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Scattering for the Defocusing, Nonlinear Schrödinger Equation With Initial Data in a Critical Space

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In this note, we prove scattering for a defocusing nonlinear Schrödinger equation with initial data lying in a critical Besov space. In addition, we obtain polynomial bounds on the scattering size as a function of the critical Besov norm.

1 Introduction

The qualitative long time behavior for the defocusing, nonlinear Schrödinger equation

$$iu_t + \Delta u = |u|^{p-1}u, \qquad u(0,x) = u_0, \tag{1.1}$$

is completely worked out in the mass-critical ($p = \frac{4}{d} + 1$) and energy-critical ($p = \frac{4}{d-2} + 1$) cases. In general, the critical L^2-based Sobolev space for (1.1) is $\dot{H}^{s_c}(\mathbb{R}^d)$, where

$$s_c = \frac{d}{2} - \frac{2}{p-1}. \tag{1.2}$$

The critical exponent (1.2) arises from the fact that if $u(t,x)$ solves (1.1), then for any $\lambda > 0$,

$$u(t,x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \tag{1.3}$$

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also solves (1.1), and the $\dot{H}^{s_c}$ norm of the initial data is invariant under (1.3). On the other hand, well-posedness fails for $s < s_c$, see [2]. Global well-posed and scattering has been established for any initial data $u_0 \in \dot{H}^{s_c}(\mathbb{R}^d)$ in the mass-critical ([4], [5], [6], [7], [26], [16], [18]) and energy-critical cases ([1], [3], [23], [27], [17], [24]).

Definition 1 (Global well-posedness and scattering). In this paper, global well-posedness refers to the existence of a global strong solution, that is, a solution that satisfies Duhamel's principle

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}|u(\tau)|^{p-1}u(\tau)\mathrm{d}\tau, \tag{1.4}$$

that is continuous in time, and depends continuously on the initial data. Scattering refers to the existence of $u_+, u_- \in \dot{H}^{s_c}(\mathbb{R}^d)$ such that

$$\lim_{t\nearrow\infty} \|u(t) - e^{it\Delta}u_+\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = 0, \tag{1.5}$$

and

$$\lim_{t\searrow-\infty} \|u(t) - e^{it\Delta}u_-\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = 0. \tag{1.6}$$

See Chapter three of [25] for a detailed treatment of global well-posedness and scattering for dispersive partial differential equations in general.

The case when $p = \frac{4}{d} + 1$, ($s_c = 0$), is called mass-critical because a solution to (1.1) preserves the mass, or L^2 norm of a solution,

$$M(u(t)) = \int |u(t,x)|^2\mathrm{d}x = M(u(0)). \tag{1.7}$$

Likewise, the case when $p = \frac{4}{d-2} + 1$, ($s_c = 1$) is called energy critical because a solution to (1.1) preserves the energy,

$$E(u(t)) = \int [\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+1}|u(t,x)|^{p+1}]\mathrm{d}x = E(u(0)). \tag{1.8}$$

The conserved quantities (1.7) and (1.8) imply that the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is uniformly bounded for the entire time of existence of the solution to (1.1) in the mass-critical and energy-critical cases. Thus, in the mass-critical and energy-critical cases, the proof of global well-posedness and scattering reduces to proving global well-posedness and scattering for a solution to (1.1) with uniformly bounded $\dot{H}^{s_c}(\mathbb{R}^d)$ norm, which has been done.

Remark 1. In the energy-critical case, the Sobolev embedding theorem implies that $E(u(0)) < \infty$ when $u_0 \in \dot{H}^1(\mathbb{R}^d)$.

It is conjectured that global well-posedness and scattering also hold for (1.1) when $0 < s_c < 1$. In this case, there is no known conserved quantity that gives uniform bounds on the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm of a solution to (1.1). Therefore, there are two possible ways in which a solution to (1.1) might fail to scatter, which are called type one blowup and type two blowup. A solution to (1.1) is called a type one blowup solution to (1.1) if the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is not uniformly bounded. Since $e^{it\Delta}$ is a unitary operator, an unbounded $\dot{H}^{s_c}(\mathbb{R}^d)$ norm automatically precludes (1.5) or (1.6) from occurring. A blowup solution to (1.1) is called a type two blowup solution if the solution fails to scatter, but the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is uniformly bounded for the entire time of its existence.

Type one blowup is known to occur for solutions to (1.1) for some d and $s_c > 1$, see [19]. Interestingly, the solutions obtained in [19] have good regularity and good decay. Specifically, they would belong to the critical Besov spaces considered here. By comparison, when $0 < s_c < 1$, if $u_0 \in H_x^1(\mathbb{R}^d)$, where H_x^1 is an inhomogeneous Sobolev space, then (1.7), (1.8), and interpolation imply a uniform bound on the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm when $0 < s_c < 1$. In fact, global well-posedness and scattering is known for a solution to (1.1) when $u_0 \in H_x^1(\mathbb{R}^d)$ and $0 < s_c < 1$, see [12] and [11].

Type two blowup has been precluded in many cases for (1.1) when $0 < s_c < 1$. One particularly important case is the cubic nonlinear Schrödinger equation in three dimensions (see [10]),

$$iu_t + \Delta u = |u|^2 u, \qquad u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}, \qquad u(0,x) = u_0. \tag{1.9}$$

In this case, $s_c = \frac{3}{2} - \frac{2}{2} = \frac{1}{2}$. In a breakthrough result, [14] proved that any solution to (1.9) with $\|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ uniformly bounded on its entire interval of existence must be globally well-posed and scattering. Thus, the obstacle to proving scattering for (1.9) with generic initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$ is the absence of a conservation law that controls the $\dot{H}^{1/2}$ norm of a solution to (1.9) with initial data in $\dot{H}^{1/2}$. Observe that the momentum

$$P(u(t)) = \int Im[\overline{u(t,x)} \nabla u(t,x)] dx, \tag{1.10}$$

is conserved and scales like the $\dot{H}^{1/2}$ norm, but does not control the $\dot{H}^{1/2}$ norm of a solution to (1.1).

The papers [1], [13], and [3] were key in developing the concentration compactness method for the nonlinear Schrödinger equation in the energy-critical case. Type two blowup was later precluded for a great many cases of (1.1) when $0 < s_c < 1$, see [22], [21], and [20]. Since the mass-critical and energy-critical problems reduce to type two blowup questions, the same techniques are useful for both problems.

In this paper, we continue the study of (1.1), $0 < s_c < 1$, with initial data lying in a subspace of the critical Sobolev space. Previously, in [9], we proved the following:

Theorem 1. The cubic nonlinear Schrödinger equation, (1.1) with $p = 3$ and $d = 3$, is globally well-posed for initial data $u_0 \in \dot{W}_x^{\frac{7}{6},\frac{11}{7}}(\mathbb{R}^3)$. No symmetry assumption is made on the initial data, but we did not prove scattering.

Here we prove scattering for the cubic problem with initial data in $\dot{B}^2_{1,1}(\mathbb{R}^3)$. Observe that by the Sobolev embedding theorem,

$$\dot{B}^2_{1,1}(\mathbb{R}^3) \subset \dot{W}_x^{\frac{7}{6},\frac{11}{7}}(\mathbb{R}^3) \subset \dot{H}^{1/2}(\mathbb{R}^3). \tag{1.11}$$

Remark 2. Throughout this paper, a Besov space always refers to a homogeneous Besov space, and a $W^{s,p}$ Sobolev space always refers to a homogeneous Sobolev space when $p \neq 2$. Therefore, for those cases (but not for an L^2-based Sobolev space), the dot will be dropped. Thus, we understand that $B^s_{p,q}$ refers to $\dot{B}^s_{p,q}$ and $W^{s,p}$ refers to $\dot{W}^{s,p}$ when $p \neq 2$.

Theorem 2. The initial value problem

$$iu_t + \Delta u = |u|^2 u, \qquad u(0,x) = u_0, \tag{1.12}$$

with radially symmetric initial data $u_0 \in B^2_{1,1}(\mathbb{R}^3)$ has a global solution that scatters. That is,

$$\|u\|_{L_t^8 L_x^4(\mathbb{R}\times\mathbb{R}^3)} < \infty. \tag{1.13}$$

Sketch of proof of Theorem 2. Previously, in [9], we proved global well-posedness for the cubic problem in three dimensions, (1.1) with $d = 3$ and $p = 3$ with initial data

$u_0 \in W_x^{\frac{7}{6},\frac{11}{7}}$,

$$\||\nabla|^{\frac{11}{7}} u_0\|_{L^{7/6}} < \infty. \tag{1.14}$$

By the Sobolev embedding theorem, $B^2_{1,1} \subset W^{\frac{7}{6},\frac{11}{7}}$, so global well-posedness follows from Theorem 1. Moreover, the solution to the cubic problem for $t > 1$ is of the form

$$u(t) = e^{i(t-1)\Delta} u(1) + v(t), \qquad \|v\|_{L_t^\infty \dot{H}^1([1,\infty)\times\mathbb{R}^3)} < \infty. \tag{1.15}$$

Remark 3. It is possible to prove similar results for all equations described in Theorem 3. No radial symmetry assumptions are needed to prove global well-posedness. ■

The proof of scattering in the cubic case uses the conformal energy of v,

$$\mathcal{E}(t) = \|(x + 2it\nabla)v\|_{L^2}^2 + 2t^2\|v\|_{L^4}^4. \tag{1.16}$$

where v is obtained by localizing $u(1)$ in frequency and space.

