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A LIOUVILLE THEOREM FOR THE CHERN–SIMONS–SCHRÖDINGER EQUATION

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Abstract. In this paper we prove a Liouville theorem for the Chern–Simons–Schrödinger equation. This result is consistent with the soliton resolution conjecture for initial data that does not lie in a weighted space. See [10] for the soliton resolution result in a weighted space.

1. **Introduction.** The self-dual Chern–Simons–Schrödinger equation with m-equivariance is

$$i(\partial_t + iA_t[u])u + \partial_r^2 u + \frac{1}{r}\partial_r u - (\frac{m + A_\theta[u]}{r})^2 u + |u|^2 u = 0, \tag{1}$$

where

$$A_t[u] = -\int_r^\infty (m + A_\theta[u])|u|^2 \frac{dr'}{r'}, \qquad \text{and} \qquad A_\theta[u] = -\frac{1}{2}\int_0^r |u|^2 r' dr'. \tag{2}$$

The Chern–Simons–Schrödinger equation was introduced in [4] as a nonrelativistic planar quantum electromagnetic model that exhibits self-duality. It is a gauge covariant nonlinear Schrödinger equation on $\mathbb{R}^2$. See also [2], [5], [6]. The model (1) is derived after fixing the Coulomb gauge condition and imposing the equivariant symmetry on the scalar field ϕ:

$$\phi(t, x) = u(t, r)e^{im\theta}. \tag{3}$$

See [9], [7], and [8].

Remark 1.1. The non-equivariant Chern–Simons–Schrödinger equation will not be discussed here. See [1], [3], [11], [14], and [17] for more information.

The solution to (1) has the conserved quantities mass and energy.

$$\begin{aligned} E[u] &= \int \frac{1}{2}|\partial_r u|^2 + \frac{1}{2}(\frac{m + A_\theta[u]}{r})^2|u|^2 - \frac{1}{4}|u|^4 dx, \\ M[u] &= \int |u|^2 dx, \end{aligned} \tag{4}$$

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where we denote $\int f(r) = 2\pi \int f(r) r dr$. Indeed, (1) is the Hamiltonian PDE for the energy in (4). We also use the inner product

$$\langle f, g \rangle = Re \int f(r)\overline{g(r)} r dr. \tag{5}$$

Integrating by parts,

$$\begin{aligned}
\frac{d}{dt}E[u] &= \int Re(\partial_r \bar{u})(\partial_r u_t) r dr + \int (\frac{m + A_\theta[u]}{r})^2 Re(\bar{u} u_t) r dr \\
&\quad - \int |u|^2 Re(\bar{u} u_t) r dr - \int \frac{m + A_\theta[u]}{r} |u|^2 \int_0^r Re(\bar{u} u_t) r' dr' dr \\
&= -\langle (\partial_{rr} + \frac{1}{r}\partial_r)\bar{u}, u_t \rangle + \langle (\frac{m + A_\theta[u]}{r})^2 u, u_t \rangle \\
&\quad - \langle |u|^2 u, u_t \rangle - \int_0^\infty |u|^2 \int_r^\infty \frac{m + A_\theta[u]}{r} Re(\bar{u} u_t) dr r' dr' \\
&= -\langle (\partial_{rr} + \frac{1}{r}\partial_r)\bar{u}, u_t \rangle + \langle (\frac{m + A_\theta[u]}{r})^2 u, u_t \rangle \\
&\quad - \langle |u|^2 u, u_t \rangle + \langle A_t[u]\bar{u}, u_t \rangle = \langle i u_t, u_t \rangle = 0.
\end{aligned} \tag{6}$$

Thus,

$$\partial_t u = -i \nabla E[u], \tag{7}$$

where ∇ (acting on a functional) is the Frechet derivative with respect to the inner product $\langle \cdot, \cdot \rangle$. Also,

$$\partial_t M[u] = Re \int \bar{u} u_t = Re \int \bar{u}(i \partial_r^2 u + \frac{i}{r}\partial_r u) = 0. \tag{8}$$

The energy functional can be written in the self-dual form

$$E[u] = \int \frac{1}{2}|D_u u|^2, \tag{9}$$

where D_u is the covariant Cauchy–Riemann operator defined by

$$D_u f = \partial_r f - \frac{m + A_\theta[u]}{r} f. \tag{10}$$

Indeed,

$$\begin{aligned}
&- Re \int (\partial_r \bar{f}) \cdot (\frac{m + A_\theta[u]}{r}) f(r) r dr = -\frac{1}{2}\int (m + A_\theta[u]) \partial_r(|f|^2) dr \\
&= \frac{1}{2}\int \partial_r A_\theta[u] |f|^2 = -\frac{1}{4}\int_0^\infty |f|^4 r dr.
\end{aligned} \tag{11}$$

Definition 1.2 (Bogomol'nyi operator)**.** The operator $u \mapsto D_u$ is called the Bogomol'nyi operator. Due to (9) and the Hamiltonian structure, any static solutions to (1) are given by solutions to the Bogomol'nyi equation

$$D_Q Q = 0. \tag{12}$$

For $m \geq 0$, there is an explicit m-equivariant static solution to the Bogomol'nyi equation, the Jackiw–Pi vortex. This solution is unique up to the symmetries of the equation ([12]):

$$Q(r) = \sqrt{8}(m+1)\frac{r^m}{1 + r^{2m+2}}, \qquad m \geq 0. \tag{13}$$

Equation (1) has the pseudoconformal transform $\mathcal{C}$,

$$[\mathcal{C}u](t,r) = \frac{1}{|t|}u(-\frac{1}{t},\frac{r}{|t|})e^{ir^2/4t}, \qquad \forall t \neq 0. \tag{14}$$

Since the soliton solution is non-scattering, applying the pseudoconformal transform to Q gives an explicit, finite-time blowup solution,

$$S(t,r) = \frac{1}{|t|}Q(\frac{r}{|t|})e^{-i\frac{r^2}{4t}}, \qquad t < 0. \tag{15}$$

It is conjectured that any blowup solution must contain either (13) or (15). Indeed, [13] proved global well-posedness and scattering for (1) with initial data with mass below the mass of the ground state,

$$\|u_0\|_{L^2} < \|Q\|_{L^2}. \tag{16}$$

Theorem 1.3. *Let $m \in \mathbb{Z}_{\geq 0}$. Let $\phi_0 \in L^2_m$ with $\|\phi_0\|_{L^2_m}$ and*

$$\|u_0\|^2_{L^2} < 8\pi(m+1). \tag{17}$$

Then (1) is globally well-posed in L^2_m and scatters both forward and backward in time.

