Global well-posedness for the radial, defocusing, nonlinear wave equation for 3 < *p* < 5

Benjamin Dodson

American Journal of Mathematics, Volume 146, Number 1, February 2024, pp. 1-46 (Article)

Published by Johns Hopkins University Press
DOI: *https://doi.org/10.1353/ajm.2024.a917538*

GLOBAL WELL-POSEDNESS FOR THE RADIAL, DEFOCUSING, NONLINEAR WAVE EQUATION FOR $3 < p < 5$

By BENJAMIN DODSON

Abstract. In this paper we continue the study of the defocusing, energy-subcritical nonlinear wave equation with radial initial data lying in the critical Sobolev space. In this case we prove scattering in the critical norm when $3 < p < 5$.

1. Introduction. In this paper we prove global well-posedness and scattering for

$$u_{tt} - \Delta u + |u|^{p-1}u = 0, \quad u : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}, \quad u(0,x) = u_0, \quad u_t(0,x) = u_1, \tag{1.1}$$

with $3 < p < 5$ and radial initial data in the critical L^2-based Sobolev space.

The critical Sobolev space for (1.1) arises from the scaling symmetry

$$u(t,x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x). \tag{1.2}$$

The $\dot{H}^{s_c}$ norm, where s_c is the critical Sobolev exponent

$$s_c = \frac{3}{2} - \frac{2}{p-1}, \tag{1.3}$$

is preserved under this scaling. It is well known that this scaling symmetry completely determines the local theory for (1.1) (see [13]). We prove that the scaling symmetry also completely determines the global theory for radial initial data. The ill-posedness results of [13] imply that this result is sharp for radial initial data.

The proof continues the study that we began in [4, 3], where we proved,

THEOREM 1.1. *The defocusing, cubic nonlinear wave equation*

$$u_{tt} - \Delta u + u^3 = 0, \quad u(0,x) = u_0, \quad u_t(0,x) = u_1, \tag{1.4}$$

is globally well-posed and scattering for all radial initial data in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$.

In this paper we prove the corresponding result for $3 < p < 5$, or equivalently by (1.3), for $\frac{1}{2} < s_c < 1$.

Manuscript received July 25, 2020; revised May 27, 2022.
Research supported in part by NSF grant DMS-1764358.
American Journal of Mathematics 146 (2024), 1–46. © 2024 by Johns Hopkins University Press.

THEOREM 1.2. *The defocusing, nonlinear wave equation*

$$u_{tt} - \Delta u + |u|^{p-1}u = 0, \quad u(0,x) = u_0, \quad u_t(0,x) = u_1, \tag{1.5}$$

is globally well-posed and scattering for radial initial data $(u_0, u_1) \in \dot{H}^{s_c}(\mathbf{R}^3) \times \dot{H}^{s_c-1}(\mathbf{R}^3)$. *Moreover, there exists a function* $f : [0,\infty) \to [0,\infty)$ *such that if* u *solves* (1.5) *with radial initial data* $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$, *then*

$$\|u\|_{L_{t,x}^{2(p-1)}(\mathbf{R}\times\mathbf{R}^3)} \leq f(\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}). \tag{1.6}$$

There are several reasons to conjecture that such a result is true for both radial and nonradial data.

First, as we have already mentioned, critical Sobolev regularity completely determines local well-posedness.

THEOREM 1.3. *Equation* (1.5) *is locally well-posed for initial data* $(u_0, u_1) \in \dot{H}^{s_c}(\mathbf{R}^3) \times \dot{H}^{s_c-1}(\mathbf{R}^3)$ *on some interval* $[-T(u_0,u_1), T(u_0,u_1)]$, *where the time of well-posedness* $T(u_0,u_1)$ *depends on the profile of the initial data* (u_0,u_1), *not just its size. Global well-posedness and scattering hold for small* $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ *norm.*

Additional regularity is enough to give a lower bound on the time of well-posedness. Therefore, there exists some $T(\|u_0\|_{\dot{H}^s}, \|u_1\|_{\dot{H}^{s-1}}) > 0$ *for any* $s_c < s < \frac{3}{2}$.

Furthermore, equation (1.1) *is ill-posed for initial data in the Sobolev space* $\dot{H}^s \times \dot{H}^{s-1}$ *when* $s < s_c$.

Proof. See [13]. □

Local well-posedness combined with conservation of the energy

$$E(u(t)) = \frac{1}{2}\int u_t(t,x)^2 dx + \frac{1}{2}\int |\nabla u(t,x)|^2 dx + \frac{1}{p+1}\int |u(t,x)|^{p+1} dx, \tag{1.7}$$

implies global well-posedness for finite energy initial data, that is, $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ and $u_1 \in \dot{H}^{s_c-1} \cap L^2$. Indeed, by the Sobolev embedding theorem,

$$E(u(0)) \lesssim \|u_t(0)\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla u(0)\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla u(0)\|_{L^2(\mathbf{R}^3)}^2 \|u(0)\|_{\dot{H}^{s_c}(\mathbf{R}^3)}^{p-1}, \tag{1.8}$$

and therefore,

$$E(u(0)) \lesssim_{\|u_0\|_{\dot{H}^{s_c}}} \|u_t(0)\|_{L^2}^2 + \|\nabla u(0)\|_{L^2}^2. \tag{1.9}$$

By conservation of energy, $E(u(0)) = E(u(t))$, so (1.9) gives a uniform bound over the norm $\|u_t(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2$. Since (1.5) is energy-subcritical, Theorem 1.3 implies that a uniform bound over the energy is enough to ensure global well-posedness. Additionally, the results of [20] imply that for initial data of finite energy with sufficiently rapid decay at infinity, the global solution to (1.1) scatters.

Definition 1.1 (Scattering). A solution to (1.1) is said to scatter forward in time if there exist $(u_0^+, u_1^+) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ such that

$$\|(u(t), u_t(t)) - S(t)(u_0^+, u_1^+)\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \to 0, \tag{1.10}$$

as $t \to +\infty$, where $S(t)$ is the solution operator to the linear wave equation

$$u_{tt} - \Delta u = 0.$$

Scattering backward in time is defined in the corresponding way. A global solution is said to scatter if it scatters both forward and backward in time.

Remark. Conservation of energy does not imply global well-posedness or scattering for the focusing problem, which will not be discussed here. See [14] for a discussion of blowup solutions in the focusing case.

The second reason to conjecture scattering for a solution to (1.1) with generic initial data $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ is that it is known that an a priori upper bound on the $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm for a solution to (1.1) implies scattering for both radial and nonradial data.

THEOREM 1.4. *Suppose* $u_0 \in \dot{H}^{s_c}(\mathbf{R}^3)$, $u_1 \in \dot{H}^{s_c-1}(\mathbf{R}^3)$, *and* u *solves* (1.5) *on a maximal interval* $0 \in I \subset \mathbf{R}$, *with*

$$\sup_{t \in I} \|u(t)\|_{\dot{H}^{s_c}(\mathbf{R}^3)} + \|u_t(t)\|_{\dot{H}^{s_c-1}(\mathbf{R}^3)} < \infty. \tag{1.11}$$

Then $I = \mathbf{R}$ *and the solution* u *scatters both forward and backward in time.*

Proof. See [17] for the proof in the radial case and [5] for the proof in the nonradial case. □

Theorem 1.4 is called a type two scattering result, while a blowup solution to (1.1) that satisfies (1.11) would be a type two blowup solution.

Remark. The pseudoconformal transformation of a NLS soliton is an example of a type two blowup solution.

Type one blowup is a solution to (1.1) for which the bound (1.11) does not hold. Since $S(t)$ is unitary, (1.10) cannot occur if (1.11) does not hold.

The tools for type two scattering results are very well-developed, especially for the energy-critical wave equation. Observe that when $s_c = 1$, or

$$u_{tt} - \Delta u + u^5 = 0, \quad u(0,x) = u_0, \quad u_t(0,x) = u_1, \tag{1.12}$$

(1.11) automatically follows from conservation of the energy

$$E(u(t)) = \frac{1}{2}\int u_t(t,x)^2 dx + \frac{1}{2}\int |\nabla u(t,x)|^2 dx + \frac{1}{6}\int u(t,x)^6 dx, \tag{1.13}$$

reducing scattering questions for the defocusing, energy-critical problem to type two scattering questions. The qualitative behavior of (1.12) has been completely worked out, proving global well-posedness and scattering, for both the radial [6, 22] and the nonradial case [1, 2, 8, 16]. The proof relies very heavily on conservation of the energy, which ensures a uniform bound over the critical Sobolev norm, and also yields a Morawetz estimate,

$$\iint \frac{u(t,x)^6}{|x|} dxdt \lesssim E(u(0)), \tag{1.14}$$

which gives a space-time integral estimate for a solution to (1.12).

When $3 < p < 5$ there is no known conserved quantity that gives an upper bound on $\|u\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}$. Additionally, since Morawetz estimates arise from conservation laws, there is also no known Morawetz estimate at the critical Sobolev regularity. Such a Morawetz estimate would likely significantly simplify the proof of Theorem 1.4, while proving an upper bound on $\|u\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}$ would mean that Theorem 1.4 would imply Theorem 1.2 for both radial and nonradial data. The author believes that [4] and this paper are the first global well-posedness and scattering results for initial data in a critical Sobolev space when there is no known conserved quantity that controls the critical Sobolev norm.

To prove Theorem 1.2 we utilize the Fourier truncation method. The initial data is split into two pieces; a piece with small $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm and a piece with finite energy. Then, a solution u to (1.5) is shown to have the decomposition

$$u(t) = v(t) + w(t), \tag{1.15}$$

where $v(t)$ has uniformly bounded energy, and $w(t)$ is a small data scattering solution to (1.5). By Theorem 1.3, a uniform bound on the energy of $v(t)$ is enough to imply global well-posedness of (1.5).

Remark. The Fourier truncation method was used in [10] to prove global well-posedness for the cubic problem when $s > \frac{3}{4}$.

To prove scattering, the wave equation (1.5) is rewritten in hyperbolic coordinates. These coordinates were quite useful to the cubic wave equation because the hyperbolic energy scales like the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ norm. For $3 < p < 5$, the hyperbolic energy and the energy (1.13) "sandwich" the $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm, giving scattering.

Remark. Previously, [18] used hyperbolic coordinates to prove scattering for (1.5) with radial data lying in the energy space and a weighted Sobolev space. The weighted Sobolev space used in [18] also scales like the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ norm.

As in [4], energy and hyperbolic energy bounds merely give a scattering size bound for any initial data in the critical Sobolev space, but with scattering size

depending on the initial data (u_0, u_1) and not just its size. To prove a scattering size bound that depends only on the size of the initial data, we use Zorn's lemma. As in [3, 4], we use a profile decomposition to show that if $(u_0^n, u_1^n) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ is a bounded sequence, then $\|u^n\|_{L_{t,x}^{2(p-1)}(\mathbf{R}\times\mathbf{R}^3)}$ is also uniformly bounded.

Remark. The upper bound in (1.6) is completely qualitative. Concentration compactness-type arguments that proved scattering in the energy-critical case also obtained a quantitative bound. See for example [23]. Here we do not obtain any quantitative bounds at all. In the author's opinion, it would be very interesting to obtain some sort of quantitative bound.

Outline of the argument. We begin by proving global well-posedness for the $p = 4$ case in section two. This is a warm-up for section three, where we then generalize this global well-posedness result to any $3 < p < 5$. After proving global well-posedness, the hyperbolic coordinates are well-defined. In section four, we prove an estimate on the initial data, before obtaining a scattering bound in section five. We conclude with a concentration compactness argument in section six.

Acknowledgments. The author was a guest of the Institute for Advanced Study during the writing of this paper. The author is also grateful to the anonymous referee for a detailed list of many helpful suggestions.

2. Global well-posedness in the $p = 4$ case. To simplify the exposition, first consider the specific case of $p = 4$ in (1.1),

$$u_{tt} - \Delta u + |u|^3 u = 0. \tag{2.1}$$

In this case

$$s_c = \frac{3}{2} - \frac{2}{4-1} = \frac{5}{6}. \tag{2.2}$$

Global well-posedness is proved using the Fourier truncation method. Using (1.2), fix $0 < \epsilon \ll 1$ and rescale so that

$$\|(P_{>1}u_0, P_{>1}u_1)\|_{\dot{H}^{5/6}\times\dot{H}^{-1/6}} < \epsilon. \tag{2.3}$$

Remark. In sections four and five, this data will again be rescaled so that most of the critical Sobolev norm lies in a ball of radius one.

Then decompose the initial data

$$v_0 = P_{\leq 1}u_0, \quad w_0 = P_{>1}u_0, \quad v_1 = P_{\leq 1}u_1, \quad w_1 = P_{>1}u_1. \tag{2.4}$$

By Theorem 1.3, (2.1) has a local solution. Decompose the solution to (2.1), $u = v + w$, where v and w solve

$$w_{tt} - \Delta w + |w|^3 w = 0, \quad w(0,x) = w_0, \quad w_t(0,x) = w_1, \tag{2.5}$$

$$v_{tt} - \Delta v + |u|^3 u - |w|^3 w = 0, \quad v(0,x) = v_0, \quad v_t(0,x) = v_1. \tag{2.6}$$

We know from [13] that (2.5) is globally well-posed and scattering. The proof uses Strichartz estimates and small data arguments.

THEOREM 2.1 (Strichartz estimates). *Let $I \subset \mathbf{R}$, $t_0 \in I$ be an interval and let u solve the linear wave equation*

$$u_{tt} - \Delta u = F, \quad u(t_0) = u_0, \quad u_t(t_0) = u_1. \tag{2.7}$$

Then we have the estimates

$$\begin{aligned} &\|u\|_{L_t^p L_x^q (I \times \mathbf{R}^3)} + \|u\|_{L_t^\infty \dot{H}^s (I \times \mathbf{R}^3)} + \|u_t\|_{L_t^\infty \dot{H}^{s-1}(I \times \mathbf{R}^3)} \\ &\lesssim_{p,q,s,\tilde{p},\tilde{q}} \|u_0\|_{\dot{H}^s(\mathbf{R}^3)} + \|u_1\|_{\dot{H}^{s-1}(\mathbf{R}^3)} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I \times \mathbf{R}^3)}, \end{aligned} \tag{2.8}$$

whenever $s \geq 0$, $2 \leq p, \tilde{p} \leq \infty$, $2 \leq q, \tilde{q} < \infty$, and

$$\frac{1}{p} + \frac{1}{q}, \ \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} \leq \frac{1}{2}, \quad \frac{1}{p} + \frac{3}{q} = \frac{3}{2} - s = \frac{1}{\tilde{p}'} + \frac{3}{\tilde{q}'} - 2. \tag{2.9}$$

Proof. Theorem 2.1 was proved for $p = q = 4$ in [21] and then in [7] for a general choice of (p,q). □

Then,

$$\begin{aligned} &\|w\|_{L_{t,x}^6 \cap L_t^{12/5} L_x^{12} \cap L_t^\infty \dot{H}^{5/6}} \\ &\lesssim \|(w_0, w_1)\|_{\dot{H}^{5/6} \times \dot{H}^{-1/6}} + \|w\|_{L_{t,x}^6}^3 \|w\|_{L_t^{12/5} L_x^{12}} \lesssim \epsilon, \end{aligned} \tag{2.10}$$

which by (2.3) implies that w is scattering. Additionally, the radial Strichartz estimate and Bernstein's inequality implies

$$\|w\|_{L_t^\infty L_x^2} \lesssim \epsilon + \|w\|_{L_{t,x}^6}^3 \|w\|_{L_t^\infty L_x^2} \lesssim \epsilon. \tag{2.11}$$

THEOREM 2.2 (Radial Strichartz estimate). *For (u_0, u_1) radially symmetric, if u solves* (2.7) *with $F = 0$,*

$$\|u\|_{L_t^2 L_x^\infty (\mathbf{R} \times \mathbf{R}^3)} \lesssim \|u_0\|_{\dot{H}^1(\mathbf{R}^3)} + \|u_1\|_{L^2(\mathbf{R}^3)}. \tag{2.12}$$

Proof. This theorem was proved in [11]. The dual of (2.12) is that if $u_0 = u_1 = 0$, and F is radial, then

$$\|u\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_t^2 L_x^1}. \tag{2.13}$$

□

Therefore, to prove global well-posedness of (1.1) in the case when $p = 4$, it is enough to prove global well-posedness of (2.6). To that end, let $E(t)$ be the energy of v,

$$E(t) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int v_t^2 + \frac{1}{5}\int |v|^5 dx. \tag{2.14}$$

By the Sobolev embedding theorem and (2.4),

$$E(0) \lesssim (\|u_0\|_{\dot{H}^{5/6}} + \|u_1\|_{\dot{H}^{-1/6}})^2 + (\|u_0\|_{\dot{H}^{5/6}} + \|u_1\|_{\dot{H}^{-1/6}})^5. \tag{2.15}$$

To prove global well-posedness of (2.6), it is enough to prove a uniform bound on $E(t)$. Indeed, suppose that (2.6) has a solution on an interval $[0, T)$, and that

$$\sup_{t\in[0,T)} E(t) < \infty. \tag{2.16}$$

Then by Theorem 1.3, there exists some $\delta > 0$ such that for any $t_0 \in [0, T)$,

$$\tilde{v}_{tt} - \Delta\tilde{v} + |\tilde{v}|^3\tilde{v} = 0, \quad \tilde{v}(t_0, x) = v(t_0, x), \quad \tilde{v}_t(t_0, x) = v_t(t_0, x), \tag{2.17}$$

has a solution on $[t_0, t_0 + \delta]$. By (2.10) and standard perturbation theory (see Lemma 6.2), this proves that the solution to (2.6) can be continued past T.