Rewriting,

$$iv_t + \Delta v = |u|^2 u = |v|^2 v + (|u|^2 u - |v|^2 v) = |v|^2 v + \mathcal{N}, \tag{1.17}$$

$$\frac{d}{dt}\mathcal{E}(t) \leq -2\langle (x + 2it\nabla)v, i(x + 2it\nabla)\mathcal{N}\rangle - 8t^2\langle |v|^2 v, i\mathcal{N}\rangle. \tag{1.18}$$

Then we prove

$$\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 dt < \infty, \tag{1.19}$$

which implies $\|v\|_{L_t^8 L_x^4([1,\infty)\times\mathbb{R}^3)} < \infty$. The proof of this fact strongly utilizes the radial symmetry. By the radial Sobolev embedding theorem, $\|xw\|_{L^\infty} < \infty$, where $w = e^{i(t-1)\Delta}u(1)$. Thus,

$$\langle (x + 2it\nabla)v, x|w|^2 w\rangle \lesssim \|w\|_{L^4}^2 \mathcal{E}(t)^{1/2}. \tag{1.20}$$

By Fubini's theorem, the contribution of (1.20) to (1.19) is bounded by

$$\int \frac{1}{t^4}(\int_1^t \|w(s)\|_{L^4}^2 \mathcal{E}(s)^{1/2} ds)dt \leq \int \frac{1}{t^3}\int_1^t \|w(s)\|_{L^4}^4 \mathcal{E}(s)ds$$

$$= \int_1^\infty \|w(s)\|_{L^4}^4 \mathcal{E}(s)\int_s^\infty \frac{1}{t^3}dt \lesssim \int \|w(s)\|_{L^4}^4 \frac{1}{s^2}\mathcal{E}(s)ds \lesssim (\int \|w(s)\|_{L^4}^8 ds)^{1/2}(\int \frac{1}{s^4}\mathcal{E}(s)^2 ds)^{1/2}. \tag{1.21}$$

Since w is a solution to the linear equation with initial data in $\dot{H}^{1/2}$, $w \in L_t^8 L_x^4$, so the contribution of this term to (1.19) is fine.

We prove similar bounds for the other terms in (1.18), as well as proving similar integral bounds for all equations in Theorem 3.

We also prove Theorem 2 in a more general context, namely for (1.1) when $0 < s_c < 1$, $1 < p \leq 3$, and the initial data are radially symmetric and in the critical Besov space $B_{1,1}^{\frac{d}{2}+s_c}(\mathbb{R}^d)$.

Theorem 3. The initial value problem (1.1) is globally well-posed and scattering for radially symmetric initial data in the Besov space $B_{1,1}^{\frac{d}{2}+s_c}(\mathbb{R}^d)$. In addition, when $1 < p < 3$, the scattering size,

$$\|u\|_{L_t^{\frac{p+1}{1-s_c}} L_x^{p+1}(\mathbb{R}\times\mathbb{R}^d)}, \tag{1.22}$$

is bounded by a polynomial function of $\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}$.

2 Preliminaries

In this section, we discuss some preliminary information that will be needed in the rest of the paper. Nothing in this section is new.

Definition 2 (Besov space). The Besov space $B_{p,q}^s(\mathbb{R}^d)$ is given by the norm

$$\|u_0\|_{B_{p,q}^s(\mathbb{R}^d)} = (\sum_j 2^{jps}\|P_j u_0\|_{L^q}^p)^{1/p}, \tag{2.1}$$

when $1 \leq p < \infty$, with the usual modification when $p = \infty$. Here, P_j is the usual Littlewood–Paley projection operator. The Sobolev embedding theorem implies that $B_{1,1}^{\frac{d}{2}+s_c}(\mathbb{R}^d) \subset \dot{H}^{s_c}(\mathbb{R}^d)$. The $B_{1,1}^{\frac{d}{2}+s_c}(\mathbb{R}^d)$ norm is invariant under the scaling symmetry (1.3).

The fact that the bound (1.22) implies scattering is a consequence of Strichartz estimates.

Theorem 4. If u is a solution to (1.1), the bound $\|u\|_{L_t^{\frac{p+1}{1-s_c}} L_x^{p+1}(\mathbb{R}\times\mathbb{R}^d)} < \infty$ implies scattering.

Proof. For the Schrödinger equation in dimensions $d \geq 3$,

$$iu_t + \Delta u = F, \qquad u(0,x) = u_0, \qquad u : I \times \mathbb{R}^d \to \mathbb{C}, \tag{2.2}$$

we have the Strichartz estimate

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}} \cap L_t^\infty L_x^2(I\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2} + \|F\|_{L_t^1 L_x^2 + L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}. \tag{2.3}$$

See [25] and the references therein for a detailed treatment of this topic.

In particular, when $F = 0$, (2.3) implies a bound on $\|u\|_{L_t^p L_x^q}$, when (p,q) is an admissible pair, that is,

$$\frac{2}{p} = d(\frac{1}{2} - \frac{1}{q}), \qquad p \geq 2. \tag{2.4}$$

Then by the Sobolev embedding theorem, if $F = 0$,

$$\|u\|_{L_t^p L_x^r} \lesssim \|u_0\|_{\dot{H}^s}, \qquad \text{for} \qquad \frac{1}{r} = \frac{1}{q} - \frac{s}{d}, \qquad (p,q) \qquad \text{is admissible.} \tag{2.5}$$

The pair (p,r) is then said to be s-admissible. Doing some algebra, $(\frac{p+1}{1-s_c}, p+1)$ is s_c-admissible. Since $\frac{p+1}{1-s_c} < \infty$, a bound on (1.22) on $\mathbb{R}\times\mathbb{R}^d$ implies scattering for (1.1). Indeed, since $\frac{p+1}{1-s_c} < \infty$, it is possible to partition $\mathbb{R}$ into finitely many intervals I_j such that

$$\|u\|_{L_t^{\frac{p+1}{1-s_c}} L_x^{p+1}(I_j\times\mathbb{R}^d)} \leq \epsilon, \qquad \text{for some} \qquad \epsilon \ll 1. \tag{2.6}$$

Now use the Strichartz space,

$$\|u\|_{\dot{S}^{s_c}(I\times\mathbb{R}^d)} = \||\nabla|^{s_c} u\|_{L_t^2 L_x^{\frac{2d}{d-2}} \cap L_t^\infty L_x^2(I\times\mathbb{R}^d)}. \tag{2.7}$$

By (2.3), if $I_j = [a_j, b_j]$,

$$\|u\|_{\dot{S}^{s_c}(I_j\times\mathbb{R}^d)} \lesssim \|u(a_j)\|_{\dot{H}^{s_c}(\mathbb{R}^3)} + \|u\|_{\dot{S}^{s_c}(I_j\times\mathbb{R}^d)}\|u\|^{p-1}_{L_t^{\frac{p+1}{1-s_c}}L_x^{p+1}(I_j\times\mathbb{R}^d)}. \tag{2.8}$$

Therefore, by (2.6),

$$\|u\|_{\dot{S}^{s_c}(I_j\times\mathbb{R}^d)} \lesssim \|u(a_j)\|_{\dot{H}^{s_c}(\mathbb{R}^3)}. \tag{2.9}$$

In particular,

$$\|u(b_j)\|_{\dot{H}^{s_c}(\mathbb{R}^3)} \lesssim \|u(a_j)\|_{\dot{H}^{s_c}(\mathbb{R}^3)}. \tag{2.10}$$

Since there are finitely many intervals I_j, we have $\|u\|_{\dot{S}^{s_c}(\mathbb{R}\times\mathbb{R}^3)} < \infty$ and thus scattering, by taking

$$u_+ = u_0 - i\int_0^\infty e^{-it\Delta}|u|^{p-1}u\,dt, \tag{2.11}$$

and

$$u_- = u_0 - i\int_0^{-\infty} e^{-it\Delta}|u|^{p-1}u\,dt. \tag{2.12}$$

■

Finally, we conclude with the pseudoconformal conservation law.

Theorem 5 (Pseudoconformal conservation law). If u solves (1.1) on $\mathbb{R}\times\mathbb{R}^d$,

$$\|(x+2it\nabla)u(t)\|_{L^2}^2 + \frac{8t^2}{p+1}\int |u(t,x)|^{p+1}dx = \|xu_0\|_{L^2}^2 + \int_0^t 4s\left(\int \frac{4-d(p-1)}{p+1}|u(s,x)|^{p+1}dx\right)ds. \tag{2.13}$$

Proof. See, for example, Section 1.4 in [7]. ■

3 Scattering for the Cubic NLS in Three Dimensions

We begin by proving scattering for the cubic equation (1.3) with $u_0 \in B^2_{1,1}(\mathbb{R}^3)$, before moving on to the general problem. In this section, we do not prove any quantitative bounds on the scattering size as a function of the $B^2_{1,1}$ norm of the initial data.

Proof of Theorem 2. By time reversal symmetry, it suffices to prove scattering on $[0,\infty)$. In [9], we proved that the cubic nonlinear Schrödinger equation is globally well-posed for initial data $u_0 \in W^{\frac{7}{6},\frac{11}{7}}(\mathbb{R}^3)$. By the Sobolev embedding theorem, $B^2_{1,1}(\mathbb{R}^3) \subset W^{\frac{7}{6},\frac{11}{7}}(\mathbb{R}^3)$, so global well-posedness follows.

Furthermore, after rescaling the initial data, suppose that the global solution has the form

$$\|u\|_{L^5_{t,x}([0,1]\times\mathbb{R}^3)} \leq \delta, \qquad \text{which implies} \qquad \|u\|_{\dot{S}^{1/2}([0,1]\times\mathbb{R}^3)} < \infty. \tag{3.1}$$

Then for $1 \leq t < \infty$, decompose

$$u(t) = w(t) + v(t), \qquad \text{where} \qquad w(t) = e^{it\Delta}u_0^{(1)}, \tag{3.2}$$

and $u_0 = u_0^{(1)} + u_0^{(2)}$ is some decomposition of u_0 that will be specified later.

