

Cops and robbers on P_5 -free graphs

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Abstract

We prove that every connected P_5 -free graph has cop number at most two, solving a conjecture of Sivaraman. In order to do so, we first prove that every connected P_5 -free graph G with independence number at least three contains a three-vertex induced path with vertices a - b - c in order, such that every neighbour of c is also adjacent to one of a, b .

1 Introduction

We denote the t -vertex path by P_t . There are a number of well-known open questions about P_5 -free graphs (a graph G is H -free if no induced subgraph of G is isomorphic to H , and $|G|$ denotes the number of vertices of G). For instance:

- the Erdős-Hajnal [4] conjecture implies that for some $c > 0$, every P_5 -free graph G has a clique or stable set of size at least $|G|^c$ (Nguyen et al. [15] have recently written what purports to be a proof of this);
- a conjecture of Esperet [5] implies that for some $c > 0$, every P_5 -free graph G has chromatic number at most $\omega(G)^c$, where $\omega(G)$ is the clique number of G ;
- a conjecture of Hoàng et al. [9] says that for some $c > 0$, there is a function f such that for every k there is an algorithm deciding whether a P_5 -free graph G is k -colourable in time $f(k)|G|^c$.

In this paper, we study another conjecture on P_5 -free graphs, which concerns the game of cops and robbers. In this game, there are s cops, and each stands on one vertex of the graph, and so does the robber. In each turn, first each cop moves to a neighbouring vertex, or does not move; and then the robber moves to a neighbouring vertex, or does not move. The cops win if at some stage, a cop is standing on the same vertex as the robber. Cops may share vertices and this game is played with full information. Given a graph G , how few cops suffice? The *cop number* $c(G)$ is the smallest number of cops which can capture the robber on G . The game played with one cop was initially defined by Nowakowski and Winkler [16] and Quilliot [18]. The version with multiple cops was introduced Aigner and Fromme [1]; in particular they proved that the cop number of any connected planar graph is at most three.

Inspired by Andreae's result [2] that the cop number of connected graphs forbidding H as a minor is bounded for every graph H , Joret et al. [11] proved that the cop number of connected H -free graphs is bounded if and only if H is a disjoint union of paths. In particular, they showed that the cop number of connected P_t -free graphs is at most $t - 2$ for $t \geq 3$. Sivaraman [19] conjectured that two cops can win on connected P_5 -free graphs and more generally that the cop number of connected P_t -free graphs is at most $t - 3$ for $t \geq 5$.

Other questions on the cop number and forbidden induced subgraphs have also been considered, for instance relating the cop number and the independence number (in other words, tK_1 -free graphs) [17] and excluding multiple induced subgraphs [10, 14, 20].

However, the question of the cop number of P_5 -free graphs has received the most attention in this field; various weakenings of the conjecture about P_5 -free graphs have been studied. Sivaraman and Testa [21] conjectured the weaker statement that the cop number of connected $2K_2$ -free graphs is at most two ($2K_2$, also written $2P_2$, is graph obtained by the disjoint union of two edges; it can also be seen as the complement of a four-vertex cycle). This was proved by the fourth author of this paper [22]. Liu [12] proved various partial results for these problems and Gupta, Mishra and Pradhan [7] have shown that the conjecture holds for multiple subclasses of P_5 -free graphs. Masjoody [13] has conjectured the weaker statement that even if two cops perhaps cannot capture the robber on P_5 -free graphs, they can confine it to a vertex.

In this paper, we prove Sivaraman's conjecture on the cop number of P_5 -free graphs.

1.1. *If G is a connected P_5 -free graph, then $c(G) \leq 2$.*

The general strategy we employ is similar to the one used by the fourth author of this paper in [22] to bound the cop number of $2K_2$ -free graphs. First, show that any graph in the class must contain a possible winning position for two cops, that is vertices $a, b \neq c$ such that $N[c] \subseteq N[a] \cup N[b]$. Then, consider a minimal graph in the class for which two cops cannot win, and use the minimality to force the robber to move to c , after which try to eventually move the cops to a, b and show the robber cannot escape.

The first part of this strategy is accomplished by the following result, which we prove in Section 3.

1.2. *If G is connected and P_5 -free, with $\alpha(G) \geq 3$, then there is a three-vertex induced path of G with vertices a, b, c in order, such that every neighbour of c is also adjacent to one of a, b .*

Here, $\alpha(G)$ is the independence number of G , that is the cardinality of the largest stable set in G . Let $a-b-c$ be the vertices in order of a three-vertex induced path of G . We say that $a-b-c$ is *domineering*, or a *domineering 3-path*, if every neighbour of c is also adjacent to one of a, b . Thus 1.2 says that every connected P_5 -free graph with $\alpha(G) \geq 3$ has a domineering 3-path.

The condition $\alpha(G) \geq 3$ in 1.2 is needed. It is easy to see that every graph G with $\alpha(G) = 2$ is P_5 -free, and it has no domineering 3-path if and only if its complement H has diameter at most two, which gives plenty of counterexamples to 1.2 with $\alpha(G) \geq 3$ omitted.

The relative of 1.2 proved by the fourth author for $2K_2$ -free graphs [22] is the following.

