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Abstract

We define a notion of modular forms of half-integral weight on the quaternionic exceptional groups. We prove that they have a well-behaved notion of Fourier coefficients, which are complex numbers defined up to multiplication by ± 1. We analyze the minimal modular form Θ_{F_4} on the double cover of F_4, following Loke–Savin and Ginzburg. Using Θ_{F_4}, we define a modular form of weight $\frac{1}{2}$ on (the double cover of) G_2. We prove that the Fourier coefficients of this modular form on G_2 see the 2-torsion in the narrow class groups of totally real cubic fields.

1. Introduction

1.1 Main result

We introduce our main result by way of an analogy. Let $\Theta(z) = \sum_{n \in \mathbf{Z}} q^{n^2}$, where $q = e^{2\pi i z}$. As is well-known, $\Theta(z)$ is a classical holomorphic modular form of weight $\frac{1}{2}$ and level $\Gamma_1(4) \subseteq \mathrm{SL}_2(\mathbf{Z})$. Consider the weight $\frac{3}{2}$ modular form

$$E_{\mathrm{CZ}}(z) := \Theta(z)^3 = \sum_{n \geq 0} r_3(n) q^n;$$

here $r_3(n) := \#\{(n_1, n_2, n_3) \in \mathbf{Z}^3 : n = n_1^2 + n_2^2 + n_3^2\}$ is the number of ways n can be written as the sum of three squares. We have named this modular form after Cohen and Zagier, in light of their papers [Coh75, Zag75].

Recall now the following theorem of Gauss.

Theorem 1.1.1 (Gauss). *Suppose n is squarefree, $n \equiv 1, 2 \pmod 4$ and $n \geq 4$. Then $r_3(n) = 12 \cdot |\mathrm{Cl}(\mathbf{Q}(\sqrt{-n}))|$, 12 times the class number of the associated quadratic imaginary field.*

Thus, the Fourier coefficients of $E_{\mathrm{CZ}}(z)$ see the class numbers of imaginary quadratic fields. Our main result is the construction of an analogous modular form Θ_{G_2} of weight $\frac{1}{2}$ on G_2, whose Fourier coefficients see the 2-torsion in the narrow class groups of totally real cubic fields. In particular, we define a notion of modular forms of half-integral weight on certain exceptional groups, very similar to the integral weight theory [GGS02]. We prove that these modular forms,

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which are now automorphic forms on certain non-linear double covers of these exceptional groups, have a robust notion of Fourier coefficients. We then construct a particular interesting example Θ_{G_2} on G_2 and partially calculate its Fourier expansion.

To motivate our construction of Θ_{G_2}, observe that one has a commuting pair $\mathrm{SL}_2 \times \mathrm{SO}(3) \subseteq \mathrm{Sp}_6$. One can also think of $E_{\mathrm{CZ}}(z)$ as the restriction to SL_2 of a weight $\frac{1}{2}$ Siegel modular theta function: $E_{\mathrm{CZ}}(z) = \Theta_{Sp_6}(\mathrm{diag}(z,z,z))$, where

$$\Theta_{Sp_6}(Z) = \sum_{v=(n_1,n_2,n_3)\in\mathbf{Z}^3} e^{2\pi i v Z v^t}$$

and Z is in the Siegel upper half-space of degree three. Now, there is the following commutative diagram of inclusions.

$$\begin{array}{ccc} \mathrm{Sp}_6 & & F_4 \\ \bigcup & \subseteq & \bigcup \\ \mathrm{SL}_2 \times \mathrm{SO}(3) & & G_2 \times \mathrm{SO}(3) \end{array}$$

Following Loke and Savin [LS10] and Ginzburg [Gin19] we consider the automorphic minimal representation on the double cover of F_4. We show that the minimal representation can be used to define a weight $\frac{1}{2}$ modular form Θ_{F_4} on F_4, and define Θ_{G_2} as the pullback to G_2 of Θ_{F_4}.

The Fourier coefficients of modular forms φ on G_2 are parametrized by integral binary cubic forms $f(u,v) = au^3 + bu^2v + cuv^2 + dv^3$, $a,b,c,d \in \mathbf{Z}$, for which $f(u,v)$ splits into three linear factors over the real numbers. So, for each such binary cubic f, there is an associated Fourier coefficient $a_\varphi(f)$, which is a complex number well-defined up to multiplication by ± 1. Our main result is the explicit description of the Fourier coefficients of the weight $\frac{1}{2}$ modular form Θ_{G_2}. More precisely, we can explicitly compute these Fourier coefficients $a_{\Theta_{G_2}}(f)$ when the binary cubic $f(u,v)$ has $d = 1$. We explicate the special case of this result when the cubic ring $\mathbf{Z}[y]/(f(1,y))$ is a maximal order in a totally real cubic field.

Theorem 1.1.2. *There is a modular form Θ_{G_2} of weight $\frac{1}{2}$ on G_2 whose Fourier coefficients satisfy the following: suppose $f(u,v) = au^3 + bu^2v + cuv^2 + dv^3$ is an integral binary cubic form with $d = 1$, and that the cubic ring $R = \mathbf{Z}[y]/(f(1,y))$ is a maximal order in a totally real cubic field $E = R \otimes \mathbf{Q}$.*

(i) *If the inverse different $\mathfrak{d}_R^{-1}$ is not a square in the narrow class group of E, then the Fourier coefficient $a_{\Theta_{G_2}}(f) = 0$.*

(ii) *If the inverse different $\mathfrak{d}_R^{-1}$ is a square in the narrow class group of E, then the Fourier coefficient $a_{\Theta_{G_2}}(f) = \pm 24|\mathrm{Cl}_E^+[2]|$, plus or minus 24 times the size of the two-torsion in the narrow class group of E.*

Thus, in both cases of Theorem 1.1.2, the Fourier coefficient of Θ_{G_2} corresponding to the binary cubic f is ± 24 times the number of square roots of the inverse different $\mathfrak{d}_R^{-1}$ in the narrow class group Cl_E^+ of E.

1.2 Extended introduction

In this section we outline the contents of the paper.

1.2.1 *Quaternionic modular forms.* As our main results concern modular forms of *half-integral* weight on the quaternionic exceptional groups, we begin by reviewing the integral weight theory. To set the stage for these quaternionic modular forms, we first recall holomorphic modular forms.

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Suppose G is a semisimple algebraic $\mathbf{Q}$-group whose associated symmetric space is a Hermitian tube domain. Then G has a notion of *holomorphic modular forms*. These can be thought of as very special automorphic forms for G, which are closely connected to arithmetic. They have a classical Fourier expansion and Fourier coefficients, and these Fourier coefficients often encode arithmetic data.

Among the exceptional Dynkin types, only E_6 and E_7 have a real form with a Hermitian symmetric space, and only E_7 has a real form with an Hermitian tube domain. So, if one is interested in studying a class of special automorphic forms on, say, G_2, F_4 or E_8, there is not an obvious place to look for such objects. Nevertheless, beginning with work of Gross and Wallach [GW94, GW96] and developed in work of Wallach [Wal03] and Gan, Gross and Savin [GGS02], a theory of special automorphic forms on the exceptional algebraic groups began to emerge.

These special automorphic forms have been dubbed *quaternionic modular forms*. For each exceptional Dynkin type, there is a so-called *quaternionic* real form: for G_2 and F_4, this is the split real form, whereas for E_6, E_7 and E_8 this is the real form with real rank equal to four. The quaternionic modular forms are special automorphic forms on reductive groups G over $\mathbf{Q}$ for which $G(\mathbf{R})$ is a quaternionic real group.

The real quaternionic exceptional groups never have a symmetric space with complex structure. However, these groups share similar structures, and the quaternionic modular forms on these groups share similar properties. To be more specific, suppose G is an adjoint exceptional group with $G(\mathbf{R})$ quaternionic. Then the maximal compact subgroup K_G of $G(\mathbf{R})$ is of the form $(\mathrm{SU}(2) \times L)/\mu_2(\mathbf{R})$, for a compact group L that depends upon G. Let $\mathbb{V}_2$ denote the standard representation of $\mathrm{SU}(2)$ and for a positive integer ℓ let $\mathbf{V}_\ell$ denote the representation of K_G that is the representation $Sym^{2\ell}(\mathbb{V}_2)$ of the $\mathrm{SU}(2)$ factor and the trivial representation of the L-factor. A quaternionic modular form on G of weight ℓ is an automorphic function $\varphi : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbf{V}_\ell$ satisfying:

(i) $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $k \in K_G$ and $g \in G(\mathbf{A})$;
(ii) $D_\ell \varphi \equiv 0$ for a certain specific differential operator D_ℓ.

This is the definition from [Pol20], which is a slight generalization and paraphrase of the definition from [GGS02], where quaternionic modular forms are defined in terms of the quaternionic discrete series representations of the group $G(\mathbf{R})$.

To make this definition precise, of course we must specify the differential operator D_ℓ. Let the notation be as above. Write $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ for the Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of $G(\mathbf{R})$. Then, as a representation of K_G, one has $\mathfrak{p} := \mathfrak{p}_0 \otimes \mathbf{C} \simeq \mathbb{V}_2 \otimes W$ for a certain symplectic representation W of L. Let $\{X_\alpha\}_\alpha$ be a basis of $\mathfrak{p}$ and $\{X_\alpha^\vee\}_\alpha$ be the dual basis of $\mathfrak{p}^\vee$. For φ satisfying $\varphi(gk) = k^{-1} \cdot \varphi(g)$, define $\widetilde{D}_\ell \varphi = \sum_\alpha X_\alpha \varphi \otimes X_\alpha^\vee$. Here $X_\alpha \varphi$ denotes the right regular action, and $\widetilde{D}_\ell \varphi$ is valued in

$$\mathbf{V}_\ell \otimes \mathfrak{p}^\vee \simeq Sym^{2\ell+1}(\mathbb{V}_2) \boxtimes W \oplus Sym^{2\ell-1}(\mathbb{V}_2) \boxtimes W.$$

We let $\mathrm{pr} : \mathbf{V}_\ell \otimes \mathfrak{p}^\vee \to Sym^{2\ell-1}(\mathbb{V}_2) \boxtimes W$ be the K_G-equivariant projection and define $D_\ell = \mathrm{pr} \circ \widetilde{D}_\ell$.

The relationship of the definition of quaternionic modular forms with representation theory is as follows. Suppose π is an irreducible $(\mathfrak{g}_0, K_G)$-module embedded in the space of automorphic forms on $G(\mathbf{Q})\backslash G(\mathbf{A})$ via a map α. Suppose, moreover, that π has minimal K_G-type $\mathbf{V}_\ell$. Then out of $\mathbf{V}_\ell$ and α one can construct a quaternionic modular form of weight ℓ: for

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$g \in G(\mathbf{A})$ set

$$\varphi(g) = \sum_{j=-\ell}^{\ell} \alpha(x_j)(g) \otimes x_j^\vee,$$

where $\{x_j\}$ is a basis of $\mathbf{V}_\ell \subseteq \pi_\ell$ and $x_j^\vee$ is the dual basis of $\mathbf{V}_\ell^\vee \simeq \mathbf{V}_\ell$. Using the fact that $\mathbf{V}_\ell$ is the minimal K-type of π, it is easy to show that φ is a quaternionic modular form of weight ℓ. If ℓ is sufficiently large depending on G, there is a discrete series representation π_ℓ of $G(\mathbf{R})$ whose minimal K_G-type is $\mathbf{V}_\ell$, so embeddings of these discrete series representations into the space of automorphic forms on G give rise to quaternionic modular forms of weight ℓ.

Modular forms of integral weight ℓ have been studied in [GGS02], [Wei06], [Pol20, Pol22a, Pol21, Pol22c] and [Dal23]. For an introduction to what is known about them, we refer to [Pol22b]. The main result of [Pol20] is that quaternionic modular forms have a robust, semi-classical Fourier expansion, similar to the Fourier expansion of classical holomorphic modular forms on tube domains. This result generalized and refined work of Wallach [Wal03].

To explain this Fourier expansion, we recall another common feature of the quaternionic exceptional groups. While none of them has a parabolic with abelian unipotent radical, they all have a Heisenberg parabolic $P = MN$ whose unipotent radical $N \supseteq Z = [N, N] \supseteq 1$ is two-step, with one-dimensional center Z. Thus, if φ is an automorphic form on G, one can take the constant term φ_Z of φ along Z, and Fourier-expand the result along N/Z: $\varphi_Z = \sum_\chi \varphi_\chi$ where $\varphi_\chi(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \chi^{-1}(n)\varphi(ng)\,dn$. The main result of [Pol20] is an explication of this Fourier expansion for quaternionic modular forms φ of weight ℓ. Namely, it is proved in [Pol20] that there are certain completely explicit functions $W_\chi : G(\mathbf{R}) \to \mathbf{V}_\ell$ so that if φ is a weight ℓ modular form, then $\varphi_\chi(g) = a_\varphi(\chi)(g_f)W_\chi(g_\infty)$ for some locally constant function $a_\varphi(\chi) : G(\mathbf{A}_f) \to \mathbf{C}$; here $g = g_f g_\infty$ is the factorization of g into its finite-adelic and infinite parts. The complex numbers $a_\varphi(\chi)(1)$ are called the Fourier coefficients of φ. This definition is designed to mimic the classical definition of Fourier coefficients of holomorphic modular forms.

While defined in a purely transcendental way, the Fourier coefficients of a quaternionic modular form φ appear to have arithmetic significance; for evidence of this claim, see [Pol22a, Pol21, Pol22c]. One purpose of the present paper is to add to this growing evidence that quaternionic modular forms have arithmetically interesting Fourier coefficients.

1.2.2 *The double cover of quaternionic exceptional groups.* In this paper, we define and study certain quaternionic modular forms of *half-integral* weight and their Fourier coefficients. To define these notions, suppose again that G is an adjoint quaternionic exceptional group. Then, since $G(\mathbf{R})$ deformation retracts onto $K_G \simeq (\mathrm{SU}(2) \times L)/\mu_2(\mathbf{R})$, and K_G has a 2-cover $\widetilde{K}_G \simeq \mathrm{SU}(2) \times L$, the group $G(\mathbf{R})$ has a 2-cover $\widetilde{G}$. Choosing a basepoint of $\widetilde{G}$ above $1 \in G(\mathbf{R})$ makes $\widetilde{G}$ into a connected Lie group, which is a central $\mu_2(\mathbf{R})$-extension of $G(\mathbf{R})$

$$1 \to \mu_2(\mathbf{R}) \to \widetilde{G} \to G(\mathbf{R}) \to 1,$$

and $\widetilde{K}_G$ can be identified with a maximal compact subgroup of $\widetilde{G}$.

Our first result, which is perhaps of independent interest, is an explicit description of these Lie groups $\widetilde{G}$. To motivate it, let $\mathfrak{h} = \mathrm{SL}_2(\mathbf{R})/\mathrm{SO}(2)$ denote the upper half-plane and recall that one can identify the double cover of $\mathrm{SL}_2(\mathbf{R})$ with pairs (g, j_g) where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ and $j_g : \mathfrak{h} \to \mathbf{C}^\times$ is a holomorphic function that satisfies $j_g(z)^2 = cz + d$. If now G is an adjoint quaternionic exceptional group, with symmetric space $X_G = G(\mathbf{R})/K_G$, we define a factor of automorphy $j_{\mathrm{lin}} : G(\mathbf{R}) \times X_G \to \mathrm{GL}_3(\mathbf{C})$, satisfying $j_{\mathrm{lin}}(g_1 g_2, x) = j_{\mathrm{lin}}(g_1, g_2 x) j_{\mathrm{lin}}(g_2, x)$. We then consider the set

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of pairs (g, j_g) with $g \in G(\mathbf{R})$ and $j_g : X_G \to \mathrm{GL}_2(\mathbf{C})$ continuous that satisfy $Sym^2(j_g(x)) = j_{\mathrm{lin}}(g, x)$. It is easy to see that this set forms a group with multiplication $(g_1, j_{g_1}(x))(g_2, j_{g_2}(x)) = (g_1 g_2, j_{g_1}(g_2 x) j_{g_2}(x))$.

Theorem 1.2.1. *With a certain topology on the set of pairs (g, j_g) above, this set can be identified with the connected topological group $\widetilde{G}$.*

When G is a split, simply connected algebraic group, such as G_2 or F_4, Steinberg [Ste16] and Matsumoto [Mat69] have defined a 2-cover $\widetilde{G}^{(2)}(k)$ of $G(k)$ for every local field k. When $k = \mathbf{R}$ and $G = G_2$ or F_4, this 2-cover can be identified with the 2-cover $\widetilde{G}$. The group $\widetilde{G}^{(2)}(k)$ can be constructed by generators and relations [Ste16], as we recall in § 2.4. The groups $\widetilde{G}^{(2)}(\mathbf{Q}_v)$ can be glued together to produce a 2-cover $\widetilde{G}^{(2)}(\mathbf{A})$ of $G(\mathbf{A})$. It follows from the construction of $\widetilde{G}^{(2)}(\mathbf{A})$ and the global triviality of the Hilbert symbol that the group of rational points $G(\mathbf{Q})$ splits into $\widetilde{G}^{(2)}(\mathbf{A})$.

Now suppose $\ell \geq 1$ is an odd integer. Let $\mathbf{V}_{\ell/2} = Sym^{\ell}(\mathbb{V}_2)$ be the representation of $\widetilde{K}_G$ that is the ℓth symmetric power of $\mathbb{V}_2$, as a representation of $\mathrm{SU}(2)$, and is the trivial representation of L. We define a quaternionic modular form φ for G of weight $\ell/2$ to be a $\mathbf{V}_{\ell/2}$-valued automorphic function $\varphi : G(\mathbf{Q}) \backslash \widetilde{G}^{(2)}(\mathbf{A}) \to \mathbf{V}_{\ell/2}$ that satisfies:

(i) $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in \widetilde{G}^{(2)}(\mathbf{A})$ and $k \in \widetilde{K}_G$; and
(ii) $D_{\ell/2}\varphi \equiv 0$.

Here the differential operator $D_{\ell/2}$ is defined exactly as D_ℓ was above. If $U \subseteq G(\mathbf{A}_f)$ is an open compact subgroup that splits into $\widetilde{G}^{(2)}(\mathbf{A})$, and φ is stabilized by U, then we say φ has level U.

To study modular forms of half-integral weight on the group $\widetilde{G}^{(2)}(\mathbf{A})$, it helps to have explicit open compact subgroups $U \subseteq G(\mathbf{A}_f)$ together with an explicit splitting $s_U : U \to \widetilde{G}^{(2)}(\mathbf{A})$. This is accomplished in the following result in case G is F_4.

Theorem 1.2.2. *When $G = F_4$, there is an explicit, large open compact subgroup $U_{F_4}(4)$ that splits into the double cover.*

When $p > 2$, it is proved by Loke and Savin [LS10] that the hyperspecial maximal compact subgroup of $G(\mathbf{Q}_p)$ splits into $\widetilde{G}^{(2)}(\mathbf{Q}_p)$. Thus, it remains to analyze the case $p = 2$, and it is here where we do detailed computations: in § 2.5, we produce an explicit (non-maximal) compact open subgroup of $F_4(\mathbf{Q}_2)$ that splits into the double cover. Our result in this direction can be considered an extension of some work of [Kar21], who considers the simply laced case.

1.2.3 *The Fourier expansion of half-integral weight modular forms.* With the groups $\widetilde{G}^{(2)}(\mathbf{A})$ reviewed and the notion of quaternionic modular form defined, it makes sense to ask about examples and properties of quaternionic modular forms of half-integral weight. The main property we prove is the existence of a robust, semi-classical Fourier expansion, analogous to the integral-weight theory. To make sense of Fourier expansions on the covering groups $\widetilde{G}^{(2)}(\mathbf{A})$, one begins with the observation that the unipotent group $N(\mathbf{Q}_v)$ splits uniquely into $\widetilde{G}^{(2)}(\mathbf{Q}_v)$ for every place v. Consequently, one can ask about the Fourier expansion of $\varphi_Z(g)$ if $\varphi(g)$ is an automorphic function on $\widetilde{G}^{(2)}(\mathbf{A})$.

To produce the desired Fourier expansion, we analyze *generalized Whittaker functions* on the groups $\widetilde{G} \simeq \widetilde{G}^{(2)}(\mathbf{R})$. If $\chi : N(\mathbf{R}) \to \mathbf{C}^\times$ is a non-trivial unitary character, and $\ell \geq 1$ is an odd

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integer, a generalized Whittaker function of type $(N, \chi, \ell/2)$ is a smooth function $F : \widetilde{G} \to \mathbf{V}_{\ell/2}$ satisfying:

(i) $F(gk) = k^{-1} \cdot F(g)$ for all $g \in \widetilde{G}$ and $k \in \widetilde{K}_G$;
(ii) $F(ng) = \chi(n)F(g)$ for all $n \in N(\mathbf{R})$ and $g \in \widetilde{G}$;
(iii) $D_{\ell/2}F \equiv 0$.

With regard to these generalized Whittaker functions, we prove the following theorem, which is the analogue in the half-integral weight case of the main result of [Pol20]. To state the result, we recall that if G is a quaternionic exceptional group, then there is a notion of 'positive semi-definiteness' of non-trivial unitary characters χ of $N(\mathbf{R})$. We let M denote a particular fixed Levi subgroup of the Heisenberg parabolic P, to be recalled in § 2.2.

Theorem 1.2.3. *Let the notation be as above, with χ a non-trivial unitary character of $N(\mathbf{R})$.*

(i) *Suppose F is a moderate-growth generalized Whittaker function of type $(N, \chi, \ell/2)$, and χ is not positive semi-definite. Then F is identically 0.*
(ii) *Suppose χ is positive semi-definite and ℓ is fixed. There are a pair of non-zero functions $W^1_\chi(g)$ and $W^2_\chi(g)$ that satisfy the following properties:*
   (a) $W^2_\chi(g) = -W^1_\chi(g)$;
   (b) *the W^j_χ are moderate growth generalized Whittaker functions of type $(N, \chi, \ell/2)$;*
   (c) *the set $\{W^1_\chi(g), W^2_\chi(g)\}$ depends continuously on χ;*
   (d) *if r is in the derived group $[M, M](\mathbf{R})$ and $\widetilde{r}$ is a preimage of r in $\widetilde{G}$, then the set $\{W^1_\chi(\widetilde{r}g), W^2_\chi(\widetilde{r}g)\} = \{W^1_{\chi\cdot r}(g), W^2_{\chi\cdot r}(g)\}$;*
   (e) *moreover, if F is moderate growth generalized Whittaker function of type $(N, \chi, \ell/2)$, then there is a pair of complex numbers $a_{\chi,2}(F) = -a_{\chi,1}(F)$ so that $F(g) = a_{\chi,j}(F)W^j_\chi(g)$ for $j = 1, 2$.*

Note that, if ζ is the non-identity element of the preimage of $\{1\}$ in $\widetilde{G}$, then $W^1_\chi(\zeta g) = W^1_\chi(g\zeta) = -W^1_\chi(g) = W^2_\chi(g)$, so that one really needs both W^1_χ and W^2_χ to appear in property 2(d) of Theorem 1.2.3.

The Fourier expansion of quaternionic modular forms on $\widetilde{G}$ of weight $\ell/2$ follows immediately from Theorem 1.2.3.

Corollary 1.2.4. *Suppose φ is a quaternionic modular form on $\widetilde{G}^{(2)}(\mathbf{A})$ of weight $\ell/2$, and $g \in \widetilde{G}^{(2)}(\mathbf{R}) \simeq \widetilde{G}$. Then there is a lattice Λ in $(N(\mathbf{Q})/Z(\mathbf{Q}))^\vee$ so that*

$$\varphi_Z(g) = \varphi_N(g) + \sum_{1 \neq \chi \in \Lambda} a^j_\varphi(\chi) W^j_\chi(g)$$

for certain complex numbers $a^j_\varphi(\chi)$ that satisfy $a^1_\varphi(\chi)W^1_\chi(g) = a^2_\varphi(\chi)W^2_\chi(g)$.

The elements $a^j_\varphi(\chi) \in \mathbf{C}/\{\pm 1\}$ are called the Fourier coefficients of φ. Note that the Fourier coefficients are defined in terms of the restriction of φ to the group $\widetilde{G}^{(2)}(\mathbf{R})$ of real points.