Proof. See Theorem 1.3 of [13]. □

Making a u-substitution,

$$\|Q\|^2_{L^2} = 16\pi(m+1)^2\int_0^\infty \frac{r^{2m+1}}{(1+r^{2m+2})^2}dr = 8\pi(m+1)\int_0^\infty \frac{du}{(1+u)^2} = 8\pi(m+1). \tag{18}$$

Remark 1.4. A function $u_0 \in L^2_m$ if $u_0 \in L^2$ and u_0 has the form (3). A function $u_0 \in H^1_m$ if $u_0 \in H^1$ and has the form (3).

More recently, [9] and [10] proved a decomposition for finite time blowup solutions to (1) with finite energy and initial data in a weighted Sobolev space.

Theorem 1.5. *If $m \in \mathbb{Z}_+$ and u is a H^1_m-solution to (1) that blows up forward in time at $T < +\infty$, then $u(t)$ admits the decomposition*

$$u(t,\cdot) - Q_{\lambda(t),\gamma(t)} \to z^*, \qquad \text{in} \qquad L^2, \qquad \text{as} \qquad t \nearrow T. \tag{19}$$

Moreover, using the pseudoconformal transformation in (14), it is possible to obtain a similar decomposition for a solution that exists globally forward in time, but fails to scatter forward in time, for initial data that also lies in a weighted L^2-space.

Proof. See [10]. □

In this paper, we prove a Liouville theorem for solutions to (1) that are global in at least one time direction.

Theorem 1.6 (Liouville theorem)**.** *Suppose $u_0 \in H^1_m$ for some $m \geq 1$ is an initial data for (1) that has a solution on the maximal interval of existence I. Furthermore, suppose that $I = (-\infty, t_0)$, where t_0 could be $+\infty$, or (t_0,∞), where t_0 could be $-\infty$. Also suppose that for any $\eta > 0$, there exists $R(\eta) < \infty$ such that*

$$\sup_{t\in I}\int_{|x|\geq R}|u(t,x)|^2dx < \eta, \tag{20}$$

where I is the interval of existence for a solution to (1). Then u is equal to the soliton solution (13), up to the scaling symmetry,

$$u_{\lambda}(t,r) = \frac{1}{\lambda} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}), \qquad \lambda > 0, \tag{21}$$

and multiplication by $e^{i\gamma}$ for some $\gamma \in \mathbb{R}$.

Remark 1.7. The computations in the next three sections are reliant on the fact that the soliton lies in the weighted L^2 space given by the norm $\| \langle x \rangle \cdot \|_{L^2}$. See for example (89) and (109). When $m = 0$, the soliton fails to lie in this space, see (13), which creates certain technical complications. We do not consider the $m = 0$ case here, but see [9] for more information on the $m = 0$ case.

This result was inspired by the Liouville theorem of [15]. There, [15] proved that for a solution to the mass-critical generalized Korteweg de-Vries equation,

$$u_t + u_{xxx} + \partial_x(u^5) = 0, \tag{22}$$

with initial data close to the rescaled soliton in $H_x^1(\mathbb{R})$, and with $H^1(\mathbb{R})$ norm uniformly bounded, then the solution to (22) must be the soliton. For the mass-critical generalized KdV equation, it is expected that multi-soliton solutions occur, which necessitates additional constraints on the size of the initial data than we have here.

Unlike the generalized KdV equation, the structure of the self-dual Chern–Simons–Schrödinger equation is defocusing outside of a soliton. For this reason, it is unnecessary to require a uniform bound on $\|u(t)\|_{H^1}$ on I. Also, since u_0 need not be close to the soliton, we do not assume an a priori bound on $\|u_0\|_{L^2}$.

Remark 1.8. For a solution to (1), $u \in H_m^{1,1}$, that exists globally forward in time, then either $u(t)$ scatters forward in time, or $u(t)$ admits the decomposition

$$u(t,\cdot) - Q_{\lambda(t),\gamma(t)} - e^{it\Delta^{(-m-2)}} u^* \to 0, \qquad \text{in} \qquad L^2, \qquad \text{as} \qquad t \to \infty. \tag{23}$$

Here, $e^{it\Delta^{(-m-2)}} u^*$ is the solution to the free, $(-m-2)$–equivariant Schrödinger flow,

$$i\partial_t u + \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{(m+2)^2}{r^2} u = 0. \tag{24}$$

The space $H_m^{1,1}$ is the space of m-equivariant functions, (3), that lie in H^1 and the weighted L^2-space, $\| |x| u \|_{L^2} < \infty$.

If (23) could be proved for any $u_0 \in H_m^1$, then Theorem 1.6 would likely follow fairly easily, since (23) would at least imply that $u^* = 0$, and thus $\|u\|_{L^2} = \|Q\|_{L^2}$. This is due to the fact that a scattering solution, or a solution with a scattering piece could not satisfy (20).

The proof of Theorem 1.6 may be broken down into three steps. First, using a virial identity combined with (20), we prove that any solution to (1) that satisfies (20) must have the mass of the soliton,

$$\|u(t)\|_{L^2} = \|Q\|_{L^2}. \tag{25}$$

Next, using an argument analogous to the argument in [16], we prove that a solution to (1) satisfying (20) and (25) must be global in both time directions. Indeed, any finite time blowup solution with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ must be a rescaled version of (15), which clearly does not satisfy (20).

Combining (20), (25), $u_0 \in H^1$, and the virial identity shows that $u(t)$ is the soliton.

2. Mass above the soliton.

In this section, we prove that if (20), then u has the same mass as the soliton.

Theorem 2.1. *Suppose $u_0 \in H^1$ is an initial data for (1) that has a solution on the maximal interval of existence I. Also suppose that for any $\eta > 0$, there exists $R(\eta) < \infty$ such that*

$$\sup_{t \in I} \int_{|x| \geq R} |u(t,x)|^2 dx < \eta, \tag{26}$$

where I is the interval of existence for a solution to (1). Then $\|u\|_{L^2} = \|Q\|_{L^2}$, where Q is the soliton, (13).