THEOREM 2.3. *The energy $E(t)$ given by* (2.14) *is uniformly bounded for all $t \in \mathbf{R}$, and moreover,*

$$\sup_{t\in\mathbf{R}} E(t) \lesssim_{\|u_0\|_{\dot{H}^{5/6}}, \|u_1\|_{\dot{H}^{-1/6}}} E(0). \tag{2.18}$$

Proof. The proof is quite similar to the proof in [4]. By direct computation,

$$\frac{d}{dt}E(v(t)) = -\int v_t[|v+w|^3(v+w) - |w|^3 w - |v|^3 v]dx. \tag{2.19}$$

By Taylor's theorem,

$$\begin{aligned} |v+w|^3(v+w) - |v|^3 v - |w|^3 w &= 4w\int_0^1 |v+\tau w|^3 d\tau - 4w\int_0^1 |\tau w|^3 d\tau \\ &= 12wv\int_0^1\int_0^1 |sv+\tau w|(sv+\tau w)ds d\tau \\ &= 4|v|^3 w + O(|v|^2|w|^2) + O(|v||w|^3). \end{aligned} \tag{2.20}$$

By Hölder's inequality and (2.14),

$$\begin{aligned} \langle v_t, |v|^2|w|^2\rangle &\lesssim \|v_t\|_{L^2_x(\mathbf{R}^3)}\|v\|^{1/3}_{L^6_x(\mathbf{R}^3)}\|v\|^{5/3}_{L^5_x(\mathbf{R}^3)}\|w\|^2_{L^{18}_x(\mathbf{R}^3)} \\ &\lesssim E(t)\|w(t)\|^2_{L^{18}_x(\mathbf{R}^3)}, \end{aligned} \tag{2.21}$$

and

$$\langle v_t, |v||w|^3\rangle \lesssim \|v_t\|_{L_x^2(\mathbf{R}^3)}\|v\|_{L_x^6(\mathbf{R}^3)}\|w\|^3_{L_x^9(\mathbf{R}^3)} \lesssim E(t)\|w(t)\|^3_{L_x^9(\mathbf{R}^3)}. \tag{2.22}$$

Therefore,

$$\frac{d}{dt}E(t) = -4\langle v_t, |v|^3 w\rangle + E(t)O(\|w(t)\|^2_{L_x^{18}(\mathbf{R}^3)} + \|w(t)\|^3_{L_x^9(\mathbf{R}^3)}). \tag{2.23}$$

If the term $4\langle v_t, |v|^3 w\rangle$ could be dropped, then we would have

$$\frac{d}{dt}E(t) \lesssim E(t)[\|w(t)\|^2_{L_x^{18}(\mathbf{R}^3)} + \|w(t)\|^3_{L_x^9(\mathbf{R}^3)}]. \tag{2.24}$$

By radial Strichartz estimates, (2.4), and (2.10),

$$\int_{\mathbf{R}} \|w(t)\|^2_{L_x^{18}(\mathbf{R}^3)} + \|w(t)\|^3_{L_x^9(\mathbf{R}^3)} dt \lesssim \epsilon^2. \tag{2.25}$$

Indeed,

THEOREM 2.4 (Radial Strichartz estimates). *Let (u_0, u_1) be spherically symmetric, and suppose u solves* (2.7) *with $F = 0$. Then if $q > 4$ and*

$$\frac{1}{2} + \frac{3}{q} = \frac{3}{2} - s, \tag{2.26}$$

then

$$\|u\|_{L_t^2 L_x^q(\mathbf{R}\times\mathbf{R}^3)} \lesssim \|u_0\|_{\dot{H}^s(\mathbf{R}^3)} + \|u_1\|_{\dot{H}^{s-1}(\mathbf{R}^3)}. \tag{2.27}$$

Proof. This was proved in [19]. □

Then for w solving (2.5), by (2.10), the Sobolev embedding theorem, and the principle of superposition,

$$\|w\|_{L_t^2 L_x^{18} \cap L_t^3 L_x^9} \lesssim \|w_0\|_{\dot{H}^{5/6}} + \|w_1\|_{\dot{H}^{-1/6}} + \||w|^3 w\|_{L_t^1 L_x^{9/5}} \lesssim \epsilon. \tag{2.28}$$

Then by Gronwall's inequality, (2.24) and (2.25) would easily imply that

$$\sup_{t\in\mathbf{R}} E(t) \lesssim E(0). \tag{2.29}$$

Remark. In fact, it is possible to say something more than (2.28). Namely, by Duhamel's principle,

$$w(t) = S(t)(w_0, w_1) - \int_0^t S(t-\tau)(0, |w|^3 w) d\tau, \tag{2.30}$$

so since the operator P_j commutes with $S(t)$,

$$\begin{aligned}
&\Big(\sum_j \|P_j w\|^2_{L_t^2 L_x^{18}}\Big)^{1/2} \\
&\lesssim \Big(\sum_j \|P_j w_0\|^2_{\dot H^{5/6}} + \|P_j w_1\|^2_{\dot H^{-1/6}} + \|P_j(|w|^3 w)\|^2_{L_t^1 \dot H^{-1/6}}\Big)^{1/2} \\
&\lesssim \|w_0\|_{\dot H^{5/6}} + \|w_1\|_{\dot H^{-1/6}} + \||w|^3 w\|_{L_t^1 \dot H^{-1/6}} \\
&\lesssim \|w_0\|_{\dot H^{5/6}} + \|w_1\|_{\dot H^{-1/6}} + \||w|^3 w\|_{L_t^1 L_x^{9/5}} \lesssim \epsilon.
\end{aligned} \tag{2.31}$$

The contribution of $4\langle v_t, |v|^3 w\rangle$ is controlled using a Morawetz estimate in conjunction with weighted Strichartz estimates, as was done in [4]. Define

$$\mathcal{E}(t) = E(t) + cM(t) + \int |v|^3 v w dx, \tag{2.32}$$

where $M(t)$ is the Morawetz potential

$$M(t) = \int v_t \frac{x}{|x|}\cdot \nabla v dx + \int v_t \frac{1}{|x|} v dx, \tag{2.33}$$

and $c > 0$ is a small, fixed constant. By Hardy's inequality,

$$cM(t) \lesssim c\|\nabla v\|_{L^2(\mathbf{R}^3)}\|v_t\|_{L^2(\mathbf{R}^3)} \lesssim cE(t), \tag{2.34}$$

and by (2.10),

$$\int |v|^3 v w dx \lesssim \|v\|^{10/3}_{L_x^5}\|v\|^{2/3}_{L_x^6}\|w\|_{L_x^{9/2}} \lesssim \epsilon E(t). \tag{2.35}$$

Therefore, $\mathcal{E}(t) \sim E(t)$.

Next, by the product rule,

$$4\langle v_t, |v|^3 w\rangle - \frac{d}{dt}\int |v|^3 v w dx = -\langle v, |v|^3 w_t\rangle. \tag{2.36}$$

Also, by direct computation and integrating by parts, since v is radial,

$$\begin{aligned}
c\frac{d}{dt}M(t) = &-2\pi c v(t,0)^2 - \frac{3c}{5}\int \frac{|v(t,x)|^5}{|x|}dx \\
&- c\int (|v+w|^3(v+w) - |v|^3 v - |w|^3 w)\frac{x}{|x|}\cdot \nabla v dx \\
&- c\int (|v+w|^3(v+w) - |v|^3 v - |w|^3 w)\frac{1}{|x|} v dx.
\end{aligned} \tag{2.37}$$

Remark. The virial identities will be computed in more detail in the next section.

Therefore,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= -2\pi c v(t,0)^2 - \frac{3c}{5}\int \frac{|v(t,x)|^5}{|x|}dx \\ &\quad - c\int (|v+w|^3(v+w) - |v|^3 v - |w|^3 w)\frac{x}{|x|}\cdot\nabla v dx - \langle v, |v|^3 w_t\rangle \\ &\quad - c\int (|v+w|^3(v+w) - |v|^3 v - |w|^3 w)\frac{1}{|x|} v dx \\ &\quad + O(E(t)[\|w(t)\|^2_{L^{18}_x(\mathbf{R}^3)} + \|w(t)\|^3_{L^9_x(\mathbf{R}^3)}]).\end{aligned} \tag{2.38}$$

By Hardy's inequality, the Sobolev embedding theorem, and the Cauchy–Schwartz inequality,

$$\begin{aligned}&\int (|v+w|^3(v+w) - |v|^3 v - |w|^3 w)\frac{1}{|x|} v dx \\ &\lesssim \left(\int \frac{1}{|x|} v^5 dx\right)^{2/3} \cdot \left\| \frac{1}{|x|^{1/2}} v\right\|^{2/3}_{L^3_x} \|w\|_{L^9_x} + \left\| \frac{1}{|x|} v\right\|_{L^2} \|v\|_{L^6_x} \|w\|^3_{L^9_x} \\ &\lesssim \delta \left(\int \frac{1}{|x|} |v|^5 dx \right) + \frac{1}{\delta} E(t) \|w(t)\|^3_{L^9_x}.\end{aligned} \tag{2.39}$$

Also, following (2.21) and (2.22),

$$\begin{aligned}&c\int [|v|^2|w|^2 + |v||w|^3]\frac{x}{|x|}\cdot\nabla v dx \\ &\lesssim \|\nabla v\|_{L^2} \|v\|^{1/3}_{L^6} \|v\|^{5/3}_{L^5} \|w\|^2_{L^{18}_x} + \|\nabla v\|_{L^2_x} \|v\|_{L^6_x} \|w\|^3_{L^9_x} \\ &\lesssim E(t)[\|w\|^2_{L^{18}_x} + \|w\|^3_{L^9_x}].\end{aligned} \tag{2.40}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}(t) + 2\pi c v(t,0)^2 + \frac{3c}{5}\int \frac{|v(t,x)|^5}{|x|}dx \\ &+ c\int w \frac{x}{|x|}\cdot\nabla(|v|^3 v) dx + \langle v, |v|^3 w_t\rangle \\ &\lesssim \frac{1}{\delta} E(t)[\|w(t)\|^2_{L^{18}_x(\mathbf{R}^3)} + \|w(t)\|^3_{L^9_x(\mathbf{R}^3)}] + \delta\left(\int \frac{|v(t,x)|^5}{|x|}dx\right).\end{aligned} \tag{2.41}$$

Make a Littlewood–Paley decomposition,

$$\langle |v|^3 v, w_t\rangle = \sum_j \langle |v|^3 v, P_j w_t\rangle. \tag{2.42}$$

By Lemma 3.3, if P_j is a Littlewood–Paley projection operator,

$$\int \frac{1}{|x|}|P_{\leq j} v|^5 dx + \int \frac{1}{|x|}|P_{\geq j} v|^5 dx \lesssim \int \frac{1}{|x|}|v|^5 dx. \tag{2.43}$$

Therefore, by Hölder's inequality, (2.43), and the Cauchy–Schwartz inequality,

$$\begin{aligned}
&\sum_j \langle |v|^3 v - |P_{\le j} v|^3 (P_{\le j} v), P_j w_t \rangle \\
&\quad \lesssim \sum_j \| |x|^{1/10} P_{\ge j} v \|_{L_x^{5/2}} \bigg(\int \frac{1}{|x|} (|P_{\le j} v|^5 + |P_{\ge j} v|^5) dx \bigg)^{3/5} \\
&\qquad\qquad \times \| |x|^{1/2} P_j w_t \|_{L_x^\infty} \\
&\quad \lesssim \bigg(\int \frac{1}{|x|} |v|^5 dx \bigg)^{3/5} \cdot \sum_j \| |x|^{1/10} P_{\ge j} v \|_{L_x^{5/2}} \| |x|^{1/2} P_j w_t \|_{L_x^\infty}.
\end{aligned} \tag{2.44}$$

By Bernstein's inequality and the radial Sobolev embedding theorem,

$$\| |x|^{1/10} P_{\ge j} v \|_{L_x^{5/2}(\mathbf{R}^3)} \lesssim 2^{-4j/5} \| \nabla v \|_{L_x^2(\mathbf{R}^3)} \lesssim 2^{-4j/5} E(t)^{1/2}. \tag{2.45}$$

Also, by Bernstein's inequality, the radial Sobolev embedding theorem, and integrating by parts,

$$\begin{aligned}
&\langle |P_{\le j} v|^3 (P_{\le j} v), P_j w_t \rangle \\
&= \bigg\langle |P_{\le j} v|^3 (P_{\le j} v), \frac{\nabla}{\nabla} P_j w_t \bigg\rangle \sim 2^{-j} \langle |P_{\le j} v|^3 (\nabla P_{\le j} v), (P_j w_t) \rangle \\
&\lesssim 2^{-j} \| |x|^{1/10} \nabla P_{\le j} v \|_{L_x^{5/2}(\mathbf{R}^3)} \bigg(\int \frac{1}{|x|} |P_{\le j} v|^5 dx \bigg)^{3/5} \| |x|^{1/2} P_j w_t \|_{L_x^\infty(\mathbf{R}^3)} \\
&\lesssim 2^{-4j/5} E(t)^{1/2} \bigg(\int \frac{1}{|x|} |v|^5 dx \bigg)^{3/5} \| |x|^{1/2} P_j w_t \|_{L_x^\infty}.
\end{aligned} \tag{2.46}$$

The term $\int (P_j w) \frac{x}{|x|} \cdot \nabla (|v|^3 v) dx$ can be handled using a similar calculation. Indeed, by Bernstein's inequality and (2.43),

$$\begin{aligned}
&\int (P_j w) \frac{x}{|x|} \cdot \nabla (|P_{\le j} v|^3 (P_{\le j} v)) dx \\
&= \int \frac{\nabla}{\nabla} (P_j w) \frac{x}{|x|} \cdot \nabla (|P_{\le j} v|^3 (P_{\le j} v)) dx \\
&\sim 2^{-j} \int (P_j \nabla w) \frac{x}{|x|} \cdot \nabla (|P_{\le j} v|^3 (P_{\le j} v)) dx \\
&\lesssim 2^{-j} \| |x|^{1/2} P_j \nabla w \|_{L_x^\infty} \| |x|^{1/10} \nabla P_{\le j} v \|_{L_x^{5/2}} \bigg(\frac{1}{|x|} |P_{\le j} v|^5 dx \bigg)^{3/5} \\
&\lesssim 2^{-4j/5} E(t)^{1/2} \bigg(\int \frac{1}{|x|} |v|^5 dx \bigg)^{3/5} \| |x|^{1/2} P_j \nabla w \|_{L_x^\infty}.
\end{aligned} \tag{2.47}$$

Meanwhile, integrating by parts,

$$\begin{aligned}
&\int (P_j w) \frac{x}{|x|} \cdot \nabla (|v|^3 v - |P_{\leq j} v|^3 (P_{\leq j} v)) dx \\
&= - \int (P_j \nabla w) \cdot \frac{x}{|x|} (|v|^3 v - |P_{\leq j} v|^3 (P_{\leq j} v)) dx \\
&\quad - 2 \int (P_j w) \frac{1}{|x|} (|v|^3 v - |P_{\leq j} v|^3 (P_{\leq j} v)) dx.
\end{aligned} \tag{2.48}$$

The term

$$- \int (P_j \nabla w) \cdot \frac{x}{|x|} (|v|^3 v - |P_{\leq j} v|^3 (P_{\leq j} v)) dx \tag{2.49}$$

may be handled in a manner identical to (2.44)–(2.46), giving

$$(2.49) \lesssim 2^{-4j/5} E(t)^{1/2} \Big(\int \frac{1}{|x|} |v|^5 dx \Big)^{3/5} \| |x|^{1/2} P_j \nabla w \|_{L_x^\infty}. \tag{2.50}$$

Meanwhile, by (2.43), the Sobolev embedding theorem, the Cauchy–Schwartz inequality, Young's inequality, Bernstein's inequality, and the Littlewood–Paley theorem,

$$\begin{aligned}
&- 2 \int (P_j w) \frac{1}{|x|} (|v|^3 v - |P_{\leq j} v|^3 (P_{\leq j} v)) dx \\
&\lesssim \int \frac{1}{|x|} |P_j w| |P_{\geq j} v| (|P_{\leq j} v|^3 + |P_{\geq j} v|^3) dx \\
&\lesssim \int \Big(\sum_j |P_j w|^2 \frac{2^{-j/3 - j/5}}{|x|^{6/5}} (|P_{\leq j} v|^6 + |P_{\geq j} v|^6) \Big)^{1/2} \\
&\qquad \times \Big(\sum_j 2^{j/3 + j/5} |P_{\geq j} v|^2 \frac{1}{|x|^{4/5}} \Big)^{1/2} dx \\
&\lesssim \Big(\sum_j 2^{-j/3 - j/5} \| P_j w \|_{L^\infty}^2 \Big)^{1/2} \sup_j \Big(\int \frac{1}{|x|} |P_{\geq j}|^5 + \frac{1}{|x|} |P_{\leq j} v|^5 dx \Big)^{3/5} \\
&\qquad \times \Big\| \Big(\sum_j |P_j v|^2 \Big)^{1/2} \Big\|_{L^5}^{1/3} \Big(\sum_j 2^{4j/5} \Big\| \frac{1}{|x|^{3/5}} |P_j v| \Big\|_{L^2}^2 \Big)^{1/3} \\
&\lesssim \delta \Big(\int \frac{1}{|x|} |v|^5 dx \Big) + \frac{1}{\delta} E(t) \Big(\sum_j 2^{-j/3 - j/5} \| P_j w \|_{L^\infty}^2 \Big)^{5/4} \\
&\lesssim \delta \Big(\int \frac{1}{|x|} |v|^5 dx \Big) + \frac{1}{\delta} E(t) \| w(t) \|_{L^{45/4}}^{5/4}.
\end{aligned} \tag{2.51}$$

Remark. We use Young's inequality to show

$$\begin{aligned}
&\Big\| \Big(\sum_j 2^{j/3+j/5} |P_{\geq j} v|^2 \frac{1}{|x|^{4/5}} \Big)^{1/2} \Big\|_{L_x^{5/2}} \\
&\quad = \Big\| \Big(\sum_j 2^{j/3+j/5} \Big| \sum_{k \geq j} P_k v \Big|^2 \frac{1}{|x|^{4/5}} \Big)^{1/2} \Big\|_{L_x^{5/2}} \\
&\quad \lesssim \Big\| \Big(\sum_j 2^{j/3+j/5} |P_j v|^2 \frac{1}{|x|^{4/5}} \Big)^{1/2} \Big\|_{L_x^{5/2}}.
\end{aligned} \tag{2.52}$$

Therefore, by (2.41)–(2.51),

$$\begin{aligned}
&\frac{d}{dt} \mathcal{E}(t) + 2\pi c v(t,0)^2 + \frac{3c}{5} \int \frac{|v(t,x)|^5}{|x|} dx \\
&\quad \lesssim \frac{1}{\delta} E(t) [\|w(t)\|_{L_x^{18}(\mathbf{R}^3)}^2 + \|w(t)\|_{L_x^9(\mathbf{R}^3)}^3] + \delta \Big(\int \frac{|v(t,x)|^5}{|x|} dx \Big) \\
&\qquad + \frac{1}{\delta} E(t)^{5/4} \Big(\sum_j 2^{-4j/5} \| |x|^{1/2} P_j \nabla_{t,x} w \|_{L_x^\infty} \Big)^{5/2}.
\end{aligned} \tag{2.53}$$

For $\delta > 0$ small, but fixed,

$$\delta \Big(\int \frac{1}{|x|} |v|^5 dx \Big) \tag{2.54}$$

may be absorbed into the left-hand side of (2.53).