1.3. *If G is connected and $2K_2$ -free, with $|G| \geq 3$ and G not a cycle of length five, then there exist distinct vertices a, b, c such that ab and bc are edges (possibly ac is also an edge) and every neighbour of c is adjacent to one of a, b .*

This differs from 1.2 in three ways, two weakenings and a strengthening. First, it of course assumes that G is $2K_2$ -free, instead of P_5 -free. Second, ac might be an edge. But third, it does not need the assumption $\alpha(G) \geq 3$. What if we try to modify 1.3, asking for a domineering 3-path in a $2K_2$ -free graph G ? Then again, it is false, but there are not so many counterexamples; every counterexample G satisfies $\alpha(G) \leq 2$, and it is easy to see that the counterexamples are the complements of Moore graphs of girth five. These are graphs of diameter two, with girth five, in which every vertex has the same degree d ; such a graph exists only when $d = 2, 3, 7$ and possibly 57.

In order to prove 1.2, we will need the following definition. Let us say a graph G is *bijointed* if

- for every two nonadjacent vertices u, v of G , there are exactly two vertices adjacent to both u, v , and they are adjacent to each other, and
- G has no clique of cardinality four.

It is said to be *nontrivial* if $|G| > 1$. If a nontrivial bijointed graph has a complement graph that is connected, then that complement would be a counterexample to 1.2, so we care about bijointed graphs. Indeed, we will show in Section 3 that any counterexample to 1.2 has a connected induced subgraph whose complement is a nontrivial bijointed graph; so the whole question boils down to showing that there is no such graph. That is proved in Section 2.

The second part of the strategy, using 1.2 to capture the robber with two cops, is accomplished in Section 4. One important difference between the proofs for the $2K_2$ -free case and the P_5 -free case is that once the robber is on c , in the former we can ensure the robber never leaves c , which is not possible in the latter.

Although this paper concerns only the $t = 5$ case of Sivaraman's conjecture, we note that it might be possible to generalize our approach to prove the conjecture for larger t .

Let us complete this section with some notation. Suppose G is a graph, which we always consider to be simple and finite. For $v \in V(G)$, we write $N(v)$ for the *neighbourhood* of v (the set of vertices adjacent to v), $N[v] = N(v) \cup \{v\}$ for its *closed neighbourhood* of v , and $M(v) = V(G) \setminus N[v]$ for the set of vertices distinct from and not adjacent to v . A vertex of a graph G is *universal* if it is adjacent to every other vertex.

If $X, Y \subseteq V(G)$ are disjoint, we say X is *complete* to Y if every vertex in X is adjacent to every vertex in Y , and X is *anticomplete* to Y if there are no edges between X, Y . If $X = \{x\}$, we say x is *complete* to Y if $\{x\}$ is complete to Y , and so on.

If $X \subseteq V(G)$, we write $G[X]$ for the subgraph of G induced on X and $G \setminus X$ for $G[V(G) \setminus X]$. If $X = \{v\}$, we write $G \setminus v$ for $G \setminus \{v\}$.

We will often write $x_1-x_2-\dots-x_k$ to represent a path with vertices $x_1, x_2, \dots, x_k$ in order.

2 Bijointed graphs

We first note that bijointed graphs exist; for instance, if H is a graph of girth at least five in which every two nonadjacent vertices have exactly one common neighbour, and we add a universal vertex to H , we obtain a bijointed graph. We will show that no graphs are bijointed other than these, and in particular, no nontrivial bijointed graph has a connected complement graph.

We need the following well-known lemma:

2.1. *Let H be a graph with girth at least five, such that for every two nonadjacent vertices u, v there is exactly one vertex adjacent to both u, v . If the complement of H is connected, then every two vertices of H have the same degree.*

Proof. Since the complement of H is connected, it suffices to show that every two nonadjacent vertices of H have the same degree. Thus, let u, w be nonadjacent. Let $N = \{v_1, \dots, v_k\}$ be the set of neighbours of u , and for $1 \leq i \leq k$ let N_i be the set of neighbours of v_i different from u . Thus, the sets $\{u\}, N, N_1, \dots, N_k$ are pairwise disjoint and have union $V(H)$. Let $w \in N_k$ say. For all $j \in \{1, \dots, k-1\}$, w has a neighbour in N_j (because it has distance two from v_j), and has exactly one such neighbour (since H has girth at least five); and has exactly one neighbour in $N \cup \{u\} \cup N_k$ (that is, v_k); and so w has degree exactly k . This proves 2.1. ■

We also need some results about strongly regular graphs. A graph is *strongly regular* with parameters (n, k, a, c) if it has n vertices, every vertex has degree k , every two adjacent vertices have exactly a common neighbours, and every two nonadjacent vertices have exactly c common neighbours. Thus, Moore graphs are the strongly regular graphs that have parameters $(n, k, 0, 1)$ for some n, k . First, we need a result about Moore graphs mentioned earlier, due to Hoffman and Singleton [8]:

2.2. *A strongly regular graph with parameters $(n, k, 0, 1)$ exists only when $n = k^2 + 1$ and $k = 2, 3, 7$ and possibly 57.*

Second, we need the following (see Lemmas 10.3.2 and 10.3.3 of Godsil and Royle [6]):

2.3. *If a strongly regular graph exists with parameters (n, k, a, c) , then either $2k = (n-1)(c-a)$ or $(a-c)^2 + 4(k-c)$ is a perfect square.*

Now we prove:

2.4. *If G is a bijoined and non-null graph, then G has a universal vertex.*

Proof. If u, v are distinct, let $R(u, v)$ be the set of all vertices adjacent to both u, v ; thus, if u, v are nonadjacent then $|R(u, v)| = 2$ and its two members are adjacent. For convenience, we say “ $R(u, v)$ is an edge”. We observe that no induced cycle of G has length four (because for two opposite vertices u, v of such a cycle, $R(u, v)$ is not an edge). Thus G is C_4 -free, where C_4 is the cycle of length four. We suppose that G has no universal vertex, for a contradiction. A *4-clique* means a clique of cardinality four.

