Secondary Mathematics Teachers' Anticipations of Student Responses to Cognitively Demanding Tasks

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Abstract

This study examines secondary mathematics teachers' anticipations of student responses related to a series of cognitively demanding mathematics tasks from multiple mathematical domains presented in the context of voluntary and asynchronous online professional development modules. We analyze 283 anticipations made by 127 teachers to 17 mathematics tasks and present four distinct foci of teachers' anticipations. Teachers focused on actions students might take, ways they might think about the task, how they might react emotionally, and what actions they might take in advance or in response to their anticipations. We conclude with a discussion of ways our results can inform efforts to support improvements in mathematics teachers' practice of anticipating.

Keywords

Secondary mathematics teachers; Anticipating; Instructional practices; Professional development

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Over the last two decades, researchers have identified a number of practices to support students in learning mathematics in powerful ways. Characterized as high leverage (Ball & Forzani, 2009), core (McDonald et al., 2013), high quality (Cobb & Jackson, 2015), or ambitious and equitable (Jackson & Cobb, 2010; Lampert et al., 2013), these practices have become an increasingly central part of teachers' instruction and an expectation of policies aimed at improving mathematics teaching and learning at scale. Many of these practices are viewed as a "standard" part of mathematics teaching and teacher preparation (National Council of Teachers of Mathematics, 2014; Association of Mathematics Teacher Educators, 2019) and have been integrated into systems of teacher evaluation (National Board of Professional Teaching Standards, 2010; Stanford Center for Assessment, Learning, & Equity, 2015;).

One of the most prominent frameworks for instructional practices used in mathematics teacher preparation and professional development over the last decade is Smith and Stein's (2011) *5 Practices for Orchestrating Productive Mathematics Discussions*. Created as a tool to scaffold teachers who are working to center their instruction on students' mathematical ideas (Stein et al., 2008), the framework outlines a set of practices to assist teachers in preparing for and leading discussions where students deepen their understanding of mathematics through engagement with one another's ideas. Stein and her colleagues (2008) conceptually develop the sequential practices of *anticipating* students' responses, *monitoring* students' progress, *selecting* and *sequencing* students' approaches for discussion, and *connecting* the salient mathematical ideas within students' shared work, and they argue that teachers' enactments of each of these practices across a lesson are enabled or constrained by the previous one. For instance, the quality and types of connections a teacher is able to facilitate in a discussion depends on the number of approaches they have selected and sequenced for the class to consider. Likewise, the extent to which the approaches and ideas identified are likely to be pedagogically productive and accessible to the class depends on the nature and quality of ideas a teacher fosters while monitoring students' explorations of the task. The ability to quickly recognize and understand students' approaches and make decisions about the kinds of support most likely to be beneficial for individual and collective learning is directly linked to teachers' anticipations of how students will engage with the mathematical task. Thus, a teacher's anticipations can significantly support or impede their ability to enact other high quality instructional practices and, as a result, their students' learning.

While influential in practice, research on how teachers learn and enact the *5 Practices* is only beginning to emerge (Heck et al., 2019; Hewitt, 2020). This is particularly true for the practice of anticipating, with relatively few studies examining secondary mathematics teachers' anticipations. This study aims to contribute to this emerging knowledge base by investigating secondary mathematics teachers' anticipations for a series of cognitively demanding mathematics tasks presented in the context of voluntary and unmoderated online professional learning modules. Specifically, we ask: *What do practicing secondary mathematics teachers anticipate students will do when asked to focus on the mathematics of a cognitively demanding task?*

In what follows, we present a conceptualization of teachers' anticipations that guided our investigation, review the literature on teachers' anticipations in mathematics and science education, and identify two limitations of the knowledge base. Next, we detail the context of our examination of secondary mathematics teachers' anticipations and describe our research design to address these limitations. We then present four distinct foci of teachers' anticipations for 17 cognitively demanding mathematics tasks from algebra and functions, geometry, and statistics and probability and then conclude with a discussion of the ways these foci can inform efforts to support improvements in the practice of anticipating.

Anticipating Student Responses and Teachers' Anticipations

Our conception of teachers' anticipations is grounded in Stein and colleagues' (2008) description of the practice of anticipating students' responses. They define the practice of anticipating as:

> developing considered expectations about how students might mathematically interpret a problem, the array of strategies—both correct and incorrect—they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn (Stein et al., 2008, p. 322).

Here, we distinguish *anticipating student response*s—the process (i.e., practice) of developing and articulating expectations of students' interpretations, approaches, and their relationships to specific mathematical learning goals—from *teachers' anticipations*, which we consider to be the articulated expectations resulting from the process of anticipating student responses. From this perspective, teachers' anticipations may be broader than typical definitions of anticipating student responses, possibly including predictions (Carpenter et al., 1988) or expected reactions related to a specific student or group of students (Fernandez & Yoshida, 2012; Lewis & Hurd, 2011).