Let $\mathcal{E}(t)$ denote the conformal energy of v,

$$\mathcal{E}(t) = \|(x+2it\nabla)v\|_{L^2}^2 + 2t^2\|v\|_{L^4}^4 = \|xv\|_{L^2}^2 + 2\langle xv, 2it\nabla v\rangle_{L^2} + 8t^2E(t), \tag{3.3}$$

where $E(t)$ is the energy in (1.8),

$$E(t) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{4}\|v\|_{L^4}^4. \tag{3.4}$$

When $w = 0$,

$$\frac{d}{dt}\mathcal{E}(t) = -2t\|v\|_{L^4}^4, \tag{3.5}$$

which implies $\|v\|_{L^4_x(\mathbb{R}^3)}^4 \lesssim \frac{1}{t^2}$. Therefore,

$$\|u\|_{L^8_t L^4_x([1,\infty)\times\mathbb{R}^3)} = \|v\|_{L^8_t L^4_x([1,\infty)\times\mathbb{R}^3)} < \infty. \tag{3.6}$$

For a general $u_0^{(1)} \in \dot{H}^{1/2}(\mathbb{R}^3)$, Strichartz estimates imply that

$$\|w\|_{L^8_t L^4_x(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)}, \tag{3.7}$$

so to prove scattering, it suffices to prove

$$\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 dt < \infty. \tag{3.8}$$

Indeed, by (3.3),

$$\int_1^\infty \|v\|_{L^4}^8 \mathrm{d}t \lesssim \int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 \mathrm{d}t. \tag{3.9}$$

By Duhamel's principle,

$$v(1) = -i\int_0^1 e^{i(1-\tau)\Delta}|u|^2 u \mathrm{d}\tau + e^{i1\Delta}u_0^{(2)}. \tag{3.10}$$

By direct computation,

$$(x+2i\nabla)\int_0^1 e^{(1-\tau)\Delta}|u|^2 u \mathrm{d}\tau = \int_0^1 (x+2i(1-\tau)\nabla)e^{i(1-\tau)\Delta}|u|^2 u \mathrm{d}\tau + \int_0^1 2i\tau\nabla e^{i(1-\tau)\Delta}|u|^2 u \mathrm{d}\tau. \tag{3.11}$$

By examining the kernel,

$$e^{it\Delta}f = \frac{C}{t^{3/2}}\int e^{-i\frac{|x-y|^2}{4t}} f(y)\mathrm{d}y, \qquad (x+2it\nabla)e^{it\Delta}f = e^{it\Delta}xf, \tag{3.12}$$

so using the radial Sobolev embedding theorem and the computations in [9],

$$\begin{aligned}\|\int_0^1 (x+2i(1-\tau)\nabla)e^{i(1-\tau)\Delta}|u|^2 u \mathrm{d}\tau\|_{L^2} &\lesssim \|x|u|^2u\|_{L_t^1 L_x^2} \lesssim \|xu\|_{L_{t,x}^\infty}\|u\|_{L_t^8 L_x^4}^2 \\ &\lesssim \|u\|_{L_t^\infty B_{1,2}^{1/2}}\|u\|_{\dot{S}^{1/2}}^2 \lesssim \|u_0\|_{B_{1,1}^2}^3.\end{aligned} \tag{3.13}$$

Next, recall from [9] that for any $0 < t < 1$,

$$u = u_1 + u_2, \qquad \text{where} \qquad \|\nabla u_1\|_{L^2} \lesssim t^{-1/4}, \qquad \|\nabla u_2\|_{L^6} \lesssim t^{-3/4}, \tag{3.14}$$

with constant independent of t. Therefore, by Strichartz estimates,

$$\|\int_0^1 2i\tau\nabla e^{i(1-\tau)\Delta}|u|^2 u \mathrm{d}\tau\|_{L^2} \lesssim \|\tau\nabla u_1\|_{L_\tau^\infty L_x^2}\|u\|_{L_t^4 L_x^6}^2 + \|\tau\nabla u_2\|_{L_\tau^\infty L_x^6}\|u\|_{L_t^4 L_x^6}^2 \lesssim_{\|u_0\|_{B_{1,1}^2}} 1. \tag{3.15}$$

Now decompose the initial data. Let $\chi \in C_0^\infty(\mathbb{R}^3)$, $\chi(x) = 1$ on $|x| \leq 1$, $\chi(x)$ is supported on $|x| \leq 2$, and let $R(\epsilon, u_0) < \infty$ be a constant sufficiently large so that

$$\sum_j 2^{j/2} \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^2} \leq \epsilon, \qquad \text{and} \qquad \sum_j 2^{2j} \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^1} \leq \epsilon. \tag{3.16}$$

By Hölder's inequality and the Sobolev embedding theorem,

$$\| \nabla ((1 - \chi(\frac{x}{R})) P_j u_0) \|_{L^2} \lesssim \| P_j u_0 \|_{\dot{H}^1}, \qquad \text{and} \qquad \| \nabla^2 ((1 - \chi(\frac{x}{R})) P_j u_0) \|_{L^1} \lesssim 2^{2j} \| P_j u_0 \|_{L^1}. \tag{3.17}$$

Then,

$$\sum_j \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^2}^{1/2} \| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{\dot{H}^1}^{1/2} \lesssim \epsilon^{1/2} \| u_0 \|_{B^2_{1,1}}^{1/2}. \tag{3.18}$$

Therefore, by the radial Sobolev embedding theorem, if $\epsilon \leq \| u_0 \|_{B^2_{1,1}}^{-2}$,

$$\| |x| e^{it\Delta} (1 - \chi(\frac{x}{R})) u_0 \|_{L^\infty} \lesssim \epsilon^{1/4}. \tag{3.19}$$

Also, by Hölder's inequality,

$$\| x \chi(\frac{x}{R}) u_0 \|_{L^2} \lesssim R^{3/2} \| u_0 \|_{L^3}, \tag{3.20}$$

so (3.12) implies

$$\| (x + 2i\nabla) v(1) \|_{L^2} \lesssim R^{3/2} \| u_0 \|_{B^2_{1,1}}. \tag{3.21}$$

The computations in [9] also imply

$$\| v(1) \|_{L^4}^4 \lesssim 1, \tag{3.22}$$

and therefore,

$$\mathcal{E}(1) \lesssim_{\| u_0 \|_{B^2_{1,1}}, R} 1. \tag{3.23}$$

To obtain the bound (3.8), observe that v solves

$$iv_t + \Delta v = |u|^2 u, \qquad v(1,x) = (3.10), \tag{3.24}$$

and w solves

$$iw_t + \Delta w = 0, \qquad w(1,x) = e^{i1\Delta} u_0^{(1)}, \tag{3.25}$$

on $[1,\infty)$.

Rearranging (3.24),

$$-\Delta v + |v|^2 v = iv_t - F, \qquad F = 2|v|^2 w + v^2 \bar{w} + 2|w|^2 v + w^2 \bar{v} + |w|^2 w = F_1 + F_2 + F_3. \tag{3.26}$$

Integrating by parts,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= 16tE(v) + 8t^2\langle v_t, -\Delta v + |v|^2 v\rangle + 4\langle xv, i\nabla v\rangle + 4t\langle xv_t, i\nabla v\rangle + 4t\langle xv, i\nabla v_t\rangle \\ &+2\langle ix\Delta v, xv\rangle - 2\langle ixF, xv\rangle = -2t\|v\|_{L^4}^4 + 8t^2\langle v_t, F\rangle - 4t\langle xF, \nabla v\rangle + 4t\langle xv, \nabla F\rangle - 2\langle ixF, xv\rangle.\end{aligned} \tag{3.27}$$

Integrating by parts and plugging in (3.26), with $F_3 = |w|^2 w$,

$$\begin{aligned}&8t^2\langle v_t, F_3\rangle - 4t\langle xF_3, \nabla v\rangle + 4t\langle xv, \nabla F_3\rangle - 2\langle ixF_3, xv\rangle \\ &= 2\langle (x+2it\nabla)|w|^2 w, i(x+2it\nabla)v\rangle_{L^2} + O(t^2\langle |v|^3 + |w|^3, |w|^3\rangle) \\ &\lesssim \|(x+2it\nabla)v\|_{L^2}\|xw\|_{L^\infty}\|w\|_{L^4}^2 + \|(x+2it\nabla)v\|_{L^2}\|t\nabla w\|_{L^\infty}\|w\|_{L^4}^2 \\ &\quad + t^2\|v\|_{L^4}^3\|w\|_{L^4}\|w\|_{L^\infty}^2 + t^2\|w\|_{L^6}^6 \\ &\lesssim \mathcal{E}(t)^{1/2}\|w\|_{L^4}^2(\|xw\|_{L^\infty} + \|t\nabla w\|_{L^\infty}) + t^{3/8}\mathcal{E}(t)^{3/4}\|w\|_{L^\infty}^2 + t^{3/2}\|w\|_{L^\infty}^2.\end{aligned} \tag{3.28}$$

Also, integrating by parts and plugging in (3.26) with $F_2 = 2|w|^2 v + w^2\bar{v}$,

$$\begin{aligned}&8t^2\langle v_t, F_2\rangle - 4t\langle xF_2, \nabla v\rangle + 4t\langle xv, \nabla F_2\rangle - 2\langle ixF_2, xv\rangle \\ &= 2\langle (x+2it\nabla)F_2, i(x+2it\nabla)v\rangle_{L^2} + O(t^2\langle |v|^3 + |w|^3, |w|^2|v|\rangle) \\ &\lesssim \|(x+2it\nabla)v\|_{L^2}\|xw\|_{L^\infty}\|w\|_{L^4}\|v\|_{L^4} + \|(x+2it\nabla)v\|_{L^2}\|t\nabla w\|_{L^\infty}\|w\|_{L^4}\|v\|_{L^4} \\ &\quad + \|(x+2it\nabla)v\|_{L^2}^2\|w\|_{L^\infty}^2 + t^2\|v\|_{L^4}^4\|w\|_{L^\infty}^2 + t^2\|w\|_{L^6}^6 \\ &\lesssim t^{-1/2}\mathcal{E}(t)^{3/4}\|w\|_{L^4} + t^{3/2}\|w\|_{L^\infty}^2 + \mathcal{E}(t)\|w\|_{L^\infty}^2.\end{aligned} \tag{3.29}$$