(1) *For each $v \in V(G)$, the complement of $G[N(v)]$ is connected.*

Suppose not. Then there is a partition (X, Y) of $N(v)$ with $X, Y \neq \emptyset$, such that X is complete to Y . Since G is C_4 -free, one of X, Y is a clique, say X ; let $x \in X$. Then x is adjacent to all other vertices in $N(v)$, so we may assume that $X = \{x\}$ and $Y = N(v) \setminus \{x\}$. Since G has no 4-clique, Y is a stable set. Since x is not universal, there exists $z \in V(G)$ nonadjacent to x . Consequently $z \notin N[v]$. But then $R(v, z)$ is a subset of Y and so not an edge, a contradiction. This proves (1).

(2) *For each $v \in V(G)$, all vertices of $G[N(v)]$ have the same degree in $G[N(v)]$.*

$G[N(v)]$ has no cycle of length three, since G is K_4 -free; and $G[N(v)]$ is C_4 -free since G is C_4 -free. Thus, $G[N(v)]$ has girth at least five. If $u, w \in N(v)$ are nonadjacent, then $R(u, w)$ consists of v and exactly one vertex of $N(v)$; and so u, w have exactly one common neighbour in $G[N(v)]$. From (1) and 2.1, this proves (2).

By (2) and since G is bijoined, for every vertex v there exists k_v such that $G[N(v)]$ is a Moore graph with parameters $((k_v)^2 + 1, k_v, 0, 1)$. In particular, $|R(u, v)| = k_v$ for every neighbour u of v . It follows that $k_u = k_v$ for every pair of neighbours u and v , and since G is connected, there exists k such that $k_v = k$ for every v . So G is a strongly regular graph with parameters $(n, k^2 + 1, k, 2)$ for some n (which is determined by k but does not matter), where $k \in \{2, 3, 7, 57\}$ by 2.2. Let us apply 2.3, and deduce that one of the following holds:

- $2(k^2 + 1) = (n - 1)(2 - k)$; but this is impossible since $k \geq 2$.
- $(k - 2)^2 + 4((k^2 + 1) - 2) = k(5k - 4)$ is a perfect square; but this is not the case when $k = 2, 3, 7, 57$.

This contradiction proves 2.4. ■

3 Finding a domineering 3-path

Let us prove 1.2, which we restate as follows.

3.1. *If G is a connected P_5 -free graph with $\alpha(G) \geq 3$, then there exists a domineering 3-path in G .*

Proof. We assume that G is a counterexample to the theorem with G minimal. Thus, G is connected and P_5 -free, with $\alpha(G) \geq 3$, and there is no domineering 3-path in G , and no proper induced subgraph has these properties. We will prove that the complement of G is bijoined, which we will show is impossible. We begin with:

(1) *No two adjacent vertices u, v satisfy $N(u) \subseteq N[v]$.*

Suppose that there are two such vertices u, v . If there is a vertex w adjacent to v and not to u , then $w-v-u$ is domineering, a contradiction; so $N[u] = N[v]$. Let G' be obtained by deleting u . Then G' is connected, P_5 -free, and satisfies $\alpha(G') \geq 3$, and so from the minimality of G , there is a domineering 3-path $a-b-c$ of G' . This is not domineering in G , and so u is adjacent to c and nonadjacent to a, b . But then $v \neq a, b, c$, and so v is adjacent to c and not to a, b , contradicting that $a-b-c$ is domineering in G' . This proves (1).

(2) *For each $v \in V(G)$ and every component C of $G[M(v)]$, no vertex $u \in N(v)$ is complete to $V(C)$.*

Because if u is such a vertex, choose $w \in V(C)$; then $v-u-w$ is domineering, a contradiction. This proves (2).

(3) *For each $v \in V(G)$, $G[M(v)]$ is non-null and connected. Moreover, every vertex in $N(v)$ has a neighbour in $M(v)$.*

By (1), $M(v) \neq \emptyset$. Suppose that C_1, C_2 are distinct components of $G[M(v)]$. Since G is connected, for $i = 1, 2$ there exists $u_i \in N(v)$ with a neighbour in $V(C_i)$. By (2), for $i = 1, 2$, u_i has a neighbour and a non-neighbour in $V(C_i)$, and since C_i is connected, there is an edge $a_i b_i$ of C_i such that u_i is adjacent to a_i and not to b_i . If u_1 has a neighbour in $V(C_2)$, we may assume that $u_1 = u_2$, but then $b_1-a_1-u_1-a_2-b_2$ is a copy of P_5 , a contradiction. Thus u_1 has no neighbour in $V(C_2)$, and similarly u_2 has no neighbour in $V(C_1)$. If u_1, u_2 are nonadjacent then $b_1-a_1-u_1-v-u_2$ is a copy of P_5 , and if u_1, u_2 are adjacent then $b_1-a_1-u_1-u_2-a_2$ is a copy of P_5 , in either case a contradiction. This proves the first assertion. For the second, let $u \in N(v)$; then u has a neighbour in $M(v)$ by (1). This proves (3).