Several investigations have provided evidence for Stein and colleagues' (2008) argument that teachers' anticipations support other practices of ambitious teaching. In their study of nine primary grades teachers' participation in Lesson Study, Vale and colleagues (2019) showed that a teacher's anticipations supported them in monitoring students' engagement as they worked on an instructional task and in purposefully selecting and sequencing students' approaches for discussion. Other researchers have documented that some of the solutions anticipated by teachers became a part of the classroom discussion when teaching the lesson (Akyuz et al., 2013; Janike, 2019). Research also suggests that although anticipating student responses can be difficult for teachers, professional learning opportunities can lead to improvements (Nickerson & Masarik, 2010). For example, researchers focused on the ways teachers learn and use frameworks of students' mathematical thinking have documented how such frameworks assist teachers in anticipating a variety of strategies, representations, and solutions (Edgington, 2012; Krause et al., 2016; Wilson et al., 2015). For example, Wilson et al. (2015) described how 19 elementary grades teachers participating in their yearlong professional development program on learning trajectories used their knowledge of a trajectory and Smith and Stein's (2011) *5 Practices* to enact high quality lessons. Teachers in their study used the trajectory to anticipate multiple strategies and hypothesize what these approaches suggested about what students might know and be ready to learn.

Though these studies suggest that anticipating student responses can assist teachers in enacting other pedagogical practices that are responsive to students and their thinking, research on what teachers anticipate is surprisingly limited. Of the existing empirical research of teachers' anticipations in the mathematics education literature, a large proportion of studies focus on teachers' anticipations of particular aspects of student work such as their solutions, strategies, and other specific and observable details of students' mathematical work on a particular task or a collection of tasks within a specific mathematical domain (Akyuz et al., 2013; Didaş Kabar &

Erbaş, 2021; Hughes, 2007; Kartal et al., 2020; Krause et al., 2016; Morrissey et al., 2019; Rupe, 2019; Şen Zeytun et al., 2010; Vale et al., 2019). For example, Krause et al. (2016) examined the number of distinct and valid strategies to equal-sharing fraction problems anticipated by 18 upper elementary grade teachers in the context of a multi-year professional development program. Similarly, Didaş Kabar & Erbaş (2021) investigated the depth of detail, number, and focus of 25 secondary preservice teachers' anticipations of student responses to a series of modeling problems in a one-semester undergraduate elective course. Some investigations report that teachers also anticipate student errors and incorrect solutions (Akyuz et al., 2013; Didaş Kabar & Erbaş, 2021; Hughes, 2006; Janike, 2019; Morrissey et al., 2019; Şen Zeytun et al., 2010) or various representations that students will use (Akyuz et al., 2013; Kartal et al., 2020). For example, Akyuz et al. (2013) reported that anticipating incorrect solutions and student difficulties was a key part of an expert middle grades mathematics teachers' planning.

In addition to anticipating observable aspects of students' written work, researchers report that some teachers also attend to students' ways of thinking when they anticipate. In these studies, teachers describe alternative conceptions that students may hold (Didaş Kabar & Erbaş, 2021; Şen Zeytun et al., 2010) or difficulties and unexpected ways of thinking (Morrissey et al., 2019; Nickerson & Masarik, 2010) they believe students will use or have when engaging with a particular task. In their study of the relationship between five secondary mathematics teachers' covariational reasoning and predictions about their students' engagement in modeling tasks for example, Şen Zeytun et al. (2010) reported that all teachers believed the modeling tasks would be difficult for their students, highlighting potential challenges for students in creating representations, identifying independent and dependent variables, and making assumptions about the problem context.

In addition to considerable variations in the focus of teachers' anticipations, research also suggests the quantity and depth of anticipations varies (Hughes, 2006; Kartal et al., 2020; Krause et al., 2016; Morrissey et al., 2019; Nickerson & Masarik, 2010; Şen Zeyton et al., 2010). In their investigation of 88 elementary preservice teachers' anticipations of student responses to cognitively demanding tasks for example, Kartal and colleagues (2020) found that though 69 teachers anticipated solutions that entailed evidence of both conceptual and procedural understandings, only 25 anticipated strategies that were mathematically distinct. Similarly, Didaş Kabar & Erbaş (2021) report that though the majority of the 25 secondary preservice teachers in their study anticipated both student strategies and potential difficulties when solving four modeling problems, their anticipations varied in their detail and awareness of possible differences in student thinking across tasks.

While these studies have begun to map the focus of teachers' anticipations, we note two limitations of this emerging body of research. First, the majority of studies have examined a relatively small number of elementary and middle grades teachers' anticipations, and the investigations of secondary mathematics teachers' anticipations almost exclusively focus on prospective teachers. Additional research with a larger number of practicing secondary teachers could provide insights on the relative similarities and differences in teachers' anticipations across grade bands and with different levels of experience and inform efforts to design tools and environments for mathematics teacher educators. Second, research to date has predominantly occurred in specific and highly facilitated contexts such as university coursework or professional development programs, many of which had introduced the practice of anticipating student responses as a focus of professional learning. Additional studies documenting teachers' anticipations in less formal settings and in the absence of an explicit focus or support for the practice could provide mathematics teacher educators and researchers with a better understanding of what teachers understand the practice of anticipating to entail and how they envision it informing their planning. Such insight could assist mathematics teacher educators to design learning experiences using teachers' intuitive ideas about the practice as a foundation upon which they might build resources and learning experiences.

