_1 in C. Now let $e' = \{u,v\}$ be an edge of P, and extend the collection of edges $(C\setminus e_1)\cup(P\setminus e')$ to a spanning tree T of G. We will show that

$$\rho_q([(z)-(q)],T) \neq \beta_q([(z)-(q)],T)\,.$$

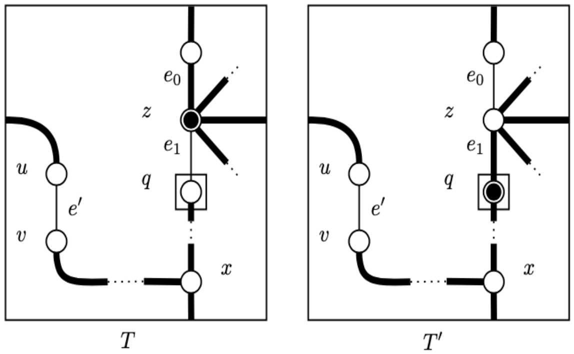


FIG. 6.1. *Rotor-routing process applied to the spanning tree T, thickened at the left, as in the proof of Lemma* 6.1. *After one step, the chip is moved from z to q, and the process is terminated, yielding the thickened tree T' at the right.*

In particular, denoting by T' the spanning tree $\rho_q([(z)-(q)],T)$, we show that the divisor classes $[(z)-(q)]+[\beta_{q,e_1}(T)]$ and $[\beta_{q,e_1}(T')]$ have distinct break divisor representatives. By the definition of the Bernardi action β_q, we immediately obtain $\rho_q \neq \beta_q$.

Acting on T with $\rho_q([(z)-(q)])$ entails only a single step of the rotor-routing process: changing the rotor at z from e_0 to e_1 and moving the chip to q, terminating the process. As a result, the spanning trees T and T' differ only by a single edge. In particular, $T' = (T \setminus e_0) \cup e_1$. This situation is depicted in Figure 6.1.

Consider the divisors $(z)-(q)+\beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$. By the definition of the Bernardi bijection, we know that $\beta_{(q,e_1)}(T')$ is a break divisor for T'. On the other hand, the divisor $(z)-(q)+\beta_{(q,e_1)}(T)$ is also a break divisor for T', distinct from $\beta_{(q,e_1)}(T')$. Since a break divisor for a spanning tree is a formal sum of endpoints of edges outside that tree, we can see the following:

$$(z)-(q)+\beta_{(q,e_1)}(T) = (z)-(q)+(q)+\sum_{\substack{e\notin T\\ e\neq e_1}} s(e) = (z)+\sum_{\substack{e\notin T'\\ e\neq e_0}} s(e)\,.$$

Recall from section 4 that $s(e)$ denotes either endpoint of the edge e. Since z is an endpoint of e_0, this shows that $(z)-(q)+\beta_{(q,e_1)}(T)$ is a break divisor for T'.

We are reduced to showing that $(z)-(q)+\beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ are distinct break divisors. By the construction of T, the collection of edges $P\cap T$ consists of two connected components P_z and P_x, containing z and x, respectively. In the Bernardi tour of T, P_x is traversed before P_z, while the opposite is true for the tour of T'. This results in different placement of the chip associated to the edge e' in the two tours: Without loss of generality, the T tour places a chip at v, while the T' tour places a chip at u; this difference is depicted in Figure 6.2.

It now remains to show that the difference in placement of the chip associated to e' yields a difference between $(z)-(q)+\beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ as divisors. Suppose for the sake of contradiction that they are the same despite their difference in choice of $s(e')$. Since e' is cut at v in the tour of T, the equality of $(z)-(q)+\beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ at u implies that there must be some other edge $f_0=\{u,w_1\}$, excluded from both T and T', which is cut at u in the tour of T and at its other endpoint w_1 in the tour of T'. Similarly, for $(z)-(q)+\beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ to agree at w_1,

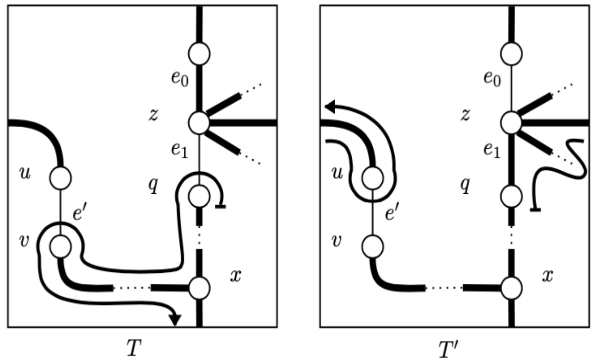


FIG. 6.2. *Bernardi process with initial data (q, e_1) applied to the spanning trees T and T' as in the proof. For simplicity, only the relevant parts of the tours are included in the figure. At the left, the tour cuts e' at u, while at the right, e' is cut at v.*