1.2.4 *The automorphic minimal representation.* One of the first examples of quaternionic modular forms of integral weight is given by the automorphic minimal representation on quaternionic E_8, which was produced by Gan [Gan00], see also [Pol22a]. The double cover of F_4 has an automorphic minimal representation; this representation was defined and studied by Loke and Savin [LS10] and further analyzed by Ginzburg [Gin19]. Our first example of a modular form of

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half-integral weight, in fact of weight $\frac{1}{2}$, comes from this automorphic minimal representation on $\widetilde{F}_4^{(2)}(\mathbf{A})$.

The following is our main result concerning the automorphic minimal representation on $\widetilde{F}_4^{(2)}(\mathbf{A})$. To state the result, let $J_0 = Sym^2(\mathbf{Z}^3)$ denote the 3×3 integral symmetric matrices, and let $J_0^\vee$ be the dual lattice with respect to the trace pairing, so that $J_0^\vee$ is the set of half-integral symmetric 3×3 matrices. If N denotes the unipotent radical of the Heisenberg parabolic of F_4, then there is an embedding of the lattice $W(\mathbf{Z})^\vee = \mathbf{Z} \oplus J_0^\vee \oplus J_0^\vee \oplus \mathbf{Z}$ in the space of characters $W(\mathbf{Q})^\vee = (N(\mathbf{Q})/Z(\mathbf{Q}))^\vee = W(\mathbf{Z})^\vee \otimes \mathbf{Q}$.

Theorem 1.2.5. *Let $\Pi_{\min} = \Pi_{\min,f} \otimes \Pi_{\min,\infty}$ denote the automorphic minimal representation of $\widetilde{F}_4^{(2)}(\mathbf{A})$. The minimal $\widetilde{K}_{F_4}$-type of $\Pi_{\min,\infty}$ is $\mathbb{V}_2 = \mathbf{V}_{1/2}$. Consequently, if $v_f \in \Pi_{\min,f}$, there is an associated quaternionic modular form $\theta(v_f)$ of weight $\frac{1}{2}$ on $\widetilde{F}_4^{(2)}(\mathbf{A})$. Moreover:*

(i) *the $(a,b,c,d) \in W(\mathbf{Q})^\vee$ Fourier coefficient of $\theta(v_f)$ is zero unless (a,b,c,d) is 'rank one';*
(ii) *the vector v_f can be chosen so that $\theta(v_f)$ (cf. Theorem 1.2.2) has level $U_{F_4}(4)$ and has non-zero $(0,0,0,1) \in W(\mathbf{Z})^\vee$ Fourier coefficient.*

The fact that the minimal $\widetilde{K}_{F_4}$-type of π_∞ is $\mathbf{V}_{1/2}$ follows easily from work of [ABP+07]. As explained above, this implies that there are associated weight-$\frac{1}{2}$ modular forms $\theta(v_f)$ on F_4. The statement that the Fourier coefficients of $\theta(v_f)$ vanish unless (a,b,c,d) is rank one is the result [Gin19, Proposition 3] of Ginzburg, imported into our language. Where we work hard is the last statement, that v_f can be chosen so that $\theta(v_f)$ has large level and non-zero $(0,0,0,1)$-Fourier coefficient.

To prove this result about level and Fourier coefficients, we make some detailed computations of certain twisted Jacquet modules of the automorphic minimal representation π, especially at the 2-adic place. To do these computations, we bootstrap off of twisted Jacquet module computations in [GP80], which concerns the Weil representation of a double cover of $\mathrm{GL}_2(\mathbf{Q}_p)$.

1.2.5 *A modular form on G_2.* Let Θ_{F_4} denote a weight $\frac{1}{2}$, level $U_{F_4}(4)$-modular form on $\widetilde{F}_4^{(2)}(\mathbf{A})$, with non-zero $(0,0,0,1)$-Fourier coefficient, as guaranteed by Theorem 1.2.5. We normalize Θ_{F_4} so that its $(0,0,0,1)$-Fourier coefficient is ± 1. There is an embedding $\widetilde{G}_2^{(2)}(\mathbf{A}) \subseteq \widetilde{F}_4^{(2)}(\mathbf{A})$, compatible with the splittings on the rational points. Denote by Θ_{G_2} the pullback to $\widetilde{G}_2^{(2)}(\mathbf{A})$ of Θ_{F_4}. Then we check that Θ_{G_2} is a quaternionic modular form of weight $\frac{1}{2}$. Our main result concerns the Fourier coefficients of Θ_{G_2}.

To describe these Fourier coefficients, first note that if N is the unipotent radical of the Heisenberg parabolic of G_2, then $(N(\mathbf{Q})/Z(\mathbf{Q}))^\vee$ can be identified with the rational binary cubic forms $f(u,v) = au^3 + bu^2v + cuv^2 + dv^3$. It is easy to show that the Fourier coefficients of Θ_{G_2} vanish outside the lattice of integral binary cubic forms. We give a formula for the Fourier coefficient $a_{\Theta_{G_2}}(f)$ for every integral binary cubic form f with $d = 1$.

To state (the main part) of this formula, we introduce a notation concerning cubic rings, following Swaminathan [Swa21]. Let R be an order in a totally real cubic field $E = R \otimes \mathbf{Q}$. Let $\mathfrak{d}_R^{-1}$ be the inverse different of R, i.e. the fractional R ideal consisting of those $x \in E$ for which $\mathrm{tr}_E(x\lambda) \in \mathbf{Z}$ for all $\lambda \in R$. Say that a pair (I, μ) of a fractional R ideal I and a totally positive unit $\mu \in E^\times_{>0}$ is *balanced* if:

(i) $\mu I^2 \subseteq \mathfrak{d}_R^{-1}$;
(ii) $N(\mu)N(I)^2\mathrm{disc}(R/\mathbf{Z}) = 1$.

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Thus, if R is the maximal order in E, (I,μ) is balanced if and only if $\mu I^2 = \mathfrak{d}_R^{-1}$. Here $N(\mu)$ is the norm of μ and $N(I)$ (well-defined up to multiplication by ± 1) is the determinant of a linear transformation of E that takes a $\mathbf{Z}$-basis of R to a $\mathbf{Z}$-basis of I.

Let Q_R be the set of balanced pairs (I,μ) up to equivalence, where we say (I,μ) is equivalent to (I',μ') if there exists $\beta \in E^\times$ such that $I' = \beta I$, $\mu' = \beta^{-2}\mu$. The set Q_R is always finite and sometimes empty. If R is the maximal order and Q_R is nonempty, then we show in § 3.5 that $|Q_R| = |\mathrm{Cl}_E^+[2]|$ where $\mathrm{Cl}_E^+[2]$ is the 2-torsion in the narrow class group of E.

Theorem 1.2.6. *Let the notation be as above, and suppose the binary cubic form $f(u,v)$ has $d = 1$. Denote by $R = \mathbf{Z}[y]/(f(1,y))$, and suppose that $R \otimes \mathbf{Q}$ is a totally real cubic field. The weight $\frac{1}{2}$ modular form Θ_{G_2} on G_2 has Fourier coefficient $a_{\Theta_{G_2}}(f) = \pm 24|Q_R|$.*

We also give an arithmetic interpretation of the Fourier coefficients of Θ_{G_2} in the case that $R \otimes \mathbf{Q}$ is of the form $\mathbf{Q} \times K$ for K a real quadratic field. See § 3.5.2.

2. Group theory

In this section, we work out many of the group-theoretic aspects of this paper. We prove Theorems 1.2.1 and 1.2.2 of the introduction.

2.1 Central extensions: the general picture and conventions

Quaterionic modular forms of half-integral weight live on certain central extensions of adjoint forms of exceptional groups. We therefore begin by discussing some generalities about extensions of the group of points of algebraic groups and setting certain conventions. The theory is much more transparent in the simply connected case (which is also our setting when $G = G_2$, F_4 or E_8), so we recall this setting first. We will only work over $\mathbf{Q}$ and its localizations, so we restrict our discussion to this case. Let p be a place of $\mathbf{Q}$ and let $\mathbf{Q}_p$ be the associated local field; we set $\mathbf{Q}_\infty = \mathbf{R}$.

Assume that G is a simply connected, simple linear algebraic group over $\mathbf{Q}$ and consider the topological group $G(\mathbf{Q}_p)$ for $p \leq \infty$. In [Del96], Deligne constructs a canonical extension

$$1 \longrightarrow H^2(\mathbf{Q}_p, \mu_n^{\otimes 2}) \longrightarrow \widetilde{G}^{(n)}(\mathbf{Q}_p) \longrightarrow G(\mathbf{Q}_p) \longrightarrow 1$$

for any $n \in \mathbb{N}$. This construction relies heavily on the cohomology of the classifying space BG and on the construction of the *Galois symbol* by Tate [Tat76]; we will not review this construction further.

It is known [Del96, MS82] that if N is the number of roots of unity in $\mathbf{Q}_p$, then

$$H^2(\mathbf{Q}_p, \mu_n^{\otimes 2}) \cong \mathbb{K}_2(\mathbf{Q}_p)/(n,N)\mathbb{K}_2(\mathbf{Q}_p) \cong \mu_{(n,N)}(\mathbf{Q}_p), \tag{1}$$

where $\mathbb{K}_2(\mathbf{Q}_p)$ is the Milnor K-theory of $\mathbf{Q}_p$. In particular, for any $p \leq \infty$, we obtain a canonical double cover

$$1 \longrightarrow \mu_2(\mathbf{Q}_p) \longrightarrow \widetilde{G}(\mathbf{Q}_p) := \widetilde{G}^{(2)}(\mathbf{Q}_p) \longrightarrow G(\mathbf{Q}_p) \longrightarrow 1, \tag{2}$$

which satisfies the following properties:

(i) when $p = \infty$ and $G(\mathbf{R})$ is not topologically simply connected, then $\widetilde{G}$ is the unique connected topological double cover of $G(\mathbf{R})$ (note that $\pi_1(G(\mathbf{R}))$ is either $\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$, so this is well-defined);

(ii) when G is $\mathbf{Q}$-split, then for all p the group $\widetilde{G}(\mathbf{Q}_p)$ agrees with the topological double cover constructed by Steinberg and Matsumoto via generators and relations.

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Both of these facts are relevant to us: in § 2.3 we give an explicit construction for $\widetilde{G}(\mathbf{R})$ for quaternionic exceptional groups that is amenable to the definition of generalized Whittaker functions. On the other hand, our main applications to modular forms involve only the split groups F_4 and G_2. In order to make certain local calculations, we recall the Steinberg–Matsumoto presentation of $\widetilde{G}(\mathbf{Q}_p)$ in § 2.4.

If $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$ is the adele ring, Deligne similarly constructs a canonical central extension of $G(\mathbf{A})$ by $\mu_2(\mathbf{Q})$, so that we have a short exact sequence of locally compact topological groups

$$1 \longrightarrow \mu_2(\mathbf{Q}) \longrightarrow \widetilde{G}(\mathbf{A}) \longrightarrow G(\mathbf{A}) \longrightarrow 1. \tag{3}$$

This central extension splits canonically over $G(\mathbf{Q})$, allowing for the definition of automorphic forms on this group. There is a decomposition $\widetilde{G}(\mathbf{A}) = \prod_p \widetilde{G}(\mathbf{Q}_p)/\mu_2^+$, where $\widetilde{G}(\mathbf{Q}_p)$ is the local cover (2) and μ_2^+ denotes the subgroup of $\bigoplus_p \mu_2(\mathbf{Q}_p)$ with product of terms being 1. When G is a simply connected, semisimple group over $\mathbf{Q}$ or $\mathbf{Q}_p$ for $p \leq \infty$ (in particular, when G is of type G_2, F_4 or E_8), we always consider this canonical double cover of Deligne.

When our reductive group G is no longer semisimple and simply connected, such as the adjoint forms of E_6 and E_7 or for Levi subgroups, there is no canonical central extension of $\widetilde{G}(\mathbf{Q}_p)$ by $\mu_2(\mathbf{Q}_2)$; indeed, we will deal with two distinct double covers of $\mathrm{GL}_2(\mathbf{Q}_p)$ in § 4.3. The classification of a large class of central extensions (known as Brylinski–Deligne covers) is given in [BD01], where the authors classify extensions of G by the Milnor K-theory sheaf $\mathbb{K}_2$, viewed as sheaves of groups on the big Zariski site over $\mathbf{Q}_p$. Given such a central extension of sheaves of groups over $\mathrm{Spec}(\mathbf{Q}_p)$

$$\mathbb{K}_2 \longrightarrow \overline{G} \longrightarrow G,$$

one obtains a topological double cover by taking $\mathbf{Q}_p$-points and pushing out by the Hilbert symbol as follows.

$$\begin{array}{ccccc} \mathbb{K}_2(\mathbf{Q}_p) & \longrightarrow & \overline{G}(\mathbf{Q}_p) & \longrightarrow & G(\mathbf{Q}_p) \\ \downarrow{\scriptstyle (\cdot,\cdot)_2} & & \downarrow & & \downarrow{\scriptstyle =} \\ \mu_2(\mathbf{Q}_p) & \longrightarrow & \widetilde{G}(\mathbf{Q}_p) & \longrightarrow & G(\mathbf{Q}_p) \end{array}$$

Working globally, Brylinski and Deligne also extend the adelic formulation (3) to this more general setting. The connection between Deligne's cover and extensions by $\mathbb{K}_2$ may be seen in the identification (1). Indeed, when G is semisimple and simply connected, it is shown in [BD01, § 4] that for any p, there exists a central extension of sheaves of groups over $\mathrm{Spec}(\mathbf{Q}_p)$ such that the bottom row of the above diagram recovers the sequence (2).

Suppose now that G is an adjoint exceptional group over $\mathbf{Q}$ of type E_6 or E_7 such that $G(\mathbf{R})$ is quaternionic (recalled in the next section). In this setting, we construct a double cover $\widetilde{G}$ of $G(\mathbf{R})$ in § 2.3. Our convention is that we assume that $\overline{G}$ is a given Brylinski–Deligne cover of G satisfying that the induced double cover of $G(\mathbf{R})$ agrees with our construction up to isomorphism. This is automatic if the pushout $\widetilde{G}(\mathbf{R})$ is connected and non-linear.

Finally, suppose that k is either a localization of $\mathbf{Q}$ or $k = \mathbf{A}$ and let $\widetilde{G}(k)$ be a given topological double cover of $G(k)$. If S is a subset of $G(k)$, we denote by $\widetilde{S}$ its inverse image in $\widetilde{G}(k)$. If $U \subset G$ is a unipotent subgroup, then it is known that $\widetilde{G}(k)$ splits canonically over $U(k)$; we use a standard abuse of notation and simply denote by $U(k) \subset \widetilde{U}(k)$ the corresponding subgroup of $\widetilde{G}(k)$.

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Table 1. The Freudenthal magic square, $J = H_3(C)$.

The group	$\dim C = 1$	$\dim C = 2$	$\dim C = 4$	$\dim C = 8$
A_J	A_1	A_2	C_3	F_4
M_J	A_2	$A_2 \times A_2$	A_5	E_6
H_J	C_3	A_5	D_6	E_7
G_J	F_4	E_6	E_7	E_8

2.2 Review of quaternionic exceptional groups

In this section, we review notation and constructions from [Pol20] concerning quaternionic exceptional groups. For more details, we refer the reader to [Pol20, §§ 2–4].

First recall the notion of a cubic norm structure J. This is a finite-dimensional vector space J over a field k that comes equipped with a homogeneous degree-three norm map $N_J : J \to k$, a non-degenerate trace pairing $(\ ,\ ) : J \otimes J \to k$, a distinguished element $1_J \in J$, and a quadratic map $\# : J \to J^\vee \simeq J$. The relevant examples of cubic norm structures for this paper are $J = k$ and $J = H_3(C)$, the 3×3 hermitian matrices over a composition k-algebra C.

Out of a cubic norm structure J, one can create various algebraic groups. First, denote by M_J the identity component of the algebraic group of linear transformations of J that preserve the norm N_J up to scaling. Let M_J^1 denote the subgroup of M_J with scaling factor equal to 1, and let A_J be the subgroup of M_J^1 that fixes the element 1_J of J.

We next discuss the so-called Freudenthal construction. If J is defined over the field k of characteristic 0, define $W_J = k \oplus J \oplus J^\vee \oplus k$, another vector space over k. One puts on W_J a certain non-degenerate symplectic form $\langle\ ,\ \rangle$ and a quartic form $q : W_J \to k$. The algebraic group H_J is defined to be the identity component of the set of pairs $(g, \nu(g)) \in \mathrm{GL}(W_J) \times \mathrm{GL}_1$ that satisfy $\langle gw_1, gw_2\rangle = \nu(g)\langle w_1, w_2\rangle$ and $q(gw) = \nu(g)^2 q(w)$. The map $\nu : H_J \to \mathrm{GL}_1$ is called the similitude, and H_J^1 is defined to be the kernel of ν.

The next algebraic structure defined out of J is a Lie algebra $\mathfrak{g}(J)$. There are two equivalent ways to define $\mathfrak{g}(J)$. In the first way, one defines

$$\mathfrak{g}(J) = \mathfrak{sl}_3 \oplus \mathfrak{m}_J^0 \oplus V_3 \otimes J \oplus (V_3 \otimes J)^\vee.$$

Here $\mathfrak{m}_J^0$ is the Lie algebra of M_J^1 and V_3 is the standard three-dimensional representation of $\mathfrak{sl}_3$. A Lie bracket can be put on $\mathfrak{g}(J)$; see [Pol20, § 4.2.1]. We refer to this way of thinking about $\mathfrak{g}(J)$ as the '$\mathbf{Z}/3$-model'. Let E_{ij} be the 3×3 matrix with a 1 in the (i, j) position and zeros elsewhere. If $X = \sum_{i,j} a_{ij} E_{ij}$ has trace 0, we will sometimes consider X as an element of $\mathfrak{g}(J)$ via the inclusion $\mathfrak{sl}_3 \subseteq \mathfrak{g}(J)$.

In the second way to define $\mathfrak{g}(J)$, one puts

$$\mathfrak{g}(J) = \mathfrak{sl}_2 \oplus \mathfrak{h}_J^0 \oplus V_2 \otimes W_J.$$

Here $\mathfrak{h}_J^0$ is the Lie algebra of H_J^1 and V_2 is the standard two-dimensional representation of $\mathfrak{sl}_2$. We refer to this way of looking at $\mathfrak{g}(J)$ as the $\mathbf{Z}/2$-model. An explicit isomorphism between the $\mathbf{Z}/3$-model and the $\mathbf{Z}/2$-model is given in [Pol20, § 4.2.4]. An algebraic group G_J can now be defined as $Aut^0(\mathfrak{g}(J))$, the identity component of the automorphisms of the Lie algebra $\mathfrak{g}(J)$.

The algebraic groups A_J, M_J, H_J, G_J fit into the Freudenthal magic square, as $J = H_3(C)$ varies with $\dim C = 1, 2, 4, 8$. In Table 1, we list the absolute Dynkin types of the above groups. The magic square can be extended to a magic triangle, which was studied in [DG02]. We refer the reader to [DG02] for properties of this triangle.

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In the algebraic group G_J we fix a specific parabolic subgroup P_J, called the Heisenberg parabolic; see [Pol20, § 4.3.2]. The subgroup P_J can be defined as the stabilizer of the line $kE_{13} \subseteq \mathfrak{g}(J)$. It has H_J as a Levi subgroup and unipotent radical $N_J \supseteq Z \supseteq 1$ which is two-step. Here $Z = [N_J, N_J]$ is the exponential of the line kE_{13}, and one can identify N_J/Z with W_J, as a representation of H_J.

Suppose now that $k = \mathbf{R}$ and the trace pairing on J is positive definite. Then the associated real groups in each row of the magic square share similar properties: the groups A_J are all anistropic, while the groups M_J have real root system of type A_2, with root spaces that can be naturally identified with the composition algebra C.

In this setting, the groups H_J all have a real root system of type C_3, with short root spaces identified with C and long root spaces one-dimensional. Denote by H_J^+ the identity component of $H_J(\mathbf{R})$. The group H_J^1 or H_J^+ (which contains H_J^1) has a hermitian symmetric domain. More specifically, let $\mathcal{H}_J = \{Z = X + iY : X, Y \in J, Y > 0\}$. Identify $\mathcal{H}_J$ with a subset of $W_J \otimes \mathbf{C}$ via $Z \mapsto r_0(Z) := (1, -Z, Z^{\#}, -N_J(Z))$. Then one proves (see [Pol20, Proposition 2.3.1]) that given $g \in H_J^+$ and $Z \in \mathcal{H}_J$, there exists $j(g, Z) \in \mathbf{C}^\times$ so that $g \cdot r_0(Z) = j(g, Z) r_0(gZ)$, for an element $gZ \in \mathcal{H}_J$. This simultaneously defines an action of H_J^+ on $\mathcal{H}_J$ and the factor of automorphy $j(g, Z)$.

Still assuming that $k = \mathbf{R}$ and the trace pairing on J is positive-definite, the group G_J is called a quaternionic group. The groups G_J in the final row of the Freudenthal magic square now all have real root system of type F_4, with short root spaces identified with C and long root spaces one-dimensional. When $J = \mathbf{R}$ instead of $H_3(C)$, the group G_J is G_2. We refer to these cases by saying that G_J is a quaternionic adjoint exceptional group. In these cases, the group $G_J(\mathbf{R})$ is connected [Tha00].

Suppose G_J is an adjoint quaternionic exceptional group. Then a specific Cartan involution on its Lie algebra $\mathfrak{g}(J)$ is defined in [Pol20, § 4.2.3]. We denote by K_J the associated maximal compact subgroup of $G_J(\mathbf{R})$. The group K_J is of the form $(\mathrm{SU}(2) \times L^0(J))/\mu_2(\mathbf{R})$, for a certain compact group $L^0(J)$.

In [Pol20, § 5.1], a specific $\mathfrak{sl}_2$-triple (e_ℓ, h_ℓ, f_ℓ) of the complexified Lie algebra of the SU(2) factor of K_J is defined. We now recall this $\mathfrak{sl}_2$-triple. Let $e = (1, 0)^t$ and $f = (0, 1)^t$ denote the standard basis of the two-dimensional representation of $\mathfrak{sl}_2 \subseteq \mathfrak{g}(J) = \mathfrak{sl}_2 \oplus \mathfrak{h}_J^0 \oplus V_2 \otimes W_J$. One sets $e_\ell = \frac{1}{4}(ie + f) \otimes r_0(i \cdot 1_J)$, $f_\ell = -\overline{e_\ell}$ and $h_\ell = [e_\ell, f_\ell]$. Here 1_J is the identity element of the cubic norm structure J.

For $\ell \in \frac{1}{2}\mathbf{Z}_{\geq 0}$, set $\mathbb{V}_2 = \mathbf{C}^2$ and $\mathbf{V}_\ell = Sym^{2\ell}(\mathbb{V}_2)$, a representation of the Lie algebra of K_J via the projection to the SU(2) factor. Using the above $\mathfrak{sl}_2$-triple, we fix a basis of $\mathbf{V}_\ell$, as follows. First, let x, y denote a weight basis of $\mathbb{V}_2$ for h_ℓ with $y = f_\ell x$. Then we let the monomials $x^i y^j$ for $i + j = 2\ell$ be our fixed basis of $\mathbf{V}_\ell$. When ℓ is an integer, the representation $\mathbf{V}_\ell$ exponentiates to a representation of K_J.

2.3 The cover in the archimedean case

In this section, we describe an explicit construction of a connected topological double cover of the quaternionic adjoint groups $G_J(\mathbf{R})$. This gives the unique non-linear double cover of these groups.

2.3.1 *Preliminaries.* Now let J be a cubic norm structure over the real numbers $\mathbf{R}$, with positive-definite trace pairing. We assume $J = \mathbf{R}$ or $J = H_3(C)$ with C a composition algebra over $\mathbf{R}$ with positive-definite norm.

Fix the $\mathfrak{sl}_2$-triple e_ℓ, h_ℓ, f_ℓ of $\mathfrak{g}_J \otimes \mathbf{C}$, recalled above. Identify $Span(e_\ell, h_\ell, f_\ell)$ with $Sym^2(\mathbb{V}_2)$ by sending $e_\ell \mapsto x^2$, $h_\ell \mapsto -2xy$, $f_\ell \mapsto -y^2$. This identification is K_J-equivariant; see the text immediately before Lemma 9.0.2 in [Pol20].