Finally, take

$$8t^2\langle v_t, F_1\rangle - 4t\langle xF_1, \nabla v\rangle + 4t\langle xv, \nabla F_1\rangle - 2\langle ixF_1, xv\rangle, \tag{3.30}$$

with $F_1 = 2|v|^2 w + v^2\bar{w}$. This term will be handled slightly differently from (3.28) and (3.29). By (3.19),

$$-4t\langle xF_1, \nabla v\rangle - 2\langle ixF_1, xv\rangle = -2\langle ixF_1, (x+2it\nabla)v\rangle \lesssim \|(x+2it\nabla)v\|_{L^2}\|v\|_{L^4}^2\|xw\|_{L^\infty} \lesssim \frac{\epsilon^{1/4}}{t}\mathcal{E}(t). \tag{3.31}$$

Next, integrating by parts,

$$\begin{aligned} 4t\langle xv, \nabla F_1\rangle &= -4t\langle x\nabla v, F_1\rangle - 12t\langle v, F_1\rangle = -4t\langle xw, \nabla(|v|^2 v)\rangle - 12t\langle v, F_1\rangle \\ &\lesssim t\|w\|_{L^4}\|v\|_{L^4}^3 + 4t\langle \nabla w, x|v|^2 v\rangle \lesssim t^{-1/2}\|w\|_{L^4}\mathcal{E}(t)^{3/4} + 4t\langle \nabla w, x|v|^2 v\rangle. \end{aligned} \tag{3.32}$$

Then by the product rule, integrating by parts, and (3.17),

$$\begin{aligned} 4t\langle \nabla w, x|v|^2 v\rangle &= 8t\langle \nabla w, |v|^2(x+2it\nabla)v\rangle - 4t^2\langle \nabla w, v^2(x-2it\nabla)\bar{v}\rangle - 8t\langle \nabla w, i\nabla(|v|^2 v)\rangle \\ &\lesssim t\|\nabla w\|_{L^\infty}\|(x+2it\nabla)v\|_{L^2}\|v\|_{L^4}^2 - 8t^2\langle i\Delta w, |v|^2 v\rangle \\ &\lesssim \|\nabla w\|_{L^\infty}\mathcal{E}(t) + t\|\Delta w\|_{L^\infty}\|v\|_{L^2}\mathcal{E}(t)^{1/2} \lesssim \|\nabla w\|_{L^\infty}\mathcal{E}(t) + t^{-1/2}\|v\|_{L^2}\mathcal{E}(t)^{1/2}. \end{aligned} \tag{3.33}$$

Meanwhile, integrating by parts in t,

$$\int_1^T 8t^2\langle v_t, F_1\rangle dt = 8t^2\langle |v|^3, |w|\rangle|_1^T - \int_1^T 8t^2\langle |v|^2 v, w_t\rangle - \int_1^T 16t\langle |v|^2 v, w\rangle dt. \tag{3.34}$$

First observe that

$$8t^2\langle |v|^3, |w|\rangle|_1^T \lesssim t^{1/2}\|w\|_{L^4}\mathcal{E}(t)^{3/4}|_1^T. \tag{3.35}$$

Also compute

$$\begin{aligned} 8t^2\langle |v|^2 v, w_t\rangle &\lesssim t\|\Delta w\|_{L^\infty}\|v\|_{L^2}\mathcal{E}(t)^{1/2} \lesssim t^{-1/2}\|v\|_{L^2}\mathcal{E}(t)^{1/2}, \\ \text{and} \qquad 16t\langle |v|^2 v, w\rangle &\lesssim t^{-1/2}\mathcal{E}(t)^{3/4}\|w\|_{L^4}. \end{aligned} \tag{3.36}$$

Therefore,

$$\mathcal{E}(t) \lesssim \int_1^t [\mathcal{E}(s)^{1/2}\|w\|_{L^4}^2 + s^{3/8}\mathcal{E}(s)^{3/4}\|w\|_{L^\infty}^2 + s^{3/2}\|w\|_{L^\infty}^2 + \mathcal{E}(s)\|w\|_{L^\infty}^2$$

$$+\frac{\epsilon}{s}\mathcal{E}(s) + \|\nabla w\|_{L^\infty}\mathcal{E}(s) + s^{-1/2}\mathcal{E}(s)^{1/2}\|v\|_{L^2} + s^{-1/2}\mathcal{E}(s)^{3/4}\|w\|_{L^4}]\mathrm{d}s + t^{1/2}\|w\|_{L^4}\mathcal{E}(t)^{3/4} + R. \tag{3.37}$$

By Fubini's theorem and Hölder's inequality

$$\int_1^\infty \frac{1}{t^4}(\int_1^t \mathcal{E}(s)^{1/2}\|w\|_{L^4}^2 \mathrm{d}s)^2 \mathrm{d}t \lesssim \int_1^\infty \frac{1}{t^3}(\int_1^t \mathcal{E}(s)\|w\|_{L^4}^4 \mathrm{d}s)\mathrm{d}t = \int_1^\infty \mathcal{E}(s)\|w(s)\|_{L^4}^4 \int_s^\infty \frac{1}{t^3}\mathrm{d}t\mathrm{d}s$$

$$\lesssim \int_1^\infty \frac{1}{s^2}\mathcal{E}(s)\|w\|_{L^4}^4 \mathrm{d}s \lesssim (\int_1^\infty \frac{1}{s^4}\mathcal{E}(s)^2 \mathrm{d}s)^{1/2}(\int_1^\infty \|w\|_{L^4}^8 \mathrm{d}s)^{1/2}. \tag{3.38}$$

Next, interpolating (3.16) and (3.17),

$$\|\nabla e^{it\Delta}(1-\chi(\frac{x}{R}))P_j u_0\|_{L^\infty}$$

$$\lesssim \inf\{t^{-3/2}2^{-j}\cdot 2^j\|(1-\chi(\frac{x}{R}))(P_j u_0)\|_{L^1}^{1/2}\|\nabla^2(1-\chi(\frac{x}{R}))(P_j u_0)\|_{L^1}^{1/2}, 2^j\|\nabla^2(1-\chi(\frac{x}{R}))(P_j u_0)\|_{L^1}\}, \tag{3.39}$$

which by (3.18) implies that for $\epsilon \leq \|u_0\|_{B^2_{1,1}}^{-8}$,

$$\int_0^\infty \|\nabla w\|_{L^\infty}\mathrm{d}t \lesssim \epsilon^{1/4}\|u_0\|_{B^2_{1,1}}^{3/4} \lesssim \epsilon^{5/32}. \tag{3.40}$$

Similar computations also show that

$$\|w\|_{L_t^2 L_x^\infty} \lesssim \epsilon^{3/8}, \qquad \text{and} \qquad \|w\|_{L^\infty} \lesssim \frac{\epsilon^{3/8}}{s^{1/2}}. \tag{3.41}$$

Therefore,

$$
\begin{aligned}
\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 dt &\lesssim \int_1^\infty \frac{R^2}{t^4} dt + \int_1^\infty \frac{1}{t^3}\mathcal{E}(t)^{3/2}\|w\|_{L^4}^2 dt + \int_1^\infty \frac{1}{t^4}(\int_1^t \mathcal{E}(s)^{1/2}\|w\|_{L^4}^2 ds)^2 dt \\
&+ \int_1^\infty \frac{1}{t^4}(\int_1^t s^{3/8}\mathcal{E}(s)^{3/4}\|w\|_{L^\infty}^2 + s^{3/2}\|w\|_{L^\infty}^2 + s^{1/4}\mathcal{E}(s)^{1/2}\|w\|_{L^\infty} ds)^2 dt \\
&+ \int_1^\infty \frac{1}{t^4}(\int_1^t \mathcal{E}(s)\|w\|_{L^\infty}^2 + \frac{\epsilon}{s}\mathcal{E}(s) + s^{-1/2}\mathcal{E}(s)^{1/2}\|v\|_{L^2} + s^{-1/2}\mathcal{E}(s)^{3/4}\|w\|_{L^4} ds)^2 dt \\
&\lesssim R^2 + (\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 dt)^{3/4}(\int_1^\infty \|w\|_{L^4}^8 dt)^{1/4} + (\int_1^\infty \frac{1}{s^4}\mathcal{E}(s)^2 ds)^{1/2}(\int_1^\infty \|w\|_{L^4}^8 ds)^{1/2} \\
&+ (\int_1^\infty \frac{1}{s^4}\mathcal{E}(s)^2 ds)^{3/4}(\int_1^\infty s^7\|w\|_{L^\infty}^{16} ds)^{1/4} + (\int_1^\infty s\|w\|_{L^\infty}^4 ds) \\
&+ (\int_1^\infty \frac{1}{s^4}\mathcal{E}(s)^2 ds)^{1/2}(\int_1^\infty \|w\|_{L^\infty}^4 ds)^{1/2} \\
&+ \epsilon^{5/16}(\int_1^\infty \frac{1}{s^4}\mathcal{E}(s)^2 ds) + (\int_1^\infty \frac{1}{s^4}\mathcal{E}(s)^2 ds)^{1/2}(\int_1^\infty \frac{1}{s^2}\|v(s)\|_{L^2}^4 ds)^{1/2}.
\end{aligned}
\tag{3.42}
$$

Therefore,

$$
\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 dt \lesssim R^2 + \int_1^\infty \|w\|_{L^4}^8 dt + \int_1^\infty \|w\|_{L^\infty}^2 dt + (\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 dt)^{1/2}(\int_1^\infty \frac{1}{t^2}\|v(t)\|_{L^2}^4 dt)^{1/2}.
\tag{3.43}
$$

Now since v solves (3.24),

$$
\frac{d}{dt}\|v\|_{L^2}^2 \lesssim \|w\|_{L^\infty}\|v\|_{L^4}^2\|v\|_{L^2}^2 + \|w\|_{L^\infty}\|w\|_{L^4}^2\|v\|_{L^2}, \qquad \|v(1)\|_{L^2}^2 \lesssim R.
\tag{3.44}
$$