Next, recall Corollary 3.3 from [4].

COROLLARY 2.5. *For any $j \in \mathbf{Z}$, if w solves the linear wave equation*

$$w_{tt} - \Delta w = 0, \quad w(0,x) = P_j w_0, \quad w_t(0,x) = P_j w_1, \tag{2.55}$$

then for any $2 < q < \infty$,

$$\| |x|^{1/2} w \|_{L_t^q L_x^\infty(\mathbf{R} \times \mathbf{R}^3)} \lesssim \|P_j w_0\|_{\dot{H}^{1/q'}} + \|P_j w_1\|_{\dot{H}^{1/q'-1}}. \tag{2.56}$$

In this case $q = \frac{5}{2}$, so by Corollary 2.5 and (2.30),

$$\| |x|^{1/2} P_j w \|_{L_t^{5/2} L_x^\infty} \lesssim \|P_j w_0\|_{\dot{H}^{3/5}} + \|P_j w_1\|_{\dot{H}^{-2/5}} + \|P_j(|w|^3 w)\|_{L_t^1 \dot{H}^{-2/5}}. \tag{2.57}$$

Also,

$$\begin{aligned}
&\| |x|^{1/2} P_j \nabla_{t,x} w \|_{L_t^{5/2} L_x^\infty} \\
&\quad \lesssim \|P_j w_0\|_{\dot{H}^{8/5}} + \|P_j w_1\|_{\dot{H}^{3/5}} + \|P_j(|w|^3 w)\|_{L_t^1 \dot{H}^{3/5}}.
\end{aligned} \tag{2.58}$$

Remark. The estimate (2.58) in the case of $\nabla_x w$ follows easily from (2.57) using the Littlewood–Paley decomposition. For $\partial_t w$, we use the fact that

$$\partial_t S(t)(w_0, w_1) = S(t)(w_1, \Delta w_0), \tag{2.59}$$

which by (2.57) implies

$$\begin{aligned} \| |x|^{1/2} P_j \nabla_t S(t)(w_0, w_1) \|_{L_t^{5/2} L_x^\infty} &= \| |x|^{1/2} P_j S(t)(w_1, \Delta w_0) \|_{L_t^{5/2} L_x^\infty} \\ &\lesssim \| P_j w_1 \|_{\dot{H}^{3/5}} + \| P_j \Delta w_0 \|_{\dot{H}^{-2/5}} \\ &= \| P_j w_1 \|_{\dot{H}^{3/5}} + \| P_j w_0 \|_{\dot{H}^{8/5}}. \end{aligned} \tag{2.60}$$

Utilizing (2.60) in the integral term in (2.30),

$$\begin{aligned} &\left\| |x|^{1/2} P_j \nabla_t \int_0^t S(t-\tau)(0, |w|^3 w) d\tau \right\|_{L_t^{5/2} L_x^\infty} \\ &\quad \lesssim \int \| P_j (|w|^3 w) \|_{\dot{H}^{3/5}} d\tau = \| P_j (|w|^3 w) \|_{L_t^1 \dot{H}^{3/5}}. \end{aligned} \tag{2.61}$$

Using the computations in (2.31), for any $\sigma > 0$ small, $\sigma = \frac{1}{60}$ will do, by Bernstein's inequality,

$$\begin{aligned} &\sum_{j \geq 0} 2^{-4j/5} \| |x|^{1/2} P_j \nabla_{t,x} w \|_{L_t^{5/2} L_x^\infty(\mathbf{R} \times \mathbf{R}^3)} \\ &\quad \lesssim_\sigma \Big(\sum_{j \geq 0} 2^{-8j/5 + \sigma} \| |x|^{1/2} P_j \nabla_{t,x} w \|^2_{L_t^{5/2} L_x^\infty(\mathbf{R} \times \mathbf{R}^3)} \Big)^{1/2} \\ &\quad \lesssim \Big(\sum_{j \geq 0} 2^{-8j/5 + \sigma} \| P_j w_0 \|^2_{\dot{H}^{8/5}(\mathbf{R}^3)} + 2^{-8j/5 + \sigma} \| P_j w_1 \|^2_{\dot{H}^{3/5}(\mathbf{R}^3)} \Big)^{1/2} \\ &\qquad + \Big(\sum_{j \geq 0} 2^{-8j/5 + \sigma} \| P_j (|w|^3 w) \|^2_{L_t^1 \dot{H}^{3/5}} \Big)^{1/2} \\ &\quad \lesssim \| w_0 \|_{\dot{H}^{5/6}} + \| w_1 \|_{\dot{H}^{-1/6}} + \| |w|^3 w \|_{L_t^1 L_x^{9/5}} \\ &\quad \lesssim \epsilon. \end{aligned} \tag{2.62}$$

Since w_0 and w_1 have the Littlewood–Paley support $P_{\geq 0}$, by (2.11),

$$\begin{aligned} \| w \|_{L_t^\infty L_x^2} &\lesssim \| w_0 \|_{L^2} + \| w_1 \|_{\dot{H}^{-1}} + \| w \|_{L_t^\infty L_x^2} \| w \|^3_{L^6_{t,x}} \\ &\lesssim \| w_0 \|_{\dot{H}^{5/6}} + \| w_1 \|_{\dot{H}^{-1/6}} + \| w \|_{L_t^\infty L_x^2} \| w \|^3_{L^6_{t,x}} \\ &\lesssim \epsilon. \end{aligned} \tag{2.63}$$

Therefore, making a computation similar to (2.62),

$$\begin{aligned}
&\sum_{j\le 0} 2^{-4j/5} \| |x|^{1/2} P_j \nabla_{t,x} w\|_{L_t^{5/2} L_x^\infty(\mathbf{R}\times\mathbf{R}^3)} \\
&\quad \lesssim \sum_{j\le 0} 2^{-4j/5} \|P_j w_0\|_{\dot H^{8/5}} \\
&\qquad + 2^{-4j/5} \|P_j w_1\|_{\dot H^{3/5}} + 2^{-4j/5} \|P_j(|w|^3 w)\|_{L_t^1 \dot H^{3/5}} \\
&\quad \lesssim_\sigma \Big(\sum_{j\le 0} 2^{-8j/5-\sigma} \|P_j w_0\|_{\dot H^{8/5}}^2 + 2^{-8j/5-\sigma} \|P_j w_1\|_{\dot H^{3/5}}^2 \\
&\qquad\qquad + 2^{-8j/5-\sigma} \|P_j(|w|^3 w)\|_{L_t^1 \dot H^{3/5}}^2 \Big)^{1/2} \\
&\quad \lesssim \|w_0\|_{L^2} + \|w_1\|_{\dot H^{-1}} + \| |w|^3 w\|_{L_t^1 L_x^{6/5}} \\
&\quad \lesssim \|w_0\|_{\dot H^{5/6}} + \|w_1\|_{\dot H^{-1/6}} + \|w\|_{L_t^3 L_x^9}^3 \|w\|_{L_t^\infty L_x^2} \lesssim \epsilon.
\end{aligned} \tag{2.64}$$

Therefore, by (2.15), (2.25), (2.62), (2.64), and Gronwall's inequality, for

$$\epsilon(\|u_0\|_{\dot H^{5/6}}, \|u_1\|_{\dot H^{-1/6}}) > 0$$

sufficiently small, (2.18) holds, proving Theorem 2.3. □

3. Global well-posedness for general p. Now prove global well-posedness of (1.1) for any $3 < p < 5$.

THEOREM 3.1. *The nonlinear wave equation*

$$u_{tt} - \Delta + |u|^{p-1} u = 0, \quad u(0,x) = u_0, \quad u_t(0,x) = u_1, \tag{3.1}$$

with radial initial data $u_0 \in \dot H^{s_c}(\mathbf{R}^3)$, $u_1 \in \dot H^{s_c-1}(\mathbf{R}^3)$, $s_c = \frac{3}{2} - \frac{2}{p-1}$, $3 < p < 5$, is globally well-posed.

Proof. The proof is a generalization of the argument in the $p = 4$ case.

First prove a generalized Morawetz inequality.

THEOREM 3.2 (Morawetz inequality). *If u solves* (3.1) *on an interval I, then*

$$\int_I \int \frac{|u(t,x)|^{p+1}}{|x|} dx dt \lesssim E(u), \tag{3.2}$$

where E is the conserved energy (1.7).

Proof. Define the Morawetz potential

$$M(t) = \int u_t u_r r^2 dr + \int u_t u r dr. \tag{3.3}$$

By direct computation,

$$\frac{d}{dt}M(t) = -\frac{1}{2}u(t,0)^2 - \frac{p-1}{p+1}\int |u|^{p+1} r dr. \tag{3.4}$$

Then (3.2) holds by the fundamental theorem of calculus and Hardy's inequality. □

The Morawetz estimate commutes very well with Littlewood–Paley projections.

LEMMA 3.3. *For any j,*

$$\int \frac{1}{|x|}|P_{\leq j}v|^{p+1}dx + \int \frac{1}{|x|}|P_{\geq j}v|^{p+1}dx \lesssim \int \frac{1}{|x|}|v|^{p+1}dx. \tag{3.5}$$

Proof. Let ψ be the Littlewood–Paley kernel.

$$\frac{1}{|x|^{\frac{1}{p+1}}}P_{\leq j}v(x) = \frac{1}{|x|^{\frac{1}{p+1}}}\int 2^{3j}\psi(2^j(x-y))v(y)dy. \tag{3.6}$$

When $|y| \lesssim |x|$,

$$\frac{1}{|x|^{\frac{1}{p+1}}}2^{3j}\psi(2^j(x-y)) \lesssim 2^{3j}\psi(2^j(x-y))\frac{1}{|y|^{\frac{1}{p+1}}}. \tag{3.7}$$

When $|y| \gg |x|$ and $|x| \geq 2^{-j}$, since ψ is rapidly decreasing, for any N,

$$\begin{aligned}\frac{1}{|x|^{\frac{1}{p+1}}}2^{3j}\psi(2^j(x-y)) &\lesssim_N \frac{1}{|x|^{\frac{1}{p+1}}}\frac{2^{3j}}{(1+2^j|x-y|)^N} \\ &\lesssim \frac{1}{|x|^{\frac{1}{p+1}}2^j|y|}\frac{2^{3j}}{(1+2^j|x-y|)^{N-1}} \\ &\lesssim \frac{1}{|y|^{\frac{1}{p+1}}}\frac{2^{3j}}{(1+2^j|x-y|)^{N-1}}.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8),

$$\left\| \frac{1}{|x|^{\frac{1}{p+1}}}|P_{\leq j}v| \right\|_{L^{p+1}(|x|\geq 2^{-j})} \lesssim \left\| \frac{1}{|x|^{\frac{1}{p+1}}}v \right\|_{L^{p+1}(\mathbf{R}^3)}. \tag{3.9}$$

When $|y| \gg |x|$ and $|x| \leq 2^{-j}$, since ψ is rapidly decreasing, for any N,

$$\begin{aligned}\frac{1}{|x|^{\frac{1}{p+1}}}2^{3j}\psi(2^j(x-y)) &\lesssim_N \frac{1}{|x|^{\frac{1}{p+1}}}\frac{2^{3j}}{(1+2^j|x-y|)^N} \\ &\lesssim \frac{1}{|x|^{\frac{1}{p+1}}}\frac{2^{3j}}{(1+2^j|x-y|)^{N-\frac{1}{p+1}}}\frac{1}{2^{\frac{j}{p+1}}|y|^{\frac{1}{p+1}}}.\end{aligned} \tag{3.10}$$

By direct computation,

$$\left\| \frac{2^{3j-\frac{j}{p+1}}}{(1+2^j|x-y|)^{N-\frac{1}{p+1}}} \right\|_{L^{\frac{p+1}{p}}(\mathbf{R}^3)} \lesssim 2^{\frac{2j}{p+1}}. \tag{3.11}$$

Therefore, by (3.10), (3.11), Young's inequality, and Hölder's inequality,

$$\begin{aligned} &\left\| \frac{1}{|x|^{\frac{1}{p+1}}} |P_{\le j} v| \right\|_{L^{p+1}(|x|\le 2^{-j})} \\ &\lesssim \left\| \frac{1}{|x|^{\frac{1}{p+1}}} \right\|_{L^{p+1}(|x|\le 2^{-j})} \left\| \frac{2^{3j-\frac{j}{p+1}}}{(1+2^j|x|)^{N-\frac{1}{p+1}}} \right\|_{L^{\frac{p+1}{p}}} \left\| \frac{1}{|y|^{\frac{1}{p+1}}} v \right\|_{L^{p+1}} \\ &\lesssim \left\| \frac{1}{|x|^{\frac{1}{p+1}}} v \right\|_{L^{p+1}(\mathbf{R}^3)}. \end{aligned} \tag{3.12}$$

This proves (3.5). □

Next, split a local solution (3.1), $u = v + w$, where w solves

$$w_{tt} - \Delta w + |w|^{p-1} w = 0, \quad w(0,x) = w_0, \quad w_t(0,x) = w_1, \tag{3.13}$$

and v solves

$$v_{tt} - \Delta v + |u|^{p-1} u - |w|^{p-1} w = 0, \quad v(0,x) = v_0, \quad v_t(0,x) = v_1. \tag{3.14}$$

Again use the rescaling (1.2) so that $v_0 = P_{\le 1} u_0$, $v_1 = P_{\le 1} u_1$, $w_0 = P_{>1} u_0$, $w_1 = P_{>1} u_1$, and

$$\|w_0\|_{\dot{H}^{s_c}(\mathbf{R}^3)} + \|w_1\|_{\dot{H}^{s_c-1}(\mathbf{R}^3)} < \epsilon. \tag{3.15}$$

As in (2.15),

$$E(0) \lesssim (\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}})^2 + (\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}})^{p+1}. \tag{3.16}$$

By small data arguments, (3.13) is globally well-posed and scattering for $\epsilon > 0$ sufficiently small. Indeed,

$$\|w\|_{L_{t,x}^{2(p-1)} \cap L_t^{\frac{2}{s_c}} L_x^{\frac{2}{1-s_c}}} \lesssim \|w_0\|_{\dot{H}^{s_c}} + \|w_1\|_{\dot{H}^{s_c-1}} + \|w\|_{L_{t,x}^{2(p-1)}}^{p-1} \|w\|_{L_t^{\frac{2}{s_c}} L_x^{\frac{2}{1-s_c}}} < \epsilon, \tag{3.17}$$

and as in (2.11),

$$\|w\|_{L_t^\infty L_x^2} \lesssim \|w_0\|_{L^2} + \|w_1\|_{\dot{H}^{-1}} + \|w\|_{L_t^\infty L_x^2} \|w\|_{L_{t,x}^{2(p-1)}}^{p-1} < \epsilon. \tag{3.18}$$

Now define the energy of v,

$$E(t) = \frac{1}{2} \int |\nabla v|^2 + \frac{1}{2} \int v_t(t,x)^2 dx + \frac{1}{p+1} \int |v(t,x)|^{p+1} dx, \tag{3.19}$$

and let

$$\mathcal{E}(t) = E(t) + cM(t) - \int |v|^{p-1} v w dx, \tag{3.20}$$

where $c > 0$ is a small constant and $M(t)$ is given by (3.3), with u replaced by v. Then by (2.19) and (3.4),

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}(t) + \frac{c}{2}v(t,0)^2 + c\left(1 - \frac{2}{p+1}\right)\int \frac{|v(t,x)|^{p+1}}{|x|}dx \\ &= -\langle v_t, |v+w|^{p-1}(v+w) - |v|^{p-1}v - |w|^{p-1}w\rangle + \frac{d}{dt}\int |v|^{p-1}vwdx \\ &\quad - c\int [|v+w|^{p-1}(v+w) - |v|^{p-1}v - |w|^{p-1}w]\frac{x}{|x|}\cdot \nabla v dx \\ &\quad - c\int [|v+w|^{p-1}(v+w) - |v|^{p-1}v - |w|^{p-1}w]\frac{1}{|x|}v dx. \end{aligned} \tag{3.21}$$

By (2.39), Hardy's inequality, and the Cauchy–Schwarz inequality,

$$\begin{aligned} &\int [|v+w|^{p-1}(v+w) - |v|^{p-1}v - |w|^{p-1}w]\frac{1}{|x|}v dx \\ &\lesssim \left(\int \frac{1}{|x|}|v|^{p+1}dx\right)^{\frac{p-2}{p-1}} \left\| \frac{1}{|x|^{1/2}} v \right\|_{L^3_x}^{\frac{2}{p-1}} \|w\|_{L^{3(p-1)}} \\ &\quad + \left\| \frac{1}{|x|} v \right\|_{L^2} \|v\|_{L^6} \|w\|_{L^{3(p-1)}}^{p-1} \\ &\lesssim \delta \left(\int \frac{1}{|x|}|v|^{p+1}dx\right) + \frac{1}{\delta} E(t) \|w\|_{L^{3(p-1)}}^{p-1}. \end{aligned} \tag{3.22}$$