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We recall an Iwasawa decomposition for the group $G_J(\mathbf{R})$. Let $P_J = H_J N_J$ be the Heisenberg parabolic of G_J. Let Q_J be the parabolic subgroup associated to the cocharacter $t \mapsto \mathrm{diag}(t, t, t^{-2}) \in \mathrm{SL}_3 \to G_J$. The Lie algebra of Q_J contains the root spaces where $E_{11} + E_{22} - 2E_{33}$ acts by the weights $0, 1, 2$ or 3. Moreover, Q_J stabilizes $Span(E_{13}, E_{23})$ in the $\mathbf{Z}/3$-model of $\mathfrak{g}_J$, as one sees by checking this on the Lie algebra level. Define $R_J = P_J \cap Q_J$ and denote by R_J^+ the connected component of the identity of $R_J(\mathbf{R})$. Recall that K_J denotes the maximal compact subgroup of $G_J(\mathbf{R})$ associated to the Cartan involution described in [Pol20].

Proposition 2.3.1. *Every $g \in G_J(\mathbf{R})$ can be written as $g = rk$ with $r \in R_J^+$ and $k \in K_J$. Moreover, if $k \in R_J^+ \cap K_J$, then k acts trivially on* $\mathrm{Span}(e_\ell, h_\ell, f_\ell)$.

Proof. The first part follows from the usual Iwasawa decomposition of G_J.

For the second part, let $M(R_J)$ denote the standard Levi subgroup of R_J, so that $M(R_J)$ is the subgroup of H_J that is the centralizer of the cocharacter defined above. Then $R_J(\mathbf{R}) \cap K_J = M(R_J)(\mathbf{R}) \cap K_J$. Thus, $R_J(\mathbf{R}) \cap K_J$ stabilizes the lines $\mathbf{R}E_{13}$ and $\mathbf{R}E_{23}$ in the Lie algebra $\mathfrak{g}(J)$. We claim that $R_J^+ \cap K_J$ acts trivially on these lines. To see this, observe that $R_J^+ \cap K_J = M(R_J)^+ \cap K_J$ is connected as it is a maximal compact subgroup of a real connected reductive group. The triviality of the action of $R_J^+ \cap K_J$ on E_{13} and E_{23} follows.

Recall that H_J^1 denotes the similitude equal one subgroup of the Freudenthal group H_J. One has $H_J^1(\mathbf{R}) \cap K_J$ acts by the scalar $j(k, i \cdot 1_J)$ on e_ℓ; see Lemma 9.0.1 of [Pol20]. Because $R_J^+ \cap K_J \subseteq H_J^1(\mathbf{R}) \cap K_J$, $R_J^+ \cap K_J$ acts by a scalar on e_ℓ. Because $R_J^+ \cap K_J$ acts trivially on E_{23}, this scalar is 1. We deduce that $R_J^+ \cap K_J$ acts trivially on e_ℓ, from which it follows that it also acts trivially on f_ℓ and h_ℓ.

Note that for the second part, one cannot replace R_J^+ with $R_J(\mathbf{R})$ as some elements of $R_J(\mathbf{R}) \cap K_J$ act non-trivially on $\mathrm{Span}(e_\ell, h_\ell, f_\ell)$. □

2.3.2 *The double cover.* For $k \in K_J$, denote by $Ad(k)$ the action of k on the space $\mathrm{Span}(e_\ell, h_\ell, f_\ell) = Sym^2(\mathbb{V}_2)$. Fix an $\mathbf{R}_{>0}^\times$-valued character χ of R_J^+, to be specified later. We define

$$f_{\mathrm{lin}} : G_J(\mathbf{R}) \to Aut_{\mathbf{C}}(Sym^2(\mathbb{V}_2)) \simeq \mathrm{GL}_3(\mathbf{C})$$

as $f_{\mathrm{lin}}(g) = \chi(r)Ad(k)$ if $g = rk$ with $r \in R_J^+$ and $k \in K_J$. By Proposition 2.3.1, f_{lin} is well-defined, because $\chi(R_J^+ \cap K_J) = 1$ as the image is a compact subgroup of $\mathbf{R}_{>0}^\times$.

Now, consider the symmetric space $X_J = G_J(\mathbf{R})/K_J$; it is connected and contractible. Define $j_{\mathrm{lin}}(g, x)$ for $x \in X_J$ and $g \in G_J(\mathbf{R})$ as $f_{\mathrm{lin}}(gh)f_{\mathrm{lin}}(h)^{-1}$ if $x = hK_J$. Note that j_{lin} is well-defined.

One has the following proposition, whose proof we omit; it follows from the fact that the Iwasawa decomposition of $G_J(\mathbf{R})$ is smooth.

Proposition 2.3.2. *The maps*

$$f_{\mathrm{lin}} : G_J(\mathbf{R}) \longrightarrow Aut_{\mathbf{C}}(Sym^2(\mathbb{V}_2)) \quad \text{and} \quad j_{\mathrm{lin}} : G_J(\mathbf{R}) \times X_J \longrightarrow Aut_{\mathbf{C}}(Sym^2(\mathbb{V}_2))$$

are smooth.

We may now define $\widetilde{G}_J$.

Definition 2.3.3. Let $\widetilde{G}_J$ be the set of pairs (g, j_g) with $g \in G_J(\mathbf{R})$ and

$$j_g : X_J \to Aut_{\mathbf{C}}(\mathbb{V}_2)$$

continuous so that $Sym^2(j_g(x)) = j_{\mathrm{lin}}(g, x)$. A multiplication is defined as

$$(g_1, j_1(x))(g_2, j_2(x)) = (g_1 g_2, j_1(g_2 x) j_2(x)).$$

The identity is the element $(1, e)$ where $e(x) = 1$ for all x.

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With these definitions, it is easily checked that $\widetilde{G}_J$ is a group.

A topology can be put on $\widetilde{G}_J$ as follows. Let $x_0 = 1K_J \in X_J$ be the basepoint determined by K_J. Now, note that given $g \in G_J(\mathbf{R})$, there are exactly two continuous lifts $X_J \to Aut_{\mathbf{C}}(\mathbb{V}_2)$ of $j_{\mathrm{lin}}(g,-) : X_J \to Aut_{\mathbf{C}}(Sym^2(\mathbb{V}_2))$, and that these lifts are determined by their value at x_0. Thus, there is an injective map of sets $\widetilde{G}_J \to G_J(\mathbf{R}) \times \mathrm{GL}_2(\mathbf{C})$ given by $(g, j_g(x)) \mapsto (g, j_g(x_0))$. We give $\widetilde{G}_J$ the subspace topology of $G_J(\mathbf{R}) \times \mathrm{GL}_2(\mathbf{C})$ via this map.

For $g' = (g, j_g(x)) \in \widetilde{G}_J$, we write $j_{1/2}(g', x) := j_g(x)$.

PROPOSITION 2.3.4. *With the above topology, $\widetilde{G}_J$ is a connected topological group. The canonical map $\widetilde{G}_J \to G_J(\mathbf{R})$ is a covering map with central μ_2 kernel.*

Proof. One first proves that $\widetilde{G}_J$ is a topological group and $\widetilde{G}_J \to G_J(\mathbf{R})$ is a covering space. This is an exercise in covering space theory, so we omit it.

Let us explain the connectedness of $\widetilde{G}_J$. We will check that $(1, e(x))$ and $(1, -e(x))$ are connected by a path. Given the other claims, this suffices.

To see that $(1, e(x))$ is connected to $(1, -e(x))$, we consider $h_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2 \subseteq \mathfrak{g}_J = \mathfrak{sl}_2 \oplus \mathfrak{h}_J^0 \oplus \mathbb{V}_2 \otimes W_J$. Now, by our formulas for the Cartan decomposition, h_0 is in the Lie algebra of K_J, so $\exp(th_0)$ is in $K_J \subseteq G_J(\mathbf{R})$. One computes that $\exp(th_0)$ acts on e_ℓ, h_ℓ, f_ℓ as:

- $e_\ell \mapsto e^{-it} e_\ell$;
- $h_\ell \mapsto h_\ell$;
- $f_\ell \mapsto e^{it} f_\ell$.

Now consider the path $[0, 2\pi] \to K_J \subseteq G_J(\mathbf{R})$ given by $t \mapsto \exp(th_0)$. This path is a loop, with $2\pi \mapsto 1$. Because $\widetilde{G}_J \to G_J(\mathbf{R})$ is a covering space, it lifts to a path $\widetilde{\gamma} : [0, 2\pi] \to \widetilde{G}_J$ satisfying $\widetilde{\gamma}(0) = 1$. Thus, $j_{1/2}(\widetilde{\gamma}(t), x_0) \in \mathrm{GL}_2(\mathbf{C})$ satisfies that its symmetric square is the action on e_ℓ, h_ℓ, f_ℓ given above. Because it is continuous and the identity at $t = 0$, $j_{1/2}(\widetilde{\gamma}(t), x_0) = \mathrm{diag}(e^{-it/2}, e^{it/2})$. Consequently, $j_{1/2}(\widetilde{\gamma}(2\pi), x_0) = -1$. This proves our assertion. □

Note that since K_J is itself connected and the path $\widetilde{\gamma}$ stays in $\widetilde{K}_J$, we see that the inverse image $\widetilde{K}_J$ of K_J is a connected compact Lie group.

Because $\widetilde{G}_J \to G_J(\mathbf{R})$ is a covering space, $\widetilde{G}_J$ is uniquely a Lie group. Note also that the map $j_{1/2}(\,, x_0) : \widetilde{K}_J \to Aut_{\mathbf{C}}(\mathbb{V}_2)$ is a group homomorphism. Finally, we remark that R_J^+ splits into $\widetilde{G}_J$ as $r \mapsto (r, j_r(x))$ with $j_r(x) = \chi(r)^{1/2}$ for all $x \in X_J$.

2.3.3 *An application.* Define $\nu : R_J \to \mathrm{GL}_1$ as $rE_{13} = \nu(r)E_{13}$ and $\lambda : R_J \to \mathrm{GL}_1$ as $rE_{23} = \lambda(r)E_{23} + *E_{13}$. In other words, if det is the determinant of the action of R_J on $\mathrm{Span}(E_{13}, E_{23})$, then $\lambda = \det(\cdot)\nu^{-1}$. Define χ, the character defining f_{lin} as $\chi = \nu\lambda^{-1} = \nu^2 \det(\cdot)^{-1}$. With this choice, which we will make from now on, one has the following lemma. Let $K_H = H_J^1(\mathbf{R}) \cap K_J$ be a maximal compact subgroup of $H_J^1(\mathbf{R})$.

LEMMA 2.3.5. *With $h \in H_J^+$, one has $j_{\mathrm{lin}}(h, x_0) = \mathrm{diag}(j(h,i), 1, \overline{j(h,i)})$ via the action on x^2, xy, y^2. Thus, if $z \in \mathcal{H}_J = H_J^1(\mathbf{R})/K_H \subseteq G_J/K_J$, then $j_{\mathrm{lin}}(h, z) = \mathrm{diag}(j(h,z), 1, \overline{j(h,z)})$. Consequently, the $(1,1)$-coordinate of $j_{1/2} : \widetilde{H_J^+} \to \mathrm{GL}_2(\mathbf{C})$ defines a squareroot of $j(h,z)$.*

Proof. Let P_S denote the Siegel parabolic of H_J, so that $P_S = H_J \cap R_J$. Suppose $h \in H_J^+$ is $h = pk$ with $p \in P_S(\mathbf{R})^+$ and $k \in K_H \subseteq H_J^1(\mathbf{R})$. Then

$$j(p, i) = \langle pr_0(i), E_{23} \rangle = \nu(p) \langle r_0(i), p^{-1} E_{23} \rangle = \chi(p).$$

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Moreover, essentially by the definition of j, $Ad(k) = \mathrm{diag}(j(k,i), 1, \overline{j(k,i)})$. As $j(h,i) = j(pk,i) = j(p,i)j(k,i)$, one obtains $j_{\mathrm{lin}}(h,x_0) = \mathrm{diag}(j(h,i),1,\overline{j(h,i)})$.

For the second statement, suppose $h_z \in H_J^+$ satisfies $h_z \cdot i = z$. Then

$$\begin{aligned} j_{\mathrm{lin}}(h,z) &= f_{\mathrm{lin}}(hh_z) f_{\mathrm{lin}}(h_z)^{-1} = \mathrm{diag}(j(hh_z,i),1,\overline{j(hh_z,i)})\,\mathrm{diag}(j(h_z,i),1,\overline{j(h_z,i)})^{-1} \\ &= \mathrm{diag}(j(h,z),1,\overline{j(h,z)}). \end{aligned}$$

The proposition follows. □

2.4 Steinberg generators and relations

In this section, we let k be a local field of characteristic zero and assume that G is a simply connected simple group over k. In this setting, Deligne's double cover (2) coincides with the Steinberg–Matsumoto cover. We thus recall this construction for the purposes of certain p-adic calculations in later sections.

Suppose that Φ is a simple root system and Δ a set of simple roots. We let (α,β) denote the pairing on Φ normalized so that $(\alpha,\alpha) = 2$ for a long root (when the root system is simply laced, we assert that all roots are long). Suppose that $\mathfrak{g}$ is the associated split, simple Lie algebra over $\mathbf{Q}$ and G the associated split, simply connected group. Steinberg [Ste16] gives a presentation for the group $G(k)$ in terms of generators and relations. One has generators $x_\alpha(u)$ for all roots α and $u \in k$, subject to the following relations:

(i) $x_\alpha(u)x_\alpha(v) = x_\alpha(u+v)$;
(ii) if α,β are roots with $\alpha+\beta \neq 0$, then the commutator

$$\{x_\alpha(u), x_\beta(v)\} = \prod_{i\alpha+j\beta\in\Phi, i,j\in\mathbf{Z}_{>0}} x_{i\alpha+j\beta}(C_{ij}u^i v^j)$$

for integers $C_{i,j}$ that depend upon the order in the product but are independent of u,v;
(iii) for $t \in k^\times$ set $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ and $h_\alpha(t) = w_\alpha(t)w_\alpha(-1)$. Then $h_\alpha(t)h_\alpha(s) = h_\alpha(ts)$;
(iv) when Φ is of type A_1, then $w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$.

Following Steinberg [Ste16, Theorem 12] (see also [LS10, § 2]), a topological double cover of $G(k)$ can now be defined as follows. Recall the Hilbert symbol $(\cdot,\cdot)_2 : k^\times \times k^\times \to \mu_2(k)$. One takes as generators elements $x_\alpha(u)$ and $\{1,\zeta\} = \mu_2$ satisfying relations (i), (ii) and (iv), along with:

(v) the elements $1,\zeta$ are in the center;
(vi) for $t \in k^\times$ set

$$\widetilde{w}_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \quad \text{and} \quad \widetilde{h}_\alpha(t) = \widetilde{w}_\alpha(t)\widetilde{w}_\alpha(-1);$$

then $\widetilde{h}_\alpha(t)\widetilde{h}_\alpha(s) = \widetilde{h}_\alpha(ts)(t,s)_2^{2/(\alpha,\alpha)}$.

From [LS10, § 3, p. 4904], who cite [Mat69, Lemme 5.4], one has

$$\{\widetilde{h}_\alpha(s), \widetilde{h}_\beta(t)\} = (s,t)_2^{(\alpha^\vee,\beta^\vee)}, \tag{4}$$

where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. We let $\widetilde{G}(k)$ denote the double cover of $G(k)$ constructed here, where the projection $p : \widetilde{G}(k) \longrightarrow G(k)$ is given by sending generators to the analogous generators in $G(k)$. As noted previously, this construction recovers Deligne's cover (2) in the split case. In particular, if $J = \mathbf{R}$ or $H_3(\mathbf{R})$, so that $G = G_J$ is the split group of type G_2 or F_4, respectively, then $\widetilde{G}(\mathbf{R}) \cong \widetilde{G}_J$.

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2.5 2-adic subgroups of F_4

We now specialize to $k = \mathbf{Q}_2$ and G the split group of type F_4. We enumerate the 4 simple roots in the usual way, so that the Dynkin diagram

$$\circ - - - \circ =>= \circ - - - \circ$$

has labels α_1 through α_4 from left to right. In this section, we define certain compact open subgroups $K_R^*(4)$ and $K_R'(4)$ of $\widetilde{F}_4(\mathbf{Q}_2)$ that we prove inject into $F_4(\mathbf{Q}_2)$. This first group is the natural analogue in $F_4(\mathbf{Q}_2)$ of the classical compact open subgroup

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_6(\mathbb{Z}_2) : C \equiv 0 \ (\mathrm{mod} 4),\ \det(A), \det(D) \equiv 1 \ (\mathrm{mod} 4) \right\}$$

that arises in the theory of Siegel theta functions of half-integral weight; indeed, $K_R^*(4)$ essentially intersects the standard GSp_6-Levi subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ in this group.

For global purposes, it is better to pass to a certain conjugate of this compact open subgroup, denoted $K_R'(4)$. While we do not use the subgroup $K_R^*(4)$ in the sequel, it is nevertheless more natural to define and prove properties about (splitting, Iwahori decomposition, etc.). Thus, we consider the case of $K_R^*(4)$ first, then pass to the conjugate $K_R'(4)$ in § 2.5.4. In § 4.2.1, we use the group $K_R'(4)$ to construct the quaternionic modular forms of half-integral weight described in Theorem 1.2.6.

Remark 2.5.1. We remark that one can also construct quaternionic modular forms of level $K_R^*(4)$. However, it is unclear whether their Fourier coefficients are as interesting.

2.5.1 *Preliminaries.* To begin, we record the following slight extension of [Kar21, Lemma 3.1].

Lemma 2.5.2. *Let k be a local field of characteristic zero. Suppose that Φ is a simple root system and $G(k)$ is the corresponding simply connected group. For any $\alpha \in \Phi$ and $s, t \in k$ such that $1 + st \neq 0$, in the double cover $\widetilde{G}(k)$ we have*

$$x_\alpha(t) x_{-\alpha}(s) = \left(1 + st, \frac{t}{1+st}\right)_2^{-2/(\alpha,\alpha)} x_{-\alpha}\left(\frac{s}{1+st}\right) \widetilde{h}_\alpha(1+st) x_\alpha\left(\frac{t}{1+st}\right).$$

Proof. This follows from [Ste73, Proposition 2.7]. □

Corollary 2.5.3. *With notation as above, now let $k = \mathbf{Q}_2$ and let $\alpha \in \Phi$ and $s, t \in \mathbf{Q}_2$.*

(i) *If Φ is doubly laced and α is a short root, then*

$$x_\alpha(t) x_{-\alpha}(s) = x_{-\alpha}\left(\frac{s}{1+st}\right) \widetilde{h}_\alpha(1+st) x_\alpha\left(\frac{t}{1+st}\right).$$

(ii) *Let Φ be of any type. If $\mathrm{val}_2(s) \geq 2$ and $\mathrm{val}_2(t) \geq 0$, then*

$$x_\alpha(t) x_{-\alpha}(s) = x_{-\alpha}\left(\frac{s}{1+st}\right) \widetilde{h}_\alpha(1+st) x_\alpha\left(\frac{t}{1+st}\right).$$

Proof. The proof of the first claim is immediate from the lemma and our normalization that $(\beta, \beta) = 2$ for long roots, so that $(\alpha, \alpha) = 1$ for our short root. The second claim follows precisely as in the proof of [Kar21, Lemma 3.1] with $\lambda = 0$. □

We now return to $G = F_4$. The inclusion of rational Lie algebras $\mathfrak{m}_J^0 \to \mathfrak{g}(J)$ discussed in § 2.2 gives rise to an embedding of algebraic groups $\mathrm{SL}_3 \to F_4$ when $J = H_3(\mathbf{Q})$. In terms of roots, the image corresponds to the subroot system with simple roots $\{\alpha_3, \alpha_4\}$. When k is a

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local field, note that this embedding lifts to a splitting $s : \mathrm{SL}_3(k) \to \widetilde{F}_4(k)$. Indeed, the subgroup $\mathrm{SL}_3(k)$ of $F_4(k)$ is generated by the elements $x_\beta(u)$ for β lying in the subroot system generated by $\{\alpha_3, \alpha_4\}$. We may define this $\mathrm{SL}_3(k)$ via generators and relations as in § 2.4, and the relations defining it continue to be satisfied in $\widetilde{F}_4(k)$ due to Corollary 2.5.3.

Lemma 2.5.4. *Let $\mathrm{SL}_3 \subset F_4$ be the $\mathbf{Q}$-subgroup just described. For any local field k, the double cover $\widetilde{F}_4(k)$ splits uniquely over $\mathrm{SL}_3(k)$.*

2.5.2 *The case of $K_R^*(4)$.* Recall that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the simple roots of F_4, with α_1, α_2 long and α_3, α_4 short. Let $R = M_R U_R$ be the standard non-maximal parabolic subgroup of F_4 with simple roots α_3, α_4 in the Levi M_R. The notation R here refers to the non-maximal parabolic R_J from § 2.3 as these two parabolic subgroups agree when $G = G_J$ is of type F_4. Set

$$\Phi^+_{M_R} = \{\alpha_3, \alpha_4, \alpha_3 + \alpha_4\},$$

set $\Phi^-_{M_R} = -\Phi^+_{M_R}$ and $\Phi_{M_R} = \Phi^+_{M_R} \cup \Phi^-_{M_R}$. Let $\Phi^+_{U_R} = \Phi^+ \setminus \Phi^+_{M_R}$, so that $\Phi^+_{U_R}$ contains the roots in the unipotent radical U_R of R.

Set $K^*_{M_R}(4)$ to be the subgroup $\widetilde{M}_R(\mathbf{Q}_2)$ generated by $\widetilde{h}_{\alpha_i}(1 + 4\mathbf{Z}_2)$ for $i = 1, 2$ and $x_\beta(\mathbf{Z}_2)$ for $\beta \in \Phi_{M_R}$. Let $U^+_R(\mathbf{Z}_2)$ be the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $x_\beta(\mathbf{Z}_2)$ for all $\beta \in \Phi^+_{U_R}$, and let $U^-_R(4\mathbf{Z}_2)$ be the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $x_{-\beta}(4\mathbf{Z}_2)$ for all $\beta \in \Phi^+_{U_R}$. Finally, let $K^*_R(4)$ be the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $U^-_R(4\mathbf{Z}_2)$, $K^*_{M_R}(4)$ and $U^+_R(\mathbf{Z}_2)$. We have the following theorem.

Theorem 2.5.5. *Let the notation be as above.*

(i) *One has $K^*_R(4) = U^-_R(4\mathbf{Z}_2) K^*_{M_R}(4) U^+_R(\mathbf{Z}_2)$.*
(ii) *The map $K^*_R(4) \to F_4(\mathbf{Q}_2)$ is injective.*

We will prove this theorem below. While the statement is natural, the proof is technical due to the lack of uniqueness of sections over various tori in $F_4(\mathbf{Q}_2)$. As a result, we cannot simply rely on the Iwahori factorization of the image of $K^*_R(4)$.

It is easy to deduce the following corollary of Theorem 2.5.5.

Corollary 2.5.6. *The group $K^*_R(4)$ has an Iwahori decomposition with respect to any standard parabolic subgroup containing R.*

Recall the subgroup $\mathrm{SL}_3 \subset F_4$ from the previous subsection. The subgroup $s(\mathrm{SL}_3(k))$ of $\widetilde{F}_4(k)$ is that which is generated by the elements $x_\beta(u)$ for $\beta \in \Phi_{M_R}$. Using Lemma 2.5.4, we now observe the following.

Lemma 2.5.7. *The map $K^*_{M_R}(4) \to F_4(\mathbf{Q}_2)$ is injective.*

Proof. If $g \in K^*_{M_R}(4)$, it is easy to see that one can express g as a product $g = t_1 t_2 s(g')$ with $t_j \in h_{\alpha_j}(1 + 4\mathbf{Z}_2)$ and $g' \in \mathrm{SL}_3(\mathbf{Q}_2)$. Consequently, if $g \mapsto 1$ in $F_4(\mathbf{Q}_2)$, then $t_1 = t_2 = 1$ and $g' = 1$, proving that $g = 1$. □

We will prove part (i) of Theorem 2.5.5 in § 2.5.3. Let us observe now that part (i) implies part (ii). Indeed, suppose $g = n_1 m n_2$ is in $K^*_R(4)$ with $n_1 \in U^-_R(4\mathbf{Z}_2)$, $m \in K^*_{M_R}(4)$ and $n_2 \in U^+_R(\mathbf{Z}_2)$. If $g \mapsto 1$ in $F_4(\mathbf{Q}_2)$, then we see easily that $n_1 = 1$ and $n_2 = 1$. Thus, $m \mapsto 1$, hence $m = 1$ by Lemma 2.5.7.