Proof. We prove this using the virial identity

$$\partial_t \int Im(\bar{u} \cdot r\partial_r u) = 4E[u]. \tag{27}$$

Lemma 2.2. *For any solution u, $0 < R < \infty$,*

$$\sup_{t \in I} \int \psi(\frac{r}{R}) Im[\bar{u} \cdot r\partial_r u] \lesssim_{M[u],E[u]} R. \tag{28}$$

Here $\psi(r) \in C^\infty(\mathbb{R}^2)$ is a radially symmetric function, $\psi(r) = 1$ for $r \leq 1$, $\psi(r) = \frac{3}{2r}$ for $r \geq 2$. Moreover,

$$\partial_r(\psi(r)r) = \phi(r), \tag{29}$$

where $\phi(r)$ is a positive, smooth function, $\phi(r) = 1$ for $r \leq 1$, $\phi(r)$ supported on $r \leq 2$, and $\phi(r) = \chi(r)^2$ for some $\chi \in C_0^\infty(\mathbb{R}^2)$.

Proof of Lemma. Consider two cases separately, when $\|u(t)\|_{H^1}$ is uniformly bounded, and the case when $\|u(t)\|_{H^1}$ is not uniformly bounded.

Case 1.

$$\sup_{t \in I} \|\nabla u(t)\|_{L^2} < \infty. \tag{30}$$

In this case, $I = \mathbb{R}$. Now then,

$$\int \psi(\frac{r}{R}) Im[\bar{u} \cdot r\partial_r u] \lesssim \sup_{t \in I} \|\nabla u\|_{L^2} \|\psi(\frac{r}{R}) ru\|_{L^2} \lesssim_{M[u]} R \sup_{t \in I} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \lesssim R. \tag{31}$$

Case 2. Since $\|\nabla u(t)\|_{L^2}$ is continuous in time, if $\sup_{t \in I} \|\nabla u(t)\|_{L^2} = \infty$, then there exists a sequence t_n such that

$$\|\nabla u(t_n)\|_{L^2} = n. \tag{32}$$

Set

$$\lambda(t_n) = \frac{\|\nabla u(t_n)\|_{L^2}}{\|\nabla Q\|_{L^2}}. \tag{33}$$

Plugging $\lambda(t_n)$ into (21), let

$$v(t_n, x) = \frac{1}{\lambda(t_n)} u(t_n, \frac{x}{\lambda(t_n)}). \tag{34}$$

By direct computation,

$$E[v(t_n, x)] = \frac{1}{\lambda(t_n)^2} E[u(t_n)] = \frac{1}{\lambda(t_n)^2} E[u_0]. \tag{35}$$

Now, recall Proposition 4.1 from [10].

Proposition 2.3 (Decomposition)**.** *Let $Z_1, Z_2 \in C^\infty_{c,m}$ be profiles that satisfy*

$$det \begin{pmatrix} (\Lambda Q, Z_1)_r & (iQ, Z_1)_r \\ (\Lambda Q, Z_2)_r & (iQ, Z_2)_r \end{pmatrix} \neq 0. \tag{36}$$

Here, Λ is the operator $r\partial_r + 1$. Then for any $M < \infty$, there exists $0 < \alpha^* \ll 1$ such that the following properties hold for all $u \in H^1_m$ with $\|u\|_{L^2} \leq M$ satisfying the small energy condition $\sqrt{E[u]} \leq \alpha^* \|u\|_{H^1_m}$.

There exists a unique $(\lambda, \gamma) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ such that $\epsilon \in H^1_m$, defined by the relation

$$u = [Q + \epsilon]_{\lambda,\gamma}, \tag{37}$$

satisfies the orthogonality conditions,

$$(\epsilon, Z_1)_r = (\epsilon, Z_2)_r = 0, \tag{38}$$

and smallness

$$\|\epsilon\|_{\dot{\mathcal{H}}^1_m} \sim_M \lambda \sqrt{E[u]}. \tag{39}$$

Remark 2.4. The space $\mathcal{H}^1_m$ is a function space adapted to the linear coercivity of the energy. When $m \geq 1$, as is true in this paper, the spaces $\mathcal{H}^1_m$ and H^1_m are equivalent.

Proof. The proof in [10] relies on the uniqueness of the soliton as a function with zero energy, the nonlinear coercivity of energy in [10], and the implicit function theorem.

Proposition 2.5 (Nonlinear coercivity of energy)**.** *For any $M > 0$, there exists $\eta > 0$ such that the nonlinear coercivity*

$$E[Q + \epsilon] \sim_M \|\epsilon\|^2_{\dot{\mathcal{H}}^1_m}, \tag{40}$$

holds for any $\epsilon \in H^1_m$ with $\|\epsilon\|_{L^2} \leq M$ satisfying the orthogonality conditions (38) and smallness $\|\epsilon\|_{\dot{\mathcal{H}}^1_m} \leq \eta$.

Proof of Proposition 2.5. We follow the argument in [10]. Observe that, by (10),

$$2E[Q + \epsilon] = \|D_{Q+\epsilon}(Q + \epsilon)\|^2_{L^2} = \|\partial_r(Q + \epsilon) - \frac{m + A_\theta[Q + \epsilon]}{r}(Q + \epsilon)\|^2_{L^2}. \tag{41}$$

Now then, since $D_Q Q = 0$,

$$\begin{aligned} &\partial_r(Q + \epsilon) - \frac{m + A_\theta[Q + \epsilon]}{r}(Q + \epsilon) \\ &= \partial_r Q - \frac{m + A_\theta[Q]}{r} Q + \partial_r \epsilon - \frac{m + A_\theta[Q]}{r}\epsilon \\ &\quad - \frac{2A_\theta[Q, \epsilon]}{r} Q - \frac{A_\theta[\epsilon]}{r} Q - \frac{2A_\theta[Q, \epsilon]}{r}\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon \\ &= D_Q \epsilon - \frac{2A_\theta[Q, \epsilon]}{r} Q - \frac{A_\theta[\epsilon]}{r} Q - \frac{2A_\theta[Q, \epsilon]}{r}\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon \\ &= L_Q \epsilon - \frac{A_\theta[\epsilon]}{r} Q - \frac{2A_\theta[Q, \epsilon]}{r}\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon, \end{aligned} \tag{42}$$

Here,

$$A_\theta[\psi_1, \psi_2] = -\frac{1}{2}\int_0^r Re(\bar{\psi_1}\psi_2) r' dr', \tag{43}$$

and

$$L_Q\epsilon = D_Q\epsilon - \frac{2A_\theta[Q,\epsilon]}{r}Q. \tag{44}$$

Using the coercivity of L_Q proved in [9], [7],

Lemma 2.6 (Coercivity of L_Q)**.** *Let $m \geq 0$. Let $\mathcal{Z}_1$, $\mathcal{Z}_2 \in C^\infty_{c,m}$ satisfy (36). Then,*

$$\|L_Q f\|_{L^2} \sim \|f\|_{\dot{\mathcal{H}}^1_m}, \qquad \forall f \in \dot{\mathcal{H}}^1_m \qquad \textit{with} \qquad (f,\mathcal{Z}_1) = (f,\mathcal{Z}_2) = 0. \tag{45}$$