(4) *For each $v \in V(G)$, $G[M(v)]$ has no domineering 3-path. Consequently $\alpha(G) = 3$.*

Suppose that $a-b-c$ is a domineering 3-path of $G[M(v)]$. We claim that $a-b-c$ is also domineering in G . To show this, it suffices to show that every neighbour u of c not in $M(v)$ is adjacent to one of a, b . But if not, then $a-b-c-u-v$ is a copy of P_5 , a contradiction. This proves the first assertion of (4). For the second, suppose that $\alpha(G) \geq 4$, and choose $v \in V(G)$ that belongs to a stable set of size four. Then $G[M(v)]$ has a stable set of size three, and it is connected by (3), and has no domineering 3-path as we just showed, contrary to the minimality of $|G|$. This proves (4).

(5) *For each edge uv , if $w \in N(u) \setminus N[v]$ and C is a component of $G \setminus (N(u) \cup N(v))$, then w is complete or anticomplete to $V(C)$.*

Suppose not; then there is an edge ab of C such that w is adjacent to a and not to b . But then $v-u-w-a-b$ is a copy of P_5 , a contradiction. This proves (5).

(6) *If v belongs to a stable set of size three, then for each edge uv , $G \setminus (N(u) \cup N(v))$ has exactly two components, both complete graphs.*

Certainly it has at most two components, since $\alpha(G) = 3$, and for the same reason, if $G \setminus (N(u) \cup N(v))$ has two components then they are both complete graphs. Thus we just need to show that $G \setminus (N(u) \cup N(v))$ has at least two components.

Let $C = G \setminus (N(u) \cup N(v))$. Suppose that C has at most one component. By (1), $N(u) \setminus N[v]$ is nonempty. Let $w \in N(u) \setminus N[v]$; then since $v-u-w$ is not domineering, it follows that w has a neighbour in $V(C)$, and hence is complete to $V(C)$ by (5). So C is non-null, and $N(u) \setminus N[v]$ is complete to $V(C)$. Now $M(v) = V(C) \cup (N(u) \setminus N[v])$, and $V(C), N(u) \setminus N[v]$ are both nonempty. If there is an induced path $a-b-c$ with $a, c \in N(u) \setminus N[v]$ and $b \in V(C)$, it follows that $a-b-c$ is domineering in $G[M(v)]$, contrary to (4); and similarly there is no induced path $a-b-c$ with $b \in N(u) \setminus N[v]$ and $a, c \in V(C)$. Thus, $V(C) \cup (N(u) \setminus N[v])$ is a clique, contradicting that v belongs to a stable set of size three. That proves (6).

(7) *Every vertex belongs to a stable set of size three; and so for every edge uv , $G \setminus (N(u) \cup N(v))$ has exactly two components, both complete graphs.*

Let X be the union of all stable sets of size three. If $X \neq V(G)$, then since G is connected, there is an edge uv with $u \notin X$ and $v \in X$. But then by (6), $G \setminus (N(u) \cup N(v))$ has exactly two components, both complete graphs, and consequently u belongs to a stable set of size three, a contradiction. This proves (7).

(8) *For every edge uv , $G \setminus (N(u) \cup N(v))$ consists of two nonadjacent vertices.*

By (7), $G \setminus (N(u) \cup N(v))$ has exactly two components C_1, C_2 , both complete graphs. For $i = 1, 2$, let X_i be the set of vertices in $N(u) \cup N(v)$ that have a neighbour in $V(C_i)$. Suppose that $c_1, c'_1 \in V(C_1)$ are distinct. From (7) applied to the edge $c_1 c'_1$, it follows that $X_2 \subseteq X_1$ (since otherwise the set of vertices nonadjacent to both c_1, c'_1 induces a connected subgraph). Also, again by (7) applied to the same edge, $N(u) \setminus N[v] \subseteq X_1$ (since for each $w \in N(u) \setminus N[v]$, if $w \notin X_1$ then $\{v, u, w\}$ induces a three-vertex path, contrary to (7)) and similarly $N(v) \setminus N[u] \subseteq X_1$.

Suppose that some $w \in N(u) \setminus N[v]$ belongs to X_2 (and X_1). Then w is complete to $V(C_1)$ and $V(C_2)$, both by (5), and the set of vertices nonadjacent to both w, c_1 is a subset of $N[v]$ including v (because w is complete to $V(C_1) \cup V(C_2)$, and c_1 is complete to $N(u) \setminus N[v]$); and so this subset induces a connected subgraph, contrary to (7). Similarly, there is no $w \in N(v) \setminus N[u]$ which belongs to X_2 . Thus $X_2 \subseteq N(u) \cap N(v)$.

Suppose that $c_2, c'_2 \in V(C_2)$ are distinct. Let $w \in N(u) \setminus N[v]$, which exists by (1). Then the set of vertices nonadjacent to both c_2, c'_2 includes u, v, w , and these three vertices induce a path, contrary to (7). So $|C_2| = 1$, say $C_2 = \{c_2\}$. Choose $d \in N(u) \cap N(v)$ adjacent to c_2 (which necessarily exists, as c_2 is not an isolated vertex); then $u-d-c_2$ is domineering, a contradiction. This proves (8).

From (8) and since $\alpha(G) = 3$, it follows that the complement of G is bijoined; and so from 2.4, G has a vertex of degree zero, contradicting that it is connected. This proves 3.1. ■

4 Bounding the cop number

In order to prove our main result, we need the following definition and two lemmas regarding it.