Therefore, by Hölder's inequality,

$$
\begin{aligned}
\|v(t)\|_{L^2}^4 &\lesssim R^2 + (\int_1^t \|w\|_{L^\infty}\|v\|_{L^4}^2\|v\|_{L^2} + \|w\|_{L^\infty}\|w\|_{L^4}^2\|v\|_{L^2} dt)^2 \\
&\lesssim R^2 + \|w\|_{L_t^2 L_x^\infty}^2(\int_1^t \|v\|_{L^4}^4\|v\|_{L^2}^2 + \|w\|_{L^4}^4\|v\|_{L^2}^2).
\end{aligned}
\tag{3.45}
$$

Therefore, by Fubini's theorem,

$$\begin{aligned}\int_1^\infty \frac{1}{t^2}\|v(t)\|_{L^2}^4 \mathrm{d}t \lesssim R^2 + \epsilon^2(\int_1^\infty \|v\|_{L^4}^8 \mathrm{d}t)^{1/2}(\int_1^\infty \frac{1}{t^2}\|v(t)\|_{L^2}^4 \mathrm{d}t)^{1/2} \\ +\epsilon^2\|w\|_{L_t^8 L_x^4}^4(\int_1^\infty \frac{1}{t^2}\|v(t)\|_{L^2}^4 \mathrm{d}t)^{1/2} \\ \lesssim R^2 + \epsilon^2(\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 \mathrm{d}t)^{1/2}(\int_1^\infty \frac{1}{t^2}\|v(t)\|_{L^2}^4 \mathrm{d}t)^{1/2} \\ +\epsilon^2\|w\|_{L_t^8 L_x^4}^4(\int_1^\infty \frac{1}{t^2}\|v(t)\|_{L^2}^4 \mathrm{d}t)^{1/2}.\end{aligned} \tag{3.46}$$

Therefore,

$$\int_1^\infty \frac{1}{t^2}\|v(t)\|_{L^2}^4 \mathrm{d}t \lesssim R^2 + \epsilon^2(\int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 \mathrm{d}t) + \epsilon^2 \int_1^\infty \|w\|_{L^4}^8 \mathrm{d}t. \tag{3.47}$$

Plugging (3.47) into (3.43),

$$\int_1^\infty \|v(t)\|_{L^4}^8 \mathrm{d}t \lesssim \int_1^\infty \frac{1}{t^4}\mathcal{E}(t)^2 \mathrm{d}t \lesssim R^2 + \int_1^\infty \|w\|_{L^4}^8 \mathrm{d}t + \int_1^\infty \|w\|_{L^\infty}^2 \mathrm{d}t. \tag{3.48}$$

Therefore, scattering follows. ■

4 Concentration Compactness in the Cubic Case

The proof of Theorem 4 implies that for a solution u to (1.9),

$$\|u\|_{L_t^8 L_x^4(\mathbb{R}\times\mathbb{R}^3)} < \infty, \qquad \text{is equivalent to} \qquad \|u\|_{L_{t,x}^5(\mathbb{R}\times\mathbb{R}^3)} < \infty. \tag{4.1}$$

Thus, Theorem 2 implies that for $u_0 \in B_{1,1}^2$, (1.9) has a global solution satisfying $\|u\|_{L_{t,x}^5(\mathbb{R}\times\mathbb{R}^3)} < \infty$. However, since R depends on $\epsilon > 0$ and u_0, not just the norm $\|u_0\|_{B_{1,1}^2}$, (3.48) does not directly give a uniform bound on

$$\|u\|_{L_{t,x}^5(\mathbb{R}\times\mathbb{R}^3)}, \qquad \text{when} \qquad \|u_0\|_{B_{1,1}^2} \leq A < \infty. \tag{4.2}$$

Such a bound follows from a concentration compactness argument, as in [8] for the nonlinear wave equation.

Following by now standard concentration compactness techniques, see for example [15],

Lemma 1. Let u_n be a bounded sequence in $\dot{H}^{1/2}$,

$$\sup_n \|u_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq A < \infty, \tag{4.3}$$

which is radially symmetric. After passing to a subsequence, assume that

$$\lim_{n\to\infty} \|u_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = A. \tag{4.4}$$

Then passing to a further subsequence, for any $1 \leq J < \infty$, there exist $\phi^1,\ldots,\phi^J \in \dot{H}^{1/2}$ such that

$$u_n = \sum_{j=1}^{J} e^{it_n^j(\lambda_n^j)^2\Delta} \frac{1}{\lambda_n^j}\phi^j(\frac{x}{\lambda_n^j}) + w_n^J, \tag{4.5}$$

where

$$\sum_{j=1}^{J} \|\phi^j\|_{\dot{H}^{1/2}}^2 + \lim_{n\to\infty} \|w_n^J\|_{\dot{H}^{1/2}}^2 = A^2, \tag{4.6}$$

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e^{it\Delta} w_n^J\|_{L_{t,x}^5(\mathbb{R}\times\mathbb{R}^3)} = 0, \tag{4.7}$$

and for $j \neq k$,

$$\lim_{n\to\infty} |\ln(\frac{\lambda_n^j}{\lambda_n^k})| + |t_n^j - t_n^k| = \infty. \tag{4.8}$$

Now let u_n be a sequence in $B_{1,1}^2(\mathbb{R}^3)$ with the uniform bound

$$\|u_n\|_{B_{1,1}^2} \leq A. \tag{4.9}$$

Then by the Sobolev embedding theorem,

$$\|u_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \lesssim A, \tag{4.10}$$

so apply Lemma 1, and observe that for any J,

$$u_n = \sum_{j=1}^{J} e^{it_n^j(\lambda_n^j)^2\Delta} \frac{1}{\lambda_n^j}\phi^j(\frac{x}{\lambda_n^j}) + w_n^J. \tag{4.11}$$

Next, observe that Lemma 1 implies that for any fixed j,

$$e^{-it_n^j\Delta}(\lambda_n^j u_n(\lambda_n^j\cdot)) \rightharpoonup \phi^j, \qquad \text{weakly in} \qquad \dot{H}^{1/2}(\mathbb{R}^3). \tag{4.12}$$

Using dispersive estimates, for any $t \in \mathbb{R}$, since $B^2_{1,1}$ is invariant under the scaling symmetry (1.3),

$$\|e^{it\Delta}e^{-it_n^j\Delta}(\lambda_n^j u_n(\lambda_n^j\cdot))\|_{L^\infty} \lesssim \frac{1}{|t-t_n^j|^{1/2}}\|u_n\|_{B^2_{1,1}}, \tag{4.13}$$

in particular, if $t_n^j \to \pm\infty$ along a subsequence, interpolating (4.13) and the Sobolev embedding theorem $\dot{H}^{1/2} \hookrightarrow L^3$,

$$\|e^{it\Delta}e^{-it_n^j\Delta}(\lambda_n^j u_n(\lambda_n^j\cdot))\|_{L^5_{t,x}([-T,T]\times\mathbb{R}^3)} = 0, \tag{4.14}$$

for any fixed $0 < T < \infty$. Since $u_n \rightharpoonup \phi$ weakly in $\dot{H}^{1/2}$ implies

$$e^{it\Delta}u_n \rightharpoonup e^{it\Delta}\phi, \qquad \text{weakly in} \qquad L^5_{t,x}, \tag{4.15}$$

(4.14) implies that $\phi^j = 0$ if $t_n^j \to \pm\infty$ along a subsequence.

Remark 4. The fact that weak convergence implies (4.15) follows from Strichartz estimates and approximating a function in $L^{5/4}_{t,x}$ with a smooth, compactly supported function and a small remainder.

Therefore, the t_n^j's must be uniformly bounded for any j, and after passing to a subsequence, $t_n^j \to t^j \in \mathbb{R}$ for any j. Since

$$e^{it_n^j(\lambda_n^j)^2\Delta}\frac{1}{\lambda_n^j}\phi^j(\frac{x}{\lambda_n^j}) = \frac{1}{\lambda_n^j}(e^{it_n^j\Delta}\phi^j)(\frac{x}{\lambda_n^j}), \tag{4.16}$$

replacing ϕ^j with $e^{it^j\Delta}\phi^j$ and absorbing the remainder into w_n^J, it is possible to set $t_n^j \equiv 0$ for all j in (4.11). Therefore,

$$u_n = \sum_{j=1}^{J}\frac{1}{\lambda_n^j}\phi^j(\frac{x}{\lambda_n^j}) + w_n^J. \tag{4.17}$$

By Theorem 2, for any j, let u^j be the solution to (1.12) with initial data ϕ^j. Since $\lambda_n^j u_n(\lambda_n^j x) \rightharpoonup \phi^j$ weakly in $\dot{H}^{1/2}$, the Sobolev embedding theorem implies that

$\lambda_n^j u_n(\lambda_n^j x) \rightharpoonup \phi^j$ weakly in L^3. Compactly supported distributions are dense in $\dot{H}^{-1/2}$, and since $L^3 \subset L^1$ on a compact set, $\lambda_n^j u_n(\lambda_n^j x)$ converges weakly in L^1 on a compact set. Therefore, for any $k \in \mathbb{Z}$,

$$\|P_k \phi^j\|_{L^1} < \infty, \tag{4.18}$$

where P_k is the usual Littlewood–Paley projection operator, and

$$\sum_k 2^{2k} \|P_k \phi^j\|_{L^1} \lesssim \sup_n \|u_n\|_{B^2_{1,1}}. \tag{4.19}$$

Therefore, for any j, by Theorem 2,

$$\|u^j\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} < \infty. \tag{4.20}$$

Furthermore, (4.6), (4.7), (4.8), and small data arguments imply that if $u^{(n)}(t,x)$ is the solution to (1.12) with initial data $u_n(x)$,

$$\lim_{n\to\infty} \|u^{(n)}\|^5_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \leq \sum_{j=1}^{\infty} \|u^j\|^5_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} < \infty. \tag{4.21}$$

For all but finitely many j's, say all but j_0, $\|u^j\|_{L^\infty_t \dot{H}^{1/2}} \leq \epsilon$, so by small data arguments and (4.6),

$$\sum_{j \geq j_0} \|u^j\|^2_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim A. \tag{4.22}$$

Therefore, there exists a function $f : [0,\infty) \to [0,\infty)$ such that if $\|u_0\|_{B^2_{1,1}} \leq A$ is radial, then (1.12) has a global solution that satisfies the bound

$$\|u\|_{L^5_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \leq f(A) < \infty. \tag{4.23}$$

Observe that (4.23) gives no explicit bound on the scattering size. In general, the bounds obtained from a concentration compactness argument are likely far from optimal. For example, in [24],

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} \leq C \exp(CE^C), \tag{4.24}$$

where $C(d)$ is a large constant, E is the energy (1.8), and u is a solution to the energy-critical problem ($s_c = 1$) with radially symmetric initial data. In the next two sections, we will do much better with data in a subspace of the critical Sobolev space when $1 < p < 3$.