Now then, by Hardy's inequality,

$$\|\frac{2}{r}A_\theta[Q,\epsilon]\epsilon\|_{L^2} \lesssim (\int_0^\infty Q\langle r\rangle|\epsilon|dr)\cdot\|\frac{1}{\langle r\rangle}\epsilon\|_{L^2} \lesssim \|\epsilon\|^{3/2}_{\dot{\mathcal{H}}^1_m}\|\epsilon\|^{1/2}_{L^2} \lesssim_M \|\epsilon\|^{3/2}_{\dot{\mathcal{H}}^1_m}. \tag{46}$$

Also, by Hardy's inequality,

$$\|\frac{1}{r}A_\theta[\epsilon]Q\|_{L^2} \lesssim (\int_0^\infty \frac{1}{\langle r\rangle^{1/2}}|\epsilon|^2dr)\cdot\|\langle r\rangle^{1/2}Q\|_{L^2} \lesssim \|\epsilon\|^{1/2}_{L^2}\|\epsilon\|^{3/2}_{\dot{\mathcal{H}}^1_m} \lesssim_M \|\epsilon\|^{3/2}_{\dot{\mathcal{H}}^1_m}. \tag{47}$$

Therefore, we have proved

$$2E[Q+\epsilon] = \|L_Q\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon\|^2_{L^2} + O_M(\|\epsilon\|^3_{\dot{H}^1_m}). \tag{48}$$

Now decompose $\epsilon = \chi_R\epsilon + (1-\chi_R)\epsilon$, where $\chi_R(r) = \chi(\frac{r}{R})$ is the function defined in Lemma 2.2. Then decompose

$$L_Q\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon = L_Q\epsilon - \frac{A_\theta[\epsilon]}{r}(1-\chi_R)\epsilon - \frac{A_\theta[\epsilon]}{r}\chi_R\epsilon. \tag{49}$$

By direct computation,

$$\|\frac{A_\theta[\epsilon]}{r}\chi_R\epsilon\|_{L^2} \lesssim (\int_0^R |\epsilon|^2rdr)\|\frac{\epsilon}{r}\|_{L^2} \lesssim R\|\epsilon\|_{L^2}\|\epsilon\|^2_{\dot{\mathcal{H}}^1_m} \lesssim RM\|\epsilon\|^2_{\dot{\mathcal{H}}^1_m}. \tag{50}$$

Next, decompose

$$L_Q\epsilon = D_Q(\chi_R\epsilon) + D_Q((1-\chi_R)\epsilon) - \frac{2A_\theta[Q,\chi_R\epsilon]}{r}Q - \frac{2A_\theta[Q,(1-\chi_R)\epsilon]}{r}Q. \tag{51}$$

Using the decay of Q,

$$\|\frac{2A_\theta[Q,(1-\chi_R)\epsilon]}{r}Q\|_{L^2} \lesssim \frac{1}{R}\|\epsilon\|_{\dot{\mathcal{H}}^1_m}. \tag{52}$$

Therefore,

$$\|L_Q\epsilon - \frac{A_\theta[\epsilon]}{r}\epsilon\|_{L^2} = \|L_Q(\chi_R\epsilon) + (D_Q - \frac{A_\theta[\epsilon]}{r})(1-\chi_R)\epsilon\|_{L^2} + \frac{1}{R}\|\epsilon\|_{\dot{\mathcal{H}}^1_m} + RM\|\epsilon\|^2_{\dot{\mathcal{H}}^1_m}. \tag{53}$$

Decompose

$$\begin{aligned}\|L_Q(\chi_R\epsilon) + (D_Q - \frac{A_\theta[\epsilon]}{r})(1-\chi_R)\epsilon\|^2_{L^2} &= \|L_Q(\chi_R\epsilon)\|^2_{L^2} \\ +\|(D_Q - \frac{A_\theta[\epsilon]}{r})(1-\chi_R)\epsilon\|^2_{L^2} &+ 2\langle L_Q(\chi_R\epsilon), (D_Q - \frac{A_\theta[\epsilon]}{r})(1-\chi_R)\epsilon\rangle.\end{aligned} \tag{54}$$

By the support of χ_R and $1-\chi_R$,

$$\langle L_Q(\chi_R\epsilon), (D_Q - \frac{A_\theta[\epsilon]}{r})(1-\chi_R)\epsilon\rangle \lesssim \||\partial_r\epsilon| + \frac{1}{r}\epsilon\|^2_{L^2(\frac{R}{2}\leq r\leq R)}. \tag{55}$$

Using the nonlinear Hardy inequality in [10],

$$\|(D_Q - \frac{A_\theta[\epsilon]}{r})(1-\chi_R)\epsilon\|^2_{L^2} \sim_M \|(1-\chi_R)\epsilon\|^2_{\dot{\mathcal{H}}^1_m}. \tag{56}$$

Therefore, we have proved

$$2E[Q+\epsilon] \sim \|\chi_R \epsilon\|_{\dot{\mathcal{H}}^1_m}^2 + \|(1-\chi_R)\epsilon\|_{\dot{\mathcal{H}}^1_m}^2 + R^2 M^2 \|\epsilon\|_{\dot{\mathcal{H}}^1_m}^4 + \|\epsilon\|_{\dot{\mathcal{H}}^1_m}^3 + \frac{1}{R}\|\epsilon\|_{\dot{\mathcal{H}}^1_m} + \||\partial_r \epsilon| + \frac{1}{r}\epsilon\|_{L^2(\frac{R}{2} \leq r \leq R)}^2. \tag{57}$$

Taking $R \ll \frac{1}{\|\epsilon\|_{\dot{H}^1_m}}$ and averaging over $\log(\|\epsilon\|_{\dot{\mathcal{H}}^1_m})$ intervals of the form $\frac{R}{2} \leq r \leq R$, the proof of Proposition 2.5 is complete. □

Now then, suppose there exists a sequence $\|v_n\|_{\mathcal{H}^1_m}$ constant, $E[v_n] \to 0$. By the uniqueness of Q (up to scaling) as a solution to $E[Q] = 0$ and the fact that $E[u] \geq 0$, and thus Q is an energy minimizer, v_n converges in $\mathcal{H}^1_m$, and thus, $v_n \to Q$ in $\mathcal{H}^1_m$. Therefore, $\|v_n - Q\|_{\dot{\mathcal{H}}^1_m} \to 0$ as $n \to \infty$.