Also by (2.20),

$$\begin{aligned} &|v+w|^{p-1}(v+w) - |v|^{p-1}v - |w|^{p-1}w \\ &\quad = p|v|^{p-1}w + O(|v|^{p-2}|w|^2) + O(|v||w|^{p-1}). \end{aligned} \tag{3.23}$$

By Hölder's inequality,

$$\begin{aligned} &\int [O(|v|^{p-2}|w|^2) + O(|v||w|^{p-1})]\frac{x}{|x|}\cdot \nabla v dx \\ &\lesssim \|\nabla v\|_{L^2} \|v\|_{L^6} \|w\|_{L^{3(p-1)}}^{p-1} + E(t)\|w\|_{L^{\frac{3}{1-s_c}}}^2 \\ &\lesssim E(t)\big[\|w\|_{L^{3(p-1)}}^{p-1} + \|w\|_{L^{\frac{3}{1-s_c}}}^2\big], \end{aligned} \tag{3.24}$$

and

$$\langle v_t, [O(|v|^{p-2}|w|^2) + O(|v||w|^{p-1})]\rangle \lesssim E(t)\big[\|w\|_{L^{3(p-1)}}^{p-1} + \|w\|_{L^{\frac{3}{1-s_c}}}^2\big]. \tag{3.25}$$

Next, by the product rule,

$$p\langle v_t, |v|^{p-1}w\rangle - \frac{d}{dt}\int |v|^{p-1}vwdx = \langle |v|^{p-1}v, w_t\rangle. \tag{3.26}$$

Following (2.44),

$$\begin{aligned}
&\sum_j \langle |v|^{p-1}v - |P_{\le j}v|^{p-1}(P_{\le j}v), P_j w_t\rangle \\
&\quad\lesssim \Big(\int \frac{1}{|x|}|P_{\le j}v|^{p+1} + \frac{1}{|x|}|P_{\ge j}v|^{p+1}\Big)^{\frac{p-1}{p+1}} \\
&\qquad \times \sum_j \||x|^{1/2}P_j w_t\|_{L^\infty}\big\||x|^{\frac{p-3}{2(p+1)}}|P_{\ge j}v|\big\|_{L^{\frac{p+1}{2}}} \\
&\quad\lesssim \sum_j 2^{-\frac{4j}{p+1}}E(t)^{1/2}\Big(\int \frac{1}{|x|}|v|^{p+1}dx\Big)^{\frac{p-1}{p+1}}\||x|^{1/2}P_j w_t\|_{L^\infty} \\
&\quad\lesssim \delta\Big(\int \frac{1}{|x|}|v|^{p+1}dx\Big) + \frac{1}{\delta}E(t)^{\frac{p+1}{4}}\Big(\sum_j 2^{-\frac{4j}{p+1}}\||x|^{1/2}P_j w_t\|_{L^\infty}\Big)^{\frac{p+1}{2}}.
\end{aligned} \tag{3.27}$$

Meanwhile, integrating by parts as in (2.47),

$$\begin{aligned}
&\sum_j \langle |P_{\le j}v|^{p-1}(P_{\le j}v), P_j w_t\rangle \\
&\quad\lesssim \delta\Big(\int \frac{1}{|x|}|v|^{p+1}dx\Big) + \frac{1}{\delta}E(t)^{\frac{p+1}{4}}\Big(\sum_j 2^{-\frac{4j}{p+1}}\|P_j w_t\|_{L^\infty}\Big)^{\frac{p+1}{2}}.
\end{aligned} \tag{3.28}$$

Using (3.17) and (3.18) in place of (2.10) and (2.11),

$$E(0)^{\frac{p+1}{4}}\int_{\mathbf{R}}\Big(\sum_j 2^{-\frac{4j}{p+1}}\|P_j w_t\|_{L^\infty}\Big)^{\frac{p+1}{4}}dt \lesssim E(0)^{\frac{p+1}{4}}\epsilon^{\frac{p+1}{4}}. \tag{3.29}$$

The contribution of

$$-c\int w\frac{x}{|x|}\cdot\nabla(|v|^{p-1}v)dx \tag{3.30}$$

may also be handled by splitting

$$\begin{aligned}
w\frac{x}{|x|}\cdot\nabla(|v|^{p-1}v) &= \sum_j P_j w\frac{x}{|x|}\cdot\nabla|v|^{p-1}v \\
&= \sum_j (P_j w)\frac{x}{|x|}\cdot\nabla(|P_{\le j}v|^{p-1}(P_{\le j}v)) \\
&\quad + \sum_j (P_j w)\frac{x}{|x|}\cdot\nabla((|v|^{p-1}v - |P_{\le j}v|^{p-1}(P_{\le j}v))),
\end{aligned} \tag{3.31}$$

integrating by parts, and summing up.

Then arguing as in the $p = 4$ case, (3.25) and (3.29) imply that

$$\sup_{t \in \mathbf{R}} \mathcal{E}(t) \lesssim \mathcal{E}(0), \tag{3.32}$$

which completes the proof of Theorem 3.1. □

4. Scattering: Estimates on initial data. To prove scattering, let $\phi(x)$ be a radial, smooth function supported on $|x| \leq 1$ and $\phi(x) = 1$ on $|x| \leq \frac{1}{2}$. Then for $R(u_0, u_1) > 0$ sufficiently large,

$$\left\| \left(1 - \phi\left(\frac{x}{R}\right)\right) u_0 \right\|_{\dot{H}^{s_c}(\mathbf{R}^3)} + \left\| \left(1 - \phi\left(\frac{x}{R}\right)\right) u_1 \right\|_{\dot{H}^{s_c - 1}(\mathbf{R}^3)} < \epsilon. \tag{4.1}$$

Then rescale according to (1.2),

$$u_0(x) \mapsto (2R)^{\frac{2}{p-1}} u_0(2Rx), \quad u_1(x) \mapsto (2R)^{\frac{p+1}{p-1}} u_1(2Rx). \tag{4.2}$$

By (2.3), if n is an integer such that $2^n > 2R$, then abusing notation and letting (u_0, u_1) denote the data given by the scaling (4.2),

$$\begin{aligned} &\|(1 - \phi(x)) u_0\|_{\dot{H}^{s_c}(\mathbf{R}^3)} + \|(1 - \phi(x)) u_1\|_{\dot{H}^{s_c - 1}(\mathbf{R}^3)} \\ &\quad + \|\phi(x) P_{>n} u_0\|_{\dot{H}^{s_c}(\mathbf{R}^3)} + \|\phi(x) P_{>n} u_1\|_{\dot{H}^{s_c - 1}(\mathbf{R}^3)} \lesssim \epsilon. \end{aligned} \tag{4.3}$$

By small data arguments, (4.1) implies that

$$\|u\|_{L_{t,x}^{2(p-1)}([0,\infty) \times \{x : |x| \geq \frac{1}{2} + t\})} \lesssim \epsilon, \tag{4.4}$$

if u is the solution to (1.1) with initial data (u_0, u_1). Translating the initial data in time from $t = 0$ to $t = 1$,

$$\|u\|_{L_{t,x}^{2(p-1)}([1,\infty) \times \{x : |x| \geq t - \frac{1}{2}\})} \lesssim \epsilon. \tag{4.5}$$

As in [4, 18], the proof of

$$\|u\|_{L_{t,x}^{2(p-1)}([1,\infty) \times \{x : |x| \leq t - \frac{1}{2}\})} < \infty, \tag{4.6}$$

will make use of the hyperbolic change of coordinates,

$$\tilde{u}(\tau, s) = \frac{e^{\tau} \sinh s}{s} u(e^{\tau} \cosh s, e^{\tau} \sinh s). \tag{4.7}$$

If u solves (1.1) and is radial, then $\tilde{u}(\tau, s)$ solves

$$\left(\partial_{\tau\tau} - \partial_{ss} - \frac{2}{s}\partial_s\right) \tilde{u}(\tau, s) + e^{-(p-3)\tau} \left(\frac{s}{\sinh s}\right)^{p-1} |\tilde{u}(\tau, s)|^{p-1} \tilde{u}(\tau, s) = 0. \tag{4.8}$$

The hyperbolic energy is given by

$$E(\tilde{u}) = \frac{1}{2}\int (\partial_s \tilde{u}(\tau,s))^2 s^2 ds + \frac{1}{2}\int (\partial_\tau \tilde{u}(\tau,s))^2 s^2 ds + \frac{1}{p+1}\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} |\tilde{u}(\tau,s)|^{p+1} s^2 ds. \tag{4.9}$$

By direct computation,

$$\frac{d}{d\tau}E(\tilde{u})(\tau) = -\frac{p-3}{p+1}\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} |\tilde{u}(\tau,s)|^{p+1} s^2 ds \le 0, \tag{4.10}$$

which implies that the energy of $\tilde{u}$ is non-increasing.

We also have a Morawetz estimate.

THEOREM 4.1. *If $\tilde{u}$ solves* (4.8) *on any interval $I = [0,T]$, then*

$$\int_I \int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\Big(\frac{\cosh s}{\sinh s}\Big)|\tilde{u}(\tau,s)|^{p+1} s^2 ds d\tau \lesssim E(\tilde{u}(0)). \tag{4.11}$$

Proof. Using the Morawetz potential in (3.3),

$$M(\tau) = \int \tilde{u}_s(\tau,s)\tilde{u}_\tau(\tau,s)s^2 ds + \int \tilde{u}_\tau(s,\tau)\tilde{u}(\tau,s) s ds. \tag{4.12}$$

Then by direct computation,

$$\frac{d}{d\tau}M(\tau) = -\frac{1}{2}\tilde{u}(\tau,0)^2 - \frac{p-1}{p+1}\int \Big(\frac{\cosh s}{\sinh s}\Big)\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde{u}(\tau,s)|^{p+1} s^2 ds. \tag{4.13}$$

Then by (4.10) and the fundamental theorem of calculus, the proof is complete. □

Previously, in [4], for the cubic wave equation, the initial data was split into a $(\tilde{v}_0,\tilde{v}_1) \in \dot{H}^1 \times L^2$ component and a $(\tilde{w}_0,\tilde{w}_1) \in \dot{H}^{1/2}\times \dot{H}^{-1/2}$. Here, it would be nice if we could do something similar, only with $\dot{H}^{1/2}$ replaced by $\dot{H}^{s_c}$. However, the hyperbolic energy scales like the $\dot{H}^{1/2}$ norm, and thus is not invariant under the general scaling (1.2). Instead, what we will do is place $(\tilde{v}_0,\tilde{v}_1) \in \dot{H}^1 \times L^2$, but $(\tilde{w}_0,\tilde{w}_1)$ will merely lie in a Sobolev space after multiplying by exponential weights. The weights in the nonlinear part of the energy (4.9) will then be used in conjunction with the weights for the Sobolev space to bound the growth of the energy of $\tilde{v}$.

THEOREM 4.2. *There exists a decomposition*

$$\tilde{u}_0 = \tilde{v}_0 + \tilde{w}_0, \quad \tilde{u}_1 = \tilde{v}_1 + \tilde{w}_1, \tag{4.14}$$

satisfying

$$\|\tilde{v}_0\|_{\dot{H}^1} + \|\tilde{v}_1\|_{L^2} \lesssim R^{(1-s_c)}(\|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}), \tag{4.15}$$

where R is given in (4.2)*, and*

$$\begin{aligned} &\sum_{k\geq 1} e^{(-2s_c+1)k}\|\chi(s-k)\tilde{w}_0\|^2_{\dot{H}^{s_c}\cap\dot{H}^1} \\ &\quad + \sum_{k\geq 1} e^{(-2s_c+1)k}\|\chi(s-k)\tilde{w}_1\|^2_{\dot{H}^{s_c-1}\cap L^2} \\ &\quad + \|\phi(e^s-1)\tilde{w}_0\|^2_{\dot{H}^{s_c}\cap\dot{H}^1} + \|\phi(e^s-1)\tilde{w}_1\|^2_{\dot{H}^{s_c-1}\cap L^2} \lesssim \epsilon^2. \end{aligned} \tag{4.16}$$

Here, $\chi(s-k)$ is given by the partition of unity,

$$1 = \sum_{k\geq 1}\chi(s-k) + \phi(e^s-1), \tag{4.17}$$

where $\chi \in C_0^\infty(\mathbf{R})$, and χ is supported on $-1 \leq s \leq 1$.

Proof. To calculate

$$\tilde{u}(\tau,s)|_{\tau=0} = \frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s)|_{\tau=0}, \tag{4.18}$$

use Duhamel's principle,

$$u(t) = S(t-1)(u_0,u_1) - \int_1^t S(t-s)(0,|u|^{p-1}u)ds. \tag{4.19}$$

The analysis will be split into three pieces, analysis of the zero velocity linear solution, $S(t-1)(u_0,0)$, analysis of the zero initial data linear solution $S(t-1)(0,u_1)$, and analysis of the Duhamel term, $\int_1^t S(t-s)(0,|u|^{p-1}u)ds$. □

Zero velocity term. In the zero velocity case, the contribution of $S(t)(u_0,0)$ to $\tilde{u}_0$ will be decomposed as follows:

$$\begin{aligned} \tilde{v}_0 &= \phi(e^s-1)(P_{\leq n}u_0)(e^s-1)\cdot(e^s-1) \\ &\quad + \phi(e^s-1)(P_{\leq n}u_0)(1-e^{-s})\cdot(1-e^{-s}) \\ &\quad + \sum_{k\geq 1}\chi(s-k)(P_{\leq n+\frac{k}{\ln(2)}}u_0)(1-e^{-s})\cdot(1-e^{-s}), \end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\tilde{w}_0 = &\phi(e^s-1)(P_{>n}u_0)(e^s-1)\cdot(e^s-1)\\ &+\phi(e^s-1)(P_{>n}u_0)(1-e^{-s})\cdot(1-e^{-s})\\ &+\sum_{k\geq 1}\chi(s-k)(P_{>n+\frac{k}{\ln(2)}}u_0)(1-e^{-s})\cdot(1-e^{-s})\\ &+\sum_{k\geq 1}\chi(s-k)u_0(e^s-1)\cdot(e^s-1),\end{aligned} \tag{4.21}$$

and the corresponding ∂_τ derivatives are the contributions to $\tilde{v}_1$ and $\tilde{w}_1$.

Indeed, setting $u_1 = 0$ and ignoring the contribution of the Duhamel term,

$$\begin{aligned}s\tilde{u}(\tau,s) &= e^\tau \sinh s\cdot S(t-1)(u_0,0)(e^\tau\cosh s, e^\tau \sinh s)\\ &= \frac{1}{2}[u_0(e^{\tau+s}-1)\cdot(e^{\tau+s}-1)+u_0(1-e^{\tau-s})\cdot(1-e^{\tau-s})].\end{aligned} \tag{4.22}$$

By direct computation,

$$\begin{aligned}&\|\partial_s[\phi(e^{\tau+s}-1)(P_{\leq n}u_0)(e^{\tau+s}-1)\cdot(e^{\tau+s}-1)]|_{\tau=0}\|_{L^2([0,\infty))}\\ &\lesssim R^{(1-s_c)}\|u_0\|_{\dot{H}^{s_c}(\mathbf{R}^3)},\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}&\left\|\left[\phi(e^{\tau+s}-1)(P_{\leq n}u_0)(e^{\tau+s}-1)\cdot\left(\frac{e^{\tau+s}-1}{s}\right)\right]\Big|_{\tau=0}\right\|_{L^2([0,\infty))}\\ &\lesssim R^{(1-s_c)}\|u_0\|_{\dot{H}^{s_c}(\mathbf{R}^3)}.\end{aligned} \tag{4.24}$$

Meanwhile, by (4.3),

$$\left\|\phi(e^{\tau+s}-1)(P_{>n}u_0)(e^{\tau+s}-1)\cdot\left(\frac{e^{\tau+s}-1}{s}\right)\Big|_{\tau=0}\right\|_{\dot{H}^{s_c}(\mathbf{R}^3)}\lesssim\epsilon. \tag{4.25}$$

Similar calculations also hold for

$$\phi(1-e^{\tau-s})u_0(1-e^{\tau-s})\cdot(1-e^{\tau-s})|_{\tau=0}. \tag{4.26}$$

Remark. Since $\partial_\tau f = \pm\partial_s f$ for the components of (4.22), the same estimates also hold for $\partial_\tau\tilde{w}(\tau,s)|_{\tau=0}$. Here we make use of

$$\left\|\frac{\partial_s(su)}{s}\right\|_{\dot{H}^{s_1-1}(\mathbb{R}^3)}\lesssim\|u\|_{\dot{H}^{s_1}(\mathbb{R}^3)},\quad\text{for any } 0\leq s_1\leq 1. \tag{4.27}$$

Turning now to the $\chi(s-k)$ terms, since $u_0 \in \dot{H}^{s_c}$, $s_c > \frac{1}{2}$, using summation by parts,

$$\begin{aligned}
&\Big\| \partial_s \Big[\sum_{k \geq 1} \chi(s-k) (P_{\leq n + \frac{k}{\ln(2)}} u_0)(1-e^{\tau - s}) \cdot (1-e^{\tau - s}) \Big] \Big|_{\tau = 0} \Big\|_{L^2([0,\infty))} \\
&= \Big\| \partial_s \Big[\sum_{k \geq 1} \chi(s-k) (P_{\leq n} u_0)(1-e^{\tau - s}) \cdot (1-e^{\tau - s}) \\
&\qquad + \sum_{k \geq 1} \chi(s-k) \sum_{1 \leq l \leq k} (P_{n + \frac{l}{\ln(2)}} u_0)(1-e^{\tau - s}) \cdot (1-e^{\tau - s}) \Big] \Big\|_{L^2} \\
&= \Big\| \partial_s \Big[\sum_{l \geq 1} \Big[\sum_{k \geq l} \chi(s-k) \Big] (P_{n + \frac{l}{\ln(2)}} u_0)(1-e^{\tau - s}) \cdot (1-e^{\tau - s}) \\
&\qquad + \sum_{k \geq 1} \chi(s-k) (P_{\leq n} u_0)(1-e^{\tau - s}) \cdot (1-e^{\tau - s}) \Big] \Big\|_{L^2} \qquad (4.28) \\
&\lesssim 2^{n(1-s_c)} \sum_{k \geq 1} \| u_0 \|_{\dot{H}^{s_c}} e^{-k/2} e^{k(1-s_c)} + \sum_{k \geq 1} \| P_{n + \frac{k}{\ln(2)}} u_0 \|_{\dot{H}^{1/2}} \\
&\lesssim R^{(1-s_c)} \| u_0 \|_{\dot{H}^{s_c}}.
\end{aligned}$$