5 A Local Result for (1.1) When $1 < p < 3$

In the second part of the paper, we will prove explicit bounds on the scattering size of a solution to (1.1) with radially symmetric initial data in $B_{1,1}^{\frac{d}{2}+s_c}$, when $0 < s_c < 1$ and $1 < p < 3$. Note that the restrictions on s_c and p require $d \geq 3$.

As in the cubic case, the first step is to rescale and obtain good bounds on the interval $[0, 1]$. The space $L_{t,x}^{\frac{(d+2)(p-1)}{2}}(\mathbb{R} \times \mathbb{R}^d)$ is also invariant under the rescaling (1.3), so rescale the initial data so that

$$\|u\|_{L_{t,x}^{\frac{(d+2)(p-1)}{2}}([0,1]\times\mathbb{R}^d)} \leq \delta, \tag{5.1}$$

for some $\delta \ll 1$.

Lemma 2. If u is a solution to (1.1) on $[0, 1]$ with initial data $u_0 \in B_{1,1}^{\frac{d}{2}+s_c}$, and u satisfies (5.1), then for any $j \in \mathbb{Z}_{<0}$,

$$\|\nabla u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)} \lesssim 2^{j\frac{s_c-1}{2}} \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}(\mathbb{R}^d)}. \tag{5.2}$$

Proof. The local solution may be obtained by showing that the operator

$$\Phi(u(t)) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}|u(\tau)|^{p-1}u(\tau)\mathrm{d}\tau, \tag{5.3}$$

has a unique fixed point in $\dot{S}^{s_c}([0, 1] \times \mathbb{R}^d)$.

Interpolating the Sobolev embedding theorem,

$$\|P_k e^{it\Delta}u_0\|_{L^\infty} \lesssim 2^{k\frac{2}{p-1}} 2^{k(d-\frac{2}{p-1})} \|P_k u_0\|_{L^1}, \tag{5.4}$$

with the dispersive estimate,

$$\|P_k e^{it\Delta}u_0\|_{L^\infty} \lesssim t^{-d/2} 2^{-k(d-\frac{2}{p-1})} 2^{k(d-\frac{2}{p-1})} \|P_k u_0\|_{L^1}, \tag{5.5}$$

where P_k is the usual Littlewood–Paley projection operator for any $k \in \mathbb{Z}$,

$$\|e^{it\Delta}u_0\|_{L^\infty} \lesssim t^{-\frac{1}{p-1}}\|u_0\|_{B^{\frac{d}{2}+s_c}_{1,1}}, \tag{5.6}$$

and

$$\|\nabla e^{it\Delta}u_0\|_{L^\infty} \lesssim t^{-\frac{1}{p-1}-\frac{1}{2}}\|u_0\|_{B^{\frac{d}{2}+s_c}_{1,1}}. \tag{5.7}$$

Interpolating (5.5) with the Sobolev embedding theorem,

$$\|\nabla P_k e^{it\Delta}u_0\|_{L^2} \lesssim 2^{k(1-s_c)}2^{k(\frac{d}{2}+s_c)}\|P_k u_0\|_{L^1}, \tag{5.8}$$

and

$$\|\nabla P_k e^{it\Delta}u_0\|_{L^{\frac{2d}{d-2}}} \lesssim 2^{-ks_c}\frac{1}{t}2^{k(\frac{d}{2}+s_c)}\|P_k u_0\|_{L^1}. \tag{5.9}$$

Interpolating this bound with

$$\|\nabla P_k e^{it\Delta}u_0\|_{L^{\frac{2d}{d-2}}} \lesssim 2^{k(2-s_c)}2^{k(\frac{d}{2}+s_c)}\|P_k u_0\|_{L^1}, \tag{5.10}$$

we obtain

$$\|\nabla e^{it\Delta}u_0\|_{L^{\frac{2d}{d-2}}} \lesssim t^{-\frac{1}{2}-\frac{1-s_c}{2}}\|u_0\|_{B^{\frac{d}{2}+s_c}_{1,1}}. \tag{5.11}$$

Therefore, for any $j \in \mathbb{Z}_{<0}$,

$$\|\nabla e^{it\Delta}u_0\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)} \lesssim 2^{j\frac{s_c-1}{2}}\|u_0\|_{B^{\frac{d}{2}+s_c}_{1,1}}. \tag{5.12}$$

By Strichartz estimates, for any $t \in [2^j, 2^{j+1}]$, let j_δ be the integer closest to $\log_2(\delta 2^j)$. By Strichartz estimates, the chain rule, and (5.1),

$$\begin{aligned} &2^{j\frac{1-s_c}{2}}\|\nabla\int_{2^{j_\delta}}^t e^{i(t-\tau)\Delta}|u(\tau)|^{p-1}u(\tau)\mathrm{d}\tau\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)} \\ &\lesssim \delta^2 2^{j\frac{1-s_c}{2}}\|\nabla u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^{j_\delta},2^{j+1}]\times\mathbb{R}^d)} \lesssim \delta^{\frac{s_c-1}{2}}\delta^2\log(\delta)\sup_{j<0}\|\nabla u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)}. \end{aligned} \tag{5.13}$$

Meanwhile, the dispersive estimate combined with the Sobolev embedding theorem $\|u\|_{L_t^\infty L_x^{\frac{d(p-1)}{2}}([0,1]\times\mathbb{R}^d)} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}$, and

$$\|\nabla \int_0^{2^{j}\delta} e^{i(t-\tau)\Delta}|u(\tau)|^{p-1}u(\tau)d\tau\|_{L^{\frac{2d}{d-2}}} \lesssim 2^{-j}\|u\|^{p-1}_{L_t^\infty L_x^{\frac{d(p-1)}{2}}([0,1]\times\mathbb{R}^d)} \|\nabla u\|_{L_t^1 L_x^{\frac{2d}{d-2}}([0,2^{j}\delta]\times\mathbb{R}^d)}. \tag{5.14}$$

Therefore, by dispersive estimates and Hölder's inequality, for any $0 < s_c < 1$,

$$\begin{aligned} &2^{j\frac{1-s_c}{2}}\|\nabla \int_0^{2^{j}\delta} e^{i(t-\tau)\Delta}|u(\tau)|^{p-1}u(\tau)d\tau\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}])} \\ &\lesssim 2^j 2^{-\frac{js_c}{2}} \sup_{t\in[2^j,2^{j+1}]} \|\nabla \int_0^{2^{j}\delta} e^{i(t-\tau)\Delta}|u(\tau)|^{p-1}u(\tau)d\tau\|_{L^{\frac{2d}{d-2}}} \\ &\lesssim 2^{-j\frac{s_c}{2}}\|u\|^{p-1}_{L_t^\infty L_x^{\frac{d(p-1)}{2}}([0,1]\times\mathbb{R}^d)} \|\nabla u\|_{L_t^1 L_x^{\frac{2d}{d-2}}([0,2^{j}\delta]\times\mathbb{R}^d)} \\ &\lesssim \delta\|u\|^{p-1}_{L_t^\infty L_x^{\frac{d(p-1)}{2}}([0,1]\times\mathbb{R}^d)} \sup_{j<0} 2^{j\frac{1-s_c}{2}}\|\nabla u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)}. \end{aligned} \tag{5.15}$$

Therefore, for $0 < \epsilon \ll 1$, for $\delta(\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}, \epsilon) > 0$ sufficiently small,

$$\sup_{j<0} 2^{j\frac{1-s_c}{2}}\|\nabla\Phi(u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}} + \epsilon\cdot\sup_{j<0} 2^{j\frac{1-s_c}{2}}\|\nabla u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)}, \tag{5.16}$$

for some $\epsilon > 0$. Thus, (5.2) holds. ∎

Now suppose (1.1) with initial data u_0 has a solution on the maximal interval $[0, T)$, where $1 < T \le \infty$. Again decompose $u = v + w$, where v and w solve

$$iv_t + \Delta v = |v+w|^{p-1}(v+w), \qquad v(0) = 0, \tag{5.17}$$

and

$$iw_t + \Delta w = 0, \qquad w(0) = u_0, \tag{5.18}$$

on $[0,\infty)$. Let $\mathcal{E}(t)$ denote the pseudoconformal energy of v,,

$$\mathcal{E}(t)=\|(x+2it\nabla)v\|_{L^2}^2+\frac{8}{p+1}t^2\|v\|_{L^{p+1}}^{p+1}=\|xv\|_{L^2}^2+2\langle xv,2it\nabla v\rangle+8t^2E(t). \tag{5.19}$$

Lemma 3. If $u_0\in B_{1,1}^{\frac{d}{2}+s_c}$ is radially symmetric, and (1.1) has a local solution satisfying (5.1), then $\mathcal{E}(1)\lesssim 1$ for $\delta(\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}})>0$ sufficiently small.

Proof. Observe that the proof of Lemma 2 also implies

$$\|\nabla v(1)\|_{L^2+L^{\frac{2d}{d-2}}}\lesssim\epsilon\cdot\sup_{j<0}2^{j\frac{1-s_c}{2}}\|\nabla u\|_{L_t^2L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)}. \tag{5.20}$$

Interpolating (5.20) with the bound

$$\|v(1)\|_{\dot{H}^{s_c}}\lesssim\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}, \tag{5.21}$$

implies $\|v(1)\|_{L^{p+1}}\lesssim 1$ for $\delta>0$ sufficiently small, since $\epsilon=\epsilon(\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}},p,d,\delta)$.