Next, since

$$\frac{\partial}{\partial \lambda} \lambda Q(\frac{x}{\lambda})|_{\lambda=1} = \Lambda Q, \tag{58}$$

and

$$\frac{\partial}{\partial \gamma} e^{i\gamma} Q|_{\gamma=0} = iQ, \tag{59}$$

combining the implicit function theorem with (36), for $E[v_n]$ sufficiently small, we can find a unique λ and γ such that (38) holds.

Then by Proposition 2.5, (39) holds. □

Therefore, for $n(M)$ sufficiently large,

$$\sqrt{E[v](t_n)} \leq \alpha^* \|v(t_n)\|_{\dot{H}^1_m} \sim \alpha^*, \tag{60}$$

and we can make the decomposition

$$v(t_n) = [Q+\epsilon]_{\lambda(t_n),\gamma(t_n)}. \tag{61}$$

Now, note that (21) implies

$$\int \psi(\frac{x}{R}) Im[\bar{u} \cdot r\partial_r u](t_n) = \int \psi(\frac{x\lambda(t_n)}{R}) Im[\bar{v} \cdot r\partial_r v](t_n) \tag{62}$$

$$= \int \psi(\frac{x\lambda(t_n)}{R}) Im[\bar{Q} \cdot r\partial_r Q] \tag{63}$$

$$+ \int \psi(\frac{x\lambda(t_n)}{R}) Im[\bar{Q} \cdot r\partial_r \epsilon](t_n) + \int \psi(\frac{x\lambda(t_n)}{R}) Im[\bar{\epsilon} \cdot r\partial_r Q](t_n) \tag{64}$$

$$+ \int \psi(\frac{x\lambda(t_n)}{R}) Im[\bar{\epsilon} \cdot r\partial_r \epsilon](t_n). \tag{65}$$

Since Q is real-valued, (63) $= 0$. Next,

$$(65) \lesssim \frac{R}{\lambda(t_n)} \|\nabla \epsilon_{\lambda,\gamma}\|_{L^2} \lesssim_{M[u]} \frac{R}{\lambda(t_n)} \sqrt{E[v(t_n)]} \lesssim_{M[u]} R\sqrt{E[u_0]}. \tag{66}$$

Next,

$$\int \psi(\frac{x\lambda(t_n)}{R}) Im[\bar{\epsilon} \cdot r\partial_r Q] \lesssim \|\epsilon_{\lambda,\gamma}\|_{L^2} \|r\partial_r Q_{\lambda,\gamma}\|_{L^2} \lesssim 1. \tag{67}$$

Finally, integrating by parts,

$$\int \psi(\frac{x\lambda(t_n)}{R}) Im[\bar{Q}_{\lambda,\gamma} \cdot r\partial_r \epsilon_{\lambda,\gamma}] = -\int \psi(\frac{x\lambda(t_n)}{R}) Im[r\partial_r \bar{Q}_{\lambda,\gamma} \cdot \epsilon_{\lambda,\gamma}] - \int \partial_r(\psi(\frac{r\lambda(t_n)}{R})r) Im[\bar{Q}_{\lambda,\gamma} \cdot \epsilon_{\lambda,\gamma}] \lesssim \|\epsilon\|_{L^2} \lesssim 1. \tag{68}$$

This proves the Lemma. □

Now then, let $V(t)$ denote a truncated virial-type quantity, and compute

$$\begin{aligned}\frac{d}{dt}V(t) &= \int \psi(\frac{r}{R})Re[\bar{u}\cdot r\partial_r \Delta u] - \int \psi(\frac{r}{R})Re[\Delta\bar{u}\cdot r\partial_r u] \\ &- \int \psi(\frac{r}{R})Re[\bar{u}\cdot r\partial_r(A_t[u]u)] + \int \psi(\frac{r}{R})Re[A_t[u]\bar{u}\cdot r\partial_r u] \\ - \int \psi(\frac{r}{R})Re[\bar{u}\cdot r\partial_r((\frac{m+A_\theta[u]}{r})^2 u)] &+ \int \psi(\frac{r}{R})Re[(\frac{m+A_\theta[u]}{r})^2\bar{u}\cdot r\partial_r u] \\ &\int \psi(\frac{r}{R})Re[\bar{u}\cdot r\partial_r(|u|^2 u)] - \int \psi(\frac{r}{R})Re[|u|^2\bar{u}\cdot r\partial_r u]\end{aligned} \tag{69}$$

$$\begin{aligned}&= \int \phi(\frac{r}{R})|\partial_r u|^2 - \frac{1}{4}\int \frac{1}{R^2}\phi''(\frac{r}{R})|u|^2 \\ &- \int \psi(\frac{r}{R})|u|^2(m+A_\theta[u]) + \int \psi(\frac{r}{R})|u|^2 r(\frac{m+A_\theta[u]}{r}) \\ &+2\int \psi(\frac{r}{R})|u|^2(\frac{m+A_\theta[u]}{r})^2 - \frac{1}{4}\int \phi(\frac{r}{R})|u|^4 - \int \psi(\frac{r}{R})|u|^4 \\ &= 2\int \phi(\frac{r}{R})|\partial_r u|^2 - \frac{1}{2R^2}\int \phi''(\frac{r}{R})|u|^2 + 2\int \psi(\frac{r}{R})|u|^2(\frac{m+A_\theta[u]}{r})^2 \\ &- \int \phi(\frac{r}{R})|u|^4 - \frac{1}{2}\int [\psi(\frac{r}{R}) - \phi(\frac{r}{R})]|u|^4.\end{aligned} \tag{70}$$

Integrating by parts,

$$\int \phi(\frac{r}{R})|\partial_r u|^2 = \int |\partial_r(\chi(\frac{r}{R})u)|^2 + O(\int_{r\geq R}\frac{1}{R^2}|u|^2). \tag{71}$$