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2.5.3 *Iwahori decomposition.* For a non-negative integer m, let $U_R^+(2^m\mathbf{Z}_2)$ be the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $x_\beta(2^m\mathbf{Z}_2)$ for all $\beta \in \Phi_{U_R}^+$, and let $U_R^-(2^m\mathbf{Z}_2)$ be the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $x_{-\beta}(2^m\mathbf{Z}_2)$ for all $\beta \in \Phi_{U_R}^+$.

We begin with the following lemma. Let U_B be the unipotent radical of the standard Borel of $\widetilde{F}_4(\mathbf{Q}_2)$.

LEMMA 2.5.8. *Recall that $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ are the simple roots.*

(i) *The unipotent group $U_B(\mathbf{Q}_2)$ is generated by the $x_{\alpha_i}(\mathbf{Q}_2)$;*
(ii) *Let U_s be the subgroup of $U_B(\mathbf{Q}_2)$ generated by the $x_{\alpha_i}(\mathbf{Z}_2)$. Then U_s contains $U_R^+(2^A)$ for some $A \gg 0$.*

Proof. The first part of the lemma is standard. For the second part, suppose $\alpha \in \Phi_{U_R}^+$. By the first part, there exists a finite word u in elements of the form $x_{\alpha_i}(r_i)$ with $r_i \in \mathbf{Q}_2$, so that $u = x_\alpha(1)$. Let T^{++} denote the subgroup of $t \in T$ with $|\alpha_i(t)| < 1$ for all i. Conjugating by a sufficiently deep $t \in T^{++}$, one finds that there exists a non-zero $r_\alpha \in \mathbf{Z}_2$ so that $x_\alpha(r_\alpha) \in U_s$. Now, for $t \in \mathbf{Z}_2^\times$, consider the commutator $\{h_\alpha(t), x_\alpha(r_\alpha)\}$. On the one hand, because $t \in \mathbf{Z}_2^\times$, this commutator is in U_s. On the other hand, this commutator is $x_\alpha((t^2-1)r_\alpha)$. As t varies in $\mathbf{Z}_2^\times$, t^2-1 fills out $8\mathbf{Z}_2$. Thus, there is $N_\alpha \gg 0$ so that $x_\alpha(2^{N_\alpha}\mathbf{Z}_2) \subseteq U_s$. The lemma follows. □

Let U be the set of products of the form $U_R^-(4\mathbf{Z}_2)K_{M_R}^*(4)U_R^+(\mathbf{Z}_2)$. Let $K_R^*(4, 2^m)$ be the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $U_R^-(4\mathbf{Z}_2)$, $K_{M_R}^*(4)$ and $U_R^+(2^m\mathbf{Z}_2)$, so that $K_R^*(4) = K_R^*(4,1)$. In order to prove Theorem 2.5.5, we need to check that $K_R^*(4) \cdot U = U$. We will do this by proving $K_R^*(4, 2^A) \cdot U = U$ for $A \gg 0$, then inducting down on A to obtain $K_R^*(4,1) \cdot U = U$.

We start with the following lemma.

LEMMA 2.5.9. *One has $U_R^-(4\mathbf{Z}_2) \cdot U = U$, and $K_{M_R}^*(4) \cdot U = U$.*

Proof. That $U_R^-(4\mathbf{Z}_2)U = U$ is trivial. For the multiplication by $K_{M_R}^*(4)$, one uses that if $\beta \in \Phi_{U_R}^-$, $\alpha \in \Phi_{M_R}$, and a, b are positive integers, then if $\gamma = a\alpha + b\beta$ is a root, then $\gamma \in \Phi_{U_R}^-$. The lemma then follows easily by applying the commutator formula. □

Now we have the following.

PROPOSITION 2.5.10. *There is $A \gg 0$ such that $U_R^+(2^A) \cdot U \subseteq U$.*

Proof. By Lemma 2.5.8, it suffices to show that $x_{\alpha_i}(\mathbf{Z}_2)U \subseteq U$ for all simple roots α_i. By Lemma 2.5.9, we must only check this for $i = 1, 2$.

Thus suppose that α_i is a simple root, $i = 1, 2$, and $\alpha \in \Phi_{U_R}^+$. Note that if a, b are positive integers, and $\alpha \neq \alpha_i$, then if $\gamma = a\alpha_i - b\alpha$ is a root, then $\gamma \in \Phi^-$. Indeed, $a\alpha_i = \gamma + b\alpha$, so that if γ were positive, we would have that both γ and α are proportional to α_i, a contradiction. It follows that, for such α_i and α and $u \in \mathbf{Z}_2$, $u' \in 4\mathbf{Z}_2$, the commutator $\{x_{\alpha_i}(u), x_{-\alpha}(u')\} \in U_B^-(4\mathbf{Z}_2)$. Here $U_B^-(4\mathbf{Z}_2)$ is the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $x_\beta(4\mathbf{Z}_2)$ for β a negative root.

Let us also note that $x_{\alpha_i}(u)x_{-\alpha_i}(u') = x_{-\alpha_i}(u'/(1+uu'))\widetilde{h}_{\alpha_i}(1+uu')x_{\alpha_i}(u/(1+uu'))$. Combining these two facts, we obtain the following: if $g = n_1mn_2$ is in U, then $x_{\alpha_i}(u)g = n_1' x_{\alpha_i}(u)m'n_2$ with $n_1' \in U_B^-(4\mathbf{Z}_2)$ and $m' \in K_{M_R}^*(4)$.

Now, one verifies easily that if $m' \in K_{M_R}^*(4)$ and $u \in U_R^+(\mathbf{Z}_2)$, then $(m')^{-1}um' \in U_R^+(\mathbf{Z}_2)$. Consequently, $x_{\alpha_i}(u)g = n_1'm'n_2'$ is in $U_B^-(4\mathbf{Z}_2) \cdot U$. The proposition follows from Lemma 2.5.9. □

It follows from Proposition 2.5.10 and Lemma 2.5.9 that $K_R^*(4, 2^A) \cdot U \subseteq U$ for $A \gg 0$. As mentioned, we will now induct downward on A to obtain $K_R^*(4) \cdot U = U$.

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We require the following lemma.

Lemma 2.5.11. *Let the notation be as usual.*

(i) *The sets* $\widetilde{h}_{\alpha_i}(1+4\mathbf{Z}_2)$ *are subgroups, and they commute with each other.*
(ii) *Suppose* $t \in 1+4\mathbf{Z}_2$ *and* $\beta \in \Phi$. *Then there are* $t_1, \dots, t_4 \in 1+4\mathbf{Z}_2$ *so that* $\widetilde{h}_\beta(t) = \prod_{i=1}^4 \widetilde{h}_{\alpha_i}(t_i)$.

Proof. The first part of the lemma follows from the usual multiplication formulas, together with the fact that the Hilbert symbol is trivial when restricted to $1+4\mathbf{Z}_2$. For the second part of the lemma, we mimic the proof of [Ste16, Lemma 38(b)]. Thus, suppose $\beta = w\alpha_i$ with α_i a simple root. Write $w = w_\alpha w'$ where $length(w') = length(w) - 1$. Set $\gamma = w_\alpha \beta$ so that $\beta = w_\alpha \gamma$. Now [Ste16, Lemma 37(c)] yields that $\widetilde{w_\alpha}(1)\widetilde{h}_\gamma(t)\widetilde{w_\alpha}(-1) = \widetilde{h}_{w_\alpha\gamma}(t)(c,t)$ for some $c = \pm 1$. However, because $t \in 1+4\mathbf{Z}_2$ and $c = \pm 1$, $(c,t) = 1$. Thus, $\widetilde{h}_\beta(t) = \widetilde{w_\alpha}(1)\widetilde{h}_\gamma(t)\widetilde{w_\alpha}(-1)$, from which we obtain

$$\widetilde{h}_\beta(t) = \widetilde{h}_\gamma(t)(\widetilde{h}_\gamma(t)^{-1}\widetilde{w_\alpha}(1)\widetilde{h}_\gamma(t))\widetilde{w_\alpha}(-1) = \widetilde{h}_\gamma(t)\widetilde{w_\alpha}(t^{-\langle\alpha,\gamma\rangle})\widetilde{w_\alpha}(-1),$$

using [Ste16, Lemma 37(e)] for the second equality. But this is $\widetilde{h}_\gamma(t)\widetilde{h}_\alpha(t^{-\langle\alpha,\gamma\rangle})$. The lemma follows by induction on the length of w. □

Proposition 2.5.12. *For every non-negative integer* m, *one has* $K^*(4,2^m)\cdot U \subseteq U$.

Proof. As just noted, Proposition 2.5.10 implies $K_R^*(4,2^A)\cdot U \subseteq U$ for $A \gg 0$. We will induct downward on N to obtain the proposition.

Thus, suppose that we have proved $K_R^*(4,2^{m+1})\cdot U \subseteq U$ for a non-negative integer m. We wish to verify that $K_R^*(4,2^m)\cdot U \subseteq U$. To do this, it suffices to check that $U_R^+(2^m\mathbf{Z}_2)\cdot U \subseteq U$. Thus, suppose $u = x_\alpha(2^m s) \in U_R^+(2^m\mathbf{Z}_2)$ and $x = n_1\, mn_2 \in U$. We have $ux = (un_1u^{-1})m(m^{-1}um)n_2$. It is easy to see that $(m^{-1}um)n_2 \in U_R^+(\mathbf{Z}_2)$. We claim that $un_1u^{-1} \in K_R^*(4,2^{m+1})$. Granted this claim, the proposition follows.

To prove the claim, suppose $n_1 = v_1\cdots v_r$ with each v_i of the form $x_{-\beta_i}(4s_i)$ with $s_i \in \mathbf{Z}_2$ and $\beta_i \in \Phi_{U_R}^+$. The commutator formula gives $uv_ju^{-1} = k'$ with $k' \in K_R^*(4,2^{m+1})$. Indeed, if $\alpha \neq \beta_i$ this follows from the commutator formula. If $\alpha = \beta_i$, this follows from the formula

$$x_\alpha(t)x_{-\alpha}(s) = x_{-\alpha}(s/(1+ts))\widetilde{h}_\alpha(1+st)x_\alpha(t/(1+st)),$$

which implies

$$x_\alpha(t)x_{-\alpha}(s)x_\alpha(-t) = x_{-\alpha}(s/(1+ts))\widetilde{h}_\alpha(1+st)x_\alpha(-st^2/(1+st)).$$ □

2.5.4 *The case of* $K_R'(4)$. We now define a new subgroup, $K_R'(4) \subseteq \widetilde{F}_4(\mathbf{Q}_2)$, by conjugating $K_R^*(4)$ by a certain element of $H_J(\mathbf{Q}_2)$. This has the effect of changing which root groups are generated by entries in $\mathbf{Z}_2$ or $4\mathbf{Z}_2$. We verify that this conjugate has an appropriate Iwahori factorizations; that it maps injectively to the linear group $F_4(\mathbf{Q}_2)$ is immediate. Our motivation is that this new group gives a useful compact open subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ for global constructions.

We need to introduce a bit more notation. Recall that $P_S = H_J \cap R$ is the Siegel parabolic subgroup of H_J; it has Levi decomposition $P_S = M_R N_S$. We set $Q = LU_Q$ denote the standard maximal parabolic of F_4 associated to the simple root α_2. Recalling the notation in § 2.3, this is the parabolic Q_J of $G_J = F_4$ when $J = H_3(\mathbf{Q}_p)$. Let $w_0 \in H_J(\mathbf{Z}) \subset H_J(\mathbf{Z}_2)$ be a representative of the unique minimal-length Weyl group element for H_J which normalizes the M_R and conjugates the Siegel parabolic P_S to its opposite.

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Let Φ_N^+ be the set of roots in the unipotent radical N of the Heisenberg parabolic P. Let $\Phi_{N_S}^+$ be the set of roots in the unipotent group N_S. These are the roots $\sum_i m_i\alpha_i$ with $m_1 = 0$ and $m_2 = 1$. Note that $\Phi_{U_R}^+ = \Phi_N^+ \sqcup \Phi_{N_S}^+ = \{\alpha_1\} \sqcup \Phi_{U_Q}^+$.

We let $N_S^+(2^m\mathbf{Z}_2)$ be the subgroup of $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $x_\alpha(2^m\mathbf{Z}_2)$ for all $\alpha \in \Phi_{N_S}^+$ and let $N^+(2^m\mathbf{Z}_2)$ be the subgroup $\widetilde{F}_4(\mathbf{Q}_2)$ generated by $x_\alpha(2^m\mathbf{Z}_2)$ for all $\alpha \in \Phi_N^+$. Similarly, define $N^-(2^m\mathbf{Z}_2)$ and $N_S^-(2^m\mathbf{Z}_2)$.

Set $U_R^+(4,1)$ to be the subgroup generated by $N_S^+(4\mathbf{Z}_2)$ and $N^+(\mathbf{Z}_2)$. Let $U_R^-(1,4)$ denote the subgroup generated by $N_S^-(\mathbf{Z}_2)$ and $N^-(4\mathbf{Z}_2)$. Finally, we define $K_R'(4)$ to be the subgroup generated by $U_R^-(1,4)$, $K_{M_R}^*(4)$ and $U_R^+(4,1)$.

The goal of this section is to prove the following theorem.

THEOREM 2.5.13. *Let the notation be as above.*

(i) *One has* $K_R'(4) = U_R^-(1,4) \cdot K_{M_R}^*(4) \cdot U_R^+(4,1)$.
(ii) *The map* $K_R'(4) \to F_4(\mathbf{Q}_2)$ *is injective.*

Proof. We first show that $K_R'(4) = w_0 K_R^* w_0^{-1}$ by showing that w_0 sends the generators of $K_R^*(4)$ to those of $K_R'(4)$. This is a straightforward calculation on the level of roots groups in $F_4(\mathbf{Q}_2)$, so we need only ensure the claim with our choice of lifts in the cover. Note that the conjugation action depends only on the element in $F_4(\mathbf{Q}_2)$ and not a choice of lift.

Recall that $K_R^*(4)$ is generated by $U_R^-(4\mathbf{Z}_2) = N^-(4\mathbf{Z}_2)N_S^-(4\mathbf{Z}_2)$, $K_{M_R}^*(4)$ and $U_R^+(\mathbf{Z}_2) = N^+(\mathbf{Z}_2)N_S^+(\mathbf{Z}_2)$. Since the cover splits canonically over unipotent subgroups, the action of w_0 on the unipotent generators is uniquely determined by the corresponding conjugation in $F_4(\mathbf{Q}_2)$, where one readily verifies that

$$w_0 N_S^-(4\mathbf{Z}_2) w_0^{-1} = N_S^+(4\mathbf{Z}_2), \quad w_0 N^-(4\mathbf{Z}_2) w_0^{-1} = N^-(4\mathbf{Z}_2),$$

and similarly for the factors of U_R^+. In addition, w_0 permutes the root groups $x_\beta(\mathbf{Z}_2)$ for $\beta \in \Phi_{M_R}$. Thus, we need only consider the torus generators $\widetilde{h}_{\alpha_i}(1+4\mathbf{Z}_2)$ with $i = 3,4$ of $K_{M_R}^*(4)$.

Suppose that $w_0 = s_1 s_2 \cdots s_6$ be a minimal word decomposition of the associated Weyl group element in terms of simple root reflections. For any root α, let $\widetilde{h}_\alpha(t)$ be the distinguished lift of the corresponding coroot $h_\alpha(t)$. Then [Gao17, 2.1(3)] tells us that for any simple reflection s_β,

$$s_\beta \widetilde{h}_\alpha(t) s_\beta^{-1} = \widetilde{h}_\alpha(t)\widetilde{h}_\beta(t^{-\langle\check{\alpha},\beta\rangle}).$$

In particular, this implies that for any $t \in 1+4\mathbf{Z}_2$, $w_0\widetilde{h}_\alpha(t)w_0^{-1}$ is a product of (commuting) elements of the form $\widetilde{h}_\beta(s)$, where s is a power of t and β ranges over the simple roots appearing in the word decomposition. In particular, for each $i = 3,4$, we see that $w_0\widetilde{h}_{\alpha_i}(t)w_0^{-1} \in K_{M_R}^*(4)$, showing that $w_0 K_{M_R}^*(4) w_0^{-1} = K_{M_R}^*(4)$.

On the other hand, we may also compute this conjugation in the group $H_J(\mathbf{Q}_2) \cong \mathrm{GSp}_6(\mathbf{Q}_2)$, where it is easy to see that for both $i = 3,4$, $w_0\widetilde{h}_{\alpha_i}(t)w_0^{-1}$ projects to $h_{\alpha_i}(t^{-1})$. This forces $w_0\widetilde{h}_{\alpha_i}(t)w_0^{-1} = \epsilon(t)\widetilde{h}_{\alpha_i}(t^{-1})$ for some central sign character $\epsilon : 1+4\mathbf{Z}_2 \longrightarrow \mu_2(\mathbf{Q}_2)$. Since $w_0\widetilde{h}_{\alpha_i}(t)w_0^{-1} \in K_{M_R}^*(4)$, Theorem 2.5.5 forces $w_0\widetilde{h}_{\alpha_i}(t)w_0^{-1} = \widetilde{h}_\alpha(t^{-1})$.

Thus, $K_R'(4) = w_0 K_R^* w_0^{-1}$. Theorem 2.5.13(ii) immediately follows from the corresponding statement in Theorem 2.5.5.

For the Iwahori decomposition, let $g' \in K_R'(4)$ be arbitrary and set $g = w_0^{-1}g'w_0 \in K_R^*(4)$. Recall that $P_J = H_J N_J$ is the Heisenberg parabolic subgroup. Set $K_J^*(4) := K_R^*(4) \cap H_J(\mathbf{Q}_2)$. Then Corollary 2.5.6 implies that $g \in K_R^*(4)$ can be written uniquely as $g = nmu$, with $n \in N_J^-(4\mathbf{Z}_2), u \in N_J(\mathbf{Z}_2)$, and $m \in K_J^*(4)$. Note that a simple corollary of the uniqueness in

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Theorem 2.5.5 is that $K_J^*(4)$ possesses the Iwahori decomposition

$$K_J^*(4) = N_S^-(4\mathbf{Z}_2)K_{M_R}^*(4)N_S^+(\mathbf{Z}_2). \tag{5}$$

Conjugating by w_0,

$$w_0 g w_0^{-1} = (w_0 n w_0^{-1})(w_0 m w_0^{-1})(w_0 u w_0^{-1}), \tag{6}$$

where now $w_0 u w_0^{-1} \in N_J^-(4\mathbf{Z}_2)$ and $w_0 u w_0^{-1} \in N_J(\mathbf{Z}_2)$. Using the group structure and Iwahori decomposition (5), we may write $m^{-1} = u_1^{-1} m_1^{-1} n_1^{-1}$ where $n_1 \in N_S(\mathbf{Z}_2)$, $u_1 \in N_S^-(4\mathbf{Z}_2)$ and $m_1 \in K_{M_R}^*(4)$. Inverting, we get

$$m = n_1 m_1 u_1 \in N_S(\mathbf{Z}_2)K_{M_R}^*(4)N_S^-(4\mathbf{Z}_2).$$

We can now conjugate by w_0 to get

$$w_0 m w_0^{-1} = (w_0 n_1 w_0^{-1})(w_0 m_1 w_0^{-1})(w_0 u_1 w_0^{-1}),$$

with $w_0 u_1 w_0^{-1} \in N_S(4\mathbf{Z}_2)$, $w_0 n_1 w_0^{-1} \in N_S^-(\mathbf{Z}_2)$ and $w_0 m_1 w_0^{-1} \in K_{M_R}^*(4)$ since $w_0 K_{M_R}^*(4) w_0^{-1} = K_{M_R}^*(4)$.

Combining this with the decomposition (6), we obtain a unique expression

$$g' = w_0 g w_0^{-1} = n' m' u'$$

where $n' = w_0 n n_1 w_0^{-1} \in U^-(1,4)$, $u' = w_0 u_1 u w_0^{-1} \in U^+(4,1)$ and $m' = w_0 m_1 w_0^{-1} \in K_{M_R}^*(4)$. □

We now state a corollary of Theorem 2.5.13 that we will need. Denote by $\Phi_{1,1}^+$ the roots $\sum_i m_i \alpha_i$ with both $m_1, m_2 > 0$. Then Φ_N^+ is a disjoint union of $\{\alpha_1\}$ and $\Phi_{1,1}^+$. Set $U_{1,1}^+(\mathbf{Z}_2)$ the subgroup generated by $x_\alpha(\mathbf{Z}_2)$ for all $\alpha \in \Phi_{1,1}^+$. Define $U_{1,1}^-(4\mathbf{Z}_2)$ similarly.

Corollary 2.5.14. *The group $K_R'(4)$ has an Iwahori factorization with respect to Q.*

Proof. Suppose $g \in K_R'(4)$. By Theorem 2.5.13, we have $g = n_1 k n_2$ with $n_1 \in U_R^-(1,4)$, $k \in K_{M_R}^*(4)$ and $n_2 \in U_R^+(4,1)$. Conjugating all terms of the form $x_{-\alpha_1}(4u)$ in n_1 to the right, one can write $n_1 = n_1' n_1''$, where n_1' in the group generated by $N_S^-(\mathbf{Z}_2)$ and $U_{1,1}^-(4\mathbf{Z}_2)$, and $n_1'' \in x_{-\alpha_1}(4\mathbf{Z}_2)$. Similarly, one can write $n_2 = n_2'' n_2'$, with n_2' in the group generated by $N_S^+(4\mathbf{Z}_2)$ and $U_{1,1}^+(\mathbf{Z}_2)$ and $n_2'' \in x_{\alpha_1}(\mathbf{Z}_2)$. This gives $g = n_1'(n_1'' k n_2'')n_2'$, which is the desired Iwahori factorization. □

2.6 Integral models

In the previous sections, we have defined integral models of the algebraic groups G_2 and F_4 using the Chevalley–Steinberg generators and relations at each finite place. To do computations in the later sections, and to coherently relate these integral models to lattices in $\widetilde{G}_J(\mathbf{R})$, we will need a somewhat explicit understanding of these integral models. In this section, we give such explicit integral models for the groups G_2 and F_4. Via the work of Steinberg, this amounts to giving a Chevalley basis of the corresponding Lie algebras, which is what we do.

2.6.1 *Type G_2.* We define $\mathfrak{g}_{2,\mathbf{Z}} := M_3(\mathbf{Z})^{\mathrm{tr}=0} \oplus V_3(\mathbf{Z}) \oplus V_3^\vee(\mathbf{Z})$. A Chevalley basis can be given by X_α being E_{ij} in $M_3(\mathbf{Z})^{\mathrm{tr}=0}$, v_1, v_2, v_3 in $V_3(\mathbf{Z})$ and $-\delta_1, -\delta_2, -\delta_3$ in $V_3^\vee(\mathbf{Z})$. Here v_1, v_2, v_3 is the standard basis of V_3 and $\delta_1, \delta_2, \delta_3$ is its dual basis.

2.6.2 *Type F_4.* First we set $J_0 = H_3(\mathbf{Z})$ to be the symmetric 3×3 matrices with integer coefficients. Let $\mathfrak{m}_J(\mathbf{Z})$ be the elements of $\mathfrak{m}_J$ that take J_0 to itself.