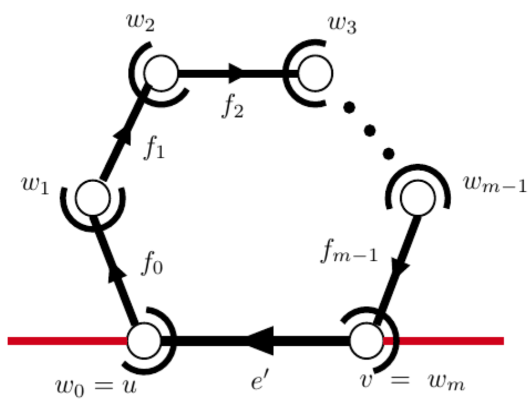


FIG. 6.3. *A depiction of the cycle D mentioned in the proof of Lemma* 6.1*. Here, the arcs surrounding vertices indicate that for this cycle, all edges are cut at their heads, in accordance with the described tour of the tree T'. The depicted edges outside the cycle are the edges of T' incident to u and v.*

we see that there must be some other edge $f_1 = \{w_1, w_2\}$, excluded from both T and T', which is cut at w_1 in the tour of T and is cut at its other endpoint w_2 in the tour of T'.

Continuing in this fashion, we see that there must exist a cycle D in G having vertices $u = w_0, w_1, \ldots, w_m = v$ and edges $f_0, \ldots, f_{m-1}, e'$, where $f_i = \{w_i, w_{i+1}\}$, such that all edges f_i are excluded from both trees T and T'. Moreover, orienting D such that e' is oriented from v to u, we see that in the tour of T, each edge of D must be cut at its tail, while in the tour of T', each edge of D must be cut at its head. This situation is depicted in Figure 6.3.

We show that the existence of such a cycle D with edges being cut in this way is impossible. To see this, we orient the cycle C in a fixed direction and partition the vertices w_i of D into two classes L and R according to whether the first edge outside C on the unique path in T from z to w_i falls on the left or right of the cycle C. Since T and T' are trees, if some vertex $w_i \in L$ is visited in the tour of T before some other vertex $w_j \in R$ (or vice versa), then all edges outside T incident to w_i will be cut before the tour visits w_j.

Consider now the tour of T. Since v is visited before u in this tour, e' is cut at its tail, and the edge f_{m-1} must be cut before u is visited. Thus, to avoid contradiction, it must be that w_{m-1} is in the same partition class as v and that f_{m-1} is cut at its tail w_{m-1}. Now, applying the same reasoning, it must be that w_{m-2} is in the same partition class as w_{m-1} and that f_{m-2} is cut at its tail w_{m-2}. Following this pattern around the cycle D, it must be that $w_0 = u$ is in the same class as w_1 and that the edge f_0 is cut at its tail $w_0 = u$. But this is impossible since, by construction, u and v cannot be in the same partition class.

Thus, it cannot be that $(z) - (q) + \beta_{(q,e_1)}(T) = \beta_{(q,e_1)}(T')$. Since the divisors $(z)-(q)+\beta_{(q,e_1)}(T)$ and $\beta_{(q,e_1)}(T')$ are necessarily distinct break divisors, they cannot be representative of the same element of $\mathrm{Pic}^g(G)$, and so $\rho_q \neq \beta_q$. ∎

For a ribbon graph G admitting a proper witness pair, either we have z, e_0, and e_1, as in Lemma 6.1 above, or there are some edges $f_1, \ldots, f_k$ incident to z such that

$$e_0 \prec f_1 \prec \cdots \prec f_k \prec e_1 \,.$$

We now prove that in the latter case, we can still find vertices q' and z' and an edge e for which the break divisors $\beta_{(q',e)}(T')$ and $(z') - (q') + \beta_{(q',e)}(T)$ still differ as described.

THEOREM 6.2. *For any nonplanar ribbon graph G admitting a proper witness pair (C, P), there exists a vertex q for which $\rho_q \neq \beta_q$.*

Proof. Suppose that the proper witness pair (C, P) is such that the path P intersects C in vertices x and z. If this witness pair satisfies $e_0 \prec e_1$ in the ribbon structure about z, then we are finished. Suppose also that the pair (C, P) fails to meet the conditions of Lemma 6.1. In particular, there exists an edge $f = \{z, v\}$ such that $f \in [e_0, e_1]$ and $f \neq e_0, e_1$. Let G' be the subgraph of G obtained by deleting the vertex z and all edges to which it is incident. Denote by F the connected component of the subgraph G' containing v. See Figure 6.4. Note in particular that F may intersect the cycle C or the path P.