We say that a subgraph H of a graph G is P_3 -connected if H is connected, and for every pair of edges $e, f \in E(H)$ there exists a sequence of edges $e = e_0, e_1, \dots, e_k = f$ such that e_i and e_{i+1} are two edges of an induced P_3 in G for every $0 \leq i \leq k-1$. Note that the property of being P_3 -connected is not an intrinsic property of H , but depends on G : even though H is not required to be an induced subgraph of G , the pairs of edges in the definition must form induced paths in G , not just in H .

4.1. *If H is a P_3 -connected subgraph of a P_5 -free graph G and $u, v \in V(G) \setminus V(H)$ are such that $uv \in E(G)$, u is anticomplete to $V(H)$ and the endpoints of some edge $e \in E(H)$ are non-neighbours of v , then v is anticomplete to $V(H)$.*

Proof. Suppose for a contradiction that v has a neighbour in $V(H)$. As H contains at least two vertices (as it contains e) and is connected, there exists $f \in E(H)$ such that one of its endpoints is a neighbour of v . Let $e = e_0, e_1, \dots, e_k = f$ be the sequence of edges from the definition of P_3 -connectedness (necessarily, $k \geq 1$). As v is not adjacent to the endpoints of e_0 but is adjacent to at least one endpoint of e_k , there exist e_i, e_{i+1} such that v is not adjacent to the endpoints of e_i (say, x, y) but is adjacent to the other end of e_{i+1} (say, z). By the choice of the sequence, x is not adjacent to z . Then, $u-v-z-y-x$ is an induced P_5 in G , which is a contradiction. This completes the proof of 4.1. ■

4.2. *If H is a P_3 -connected subgraph of a connected graph G and $v \in V(G)$, then either*

- (4.2.1) *there exists a P_3 -connected subgraph H' of G such that $H \subseteq H'$ and $v \in V(H')$, or*
- (4.2.2) *there exists a P_3 -connected subgraph H' of G such that $v \in V(H')$ and some $u \in V(H')$ is complete to $V(H)$ in H' , or*
- (4.2.3) *v is complete to $V(H)$.*

Proof. Let Q be a shortest path in G with one end v and another end in $V(H)$. Let $v = v_0-v_1-\dots-v_\ell$ be the vertex set of Q , where $v_\ell \in V(H)$. If $\ell = 0$, that is $v \in V(H)$, then $H' = H$ satisfies (4.2.1), and so we assume $\ell \geq 1$.

Suppose now that $v_{\ell-1}$ has a non-neighbour in $V(H)$. Since H is connected, we may suppose that Q, v_ℓ are chosen such that for some $w \in V(H)$ we have $v_\ell w \in E(H)$ and $v_{\ell-1}w \notin E(G)$. We claim that in this case $H' = Q \cup H$ satisfies (4.2.1); let us verify that H' is P_3 -connected. As Q is a shortest path, it is necessarily induced, and so Q is P_3 -connected. We also know that H is P_3 -connected. As $v_{\ell-1}v_\ell \in E(Q)$ and $v_\ell w \in E(H)$ are two edges of an induced P_3 in G it follows that $Q \cup H$ (that is, the graph with vertex set $V(Q) \cup V(H)$ and edge set $E(Q) \cup E(H)$) is P_3 -connected, as claimed.

It remains to consider the case when $v_{\ell-1}$ is complete to $V(H)$. If $\ell = 1$, then $v = v_{\ell-1}$ and (4.2.3) holds, so we may assume $\ell \geq 2$. Let $u = v_{\ell-1}$ and let H' be a subgraph of G with $V(H') = V(Q) \cup V(H)$ and $E(H') = E(Q) \cup \{uw : w \in V(H)\}$. We claim that H' satisfies (4.2.2). Indeed, u is complete to $V(H)$ in H' and H' is P_3 -connected since Q is P_3 -connected and every edge $uw \in E(H') \setminus E(Q)$ forms a induced P_3 in G with the edge $v_{\ell-2}u = v_{\ell-2}v_{\ell-1} \in E(Q)$. This finishes the proof of 4.2. ■

We are now ready to prove 1.1.

Proof. Suppose for a contradiction that there exists a connected P_5 -free graph G on which the robber has a winning strategy to evade two cops, and choose such G with $|V(G)|$ minimum. In a series of claims we will obtain properties of this graph which will yield the desired contradiction.

(1) $\alpha(G) > 2$.

To show (1), it suffices to see that if $\alpha(G) \leq 2$, there exists a dominating set of size at most two, on which two cops may start the game and win at the following turn, which is a contradiction.

(2) *There exists a domineering path $a-b-r$ in G .*

This follows directly from 1.2 and (1). In the rest of the proof, we will always refer to a fixed domineering path $a-b-r$.

(3) *For every $v \in V(G)$, no vertex of $N(v)$ is complete to a component of $G[M(v)]$.*

Suppose otherwise that there exists $u \in N(v)$ and a component C of $G[M(v)]$ such that u is adjacent to every vertex of C . First note that $G \setminus C$ is connected as all vertices in $V(G) \setminus C$ adjacent to C are in $N(v)$. By minimality of G , there exists a winning strategy for two cops on $G \setminus C$. We will use this strategy to define a winning strategy for two cops on G , which will be a contradiction. When playing on G , we say the robber's *shadow* is on x if the robber is on $x \in V(G) \setminus C$, and is on v if the robber is on a vertex of C . In particular, the robber's shadow is always in $G \setminus C$. We show that any move of the robber yields a valid move for the robber's shadow in the sense that at every turn of the game the shadow either stays on its current vertex or moves to an adjacent vertex. At a given turn, suppose the robber moves from x_1 to x_2 (in particular, $x_1x_2 \in E(G)$).