In what follows, we describe our study of practicing secondary mathematics teachers' anticipations that addresses some of the limitations in the literature summarized in the previous paragraph. Using data collected from a series of voluntary and unmoderated online professional learning modules created to support mathematics teachers' implementation of new state mathematics standards, we examine 127 secondary mathematics teachers' anticipations related to 17 cognitively demanding instructional tasks spanning the domains of algebra and functions, geometry, and statistics and probability. We present four foci of teachers' anticipations and then discuss ways in which our findings may inform mathematics teacher educators' efforts to support mathematics teachers in developing their practice of anticipating student responses.

Methods

To investigate our research question, we utilized an instrumental case study design (Yin, 2009). Instrumental case studies provide insight into a particular phenomenon. In this case, the phenomenon is how secondary teachers respond when asked to anticipate student responses to cognitively demanding tasks outside of the context of lesson planning (i.e., tasks were presented, not selected by the teachers, not in the context of planning for their own students in their own classrooms). This particular case is bound by context and time—teachers' anticipations within the context of unmoderated online professional development modules between February 2017 and March 2019. This context, which we detail in the following section, is important as we recognize that anticipating student responses to a mathematics task that one did not select and outside of the context of a specific group of students might be different from anticipating student thinking in the context of planning a lesson for a particular group of students. However, given the limited research focused on secondary mathematics teachers' practice of anticipating, the broad participation of teachers in the professional learning module discussion boards has the potential to provide much needed insight into this phenomenon. In the following sections, we provide details regarding the context, participants, data collection, and data analysis.

Context

This study occurred within the context of implementing newly revised state high school mathematics standards for a southeastern state in the United States. As part of the implementation process, 20 professional learning modules were created within a Learning Management System (LMS) and made freely available to every high school mathematics teacher in the state. The modules were made available in January of 2017 and aimed to support teachers in making sense of the newly revised standards. Each module was designed to facilitate an exploration of specific content and mathematical practice standards for a unit within each of the state's three integrated high school mathematics courses. While the modules were focused on teacher learning with respect to the standards, the professional learning activities within each module were aligned with a particular view of teaching—one that is student centered, emphasizes student sense making, and one in which mathematical discourse is valued. To that end, the professional learning activities in each module were developed around one cognitively demanding mathematics task that served as a context for examinations of video, asynchronous discussions, and analyses of student work. For example, *Floating Down the River* (see Figure 1) served as the focal task for the module developed for the Equations and Introduction to Functions unit of the first course.

Figure 1

Example of a Module Focus Task - Floating Down the River

Floating the River (adapted from the Mathematics Vision Project)
Unit 1: Equations and Introduction to Functions

Three friends floating down the South Fork of the New River in Ashe County. The last time they went, they noticed that when they were between the rapids, they weren't going very fast and the water was deep. However, the water was much more shallow when they were going fast. They decided to collect measurements every ten minutes of the river's depth, their speed, and how far they had traveled down the river.

- Friend 1 measured the depth of the river with a fishing weight tied to a string
- Friend 2 used a waterproof GPS to measure their speed in miles per hour
- Friend 3 used a running app on a smartphone to measure the total distance they had traveled in miles.

Given different representations of these measurements, describe their trip down the river. Do you agree or disagree with their observation?

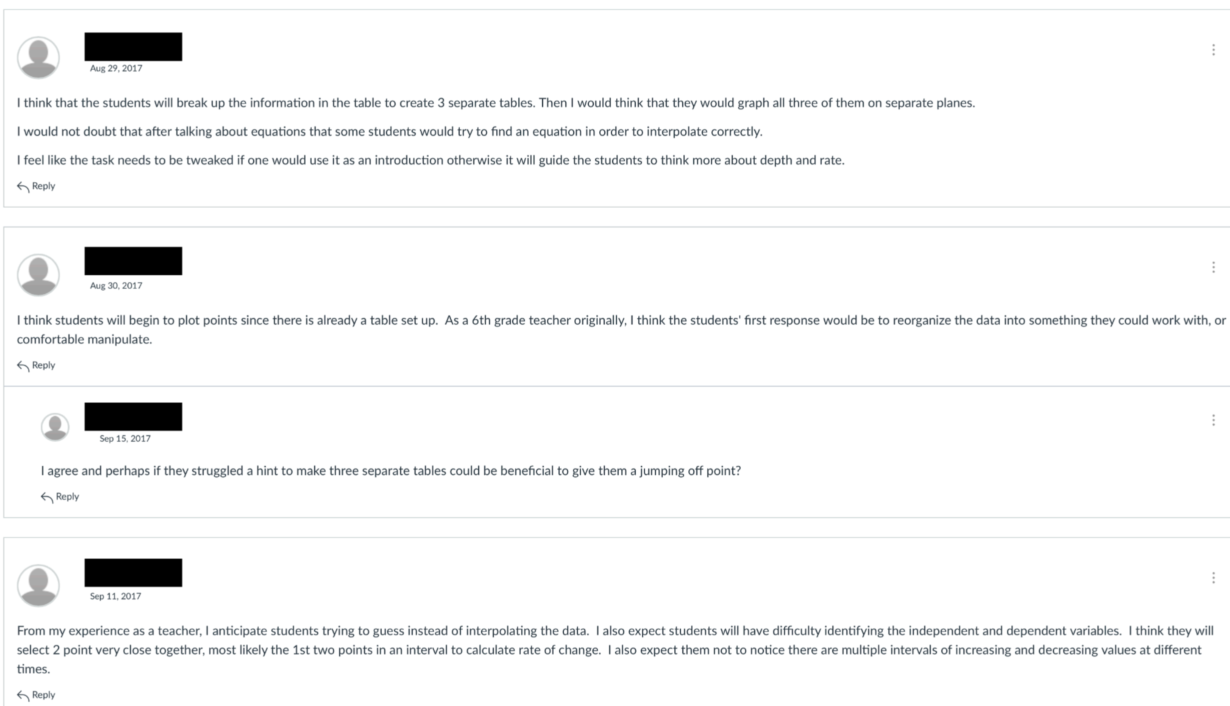
Time (min)	Depth (ft)	Rate (miles per hour)	Distance (miles)
0	4	2	0
10	4	2	0.33
20	3	2.5	0.74
30	2	4	1.15
40	2	4	1.82
50	2	4	2.49
60	3	2.5	3.00
70	3	2.5	3.42
80	3	2.5	3.84
90	3	2.5	4.25
100	1	8	4.93
110	1	8	6.27
120	8	1	6.82
130	8	1	6.99
140	8	1	7.17