If the component F meets neither C nor P, then one can ignore the edge f, applying the same reasoning as in Lemma 6.1: The rotor-routing process carried out

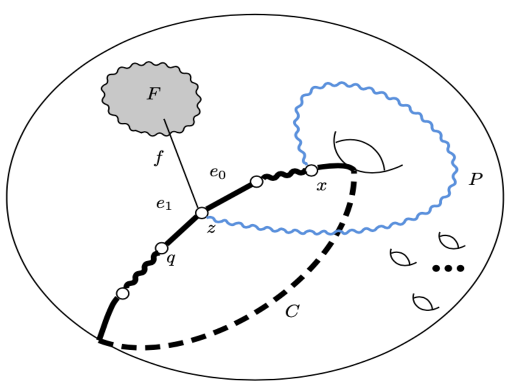


FIG. 6.4. *Proper witness pair (C, P) in a graph embedded on a surface of positive genus. Supposing that there is an edge f incident to z with $e_0 \prec f \prec e_1$, we can identify the component F of $G \setminus z$ and consider its possible intersection with the cycle C and the path P. Wavy lines represent paths that may contain additional vertices and edges.*

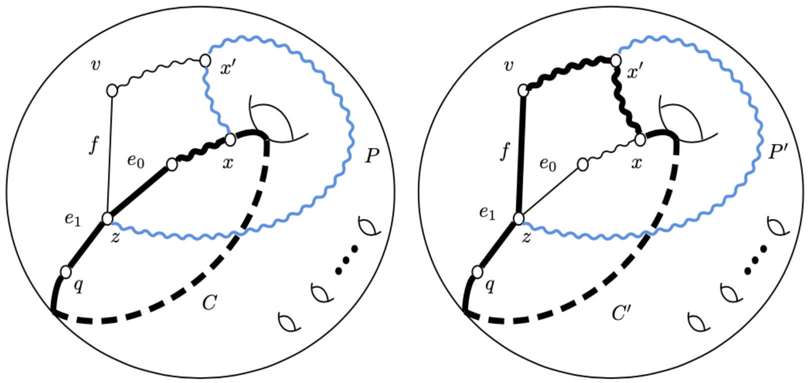


FIG. 6.5. *Situation in the proof of Theorem* 6.2 *where the intruding edge f leads to a path to P. The new proper witness pair (C', P) is shown in the right figure.*

in the subgraph F will not influence the rotors on the path P, so the break divisors $(z)-(q)+\beta_{q,e_1}(T)$ and $\beta_{q,e_1}(T')$ will still differ as described in the proof. We therefore assume that there is no such edge f with $F\cap C$ and $F\cap P$ empty.

Suppose that the interval (e_0, e_1) about z contains some positive number of edges f. We illustrate a method of finding a new witness pair (C', P') and reassigning the roles of e_0, e_1, z, and q such that the new interval I serving as the replacement of (e_0, e_1) has strictly fewer edges than before. This means that we can eventually find a witness pair (C'', P'') such that the interval between the two edges of C'' incident to z is empty, and therefore Lemma 6.1 can be applied.

Suppose that there is some edge in (e_0, e_1) that can be extended to a path from z to a vertex of P. Let f be the maximal such edge, and suppose it is extended to a path from $Q_{z,x'}$ through F connecting z to some vertex $x'\in V(P)$. Then, necessarily, there is a path $C_{z,x}$ from z to x in C which does not contain q, and there is a path $P_{x,x'}$ from x to x' in P. In this case, we replace the nonseparating cycle C by the new cycle $C' = (C\setminus C_{z,x'})\cup Q_{z,x'}\cup P_{x,x'}$ and the path P by the new path $P' = P\setminus P_{x,x'}$. Following this, we repeat this process, replacing e_0 with f and replacing x with x'. Doing so, we see that the new interval (f, e_1) has cardinality strictly less than that of the original interval (e_0, e_1). This process can be seen in Figure 6.5.

Suppose that there is some edge in (e_0, e_1) that can be extended to a path from z to a vertex of C. Let f be the minimal such edge, and suppose it is extended to a path $Q_{z,c}$ through F connecting z to some vertex $c\in V(C)$. We have the following two cases:

- If $c = x$ or if c lies on the path from z to x in C, denote by $C_{z,c}$ the path from z to c in C which avoids q. Replace the nonseparating cycle C by the new cycle $C' = (C\setminus C_{z,c})\cup Q_{z,c}$, and leave the path P unchanged. We now let f fill the role of e_0 and see that the interval (f, e_1) has cardinality strictly less than that of (e_0, e_1).
- If the vertex c lies on the path from z to x in C which passes through q or if $c = q$, let $C_{z,c}$ denote the path from z to c in C passing through (or stopping at) q. Replace the nonseparating cycle C by the new cycle $C' = (C\setminus C_{z,c})\cup Q_{z,c}$, and leave the path P unchanged. We now let f fill the role of e_1 and see that the interval (e_0, f) has cardinality strictly less than that of (e_0, e_1). This process can be seen in Figure 6.6.