Using the computations in (3.11),

$$\begin{aligned}&\|(x+2i1\nabla)\int_0^1 e^{i(1-\tau)\Delta}|u|^{p-1}u\,d\tau\|_{L_x^2}\\ &\lesssim\|x|u|^{p-1}u\|_{L_t^1L_x^2([0,1]\times\mathbb{R}^d)}+\|t\nabla u\|_{L_t^2L_x^{\frac{2d}{d-2}}([0,1]\times\mathbb{R}^d)}\|u\|_{L_{t,x}^{\frac{(d+2)(p-1)}{2}}([0,1]\times\mathbb{R}^d)}^{p-1}.\end{aligned} \tag{5.22}$$

Then by Lemma 2,

$$\|t\nabla u\|_{L_t^2L_x^{\frac{2d}{d-2}}([0,1]\times\mathbb{R}^d)}\|u\|_{L_{t,x}^{\frac{(d+2)(p-1)}{2}}([0,1]\times\mathbb{R}^d)}^{p-1}\lesssim\delta^{p-1}\cdot\sup_{j<0}2^{j\frac{1-s_c}{2}}\|\nabla u\|_{L_t^2L_x^{\frac{2d}{d-2}}([2^j,2^{j+1}]\times\mathbb{R}^d)}. \tag{5.23}$$

To handle the first term in (5.22), consider the cases $\frac{1}{2}\leq s_c<1$ and $0<s_c<\frac{1}{2}$ separately. When $\frac{1}{2}\leq s_c<1$, the radial Sobolev embedding theorem implies

$$\|x|u|^{p-1}u\|_{L_t^1L_x^2}\lesssim\|x^{\frac{2}{p-1}}u\|_{L_{t,x}^{\infty}}^{\frac{p-1}{2}}\|u\|_{L_t^\infty L_x^{\frac{d}{2}(p-1)}}^{\frac{p-1}{2}+(1-c)}\|u\|_{L_{t,x}^{\frac{(d+2)(p-1)}{2}}}^{c}\lesssim\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{p-c}\delta^c, \tag{5.24}$$

where $c\searrow 0$ as $s_c\nearrow 1$.

When $0 < s_c < \frac{1}{2}$, using the radial Strichartz estimates,

$$\||x|u|^{p-1}u\|_{L_t^1 L_x^2([0,1]\times\mathbb{R}^d)} \lesssim \|x^{\frac{d-1}{2}}u\|^{\frac{2}{d-1}}_{L_t^{\frac{2}{\frac{1}{2}-s_c}} L_x^\infty} \|u\|^{p-1+(1-\frac{2}{d-1})-c}_{L_t^\infty L_x^{\frac{d}{2}(p-1)}} \|u\|^{c}_{L_{t,x}^{\frac{(d+2)(p-1)}{2}}} \lesssim \|u_0\|^{p-c}_{B_{1,1}^{\frac{d}{2}+s_c}} \delta^c, \tag{5.25}$$

where $c > 0$ for all $0 < s_c < \frac{1}{2}$ and $d \geq 3$, with appropriate p.

This proves the Lemma. ■

6 Scattering for (1.1) When $1 < p < 3$ and $0 < s_c < 1$

Having obtained good bounds on the interval $[0, 1]$, we can use the pseudoconformal conservation of energy to extend these bounds to $[1, \infty)$.

Theorem 6. The initial value problem

$$iu_t + \Delta u = |u|^{p-1}u, \qquad u(0, x) = u_0 \in B_{1,1}^{\frac{d}{2}+s_c}(\mathbb{R}^d), \qquad u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}, \tag{6.1}$$

is globally well-posed and scattering when u_0 is radially symmetric. Moreover,

$$\|u\|_{L_t^{\frac{p+1}{1-s_c}} L_x^{p+1}(\mathbb{R}\times\mathbb{R}^d)} \leq C(1 + \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}})^r, \tag{6.2}$$

for some C that does not depend on $\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}$ and $r(d, s_c) < \infty$.

Remark 5. When $\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}$ is small,

$$\|u\|_{L_t^{\frac{p+1}{1-s_c}} L_x^{p+1}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{\dot{H}^{s_c}} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}. \tag{6.3}$$

So it suffices to consider $\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}} \gtrsim 1$.

Proof of Theorem 6. If v solves (5.17) on $[1, \infty)$ with $w = 0$, $0 < s_c < 1$, and $\mathcal{E}(1) < \infty$, where $\mathcal{E}(t)$ is given by (5.19), then by direct computation,

$$\frac{d}{dt}\mathcal{E}(t) = -\frac{4}{p+1} t\|v\|^{p+1}_{L^{p+1}} < 0, \tag{6.4}$$

which implies

$$\|v\|_{L^{p+1}}^{p+1} \lesssim \frac{1}{t^2}. \tag{6.5}$$

Equation (6.5) implies that the left hand side of (6.2) is finite, which implies scattering.

Now compute $\frac{d}{dt}\mathcal{E}(t)$ when w need not be zero, but w solves (5.18) with $u_0 \in B_{1,1}^{\frac{d}{2}+s_c}$, radially symmetric. Then by direct computation,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) = -\frac{4}{p+1}t\|v\|_{L^{p+1}}^{p+1} - 2\langle(x+2it\nabla)v, i(x+2it\nabla)(|v+w|^{p-1}(v+w) - |v|^{p-1}v)\rangle \\ -8t^2\langle|v|^{p-1}v, i(|v+w|^{p-1}(v+w) - |v|^{p-1}v)\rangle.\end{aligned} \tag{6.6}$$

Doing some linear algebra,

$$\langle(x+2it\nabla)f, (x+2it\nabla)g\rangle = \langle(x+2it\nabla)f, xg\rangle + \langle 2it\nabla f, 2it\nabla g\rangle + \langle xf, 2it\nabla g\rangle. \tag{6.7}$$

Apply the linear algebra in (6.7) to (6.6) and compute term by term. First, when $\frac{1}{2} \le s_c < 1$, by the radial Sobolev embedding theorem, since $\frac{p-1}{2} < 1$,

$$\begin{aligned}&-2\langle(x+2it\nabla)v, ix(|v+w|^{p-1}(v+w) - |v|^{p-1}v)\rangle \\ &\lesssim \|(x+2it\nabla)v\|_{L^2}\|x^{\frac{2}{p-1}}w\|_{L^\infty}^{\frac{p-1}{2}}\|w\|_{L^{p+1}}^{1-\frac{p-1}{2}}(\|v\|_{L^{p+1}}^{p-1} + \|w\|_{L^{p+1}}^{p-1}) \\ &\lesssim \mathcal{E}(t)^{1/2}\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{\frac{p-1}{2}}\|w\|_{L^{p+1}}^{1-\frac{p-1}{2}}(\|v\|_{L^{p+1}}^{p-1} + \|w\|_{L^{p+1}}^{p-1}).\end{aligned} \tag{6.8}$$

When $0 < s_c < \frac{1}{2}$, split

$$xw = (x+2it\nabla)w - 2it\nabla w. \tag{6.9}$$

Again by (3.12) and the radial Sobolev embedding theorem, for $s_c < \frac{d}{2} - 1$, $\|yu_0\|_{\dot{H}^{s_c+1}} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}$, so interpolating the Strichartz estimate,

$$\|e^{it\Delta}u_0\|_{L_t^{\frac{p+1}{1-s_c}}L_x^{p+1}} \lesssim \|u_0\|_{\dot{H}^{s_c}}, \tag{6.10}$$

with the Littlewood–Paley projection estimate

$$\|P_j e^{it\Delta}u_0\|_{L_{t,x}^\infty} \lesssim \|P_j u_0\|_{\dot{H}^{d/2}}, \tag{6.11}$$

implies that

$$\|(x+2it\nabla)w\|_{L_t^{\frac{2(p+1)}{3-p}\frac{1}{1-s_c}} L_x^{\frac{2(p+1)}{3-p}}} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}. \tag{6.12}$$

Now take $\frac{1}{q} = s_c$. By the Sobolev embedding theorem, dispersive estimates, $s_c = \frac{d}{2} - \frac{2}{p-1}$, and $\frac{2}{p-1} < \frac{d}{2}$,

$$\||\nabla|^{\frac{2}{p-1}} e^{it\Delta} u_0\|_{L_x^\infty} \lesssim \|e^{it\Delta} u_0\|_{B_{1,q}^{\frac{d}{2}}} \lesssim \frac{1}{t^{\frac{2}{p-1}}} \|u_0\|_{B_{1,q'}^{\frac{d}{2}}} \lesssim \frac{1}{t^{\frac{2}{p-1}}} \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}. \tag{6.13}$$

Therefore,

$$\|t^{\frac{2}{p-1}} |\nabla|^{\frac{2}{p-1}} w\|_{L^\infty}^{\frac{p-1}{2}} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{\frac{p-1}{2}}. \tag{6.14}$$

Interpolating (6.10) and (6.14),

$$\|2it\nabla w\|_{L_t^{\frac{2(p+1)}{3-p}\cdot\frac{1}{1-s_c}} L_x^{\frac{2(p+1)}{3-p}}} \lesssim \|t^{\frac{2}{p-1}} |\nabla|^{\frac{2}{p-1}} w\|_{L_{t,x}^\infty}^{\frac{p-1}{2}} \|w\|_{L_t^{\frac{p+1}{1-s_c}} L_x^{p+1}}^{\frac{3-p}{2}} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}. \tag{6.15}$$

Therefore, we have proved

$$\|xw\|_{L_t^{\frac{2(p+1)}{3-p}\frac{1}{1-s_c}} L_x^{\frac{2(p+1)}{3-p}}} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}. \tag{6.16}$$

Next, integrating by parts,

$$\begin{aligned} -2\langle(2it\nabla)v, i(2it\nabla)(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle &= -8t^2\langle\nabla v, i\nabla(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle \\ &= 8t^2\langle\Delta v, i(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle. \end{aligned} \tag{6.17}$$