Therefore,

$$\begin{aligned}(70) = 2\int |\partial_r(\chi(\frac{r}{R})u)|^2 &+ 2\int |\chi(\frac{r}{R})u|^2(\frac{m+A_\theta[\chi(\frac{r}{R})u]}{r})^2 - \int |\chi(\frac{r}{R})u|^4 \\ &+O(\int_{r\geq R}\frac{1}{R^2}|u|^2) + 2\int (\psi(\frac{r}{R}) - \phi(\frac{r}{R}))(\frac{m+A_\theta[u]}{r})^2|u|^2 \\ &+ \int |\chi(\frac{r}{R})u|^2\cdot\{(\frac{m+A_\theta[u]}{r})^2 - (\frac{m+A_\theta[\chi u]}{r})^2\} \\ &- \int [\chi(\frac{r}{R})^2 - \chi(\frac{r}{R})^4]|u|^4 - \frac{1}{2}\int [\psi(\frac{r}{R}) - \phi(\frac{r}{R})]|u|^4.\end{aligned} \tag{72}$$

Using (4),

$$2\int |\partial_r(\chi(\frac{r}{R})u)|^2 + 2\int |\chi(\frac{r}{R})u|^2(\frac{m+A_\theta[\chi(\frac{r}{R})u]}{r})^2 - \int |\chi(\frac{r}{R})u|^4 = 4E[\chi_R u]. \tag{73}$$

Next, since $A_\theta[u] \lesssim M^2$,

$$O(\int_{r\geq R}\frac{1}{R^2}|u|^2) + 2\int (\psi(\frac{r}{R}) - \phi(\frac{r}{R}))(\frac{m+A_\theta[u]}{r})^2|u|^2 \lesssim \frac{1}{R^2}\int_{r\geq R}|u|^2. \tag{74}$$

By direct computation,

$$|A_\theta[u] - A_\theta[\chi_R u]| = \int_0^r [|u|^2 - |\chi_R u|^2]r'dr', \tag{75}$$

so $|A_\theta[u] - A_\theta[\chi_R u]|$ is supported on $r \geq R$. Therefore,

$$\int |\chi(\frac{r}{R})u|^2 \cdot \{(\frac{m + A_\theta[u]}{r})^2 - (\frac{m + A_\theta[\chi_R u]}{r})^2\} \lesssim \frac{1}{R^2} \int_{r \geq R} |u|^2. \tag{76}$$

Finally, for

$$-\int [\chi(\frac{r}{R})^2 - \chi(\frac{r}{R})^4]|u|^4 - \frac{1}{2}\int [\psi(\frac{r}{R}) - \phi(\frac{r}{R})]|u|^4, \tag{77}$$

consider two cases separately, as in Lemma 2.2. For $\|u\|_{\dot{H}^1} \lesssim 1$, by interpolation,

$$\begin{aligned} &-\int [\chi(\frac{r}{R})^2 - \chi(\frac{r}{R})^4]|u|^4 - \frac{1}{2}\int [\psi(\frac{r}{R}) - \phi(\frac{r}{R})]|u|^4 \\ &\leq \int_{r \geq R} |u|^4 \lesssim (\int_{r \geq R} |u|^2)\|u\|_{\dot{H}^1}^2 \leq o_R(1), \end{aligned} \tag{78}$$

where $o_R(1)$ is a quantity that approaches 0 as $R \to \infty$. For the last step, (26) is used. When $\|u\|_{\dot{H}^1} \gg 1$, then by Proposition 2.5,

$$u = Q_{\lambda(t),\gamma(t)} + \epsilon(t,x) = \lambda(t)e^{i\gamma(t)}Q(\lambda(t)x) + \epsilon(t,x), \tag{79}$$

with $\lambda(t) \gg 1$. Then by direct computation,

$$\int_{r \geq R} \lambda(t)^4 Q(\frac{x}{\lambda(t)})^4 dx \leq o_R(1). \tag{80}$$

Also, by Proposition 2.5 and (26),

$$\int_{r \geq R} |\epsilon(t,x)|^4 dx \lesssim \|\epsilon\|_{\dot{H}^1}^2 \|\epsilon\|_{L^2(r \geq R)}^2 \leq o_R(1). \tag{81}$$

Therefore,

$$\frac{d}{dt}V(t) = E[\chi_R u] + o_R(1). \tag{82}$$

Therefore,

$$\int_0^T E[\chi_R u]dt \lesssim R + T o_R(1). \tag{83}$$

Taking $R = T^{1/3}$, there exists a sequence $t_n' \to \infty$, $R_n \to \infty$, satisfying

$$E[\chi(\frac{r}{R_n})u(t_n')] \to 0. \tag{84}$$

Therefore,

$$\chi(\frac{r}{R_n})u(t_n') = [Q + \epsilon]_{\lambda_n,\gamma_n}, \tag{85}$$

and for n sufficiently large,

$$\|\epsilon\|_{\dot{H}_m^1}^2 \sim E[\chi(\frac{r}{R_n})u(t_n')]. \tag{86}$$

Therefore, $\|\epsilon\|_{L^2(|x| \leq R_n')} \to 0$ for some $R_n' \to \infty$, and

$$\|u\|_{L^2(|x| \geq R_n')} \to 0. \tag{87}$$

The bounds (26) on the mass implies that there exists $\lambda_0 > 0$ such that $\lambda(t_n') \geq \lambda_0 > 0$. Therefore,

$$\|Q_{\lambda(t_n'),\gamma(t_n')}\|_{L^2(|x| \leq R_n')} \to \|Q\|_{L^2}, \tag{88}$$

as $n \to \infty$. □

Remark 2.7. Note that the estimate in (83) utilizes that $T \to \infty$. We cannot use this argument if u blows up in finite time in both directions.

3. **Rigidity for finite time blowup solutions at the soliton.** Now we duplicate the result of [16] for the Chern–Simons–Schrödinger equation, showing that if u is a blowup solution to (1) with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and $u_0 \in H^1$, the soluton u must be of the form (15). Such a solution would violate (26) in the scattering time direction.

Theorem 3.1. *When $m \geq 1$, if u is a finite time blowup solution with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and $u_0 \in H^1$, then u is equal to a pseudoconformal transformation of a soliton.*

Proof. By time translation symmetry and the scaling symmetry, suppose that u blows up at time $t = 0$, and let u_0 be the data for $u(t, r)$ at $t = -1$.