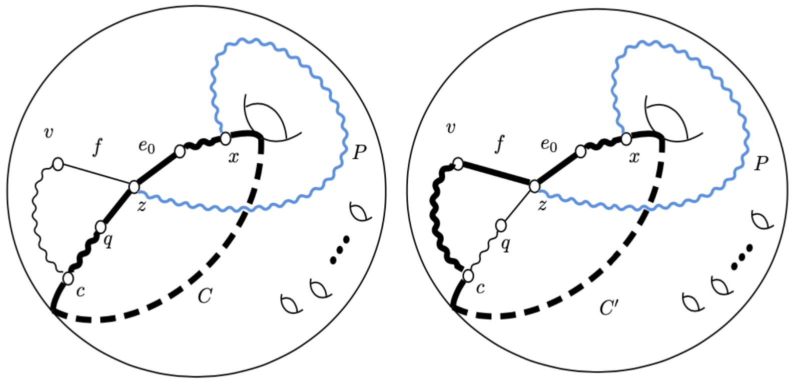


FIG. 6.6. *Situation in the proof of Theorem* 6.2 *where the intruding edge f leads to a path to C. The new proper witness pair (C', P) is shown in the image at the right.*

Using the above approaches, it follows that in the presence of a proper witness pair (C, P) in G, we can always recover a witness pair (C'', P'') which either satisfies the conditions of Lemma 6.1 or is such that no edges incident to z inhibit the proof of Lemma 6.1. Thus, we conclude that if G is a nonplanar ribbon graph admitting a proper witness pair, then there is some vertex q of G such that $\rho_q \neq \beta_q$. ∎

Considering Proposition 3.3 alongside Theorem 6.2, we immediately obtain the following corollary.

COROLLARY 6.3. *For a nonplanar graph G endowed with any ribbon structure, there exists some vertex q of G such that $\rho_q \neq \beta_q$.*

7. Simple graphs without proper witnesses. We next show that for nonplanar ribbon graphs G without multiple edges or loops that do not contain a proper witness pair, there is some vertex q for which $\rho_q \neq \beta_q$. This completes the proof of the Baker–Wang conjecture.

THEOREM 7.1. *Let G denote a nonplanar ribbon graph with no multiple edges and no loops such that G does not admit a proper witness pair. Then there is some vertex q of G such that $\rho_q \neq \beta_q$.*

Proof. As in the proof of Lemma 6.1, we will construct a spanning tree T such that the Bernardi and rotor-routing torsors disagree on T.

In order to do so, we must first fix some terminology regarding our construction. It is recommended that the reader reference Figure 7.1 while we introduce the required terminology.

1. Since G is a nonplanar ribbon graph with no proper witness pair, there exists at least one tight witness pair (Definition 3.5). Suppose that (C, P) is one such pair and that z is the unique vertex in the intersection of P and C.
2. Suppose that q is a neighbor of z in C. Orient $\{z, q\}$ from q to z, and extend this orientation to all of C. We may assume that the interval on the left of C is the minimal interval described in the definition of a tight witness pair. Let e be the edge $\{z, q\}$, and let e' be the edge immediately succeeding e in the ribbon structure about q.
3. Partition all neighbors y of z outside the cycle C into two classes L and R so that $y \in L$ if and only if $\{y, z\}$ is on the left of C and $y \in R$ if and only if $\{y, z\}$ is on the right of C.

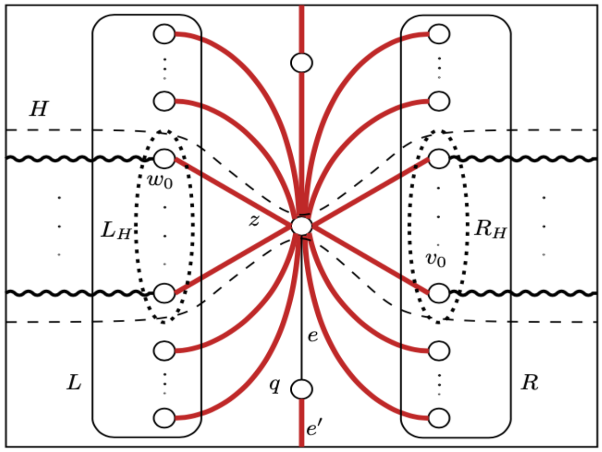


FIG. 7.1. *The situation in the proof of Theorem* 7.1. *The tree* T *is indicated with thickened edges, while the subgraph* H *of* G *is indicated using a dashed line. The vertex sets* L_H *and* R_H *are shown within the dotted regions within* H. *For simplicity, we have depicted the edge* e' *as an edge of the cycle* C.