- If $x_1, x_2 \in C$, then the robber's shadow stays on v , which is a valid move.
- If $x_1, x_2 \notin C$, then the robber's shadow also moves from x_1 to x_2 , which is a valid move
- If $x_1 \in C$ and $x_2 \notin C$, we note that necessarily $x_2 \in N(v)$. Hence, the shadow moving from v to x_2 is a valid move.
- If $x_1 \notin C$ and $x_2 \in C$, then $x_1 \in N(v)$ and so the shadow moving from x_1 to v is a valid move.

Consider the following strategy for the cops on G . First, the cops follow the winning strategy on $G \setminus C$ to capture the robber's shadow. Once the cops have captured the robber's shadow, either they have captured the robber (if the robber and its shadow are on the same vertex) or the robber is in C and its shadow is on v . In the latter case, having captured the shadow, at least one of the cops is on v . The other cop may then eventually move to u , and then capture the robber since u is complete to C . Note that in the meantime, the robber cannot leave C as it would be immediately captured by the cop on v . This proves (3).

Note that the proof of (3) is a retract (special type of homomorphism) argument which is a standard tool in the study of the game of cops and robbers. A quite general version, which is close to the one presented here, was proved by Berarducci and Intriglia [3].

(4) *For every $v \in V(G)$, $G[M(v)]$ is connected.*

Using (3), this is exactly the first part of (3) in the proof of the existence of a domineering 3-path.

(5) *There exists a strategy for two cops on G to guarantee that, in order to avoid capture, the robber moves to r .*

By (2) we have that $N[r] \subseteq N[a] \cup N[b]$. Since $ab \in E(G)$, this implies that $G \setminus r$ is a connected graph. By minimality of G , there exists a winning strategy for two cops on $G \setminus r$. The strategy for cops in (5) is to play this strategy on G as long as the robber has not entered r . If the robber never

enters r then it is eventually captured since this strategy is winning when restricted to $G \setminus r$. Hence, the robber eventually moves to r (or chooses r as its initial position). This proves (5).

In the rest of the proof, we will construct a strategy for two cops to attempt to capture the robber on G . However, since $c(G) > 2$, we may assume that the robber has, and is playing, a strategy to avoid capture. The cops' strategy will begin by employing the strategy from (5), and let $c_1, c_2 \in V(G)$ be the positions of the cops after the robber moves to r . Note that it is possible that $c_1 = c_2$.

(6) $c_1, c_2 \in M(r)$.

If (6) did not hold, one of the two cops could capture the robber at the next turn, which would be a contradiction.

(7) For each $i \in \{1, 2\}$, if $c_i \in N[a]$, then $c_{3-i} \notin N(b)$.

Suppose otherwise that for some $i \in \{1, 2\}$, $c_i \in N[a]$ and $c_{3-i} \in N(b)$. Then the cops can move in one turn from $\{c_1, c_2\}$ to $\{a, b\}$. Being on r at the start of the turn, the robber necessarily will be in $N[r]$ following its turn. As $N[r] \subseteq N[a] \cup N[b]$ by (2), the cops may then capture the robber at the following turn. This contradiction proves (7).

We say that a subgraph H of G is a *snare* if

- (H1) H is P_3 -connected,
- (H2) $V(H) \subseteq M(r)$,
- (H3) there exists $d_1 d_2 \in E(H)$ such that $d_1 \in N[c_1]$ and $d_2 \in N[c_2]$,
- (H4) $a \in V(H)$, and
- (H5) $V(H) \cap M(b) \neq \emptyset$.

We finish the proof by showing that if a snare exists then the two cops can capture the robber, and finally that a snare exists.

(8) *There is no snare.*

Suppose for a contradiction there exists a snare H . The cops are currently on c_1, c_2 . The cops move to the endpoints of an edge $d_1 d_2 \in E(H)$ as in (H3). Since H is connected, has at least one edge and $a \in V(H)$, there is an edge $f \in E(H)$ incident with a . By (H1), there exists a sequence of edges $d_1 d_2 = e_0, e_1, e_2, \dots, e_k = f$ of H such that e_i and e_{i+1} are two edges of an induced P_3 in G for every $0 \leq i \leq k-1$. We may suppose without loss of generality that none of $e_0, \dots, e_{k-1}$ is incident with a . Over the next k turns, the cops will follow this sequence of edges. In other words, in i turns (for $0 \leq i \leq k$) the cops will be on distinct endpoints of edge e_i . Finally, the cops will move to $\{a, b\}$. Note that by (H2), $b \notin V(H)$. We remark that, since the robber is following a strategy which will avoid capture, it never enters the closed neighbourhood of a cop.

Let r' be the first vertex visited by the robber in $M(b)$, and let R be the set of all the previous positions of the robber. Note that $R \subseteq N[b]$. Such an r' exists as once one of the cops is on b (which always happens in the strategy described above), the robber must move to $M(b)$ as it would otherwise be captured at the next turn.