Lemma 3.2. *Fix $R > 0$ large. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be a smooth cut-off, $\phi(x) = 1$ for $|x| \leq 1$, and $\phi(x) = 0$ for $|x| > 2$. Then,*

$$\lim_{t \nearrow 0} \|\phi(\frac{x}{R})|x|u(t,x)\|_{L^2} = 0. \tag{89}$$

Proof. If u blows up in finite time, $\lim_{t \nearrow 0} \|u(t)\|_{\dot{H}^1} = \infty$. Now let

$$\lambda(t) = \frac{\|u\|_{\dot{H}^1}}{\|Q\|_{\dot{H}^1}}, \tag{90}$$

and let

$$v(t,x) = \frac{1}{\lambda(t)} u(t, \frac{x}{\lambda(t)}). \tag{91}$$

Then by (35), $E(v(t)) \to 0$, and $\|v(t)\|_{\dot{H}^1}$ and $\|v(t)\|_{L^2}$ are uniformly bounded for $-1 < t < 0$. Therefore, by Proposition 2.3, for t sufficiently close to 0,

$$v(t) = [Q + \epsilon]_{\tilde{\lambda}(t), \tilde{\gamma}(t)}, \tag{92}$$

and furthermore,

$$\|\epsilon\|_{\dot{H}^1_m} \to 0. \tag{93}$$

By (92), (93), and $\|v\|_{\dot{H}^1} = 1$, $\tilde{\lambda}(t) \sim 1$, so

$$e^{-i\tilde{\gamma}(t)} v(t) \rightharpoonup Q, \qquad \text{in} \qquad L^2, \tag{94}$$

and since $\|v\|_{L^2} = \|Q\|_{L^2}$, (94) can be upgraded to convergence in L^2. Since $\lambda(t) \nearrow \infty$ as $t \nearrow 0$, (89) holds. □

Lemma 3.3. *For any $R > 0$,*

$$\lim_{t \nearrow 0} \int \phi(\frac{x}{R}) Im(\bar{u} \cdot r\partial_r u) dx = 0. \tag{95}$$

Proof. The argument is identical to the argument proving Lemma 2.2, except that now, insert $\|\epsilon(t)\|_{L^2} \to 0$ into (63)–(65), proving (95). □

Returning to the proof of Theorem 1.3, by direct computation,

$$\frac{d}{dt} \int r^2 |u|^2 dx = 4 \int Im(\bar{u} \cdot r\partial_r u) dx. \tag{96}$$

Integrating by parts,

$$\frac{d}{dt} \int \phi(\frac{r}{R})^2 r^2 |u|^2 dx = 4 \int \phi^2(\frac{x}{R}) Im(\bar{u} \cdot r\partial_r u) dx + \frac{8}{R} \int \phi(\frac{r}{R}) \phi'(\frac{r}{R}) r Im(\bar{u} \cdot r\partial_r u) dx. \tag{97}$$

Also, by direct computation and integrating by parts,

$$\frac{d}{dt}\int \phi(\frac{r}{R})^2 Im(\bar{u}\cdot r\partial_r u)dx = 2\int \phi(\frac{r}{R})^2[|\partial_r u|^2 + (\frac{m + A_\theta[u]}{r})^2|u|^2 - \frac{1}{2}|u|^4]dx + \frac{C}{R}\int \phi'(\frac{r}{R})\phi(\frac{r}{R})\{r|u_r|^2 + \frac{1}{r}|u|^2\}dx. \tag{98}$$

Therefore, by Lemmas 3.2 and 3.3, taking $R \nearrow \infty$, for any $-1 < t < 0$,

$$\int |x|^2|u(t,x)|^2 dx = 8E[u]t^2. \tag{99}$$

Making a pseudoconformal transformation of the solution, let

$$v(t,r) = \frac{1}{|t|}u(-\frac{1}{t}, \frac{r}{|t|})e^{ir^2/4t}. \tag{100}$$

By (100),

$$\frac{1}{2}\int |\partial_r v|^2 dx = \frac{1}{2t^2}\|u_r(-\frac{1}{t}, \cdot)\|_{L^2}, \qquad \frac{1}{2}\int \frac{r^2}{4t^2}|u(-\frac{1}{t}, \frac{r}{t})|^2 r dr = \frac{1}{t^2}E[u], \tag{101}$$

and by (97),

$$Re(\int \frac{1}{t^2}\overline{u_r(-\frac{1}{t}, \frac{r}{t})}\cdot \frac{ir}{2t^2}u(-\frac{1}{t}, \frac{r}{t})dx) = -\frac{2}{t^2}E[u]. \tag{102}$$

Therefore, $E[v] = 0$, and thus, v is a soliton, so u is a pseudoconformal transformation of a soliton. □

Remark 3.4. When $m = 0$, the pseudoconformal transformation of the soliton does not lie in H^1_m.

4. **The Liouville theorem.** Now we have proved that the only solution to (1) that satisfies (26) has mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and is global in both time directions. Then we complete the proof of the Liouville theorem by showing that u is a soliton.

Theorem 4.1. *The solution satisfying $\|u\|_{L^2} = \|Q\|_{L^2}$ and (21) is the soliton.*

Proof. We again use the virial identity in (69) and (70), only this time we integrate from $-T$ to T. Integrating by parts,

$$\int \psi(\frac{r}{R})Re[\bar{u}\cdot r\partial_r \Delta u] - \int \psi(\frac{r}{R})Re[\Delta\bar{u}\cdot r\partial_r u] = 2\int \phi(\frac{r}{R})|\partial_r u|^2 + O(\frac{1}{R^2}\int_{r\geq R}|u|^2), \tag{103}$$

where again $\phi(r) = \partial_r(r\psi(r))$ and $\phi(r) = \chi(r)^2$ for some $\chi \in C_0^\infty$. Now then,

$$\frac{1}{R^2}\int_{r\geq R}\lambda^2 Q(\frac{x}{\lambda})^2 dx = \frac{\lambda}{R^2}\int_R^\infty \frac{(\lambda r)^{2m+1}}{(1+(\lambda r)^{2m+2})^2}dr \lesssim \frac{1}{R^4\lambda^2}. \tag{104}$$

Next, for

$$u = \lambda Q(\lambda x) + \lambda\epsilon(\lambda x), \tag{105}$$

since $\|u\|_{L^2} = \|Q\|_{L^2}$, by standard linear algebra,

$$\|\epsilon\|_{L^2}^2 = -2\langle \lambda Q(\lambda r), \lambda\epsilon(\lambda r)\rangle. \tag{106}$$

Therefore,

$$\|\epsilon\|_{L^2}^2 = 2|\langle\epsilon, Q\rangle| \leq 2|\langle\chi(\frac{r}{R})\lambda\epsilon(\lambda x), \lambda Q(\lambda x)\rangle| + 2|\langle(1-\chi(\frac{r}{R}))\lambda\epsilon(\lambda x), \lambda Q(\lambda x)\rangle|. \tag{107}$$