4. Let H be the subgraph of G induced by all witness paths for C that intersect C at z. Further, let $L_H \subseteq L$ and $R_H \subseteq R$ denote the vertices of H which lie in L and R, respectively.
5. Let $Z \subseteq E(G)$ denote the collection of edges incident to z with endpoints in L or R. We can extend the collection of edges $C \cup Z \setminus \{e\}$ to a spanning tree T so that all edges incident to z except e are included in the tree T and such that all vertices in R_H have degree 1 in the subgraph $T \cap H$. This is because, by definition, all neighbors of z in R_H lie on paths connecting to z through vertices of L_H. Note that the vertices in R_H may have degree greater than 1 in T, but this situation arises only when there are other vertices of G whose only path to z passes through a vertex of R_H.
6. Let w_0 denote the vertex in L_H such that $\{z, w_0\}$ is the first edge in H to the left of C. Likewise, let v_0 denote the vertex in R_H such that $\{z, v_0\}$ is the first edge to the right of C.
7. Consider the vertices of $w \in L \setminus L_H$. As in the proof of Theorem 6.2, let G' denote the subgraph of G obtained by removing the vertex z and all edges incident to z. Let W denote the connected component of G' containing w. Since (C, P) is a tight witness pair, W does not intersect the cycle C. Indeed, if W intersects C, then there are some vertices $w, w' \in W$ and $c \in C$ with $c \neq z$ such that $\{w, z\}$ and $\{w', c\}$ are edges in G. In this case, we may form a new cycle C' which agrees with C on the (oriented) path from z to c in C but which contains the edges $\{w, z\}$ an $\{w', c\}$. Doing so then yields a witness pair (C', P) such that the length of the ribbon structure interval about z on the left of C' is strictly less than the length of the ribbon structure interval about z on the left of C. This is precisely what the tightness of the witness pair (C, P) prohibits, and so such a situation is impossible.

Consider the rotor-routing process taking T to $T' := \rho_q([(z) - (q)], T)$. At some point, the rotor at z shifts to the edge $\{z, w_0\}$, forming a unique cycle in the subgraph of H defined by the rotors at vertices of H. Restricting the rotor configuration to the graph H, we obtain a unicycle $U = (\sigma, z)$ in H. Further, by the assumption that G has no proper witness, we see that all paths from any vertex of H other than z to any vertex on the path C must pass through z. In particular, combining this with the fact that (C, P) is a tight witness pair ensures that no rotors outside of H or the subgraphs W will move until the rotor at z shifts to e.

Consider now the rotor-routing process in the graph H beginning from starting configuration U. By Lemma 5.3, for all neighbors v of z in R_H, the rotor at z shifts to $\{z, v\}$ before the rotor at any $v \in R_H$ completes a full rotation. Considering the implication of this statement in the graph G, we see that the rotor at z must shift to e before the rotors at any $v \in R_H$ complete a full rotation, and so the entire rotor-routing process taking T to T' terminates before the rotor of any vertex in R_H completes a full rotation.

Now, either the rotor-routing process associated to $\rho_q([(z) - (q)], T)$ never deposited a chip at any vertex in R_H, and so all edges from z to vertices in R_H lie in the tree T', or this rotor-routing process deposited a chip at some vertex of R_H, and so the edge joining z to this vertex lies outside the tree T'.

In the former case, since no $v \in R_H$ received a chip, all vertices of R_H have degree 1 in the subgraph $T' \cap H$. This means in particular that on the Bernardi tour associated to T', all vertices of R_H receive at least one chip from their neighbors in H in addition to whatever chips they receive from edges to neighbors outside H. However, on the Bernardi tour associated to T, all edges of H incident to vertices in R_H will be cut at their other endpoint. Therefore, no vertex of R_H will receive any chips from neighbors in H on this tour, while each such vertex receives the same number of chips as in the tour of T' from neighbors outside H. This means in particular that the break divisors $\beta_{(q,e)}(T')$ and $(z) - (q) + \beta_{(q,e)}(T)$ differ at all vertices of R_H and so cannot be representative of the same element of $\mathrm{Pic}^g(G)$.

In the latter case, suppose that the rotor-routing process on T deposited a chip at the vertex v, and so $\{z, v\}$ lies outside T'. Then, in this case, the chip must have returned to z across an edge connecting z to some vertex ℓ in L_H. Indeed, all edges incident to z on the right of C are included in the tree T, so in order for the chip to return to z across an edge connecting z to a vertex in R_H, the rotor at that vertex must complete a full rotation, which is impossible by Lemma 5.3. Further, by appealing to this same lemma, we also see that the edge $\{\ell, z\}$ across which the chip returned to z must be contained in T'. This follows from the fact that $\{\ell, z\}$ is in the tree T by construction, and so 5.3 ensures that the rotor at z reaches $\{\ell, z\}$ before the chip returns across this edge. In particular, the rotor at z continues to the next edge in the ribbon structure at z, and so the rotor at ℓ remains $\{\ell, z\}$ at the termination of the process.

Thus, for all $v \in R_H$ which received a chip during the rotor-routing process, the unique path from v to z in T' passes through a vertex of L_H. Note in particular that this need not be the case if G has multiple edges. Indeed, in the event that there is some $v \in R_H$ such that there are parallel edges between z and v, v may receive a chip in the rotor-routing process and eventually terminate in a configuration such that one of the edges $\{v, z\}$ is still in the tree T'.