Summing, by (5.17),

$$\begin{aligned} 8t^2\langle\Delta v, i(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle - 8t^2\langle|v|^{p-1}v, i(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle \\ = 8t^2\langle -iv_t, i(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle \\ -8t^2\langle(|v+w|^{p-1}(v+w)-|v|^{p-1}v), i(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle \\ = -8t^2\langle v_t, (|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle. \end{aligned} \tag{6.18}$$

Next, integrating by parts,

$$-2\langle xv, i(2it\nabla)(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle = 4t\langle xv, \nabla(|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle$$
$$= -4td\langle v, (|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle - 4t\langle x\cdot\nabla v, (|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle. \tag{6.19}$$

Integrating the second term in (6.19) by parts again,

$$-4t\langle x\cdot\nabla v, (|v+w|^{p-1}(v+w)-|v|^{p-1}v)\rangle = \frac{4dt}{p+1}(\|v+w\|_{L^{p+1}}^{p+1} - \|v\|_{L^{p+1}}^{p+1})$$
$$+4t\langle x\cdot\nabla w, |v+w|^{p-1}(v+w)\rangle. \tag{6.20}$$

Now then, summing,

$$4t\langle x\cdot\nabla w, |v+w|^{p-1}(v+w)\rangle = 4t\langle (x+2it\nabla)\cdot\nabla w, |v+w|^{p-1}(v+w)\rangle$$
$$-8t^2\langle i\Delta w, |v+w|^{p-1}(v+w)\rangle. \tag{6.21}$$

Summing (6.18) and (6.21), since $w_t = i\Delta w$,

$$(6.18)+(6.21) = 4t\langle (x+2it\nabla)\cdot\nabla w, |v+w|^{p-1}(v+w)\rangle - 8t^2\langle v_t + w_t, |v+w|^{p-1}(v+w)\rangle$$
$$+8t^2\langle v_t, |v|^{p-1}v\rangle. \tag{6.22}$$

By the radial Sobolev embedding theorem, (3.12), and the fact that $\frac{d}{2} > 1$,

$$\|(x+2it\nabla)\cdot\nabla w\|_{L_t^{\frac{p+1}{1-s_c}} L_x^{p+1}} \lesssim \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}. \tag{6.23}$$

Therefore, plugging these computations back into (6.6),

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) \lesssim -\frac{4}{p+1}t\|v\|_{L^{p+1}}^{p+1} - \frac{8t^2}{p+1}\frac{d}{dt}\|v+w\|_{L^{p+1}}^{p+1} + \frac{8t^2}{p+1}\frac{d}{dt}\|v\|_{L^{p+1}}^{p+1}\\ +t\|(x+2it\nabla)\cdot\nabla w\|_{L^{p+1}}\|v+w\|_{L^{p+1}}^{p}\\ +t\|v\|_{L^{p+1}}\|w\|_{L^{p+1}}(\|v\|_{L^{p+1}}^{p-1}+\|w\|_{L^{p+1}}^{p-1}) + t\|w\|_{L^{p+1}}^{p+1}\\ +\mathcal{E}(t)^{1/2}\|(x+2it\nabla)w\|_{L^{\frac{2(p+1)}{3-p}}}(\|v\|_{L^{p+1}}^{p-1}+\|w\|_{L^{p+1}}^{p-1})\\ +\mathcal{E}(t)^{1/2}\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{\frac{p-1}{2}}\|w\|_{L^{p+1}}^{1-\frac{p-1}{2}}(\|v\|_{L^{p+1}}^{p-1}+\|w\|_{L^{p+1}}^{p-1}).\end{aligned} \tag{6.24}$$

Then by the product rule,

$$\begin{aligned}&\frac{d}{dt}[\mathcal{E}(t)+\frac{8t^2}{p+1}\|v+w\|_{L^{p+1}}^{p+1}-\frac{8t^2}{p+1}\|v\|_{L^{p+1}}]\\ &\lesssim -\frac{4}{p+1}t\|v\|_{L^{p+1}}^{p+1}+t\|(x+2it\nabla)\cdot\nabla w\|_{L^{p+1}}\|v+w\|_{L^{p+1}}^{p}+t\|v\|_{L^{p+1}}^{p}\|w\|_{L^{p+1}}+t\|w\|_{L^{p+1}}^{p+1}\\ &+\mathcal{E}(t)^{1/2}\|(x+2it\nabla)w\|_{L^{\frac{2(p+1)}{3-p}}}(\|v\|_{L^{p+1}}^{p-1}+\|w\|_{L^{p+1}}^{p-1})+\mathcal{E}(t)^{1/2}\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{\frac{p-1}{2}}\|w\|_{L^{p+1}}^{1-\frac{p-1}{2}}(\|v\|_{L^{p+1}}^{p-1}+\|w\|_{L^{p+1}}^{p-1}).\end{aligned} \tag{6.25}$$

Since $\|v(1)\|_{L^{p+1}}\lesssim 1,\mathcal{E}(1)\lesssim 1$, and dispersive estimates imply that $\|w(1)\|_{L^{p+1}}\lesssim 1$, the Cauchy–Schwartz inequality and (6.25) imply that

$$\begin{aligned}\frac{1}{t^2}\mathcal{E}(t)&\lesssim\frac{1}{t^2}+\|w(t)\|_{L^{p+1}}(\|v(t)\|_{L^{p+1}}^{p}+\|w(t)\|_{L^{p+1}}^{p})+\frac{1}{t^2}\int_1^t\tau\|(x+2it\nabla)\cdot\nabla w\|_{L^{p+1}}^{p+1}\mathrm{d}\tau\\ &+\frac{1}{t^2}\int_1^t\tau\|w\|_{L^{p+1}}^{p+1}+\frac{1}{t^2}\int_1^t\frac{\mathcal{E}(t)^{1/2}}{\tau}\cdot\tau\|(x+2it\nabla)w\|_{L^{\frac{2(p+1)}{3-p}}}(\|v\|_{L^{p+1}}^{p-1}+\|w\|_{L^{p+1}}^{p-1})\mathrm{d}\tau\\ &+\frac{1}{t^2}\int_1^t\frac{\mathcal{E}(\tau)^{1/2}}{\tau}\cdot\tau\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{\frac{p-1}{2}}\|w\|_{L^{p+1}}^{1-\frac{p-1}{2}}(\|v\|_{L^{p+1}}^{p-1}+\|w\|_{L^{p+1}}^{p-1})\mathrm{d}\tau,\end{aligned} \tag{6.26}$$

with implicit constants depending only on p and d. Then choosing $0<\delta(p,d)\ll 1$ sufficiently small, by the Cauchy–Schwartz inequality,

$$\begin{aligned}\frac{1}{t^2}\mathcal{E}(t)&\lesssim\frac{1}{t^2}+\|w(t)\|_{L^{p+1}}(\|v(t)\|_{L^{p+1}}^{p}+\|w(t)\|_{L^{p+1}}^{p})+\frac{1}{t^2}\int_1^t\tau\|(x+2it\nabla)\cdot\nabla w\|_{L^{p+1}}^{p+1}\mathrm{d}\tau\\ &+\frac{1}{t^2}\int_1^t\tau\|w\|_{L^{p+1}}^{p+1}+\frac{\delta}{t^2}\int_1^t\frac{\mathcal{E}(t)}{\tau^2}\tau\mathrm{d}\tau+\frac{1}{\delta t^2}\int_1^t\tau\|(x+2it\nabla)w\|_{L^{\frac{2(p+1)}{3-p}}}^2(\|v\|_{L^{p+1}}^{2(p-1)}+\|w\|_{L^{p+1}}^{2(p-1)})\mathrm{d}\tau\\ &+\frac{1}{\delta t^2}\int_1^t\tau\|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{p-1}\|w\|_{L^{p+1}}^{3-p}(\|v\|_{L^{p+1}}^{2(p-1)}+\|w\|_{L^{p+1}}^{2(p-1)})\mathrm{d}\tau.\end{aligned} \tag{6.27}$$

Therefore, by Young's inequality,

$$\begin{aligned} &\| \frac{1}{t^2} \mathcal{E}(t) \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))} \\ &\lesssim 1 + \| \|v(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))}^{\frac{p}{p+1}} \| \|w(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))}^{\frac{1}{p+1}} + \| \|w(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))} \\ &+ \| \|(x+2it\nabla) \cdot \nabla w\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))} + \frac{1}{\delta} \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{p-1} \| \|w(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))} \\ &+ \frac{1}{\delta} \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{p-1} \| \|w(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))}^{\frac{3-p}{p+1}} \| \|v(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))}^{\frac{2(p-1)}{p+1}} \\ &+ \frac{1}{\delta} \|(x+2it\nabla) w\|_{L_t^{\frac{2(p+1)}{3-p} \frac{1}{1-s_c}} L_x^{\frac{2(p+1)}{3-p}}}^2 \| \|v(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))}^{\frac{2(p-1)}{p+1}} \\ &+ \frac{1}{\delta} \|(x+2it\nabla) w\|_{L_t^{\frac{2(p+1)}{3-p} \frac{1}{1-s_c}} L_x^{\frac{2(p+1)}{3-p}}}^2 \| \|w(t)\|_{L^{p+1}}^{p+1} \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))}^{\frac{2(p-1)}{p+1}} . \end{aligned} \tag{6.28}$$

Then combining $\|v(t)\|_{L^{p+1}}^{p+1} \lesssim \frac{1}{t^2} \mathcal{E}(t)$, Strichartz estimates, (6.8)–(6.16), and (6.23),

$$\| \frac{1}{t^2} \mathcal{E}(t) \|_{L_t^{\frac{1}{1-s_c}}([1,\infty))} \lesssim_{p,d} 1 + \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{p+1} + \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{2p} + \|u_0\|_{B_{1,1}^{\frac{d}{2}+s_c}}^{\frac{2(p+1)}{3-p}} . \tag{6.29}$$

This proves the theorem. ■

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