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We set

$$f_{4,\mathbf{Z}} := (M_3(\mathbf{Z}) \oplus \mathfrak{m}_J(\mathbf{Z}))^{2\,\mathrm{tr}=\mu}/\mathbf{Z}(\mathbf{1}_3, 2\mathbf{1}_{J_0}) \oplus V_3(\mathbf{Z}) \otimes J_0 \oplus V_3(\mathbf{Z})^\vee \otimes J_0,$$

where the notation is as follows. Here $\mu : \mathfrak{m}_J \to \mathbf{Q}$ is the map satisfying

$$(\phi X_1, X_2, X_3) + (X_1, \phi X_2, X_3) + (X_1, X_2, \phi X_3) = \mu(\phi)(X_1, X_2, X_3).$$

A pair $(\phi_1, \phi_2) \in M_3(\mathbf{Z}) \oplus \mathfrak{m}_J(\mathbf{Z})$ is in $(M_3(\mathbf{Z}) \oplus \mathfrak{m}_J(\mathbf{Z}))^{2\,\mathrm{tr}=\mu}$ if $2\,\mathrm{tr}(\phi_1) = \mu(\phi_2)$. Note that the pair $(\mathbf{1}_3, 2\mathbf{1}_{J_0})$ is in $(M_3(\mathbf{Z}) \oplus \mathfrak{m}_J(\mathbf{Z}))^{2\,\mathrm{tr}=\mu}$ and we quotient out by the integer multiples of this pair.

We identify the quotient $(M_3(\mathbf{Z}) \oplus \mathfrak{m}_J(\mathbf{Z}))^{2\,\mathrm{tr}=\mu}/\mathbf{Z}(\mathbf{1}_3, 2\mathbf{1}_{J_0})$ with a subset of $\mathfrak{sl}_3 \oplus \mathfrak{m}_J^0$ via

$$(\phi_1, \phi_2) \mapsto \phi_1 + \phi_2 - tr(\phi_1)\mathbf{1} := \left(\phi_1 - \frac{\mathrm{tr}(\phi_1)}{3}\mathbf{1}_3\right) + \left(\phi_2 - \frac{\mu(\phi_2)}{3}\mathbf{1}_{J_0}\right) \in \mathfrak{sl}_3 + \mathfrak{m}_J^0.$$

It is easy to see that this element acts on $V_3(\mathbf{Z}) \otimes J_0$ and $V_3(\mathbf{Z})^\vee \otimes J_0$ preserving these integral structures.

One has the following proposition.

Proposition 2.6.1. *The lattice $f_{4,\mathbf{Z}}$ is closed under the bracket.*

Now, we observe that because $J_0 = H_3(\mathbf{Z})$, $\mathfrak{m}_J = M_3(\mathbf{Q})$ with $X \in M_3(\mathbf{Z})$ acting on $Y \in H_3(\mathbf{Q})$ as $XY + YX^t$. Moreover, one can check by easy explicit calculation, $M_3(\mathbf{Z}) = \{X \in m_J(\mathbf{Z}) : \mu(X) \in 2\mathbf{Z}\}$.

Consequently, we have

$$f_{4,\mathbf{Z}} = (M_3(\mathbf{Z}) + M_3(\mathbf{Z}))^{\mathrm{tr}_1=\mathrm{tr}_2}/\mathbf{Z}(\mathbf{1}, \mathbf{1}) + V_3(\mathbf{Z}) \otimes J_0 + V_3(\mathbf{Z})^\vee \otimes J_0.$$

For the Chevalley basis, we take the usual bases of $X_\alpha = E_{ij}$ of the two copies of $M_3(\mathbf{Z})$. Now J_0 is the $\mathbf{Z}$-span of

$$\{e_{11}, e_{22}, e_{33}, x_1 = e_{23} + e_{32}, x_2 = e_{31} + e_{13}, x_3 = e_{12} + e_{21}\},$$

where e_{ij} denotes the element of $M_3(\mathbf{Z})$ with a 1 in the (i, j) location and zeros elsewhere. For the rest of the Chevalley basis, we take the elements $v_j \otimes x_k, v_j \otimes e_{kk}, -\delta_j \otimes x_k$ and $-\delta_j \otimes e_{kk}$.

2.7 Splittings

We may now combine our local results to construct splittings of certain congruence subgroups of $G_2(\mathbf{R})$ and $F_4(\mathbf{R})$ into the double cover.

Recall that when $p > 2$ is odd, we have the hyperspecial maximal compact subgroup $K_p = G(\mathbf{Z}_p)$ of $G(\mathbf{Q}_p)$ induced by our integral model.

Lemma 2.7.1 [LS10, Proposition 2.1]. *The central extension $\widetilde{G}(\mathbf{Q}_p)$ splits over K_p. The splitting homomorphism $s_p : K_p \longrightarrow \widetilde{G}(\mathbf{Q}_p)$ is unique, and we denote its image by K_p^*.*

We now define a congruence subgroup $\Gamma_{F_4}(4) \subseteq F_4(\mathbf{R})$ and explain that this subgroup splits into $\widetilde{F}_4(\mathbf{R})$. Let $K_R(4)$ be the image in $F_4(\mathbf{Q}_2)$ of the subgroup $K'_R(4)$, and let $s_2 : K_R(4) \to \widetilde{F}_4(\mathbf{Q}_2)$ be the induced splitting. Define now

$$\Gamma_{F_4}(4) := F_4(\mathbf{Q}) \cap K_R(4) \prod_{p>2} K_p \subset F_4(\mathbf{Z}). \tag{7}$$

To construct a splitting of $\Gamma_{F_4}(4)$ into $\widetilde{F}_4$, we will use the following lemma.

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Lemma 2.7.2. *Suppose A, B are two groups containing a central μ_2, and $\Gamma \subseteq A/\mu_2, B/\mu_2$. Let $s : \Gamma \to (A \times B)/\mu_2^{\Delta}$ and $s_A : \Gamma \to A$ be given splittings. Then there exists a unique splitting $s_B : \Gamma \to B$ so that $s(\gamma) = (s_A(\gamma), s_B(\gamma))\mu_2^{\Delta}$ for all $\gamma \in \Gamma$.*

Proof. By assumption, for each $\gamma \in \Gamma$ one has $s(\gamma) = \pm(s_A(\gamma), s_B(\gamma))$ for a unique $s_B(\gamma) \in B$. This uniquely determines the map $s_B : \Gamma \to B$, and one checks that it is a group homomorphism. □

Using the inclusion $\Gamma_{F_4}(4) \subset F_4(\mathbf{Q}) \subset F_4(\mathbf{R})$, we obtain a splitting $s_\Gamma : \Gamma_{F_4}(4) \to \widetilde{F}_4(\mathbf{R})$ by applying Lemma 2.7.2 with $\Gamma = \Gamma_{F_4}(4)$, $A = \widetilde{F}_4(\mathbf{A}_f)$ and $B = \widetilde{F}_4(\mathbf{R})$. Let $s_f : \Gamma_{F_4}(4) \to \widetilde{F}_4(\mathbf{A}_f)$ be the section induced from the local sections s_p from Lemma 2.7.1 and Theorem 2.5.5. With this notation, we have obtained the following.

Proposition 2.7.3. *There is a unique splitting $s_\Gamma : \Gamma_{F_4}(4) \to \widetilde{F}_4(\mathbf{R})$ characterized by the fact that $s_{\mathbf{Q}}(\gamma) = \pm(s_f(\gamma), s_\Gamma(\gamma))$ for all $\gamma \in \Gamma_{F_4}(4)$.*

Below we will need the following proposition.

Proposition 2.7.4. *For all integers u, the splitting s_Γ satisfies $s_\Gamma(x_\alpha(u)) = x_\alpha(u)$ for all $\alpha \in \Phi_N^+ \cup \Phi_{N_S}^- \cup \Phi_{M_R}$ and $s_\Gamma(x_\alpha(4u)) = x_\alpha(4u)$ for all $\alpha \in \Phi_{N_S}^+ \cup \Phi_N^-$.*

Proof. Indeed, this compatibility occurs for $s_{\mathbf{Q}}$ and s_p for all $p = 2, 3, \ldots$. The proposition thus follows from the definition of s_Γ. □

In the next section, we recall the inclusion of algebraic $\mathbf{Q}$-groups $G_2 \subseteq F_4$ and prove an inclusion $\widetilde{G}_2(\mathbf{R}) \subseteq \widetilde{F}_4(\mathbf{R})$. Assuming these inclusions for the moment, we set $\Gamma_{G_2}(4) = G_2(\mathbf{R}) \cap \Gamma_{F_4}(4)$ and obtain a splitting $\Gamma_{G_2}(4) \to \widetilde{G}_2(\mathbf{R})$.

2.8 Group embeddings

We conclude this section with some remarks about the inclusion of G_2 in F_4.

2.8.1 *Algebraic groups over* $\mathbf{Q}$. We recall the following proposition from the theory of algebraic groups; see [Mil17, Theorem 25.4(c)].

Proposition 2.8.1. *Suppose k is a field of characteristic 0, H, G are algebraic groups over k, with H semisimple, connected and simply connected. Suppose $L : \mathfrak{h} \to \mathfrak{g}$ is an embedding of Lie algebras. Then there exists a unique map $H \to G$ of algebraic groups whose differential is L.*

We first work with algebraic groups over $\mathbf{Q}$. Either from the proposition or directly, one can see easily that there is a map $\mathrm{SL}_3 \to F_4$, lifting the Lie algebra embedding $\mathfrak{m}_J^0 \to \mathfrak{f}_4$ in the notation of [Pol20]. Let $\mathrm{SO}(3)$ denote the group of $g \in \mathrm{SL}_3$ with $g^t g = 1$. Composing with the map $\mathrm{SO}(3) \to \mathrm{SL}_3$, we get an embedding of $\mathrm{SO}(3)$ into F_4.

Lemma 2.8.2. *The centralizer of $\mathrm{SO}(3)$ in F_4 is a split form of type G_2.*

Proof. Denote by G' the identity component of the centralizer of $\mathrm{SO}(3)$ in F_4. We first observe that on the level of Lie algebras, we have $\mathfrak{g}_2 \to \mathfrak{f}_4$, and this $\mathfrak{g}_2$ is exactly $\mathfrak{f}_4^{\mathrm{SO}(3)}$. Consequently, the action of G' on $\mathfrak{f}_4$ induces an action of G' on $\mathfrak{g}_2$, so we obtain a map $\alpha : G' \to G_2$, because G_2 is defined as the group of automorphisms of its Lie algebra.

In the reverse direction, Proposition 2.8.1 implies the existence of a map $\beta : G_2 \to F_4$ lifting the inclusion of Lie algebras $\mathfrak{g}_2 \to \mathfrak{f}_4$. The image of this G_2 centralizes $\mathrm{SO}(3)$ by uniqueness of the lift: if $g \in \mathrm{SO}(3)$, then $g\beta(h)g^{-1}$ is another lift, so is equal to β. Consequently, β gives a map $G_2 \to G'$. The map $\alpha \circ \beta : G_2 \to G_2$ induces the identity on Lie algebras by construction, so is

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the identity. Similarly, the map $\beta \circ \alpha : G' \to G'$ induces the identity of Lie algebras, so is the identity.

Finally, we show $C_{F_4}(\mathrm{SO}(3))$ is connected. The conjugation action of any element $\tau \in C_{F_4}(\mathrm{SO}(3))(\overline{\mathbf{Q}})$ on G_2 must be inner, since $\mathrm{Out}(G_2)$ is trivial. In particular, if $C_{F_4}(\mathrm{SO}(3))$ is not connected, there must exist a finite-order element $\tau \notin G_2(\overline{\mathbf{Q}})$ centralizing both $\mathrm{SO}(3)$ and G_2. But this would imply that the Lie subalgebra $\mathfrak{so}(3) \oplus \mathfrak{g}_2 \subset \mathfrak{f}_4$ is not maximal, a contradiction. □

2.8.2 *Real Lie groups.* We now work with real Lie groups. We will explain the fact that the centralizer of $\mathrm{SO}(3)$ in $\widetilde{F}_4$ is the group $\widetilde{G}_2$; see also [HPS96].

First consider the case of the linear group F_4.

LEMMA 2.8.3. *The centralizer of* $\mathrm{SO}(3)$ *in* $F_4(\mathbf{R})$ *is* $G_2(\mathbf{R})$.

Proof. As in the proof of Lemma 2.8.2, the identity component $C_{F_4(\mathbf{R})}(\mathrm{SO}(3))^0$ maps to the connected Lie group $G_2(\mathbf{R})$. Moreover, this group has Lie algebra exactly $\mathfrak{f}_4^{\mathrm{SO}(3)} = \mathfrak{g}_2$ (it is easy to see that the Lie algebra is contained in this set, and it is surjective by considering the exponential map). It thus remains to determine which Lie group of type G_2 this is.

Because we already know $G_2 \to F_4$ as real algebraic groups, we obtain $G_2(\mathbf{R}) \to C_{F_4(\mathbf{R})}(\mathrm{SO}(3))$. Because the connected double cover of $G_2(\mathbf{R})$ does not split over $G_2(\mathbf{R})$, the identity component of the centralizer of $\mathrm{SO}(3)$ must be the linear group $G_2(\mathbf{R})$. Finally, since $F_4(\mathbf{R})$ and $G_2(\mathbf{R})$ are $\mathbf{R}$-split, an argument mirroring that in the algebraic setting shows that $C_{F_4(\mathbf{R})}(\mathrm{SO}(3))$ is connected. □

Now, for the case of covering groups. First observe that $\mathrm{SO}(3) \subseteq \mathrm{SL}_3(\mathbf{R}) \subseteq R_J^+$, so the $\mathrm{SO}(3)$ splits into $\widetilde{F}_4(\mathbf{R})$ by Lemma 2.5.4; the splitting is unique because $\mathrm{SO}(3)$ is equal to its derived group.

LEMMA 2.8.4. *The identity component of the centralizer* $C_{\widetilde{F}_4(\mathbf{R})}(\mathrm{SO}(3))^0$ *of* $\mathrm{SO}(3)$ *in* $\widetilde{F}_4(\mathbf{R})$ *is identified with* $\widetilde{G}_2(\mathbf{R})$.

Proof. Let G' be the identity component of the centralizer of this $\mathrm{SO}(3)$ in $\widetilde{F}_4(\mathbf{R})$. Then G' consists of elements $(g, j_{1/2}(g))$ where $j_{1/2}(g) : X_{F_4} \to \mathrm{GL}_2(\mathbf{C})$ is a continuous map whose symmetric square is $j_{\mathrm{lin}}(g) : X_{F_4} \to \mathrm{GL}_3(\mathbf{C})$. Every element $g \in F_4(\mathbf{R})$ occurring in such a pair commutes with $\mathrm{SO}(3)$, so that $g \in G_2(\mathbf{R})$. We thus obtain a map $G' \to G_2(\mathbf{R})$. An argument with the exponential map and Lie algebras proves that this map is surjective, because $G_2(\mathbf{R})$ is generated by the image of the exponential map.

We now construct a map $G' \to \widetilde{G}_2(\mathbf{R})$. We claim that $G_2(\mathbf{R})/K_{G_2} = X_{G_2}$ embeds into $F_4(\mathbf{R})/K_{F_4} = X_{F_4}$; this follows from the claim that the maximal compact subgroups K_{G_2} and K_{F_4} satisfy $K_{G_2} = G_2(\mathbf{R}) \cap K_{F_4}$. Granting this for a moment, if $(g, j_{1/2}(g))$ is in G', restricting $j_{1/2}(g)$ to X_{G_2} gives an element of $\widetilde{G}_2(\mathbf{R})$. We therefore obtain $G' \to \widetilde{G}_2(\mathbf{R})$, which covers the identity map on $G_2(\mathbf{R})$. Because G' is a connected Lie group with Lie algebra $\mathfrak{g}_2$, and $G_2(\mathbf{R})$ does not split into $\widetilde{G}_2(\mathbf{R})$, the map $G' \to \widetilde{G}_2$ is an isomorphism.

To see that $K_{G_2} = G_2(\mathbf{R}) \cap K_{F_4}$, first recall that K_{G_2} and K_{F_4} are the subgroups of $G_2(\mathbf{R})$, respectively $F_4(\mathbf{R})$, that also preserve the bilinear form $B_\theta(X, Y) := -B(X, \theta Y)$ on $\mathfrak{g}_2$, respectively, $\mathfrak{f}_4$, where θ is the Cartan involution on these Lie algebras. Because the Cartan involution θ on $\mathfrak{f}_4$ described in [Pol20] restricts to that on $\mathfrak{g}_2$, it is clear that $G_2(\mathbf{R}) \cap K_{F_4}$ is contained in K_{G_2}. For the reverse inclusion, one notes that K_{G_2} can be generated by the exponentials of elements of $\mathfrak{f}_4^{\mathrm{SO}(3),\theta=1} \subseteq \mathfrak{f}_4^{\theta=1}$, which are in K_{F_4}. □

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Remark 2.8.5. The fact that the Cartan involution on $\mathfrak{f}_4$ restricts to that on $\mathfrak{g}_2$ plays a useful role in verifying that the pullback of a modular form on $\widetilde{F}_4(\mathbf{R})$ to $\widetilde{G}_2(\mathbf{R})$ remains a modular form.

2.8.3 *Covering groups.* We now explain the map $\widetilde{G}_2(\mathbf{Q}_v) \to \widetilde{F}_4(\mathbf{Q}_v)$. By Proposition 2.8.1, we have an embedding of linear algebraic groups $\iota_{\mathrm{lin}} : G_2 \to F_4$.

Lemma 2.8.6. *Using the integral structures induced from* § 2.6*, for every prime p one has $\iota_{\mathrm{lin}}(G_2(\mathbf{Z}_p)) \subseteq F_4(\mathbf{Z}_p)$.*

Proof. The Lie algebra constructions of § 2.6 define the adjoint forms of groups of type G_2 and F_4. Because these groups are also simple, simply connected and have rank at least 2, the hyperspecial maximal compact subgroups of each are generated by the $x_\alpha(\mathbf{Z}_p)$ for α a root of G_2, respectively, F_4. But under the map $\mathfrak{g}_2 \to \mathfrak{f}_4$, the long root spaces of G_2 map to long roots of F_4, and the short roots of G_2 map to a sum of 3 commuting short roots of F_4. The lemma follows. □

Proposition 2.8.7. *For every place v of $\mathbf{Q}$, there is an injection $\iota_v : \widetilde{G}_2(\mathbf{Q}_v) \to \widetilde{F}_4(\mathbf{Q}_v)$. The maps ι_v glue together to give an injection $\iota : \widetilde{G}_2(\mathbf{A}) \to \widetilde{F}_4(\mathbf{A})$, that is compatible with the splittings on rational points.*

Proof. Let $\widetilde{G}_2''(\mathbf{Q}_v)$ be the inverse image in $\widetilde{F}_4(\mathbf{Q}_v)$ of $\iota_{\mathrm{lin}}(G_2(\mathbf{Q}_v)) \subseteq F_4(\mathbf{Q}_v)$. Let $G_2'(\mathbf{Q}_v)$ be the universal central extension of $G_2(\mathbf{Q}_v)$, as constructed in [Ste16, § 6]. Then $G_2'(\mathbf{Q}_v)$ is a central extension of $G_2(\mathbf{Q}_v)$ by the Milnor K-group $K_2(\mathbf{Q}_v)$; see [Ste16, § 7, Theorem 12]. On the one hand, by our definition of $\widetilde{G}_2(\mathbf{Q}_v)$ in terms of generators and relations, $\widetilde{G}_2(\mathbf{Q}_v)$ is the pushout of $G_2'(\mathbf{Q}_v)$ along the Hilbert symbol of $K_2(\mathbf{Q}_v)$. On the other hand, because $G_2'(\mathbf{Q}_v)$ is universal, there is a unique map $K_2(\mathbf{Q}_v) \to \mu_2(\mathbf{Q}_v)$ for which $\widetilde{G}_2''(\mathbf{Q}_v)$ is obtained by $G_2'(\mathbf{Q}_v)$ via pushout. But as is well-known, $K_2(\mathbf{Q}_v)/2K_2(\mathbf{Q}_v) \simeq \mu_2(\mathbf{Q}_v)$, so the only non-trivial map is given by the Hilbert symbol. Note now that the extension of $G_2(\mathbf{Q}_v)$ defined by $\widetilde{G}_2''(\mathbf{Q}_v)$ is not split, as it is already not split over the $\mathrm{SL}_3 \subseteq G_2 \subseteq F_4$ generated by the long roots of G_2. Consequently, the map $G_2'(\mathbf{Q}_v) \to \widetilde{G}_2''(\mathbf{Q}_v)$ factors through $\widetilde{G}_2(\mathbf{Q}_v)$. The induced map $\widetilde{G}_2(\mathbf{Q}_v) \to \widetilde{G}_2''(\mathbf{Q}_v)$ is clearly an isomorphism. This constructs the ι_v in the statement of the proposition.

Taking all the ι_v together, we obtain an injection $\iota : \widetilde{G}_2(\mathbf{A}) \to \widetilde{F}_4(\mathbf{A})$. By Lemma 2.8.6, the map is well-defined, i.e. respects the restricted product nature of these groups. Note that here we are using the uniqueness of the splitting in Lemma 2.7.1.

Finally, we obtain two potentially distinct splittings of $G_2(\mathbf{Q})$ into $\widetilde{F}_4(\mathbf{A})$: one via $\iota(G_2(\mathbf{Q})) \subseteq \iota(\widetilde{G}_2(\mathbf{A}))$ and the other via $\iota_{\mathrm{lin}}(G_2(\mathbf{Q})) \subseteq F_4(\mathbf{Q}) \subseteq \widetilde{F}_4(\mathbf{A})$. But every map $G_2(\mathbf{Q}) \to \mu_2(\mathbf{Q})$ is trivial, so these splittings coincide. □

3. Modular forms

In this section, we define quaternionic modular forms of half-integral weight, generalizing the integral weight theory of [Pol20] and prove the main results about their Fourier expansions and Fourier coefficients. We then assert the existence of a certain modular form Θ_{F_4} of weight $\frac{1}{2}$ on $\widetilde{F}_4(\mathbf{A})$, the proof of which we defer to § 4. Finally, we consider the pull back of Θ_{F_4} to $\widetilde{G}_2(\mathbf{A})$, proving Theorems 1.2.3 and 1.2.6 of the introduction. Along the way, we also do arithmetic invariant theory related to cubic rings and their inverse differents.

3.1 Quaternionic modular forms

We begin by studying quaternionic modular forms of half-integral weight. Suppose $\ell \geq 1$ is an odd integer and recall that $\mathbf{V}_{\ell/2} := Sym^{\ell}(\mathbb{V}_2)$. We consider $\mathbf{V}_{\ell/2}$ as a representation of $\widetilde{K}_J$ via

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the map $j_{1/2}(\cdot, x_0) : \widetilde{K}_J \to \mathrm{GL}_2(\mathbb{V}_2)$. A modular form on G_J of weight $\ell/2$ will be a certain $\mathbf{V}_{\ell/2}$-valued automorphic function.

To define the appropriate sorts of functions on $\widetilde{G}_J$ that we will be considering, we require a certain differential operator. Let $\mathfrak{g}(J) \otimes \mathbf{C} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}(J) \otimes \mathbf{C}$, which we identify with the complexified Lie algebra of $\widetilde{G}_J$. In [Pol20, § 5], an identification is given between $\mathfrak{p}$ and $\mathbb{V}_2 \otimes W_J$ over $\mathbf{C}$. Let $\{X_\alpha\}$ be a basis of $\mathfrak{p}$ and $\{X_\alpha^\vee\}$ the dual basis of the dual space $\mathfrak{p}^\vee$. Suppose now that φ is a smooth $\mathbf{V}_{\ell/2}$-valued function on $\widetilde{G}_J$ satisfying $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in \widetilde{G}_J$ and $k \in \widetilde{K}_J$. For such a function, we define $D'_{\ell/2}\varphi(g) = \sum_\alpha X_\alpha \varphi(g) \otimes X_\alpha^\vee$, which is valued in

$$\mathbf{V}_{\ell/2} \otimes \mathfrak{p}^\vee \simeq \mathrm{Sym}^{\ell-1}(\mathbb{V}_2) \otimes W_J \oplus \mathrm{Sym}^{\ell+1}(\mathbb{V}_2) \otimes W_J.$$

Let $pr : \mathbf{V}_{\ell/2} \otimes \mathfrak{p}^\vee \to \mathrm{Sym}^{\ell-1}(\mathbb{V}_2) \otimes W_J$ be the $\widetilde{K}_J$-equivariant projection and define the operator $D_{\ell/2} = pr \circ D'_{\ell/2}$.