This means that in the Bernardi tour associated to T', the edge $\{z, v\}$ will be cut at z, and so z will receive a chip from an edge to at least one neighbor in R_H. Considering also that z receives a chip from an edge in the cycle C, we see that z must have a coefficient of at least 2 in the break divisor $\beta_{(q,e)}(T')$. On the other hand, all edges incident to z except for e are included in T, so that in the divisor $\beta_{(q,e)}(T)$, the vertex z has coefficient 0. Thus, the break divisors $(z) - (q) + \beta_{(q,e)}(T)$

and $\beta_{(q,e)}(T')$ necessarily differ at z and cannot be representative of the same element of $\mathrm{Pic}^g(G)$.

Combining the above, we conclude that in either case, the rotor-routing and Bernardi torsors disagree on the tree T, so that $\rho_q \neq \beta_q$. ∎

In view of Theorems 6.2 and 7.1 we obtain the following corollary, verifying the Baker–Wang conjecture.

COROLLARY 7.2 (Baker–Wang conjecture). *Given a nonplanar ribbon graph with no multiple edges and no loops, there exists a vertex q for which $\rho_q \neq \beta_q$.*

8. Examples. We conclude with two examples to address the following questions:

1. Is the assumption of no multiple edges necessary for the Baker–Wang conjecture?
2. For nonplanar ribbon graphs, is the *difference* between the two torsor structures independent of the base vertex?

Our examples show that the answer to question (1 is yes, while the answer to question 2 is no.

Fix a labeling of the N spanning trees of the underlying graph and consider the actions ρ_q and β_q as homomorphisms:

$$\rho_q : \mathrm{Pic}^0(G) \longrightarrow S_N\,,$$
$$\beta_q : \mathrm{Pic}^0(G) \longrightarrow S_N\,.$$

The *difference* between the two torsors *on a given divisor class* $[D]$ can be computed as $\rho_q([D])^{-1}\beta_q([D]) \in S_N$. The difference between the two torsor structures is dependent on the base vertex if there exist two vertices p and q and a divisor class $[D]$ such that

$$\rho_p([D])^{-1}\beta_p([D]) \neq \rho_q([D])^{-1}\beta_q([D])\,.$$

Example 8.1. Consider the graph depicted in Figure 8.1, which we call the *rounded bowtie graph.* For the rounded bowtie graph, we have that $\rho_q = \beta_q$ for all vertices q.

We compute all Bernardi and rotor-routing torsor structures on the graph. This graph has only four spanning trees, yielding a particularly straightforward collection of computations.

For a given spanning tree T and a given choice of vertex q and incident edge $e = \{v, q\}$, we first compute the rotor-routing action, yielding a permutation $\rho_q([(v) - (q)]) \in S_n$. Following this, we calculate the permutation associated to the Bernardi action by computing break divisors $\beta_{(q,e)}(T)$ for all spanning trees T. Break divisors will be written as row vectors with coordinates in the order (a, b, c). With

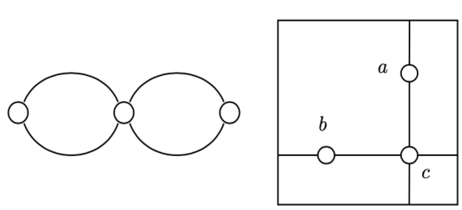
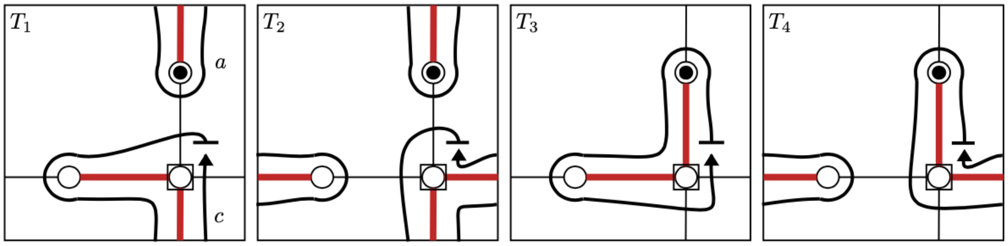


FIG. 8.1. *Rounded bowtie graph with the relevant torus embedding. As in prior figures, the ribbon structure is given by the counterclockwise orientation of the torus.*