We use Bernstein's inequality to estimate the last term. Also, by the radial Sobolev embedding and the fact that $s_c > \frac{1}{2}$,

$$\begin{aligned}
&\Big\| \sum_{k \geq 1} \chi(s-k) (P_{\leq n + \frac{k}{\ln(2)}} u_0)(1-e^{\tau - s}) \cdot \Big(\frac{1-e^{\tau - s}}{s} \Big) \Big|_{\tau = 0} \Big\|_{L^2([0,\infty))} \\
&\lesssim 2^{n(1-s_c)} \Big(\sum_{k \geq 1} \frac{1}{k^2} \Big)^{1/2} \| u_0 \|_{\dot{H}^{s_c}(\mathbf{R}^3)} \qquad (4.29) \\
&\lesssim R^{(1-s_c)} \| u_0 \|_{\dot{H}^{s_c}(\mathbf{R}^3)}.
\end{aligned}$$

Also by (4.3),

$$\Big\| \sum_{k \geq 1} \chi(s-k) (P_{> n + \frac{k}{\ln(2)}} u_0)(1-e^{\tau - s}) \cdot \Big(\frac{1-e^{\tau - s}}{s} \Big) \Big|_{\tau = 0} \Big\|_{\dot{H}^{s_c}(\mathbf{R}^3)} \lesssim \epsilon. \qquad (4.30)$$

Finally, take

$$\sum_{k \geq 1} \chi(s-k) u_0 (e^{\tau + s} - 1) \cdot \Big(\frac{e^{\tau + s} - 1}{s} \Big) \Big|_{\tau = 0}. \qquad (4.31)$$

For any $0 < s < 1$, if $g(x)$ is monotone increasing or decreasing and $g'(x) \sim M$ for all $x \in [0,\infty)$, by a change of variables and Bernstein's inequality,

$$\begin{aligned} 2^{ks}\|P_k(P_j f(g(x)))\|_{L^2} &\lesssim \frac{2^{(k-j)s}}{M^{1/2}}\|P_j f\|_{\dot H^s}, \quad \text{and,} \\ 2^{ks}\|P_k(P_j f(g(x)))\|_{L^2} &\lesssim 2^{-k}\|\nabla P_k(P_j f(g(x)))\|_{L^2} \\ &\lesssim M^{1/2}2^{k(s-1)}2^{j(1-s)}\|P_j f\|_{\dot H^s}, \\ \text{which implies} \quad \|f(g(x))\|_{\dot H^s} &\lesssim M^{s-1/2}\|f\|_{\dot H^s}. \end{aligned} \tag{4.32}$$

Therefore,

$$\begin{aligned} &\Big\|\chi(s-k)[1-\phi(e^{\tau+s}-1)]u_0(e^{\tau+s}-1)\cdot\Big(\frac{e^{\tau+s}-1}{s}\Big)\Big|_{\tau=0}\Big\|_{\dot H^{s_c}} \\ &\lesssim e^{-k/2}\bigg(\int_{e^{k-1}}^{e^{k+1}}|u_0(r)|^2r^2dr\bigg)^{1/2} \\ &\quad + e^{-k/2+k\cdot s_c}\bigg(\int_{e^{k-1}}^{e^{k+1}}||\nabla|^{s_c}u_0(r)|^2r^2dr\bigg)^{1/2} \\ &\lesssim e^{-k(\frac12-s_c)}\bigg(\int_{e^{k-1}}^{e^{k+1}}|u_0(r)|^2r^{2(1-s_c)}dr\bigg)^{1/2} \\ &\quad + e^{-k(\frac12-s_c)}\bigg(\int_{e^{k-1}}^{e^{k+1}}||\nabla|^{s_c}u_0(r)|^2r^2dr\bigg)^{1/2}. \end{aligned} \tag{4.33}$$

By (4.1), (4.2), and Hardy's inequality, this satisfies Theorem 4.2. In this case, it is not quite true that $\partial_s f = \pm\partial_\tau f$, we have the terms

$$\begin{aligned} &(\partial_s+\partial_\tau)\Big[\sum_{k\geq1}\chi(s-k)(P_{\leq n+\frac{k}{\ln(2)}}u_0)(1-e^{\tau-s})\cdot(1-e^{\tau-s})\Big]\Big|_{\tau=0} \\ &= \chi'(s-1)(P_{\leq n}u_0)(1-e^{-s})\cdot(1-e^{-s}) \\ &\quad + \sum_{l\geq1}\chi'(s-l)(P_{n+\frac{l}{\ln(2)}}u_0)(1-e^{-s})\cdot(1-e^{-s}), \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} &(\partial_s-\partial_\tau)\Big[\sum_{k\geq1}\chi(s-k)u_0(e^{\tau+s}-1)\cdot(e^{\tau+s}-1)\Big]\Big|_{\tau=0} \\ &= \chi'(s-1)u_0(e^s-1)\cdot(e^s-1). \end{aligned} \tag{4.35}$$

Since $\sum_{k\geq1}|\chi'(s-k)| \lesssim \frac1s\sum_{k\geq1}|\chi(s-k)|$, for any $s\in[0,\infty)$, using (4.29)–(4.31) completes the estimates of the zero velocity term.

Zero initial data. Turning to estimating the contribution of $S(t)(0,u_1)$, split

$$u_1 = \phi(x)P_{\leq n}u_1 + [u_1 - \phi(x)P_{\leq n}u_1]. \tag{4.36}$$

By direct calculation,

$$\left\| \partial_{\tau,s} \int_{1-e^{-s}}^{e^s-1} \phi(r) P_{\leq n} u_1(r) r dr \right\|_{L^2([0,\infty))} \lesssim 2^{n(1-s_c)} \|u_1\|_{\dot{H}^{s_c-1}}, \tag{4.37}$$

and by Hölder's inequality,

$$\left\| \frac{1}{s} \int_{1-e^{-s}}^{e^s-1} \phi(r) P_{\leq n} u_1(r) r dr \right\|_{L^2([0,\infty))} \lesssim 2^{n(1-s_c)} \|u_1\|_{\dot{H}^{s_c-1}}. \tag{4.38}$$

Turning to the contribution of $g = u_1 - \phi(x) P_{\leq n} u_1$, as in [4], observe that

$$\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g = \partial_t \left(\frac{\cos(t\sqrt{-\Delta})}{\Delta} g \right). \tag{4.39}$$

Note that (4.3) guarantees that

$$\|g\|_{\dot{H}^{s_c-1}} \lesssim \epsilon. \tag{4.40}$$

Plugging in the formula for a solution to the wave equation when $r > t$, let $w(t,r) = \cos(t\sqrt{-\Delta}) f$, where $f = \frac{g}{\Delta}$. Then,

$$\begin{aligned} &\partial_t(w(t,r)) \\ &= \frac{1}{2r} \partial_t (f(t+r)(t+r) + f(r-t)(r-t)) \\ &= \frac{1}{2r} [f(t+r) + f'(t+r)(t+r) - f(r-t) - f'(r-t)(r-t)]. \end{aligned} \tag{4.41}$$

Since $f \in \dot{H}^{s_c+1}(\mathbf{R}^3)$, the contribution of

$$f'(e^{\tau+s}-1) \cdot (e^{\tau+s}-1)|_{\tau=0}, \quad f'(1-e^{\tau-s}) \cdot (1-e^{\tau-s})|_{\tau=0} \tag{4.42}$$

may be handled in a manner identical to the contribution of the terms arising from $S(t)(u_0, 0)$.

Now consider the contribution of

$$\frac{1}{s} [f(e^{\tau+s}-1) - f(1-e^{\tau-s})]|_{\tau=0}. \tag{4.43}$$

The terms when $1 \leq k \leq n \ln(2) \frac{1-s_c}{s_c-1/2}$ will be placed in $(\tilde{v}_0, \tilde{v}_1)$ and the terms when $k > n \ln(2) \frac{1-s_c}{s_c-1/2}$ will be placed in $(\tilde{w}_0, \tilde{w}_1)$.

By a change of variables, for $k \geq 1$,

$$\int (\chi(s-k) f'(e^s-1) \cdot e^s)^2 ds \lesssim e^{2(s_c-\frac{1}{2})k} \int_{e^{k-1}}^{e^{k+1}} |f'(r)|^2 r^{2(1-s_c)} dr, \tag{4.44}$$

and by the Sobolev embedding theorem,

$$\int (\chi(s-k)f'(1-e^{-s})\cdot e^{-s})^2 ds \lesssim e^{-k}\bigg(\int_{1-e^{-k+1}}^{1-e^{-k-1}} |f'(r)|^2 dr\bigg) \lesssim e^{-2k}\|f\|^2_{\dot H^{1+s_c}(\mathbf{R}^3)}. \tag{4.45}$$

Furthermore, by the fundamental theorem of calculus,

$$|f(e^s-1)-f(1-e^{-s})| \le \int_{1-e^{-s}}^{e^s-1} |f'(r)|dr, \tag{4.46}$$

so

$$\int \chi(s-k)^2 |f(e^s-1)-f(1-e^{-s})|^2 ds \lesssim \sum_{0\le l\le k} e^l \bigg(\int_{e^{l-1}}^{e^{l+1}} |f'(r)|^2 dr\bigg). \tag{4.47}$$

Therefore,

$$\begin{aligned} &\Big\| \sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-\frac12}} \frac1s \chi(s-k)[f(e^{\tau+s}-1)-f(1-e^{\tau-s})]\Big|_{\tau=0}\Big\|_{\dot H^1} \\ &+\Big\| \sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-\frac12}} \chi(s-k)\partial_\tau[f(e^{\tau+s}-1)-f(1-e^{\tau-s})]\Big|_{\tau=0}\Big\|_{L^2} \\ &\lesssim R^{1-s_c}\|f\|_{\dot H^{s_c+1}}. \end{aligned} \tag{4.48}$$

Indeed, by the product rule and (4.47),

$$\begin{aligned} &\Big\| \sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-\frac12}} \frac1s \chi(s-k)[f(e^{\tau+s}-1)-f(1-e^{\tau-s})]\Big|_{\tau=0}\Big\|_{\dot H^1} \\ &+\Big\| \sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-\frac12}} \chi(s-k)\partial_\tau[f(e^{\tau+s}-1)-f(1-e^{\tau-s})]\Big|_{\tau=0}\Big\|_{L^2} \\ &\lesssim \sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-1/2}} \Big\|\frac1s\chi(s-k)[e^s f'(e^s-1)+e^{-s}f'(1-e^{-s})]\Big\|_{L^2} \\ &\quad+\sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-1/2}} \Big\|\frac{1}{s^2}\chi(s-k)[f(e^s-1)-f(1-e^{-s})]\Big\|_{L^2} \\ &\quad+\sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-1/2}} \Big\|\frac1s\chi(s-k)[e^s f'(e^s-1)+e^{-s}f'(1-e^{-s})]\Big\|_{L^2} \\ &\quad+\sum_{1\le k\le n\ln(2)\frac{1-s_c}{s_c-1/2}} \Big\|\frac1s\chi'(s-k)[f(e^s-1)-f(1-e^{-s})]\Big\|_{L^2} \\ &\lesssim R^{1-s_c}\|f\|_{\dot H^{s_c+1}}. \end{aligned} \tag{4.49}$$

Next, using the change of variables in (4.32), (4.46), and (4.47),

(4.50)
$$\sum_{k>n\ln(2)\frac{1-s_c}{s_c-\frac{1}{2}}} e^{-k(2s_c-1)}\Big\|\frac{1}{s}\chi(s-k)[f(e^{\tau+s}-1)-f(1-e^{\tau-s})]\Big|_{\tau=0}\Big\|_{\dot{H}^{s_c}}^2$$
$$\lesssim \sum_{k>n\ln(2)\frac{1-s_c}{s_c-\frac{1}{2}}} e^{-k(2s_c-1)}\left\|\frac{1}{s}\chi(s-k)\left(\int_{1-e^{-s}}^{e^s-1} f'(r)dr\right)\right\|_{\dot{H}^{s_c}} \lesssim \epsilon^2\sum_{k\geq 1}\frac{1}{k^2}\lesssim\epsilon^2.$$

Also, by the change of variables in (4.32) and the dual of Hardy's inequality, $\|f\|_{\dot{H}^{-s}}\lesssim \||x|^s f\|_{L^2}$ for $0\leq s\leq 1$,

$$\sum_{k>n\ln(2)\frac{1-s_c}{s_c-\frac{1}{2}}} e^{-k(2s_c-1)}\Big\|\frac{\chi(s-k)}{s}\partial_\tau[f(e^{\tau+s}-1)-f(1-e^{\tau-s})]\Big|_{\tau=0}\Big\|_{\dot{H}^{s_c-1}}^2$$
$$= \sum_{k>n\ln(2)\frac{1-s_c}{s_c-\frac{1}{2}}} e^{-k(2s_c-1)}\Big\|\frac{\chi(s-k)}{s}[f'(e^s-1)e^s+e^{-s}f'(1-e^{-s})]\Big\|_{\dot{H}^{s_c-1}}^2 \tag{4.51}$$
$$\lesssim \Big\|\frac{1}{|x|}f'\Big\|_{\dot{H}^{s_c-1}}^2\Big(\sum_{k\geq 1}\frac{1}{k^2}\Big)\lesssim \|f'\|_{\dot{H}^{s_c}}^2\lesssim\epsilon^2.$$

Finally, consider

$$f(e^{\tau+s}-1)-f(1-e^{\tau-s}), \tag{4.52}$$

when $s<1$. By direct computation,

$$\partial_\tau[f(e^{\tau+s}-1)-f(1-e^{\tau-s})]|_{\tau=0}=f'(e^s-1)\cdot e^s+f'(1-e^{-s})\cdot e^{-s}. \tag{4.53}$$

Then for $g\in\dot{H}^{1-s_c}$, by Hardy's inequality,

$$\int f'(e^s-1)\cdot e^s\cdot g(s)sds+\int f'(1-e^{-s})\cdot e^{-s}\cdot g(s)sds$$
$$\lesssim \|f\|_{\dot{H}^{1+s_c}}\|g\|_{\dot{H}^{1-s_c}}\lesssim\epsilon\|g\|_{\dot{H}^{1-s_c}}. \tag{4.54}$$

Also, by the fundamental theorem of calculus,

$$\begin{aligned} &f(e^s-1)-f(1-e^{-s})\\ &=\int_{s-\frac{s^2}{2}+\frac{s^3}{3!}-\cdots}^{s+\frac{s^2}{2}+\frac{s^3}{3!}+\cdots} f'(r)dr\\ &=\int_0^1 f'\left(s+\theta\left(\frac{s^2}{2}+\frac{s^3}{3!}+\cdots\right)\right)\cdot\left(\frac{s^2}{2}+\frac{s^3}{3!}+\cdots\right)d\theta\\ &\quad+\int_{-1}^0 f'\left(s+\theta\left(\frac{s^2}{2}-\frac{s^3}{3!}+\cdots\right)\right)\cdot\left(\frac{s^2}{2}+\frac{s^3}{3!}+\cdots\right)d\theta. \end{aligned} \tag{4.55}$$

Therefore, since $s \le 1$,

$$\left\| \frac{\phi(s)}{s}[f(e^s-1)-f(1-e^{-s})] \right\|_{\dot{H}^{s_c}} \lesssim \|f\|_{\dot{H}^{s_c+1}} \lesssim \epsilon. \tag{4.56}$$

Thus, the contribution of the zero initial data term is suitable for Theorem 4.2.

Duhamel term. Now take the Duhamel term u_{nl}. Because the curve $t^2-r^2=1$ has slope $\frac{dr}{dt} > 1$ everywhere,

$$s\tilde{u}_{nl}(\tau,s)|_{\tau=0} = \int_1^{e^\tau \cosh s} \int_{e^\tau \sinh s - e^\tau \cosh s + t}^{e^\tau \sinh s + e^\tau \cosh s - t} r|u|^{p-1}u(t,r)drdt. \tag{4.57}$$

By direct computation,

$$\begin{aligned} &\int_0^k (\partial_{s,\tau}(s\tilde{u}_{nl})|_{\tau=0})^2 ds \\ &\lesssim \int_0^k e^{2s}\left(\int_1^{\cosh s}(e^s-t)|u|^{p-1}u(t,e^s-t)dt\right)^2 ds \\ &\quad + \int_0^k e^{-2s}\left(\int_1^{\cosh s}(t-e^{-s})|u|^{p-1}u(t,t-e^{-s})dt\right)^2 ds. \end{aligned} \tag{4.58}$$

The term $e^s(\int_1^{\cosh s}(e^s-t)|u|^{p-1}u(t,e^s-t)dt)\chi_{s\in[2,\infty)}(s)$ will contribute to $(\tilde{w}_0,\tilde{w}_1)$, where $\chi_A(s)$ is the characteristic function of a set A, and

$$\begin{aligned} &e^s\left(\int_1^{\cosh s}(e^s-t)|u|^{p-1}u(t,e^s-t)dt\right)\chi_{s\in[0,2]} \\ &+\int_0^k e^{-s}\left(\int_1^{\cosh s}(t-e^{-s})|u|^{p-1}u(t,t-e^{-s})dt\right). \end{aligned} \tag{4.59}$$

will contribute to $(\tilde{v}_0,\tilde{v}_1)$.