Suppose that $\mathbf{G}_J$ is a reductive group over $\mathbf{Q}$ such that $\mathbf{G}_J(\mathbf{R})$ is an adjoint quaternionic exceptional group. Following our conventions from § 2.1, we further assume we are given a metaplectic double cover $\widetilde{\mathbf{G}}_J(\mathbf{A})$ of $\mathbf{G}_J(\mathbf{A})$ coming from the appropriate Brylinski–Deligne extension. We thus have a short exact sequence of locally compact topological groups

$$1 \longrightarrow \mu_2(\mathbf{Q}) \longrightarrow \widetilde{\mathbf{G}}_J(\mathbf{A}) \longrightarrow \mathbf{G}_J(\mathbf{A}) \longrightarrow 1,$$

which splits canonically over $G_J(\mathbf{Q})$; let $s_{\mathbf{Q}}$ denote this splitting. There is a decomposition $\widetilde{\mathbf{G}}_J(\mathbf{A}) = \prod_p \widetilde{\mathbf{G}}_J(\mathbf{Q}_p)/\mu_2^+$. Our convention implies that $\widetilde{\mathbf{G}}_J(\mathbf{R}) \cong \widetilde{G}_J$.

Then for all but finitely many odd primes p, $\mathbf{G}_J$ is unramified and contains a hyperspecial subgroup $K_p := \mathbf{G}_J(\mathbf{Z}_p)$ over which the cover $\widetilde{\mathbf{G}}_J(\mathbf{Q}_p) \to \mathbf{G}_J(\mathbf{Q}_p)$ splits [Wei18, § 7]. Let T be a finite number of primes containing 2 such that for $p \notin T$, the above statement holds. Let $K^T \subset \mathbf{G}_J(\mathbf{A}_T) := \prod_{p \in T} \mathbf{G}_J(\mathbf{Q}_p)$ be a given compact subgroup equipped with a splitting

$$\begin{array}{ccc} & & \widetilde{\mathbf{G}}_J(\mathbf{A}_T) \\ & \overset{s^T}{\nearrow} & \downarrow \\ K^T & \longrightarrow & \mathbf{G}_J(\mathbf{A}_T) \end{array}$$

where $\widetilde{\mathbf{G}}_J(\mathbf{A}_T) := \prod_{p \in T} \widetilde{\mathbf{G}}_J(\mathbf{Q}_p)/\mu_2^+$.

Setting $K_T := K^T \prod_{p \notin T} K_p$, we have a splitting $s_T : K_T \to \widetilde{\mathbf{G}}_J(\mathbf{A}_f)$; let K_T^* denote its image.

Definition 3.1.1. Suppose $\ell \geq 1$ is an odd integer. An adelic quaternionic modular form on $\widetilde{G}_J(\mathbf{A})$ of weight $\ell/2$ and level (K_T, s_T) is a smooth function

$$\varphi : G_J(\mathbf{Q}) \backslash \widetilde{\mathbf{G}}_J(\mathbf{A}) \to \mathbf{V}_{\ell/2}$$

of moderate growth satisfying:

(i) $\varphi(gk_\infty) = k_\infty^{-1} \cdot \varphi(g)$ for all $g \in \widetilde{\mathbf{G}}_J(\mathbf{A})$ and $k \in \widetilde{K}_\infty$;
(ii) $\varphi(gk) = \varphi(g)$ for all $g \in \widetilde{\mathbf{G}}_J(\mathbf{A})$ and $k \in K_T^*$;
(iii) $D_{\ell/2}\varphi \equiv 0$.

Our first main result will be to show that such a definition of quaterionic modular form of half-integral weight has a robust theory of Fourier coefficients, generalizing the integral weight theory of [Pol20] and its antecedents.

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3.2 Generalized Whittaker functions

We now investigate the so-called generalized Whittaker functions associated to quaternionic modular forms. In other words, we reproduce the main result of [Pol20] except now in the half-integral weight case. Because almost all of the proof in [Pol20] carries over, we are quite brief.

We begin with the following crucial proposition. Recall that an $\omega = (a, b, c, d) \in W_J(\mathbf{R})$ is said to be positive semi-definite if the function $p_\omega(Z) = aN(Z) + (b, Z^\#) + (c, Z) + d$ is never 0 on the upper half-space $\mathcal{H}_J = \{X + iY : X, Y \in J, Y > 0\}$.

Proposition 3.2.1. *Consider the function $g \mapsto \langle \omega, gr_0(i)\rangle$ on $H_J(\mathbf{R})^+$, and suppose ω is positive semi-definite. Then there exists a smooth genuine function $\alpha_\omega(g) : \widetilde{H_J(\mathbf{R})^+} \to \mathbf{C}$ satisfying $\alpha_\omega(g)^2 = \langle \omega, gr_0(i)\rangle$.*

Proof. Recall from [Pol20] that $\langle \omega, gr_0(i)\rangle = -j(g, i)p_\omega(g \cdot i)$. Because $\mathcal{H}_J$ is contractible and $p_\omega(Z)$ is never 0 on $\mathcal{H}_J$, $p_\omega(Z)$ has a smooth square root on $\mathcal{H}_J$. This follows from covering space theory: the map $\mathbf{C}^\times \to \mathbf{C}^\times$ via $z \mapsto z^2$ is a cover, so the map $Z \mapsto p_\omega(Z)$ from $\mathcal{H}_J \to \mathbf{C}^\times$ lifts to the first copy of $\mathbf{C}^\times$. Let us pick, arbitrarily, one of the two square roots and call it $p_\omega(Z)^{1/2}$.

Now, the function $g \mapsto j(g, i)$ on $H_J(\mathbf{R})^+$ has a genuine square root $j_{1/2}$ on $\widetilde{H_J(\mathbf{R})^+}$; such a function was constructed in the Lemma 2.3.5. Thus, $\alpha_\omega(g) = \sqrt{-1}j_{1/2}(g, x_0)p_\omega(gi)^{1/2}$ is the desired function. □

We can now state the main theorem of this section. To do so, we give some notation. First, let $n = \ell/2 \in \frac{1}{2} + \mathbf{Z}_{\geq 0}$. Suppose $\omega \in W_J(\mathbf{R})$ is positive semi-definite. Let $\alpha_\omega(g)$ be one of the two square roots of $\langle \omega, gr_0(i)\rangle$ to $\widetilde{H_J(\mathbf{R})^+}$. For $g \in \widetilde{H_J(\mathbf{R})^+}$, define

$$\mathbf{W}_{\omega,\alpha_\omega}(g) = \nu(g)^{n+1} \sum_{-n \leq v \leq n} \left(\frac{|\alpha_\omega(g)|}{\alpha_\omega(g)}\right)^{2v} K_v(|\alpha_\omega(g)|^2) \frac{x^{n+v}y^{n-v}}{(n+v)!(n-v)!}. \tag{8}$$

Here the sum is over half-integers $v \in \frac{1}{2} + \mathbf{Z}$ with $-n \leq v \leq n$. Note that:

(i) n, v are half-integers, i.e. in $\frac{1}{2} + \mathbf{Z}$, so that $n + v$ and $n - v$ are integers;
(ii) $\nu(g) > 0$ so $\nu(g)^{n+1}$ makes sense;
(iii) $2v$ is an odd integer;
(iv) one has $\mathbf{W}_{\omega,-\alpha_\omega}(g) = -\mathbf{W}_{\omega,\alpha_\omega}(g)$;
(v) for $\epsilon \in \mu_2(\mathbf{Q})$, one has $\mathbf{W}_{\omega,\alpha_\omega}(\epsilon g) = \epsilon \mathbf{W}_{\omega,\alpha_\omega}(g)$.

Let N_J be the unipotent radical of the Heisenberg parabolic of G_J. This subgroup of $G_J(\mathbf{R})$ splits uniquely into $\widetilde{G}_J$ so we also write $N_J(\mathbf{R})$ for its image in $\widetilde{G}_J$. One can extend $\mathbf{W}_{\omega,\alpha_\omega}$ to a function on all of $\widetilde{G}_J$ as

$$\mathbf{W}_{\omega,\alpha_\omega}(nmk) = e^{i\langle \omega, \overline{n}\rangle} k^{-1} \mathbf{W}_{\omega,\alpha_\omega}(m)$$

for $n \in N_J(\mathbf{R})$, $m \in \widetilde{H_J(\mathbf{R})^+}$, and $k \in \widetilde{K}_J$. One checks immediately that this is well-defined.

Recall that a generalized Whittaker function of weight n for ω is a function $\phi : \widetilde{G}_J \to Sym^{2n}(\mathbb{V}_2)$ satisfying:

(i) $\phi(gk) = k^{-1}\phi(g)$ for all $g \in \widetilde{G}_J$ and $k \in \widetilde{K}_J$;
(ii) $\phi(ug) = e^{i\langle \omega, \overline{u}\rangle}\phi(g)$ for all $g \in \widetilde{G}_J$ and $u \in N_J(\mathbf{R})$; here $\overline{u}$ is the image of $u \in W_J(\mathbf{R})$;
(iii) $D_n\phi \equiv 0$.

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Theorem 3.2.2. *Suppose $\omega \in W_J(\mathbf{R})$ is non-zero and $n \in \frac{1}{2} + \mathbf{Z}$ is positive. Suppose moreover that $\phi : \widetilde{G}_J \to Sym^{2n}(\mathbb{V}_2)$ is a moderate growth generalized Whittaker function of weight n for ω.*

(i) *If ω is not positive semi-definite, then $\phi \equiv 0$.*
(ii) *If ω is positive semi-definite, then ϕ is proportional to $\mathbf{W}_{\omega,\alpha_\omega}(g)$.*

Proof. The work is nearly identical to [Pol20], so we only sketch the proof.

Let us first review the definition of the right regular action of the Lie algebra $\mathfrak{g}(J)$ on smooth functions ϕ on $\widetilde{G}_J$. Thus, suppose $X \in \mathfrak{g}(J)$. Then for $t \in \mathbf{R}$ sufficiently small, $\exp(tX)$ is an element of $G_J(\mathbf{R})$ near the identity. Because $\widetilde{G}_J \to G_J(\mathbf{R})$ is a covering space, there is a unique lift, call it $\exp'(tX)$, of $\exp(tX)$ to $\widetilde{G}_J$ that is near the identity of $\widetilde{G}_J$. Then $(X\phi)(g) := (d/dt)\phi(g\exp'(tX))|_{t=0}$. It is a fact that this definition gives a linear action of the real Lie algebra $\mathfrak{g}(J)$ on smooth functions on $\widetilde{G}_J$. One obtains an action of $\mathfrak{g}(J) \otimes \mathbf{C}$ by complexification.

Now let $\phi = \sum_v \phi_v((x^{n+v}y^{n-v})/((n+v)!(n-v)!))$ be a generalized Whittaker function. (To make notation consistent with [Pol20], $\lambda = \omega$.) By the Iwawasa decomposition $\widetilde{G}_J = R_J^+\widetilde{K}_J$, and because ϕ is $\widetilde{K}_J$-equivariant, by definition, to determine ϕ it suffices to determine ϕ on R_J^+.

Now, recall that R_J^+ splits into $\widetilde{G}_J$. Thus, $\phi|_{R_J^+}$ can be thought of as function on the linear group $G_J(\mathbf{R})$, so we may apply [Pol20, Corollary 7.6.1] to obtain differential equations satisfied by ϕ: indeed, the proof of this corollary is to write a basis of $X \in \mathfrak{p}$ as sums $X = X_1 + X_2$, with $X_1 \in Lie(R_J^+) \otimes \mathbf{C}$ and $X_2 \in Lie(K_J) \otimes \mathbf{C} = Lie(\widetilde{K}_J) \otimes \mathbf{C}$, and use the given action of $Lie(K_J) = Lie(\widetilde{K}_J)$ on ϕ to write the differential equation $D_n\phi \equiv 0$ in explicit coordinates on R_J^+. In [Pol20, Corollary 7.6.1] recall that:

- $w \in \mathbf{R}_{>0}^\times$ is considered as an element in the center of the group $H_J(\mathbf{R})^+$ which acts on E_{13} as the real number w^2 (as opposed to w^{-2}); the element w is in R_J^+ so the group of such w's splits into $\widetilde{G}_J$;
- $\widetilde{Z} = Mr_0(i)$ and $r_0(Z) = (1, -Z, Z^\#, -n(Z))$;
- for $E \in J$, $D_{Z(E)}$ denotes the action of the Lie algebra element $\frac{1}{2}M(\Phi_{1,E}) - in_L(E)$, where $\Phi_{1,E}$ is the map $J \to J$ given by $Z \mapsto \{E, Z\}$ (see [Pol20, § 3.3.2, equation (7)]; see also [Pol20, § 3.3, equation (3)]) and $M(\Phi_{1,E})$ is defined in § 3.4.1 at the top of p. 1229 of [Pol20];
- $V(E)$ is defined in § 5.1, equation (19) of [Pol20].

Now, one solves these equations on a connected open subset U of $\mathcal{H}_J$ where $p_\omega(Z) \neq 0$. To do this, one first argues as in § 8.1 of [Pol20] that $\phi_v(w, X, Y)$ (see § 8.2, p. 1257) is of the form $w^{2n+2}Y_v(m)K_v(|\langle\omega, \widetilde{Z}\rangle|)$ for some function $Y_v(m)$ that does not depend on w. Indeed, the differential equations:

(i) $(w\partial_w - 2(n+1) + k)\phi_k = -\langle\omega, \widetilde{Z}^*\rangle\phi_{k-1}$;
(ii) $(w\partial_w - 2(n+1) - k)\phi_k = -\langle\omega, \widetilde{Z}\rangle\phi_{k+1}$;

from [Pol20, Corollary 7.6.1], taken together, imply that $w^{-2n-2}\phi_v(w, X, Y)$ satisfies Bessel's differential equation. The fact that this function must be of moderate growth as $w \to \infty$ then implies that, as a function of w, it is proportional to $K_v(|\langle\omega, \widetilde{Z}\rangle|)$.

To understand the functions $Y_v(m) = Y_v(X, Y)$, one argues as on the top of p. 1257 of [Pol20] to obtain that $\phi(w, X, Y) = \phi(w, m)$ is of the form $Y'_{1/2}(m)\mathbf{W}_{\omega,\alpha_\omega}(g)$ for some function $Y'_{1/2}(m)$ that does not depend on w. In other words, one uses the differential equations in w above again to relate $Y_v(m)$ to $Y_{v+1}(m)$ for each v. Note that the function $Y'_{1/2}(m)$ descends to the linear group.

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Now one proves that the $\mathbf{W}_{\omega,\alpha_\omega}$ are annihilated by the operator D_n, exactly as in the proof of Proposition 8.25 of [Pol20]. Note that in this proof, the term $|\alpha_\omega(g)|\alpha_\omega(g)^{-1}$ is rewritten as a product of $|\alpha_\omega(g)|^{-1}$ and a term that is annihilated by $D_{Z(E)}$. Moreover, the absolute value $|\alpha_\omega(g)|^{-1}$ descends to the linear group. This is why the manipulations of [Pol20] carry over to this half-integral weight case. In any event, it follows from this that $D_{Z(E)}(Y'_{1/2}(m)) = 0$ and $D_{Z(E)^*}(Y'_{1/2}(m)) = 0$, from which one concludes $Y'_{1/2}(m)$ is constant.

Thus, the $\mathbf{W}_{\omega,\alpha_\omega}$ are annihilated by the operator D_n, and on an open subset where $p_\omega(Z) \neq 0$, any moderate growth solution agrees with the $\mathbf{W}_{\omega,\alpha_\omega}$ up to constant multiple. The rest of the argument now follows as in the proof of Proposition 8.2.4 of [Pol20]. □

From Theorem 3.2.2 follows immediately the definition of Fourier coefficients of modular forms of weight $\ell/2$: let $Z = [N_J, N_J]$ denote the one-dimensional center of N_J. Let φ be a modular form for $\widetilde{G}_J(\mathbf{A})$ of weight $\ell/2$ and level (K_T, s_T) as in Definition 3.1.1. Set $\varphi_Z(g) = \int_{Z(\mathbf{Q})\backslash Z(\mathbf{A})} \varphi(zg)\,dz$ and

$$\varphi_N(g) = \int_{N_J(\mathbf{Q})\backslash N_J(\mathbf{A})} \varphi(ng)\,dn.$$

Then we have the following generalization of [Pol20, Corollary 1.2.3].

Corollary 3.2.3. *For each positive semi-definite for $\omega \in W_J(\mathbf{Q})$, there exist a constant $a_\varphi(\omega)$, well-defined up to multiplication by ± 1, such that for $g \in \widetilde{G}_J \subseteq \widetilde{G}_J(\mathbf{A})$,*

$$\varphi_Z(g) = \varphi_N(g) + \sum_{\omega \in W_J(\mathbf{Q})} a_\varphi(\omega)\mathbf{W}_{2\pi\omega}(g),$$

where the sum runs over positive semi-definite vectors. The function $\mathbf{W}_{2\pi\omega}(g)$ is one element of the set $\{\mathbf{W}_{2\pi\omega,\alpha_{2\pi\omega}}, -\mathbf{W}_{2\pi\omega,\alpha_{2\pi\omega}}\}$.

The complex number $a_\varphi(\omega)$ is thus well-defined up to multiplication by ± 1. These numbers $a_\varphi(\omega) \in \mathbf{C}/\{\pm 1\}$ are, by definition, the Fourier coefficients of φ.

The K-Bessel functions $K_v(z)$ in the definition of the Whittaker functions $\mathbf{W}_{\omega,\alpha_\omega}$ only occur for half-integral values of v. This is especially nice as these satisfy the following classical lemma.

Lemma 3.2.4. *the K-Bessel function satisfies the following facts.*

(i) *For any value of v,*

$$-z^v(\partial_z(z^{-v}K_v(z))) = K_{v+1}(z).$$

(ii) *For any value of v,*

$$K_{-v}(z) = K_v(z).$$

(iii) *We have*

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}.$$

Thus, the functions $\mathbf{W}_{\omega,\alpha_\omega}$ are particularly simple as functions of $\alpha_\omega(g)$ and $\nu(g)$. For example, when $l = 1$, we have

$$\mathbf{W}_{\omega,\alpha_\omega}(g) = \sqrt{\frac{\pi\nu(g)^3}{2}}\frac{e^{-|\alpha_\omega(g)|^2}}{|\alpha_\omega(g)|}\left[\left(\frac{|\alpha_\omega(g)|}{\alpha_\omega(g)}\right)x + \left(\frac{\alpha_\omega(g)}{|\alpha_\omega(g)|}\right)y\right]$$

if $g \in \widetilde{H_J(\mathbf{R})^+}$.

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3.3 The minimal modular form of $\widetilde{F}_4(\mathbf{A})$

Our first application is the existence of a particular modular form of weight $\frac{1}{2}$ on $\widetilde{F}_4(\mathbf{A})$ with exceptionally few non-zero Fourier coefficients in the sense of Lemma 3.3.2 below.

Set $U_{F_4}(4) = K'_R(4)\prod_{p>2} F_4(\mathbf{Z}_p) \subseteq F_4(\mathbf{A}_f)$.

THEOREM 3.3.1. *There exists a modular form Θ_{F_4} on $\widetilde{F}_4(\mathbf{A})$ of weight $\frac{1}{2}$ which satisfies the following properties:*

(i) *Θ_{F_4} is constructed from the automorphic minimal representation;*
(ii) *the level of Θ_{F_4} is $U_{F_4}(4)$;*
(iii) *the $(0,0,0,1)$-Fourier coefficient of Θ_{F_4} is equal to ± 1.*

The proof of this theorem is representation theoretic, relying on the analysis of the automorphic minimal representation $\Pi_{\min}$ of $\widetilde{F}_4(\mathbf{A})$, and takes up all of § 4. We defer the discussion of this representation until then. We do, however, need the following properties of Θ_{F_4}, which follow from the minimality of $\Pi_{\min}$.

To simplify notation, set $\Theta = \Theta_{F_4}$. The automorphic function Θ has Fourier expansion

$$\Theta_Z(g) = \Theta_N(g) + \sum_{\omega \in W_J(\mathbf{Q})} \Theta_\omega(g).$$

Here, for $g \in \widetilde{F}_4(\mathbf{A})$, we have

$$\Theta_\omega(g) = \int_{N_J(\mathbf{Q})\backslash N_J(\mathbf{A})} \Theta(ng)\psi^{-1}(\langle \omega, \overline{n}\rangle)\, dn.$$

Recall the notion of *rank* of an element $\omega \in W_J(\mathbf{Q})$ as defined in [Pol18, Definitions 4.2.9 and 4.3.2].

LEMMA 3.3.2. *Let the notation be as above.*

(i) *If $\gamma \in H^1_J(\mathbf{Q})$, then $\Theta_\omega(\gamma g) = \Theta_{\omega\cdot\gamma}(g)$. If $\gamma \in \Gamma_{F_4}(4) \cap H^1_J(\mathbf{R})$, and $g = g_\infty$ is in the image of $\widetilde{F}_4(\mathbf{R}) \to \widetilde{F}_4(\mathbf{A})$, then $\Theta_\omega(s_\Gamma(\gamma)g) = \Theta_{\omega\cdot\gamma}(g)$.*
(ii) *One has $\Theta_\omega \equiv 0$ unless $\mathrm{rk}(\omega) \leq 1$.*
(iii) *Suppose $g = g_\infty$ is in the image of $\widetilde{F}_4(\mathbf{R}) \to \widetilde{F}_4(\mathbf{A})$ and ω is of rank one. Then $\Theta_\omega(g) \equiv 0$ unless ω lies in the lattice $W_J(\mathbf{Z}) = \mathbf{Z} \oplus J_0 \oplus J_0 \oplus \mathbf{Z}$.*

Proof. The first part of the first claim follows easily from the usual change of variables in the integral defining Θ_ω. For the second part of the first claim, we have

$$\Theta_\omega(s_\Gamma(\gamma)g) = \Theta_\omega(s_\Gamma(\gamma)g s_f(\gamma)) = \Theta_\omega(s_{\mathbf{Q}}(\gamma)g) = \Theta_{\omega\cdot\gamma}(g)$$

using that Θ is right invariant under $s_f(\Gamma_{F_4}(4))$.

The second claim follows from the construction of Θ from $\Pi_{\min}$ in § 4 and the minimality of $\Pi_{\min}$. More specifically, the claim follows directly from Proposition 3 of [Gin19].

For the final claim, let $W_J(\mathbf{Z})^\vee$ be the dual lattice to $W_J(\mathbf{Z})$ under the symplectic form, so that $W_J(\mathbf{Z})^\vee = \mathbf{Z} \oplus J_0^\vee \oplus J_0^\vee \oplus \mathbf{Z}$. We first prove that $\Theta_\omega(g)$ vanishes unless ω is in $W_J(\mathbf{Z})^\vee$. To see this, suppose $n_0 \in W_J(\widehat{\mathbf{Z}}) = W_J(\mathbf{Z}) \otimes \widehat{\mathbf{Z}}$ and $n = \exp(n_0) \in \widetilde{F}_4(\mathbf{A}_f) \to \widetilde{F}_4(\mathbf{A})$. Then $n \in K_R(4)\prod_p K_p$, so Θ is right-invariant by n. But then

$$\Theta_\omega(g) = \Theta_\omega(gn) = \psi(\langle \omega, n_0\rangle)\Theta_\omega(g).$$

Consequently, if $\Theta_\omega(g) \neq 0$, then $\langle \omega, n_0\rangle \in \widehat{\mathbf{Z}}$ for all $n_0 \in W_J(\widehat{\mathbf{Z}})$, so $\omega \in W_J(\mathbf{Z})^\vee$.

For the stronger claim that $\Theta_\omega(g)$ vanishes unless $\omega \in W_J(\mathbf{Z}) \subseteq W_J(\mathbf{Z})^\vee$, we use the following lemma. □

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Lemma 3.3.3. *If $\omega \in W_J(\mathbf{Z})^\vee$ is of rank one, then $\omega \in W_J(\mathbf{Z})$.*

Proof. Write $\omega = (a, b, c, d)$. Then $b^\# = ac \in J_0^\vee$ and $c^\# = db \in J_0^\vee$ by [GS05, Proposition 11.2]. But an elementary check shows that if $X \in J_0^\vee$ and $X^\# \in J_0^\vee$, then, in fact, $X \in J_0$. The lemma follows. □

3.4 Pullback to G_2

We have defined an inclusion $\widetilde{G}_2(\mathbf{A}) \subseteq \widetilde{F}_4(\mathbf{A})$ in § 2.8.3 and a modular form Θ_{F_4} on the latter group. Let Θ_{G_2} be the automorphic function that is the pullback of Θ_{F_4} to $\widetilde{G}_2(\mathbf{A})$, which is evidently smooth of moderate growth and satisfies the equivariance property (i). In fact, it also satisfies the requisite differential equation.