(1) $(0,1,1)$ (2) $(0,0,2)$ (3) $(1,1,0)$ (4) $(1,0,1)$

FIG. 8.2. *Bernardi and rotor-routing actions on spanning trees of the rounded bowtie graph. The spanning trees T_i are indicated in each of the figures* (1) *through* (4) *with thickened edges. The Bernardi tours depicted are associated to the bijection $\beta_{(c,e)}$, where $e=\{a,c\}$. The associated break divisors $\beta_{(c,e)}(T_i)$ are given below each tree.*

these in hand, we translate the break divisors by the divisor $(v)-(q)$, yielding new divisors,

$$(v)-(q)+\beta_{(q,e)}(T) \tag{8.1}$$

for each spanning tree T. Finally, we can check for linear equivalence between these divisors and the other break divisors $\beta_{(q,e)}(S)$ for other spanning trees S. In particular, if some pair of spanning trees S and T satisfy

$$(v)-(q)+\beta_{(q,e)}(T)\sim\beta_{(q,e)}(S)\,,$$

then the permutation $\beta_q([(v)-(q)])$ sends the spanning tree T to S. Calculating these divisors (8.1) and performing these comparisons for all spanning trees then yields a permutation representation for $\beta_q([(v)-(q)])$.

In order to verify that $\rho_q=\beta_q$ for all vertices q of the graph in Figure 8.1, we perform this process for all choices of vertices q and v of the graph in order to understand the action of divisor classes of $\mathrm{Div}^0(G)$ generators on the trees. There are three nonequivalent choices for q and v: one in which the target vertex q has degree 4 and the vertex v has degree 2, one in which q and v both have degree 2, and one in which q has degree 2 and v has degree 4. In Figure 8.2, we have included a depiction of the four spanning trees of the rounded bowtie graph, together with their associated Bernardi tours and break divisors for this choice of $q=c$ and $v=a$.

In calculating the Bernardi action, we make use of the Bernardi bijection $\beta_{(c,e)}$, where $e=\{c,a\}$. Consider the divisor $(a)-(c)+\beta_{(c,e)}(T_1)=(1,1,0)$. This is the break divisor $\beta_{(c,e)}(T_3)$, so we conclude that $\beta_c([(a)-(c)])$ maps T_1 to T_3. Similarly, we see

$$\begin{aligned}
(a)-(c)+\beta_{(c,e)}(T_3)&=(2,1,-1)\sim(0,1,1)=\beta_{(c,e)}(T_1)\,,\\
(a)-(c)+\beta_{(c,e)}(T_2)&=(1,0,1)=\beta_{(c,e)}(T_4)\,,\\
(a)-(c)+\beta_{(c,e)}(T_4)&=(2,0,0)\sim(0,0,2)=\beta_{(c,e)}(T_2)\,.
\end{aligned}$$

Together, this implies that we have a permutation representation $\beta_c([(a)-(c)])=(13)(24)$. Likewise, checking the rotor-routing action $\rho_c([(a)-(c)])$ yields that $\rho_c([(a)-(c)])=(13)(24)=\beta_c([(a)-(c)])$. By the symmetry of the rounded bowtie graph, it follows that $\rho_c=\beta_c$. Other cases follow similarly, and so the associated work is not included.

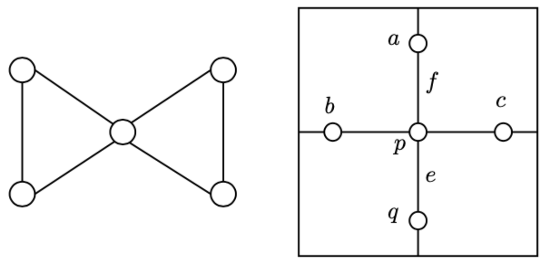


Fig. 8.3. *Pointed bowtie graph with the relevant torus embedding. Vertices are labeled* $a, b, c, p, q,$ *and edges* e, f *are relevant in Example* 8.2.

We now turn to the issue of the discrepancy between base vertices, illustrating in particular a graph, depicted in Figure 8.3, having vertices p and q such that $\rho_p = \beta_p$ while $\rho_q \neq \beta_q$.

Example 8.2. Consider the graph depicted in Figure 8.3, which we call the *pointed bowtie graph.* For the pointed bowtie graph and the two indicated vertices p and q, we have $\rho_p = \beta_p$ while $\rho_q \neq \beta_q$.

We deal with divisor classes of the form $[(z) - (q)]$ for the sake of their simplicity. In Figure 8.3, all vertices are labeled, and in addition, we have included all spanning trees of the graph, together with the labeling to be used in the cycle representation of our actions. We will show that

$$\rho_q([(a) - (p)])^{-1}\beta_q([(a) - (p)]) \neq \rho_p([(a) - (p)])^{-1}\beta_p([(a) - (p)]).$$

Before proceeding, note that by Theorem 7.1, we are guaranteed that $\rho_q \neq \beta_q$ for this choice of q. We compute now the action of the divisor $[(a)-(p)]$ on each spanning tree using the actions ρ_q and β_q. Since the base vertex of these actions is q and not p, we must first rewrite $[(a) - (p)] = [(a) - (q)] - [(p) - (q)]$, after which one can compute the actions of $[(a) - (q)]$ and $[(p) - (q)]$ under both ρ_q and β_q. Finally,

$$\begin{aligned} \rho_q([(a) - (p)]) &= \rho_q([(a) - (q)]\left[\rho_q([(p) - (q)])\right]^{-1}, \\ \beta_q([(a) - (p)]) &= \beta_q([(a) - (q)]\left[\beta_q([(p) - (q)])\right]^{-1}. \end{aligned}$$