By Hölder's inequality, since $e^s-\cosh s \sim e^s$, combined with global well-posedness in the previous section and (4.5),

$$\begin{aligned} &\int_0^k e^{2s}\left(\int_1^{\cosh s}(e^s-t)|u|^{p-1}u(t,e^s-t)dt\right)^2 ds \\ &\lesssim \int_0^k \int_1^{\cosh s} e^{3s}(e^s-t)^2|u|^{2p}(t,e^s-t)dtds \\ &\lesssim \int_0^{e^k}\int_{t^2-r^2\le 1}|u|^{2p}(t,r)r^4dtdr \\ &\lesssim \|u\|^{2(p-1)}_{L^{2(p-1)}_{t,x}(|x|>|t|)}\left\||x|^{\frac{3}{2}-s_c}u\right\|^2_{L^\infty_{t,x}} e^{2(s_c-\frac{1}{2})k} \lesssim \epsilon^2 e^{2(s_c-\frac{1}{2})k}. \end{aligned} \tag{4.60}$$

Remark. The Strichartz norms $L_{t,x}^{2(p-1)}$ and $\| |x|^{3/2-s_c} u \|_{L^\infty}$ are invariant under the scaling (4.2).

Additionally, by the radial Sobolev embedding theorem and (4.4), using the calculations in (4.60),

$$\begin{aligned} &\sum_k e^{-2(s_c-\frac{1}{2})k} \int_{k-1}^{k} e^{2s} \Big(\int_2^{\cosh s} (e^s - t) |u|^{p-1} u(t, e^s - t) dt \Big)^2 ds \\ &\lesssim \sum_k e^{-2(s_c-\frac{1}{2})k} \int_{k-1}^{k} \int_2^{\cosh s} e^{3s} (e^s - t)^2 |u|^{2p}(t, e^s - t) dt ds \\ &\lesssim \sum_k e^{-2(s_c-\frac{1}{2})k} \int_0^{e^k} \int_{t^2 - r^2 \leq 1, r \sim e^k, t \geq 2} |u|^{2p}(t,r) r^4 dt dr \lesssim \epsilon^2. \end{aligned} \tag{4.61}$$

Meanwhile, by (4.60) and the radial Sobolev embedding theorem,

$$\int_0^\infty e^{2s} \Big(\int_1^2 (e^s - t) |u|^{p-1} u(t, e^s - t) dt \Big)^2 ds \lesssim 1. \tag{4.62}$$

Also by a change of variables and Hölder's inequality, since $(t - e^{-s}) \gtrsim 1$ for $s \geq 1$ and $t \geq 1$,

$$\begin{aligned} &\int_1^\infty e^{-2s} \Big(\int_1^{\cosh s} (t - e^{-s}) |u|^{p-1} u(t, t - e^{-s}) dt \Big)^2 ds \\ &\lesssim \int_1^\infty \int_1^{\cosh s} e^{-s} (t - e^{-s})^2 |u|^{2p}(t, t - e^s) dt ds \\ &\lesssim \int_2^\infty \int_{t^2 - r^2 \leq 1} |u|^{2p}(t,r) r^2 dt dr \lesssim \|u\|_{L_{t,x}^{2(p-1)}}^{2(p-1)} \| |x|^{3/2-s_c} u \|_{L_{t,x}^\infty}^2 \lesssim \epsilon^2. \end{aligned} \tag{4.63}$$

Also, by the radial Sobolev embedding theorem and Young's inequality, since $\| |x|^{3/2-s_c} u \|_{L^\infty} \lesssim \epsilon$ outside $|x| = t$,

$$\begin{aligned} &\int_0^1 e^{-2s} \Big(\int_1^{\cosh s} (t - e^{-s}) |u|^{p-1} u(t, t - e^{-s}) dt \Big)^2 ds \\ &\lesssim \int_1^3 \Big(\int_{t^2 - r^2 \leq 1} u(t,r)^{2p} r^2 dr \Big)^{1/2} dt \lesssim \epsilon^2 \int_1^3 \frac{1}{(t-1)^{-1+\frac{s_c}{2}}} dt \lesssim \epsilon^2. \end{aligned} \tag{4.64}$$

This takes care of the nonlinear Duhamel piece, which completes the proof of Theorem 4.2.

Remark. Note that the Duhamel term is why we have the norm $\dot{H}^{s_c} \cap \dot{H}^1 \times \dot{H}^{s_c - 1} \cap L^2$ in (4.16).

5. Scattering: Virial identities. Now we are ready to prove scattering.

THEOREM 5.1. *For any radial (u_0, u_1), the global solution to* (1.1) *scatters both forward and backward in time. That is, if u is the global solution to* (1.1) *with initial data (u_0, u_1), then*

$$\|u\|_{L_{t,x}^{2(p-1)}(\mathbf{R}\times\mathbf{R}^3)} \leq M(u_0, u_1) < \infty. \tag{5.1}$$

Proof. The standard Littlewood–Paley projection operator is only known to have a rapidly decreasing weight, which when commuting with the exponentially decreasing weights in Theorem 4.2, will only be rapidly decreasing. So instead, in this section we will rely on projection operators with smooth, compactly supported kernels. Choose $\psi \in C_0^\infty(\mathbf{R}^3)$ to be a radial, decreasing function supported on $|x| \leq \frac{1}{2}$, and such that $\int \psi(x)dx = 1$. Then define the Fourier multipliers

$$\tilde{P}_0 f(x) = \int \psi(x-y) f(y) dy, \tag{5.2}$$

and for $j \geq 1$,

$$\tilde{P}_j f(x) = 2^{3j} \int \psi(2^j(x-y)) f(y) dy - 2^{3(j-1)} \int \psi(2^{j-1}(x-y)) f(y) dy. \tag{5.3}$$

Clearly,

$$f = \sum_{j\geq 0} \tilde{P}_j f. \tag{5.4}$$

Remark. If $j < 0$, then $\tilde{P}_j = 0$.

Now modify the definition of $\tilde{v}_0$ and $\tilde{w}_0$ from Theorem 4.2. Let

$$\tilde{\tilde{v}}_0 = \tilde{v}_0 + \tilde{P}_{\leq n}\phi(s)\tilde{w}_0 + \sum_{k\geq 1} \tilde{P}_{\leq n - \frac{k}{\ln 2}\cdot\frac{s_c-\frac{1}{2}}{1-s_c}} \chi(s-k)\tilde{w}_0, \quad \tilde{\tilde{w}}_0 = \tilde{u}_0 - \tilde{\tilde{v}}_0, \tag{5.5}$$

and let

$$\tilde{\tilde{v}}_1 = \tilde{v}_1 + \tilde{P}_{\leq n}\phi(s)\tilde{w}_1 + \sum_{k\geq 1} \tilde{P}_{\leq n - \frac{k}{\ln 2}\cdot\frac{s_c-\frac{1}{2}}{1-s_c}} \chi(s-k)\tilde{w}_1, \quad \tilde{\tilde{w}}_1 = \tilde{u}_1 - \tilde{\tilde{v}}_1. \tag{5.6}$$

Remark. Note that (4.15) and (4.16) still hold for the new $(\tilde{v}_0, \tilde{v}_1)$ and $(\tilde{w}_0, \tilde{w}_1)$, since the supports of $\phi(s)$ and $\chi(s-k)$ are almost disjoint, and thus the terms are almost orthogonal.

Furthermore, modifying (3.13) and (3.14), split $\tilde{u} = \tilde{v} + \tilde{w}$, where $\tilde{w}$ solves

$$\begin{gathered} \partial_{\tau\tau}\tilde{w} - \partial_{ss}\tilde{w} - \frac{2}{s}\partial_s\tilde{w} + e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} \cdot |\tilde{w}|^{p-1}\tilde{w} = 0, \\ w(0,s) = \tilde{\tilde{w}}_0(s), \quad w_\tau(0,s) = \tilde{\tilde{w}}_1, \end{gathered} \tag{5.7}$$

and $\tilde{v}$ solves

$$\begin{aligned} \partial_{\tau\tau}\tilde{v} - \partial_{ss}\tilde{v} - \frac{2}{s}\partial_s\tilde{v} + e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde{u}|^{p-1}\tilde{u} - |\tilde{w}|^{p-1}\tilde{w}] = 0, \\ \tilde{v}(0,s) = \tilde{\tilde{v}}_0, \quad \tilde{v}_\tau(0,s) = \tilde{\tilde{v}}_1. \end{aligned} \tag{5.8}$$

Equation (5.7) may be shown to be scattering using small data arguments. Indeed, by Strichartz estimates, finite propagation speed, (4.16), and the fact that $(\frac{s}{\sinh s}) \lesssim e^{-\delta s}$ for any $0 \leq \delta < 1$, if $\tilde{w}$ solves $\partial_{\tau\tau}\tilde{w} - \partial_{ss}\tilde{w} - \frac{2}{s}\partial_s\tilde{w} = 0$ with initial data $(\tilde{\tilde{w}}_0, \tilde{\tilde{w}}_1)$,

$$\Big\| e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{p-3}{p-2}}\tilde{w}\Big\|_{L^{2(p-1)}_{\tau,s}} \lesssim \epsilon. \tag{5.9}$$

Remark. See Lemma 5.2 for a more detailed calculation in a more difficult setting.

The same calculation also works for the radial Strichartz estimates,

$$\int \Big\| e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde{w}\Big\|^2_{L^{\frac{3}{1-s_c}}} d\tau \lesssim \epsilon^2. \tag{5.10}$$

Furthermore, using Strichartz estimates,

$$\begin{aligned} &\Big\| e^{-\frac{p-3}{p-1}\tau}\int_0^\tau S(\tau - t')\Big(0, e^{-(p-3)t'}\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde{w}|^{p-1}\tilde{w}\Big)dt'\Big\|_{L^{2(p-1)}_{\tau,s}} \\ &\lesssim \Big\| e^{-\frac{p-3}{p-1}\tau}e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde{w}|^{p-1}\tilde{w}\Big\|_{L^1_\tau L^{\frac{6}{5-2s_c}}_s} \\ &\lesssim \Big\| e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{p-2}{p-1}}\tilde{w}\Big\|^{p-1}_{L^{2(p-1)}_{\tau,s}} \Big\| e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde{w}\Big\|_{L^2_\tau L^{\frac{3}{1-s_c}}_s}. \end{aligned} \tag{5.11}$$

Therefore,

$$\Big\| e^{-\frac{p-3}{p-1}\tau}e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde{w}|^{p-1}\tilde{w}\Big\|_{L^1_\tau L^{\frac{6}{5-2s_c}}_s} \lesssim \epsilon^p, \tag{5.12}$$

which gives us Strichartz estimates with the appropriate weights for any admissible pair.

Now, define the modified energy

$$\begin{aligned} \mathcal{E}(\tau) &= E(\tau) + cM(\tau) \\ &+ \int \sum_{j\geq 0}[|\tilde{v}|^{p-1}\tilde{v} - |\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_j\tilde{v})]\cdot(\tilde{P}_j\tilde{w})e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^3 s^2 ds, \end{aligned} \tag{5.13}$$

where $c > 0$ is a small constant,

$$E(\tau) = \frac{1}{2}\int \tilde{v}_s(s,\tau)^2 s^2 ds + \frac{1}{2}\int \tilde{v}_\tau(s,\tau)^2 s^2 ds + \frac{1}{p+1}\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde{v}(s,\tau)|^{p+1} s^2 ds, \tag{5.14}$$

and

$$M(\tau) = \int \tilde{v}_s(s,\tau)\tilde{v}_\tau(s,\tau) s^2 ds + \int \tilde{v}(s,\tau)\tilde{v}_\tau(s,\tau) s ds. \tag{5.15}$$

As in (2.35) it is possible to show that

$$\int \sum_{j\geq 0} [|\tilde{v}|^{p-1}\tilde{v} - |\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_j\tilde{v})]\cdot(\tilde{P}_j\tilde{w}) e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^3 s^2 ds \ll E(\tau). \tag{5.16}$$

It will be convenient to take $p = 4$, since one may easily generalize the computations when $p = 4$ to any $3 < p < 5$. By the Littlewood–Paley theorem, if P_j is the standard Littlewood–Paley operator, and the boundedness of the maximal function,

$$\begin{aligned}
&\Big\| \sum_j |P_{\geq j} v|(|P_{\leq j} v|^3 + |P_{\geq j} v|^3)|P_j w|\Big\|_{L^1} \\
&\lesssim \sum_{k\geq 0} \Big\| \Big(\sum_j |P_{j+k} v|^2\Big)^{1/2} \sup_j (|P_{\leq j} v|^3 + |P_{\geq j} v|^3)\Big(\sum_j |P_j w|^2\Big)^{1/2}\Big\|_{L^1} \\
&\lesssim \sum_{k\geq 0} 2^{-2k/3}\|v\|_{L^5}^{10/3}\Big(\sum_j \|P_j w\|_{L^\infty}^2 2^{-4j/3}\Big)^{1/2}\Big(\sum_j \|P_{j+k} v\|_{L^2}^2 2^{2(j+k)}\Big)^{1/3} \\
&\lesssim \|w\|_{\dot{H}^{5/6}}\|v\|_{L^5}^{10/3}\|\nabla v\|_{L^2}^{2/3}.
\end{aligned} \tag{5.17}$$

Generalizing this computation to (5.16), observe that $\tilde{P}_j$ commutes well with $(\frac{s}{\sinh s})^{3/5}$. Indeed, for any $j \geq 0$,

$$\Big[\tilde{P}_j, \Big(\frac{s}{\sinh s}\Big)^{3/5}\Big] \lesssim 2^{-j}\Big(\frac{s}{\sinh s}\Big)^{3/5}\tilde{P}_{j-1}, \tag{5.18}$$

which gives good estimates on the contribution of the error terms

$$\Big(\frac{s}{\sinh s}\Big)^{3/5}\tilde{P}_j\tilde{v} - \tilde{P}_j\Big(\Big(\frac{s}{\sinh s}\Big)^{3/5}\tilde{v}\Big). \tag{5.19}$$

Making use of the Strichartz estimates calculations in (5.9) and (5.12),

$$\sum_{j\geq 0}\Big\| e^{-\tau/4}\Big(\frac{s}{\sinh s}\Big)^{3/5}\tilde{P}_j\tilde{w}\Big\|_{L^{9/2}}^2 \lesssim \epsilon^2, \tag{5.20}$$

so by the Sobolev embedding theorem the contribution of w may be handled in a manner identical to $P_j w$ in (5.17). Finally, by construction, for any fixed $j \geq 0$,

$$\tilde{P}_j = \sum_{k \geq 0} c(k,j) P_k, \tag{5.21}$$

where $c(k,j)$ has good decay away from j, so the Littlewood–Paley theorem may be used as in (5.17).

Now then, by (4.10) and (4.13),

$$\begin{aligned}
&\frac{d}{d\tau}\mathcal{E}(\tau) = \\
&\quad -\frac{c}{2}\tilde{v}(\tau,0)^2 \\
&\quad -c\frac{p-1}{p+1}\int \Big(\frac{\cosh s}{\sinh s}\Big) e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} |\tilde{v}(s,\tau)|^{p+1} s^2 ds \\
&\quad -\frac{p-3}{p+1}\int \Big(\frac{s}{\sinh s}\Big)^{p-1} e^{-(p-3)\tau} |\tilde{v}(s,\tau)|^{p+1} s^2 ds \\
&\quad +\frac{d}{d\tau}\int \sum_{j \geq 0} [|\tilde{v}|^{p-1}\tilde{v} - |\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_j \tilde{v})] \cdot \tilde{P}_j \tilde{w} s^2 ds
\end{aligned} \tag{5.22}$$

$$-\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} [|\tilde{u}|^{p-1}\tilde{u} - |\tilde{v}|^{p-1}\tilde{v} - |\tilde{w}|^{p-1}\tilde{w}]\tilde{v}_\tau s^2 ds \tag{5.23}$$

$$-\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} [|\tilde{u}|^{p-1}\tilde{u} - |\tilde{v}|^{p-1}\tilde{v} - |\tilde{w}|^{p-1}\tilde{w}]\tilde{v}_s s^2 ds \tag{5.24}$$

$$-\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} [|\tilde{u}|^{p-1}\tilde{u} - |\tilde{v}|^{p-1}\tilde{v} - |\tilde{w}|^{p-1}\tilde{w}]\tilde{v} s ds. \tag{5.25}$$

First, as in (3.22),

$$\begin{aligned}
&\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} [|\tilde{u}|^{p-1}\tilde{u} - |\tilde{v}|^{p-1}\tilde{v} - |\tilde{w}|^{p-1}\tilde{w}]\tilde{v} s ds \\
&\quad \lesssim \int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} [|\tilde{v}|^p |\tilde{w}| + |\tilde{w}|^{p-1}|\tilde{v}|^2] s ds \\
&\quad \lesssim \bigg(\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} |\tilde{v}|^{p+1}\Big(\frac{\cosh s}{\sinh s}\Big) s^2 ds\bigg)^{\frac{p-2}{p-1}} \\
&\qquad \times \Big\| \frac{1}{s^{1/2}}\tilde{v}\Big\|_{L^3}^{\frac{2}{p-1}} \Big\| e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde{w}\Big\|_{L^{3(p-1)}} \\
&\qquad + \Big\| e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde{w}\Big\|_{L^{3(p-1)}}^{p-1} \Big\| \frac{1}{s}\tilde{v}\Big\|_{L^2} \|\tilde{v}\|_{L^6} \\
&\quad \lesssim \delta\bigg(\int e^{-(p-3)\tau}\Big(\frac{\cosh s}{\sinh s}\Big)\Big(\frac{s}{\sinh s}\Big)^{p-1} |\tilde{v}|^{p+1} s^2 ds\bigg) \\
&\qquad + \frac{1}{\delta} E(\tau) \Big\| e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde{w}\Big\|_{L^{3(p-1)}}^{p-1}.
\end{aligned} \tag{5.26}$$

This takes care of (5.25).