We claim that R is anticomplete to $V(H)$. Suppose otherwise that at least one vertex of R has a neighbour in $V(H)$. By (H2), r has no neighbours in $V(H)$, and so there must exist consecutive

positions of the robber $r_1, r_2 \in R$ such that r_1 has no neighbours in $V(H)$ (and is not itself in $V(H)$) but r_2 has at least one neighbour in $V(H)$. Since $r_2 \in N[b]$, the cops are not on $\{a, b\}$ when the robber moves to r_2 , and so they are positioned on distinct endpoints of an edge of H . These endpoints are non-neighbours of r_2 , and so it follows from 4.1 that r_2 has no neighbours in $V(H)$. This is a contradiction, which implies our claim. Note that this implies that $r' \notin V(H)$.

If the cops are positioned on an edge of H when the robber first moves to r' , then the same argument yields that r' has no neighbour in $V(H)$, and because of (H4) we in particular have that $r'a \notin E(G)$. Otherwise, the cops are on a, b when the robber moves to r' , and so $r'a \notin E(G)$. Hence, in all cases $r'a \notin E(G)$.

As $N[r] \subseteq N[a] \cup N[b]$, it follows that $r'r \notin E(G)$. As $G[R]$ is connected, and r' has a neighbour in R there exists an induced path r_1-r_2-r' such that $r_1, r_2 \in R$. Suppose that there exists an edge $xy \in E(H)$ such that $r'x \in E(G)$, but $r'y \notin E(G)$. Then $r_1-r_2-r'-x-y$ is an induced P_5 . Hence, no such edge exists. As H is connected and $r'a \notin E(H)$ (in particular, r' is not complete to $V(H)$) it follows that r' is anticomplete to $V(H)$. Since $ab \in E(G)$ and $a \in V(H)$, it follows that b is not anticomplete to H . However, by (H5) b is not complete to H . Since H is connected, there exists $xy \in E(H)$ such that $bx \in E(G)$ but $by \notin E(G)$. Then $r'-r_2-b-x-y$ is an induced P_5 . This contradiction finishes the proof of (8).

It remains to show that there exists a snare in G , a contradiction.

(9) *There exists a snare.*

Let $G' = G[M(r)]$. By (4) G' is connected. Let P be an induced path in G' with ends c_1 and c_2 . Then P is P_3 -connected, and either $c_1 = c_2$ or P contains an edge satisfying (H3), as it has at most three edges.

Suppose first that $c_1 \in M(b)$. By 4.2 applied to G' with $H = P$ and $v = a$, either we find a subgraph H' satisfying (4.2.1) or (4.2.2), or $c_1, c_2 \in N[a]$. In the first and second cases, H' is a snare; property (H3) is the only one which takes a little effort to verify. If H' satisfies (4.2.1) and $c_1 \neq c_2$, then P contains an edge satisfying (H3) as noted above. If H' satisfies (4.2.1) and $c_1 = c_2$, then c_1x satisfies (H3) for any neighbour x of c_1 in H' ; note that x exists by connectivity of H' unless $|V(H')| = 1$, that is $a = c_1 = c_2$, which contradicts that $c_1 \in M(b)$. If H' satisfies (4.2.2) and $u \in V(H')$ is complete to $V(P)$ in H' then uc_1 satisfies (H3). In the remaining case, $c_1, c_2 \in N[a]$ and $G'[\{c_1, a\}]$ is a snare.

Thus we may assume that $c_1, c_2 \in N(b)$. By (7) we have $c_1, c_2 \notin N[a]$. By (3) there exists $d \in V(G') \cap M(b)$. Assume first that $\{c_1, c_2, d\}$ is not a clique of size three. By 4.2 applied to G' with $H = P$ and $v = d$, there exists a P_3 -connected subgraph H^* of G' such that $c_1, c_2, d \in V(H^*)$, and H^* contains a path with ends c_1 and c_2 with at most three edges. Indeed, if (4.2.1) or (4.2.2) holds then $H^* = H'$ has the required properties. If (4.2.3) holds then $H^* = G'[\{c_1, c_2, d\}]$ is either an induced P_2 if $c_1 = c_2$ or an induced P_3 , as in this case d is adjacent to c_1 and c_2 and $\{c_1, c_2, d\}$ is not a clique of size three, and so H^* is again as required.

We now apply 4.2 to G' with $H = H^*$ and $v = a$. As $c_1, c_2 \notin N[a]$ and $c_1, c_2 \in V(H^*)$, a is not complete to $V(H^*)$ and (4.2.3) does not hold. Thus (4.2.1) or (4.2.2) holds, that is there exists a P_3 -connected subgraph H' of G' such that $c_1, c_2, d, a \in V(H')$ and H' still contains a path with ends c_1 and c_2 with at most three edges. It is routine to check that H' is a snare.

It remains to consider the case when $C = \{c_1, c_2, d\}$ is a clique of size three. We proceed similarly to the proof of 4.2. Let Q be a shortest path from a to C in G' with vertices $a = v_0-v_1-\dots-v_\ell$, where

$v_\ell \in C$. As $c_1, c_2 \notin N[a]$, we have $a \notin C$ and so $\ell \geq 1$.