With these permutations in hand, we are able to form the product $\rho_q([(a) - (p)]^{-1}\beta_q([(a)-(p)])$. To start this process, we have produced in Figure 8.4 the Bernardi tours and break divisors associated to each of the spanning trees. For this example, break divisors will be written as row vectors with coordinates in the order (a, b, p, c, q). With these in hand, we can now compute the cycle representation of the permutation $\beta_q([(p) - (q)])$ using the Bernardi bijection $\beta_{(q,e)}$, where e is the edge $\{p, q\}$.

Beginning from the spanning tree T_1, computation of the translates $[(p) - (q)] + [\beta_{(q,e)}(T_i)]$ and their break divisor representatives yields that the permutation $\beta_q([(p)-(q)])$ has cycle representation (193)(278)(456). On the other hand, it can be verified from performing rotor-routing on the trees $T_1, \ldots, T_9$ that $\rho_q([(p) - (q)]) = (123)(459)(678)$. We have

$$\begin{aligned} \beta_q([(p) - (q)]) &= (193)(278)(456)\,, \\ \rho_q([(p) - (q)]) &= (123)(459)(678)\,. \end{aligned}$$

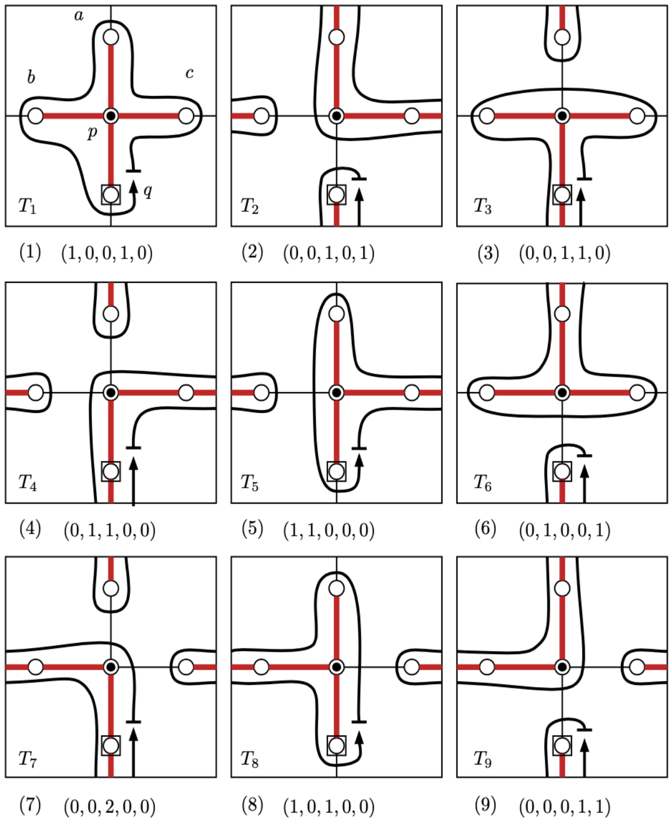


FIG. 8.4. *Bernardi and rotor-routing actions on spanning trees of the pointed bowtie graph. The spanning trees T_i are indicated in each of the figures in red. Denoting $e = \{p, q\}$, the break divisors $\beta_{(q,e)}(T_i)$ are given below the corresponding tree with the vertex ordering (a, b, p, c, q).*

Following this same reasoning, we can obtain cycle representations

$$\beta_q([(a) - (q)]) = (139)(287)(465)\,,$$
$$\rho_q([(a) - (q)]) = (132)(495)(687)\,.$$

Combining the above expressions, it follows that

$$\rho_q([(a) - (p)])^{-1}\beta_q([(a) - (p)]) = (158)(269)\,.$$

It remains to see that the torsor actions of $\beta_p([(a) - (p)])$ and $\rho_p([(a) - (p)])$ coincide. This is most easily done by taking the same approach as above, utilizing the Bernardi bijection $\beta_{(p,f)}$. These computations are more straightforward than those above, so we leave the work as an exercise. One obtains

$$\beta_p([(a) - (p)]) = (163)(245)(789)\,,$$
$$\rho_p([(a) - (p)]) = (163)(245)(789)\,.$$

In particular, $\rho_p([(a) - (p)]) = \beta_p([(a) - (p)])$. Now note that by the symmetry of the pointed bowtie graph, the coincidence of ρ_p and β_p on $[(a) - (p)]$ ensures their coincidence on all generators of $\mathrm{Div}^0(G)$. We now have $\rho_q \neq \beta_q$ while $\rho_p = \beta_p$.

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