Next, following (3.24) and (3.25),

$$\begin{aligned}
&\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde v|^{p-2}|\partial_{s,\tau}\tilde v||\tilde w|^2 s^2 ds\\
&\lesssim \Big\|e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde w\Big\|^2_{L^{\frac{3}{1-s_c}}}\\
&\quad\times\Big\|e^{-\frac{p-3}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{p-1}{p+1}}\tilde v\Big\|^{p-2-\frac{5-p}{p-1}}_{L^{p+1}}\|\partial_{s,\tau}\tilde v\|^{1+\frac{5-p}{p-1}}_{L^2}\\
&\lesssim E(\tau)\Big\|e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde w\Big\|^2_{L^{\frac{3}{1-s_c}}},
\end{aligned}\tag{5.27}$$

and

$$\begin{aligned}
&\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde w|^{p-1}|\tilde v||\partial_{s,\tau}\tilde v|s^2 ds\\
&\lesssim \Big\|e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde w\Big\|^{p-1}_{L^{3(p-1)}}\|\partial_{s,\tau}\tilde v\|^2_{L^2}\\
&\lesssim E(\tau)\Big\|e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde w\Big\|^{p-1}_{L^{3(p-1)}}.
\end{aligned}\tag{5.28}$$

By (5.9) and (5.10), these terms may be handled using Gronwall's inequality.

Thus, the only terms left to consider in (5.23) and (5.24) are

$$\begin{aligned}
&-p\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde v|^{p-1}\tilde w]\tilde v_\tau s^2 ds\\
&-p\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde v|^{p-1}\tilde w]\tilde v_s s^2 ds.
\end{aligned}\tag{5.29}$$

By the product rule,

$$\begin{aligned}
&\frac{d}{d\tau}\int\sum_j[|\tilde v|^{p-1}\tilde v-|\tilde P_{\le j}\tilde v|^{p-1}\tilde P_{\le j}\tilde v]e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\tilde P_j\tilde w s^2 ds\\
&-p\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}|\tilde v|^{p-1}\tilde v_\tau\tilde w s^2 ds\\
&=-\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde v|^{p-1}\tilde v-|\tilde P_{\le j}\tilde v|^{p-1}\tilde P_{\le j}\tilde v](\tilde P_j\tilde w_\tau)s^2 ds\\
&\quad-\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\partial_\tau[|\tilde P_{\le j}\tilde v|^{p-1}\tilde P_{\le j}\tilde v]\tilde P_j\tilde w s^2 ds\\
&\quad-(p-3)\int\sum_j[|\tilde v|^{p-1}\tilde v-|\tilde P_{\le j}\tilde v|^{p-1}(\tilde P_{\le j}\tilde v)](\tilde P_j\tilde w)\\
&\qquad\qquad\times e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}s^2 ds.
\end{aligned}\tag{5.30}$$

By the Cauchy–Schwartz inequality,

$$\begin{aligned} &-(p-3)\int |\tilde{v}|^{p-1}\tilde{v}\tilde{w}e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}s^2ds \\ &\lesssim \delta\Big(\int |\tilde{v}|^{p+1}e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}s^2ds\Big) \\ &\quad + \frac{1}{\delta}\Big\|e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde{w}\Big\|_{L^{3(p-1)}}^{p-1}\|\partial_{s,\tau}\tilde{v}\|_{L^2}^2. \end{aligned} \tag{5.31}$$

This computation may be generalized to show

$$\begin{aligned} &-(p-3)\int \sum_j [|\tilde{v}|^{p-1}\tilde{v} - |\tilde{P}_{\le j}\tilde{v}|^{p-1}(\tilde{P}_{\le j}\tilde{v})](\tilde{P}_j\tilde{w})e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}s^2ds \\ &\lesssim \delta\Big(\int |\tilde{v}|^{p+1}e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}s^2ds\Big) + \frac{1}{\delta}\|e^{-\frac{p-3}{p-1}\tau}\Big(\frac{s}{\sinh s}\Big)\tilde{w}\|_{L^{3(p-1)}}^{p-1}\|\partial_{s,\tau}\tilde{v}\|_{L^2}^2. \end{aligned} \tag{5.32}$$

using the same arguments as in (5.16)–(5.21).

Next,

$$\begin{aligned} &\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\partial_\tau[|\tilde{P}_{\le j}\tilde{v}|^{p-1}\tilde{P}_{\le j}\tilde{v}]\tilde{P}_j\tilde{w}s^2ds \\ &\lesssim \Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{\frac{p-3}{2(p+1)}}\tilde{P}_{\le j}\tilde{v}_\tau\Big\|_{L^{\frac{p+1}{2}}} \\ &\quad \times \Big(\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\Big(\frac{\cosh s}{\sinh s}\Big)|\tilde{v}|^{p+1}s^2ds\Big)^{\frac{p-1}{p+1}} \\ &\quad \times \Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}e^{-\frac{2(p-3)}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}}\tilde{P}_j\tilde{w}\Big\|_{L^\infty}. \end{aligned} \tag{5.33}$$

By the radial Sobolev embedding theorem,

$$\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{\frac{p-3}{2(p+1)}}\tilde{P}_{\le j}\tilde{v}_\tau\Big\|_{L^{\frac{p+1}{2}}} \lesssim 2^{\frac{j(p-3)}{p+1}}E(\tilde{v})^{1/2}. \tag{5.34}$$

Therefore,

$$\begin{aligned} \sum_j (5.33) &\lesssim \delta\Big(\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\Big(\frac{\cosh s}{\sinh s}\Big)|\tilde{v}|^{p+1}s^2ds\Big) \\ &\quad + \frac{1}{\delta}E(t)^{\frac{p+1}{4}}\Big(\sum_j 2^{j\frac{p-3}{p+1}}\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}e^{-\frac{2(p-3)}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}}\tilde{P}_j\tilde{w}\Big\|_{L^\infty}\Big)^{\frac{p+1}{2}}. \end{aligned} \tag{5.35}$$

LEMMA 5.2. *Using Corollary 2.5, the weights in τ and s, and the definition of $\tilde{w}_0$ and $\tilde{w}_1$,*

$$\begin{aligned}\int \Big(\sum_j 2^{j\frac{p-3}{p+1}} \Big\| \Big(\frac{\sinh s}{\cosh s}\Big)^{1/2} e^{-\frac{2(p-3)}{p+1}\tau} \Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}} \tilde{P}_j \tilde{w} \Big\|_{L^\infty} \Big)^{\frac{p+1}{2}} d\tau \\ \lesssim \epsilon^{\frac{p+1}{2}} R^{-(1-s_c)\frac{p-3}{2}}.\end{aligned} \tag{5.36}$$

Proof. First replace $\tilde{w}$ by $S(\tau)(\tilde{\tilde{w}}_0, \tilde{\tilde{w}}_1)$, where $(\tilde{\tilde{w}}_0, \tilde{\tilde{w}}_1)$ are given by (5.5) and (5.6). Observe that by Corollary 2.5, Theorem 4.2, and Bernstein's inequality,

$$\begin{aligned}&\Big(\sum_{j>n} 2^{j(\frac{p-3}{p+1})} \| s^{1/2} S(\tau)(\tilde{P}_j \phi(s)\tilde{v}_0, \tilde{P}_j \phi(s)\tilde{v}_1) \|_{L_\tau^{\frac{p+1}{2}} L_x^\infty} \Big)^{\frac{p+1}{2}} \\ &\lesssim \Big(\sum_{j>n} 2^{j(\frac{p-3}{p+1})} \| (\tilde{P}_j \phi(s)\tilde{v}_0, \tilde{P}_j \phi(s)\tilde{v}_1) \|_{\dot{H}^{\frac{p-1}{p+1}} \times \dot{H}^{\frac{p-1}{p+1}-1}} \Big)^{\frac{p+1}{2}} \\ &\lesssim R^{\frac{p-3}{2}} R^{\frac{p-1}{2}} R^{-s_c \cdot \frac{p+1}{2}} \| (\tilde{P}_{>n} \phi(s)\tilde{v}_0, \tilde{P}_{>n} \phi(s)\tilde{v}_1) \|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}^{\frac{p+1}{2}}.\end{aligned} \tag{5.37}$$

Doing some algebra using (1.3),

$$\begin{aligned}p - 2 - s_c \cdot \frac{p+1}{2} &= \frac{p-5}{2} + \frac{p+1}{2}(1-s_c) \\ &= -(p-1)(1-s_c) + \frac{p+1}{2}(1-s_c) \\ &= \frac{3-p}{2}(1-s_c).\end{aligned} \tag{5.38}$$

Then by (4.3), (5.36) holds for $\tilde{w}$ replaced by $S(\tau)(\tilde{P}_{>n}\phi(s)\tilde{w}_0, \tilde{P}_{>n}\phi(s)\tilde{w}_1)$.

Next, for $0 < k < n\ln(2) \cdot \frac{1-s_c}{s_c - 1/2}$, observe that by finite propagation speed,

$$S(\tau)(P_{>n - \frac{k}{\ln(2)} \frac{s_c - 1/2}{1-s_c}} \chi(s-k)\tilde{w}_0, P_{>n - \frac{k}{\ln(2)} \frac{s_c - 1/2}{1-s_c}} \chi(s-k)\tilde{w}_1) \tag{5.39}$$

is supported on the set $\{(\tau, s) : \tau + s \geq k - 2,\ \tau > 0,\ s > 0\}$. Therefore, for any such k, by (4.16),

$$\begin{aligned}\int \Big(\sum_{j > n - \frac{k}{\ln(2)} \frac{s_c - 1/2}{1-s_c}} 2^{j\frac{p-3}{p+1}} \Big\| \Big(\frac{\sinh s}{\cosh s}\Big)^{1/2} e^{-\frac{2(p-3)}{p+1}\tau} \Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}} \\ \times S(\tau)(\tilde{P}_j \chi(s-k)\tilde{w}_0, \tilde{P}_j \chi(s-k)\tilde{w}_1) \Big\|_{L^\infty} \Big)^{\frac{p+1}{2}} d\tau \\ \lesssim e^{-(s_c - \frac{1}{2})k} R^{-(1-s_c)\frac{p-3}{2}} \| (\tilde{w}_0, \tilde{w}_1) \|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}.\end{aligned} \tag{5.40}$$

The proof uses the algebraic fact that

$$\begin{aligned} &e^{-(p-3)k} e^{\frac{p+1}{2}(s_c-1/2)k} 2^{\frac{k}{\ln(2)}\frac{s_c-1/2}{1-s_c}\cdot\frac{p-3}{2}(1-s_c)} \\ &\quad = e^{-(p-3)k} 2^{\frac{p+1}{2}(s_c-1/2)k} e^{k(s_c-1/2)\cdot\frac{p-3}{2}} = 1. \end{aligned} \tag{5.41}$$

Furthermore, observe that on the set $\{(\tau, s) : \tau + s \geq k-2\}$,

$$e^{-2\frac{p-3}{p+1}\tau} \Big(\frac{s}{\sinh s} \Big)^{\frac{2(p-1)}{p+1}} \leq e^{-\frac{2(p-3)}{p+1}\tau} e^{-\alpha|k-\tau|}, \tag{5.42}$$

for some $\alpha(p) > 0$. Therefore, (4.16) combined with (5.42) give good summation of (5.36) in k for $S(\tau)(\tilde{\tilde{w}}_0, \tilde{\tilde{w}}_1)$, where $(\tilde{\tilde{w}}_0, \tilde{\tilde{w}}_1)$ is given by (5.5) and (5.6).

The contribution of terms for which k satisfies $n - \frac{k}{\ln(2)}\frac{s_c-1/2}{1-s_c} < 0$ are better. This is because $(\frac{\sinh s}{\cosh s})^{1/2} \lesssim \inf\{s^{1/2}, 1\}$. Therefore, it is possible to combine the weighted estimates in Corollary 2.5 with the radial Strichartz estimates in Theorem 2.1 (to handle the case when $j = 0$) to prove (5.36) when $\tilde{w}$ is replaced by $S(\tau)(\tilde{w}_0, \tilde{w}_1)$.

For the contribution of the nonlinear term, observe that the same arguments would prove

$$\Big\| e^{-\frac{2(p-3)}{p+1}\tau} \Big(\frac{s}{\sinh s} \Big)^{\frac{2(p-2)}{p+1}} \tilde{P}_j S(\tau)(\tilde{w}_0, \tilde{w}_1) \Big\|_{L^2_\tau L^q} \lesssim \epsilon R^{-(1-s_c)\frac{p-3}{p+1}} \tag{5.43}$$

when

$$q = \frac{1}{2} - \frac{5p-7}{6(p+1)}. \tag{5.44}$$

This is because $(\frac{p+1}{2}, q)$ is a $\dot{H}^{\frac{2p-4}{p+1}}$-admissible pair, and any fractional power $(\frac{s}{\sinh s})$ gives good decay at large s.

Remark. It is okay to change the $(\frac{s}{\sinh s})$ exponent from $\frac{2(p-1)}{p+1}$ to $\frac{2(p-2)}{p+1}$.

Combining inequalities (5.43), (5.9), the calculations in (5.27), radially symmetric Strichartz estimates in Theorem 2.4, and the standard Strichartz estimates in Theorem 2.1 proves

$$\Big\| e^{-\frac{2(p-3)}{p+1}\tau} e^{-(p-3)\tau} \Big(\frac{s}{\sinh s} \Big)^{p-1} |\tilde{w}|^p \Big\|_{L^1_\tau L_s^{\frac{2q}{q+2}}} \lesssim \epsilon^p R^{-(1-s_c)\frac{p-3}{p+1}}, \tag{5.45}$$

which by the principle of superposition proves the lemma. □

Returning to the proof of Theorem 5.1, by direct computation,

$$[|\tilde{v}|^{p-1}\tilde{v} - |\tilde{P}_{\leq j}\tilde{v}|^{p-1}\tilde{P}_{\leq j}\tilde{v}] = O(|\tilde{P}_{>j}\tilde{v}|(|\tilde{P}_{\leq j}\tilde{v}|^{p-1} + |\tilde{P}_{>j}\tilde{v}|^{p-1})). \tag{5.46}$$

Also, by Bernstein's inequality and the radial Sobolev embedding theorem,

$$\begin{aligned}
&\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde v|^{p-1}\tilde v-|\tilde P_{\le j}\tilde v|^{p-1}\tilde P_{\le j}\tilde v]\tilde P_j\tilde w_\tau s^2ds\\
&\lesssim\Big(\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\Big(\frac{\cosh s}{\sinh s}\Big)|\tilde v|^{p+1}s^2ds\Big)^{\frac{p-1}{p+1}}\\
&\quad\times\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{\frac{p-3}{2(p+1)}}|\tilde P_{>j}\tilde v|\Big\|_{L^{\frac{p+1}{2}}}\\
&\quad\times\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}e^{-\frac{2(p-3)}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}}\tilde P_j\tilde w_\tau\Big\|_{L^\infty}.
\end{aligned}\tag{5.47}$$

By the radial Sobolev embedding theorem and the definition of $\tilde P_j$,

$$\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{\frac{p-3}{2(p+1)}}|\tilde P_{>j}\tilde v|\Big\|_{L^{\frac{p+1}{2}}}\lesssim 2^{-\frac{4j}{p+1}}E(\tilde v)^{1/2}.\tag{5.48}$$

Now then, as in Lemma 5.2,

$$\begin{aligned}
&\int\Big(\sum_j 2^{-\frac{4j}{p+1}}\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}e^{-\frac{2(p-3)}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}}\tilde P_j\tilde w_\tau\|_{L^\infty}\Big)^{\frac{p+1}{2}}d\tau\\
&\lesssim\epsilon^{\frac{p+1}{2}}R^{-(1-s_c)\frac{p-3}{2}}.
\end{aligned}\tag{5.49}$$

First replace $\tilde w$ by $S(\tau)(\tilde w_0,\tilde w_1)$. Then by (5.21) and Lemma 5.2

(5.50)

$$\begin{aligned}
&\int\Big(\sum_j 2^{-\frac{4j}{p+1}}\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}e^{-\frac{2(p-3)}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}}\tilde P_j\partial_\tau S(\tau)(\tilde w_0,\tilde w_1)\Big\|_{L^\infty}\Big)^{\frac{p+1}{2}}d\tau\\
&=\int\Big(\sum_j 2^{-\frac{4j}{p+1}}\\
&\qquad\times\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}e^{-\frac{2(p-3)}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}}\partial_\tau S(\tau)(\tilde P_j\tilde w_0,\tilde P_j\tilde w_1)\Big\|_{L^\infty}\Big)^{\frac{p+1}{2}}d\tau\\
&=\int\Big(\sum_j 2^{-\frac{4j}{p+1}}\\
&\qquad\times\Big\|\Big(\frac{\sinh s}{\cosh s}\Big)^{1/2}e^{-\frac{2(p-3)}{p+1}\tau}\Big(\frac{s}{\sinh s}\Big)^{\frac{2(p-1)}{p+1}}S(\tau)(\tilde P_j\tilde w_1,\tilde P_j\Delta\tilde w_0)\Big\|_{L^\infty}\Big)^{\frac{p+1}{2}}d\tau\\
&\lesssim\epsilon^{\frac{p+1}{2}}R^{-(1-s_c)\frac{p-3}{2}}.
\end{aligned}$$

The contribution of the Duhamel term may be handled using the principle of superposition as in (2.60)–(2.64) combined with (5.45).

The term

$$-p\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde v|^{p-1}\tilde w]\tilde v_s s^2ds\tag{5.51}$$

may be handled in a similar manner, only integrating by parts in s. Indeed,

$$\begin{aligned} &-p\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde{v}|^{p-1}\tilde{v}_s\tilde{w}]s^2ds \\ &\quad=-\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\partial_s[|\tilde{v}|^{p-1}\tilde{v}]\tilde{w}s^2ds \\ &\quad=-\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1} \\ &\qquad\qquad\times\partial_s[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})s^2ds \\ &\qquad-\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\partial_s[|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})s^2ds. \end{aligned} \tag{5.52}$$

The contribution of

$$-\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\partial_s[|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})s^2ds \tag{5.53}$$

may be handled as in (5.33). Integrating by parts,

$$\begin{aligned} &-\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}\partial_s[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})s^2ds \\ &=\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w}_s)s^2ds \\ &\quad+2\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})sds \\ &\quad+\int\sum_j e^{-(p-3)\tau}\partial_s\Big(\Big(\frac{s}{\sinh s}\Big)^{p-1}\Big)[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})s^2ds. \end{aligned} \tag{5.54}$$

The term

$$\int e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w}_s)s^2ds \tag{5.55}$$

may be handled exactly as in (5.47). Since $\frac{\partial}{\partial s}(\frac{s}{\sinh s})^{p-1}\lesssim(\frac{s}{\sinh s})^{p-1}$, the contribution of

$$\int\sum_j e^{-(p-3)\tau}\partial_s\Big(\Big(\frac{s}{\sinh s}\Big)^{p-1}\Big)[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})s^2ds \tag{5.56}$$

may be handled as in (5.32). The term

$$2\int\sum_j e^{-(p-3)\tau}\Big(\frac{s}{\sinh s}\Big)^{p-1}[|\tilde{v}|^{p-1}\tilde{v}-|\tilde{P}_{\leq j}\tilde{v}|^{p-1}(\tilde{P}_{\leq j}\tilde{v})](\tilde{P}_j\tilde{w})sds \tag{5.57}$$

may be handled using (5.26) and (5.16)–(5.21).