One activity included in each module asked teachers to anticipate student responses to the focal task. At the beginning of each module, teachers were presented with the task, asked to solve it themselves, and then asked to respond on a discussion board to the following prompt:

Focusing on the mathematics of the task, what do you anticipate students will do? List any and all methods you think students will use. The discussions were set up so that responses were not required, and prior participant responses were visible. This setup allowed teachers to participate in several different ways including reading prior responses, making a post with or without reading others’ responses, and/or commenting or liking previous posts (see Figure 1). These posts to the Anticipating Student Responses discussion board from each of the 20 professional learning modules are the focus of our investigation.

Figure 2

Sample Discussion Board Response Thread



Participants

All secondary (9-12) mathematics teachers within this southeastern state were invited to engage with the professional learning modules and participate in the study. Approximately 3,100 had enrolled in the LMS and as of March 2019, and 1,495 had engaged with the materials in at least one of the 20 modules. Of the 1,485 teachers who had engaged, 127 had posted to at least one of the anticipating student responses discussion boards. These 127 teachers represented 18 different school districts across the state, with 13 working in urban districts, 42 in suburban, 62 in rural, and 10 from charter or independent schools.

Data Collection and Analysis

Data for this study include all initial posts to each of the 20 anticipating student responses discussion boards from their launch in January 2017 through to March 2019. We chose to exclude replies that stemmed from initial posts because replies were often incomplete ideas connected to prior posts and difficult to connect to specific anticipations. In Figure 2 for example, the first, second, and fourth posts were original anticipations from participating teachers and were thus considered data for the study. The third post was a reply of agreement with the previous teacher and was therefore not considered as data for this study. All initial posts from each of the discussion boards were copied and pasted into a document and saved as a pdf file. Three of the 20 discussion boards did not have any posts at the time. Overall, there were 283 individual posts representing teachers' anticipations from 127 unique teachers across the 17 discussion boards.

Each of the 17 documents was uploaded to a qualitative research software program. We defined an individual teacher's post to be the unit of analysis for the study and began our analysis by reviewing the data with an eye toward Smith and Stein's (2011) description of anticipating student responses, which guided the design of the discussion prompt; that is, we explored the data to determine whether identifying the various mathematical strategies, representations, solutions, and interpretations that teachers described would be useful in answering our research question. After our first pass using this approach, it was evident that the teachers had a broader view of what it meant to anticipate student responses than Smith and Stein's description. Despite an explicit focus on the mathematics of the task in the prompt, some posts included descriptions of how students might "become frustrated" or "give up." Other posts discussed ways they might respond if a particular anticipation occurred when teaching. As a result, we elected to use open coding and a constant comparative method as a team until we settled on a final set of data-driven codes (Ryan & Bernard, 2003).

As summarized in Table 1, our codes describe the foci of teachers' anticipations that we interpreted from the data. Teachers' posts included anticipations that attended to students' mathematical actions, students' mathematical thinking, students' affective responses to the task, and teachers' instructional choices during task implementation, either in preparation or response, to their anticipations. We also included an *Other* code to capture foci not represented in our codebook. Our coding process involved assigning one or more of these codes to each discussion board post. For example, one teacher anticipating student responses to the Floating the River task posted, "I also believe students will draw three separate graphs but might have a difficult time determining just what data to use to draw the graphs" (2:11). The post was assigned the *Students' Mathematical Actions* code because it referred to students drawing three graphs—an action the teacher might observe—and the *Students' Mathematical Thinking* code because of the inference that students might have "a difficult time."