Assume first that $\ell \geq 2$ and let H be the subgraph of G' defined by $V(H) = V(Q) \cup C$, and

$$E(H) = E(Q) \cup \{v_\ell u : u \in V(C), v_{\ell-1}u \notin E(G')\} \cup \{v_{\ell-1}u : u \in V(C), v_{\ell-1}u \in E(G')\}.$$

As Q is P_3 -connected, H is connected, and every edge in $E(H) \setminus E(Q)$ forms an induced P_3 in G either with the edge $v_{\ell-2}v_{\ell-1}$ or $v_{\ell-1}v_\ell$, it follows that H is P_3 -connected. Note that $V(H) \subseteq V(G') = M(r)$ and $a, d \in V(H)$. Let us show with what choice of edge (H3) holds. If at least one edge with both ends in C is in H , then pick such an edge. Otherwise, it follows from the definition of H that $v_{\ell-1}$ is necessarily complete to C , and so we can pick $c_1v_{\ell-1}$. Hence, H is a snare.

It remains to consider the case $\ell = 1$. In this case a has a neighbour in C , and so ad is the unique edge from a to C , as $c_1, c_2 \notin N[a]$. Then $H = G'[\{a, d, c_1\}]$ is an induced P_3 in G' and the edge dc_1 satisfies (H3). It follows that H is a snare in this last case.

This completes the proof of (9) and thus of the theorem. ■

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References

- [1] M. Aigner and M. Fromme. A game of cops and robbers. *Discrete Applied Mathematics*, 8(1):1 – 12, 1984. doi:10.1016/0166-218X(84)90073-8.
- [2] T. Andreae. On a pursuit game played on graphs for which a minor is excluded. *Journal of Combinatorial Theory, Series B*, 41(1):37–47, Aug. 1986. doi:10.1016/0095-8956(86)90026-2.
- [3] A. Berarducci and B. Intrigila. On the cop number of a graph. *Advances in Applied Mathematics*, 14(4):389–403, 1993. doi:10.1006/aama.1993.1019.
- [4] P. Erdős and A. Hajnal. Ramsey-type theorems. *Discrete Applied Mathematics*, 25(1-2):37–52, Oct. 1989. doi:10.1016/0166-218X(89)90045-0.
- [5] L. Esperet. *Graph colorings, flows and perfect matchings*. Habilitation thesis, Université Grenoble Alpes, 2017. URL: <https://theses.hal.science/tel-01850463/document>.
- [6] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 2001. doi:10.1007/978-1-4613-0163-9.
- [7] U. K. Gupta, S. Mishra, and D. Pradhan. Cops and robber on subclasses of P_5 -free graphs. *Discrete Mathematics*, 346(6):113353, June 2023. doi:10.1016/j.disc.2023.113353.
- [8] A. J. Hoffman and R. R. Singleton. On Moore graphs with diameters 2 and 3. *IBM Journal of Research and Development*, 4(5):497–504, 1960. doi:10.1147/rd.45.0497.

- [9] C. T. Hoàng, M. Kamiński, V. Lozin, J. Sawada, and X. Shu. Deciding k -colorability of P_5 -free graphs in polynomial time. *Algorithmica*, 57(1):74–81, May 2010. doi:10.1007/s00453-008-9197-8.
- [10] R. Javadi and A. Momeni. The game of cops and robber on (claw, even-hole)-free graphs, Jan. 2022. URL: <http://arxiv.org/abs/2112.07503>.
- [11] G. Joret, M. Kaminski, and D. O. Theis. The cops and robber game on graphs with forbidden (induced) subgraphs. *Contributions to Discrete Mathematics*, 5, Sept. 2010. doi:10.11575/cdm.v5i2.62032.
- [12] M. Liu. The cop number of graphs with forbidden induced subgraphs. *arXiv: 1908.11478*, 2019. URL: <https://arxiv.org/abs/1908.11478>.
- [13] M. Masjoody. Confining the robber on cographs. *Contributions to Discrete Mathematics*, 18(1):1–14, Apr. 2023. URL: <https://cdm.ucalgary.ca/article/view/71282>.
- [14] M. Masjoody and L. Stacho. Cops and robbers on graphs with a set of forbidden induced subgraphs. *Theoretical Computer Science*, pages 186–194, 2020. doi:10.1016/j.tcs.2020.06.032.
- [15] T. Nguyen, A. Scott, and P. Seymour. Induced subgraph density. VIII. P_5 has the Erdős-Hajnal property. In preparation.
- [16] R. Nowakowski and P. Winkler. Vertex-to-vertex pursuit in a graph. *Discrete Mathematics*, 43(2):235 – 239, 1983. doi:10.1016/0012-365X(83)90160-7.
- [17] J. Petr, J. Portier, and L. Versteegen. A note on cops and robbers, independence number, domination number and diameter. *Discrete Mathematics*, 346(1):113175, Jan. 2023. doi:10.1016/j.disc.2022.113175.
- [18] A. Quilliot. *Thèse de 3e cycle*. PhD Thesis, Université de Paris VI, 1978.
- [19] V. Sivaraman. An application of the Gyárfás path argument. *Discrete Mathematics*, 342(8):2306–2307, 2019. doi:10.1016/j.disc.2019.04.020.
- [20] V. Sivaraman. Cop number of graphs without long holes. *arXiv: 2001.00477*, Dec. 2019. URL: <http://arxiv.org/abs/2001.00477>.
- [21] V. Sivaraman and S. Testa. Cop number of $2K_2$ -free graphs. *arXiv: 1903.11484*, 2019. URL: <https://arxiv.org/abs/1903.11484>.
- [22] J. Turcotte. Cops and robbers on $2K_2$ -free graphs. *Discrete Mathematics*, 345(1):112660, Jan. 2022. doi:10.1016/j.disc.2021.112660.