By Hölder's inequality,

$$2|\langle(1-\chi(\frac{r}{R}))\lambda\epsilon(\lambda x), \lambda Q(\lambda x)\rangle| \lesssim \frac{1}{R^2\lambda^2}\|\epsilon\|_{L^2}. \tag{108}$$

Now let

$$\tilde{Q}(r) = -\int_r^\infty Q(r)dr. \tag{109}$$

Since $Q(r) \lesssim \frac{1}{r^4}$ for r large and $Q(r) \leq 1$ for all r, $\tilde{Q}(r) \in L^2(\mathbb{R}^2)$. Moreover,

$$\partial_r\tilde{Q}(r) = Q(r), \qquad \text{and} \qquad \partial_r\tilde{Q}(\lambda r) = \lambda Q(\lambda r). \tag{110}$$

Therefore,

$$\langle\chi(\frac{r}{R})\lambda\epsilon(\lambda r), \lambda Q(\lambda r)\rangle = \langle\chi(\frac{r}{R})\lambda\epsilon(\lambda r), \partial_r\tilde{Q}(\lambda r)\rangle. \tag{111}$$

Integrating by parts,

$$\begin{aligned}(111) \lesssim \frac{1}{\lambda}\|\chi(\frac{r}{R})\lambda\epsilon(\lambda r)\|_{\dot{H}^1_m} &\lesssim \frac{1}{\lambda}E[\chi(\frac{r}{R})\lambda\epsilon(\lambda r) + \lambda Q(\lambda r)]^{1/2} \\ &\lesssim \frac{1}{\lambda}E[\chi(\frac{r}{R})u]^{1/2} + \frac{1}{\lambda^3R^3}.\end{aligned} \tag{112}$$

Therefore,

$$\|\epsilon\|_{L^2}^2 \lesssim \frac{1}{\lambda}E[\chi(\frac{r}{R})u]^{1/2} + \frac{1}{\lambda^3R^3}, \tag{113}$$

and by (104) and (113),

$$\int_{r\geq R}\frac{1}{R^2}|u|^2 \lesssim \frac{1}{\lambda R^2}E[\chi(\frac{r}{R})u]^{1/2} + \frac{1}{\lambda^2R^4}. \tag{114}$$

Also by (104),

$$\int_{r\geq R}\frac{R}{r}|u|^4dx \lesssim \frac{1}{\lambda^2R^4} + \int_{r\geq R}\frac{R}{r}|\lambda\epsilon(\lambda x)|^4dx. \tag{115}$$

By standard perturbation theory and the fact that the $L^4_{t,x}$ norm is invariant under the scaling (21),

$$\int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}\int|\lambda\epsilon(t,\lambda x)|^4dxdt \lesssim \|\epsilon(t_0)\|_{L^2}^4 \lesssim \lambda(t_0)^2\int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}\|\epsilon(t)\|_{L^2}^4. \tag{116}$$

Plugging in (113), since $\lambda(t) \sim \lambda(t_0)$ for $t \in [t_0, t_0 + \frac{1}{\lambda(t_0)^2}]$,

$$\lambda(t_0)^2\int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}\|\epsilon(t)\|_{L^2}^4 \lesssim \int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}E[\chi(\frac{r}{R})u]dt + \int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}\frac{1}{R^4}dt. \tag{117}$$

Making an averaging argument, for a fixed R_0,

$$\sum_j\int_{r\geq 2^jR_0}\frac{2^jR_0}{r}|f(x)|^4dx \lesssim \|f(x)\|_{L^4}^4. \tag{118}$$

Therefore, for any $\delta > 0$ there exists some $R_0 \leq R_* \leq C(\delta)R_0$ such that

$$\begin{aligned}\int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}\int_{r\geq R_*}\frac{R_*}{r}|\lambda\epsilon(t,\lambda x)|^4dxdt &\lesssim \delta\int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}\|\epsilon(t)\|_{L^4}^4dt \\ &\lesssim \delta\int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}E[\chi(\frac{r}{R_*})u]dt + \int_{t_0}^{t_0+\frac{1}{\lambda(t_0)^2}}\frac{1}{R^4}dt.\end{aligned} \tag{119}$$

Therefore, we have proved

$$\int_{a}^{b} E[\chi(\frac{r}{R})u]dt \lesssim R\|\epsilon(a)\|_{L^2} + R\|\epsilon(b)\|_{L^2} + \int_{a}^{b} \frac{1}{R^4}dt. \tag{120}$$

Taking $R = T^{1/4}$, averaging (120), and plugging in (113), we have proved

$$\lim_{T \nearrow \infty} \inf_{t \in [0,T]} \|\epsilon(t)\|_{L^2} = 0, \qquad \lim_{T \nearrow \infty} \inf_{t \in [-T,0]} \|\epsilon(t)\|_{L^2} = 0. \tag{121}$$

Now for any $j \in \mathbb{Z}$, $j \geq 0$, let

$$t_j^+ = \inf\{t \geq 0 : \|\epsilon(t)\|_{L^2} = 2^{-j}\}, \qquad t_j^- = \sum\{t < 0 : \|\epsilon(t)\|_{L^2} = 2^{-j}\}. \tag{122}$$

Then let $T_j = t_j^+ - t_j^-$, $I_j = [t_j^-, t_j^+]$. Then $T_j \to \infty$ as $j \to \infty$. Then by (113),

$$\|\epsilon(t_j^{\pm})\|_{L^2}^4 \lesssim \frac{1}{\lambda(t_j^{\pm})^2} E[\chi(\frac{r}{R})u(t_j^{\pm})] + \frac{1}{R_j^6}. \tag{123}$$

Therefore, by (120),

$$\int_{I_j} E[\chi(\frac{r}{R})u]dt \lesssim \frac{R_j}{T_j}(\int_{I_j} E[\chi(\frac{r}{R})u]^{1/4}dt) + \frac{T_j}{R_j^4}. \tag{124}$$

Therefore,

$$\int_{I_j} E[\chi(\frac{r}{R})u]dt \lesssim \frac{R_j}{T_j^{1/4}} + \frac{T_j}{R_j^4} \sim 1. \tag{125}$$

Then, by the dominated convergence theorem, taking $j \to \infty$,

$$\int_{\mathbb{R}} E[u]dt \lesssim 1, \tag{126}$$

which implies $E[u] \equiv 0$, and thus u is a soliton. □

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