Table 1

Teachers' Anticipations Codebook

Code	Definition	Example
Students' Mathematical Actions	Responses focused on observable mathematical actions students might do, use, or say	I think that the students will break up the information in the table to create 3 separate tables. Then I would think that they would graph

		all three of them on separate planes.
Students' Mathematical Thinking	Responses focused on inferences of student thinking based on assumptions about the cognitive resources students will bring to the task	Students may not recognize the units of measure as they will get confused with "miles per hour" versus minutes
Students' Affective Responses to the Task	Responses focused on emotional and physical responses students will have to the task	I anticipate students freezing. They will need more direction such as graphing on 3 separate graphs.
Teachers' Instructional Choices During Task Implementation	Responses focused on the teachers' past, present, or future actions	*I think I might separate the tables into 3 different tables, and maybe have each group create a graph for a table given to them.*
Other	Responses focused on ideas not represented by other codes	*Interpret key features of graphs, tables, and equations of square root functions. Include appropriate domain and range, increasing and decreasing, positive and negative and end behavior. Generate different representations (equation, graph, table), by hand in simple cases and using technology for more complicated cases, to show key features (same as above). [Note: This is a restatement of a specific state math standard.]*

Our process for developing the codebook, team coding, and determining reliability was guided by DeCuir-Gunby et al.'s (2012) recommendations for codebook development and use. After finalizing our codebook, to establish reliability we randomly selected a subset of data for all five team members to code and repeated this process until the team consistently applied the same codes to the data. Once reliability was established, all data were coded independently by two members of the team, and any inconsistency was considered by the entire team until reaching consensus. After coding was complete, the team reviewed the data for emerging themes and worked collaboratively to construct a case narrative for each of the themes.

Findings

Summarized in Table 2, our analysis of the 283 posts revealed that when asked to anticipate with a focus on the mathematics of cognitively demanding tasks, teachers' anticipations had at least one of four foci: students' mathematical actions, students' mathematical thinking, students' affective responses to the task, and teachers' instructional choices during task implementation. Teachers most commonly included considerations of students' mathematical actions and students' mathematical thinking in their anticipations, which is similar to the way that anticipating student responses is often described in mathematics education (e.g., Smith & Stein, 2011). While less common, it is notable that teachers also anticipated student affective responses to mathematics tasks and included anticipations of their own practice. We do not address posts that were coded as "Other" because this code ended up capturing responses in the discussion board that did not address the prompt in any way (see the example in Table 1). In what follows, we describe each of the four foci of teachers' anticipations in greater detail and provide examples to illustrate the variation within each category.

Table 2

Distribution of n = 283 of Teachers' Anticipations with Different Foci

Students' Mathematical Actions		227 (80%)
	Mathematical Representations	147
	Mathematical Strategies	93
	Physical and Social Resources	88
Students' Mathematical Thinking		225 (80%)
	Difficulties	64

Prior Knowledge	51
Mathematical Connections	46
Students' Affective Responses to the Task	18 (6%)
Teachers' Instructional Choices During Task Implementation	70 (25%)
Preparing to Respond to Students	32
Preparing for Instruction	29
Other	5 (2%)

Students' Mathematical Actions

There were 227 posts (80% of all posts) that attended to observable actions that students might take when engaging with the task. These posts commonly included anticipations focused on creating mathematical representations, using specific mathematical strategies, and using physical or social resources.

Mathematical Representations

More than 75% of the posts that attended to students' mathematical actions included anticipations focused on creating mathematical representations (n = 147; 77%). These included anticipating the use of graphs, tables, pictures, and/or equations to make sense of the task, determining mathematical relationships, or using the representation as an intermediate step in the solving process. For example, when anticipating student actions on the Floating the River task, one teacher wrote, "I would think that the students would start by drawing 3 different graphs. I also think some students may have different graphs based on whether they use positive or negative values to represent the depth" (2:18). In this post, the teacher anticipated students would create graphical representations to make sense of the task and how their representations might differ according to how they interpreted the depth variable.

Often, teachers' posts included multiple anticipations focused on students' mathematical actions. For example, when anticipating student responses to a geometric transformations task, a teacher wrote,

> I anticipate some students will think that because S and R are rotation and reflection, that the order won't matter and that SR and RS will result in the same images.… But some students will try drawing to see what happens and will discover that isn't the case. I anticipate some students will then think perhaps they can undo RS by doing SR. But then (hopefully) they will have to get into discussion about direction of rotation, not just amount of rotation. (8:23)

Similar to the previous post, the teacher anticipated students would create a representation; however, they went further to say the drawing would be used to determine a relationship—that transformations are not necessarily commutative. They anticipated the action of drawing a picture and then using the representation to take another action. Anticipations that described creating a representation followed by some additional action using that representation were common in the data. For instance, when anticipating student responses to an algebraic pattern task, one teacher stated, "After studying the logo [the algebraic pattern], students should be able to make their own table, then use the table to make their own graph" (3:29). Again, the teacher anticipated creating a representation (a table) and then using it to take another action, in this case to create another representation (a graph).

Mathematical Strategies

In addition to attending to the representations students might create, teachers' anticipations focused on students' mathematical actions also described mathematical strategies students might use to engage with the task. These anticipations included applying a procedure, using trial and error, counting, or measuring and accounted for 41% (n = 93) of anticipations in this category. The most common among these was *applying a procedure* which included procedures like finding the area of polygons, factoring, calculating regression, using the distance formula, and the Pythagorean Theorem. Anticipations such as "I think the students will use the quadratic formula or try and factor it first" (9:9), "I think that the students would find the area of the three circles and a triangle" (17:31), and "Some students may try the distance formula" (10:2) were common among the posts focused on students' mathematical activity. As these examples suggest, many of the strategies teachers described were specific mathematical procedures closely related to the task domain (e.g., quadratic formula, distance formula).