Then by (5.12) and a Gronwall-type estimate, we have proved

$$\iint |\tilde{v}(\tau,s)|^{p+1}\Big(\frac{\cosh s}{\sinh s}\Big)\Big(\frac{s}{\sinh s}\Big)^{p-1} e^{-(p-3)\tau} s^2 ds d\tau < \infty. \tag{5.58}$$

By the radial Sobolev embedding theorem,

$$\Big(\frac{\sinh s}{\cosh s}\Big)|\tilde{v}(\tau,s)|^{p-3} \lesssim E(\tilde{v})^{\frac{p-3}{2}} < \infty. \tag{5.59}$$

Therefore, we have proved

$$\iint |v(e^\tau \cosh s, e^\tau \sinh s)|^{2(p-1)} (e^\tau \sinh s)^2 e^{2\tau} ds d\tau < \infty, \tag{5.60}$$

which by a change of variables formula implies

$$\int_1^\infty \int_{t^2-r^2\geq 1} |v(t,r)|^{2(p-1)} r^2 dr dt < \infty. \tag{5.61}$$

Also, by (5.9) and a change of variables,

$$\begin{aligned} &\iint e^{-2(p-3)\tau} |\tilde{w}(e^\tau \cosh s, e^\tau \sinh s)|^{2(p-1)} \Big(\frac{s}{\sinh s}\Big)^{2(p-2)} s^2 ds d\tau \\ &= \iint e^{2\tau} |w(e^\tau \cosh s, e^\tau \sinh s)|^{2(p-1)} (e^\tau \sinh s)^2 ds d\tau \\ &= \iint_{t^2-r^2\geq 1} |w(t,r)|^{2(p-1)} r^2 dr dt \lesssim \epsilon^{2(p-1)}. \end{aligned} \tag{5.62}$$

Combining (4.5) with (5.61) and (5.62) completes the proof of Theorem 5.1. □

6. Scattering. As in [3, 4], we use concentration compactness and a perturbative argument to obtain a uniform bound on the scattering size for initial data with bounded $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ norm.

Let (u_0^n, u_1) be a radially symmetric sequence uniformly bounded in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$,

$$\|u_0^n\|_{\dot{H}^{s_c}(\mathbf{R}^3)} + \|u_1^n\|_{\dot{H}^{s_c-1}(\mathbf{R}^3)} \leq A, \tag{6.1}$$

and let u^n be the solution to (1.1) with initial data (u_0^n, u_1^n). By Zorn's lemma, to prove (1.6), it suffices to show that

$$\|u^n\|_{L_{t,x}^{2(p-1)}(\mathbf{R}\times\mathbf{R}^3)} \tag{6.2}$$

is uniformly bounded for any such sequence.

The proof of this fact uses the profile decomposition of [15]. We prove that (u_0^n, u_1^n) must converge, at least after passing to a subsequence, and then show that this convergence implies the existence of a maximizer, which by the analysis in the previous five sections has finite $L_{t,x}^{2(p-1)}$ norm.

The argument by now is a fairly standard concentration compactness argument. See [9] for the use of this argument to prove scattering for an energy-critical nonlinear wave equation. See [12, pp. 245–269] for a detailed description of the concentration compactness method.

THEOREM 6.1 (Profile decomposition). *Suppose that there is a uniformly bounded, radially symmetric sequence such that*

$$\|u_0^n\|_{\dot{H}^{s_c}(\mathbf{R}^3)} + \|u_1^n\|_{\dot{H}^{s_c-1}(\mathbf{R}^3)} \leq A < \infty. \tag{6.3}$$

Then there exists a subsequence, also denoted $(u_0^n, u_1^n) \subset \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ *such that for any* $N < \infty$*,*

$$S(t)(u_0^n, u_1^n) = \sum_{j=1}^{N} \Gamma_n^j S(t)(\phi_0^j, \phi_1^j) + S(t)(R_{0,n}^N, R_{1,n}^N), \tag{6.4}$$

with

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S(t)(R_{0,n}^N, R_{1,n}^N)\|_{L_{t,x}^q(\mathbf{R}\times\mathbf{R}^3)} = 0. \tag{6.5}$$

$\Gamma_n^j = (\lambda_n^j, t_n^j)$ *belongs to the group* $(0,\infty) \times \mathbf{R}$*, which acts by*

$$\Gamma_n^j F(t,x) = \lambda_n^j F(\lambda_n^j (t - t_n^j), \lambda_n^j x). \tag{6.6}$$

The Γ_n^j *are pairwise orthogonal, that is, for every* $j \neq k$*,*

$$\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + (\lambda_n^j)^{1/2}(\lambda_n^k)^{1/2}|t_n^j - t_n^k| = \infty. \tag{6.7}$$

Furthermore, for every $N \geq 1$*,*

$$\begin{aligned} &\|(u_{0,n}, u_{1,n})\|_{\dot{H}^{s_c}\times\dot{H}^{s_c-1}}^2 \\ &\quad = \sum_{j=1}^{N} \|(\phi_0^j, \phi_0^k)\|_{\dot{H}^{s_c}\times\dot{H}^{s_c-1}}^2 + \|(R_{0,n}^N, R_{1,n}^N)\|_{\dot{H}^{s_c}\times\dot{H}^{s_c-1}}^2 + o_n(1), \end{aligned} \tag{6.8}$$

and for any $1 \leq j \leq N$*,*

$$(\Gamma_n^j)^{-1}(R_{0,n}^N, R_{1,n}^N) \rightharpoonup 0, \tag{6.9}$$

weakly in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$*.*

Therefore, to summarize Theorem 6.1,

$$S(t)(u_0^n,u_1^n)=\sum_{j=1}^{N}S(t-t_n^j)(\lambda_n^j\phi_0^j(\lambda_n^j x),(\lambda_n^j)^2\phi_1^j(\lambda_n^j x)) + S(t)(R_{0,n}^N,R_{1,n}^N), \tag{6.10}$$

$$S(\lambda_n^j t_n^j)\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big),\frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\rightharpoonup\phi_0^j(x), \tag{6.11}$$

weakly in $\dot H^{s_c}(\mathbf{R}^3)$, and

$$\partial_t S(t+\lambda_n^j t_n^j)\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big),\frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big|_{t=0}\rightharpoonup\phi_1^j(x) \tag{6.12}$$

weakly in $\dot H^{s_c-1}(\mathbf{R}^3)$.

First consider the case that $\lambda_n^j t_n^j$ is uniformly bounded. In this case, after passing to a subsequence, $\lambda_n^j t_n^j$ converges to some t^j. Changing (ϕ_0^j,ϕ_1^j) to $S(-t^j)(\phi_0^j,\phi_1^j)$ and absorbing the error into $(R_{0,n}^N,R_{1,n}^N)$,

$$\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big),\frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\rightharpoonup\phi_0^j(x), \tag{6.13}$$

and

$$\partial_t S(t)\Big(\frac{1}{\lambda_n^j}u_0^n\Big(\frac{x}{\lambda_n^j}\Big),\frac{1}{(\lambda_n^j)^2}u_1^n\Big(\frac{x}{\lambda_n^j}\Big)\Big)\Big|_{t=0}\rightharpoonup\phi_1^j(x). \tag{6.14}$$

Then if u^j is the solution to (1.1) with initial data (ϕ_0^j,ϕ_1^j), by Theorem 5.1,

$$\|u^j\|_{L_{t,x}^{2(p-1)}(\mathbf{R}\times\mathbf{R}^3)}\le M(\phi_j^0,\phi_j^1)<\infty. \tag{6.15}$$

Next, suppose that after passing to a subsequence, $\lambda_n^j t_n^j\nearrow+\infty$. Then a solution to (1.1) approaches a translation in time of a solution to (1.1) that scatters backward in time to $S(t)(\phi_0,\phi_1)$, that is,

$$\lim_{t\to-\infty}\|u-S(t)(\phi_0,\phi_1)\|_{\dot H^{s_c}\times\dot H^{s_c-1}}=0. \tag{6.16}$$

Indeed, by Strichartz estimates, the dominated convergence theorem, and small data arguments, for some $T<\infty$ sufficiently large, (1.1) has a solution u on $(-\infty,-T]$ such that

$$\begin{aligned}&\|u\|_{L_{t,x}^{2(p-1)}\cap L_t^{\frac{2}{s_c}}L_x^{\frac{2}{1-s_c}}((-\infty,-T]\times\mathbf{R}^3)}\lesssim\epsilon,\\ &(u(-T,x),u_t(-T,x))=S(-T)(\phi_0,\phi_1),\end{aligned} \tag{6.17}$$

and by Strichartz estimates,

$$\lim_{t \to +\infty} \| S(t)(u(-t), u_t(-t)) - (\phi_0, \phi_1) \|_{\dot{H}^{s_c} \times \dot{H}^{s_c - 1}} \lesssim \epsilon^p. \tag{6.18}$$

Then by the inverse function theorem, there exists some $(u_0(-T), u_1(-T))$ such that (1.1) has a solution that scatters backward in time to $S(t)(\phi_0, \phi_1)$. Moreover, by Theorem 5.1, this solution must also scatter forward in time. Therefore,

$$S(-t_n^j)(\lambda_n^j \phi_0^j(\lambda_n^j x), (\lambda_n^j)^2 \phi_1^j(\lambda_n^j x)) \tag{6.19}$$

converges strongly to

$$(\lambda_n^j u^j(-\lambda_n^j t_n^j, \lambda_n^j x), (\lambda_n^j)^2 u_t^j(-\lambda_n^j t_n^j, \lambda_n^j x)) \tag{6.20}$$

in $\dot{H}^{s_c} \times \dot{H}^{s_c - 1}$, where u^j is the solution to (1.1) that scatters backward in time to $S(t)(\phi_0^j, \phi_1^j)$, and the remainder may be absorbed into $(R_{0,n}^N, R_{1,n}^N)$. Let $(\tilde{\phi}_j^0, \tilde{\phi}_j^1)$ denote the initial data of such a solution. In this case as well,

$$\| u^j \|_{L_{t,x}^{2(p-1)}(\mathbf{R} \times \mathbf{R}^3)} \leq M(\tilde{\phi}_j^0, \tilde{\phi}_j^1) < \infty. \tag{6.21}$$

The proof for $\lambda_n^j t_n^j \searrow -\infty$ is similar.

Next, by (6.8), there are only finitely many j such that

$$\| \phi_0^j \|_{\dot{H}^{s_c}} + \| \phi_1^j \|_{\dot{H}^{s_c - 1}} > \epsilon.$$

For all other j, small data arguments imply

$$\| u^j \|_{L_{t,x}^{2(p-1)}(\mathbf{R} \times \mathbf{R}^3)} \lesssim \| \phi_0^j \|_{\dot{H}^{s_c}} + \| \phi_1^j \|_{\dot{H}^{s_c - 1}}. \tag{6.22}$$

Then make use of standard perturbation results for nonlinear wave equations.

LEMMA 6.2 (Perturbation lemma). *Let $I \subset \mathbf{R}$ be a time interval. Let $t_0 \in I$, $(u_0, u_1) \in \dot{H}^{s_c} \times \dot{H}^{s_c - 1}$, and suppose there exist some constants M, A, $A' > 0$. Let $\tilde{u}$ solve the equation*

$$(\partial_{tt} - \Delta) \tilde{u} = F(\tilde{u}) = e, \tag{6.23}$$

on $I \times \mathbf{R}^3$, and also suppose

$$\begin{aligned} &\sup_{t \in I} \| (\tilde{u}(t), \partial_t \tilde{u}(t)) \|_{\dot{H}^{s_c} \times \dot{H}^{s_c - 1}} \leq A, \quad \| \tilde{u} \|_{L_{t,x}^{2(p-1)}(I \times \mathbf{R}^3)} \leq M, \\ &\| (u_0 - \tilde{u}(t_0), u_1 - \partial_t \tilde{u}(t_0)) \|_{\dot{H}^{s_c} \times \dot{H}^{s_c - 1}} \leq A', \end{aligned} \tag{6.24}$$

and

$$\| e \|_{L_t^{\frac{2}{1+s_c}} L_x^{\frac{2}{2-s_c}}(I \times \mathbf{R}^3)} + \| S(t - t_0)(u_0 - \tilde{u}(t_0), u_1 - \partial_t \tilde{u}(t_0)) \|_{L_{t,x}^{2(p-1)}(I \times \mathbf{R}^3)} \leq \epsilon. \tag{6.25}$$

Then there exists $\epsilon_0(M, A, A')$ *such that if* $0 < \epsilon < \epsilon_0$ *then there exists a solution to* (1.1) *on* I *with* $(u(t_0), \partial_t u(t_0)) = (u_0, u_1)$, $\|u\|_{L_{t,x}^{2(p-1)}(I\times\mathbf{R}^3)} \leq C(M, A, A')$, *and for all* $t \in I$,

$$\|(u(t), \partial_t u(t)) - (\tilde{u}(t), \partial_t \tilde{u}(t))\|_{\dot{H}^{s_c}\times\dot{H}^{s_c-1}} \leq C(A, A', M)(A' + \epsilon). \tag{6.26}$$

Proof. This Lemma appears throughout the literature on nonlinear wave equations. □

By Lemma 6.2, the asymptotic orthogonality property (6.7), and (6.22),

$$\limsup_{n\nearrow\infty} \|u^n\|^2_{L_{t,x}^{2(p-1)}(\mathbf{R}\times\mathbf{R}^3)} \lesssim \sum_j \|u^j\|^2_{L_{t,x}^{2(p-1)}(\mathbf{R}\times\mathbf{R}^3)} < \infty. \tag{6.27}$$

This proves Theorem 1.3. □

JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218

REFERENCES

[1] H. Bahouri and P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, *Amer. J. Math.* **121** (1999), no. 1, 131–175.

[2] H. Bahouri and J. Shatah, Decay estimates for the critical semilinear wave equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998), no. 6, 783–789.

[3] B. Dodson, Global well-posedness and scattering for the radial, defocusing, cubic wave equation with initial data in a critical Besov space, *Anal. PDE* **12** (2019), no. 4, 1023–1048.

[4] ———, Scattering for the radial defocusing cubic nonlinear wave equation with initial data in the critical Sobolev space, *Duke Math. J.* **170** (2021), no. 15, 3267–3321.

[5] B. Dodson, A. Lawrie, D. Mendelson, and J. Murphy, Scattering for defocusing energy subcritical nonlinear wave equations, *Anal. PDE* **13** (2020), no. 7, 1995–2090.

[6] J. Ginibre, A. Soffer, and G. Velo, The global Cauchy problem for the critical nonlinear wave equation, *J. Funct. Anal.* **110** (1992), no. 1, 96–130.

[7] J. Ginibre and G. Velo, Generalized Strichartz inequalities for the wave equation, *J. Funct. Anal.* **133** (1995), no. 1, 50–68.

[8] M. G. Grillakis, Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity, *Ann. of Math. (2)* **132** (1990), no. 3, 485–509.

[9] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation, *Acta Math.* **201** (2008), no. 2, 147–212.

[10] C. E. Kenig, G. Ponce, and L. Vega, Global well-posedness for semi-linear wave equations, *Comm. Partial Differential Equations* **25** (2000), no. 9-10, 1741–1752.

[11] S. Klainerman and M. Machedon, Space-time estimates for null forms and the local existence theorem, *Comm. Pure Appl. Math.* **46** (1993), no. 9, 1221–1268.

[12] H. Koch, D. Tataru, and M. Vişan, *Well-Posedness for Nonlinear Dispersive Equations*, Birkhäuser, Basel, 2014.

[13] H. Lindblad and C. D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations, *J. Funct. Anal.* **130** (1995), no. 2, 357–426.

[14] F. Merle and H. Zaag, Determination of the blow-up rate for the semilinear wave equation, *Amer. J. Math.* **125** (2003), no. 5, 1147–1164.

[15] J. Ramos, A refinement of the Strichartz inequality for the wave equation with applications, *Adv. Math.* **230** (2012), no. 2, 649–698.

[16] J. Shatah and M. Struwe, Regularity results for nonlinear wave equations, *Ann. of Math. (2)* **138** (1993), no. 3, 503–518.

[17] R. Shen, On the energy subcritical, nonlinear wave equation in $\mathbb{R}^3$ with radial data, *Anal. PDE* **6** (2013), no. 8, 1929–1987.

[18] _____, Scattering of solutions to the defocusing energy subcritical semi-linear wave equation in 3D, *Comm. Partial Differential Equations* **42** (2017), no. 4, 495–518.

[19] J. Sterbenz, Angular regularity and Strichartz estimates for the wave equation. With an appendix by Igor Rodnianski, *Int. Math. Res. Not. IMRN* **2005** (2005), no. 4, 187–231.

[20] W. A. Strauss, Nonlinear scattering theory at low energy, *J. Funct. Anal.* **41** (1981), no. 1, 110–133.

[21] R. S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Math. J.* **44** (1977), no. 3, 705–714.

[22] M. Struwe, Globally regular solutions to the u^5 Klein-Gordon equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **15** (1988), no. 3, 495–513.

[23] T. Tao, Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions, *Dyn. Partial Differ. Equ.* **3** (2006), no. 2, 93–110.