Proposition 3.4.1. *The automorphic function Θ_{G_2} is a weight $\frac{1}{2}$ quaternionic modular form on $\widetilde{G}_2(\mathbf{A})$.*

Proof. This follows just as in [Pol21, Corollary 4.2.3]. □

In this section, we partially compute the Fourier expansion of Θ_{G_2}. For $g \in \widetilde{F}_4(\mathbf{R})$ we have

$$\Theta_Z(g) = \Theta_N(g) + \sum_{\substack{\omega \in W_J(\mathbf{Z}) \\ \mathrm{rk}(\omega)=1}} a(\omega; \alpha_{2\pi\omega}) \mathbf{W}_{2\pi\omega;\alpha_{2\pi\omega}}(g)$$

with $a(\omega; -\alpha_{2\pi\omega}) = -a(\omega; \alpha_{2\pi\omega})$.

Suppose $\gamma \in \Gamma_{F_4}(4) \cap H_J^1(\mathbf{R})$. Define $\alpha_{2\pi\omega}^\gamma(g) = \alpha_{2\pi\omega}(\gamma g)$. Note that

$$\alpha_{2\pi\omega}^\gamma(g)^2 = 2\pi\langle \omega, \gamma g \cdot r_0(i)\rangle = 2\pi\langle \omega \cdot \gamma, g \cdot r_0(i)\rangle,$$

so that $\alpha_{2\pi\omega}^\gamma$ is an $\alpha_{2\pi\omega\cdot\gamma}$, and $\mathbf{W}_{2\pi\omega;\alpha_{2\pi\omega}}(\gamma g) = \mathbf{W}_{2\pi\omega\cdot\gamma,\alpha_{2\pi\omega}^\gamma}(g)$.

Lemma 3.4.2. *For $\gamma \in \Gamma_{F_4}(4) \cap H_J^1(\mathbf{R})$, one has an equality of Fourier coefficients $a(\omega; \alpha_{2\pi\omega}) = a(\omega \cdot \gamma; \alpha_{2\pi\omega}^\gamma)$.*

Proof. By Lemma 3.3.2, one has $\Theta_\omega(\gamma g) = \Theta_{\omega\cdot\gamma}(g)$. Thus,

$$\begin{aligned} a(\omega; \alpha_{2\pi\omega}) \mathbf{W}_{2\pi\omega;\alpha_{2\pi\omega}}(\gamma g) = \Theta_\omega(\gamma g) = \Theta_{\omega\cdot\gamma}(g) &= a(\omega\cdot\gamma; \alpha_{2\pi\omega}^\gamma) \mathbf{W}_{2\pi\omega\cdot\gamma;\alpha_{2\pi\omega}^\gamma}(g) \\ &= a(\omega\cdot\gamma; \alpha_{2\pi\omega}^\gamma) \mathbf{W}_{2\pi\omega;\alpha_{2\pi\omega}}(\gamma g). \end{aligned}$$

Consequently, $a(\omega; \alpha_{2\pi\omega}) = a(\omega \cdot \gamma; \alpha_{2\pi\omega}^\gamma)$. □

We now consider the Fourier coefficients of $\Theta_{G_2} = \Theta_{F_4}|_{\widetilde{G}_2(\mathbf{A})}$. We require the following two lemmas. Recall that the Fourier coefficients of a modular form on G_2 are parameterized by elements of $W_{\mathbf{Q}}(\mathbf{Q})$, which may be thought of as $\mathrm{Sym}^3(\mathbf{Q}^2)$ by sending

$$(r, s, t, z) \in W_{\mathbf{Q}}(\mathbf{Q}) \longmapsto ru^3 + 3su^2v + 3tuv^2 + zv^3 \in \mathrm{Sym}^3(\mathbf{Q}^2).$$

If $\omega = (a, b, c, d) \in W_J(\mathbf{Q})$, set $\mathrm{tr}(\omega) = (a, \mathrm{tr}(b)/3, \mathrm{tr}(c)/3, d) \in \mathrm{Sym}^3(\mathbf{Q}^2)$, so that $\mathrm{tr}(\omega)$ corresponds to the binary cubic form $au^3 + \mathrm{tr}(b)u^2v + \mathrm{tr}(c)uv^2 + dv^3$. Now, for each $\omega' \in \mathrm{Sym}^3(\mathbf{Q}^2)$, fix a choice of $\alpha_{2\pi\omega'}(g)$. Note that for $\omega \in W_J(\mathbf{Q})$ the restriction of $\alpha_{2\pi\omega}(g)$ to the Heisenberg Levi in $\widetilde{G}_2(\mathbf{R}) \subset \widetilde{F}_4(\mathbf{R})$, is of the form $\epsilon(\omega; \mathrm{tr}(\omega))\alpha_{2\pi\,\mathrm{tr}(\omega)}(g)$ where $\epsilon(\omega; \mathrm{tr}(\omega)) \in \{\pm 1\}$.

Lemma 3.4.3. *Suppose φ is a modular form on $\widetilde{F}_4(\mathbf{A})$ of weight $\ell/2$, with Fourier expansion $\varphi_Z(g) = \varphi_N(g) + \sum_{\omega \in W_J(\mathbf{Q})} a(\omega; \alpha_{2\pi\omega}) \mathbf{W}_{2\pi\omega;\alpha_{2\pi\omega}}(g)$. Let φ' be the restriction of φ to $\widetilde{G}_2(\mathbf{A})$.*

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Then φ' is modular form on $\widetilde{G}_2(\mathbf{A})$ of weight $\ell/2$, with Fourier expansion

$$\varphi'_{Z'}(g) = \varphi'_{N'}(g) + \sum_{\omega' \in \mathrm{Sym}^3(\mathbf{Q}^2)} b(\omega'; \alpha_{2\pi\omega'}) \mathbf{W}_{2\pi\omega';\alpha_{2\pi\omega'}}(g),$$

where $N' \subset G_2$ denotes the unipotent radical of the Heisenberg parabolic. The Fourier coefficients $b(\omega'; \alpha_{2\pi\omega'})$ are given as follows:

$$b(\omega'; \alpha_{2\pi\omega'}) = \sum_{\omega \in W_J(\mathbf{Q}) : \mathrm{tr}(\omega) = \omega'} \epsilon(\omega; \omega') a(\omega; \alpha_{2\pi\omega}).$$

The sum, a priori *infinite, is in fact finite.*

Proof. The point is that one can simply restrict the Fourier expansion of φ to $\widetilde{G}_2(\mathbf{R})$ to obtain the Fourier expansion of φ'. In more detail, one checks that when the function $\mathbf{W}_{\omega,\alpha_{2\pi\omega}}$ on $\widetilde{F}_4(\mathbf{R})$ is restricted to $\widetilde{G}_2(\mathbf{R})$, one obtains the function $\epsilon(\omega; \mathrm{tr}(\omega)) \mathbf{W}_{2\pi\,\mathrm{tr}(\omega);\alpha_{2\pi\,\mathrm{tr}(\omega)}}$ on $\widetilde{G}_2(\mathbf{R})$. We omit the proof of the finiteness claim, as we do not really need it, but we note that it follows from the vanishing of the Fourier coefficients that are not positive semi-definite, and that a similar argument can be found in [Pol21, § 5.1]. □

In particular, if we can control the signs $\epsilon(\omega; \omega')$, we can use our knowledge of the Fourier expansion of Θ_{F_4} to obtain information about the Fourier expansion of Θ_{G_2}. The following lemma controls the signs $\epsilon(\omega; \omega')$.

Below, for $T \in J_0$, we set $\overline{n}(T) = \exp(\delta_2 \otimes T)$, which are unipotent elements of $H^1_J \subseteq F_4$.

Lemma 3.4.4. *Suppose $\gamma_1 = \overline{n}(T_1)$ and $\gamma_2 = \overline{n}(T_2)$ are such that $\det(T_1 t + 1) = \det(T_2 t + 1)$. Then $\alpha^{\gamma_1}_{2\pi(0,0,0,1)}$ and $\alpha^{\gamma_2}_{2\pi(0,0,0,1)}$ have equal (as opposed to opposite) restrictions on $\widetilde{G}_2(\mathbf{R})$.*

Proof. We have $\alpha_{2\pi\omega}(g) = \sqrt{-1} j_{1/2}(g, x_0) p_{2\pi\omega}(gi)^{1/2}$ for a fixed squareroot of $p_{2\pi\omega}(Z)$. Thus,

$$\alpha^{\gamma_i}_{2\pi(0,0,0,1)}(1) = \alpha_{2\pi(0,0,0,1)}(\gamma_i) = \sqrt{-1} j_{1/2}(\overline{n}(T_i), x_0) p_{2\pi(0,0,0,1)}(\gamma_i \cdot i)^{1/2}.$$

Note that $p_{2\pi(0,0,0,1)}(Z)^{1/2}$ is constant. We thus must analyze $j_{1/2}(\overline{n}(T_i), x_0)$. But now note that there is a unique splitting $\overline{n}(J_3(\mathbf{R})) \to \widetilde{F}_4(\mathbf{R})$, this splitting is continuous, and by Lemma 2.7.4, this continuous splitting agrees with the splitting over $\Gamma_{F_4}(4)$. Consequently $j_{1/2}(\overline{n}(T), x_0)$ is a continuous function of $T \in J_3(\mathbf{R})$ and, thus, a fixed squareroot of $\det(Ti + 1)$. Now, by Lemma 3.5.4 proved below, there is a path of $g_t \in \mathrm{SO}_3(\mathbf{R})$ (which is connected) connecting T_1 to T_2. Thus, $\det(T_1 i + 1)^{1/2}$ varies continuously to $\det(T_2 i + 1)^{1/2}$ via $\det(g_t T_1 g_t^t i + 1)^{1/2}$. But $\det(g_t T_1 g_t^t i + 1) = \det(T_1 i + 1)$ because $g_t \in \mathrm{SO}_3(\mathbf{R})$. The lemma follows. □

To describe the Fourier coefficients of Θ_{G_2}, we require the following definition.

Definition 3.4.5. Recall that $J_0 := S^2(\mathbf{Z}^3) = H_3(\mathbf{Z})$ denotes the symmetric 3×3 matrices with integer entries. If $X \in J_0$, then $\det(tI + X)$ is a monic cubic polynomial with integer coefficients. For a cubic monic polynomial p with integer coefficients, let

$$Q_p := \{X \in J_0 : \det(tI + X) = p(t)\}$$

be the set of X in J_0 with $\det(tI + X) = p(t)$.

The set Q_p is finite, and can only be nonempty when $p(t)$ has three real roots. In fact, it can be empty even when $p(t)$ has three real roots.

We now assume that Θ_{F_4} is normalized so that its $(0,0,0,1)$-Fourier coefficient is ± 1. Putting everything together, we have the following result computing a family of Fourier coefficients of Θ_{G_2}.

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Theorem 3.4.6. *The pullback Θ_{G_2} of Θ_{F_4} to $\widetilde{G}_2(\mathbf{A})$ has the following Fourier coefficients: if a, b, c are integers and $p(u,v) = au^3 + bu^2v + cuv^2 + v^3$, then the $p(u,v)$ Fourier coefficient of Θ_{G_2} is $\pm|Q_{p(1,t)}|$.*

Proof. By Lemmas 3.4.4 and 3.4.3, the Fourier coefficient of Θ_{G_2} corresponding to $p(u,v)$ is the sum of the Fourier coefficients of Θ_{F_4} corresponding to elements $(\det(T), T^{\#}, T, 1)$ in W_J with $T \in J_0$ and $\det(t1 + T) = p(1,t)$. Thus, the desired Fourier coefficient of Θ_{G_2} is given by a sign times the number of $T' \in J_0$ with $\det(tI + T') = p(1,t)$. This is $|Q_{p(1,t)}|$, as claimed. □

3.5 Arithmetic invariant theory

The purpose of this section is to do some arithmetic invariant theory related to the set Q_p. In particular, if $R = \mathbf{Z}[t]/(p(t))$, then we relate Q_p to the sets Q_R defined as follows. Set $E = R \otimes \mathbf{Q}$ and assume that $p(t)$ is such that E is an étale $\mathbf{Q}$-algebra. If I is a fractional ideal of R and $\mu \in E^\times$ is totally positive, again as before say that (I, μ) is *balanced* if:

– $\mu I^2 \subseteq \mathfrak{d}_R^{-1}$;
– $N(\mu)N(I)^2\mathrm{disc}(R/\mathbf{Z}) = 1$.

Note that this all makes sense, regardless of whether E is a field. One puts on pairs (I, μ) an equivalence relation: $(I, \mu) \sim (\beta I, \beta^{-2}\mu)$ for $\beta \in E^\times$ and lets Q_R denote the set of equivalence classes.

3.5.1 *The case of a field.* Let $R = \mathbf{Z}[t]/(p(t))$ be a monogenic order in a totally real cubic field $E = R \otimes \mathbf{Q}$. Observe that the group $\mathrm{SO}_3(\mathbf{Z})$ acts on the set Q_p by $X \mapsto gXg^t$.

Lemma 3.5.1. *Suppose $T \in J_0$ has $\det(tI + T) = p(t)$. Then $\mathrm{SO}_3(\mathbf{Z})$ acts freely on T, i.e. if $g \in \mathrm{SO}_3(\mathbf{Z})$ and $gTg^t = T$, then $g = 1$.*

Proof. Suppose $g \in \mathrm{SO}_3(\mathbf{Z})$, and $T = gTg^t = gTg^{-1}$. Then g commutes with T, so $g \in \mathbf{Q}[T]$. It follows that g is symmetric, so $1 = gg^t = g^2$. Thus, $g \in \mu_2(\mathbf{Q}[T])$. But $\mathbf{Q}[T]$ is a field by assumption, so $g = \pm 1$. Because $\det(g) = 1$, $g = 1$, proving the lemma.

Note that the lemma is false if we do not assume $R \otimes \mathbf{Q}$ is a field. □

The following lemma is well-known.

Lemma 3.5.2. *Suppose $M = \mathbf{Z}^3$ has a symmetric bilinear form on it $(\ ,\ )$ which is integral, i.e. $(v, w) \in \mathbf{Z}$ for all $v, w \in M$. Suppose moreover that the bilinear form $(\ ,\ )$ is positive-definite and of determinant one, i.e. $\det((v_i, v_j)) = 1$ for a basis v_1, v_2, v_3 of M over $\mathbf{Z}$. Then M has an orthonormal basis v_1', v_2', v_3'.*

Here is the main result of this section.

Proposition 3.5.3. *Suppose $R = \mathbf{Z}[t]/(p(t))$ is an order in a totally real cubic field $E = R \otimes \mathbf{Q}$. Then there is a bijection (to be given in the proof) between the sets Q_R and $\mathrm{SO}_3(\mathbf{Z})\backslash Q_p$. In particular, $|Q_p| = |\mathrm{SO}_3(\mathbf{Z})| \cdot |Q_R| = 24|Q_R|$.*

As mentioned in the introduction, this proposition essentially follows from the work in [Swa21]. Because [Swa21] is much more general, we give a direct proof of this simple case that we need.

Proof. Let ω be the image of t in $R = \mathbf{Z}[t]/(p(t))$. Associated to a $T \in J_0$ with $\det(tI + T) = p(t)$, we obtain a module $M = \mathbf{Z}^3$, together with a unimodular quadratic form $(\ ,\ )$ and orthonormal basis e_1, e_2, e_3. The element T defines an action of R on M, via $\omega m = -Tm$. Because T is

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symmetric, this action is symmetric for the bilinear form: $(v, \lambda w) = (\lambda v, w)$ for all $v, w \in M$ and $\lambda \in R$.

We can think of M as a fractional ideal I of $E := R \otimes \mathbf{Q}$. That is, $I = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3$ with $e_1, e_2, e_3 \in E$ such that $-\omega e_i = \sum_j T_{ij} e_j$. Moreover, because the action of R is symmetric, the bilinear form on I is of the form $(v, w) = \mathrm{tr}(\mu v w)$ for some fixed $\mu \in E^\times$. Because the bilinear form is positive-definite and because E is totally real, μ must be totally positive. We thus obtain a pair (I, μ). The choice of I is well-defined up to scalar multiple. We claim that the pair (I, μ) is balanced. To see this, first note that because our form $(v, w) = \mathrm{tr}(\mu v w)$ is integral on I, and I is a fractional ideal, we have $\mu I^2 \subseteq \mathfrak{d}_R^{-1}$. Now, one checks easily that $\det((\mathrm{tr}(\mu v_i v_j))) = N(\mu)\det((\mathrm{tr}(v_i v_j)))$. Thus,

$$1 = \det((e_i, e_j)) = N(\mu)\det(\mathrm{tr}(e_i e_j)) = N(\mu)N(I)^2 \mathrm{disc}(R/\mathbf{Z}).$$

Thus, out of $T \in Q_p$, we have constructed a class $[I, \mu]$ in Q_R. Tracing through the maps, one sees that $[I, \mu]$ is well-defined. Moreover, if $g \in \mathrm{SO}_3(\mathbf{Z})$, then $g \cdot T$ maps to the same pair $[I, \mu]$, because the action of g just changes the basis e_1, e_2, e_3 of I.

In the reverse direction, suppose given a pair (I, μ) with (I, μ) balanced. Then the pairing $(v, w) = \mathrm{tr}(\mu v w)$ on I is integral. Moreover, if v_1, v_2, v_3 is an integral basis of I, then $\det((v_i, v_j)) = \det(\mathrm{tr}(\mu v_i v_j)) = N(\mu)N(I)^2\mathrm{disc}(R/\mathbf{Z}) = 1$. By Lemma 3.5.2, I has an orthonormal basis e_1, e_2, e_3. We thus obtain $T := -((e_i, \omega e_j))_{ij}$ with $\det(tI + T) = p(t)$. The basis e_1, e_2, e_3 is well-defined up to the action of $O_3(\mathbf{Z}) = \{\pm 1\} \times \mathrm{SO}_3(\mathbf{Z})$ so the element T is well-defined in the orbit space $\mathrm{SO}_3(\mathbf{Z})\backslash Q_p$.

The maps described above are inverse bijections. Noting that $|\mathrm{SO}_3(\mathbf{Z})| = 24$, the proposition follows. □

The following lemma was used above.

LEMMA 3.5.4. *The group* $\mathrm{SO}_3(\mathbf{R})$ *acts transitively on the set of* $T \in J_0 \otimes \mathbf{R}$ *with fixed characteristic polynomial* $p(t)$.

Proof. Because $O_3(\mathbf{R}) = \{\pm 1\} \times \mathrm{SO}_3(\mathbf{R})$, it suffices to see that $O_3(\mathbf{R})$ acts transitively. But now, every real symmetric matrix can be diagonalized by an element of $O_3(\mathbf{R})$. Using the action of the symmetric group $S_3 \subseteq O_3(\mathbf{R})$ finishes the proof. □

We end this section by discussing the set Q_R when R is a maximal order in E.

PROPOSITION 3.5.5. *Suppose* R *is the maximal order in* E. *Then if* Q_R *is non-empty,* $|Q_R| = |\mathrm{Cl}_E^+[2]|$, *the size of the two-torsion in the narrow class group of* E.

To prove the proposition, we will use the following lemma. Consider the group A_R of equivalence classes of pairs (J, λ) with $\lambda J^2 = (1)$, J a fractional E-ideal and λ totally positive. That is, (J, λ) is equivalent to (J', λ') if there exists $\mu \in E^\times$ so that $J' = \mu J$ and $\lambda' = \mu^{-2}\lambda$. It is clear that Q_R, when non-empty, is a torsor for A_R. Let A'_R denote the set of such pairs (J, λ) except modulo the equivalence relation (J, λ) is equivalent to (J', λ') if there exists $\mu \in E^\times_{>0}$ so that $J' = \mu J$ and $\lambda' = \mu^{-2}\lambda$.

LEMMA 3.5.6. *One has the following exact sequences:*

$$1 \to R^\times_{>0}/(R^\times_{>0})^2 \to A'_R \to \mathrm{Cl}_E^+[2] \to 1, \tag{9}$$

and

$$1 \to E^\times/\left(\pm E^\times_{>0}\right) \to A'_R \to A_R \to 1. \tag{10}$$

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Proof. We first consider the sequence (9). The map $A'_R \to \mathrm{Cl}_E^+$ is given by sending $[J, \lambda]$ to $[J] \in \mathrm{Cl}_E^+$. Because $[J^2] = (\lambda^{-1})$ with λ totally positive, $[J] \in \mathrm{Cl}_E^+[2]$. It is clear that this map is surjective.

For the kernel, if $[J] = 1$ in Cl_E^+, then $J = (\epsilon)$ with ϵ totally positive. Consider $\lambda\epsilon^2$. This is in $R_{>0}^\times$. The element ϵ is well-defined up to multiplication by an $\epsilon_1 \in R_{>0}^\times$, so $\lambda\epsilon^2$ gives a well-defined class in $R_{>0}^\times/(R_{>0}^\times)^2$. It is checked immediately that this map gives an isomorphism of the kernel of $\{A'_R \to \mathrm{Cl}_E^+[2]\}$ with $R_{>0}^\times/(R_{>0}^\times)^2$.

Now consider the sequence (10). The map $A'_R \to A_R$ is dividing out by the courser equivalence relation. The kernel of this map is the image in A'_R of the set of pairs $((\mu), \mu^2)$ with $\mu \in E^\times$. This is trivial in A'_R precisely when there exists $\mu' \in E_{>0}^\times$ so that $((\mu), \mu^2) = ((\mu'), \mu'^2)$, which happens precisely if $\mu \in \pm E_{>0}^\times$. The lemma follows. □

Proposition 3.5.5 follows from Lemma 3.5.6 by observing that both $R_{>0}^\times/(R_{>0}^\times)^2$ and $E^\times/(\pm E_{>0}^\times)$ have size 4. Finally, again assuming that R is the maximal order in E, we remark that it follows from [Gro03, Proposition 3.1] that Q_R is non-empty if and only if every quadratic extension of E that is unramified at all finite primes is totally real. Combining Proposition 3.5.3 with Theorem 3.4.6 gives Theorem 1.2.6. Combining the result with Proposition 3.5.5 gives Theorem 1.1.2.

3.5.2 *The general case.* In the previous subsection, we discussed the arithmetic invariant theory of the sets Q_p when $E = R \otimes \mathbf{Q}$ is a field. We now make some remarks about the arithmetic invariant theory of the sets Q_p when E is just an étale cubic $\mathbf{Q}$-algebra. We omit the proofs, as they are simple generalizations of the proofs in the previous subsection.

Recall that if $p(t) \in \mathbf{Z}[t]$ is cubic and monic, then Q_p denotes the set of $T \in J_0 = Sym^2(\mathbf{Z}^3)$ such that $\det(t1_3 + T) = p(t)$.

One has the following bijection.

Proposition 3.5.7. *There is a bijection between equivalence classes of balanced pairs Q_R and the $O_3(\mathbf{Z})$ (or, equivalently, $\mathrm{SO}_3(\mathbf{Z})$) orbits on Q_p. Moreover, the stabilizer of $T \in Q_p$ under the action of $O_3(\mathbf{Z})$ is $\mu_2(\mathcal{O}_I)$, where*

$$\mathcal{O}_I = \{\alpha \in E : \alpha I \subseteq I\}.$$

As a consequence of the proposition, one obtains

$$\#Q_p = \sum_{[(I,\mu)] \text{ balanced}} \frac{\#O_3(\mathbf{Z})}{\mu_2(\mathcal{O}_I)}.$$

In particular, if R is maximal so that $\mathcal{O}_I = R$ for all I, then

$$\#Q_p = \frac{48}{\mu_2(R)} \times \#\{[(I,\mu)] \text{ balanced}\}.$$

In this maximal case, assuming that E is étale, one has that (I, μ) is balanced precisely if $\mu I^2 = \mathfrak{d}_R^{-1}$. Now one can consider the exact sequences as in Lemma 3.5.6, which become

$$1 \to R_{>0}^\times/(R_{>0}^\times)^2 \to A'_R \to \mathrm{Cl}_E^+[2] \to 1$$

and

$$1 \to E^\times/(\mu_2(E)E_{>0}^\times) \to A'_R \to A_R \to 1.$$

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TABLE 2. Numerical data associated with some of the Fourier coefficients of Θ_{G_2}.