In other cases, the strategies teachers anticipated were more general, such as counting to look for patterns or measuring to explore relationships. Descriptions of trial and error approaches were especially prevalent among these more general strategies, with teachers anticipating that students might "try and guess the exact measurement" (10:7), "use their calculators and plug numbers in until they got something to work" (1:8), "do trial and error before they would try a calculation" (11:5), or "try to solve by making tables or just plugging in random numbers" (14:1). Trial and error anticipations were frequently intertwined with inferences about how students might use this strategy, which ranged from systematic to random.

Physical and Social Resources

Beyond representations and strategies, 39% (n = 88) of teachers' anticipations that focused on students' mathematical actions described the use of physical or social resources. A common physical resource that teachers anticipated students might use to engage with the tasks was a graphing calculator. For example, one teacher noted, "I think students will go to their calculator first and graph the points" (11:10) when anticipating student responses to a congruent triangles task. Similarly, another teacher wrote "I anticipate the students turning to the calculator and finding lines of best fit for the different categories" (2:4). While calculators were the predominant physical resource described in teachers' posts, teachers also anticipated the use of graph paper (e.g., "I think some students will graph the points on a separate sheet of graph paper" (8:25)) and tracing paper (e.g., "I anticipate that students would first want to use patty paper to experiment with reflections and rotations" (12:10).).

Teachers also described how students might draw upon social resources as they solved mathematical tasks. In some instances, teachers described how students "would definitely discuss some ideas with each other" (1:12) and seek support from their peers. In other cases, teachers anticipated that students would rely on their teacher, such as waiting for explicit directions or wanting affirmation as they engaged with the task. While most of these posts described general ways that students would seek assistance from the teacher, others referred to specific aspects of the task. For example, one teacher anticipated students would ask questions about the terminology in transformational geometry task stating, "I think the students will ask questions as to what the terms mean" (8:13).

In sum, anticipations that focused on students' mathematical actions were aligned with "anticipating student strategies" as is often defined in the literature. For this group of teachers, the strategies included not only "generic" mathematical strategies (i.e., strategies that can be used for any mathematical problem) but also the specific types of representations students might create, as well as material or social resources they might seek out.

Students' Mathematical Thinking

Whereas anticipations focused on students' mathematical actions attended to what teachers might observe as students engaged with a task, anticipations focused on students' mathematical thinking described teachers' assumptions and inferences about what students might know or not know that might support or impede their engagement with the task. Approximately 80% of teachers' anticipations (n = 225) focused on student thinking, including assertions about the difficulties students might have, students' use of prior knowledge, and conjectures about the connections among mathematical ideas students might make.

Difficulties

Approximately 30% of teachers' anticipations that focused on student thinking (n = 64; 28%) described some difficulty students might experience when engaging with the task. In some cases, the difficulty was task specific and concerned common mistakes students might make in calculation. For instance, teachers asserted that students might "struggle with the scale in the graph problems" (1:25), or "run into a problem when it comes to the conversion of units" (18:42). In other cases, teachers' anticipations described particular features of the task that may pose barriers for students such as the wording of a task or unfamiliar vocabulary. Other posts took a broader lens and made claims about common challenges students experience learning particular mathematical concepts. When anticipating student responses to a transformational geometry task for example, one teacher wrote:

> Students understand translations, reflections, and dilations fairly easily. Rotations are where students get stuck. I teach rotations by making students graph the figure on a coordinate plane and then let them rotate the paper and name the new coordinates. (8:14)

Similar anticipations included "They may have difficulty with domain and range" (1:11), and "the recursive formula will be difficult for them" (3:8). In each of these instances, teachers used their past experiences to anticipate what mathematical ideas students might find challenging.

Prior Knowledge

In addition to difficulties, teachers' anticipations focused on students' mathematical thinking also attended to what they expected students to know and use when engaging with the task. Of the posts that included anticipations related to student thinking, 23% (n = 51) referenced students' prior knowledge. In most of these cases, teachers made assumptions about what students learned earlier in the year or in previous courses and how they might use that knowledge to engage with the task, such as "[students] will think about some of their prior knowledge of distance and time and graphing (domain/range)" (1:12) or "I think students who have little conceptual understanding will say that it is too hard and will not be able to think of strategies to solve." (1:7). In other cases, teachers also considered the personal experiences students might have that would help them make sense of and engage with the task. When anticipating student responses to the Floating the River task for example, teachers described how students' experiences with water and rivers might be a resource for engaging with the task. One teacher wrote, "being that most of my students love the river and do a lot of fishing and things, students would start to draw on their experience." Almost a quarter of the anticipations focused on student thinking included considering prior mathematical and contextual knowledge.

Mathematical Connections

Teachers' anticipations focused on students' mathematical thinking also included conjectures about the mathematical connections students might or should make. Approximately 20% of the posts focused on student thinking (n = 46) contained claims about relationships among mathematical ideas or representations students might recognize. In some cases, teachers anticipated that students might organically make connections among mathematical ideas. When anticipating student responses to an exponential functions task for example, one teacher wrote, "those who choose to use the tree diagram quickly learn that the function grows quickly" (4:33). Another stated, "some will see that it is a multiplication problem which means it is raised to a power" (4.38). In these cases, the teachers anticipated students will "see" connections between their representation and the rate of change of a function or between the structure of an exponential function and repeated multiplication.

In other cases, teachers anticipated that students might not make the mathematical connections they believed to be central to the task. For example, one teacher anticipated that students would notice a pattern when engaging with an algebraic pattern task but would not make a connection to the rate of change and initial value of a function modeling the pattern. They stated, "I think they will easily see the pattern is going up by 3 each time but they will not put that as the slope and I think they will forget that it starts at 5 which is the y-intercept" (3.23).

Close to half of teachers' anticipations included a focus on students' mathematical thinking. These attended to anticipated difficulties students might encounter, students' prior knowledge of the mathematical content and situational contexts, as well as mathematical connections students might make when engaged in the task.

Students' Affective Responses to the Task

In addition to focusing on students' mathematical activity and thinking, teachers' anticipations also focused on students' affective responses to a task. Approximately 6% of teachers' posts (n = 18) attended to the affective responses students might have when engaging with a task. In some instances, teachers' posts described physical manifestations of students' affect, such as anticipating that students might "freeze", "throw their arms up", and "completely shut down". Others described ways students might emotionally respond to a task, including being "intimidated", "overwhelmed," or "frustrated." When considering an exponential and logarithmic functions task for example, one teacher anticipated, "My first thought is that students will panic because this is a complicated problem" (14:2). In another post related to a Floating the River task, another teacher wrote, "My students would be overwhelmed by the amount of data and the different units" (2:9). All posts that included anticipations in this category highlighted negative emotions responses and few included anticipations focused on student mathematical activity or thinking.

Teachers' Instructional Choices During Task Implementation

Although the prompt for each discussion board explicitly focused on students, 25% of teachers' anticipations (n = 70) focused on instructional actions they might take when implementing the task. Some anticipations in this category described teachers' instructional choices in response to students such as including plans for adjusting their instruction or considerations for different forms of support for students. Others focused on actions teachers might take prior to instruction to support student learning such as setting up the learning environment, adapting the task, and launching the task.

Preparing to Respond to Students

Nearly half of teachers' anticipations focused on teachers' instructional choices (n = 32; 46%) described ways they would respond to students engaging with the task. Some of these anticipations outlined instructional choices teachers planned to make in response to students' actions. . For example, one teacher's anticipations for the Floating the River task described how they would support students engaging with the task:

> I think students will be uncertain of where to begin… I would encourage my students to come up with something by drawing on past experiences. We would put these up in the room and would revisit them as we learned more throughout the unit. As they learned

> something that I thought would impact their understanding of the River Task, we would revisit their original responses and I would allow them time to collaborate and make changes (2:24).

In this post, the teacher anticipated students might not know how to initially engage with the task and described how they would ask students to make a prediction based on their experiences that they would revisit as students progressed through the unit. Other posts detailed a sequence of instructional moves the teacher would make in response to their anticipations. For instance, after anticipating students would create two perpendicular linear equations when asked to construct a system of linear equations with a solution of (4, -1), a teacher wrote, "Then I would have them branch out and pick other lines. I think then I would have them graph it. Then I would have them come up with their own points for their partners to solve" (6:2). In these and similar anticipations that included a focus on teachers' instructional choices in response to students, teachers included their response to students as part of their anticipations.

Other anticipations addressed different forms of support students might need to engage with the task. The most common kind of support teachers described focused on the guidance they planned to offer students. Twenty one percent of teachers' anticipations (n = 15) in this category referred to how they planned to assist students as they engaged with the task such as providing "a little hint", "guid[ing]" students' mathematical explorations, or "questioning". For example, one teacher posted they would prepare a set of questions to pose in response to student difficulties stating, "Before I implement this task, I would work out a series of probing questions that I could use to push them into a mathematical solution" (2.10). In another post, a teacher wrote, "I would have to prompt them to create a table then possibly enter the table of values into the regression model of the calculator" (5.14). Material resources was another kind of support teachers anticipated offering to students. Approximately 9% (n = 6) of anticipations in this category described how teachers would be ready to provide students graph paper, graphing calculators, patty paper, etc. should they need them.

Preparing for Instruction

In addition to considering ways of responding to students when implementing the tasks, teachers' anticipations that focused on teachers' instructional choices (n = 29; 41%) also attended to ways they might prepare for instruction that could support student learning. Some anticipations described how teachers would organize their classroom space in order to implement the task, in particular the need to arrange students in small groups. For example, one teacher stated, "I think they would definitely discuss some ideas with each other if [the] classroom was set up for these to be completed in a group" (1.2). Others discussed particular adaptations to the task to avoid some of their anticipated difficulties students might have. When anticipating responses to the Floating the River task for example, one teacher stated, "I would probably break the tasks up into dealing with the different topics one at a time" (2.3). Another stated, "I feel like the task needs to be tweaked if one would use it as an introduction" (2.13).

Teachers' anticipations also included plans for how they might organize instruction based on what they believed students would do. The majority of these posts described a need for direct instruction. For example, one teacher anticipated the need for direct instruction to use a systems of linear equations task stating, "I will need to go step by step to remind them of how to solve and deconstruct each step to help them complete number 1" (6.5). In other posts, teachers described how whole or small group discussions would be beneficial for students, such as another teachers' post related to the same task. They noted, "I think we'd have to have a really solid discussion after the first question before they moved on to the others" (6.18). Across these posts, teachers' anticipations included ways they might organize their instruction to support student engagement and learning when implementing the tasks.

Though less prevalent than posts focusing on students' mathematical actions or mathematical thinking, a number of teachers' anticipations included descriptions of the instructional choices they might make when implementing tasks. In some cases, these actions focused on ways they might respond instructionally if students engaged according to their expectations. In other cases, they described ways they might prepare for the lesson to support student learning.

Discussion

Our primary goal for this study was to better understand secondary mathematics teachers' anticipations to a set of cognitively demanding tasks presented in voluntary and asynchronous online professional development modules, and our results suggest four distinct foci of anticipations. Teachers in our study attended to what they might observe students doing mathematically such as the representations they would create, the strategies they might use, and the resources they would seek. They also considered how students might be thinking mathematically including the difficulties they might have, the existing knowledge they might bring to bear on the task, and the connections they could make to other mathematical ideas. These findings are similar to many of the existing studies of elementary and middle grades prospective and practicing teachers' anticipations (e.g., Krause et al., 2016; Kartal et al., 2020) and provide additional evidence that considerations of students' mathematical actions and thinking are prominent foci when teachers anticipate.

Our results also revealed that some teachers consider more than mathematics when anticipating and attend to a range of physical and emotional reactions students might have when engaging with mathematics tasks. Although few studies have reported anticipations related to students’ affect, our findings suggest that teachers may consider disengagement, frustration, and apathy when anticipating. Given the significant influence of emotions on learning (Bransford et al., 2000; National Academies of Science, Engineering, and Medicine, 2018), additional research is needed to better understand the extent to which teachers consider student affect when anticipating student responses.

Anticipating future events allows one to act prior to that event to increase or decrease its likelihood and to be prepared to respond (von Glasersfeld, 1998), and our results suggest some teachers consider what actions they may take in preparation for or in response to anticipated students’ actions, thinking, or reactions. In some cases, teachers described ways they might introduce the task or set up their classroom to enable or prevent their anticipations related to students. In others, they developed plans for instructional support contingent on their anticipations. Although implicit in early scholarship related to anticipating (e.g., Stein et al., 2008), more recent professional learning materials have highlighted the importance of planning to notice and respond to student thinking when anticipating student responses (Smith et al., 2020). Our findings suggest that some secondary mathematics teachers already consider themselves in relation to their anticipations of student action, thinking, and affect, as well as the ways in which their instruction might create, prevent, and respond to opportunities for student learning afforded by cognitively demanding instructional tasks.

Our findings extend the knowledge base about anticipating student responses from prospective and practicing teachers in elementary and middle grades to secondary mathematics teachers. Like the research reviewed in this paper, secondary mathematics teachers in our study attended to the representations and strategies they might observe students using when engaging with a cognitively demanding mathematics task as well as what they might infer about their mathematical thinking from these actions. However, our analytic approach, which distinguished students' mathematical actions and students' mathematical thinking, revealed some teachers do not always attend to both when anticipating student responses. Anticipations focused on students' mathematical thinking, in the absence of the actions students might take as a result of their understanding, may be based on assumptions. Likewise, attending to student actions without considering what one could infer about students' understanding from those actions may not assist teachers in preparing for instruction. Similar to Cobb and Steffe (1983), we see teachers' anticipations as potential models of students' mathematical understanding where the actions one might observe are evidence of what they know and are yet to know. For mathematics teacher educators, our findings suggest that encouraging teachers to attend to both students' mathematical actions and mathematical thinking—and the plausible links between them—may be a productive approach to support teachers in developing their practice of anticipating student responses.

Teachers participating in this study anticipated responses from unspecified students to instructional tasks they did not select, and our findings should be understood in that context. Because students' race, gender, and socioeconomic status can influence the ways teachers interact with students and interpret their work (Reyes & Stanic, 1988), anticipating "generic" student responses affords mathematics teacher educators and researchers an opportunity to understand what teachers believe students might do or think without assumptions based on characteristics of particular students. At the same time, anticipating responses for unknown students does not permit teachers to use their knowledge of students, their strengths, and the resources they might bring to a task.

For mathematics teacher educators working with teachers in professional development, our findings provide insight into what teachers understand the practice of anticipating to entail. Unlike most of the research we reviewed, teachers in our study were not participating in formal and facilitated professional development that included a focus of frameworks for students' mathematical thinking or high-leverage instructional practices. Rather, teachers engaged in unmoderated online professional learning modules designed to support them in learning about new state mathematics standards. As an initial part of each module, they were asked to complete a mathematics task and then anticipate how students would respond to that task. Our examination showed they not only considered students' mathematical actions and mathematical thinking but also their affective reactions and how they would adjust their instruction in response to these anticipations. Whereas mathematical actions and thinking are the focus of many mathematics teacher educators' work with teachers, our findings suggest that there are other dimensions of teachers' anticipations that should be considered and leveraged for professional learning. Though the research or professional literature has not typically considered student affect, knowing some teachers consider motivation, frustration, and engagement when anticipating student responses can assist mathematics teacher educators in surfacing and framing such expectations of students productively in professional learning contexts. Similarly, knowing that teachers consider changes in their planned instruction to support or prevent particular learning challenges when anticipating creates opportunities for mathematics teacher educators to surface and discuss the importance of students engaging with and overcoming conceptual barriers.

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