$p(t)$	Structure	Maximal monogenic (LMFDB)	$\#Q_p$	Cl_E^+ (SAGE)
$t^3 - t^2 - 2t + 1$	Cubic field	Yes	24	1
$t^3 - 3t - 1$	Cubic field	Yes	24	1
$t^3 - t^2 - 3t + 1$	Cubic field	Yes	24	1
$t^3 - t^2 - 9t + 10$	Cubic field	Yes	48	C_4
$t^3 - t^2 - 14t + 23$	Cubic field	Yes	48	C_4
$t^3 - t^2 - 11t + 12$	Cubic field	Yes	48	C_4
$t^3 - t^2 - 12t - 1$	Cubic field	Yes	48	C_4
$t^3 - 5t - 1$	Cubic field	Yes	24	1
$t^3 - t^2 - 9t + 8$	Cubic field	Yes	0	C_6
$t^3 - 21t - 35$	Cubic field	Yes	24	C_3
$(t-1)(t^2-2)$	Quadratic	Yes	12	1
$(t-2)(t^2-3)$	Quadratic	Yes	0	C_2
$(t-3)(t^2-10)$	Quadratic	Yes	24	C_2
$t^3 - t^2 - 54t + 169$	Cubic field	Yes	96	$C_2 \times C_2$
$t^3 - t^2 - 34t - 57$	Cubic field	Yes	96	$C_4 \times C_2$

Considering the different cases separately, one sees that in all étale maximal cases, $\#A_R = \#\mathrm{Cl}_E^+[2]$. Thus, if R is maximal and E is étale, one has the formula

$$\#Q_p = \frac{48}{\mu_2(R)} |\mathrm{Cl}_E^+[2]| \times \delta_R$$

where δ_R is 0 if the inverse different $\mathfrak{d}_R^{-1}$ is not a square in Cl_E^+ and 1 if it is such a square. We state this as a proposition.

PROPOSITION 3.5.8. *Let the notation be as above, and assume that $R = \mathbf{Z}[t]/(p(t))$ is the maximal order in $E = R \otimes \mathbf{Q}$, which is assumed étale. Then $\#A_R = \#\mathrm{Cl}_E^+[2]$. Consequently, $\#Q_p = (48/\mu_2(R))|\mathrm{Cl}_E^+[2]| \times \delta_R$ where δ_R is 0 if the inverse different $\mathfrak{d}_R^{-1}$ is not a square in Cl_E^+ and 1 if it is such a square.*

Note that if $R = \mathbf{Z} \times \mathcal{O}_K$ with K real quadratic, then $\mathrm{Cl}_E^+ = \mathrm{Cl}_K^+$. For the sake of completeness, we now answer the question of when the maximal order in such a case is monogenic.

PROPOSITION 3.5.9. *Set $R = \mathbf{Z} \times \mathcal{O}_K$ with K a real quadratic field.*

(i) *If ℓ is squarefree and $\mathcal{O}_K = \mathbf{Z}[\sqrt{\ell}]$, then R is monogenic if and only if $\ell = r^2 \pm 1$ for some r in $\mathbf{Z}$. In this case, $(r, \sqrt{\ell})$ is a generator of R.*
(ii) *If $\mathcal{O}_K = \mathbf{Z}[\omega]$ with $\omega = (1 + \sqrt{4\ell + 1})/2$, then R is monogenic if and only if the equation $r(r-1) = \ell \pm 1$ has a solution, in which case (r, ω) is a generator.*

3.5.3 *Table of data.* In Table 2, we present a table of numerical data for the Fourier coefficients $|Q_p|$ of Θ_{G_2}. The rings R were checked to be maximal (monogenic) orders by the L-function and Modular Form Database (LMFDB) [LMF20]. The computer algebra system SAGE [Sag22] was used to compute the narrow class groups Cl_E^+. In the table, the notation C_n denotes the cyclic group of order n.

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4. The automorphic minimal representation

In this section, we construct and study the modular form Θ_{F_4} of weight $\frac{1}{2}$ on the double cover of F_4 and prove Theorem 3.3.1 via a careful analysis of the automorphic minimal representation of $\widetilde{F}_4(\mathbf{A})$.

4.1 Review of the construction

We begin by reviewing the construction of the automorphic minimal representation $\Pi_{\min}$ on $\widetilde{F}_4(\mathbf{A})$, following Loke and Savin [LS10], and then Ginzburg [Gin19].

Recall that we have ordered the simple roots of F_4 in the usual way, so that the Dynkin diagram

$$\circ - - - \circ =>= \circ - - - \circ$$

has labels α_1 through α_4 from left to right. Define $m_{\alpha_1} = m_{\alpha_2} = 2$ and $m_{\alpha_3} = m_{\alpha_4} = 1$. Let p be a place of $\mathbf{Q}$, allowing $p = \infty$. We begin with the following lemma.

Lemma 4.1.1. *Let $\widetilde{T}(\mathbf{Q}_p)$ denote the inverse image of the fixed split maximal torus of $F_4(\mathbf{Q}_p)$ in $\widetilde{F}_4(\mathbf{Q}_p)$, and $Z(\widetilde{T}(\mathbf{Q}_p))$ its center. Then $t \in Z(\widetilde{T}(\mathbf{Q}_p))$ if and only if $t = \pm \prod_i \widetilde{h}_{\alpha_i}(t_i^{m_i})$.*

Proof. One applies the commutator formula (4) $\{\widetilde{h}_\alpha(s), \widetilde{h}_\beta(t)\} = (s,t)^{(\alpha^\vee, \beta^\vee)}$. □

We will also have need of a maximal abelian subgroup at every local place. This is handled uniformly by the following lemma.

Lemma 4.1.2. *For any place $p \leq \infty$, the subgroup*

$$T_*(\mathbf{Q}_p) := \pm \widetilde{h}_{\alpha_1}(\mathbf{Q}_p^\times)\widetilde{h}_{\alpha_2}((\mathbf{Q}_p^\times)^2)\widetilde{h}_{\alpha_3}(\mathbf{Q}_p^\times)\widetilde{h}_{\alpha_4}(\mathbf{Q}_p^\times)$$

is a maximal abelian subgroup of $\widetilde{T}(\mathbf{Q}_p)$.

Proof. This is an easy check using the commutator formula. □

For each p, we let $B_*(\mathbf{Q}_p) = T_*(\mathbf{Q}_p)U_B(\mathbf{Q}_p)$ denote the associated subgroup of $\widetilde{B}(\mathbf{Q}_p)$.

Definition 4.1.3. A genuine character χ_p of $Z(\widetilde{T}(\mathbf{Q}_p))$ is said to be *exceptional* if for each simple root α, $\chi_p(\widetilde{h}_\alpha(t^{m_\alpha})) = |t|_v$. We let $\nu_{\text{exc}} := (1/m_\alpha)_\alpha \in X^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$ to be the associated exponent.

Lemma 4.1.1 implies that there is a unique exceptional character χ_p on the center of the covering torus of $\widetilde{F}_4(\mathbf{Q}_p)$. Let $\chi_{\text{exc}} = \prod_p \chi_p$ be the induced character on $Z(\widetilde{T}(\mathbf{A}))$. Note that χ is automatically automorphic by the product formula.

We consider the subgroup of $\widetilde{T}(\mathbf{A})$ given by

$$T_*(\mathbf{A}) := T(\mathbf{Q})Z(\widetilde{T}(\mathbf{A}));$$

this is a maximal abelian subgroup [Wei16, Theorem 4.1]. Abusing notation, write χ_{exc} for the automorphic extension of χ_{exc} from $Z(\widetilde{T}(\mathbf{A}))$ to $T_*(\mathbf{A})$. Inflating χ_{exc} to a character of $B_*(\mathbf{A}) := T_*(\mathbf{A})U_B(\mathbf{A})$, consider the induced representation

$$V_0 := \text{Ind}_{B_*(\mathbf{A})}^{\widetilde{F}_4(\mathbf{A})}(\delta_B^{1/2}\chi_{\text{exc}}),$$

where δ_B is the modular character of $B(\mathbf{A})$.

Remark 4.1.4. In their construction of this representation, Loke and Savin instead define a representation $\pi(\chi_{\text{exc}})$ of $\widetilde{T}(\mathbf{A})$, inflate to $\widetilde{B}(\mathbf{A})$, then induce to $\widetilde{F}_4(\mathbf{A})$. It follows from [LS10, Propositions 4.1 and 5.3] that their $\pi(\chi_{\text{exc}})$ is an irreducible representation of $\widetilde{T}(\mathbf{A})$ with the

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same central character as $\mathrm{Ind}_{T_*(\mathbf{A})}^{\widetilde{T}(\mathbf{A})}(\chi_{\mathrm{exc}})$, so they are isomorphic. In fact, both representations are realized as spaces of functions on $T(\mathbf{Q})\backslash\widetilde{T}(\mathbf{A})$, and we claim that they are identical. This is because there is, in the terminology of [LS10], a unique genuine representation in $AT(\mathbf{Q})\backslash\widetilde{T}(\mathbf{A})$ that is invariant under $M_s T_2^1 \prod_{p>2} T_p$; see [LS10, Corollary 5.2]. (This is true for F_4, but not true in general.)

For $\mathbf{s} = (s_1, s_2, s_3, s_4) \in \mathbf{C}^4$, define $\omega_{\mathbf{s}}$ a character of $T(\mathbf{A})$ as $\omega_{\mathbf{s}}(h_{\alpha_i}(t_i)) = |t_i|^{s_i}$. Set

$$V_s = \mathrm{Ind}_{B_*(\mathbf{A})}^{\widetilde{F}_4(\mathbf{A})}(\delta_B^{1/2}\chi_{\mathrm{exc}}\omega_s).$$

Let $f(g, \mathbf{s})$ be a flat section in this induced representation, and set

$$E(g, f, \mathbf{s}) = \sum_{\gamma \in B(\mathbf{Q})\backslash F_4(\mathbf{Q})} f(\gamma g, \mathbf{s}).$$

The automorphic minimal representation on $\widetilde{F}_4(\mathbf{A})$ is constructed as the residue of these Eisenstein series at a distinguished point.

Theorem 4.1.5 [LS10, Theorem 7.1]. *The Eisenstein series $E(g, f, \mathbf{s})$ have at worst a simple multi-pole at $\mathbf{s} = 0$. Let*

$$\theta(g, f) = \lim_{\mathbf{s}\to 0} s_1 s_2 s_3 s_4 E(g, f, \mathbf{s})$$

and $\Pi_{\min}$ be the space of these residues $\theta(g, f)$. Then $\theta(g, f)$ is a genuine, square-integrable automorphic form on $\widetilde{F}_4(\mathbf{A})$. Moreover, the representation $\Pi_{\min}$ is irreducible.

Remark 4.1.6. In [LS10], this theorem is proved for the associated automorphic representation on the double cover of all split simply connected semisimple groups over $\mathbf{Q}$. These are examples of *generalized theta representations*, which play a fundamental role in the study of automorphic representations of non-linear covering groups; see, for example, [Pat84, CFH12, BFG03, FG18, Les19] for some conjectures and aspects of this area.

Write $\Pi_{\min} = \bigotimes_p' \Pi_{\min,p}$. Then Loke–Savin also identify the representations $\Pi_{\min,p}$ in terms of principal series. To do this, extend the character χ_p of $Z(\widetilde{T}(\mathbf{Q}_p))$ to the subgroup $B_*(\mathbf{Q}_p)$, and let $I_p = Ind_{B_*(\mathbf{Q}_p)}^{\widetilde{F}_4(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_p)$.

Proposition 4.1.7 [LS10, Proposition 6.3]. *The representation I_p has a unique irreducible quotient, which is $\Pi_{\min,p}$.*

The notation $\Pi_{\min}$ references Ginzburg's theorem [Gin19, Theorem 1] that $\Pi_{\min}$ is an *automorphic minimal representation* in the sense that the set of nilpotent elements associated to non-vanishing Fourier–Whittaker coefficients of $\Pi_{\min}$ are contained in the minimal nilpotent orbit $\mathcal{O}_{\min} \subset \mathfrak{f}_4(\overline{\mathbf{Q}})$; we refer the reader to [Gin14] for the notions of Fourier–Whittaker coefficients associated to nilpotent orbits. This result plays a central role in our analysis of the Fourier expansion of Θ_{F_4}; see Lemma 3.3.2.

4.2 Archimedean aspects

Relating these generalized theta series to quaternionic modular forms requires information of the $\widetilde{K}_\infty$-types of the local representation $\Pi_{\min,\infty}$. This representation turns out to be the same as the representation Π_{GW} constructed by Gross–Wallach in [GW94].

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Proposition 4.2.1. *The representation* $\Pi_{\min,\infty}$ *is isomorphic to the minimal representation* Π_{GW} *constructed by Gross–Wallach; its* $\widetilde{K}_\infty = \mathrm{SU}(2) \times \mathrm{Sp}(6)$*-types are*

$$\bigoplus_{n=0}^{\infty} Sym^{1+n}(\mathbf{C}^2) \boxtimes \mathbf{V}(n\omega_3), \tag{11}$$

where ω_3 *is the third fundamental weight of* $\mathrm{Sp}(6)$ *and* $\mathbf{V}(n\omega_3)$ *denotes the irreducible representation of* $\mathrm{Sp}(6)$ *with highest weight* $n\omega_3$*. In particular,* $\Pi_{\min,\infty}$ *has minimal* $\widetilde{K}_\infty$*-type* $\mathbf{V}_{1/2}$*.*

Proof. Note that, from [LS10, Proposition 6.3], $\Pi_{\min,\infty}$ is the Langlands quotient of the principal series representation

$$\mathrm{Ind}_{B_*(\mathbf{R})}^{\widetilde{F}_4(\mathbf{R})}(\delta_B^{1/2}\chi_\infty) \cong \mathrm{Ind}_{\widetilde{B}}^{G}(\pi(\chi_\infty)),$$

where χ_∞ is the exceptional character and $\pi(\chi_\infty) \cong \tilde{\delta} \boxtimes \chi_\infty$ is the induced representation of $\widetilde{T}(\mathbf{R}) = \widetilde{M} \cdot T(\mathbf{R})^\circ$. Here $\widetilde{M}$ is a certain finite subgroup of $\widetilde{T}(\mathbf{R})$ and $T(\mathbf{R})^\circ$ is the connected component of the identity of the covering torus. Note we use the fact that

$$\nu_{\mathrm{exc}} = \left(\tfrac{1}{2}, \tfrac{1}{2}, 1, 1\right) = \rho - \tfrac{1}{2}(\omega_1 + \omega_2) \in \mathfrak{t}^* := X^*(T) \otimes_{\mathbf{Z}} \mathbf{R} \tag{12}$$

lies in the dominant chamber in identifying $\Pi_{\min,\infty}$ as the Langlands quotient.

Referring the reader to [ABP+07, §§ 4 and 5] for the notions of pseudospherical representations and notation, in the decomposition

$$\pi(\chi_\infty) = \tilde{\delta} \boxtimes \chi_\infty,$$

the two-dimensional representation $\tilde{\delta}$ is a *pseudospherical representation* of $\widetilde{M}$. It is easy to check that there is a unique such representation for $\widetilde{G} = \widetilde{F}_4(\mathbf{R})$, and it arises as the restriction of the $\widetilde{K}_\infty = \mathrm{SU}(2) \times \mathrm{Sp}(6)$-representation $\mathbf{V}_{1/2}$ to $\widetilde{M} \subset \widetilde{K}_\infty$. In particular, $\mathbf{V}_{1/2}$ is the unique pseudospherical $\widetilde{K}_\infty$-type for $\widetilde{G}$.

In the notation of [ABP+07], we see that $\Pi_{\min,\infty}$ is the Langlands quotient $J(\tilde{\delta}, \nu_{\mathrm{exc}})$ of the corresponding pseudospherical principal series

$$I(\tilde{\delta}, \nu_{\mathrm{exc}}) = \mathrm{Ind}_{\widetilde{B}(\mathbf{R})}^{\widetilde{G}}(\tilde{\delta} \boxtimes (\nu_{\mathrm{exc}} + \rho)).$$

By [ABP+07, Definition 5.5] and the subsequent discussion, we conclude that $\Pi_{\min,\infty}$ has the minimal $\widetilde{K}_\infty$-type $\mathbf{V}_{1/2}$. The key point, as noted in [ABP+07, § 5], is that this Langlands quotient $J(\tilde{\delta}, \nu_{\mathrm{exc}})$ is the unique irreducible representation of $\widetilde{G}$ containing the $\widetilde{K}_\infty$-type $\mathbf{V}_{1/2}$ and having infinitesimal character $\nu_{\mathrm{exc}} \in \mathfrak{t}^*/W$. This follows from the analysis of pseudospherical $\widetilde{K}_\infty$-types in [ABP+07] combined with Harish-Chandra's subquotient theorem.

On the other hand, Gross and Wallach apply cohomological techniques to construct the minimal representation Π_{GW} in [GW96]; here, minimal means the ideal of $\mathfrak{U}(\mathfrak{f}_4(\mathbf{C}))$ annihilating Π_{GW} is the Joseph ideal. In particular, they compute that the $\widetilde{K}_\infty$-types of Π_{GW} are precisely the representations occurring in the proposition [GW96, § 12]. Furthermore, as an element of $\mathfrak{t}^*/W$, the infinitesimal character of Π_{GW} is

$$\nu_{\mathrm{GW}} := \rho - \tfrac{3}{2}\omega_1,$$

where ω_1 is the first fundamental weight of F_4 (see [GW96, p. 109]). Here W denotes the Weyl group of the pair (F_4, T).

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To complete the proof, it suffices to check that there exists $w \in W$ such that $w(\nu_{\mathrm{GW}}) = \nu_\infty$. Referencing (12), this is equivalent to the existence of $w \in W$ such that

$$w \bullet \left(-\tfrac{3}{2}\omega_1\right) = -\tfrac{1}{2}(\omega_1 + \omega_2),$$

where $\bullet$ denotes the dot action of the Weyl group of F_4 on $\mathfrak{t}^*$. The existence of such an element may be verified via a computer calculation, using SAGE [Sag22] for example. By uniqueness, this proves the proposition. □

4.2.1 *Modular forms of weight* $\frac{1}{2}$. Using Proposition 4.2.1, we can now construct modular forms of weight 1/2 on $\widetilde{F}_4(\mathbf{A})$ from Π_{min}. Let x, y be our fixed weight basis of $\mathbf{V}_{1/2} = \mathbb{V}_2 \simeq \mathbb{V}_2^\vee$. Setting $\Pi_{\mathrm{min},f} = \bigotimes'_{p<\infty} \Pi_{\mathrm{min},p}$, fix a vector $v_f \in \Pi_{\mathrm{min},f}$ and let

$$\alpha : \Pi_{\mathrm{min}} = \Pi_{\mathrm{min},f} \otimes \Pi_{\mathrm{min},\infty} \to \mathcal{A}(\widetilde{F}_4(\mathbf{A}))$$

be the automorphic embedding in Theorem 4.1.5. Define

$$\theta(v_f) := \alpha(v_f \otimes x) \otimes x^\vee + \alpha(v_f \otimes y) \otimes y^\vee \in \mathcal{A}(\widetilde{F}_4(\mathbf{A})) \otimes \mathbb{V}_2^\vee. \tag{13}$$

One obtains a quaternionic modular form of weight $\frac{1}{2}$ on $\widetilde{F}_4(\mathbf{R})$. Indeed, the construction of the Schmid operator $D_{1/2}$ precisely detects the fact that the automorphic function $\theta(v_f)$ corresponds to the minimal $\widetilde{K}_\infty$-type $\mathbb{V}_2$, so that $D_{1/2}\theta(v_f) \equiv 0$ for any v_f. The other required properties are clear.

Our goal for the remainder of the section is to prove that v_f can be chosen so that $\theta(v_f)$ has $U_{F_4}(4)$ level and non-zero $(0,0,0,1)$-Fourier coefficient, as in Theorem 1.2.5.

4.3 Weil representations for $\mathbf{GL_2}$

To accomplish this goal, we will calculate a certain twisted Jacquet module of Π_{min}. For this latter calculation, we make a detour to consider the Weil representation of GL_2.

The main results of this section are Corollaries 4.3.12 and 4.3.14, asserting that if certain Whittaker functionals vanish on particular subspaces of these Weil representations, then they vanish identically. For this we need to compare a certain double cover of $\mathrm{GL}_2(\mathbf{Q}_p)$ arising in our context with other constructions in the literature. Strictly speaking, we could appeal to the results of Kazhdan and Patterson [KP84, § 1] to see that the representation theory of these various covers of $\mathrm{GL}_2(\mathbf{Q}_p)$ are related as described in Proposition 4.3.9. We have opted for a more-or-less self-contained presentation for the sake of the reader.

4.3.1 *The double cover of* $\mathrm{SL}_2(\mathbf{Q}_p)$ *and its Weil representation.* Now set $k = \mathbf{Q}_p$ for any prime p, though the results of this section hold for any local field. We recall various essentially well-known facts about the group $\widetilde{\mathrm{SL}_2}(k)$ and its Weil representation.

Let (V, q) be a quadratic space over k, and $B(x, y) = q(x + y) - q(x) - q(y)$ the associated bilinear form. We define a representation of $\widetilde{\mathrm{SL}_2}(k)$ on $S(V)$, the Schwartz space of V, which is genuine if $\dim(V)$ is odd.

We fix the additive character ψ of k. Fix the Haar measure dv on V that is self-dual with respect to the Fourier transform on V as

$$\widehat{\Phi}(v) = \int_V \psi((v, w))\Phi(w)\, dw.$$

Define $F_q(v) = \psi(q(v))$, and let $\gamma(q) \in \mathbf{C}$ be defined as

$$\gamma(q) = \lim_{L \subset V} \int_L F_q(v)\, dv, \tag{14}$$

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where the limit indicates that the value stabilizes for sufficiently large lattices L in V and we take this value.

One defines a Weil representation of $\widetilde{\mathrm{SL}}_2(k)$ on $S(V)$, via:

(i) $\zeta \cdot \Phi(v) = (-1)^{dim(V)}\Phi(v)$;
(ii) $x_\alpha(t) \cdot \Phi(v) = \psi(tq(v))\Phi(v)$;
(iii) $w_1 \cdot \Phi(v) = \gamma(q)\widehat{\Phi}(v)$, where $w_1 = \widetilde{w}_\alpha(1)$;
(iv) $\widetilde{h}_\alpha(y) \cdot \Phi(v) = |y|^{d/2}(\gamma(yq)/\gamma(q))\Phi(yv)$.

Proposition 4.3.1. *The implied action of $\widetilde{\mathrm{SL}}_2(k)$ on $S(V)$ is well-defined and gives a representation, denoted by $\omega_{\psi,q}$. This representation is genuine when* $\dim(V)$ *is odd.*

Proof. We omit the proof, which is well-known. □

Consider now the special case where $V = k$ and $q(x) = x^2$. The genuine representation $\omega_{\psi,q}$ is not irreducible: if $S^+(k)$ is the subspace of even Schwartz functions (i.e. $\Phi(-x) = \Phi(x)$), then $\widetilde{\mathrm{SL}}_2(k)$ preserves this subspace. This gives an irreducible representation, which we will denote by ω_ψ^+.

In [Gel76], Gelbart defines a double cover of $\mathrm{SL}_2(k)$ via an explicit two-cocycle, as follows. For a matrix $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ define

$$x(s) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

Define

$$\alpha(g_1, g_2) = (x(g_1), x(g_2))_2(-x(g_1)x(g_2), x(g_1g_2))_2$$

and $\widetilde{\mathrm{SL}}_2'(k)$ as the set of pairs (g, ζ) with $g \in \mathrm{SL}_2(k)$ and $\zeta \in \{\pm 1\}$ with multiplication

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1g_2, \alpha(g_1, g_2)\zeta_1\zeta_2). \tag{15}$$

Because of the uniqueness up-to-isomorphism of the non-trivial double cover of $\mathrm{SL}_2(k)$, this double cover is isomorphic to $\widetilde{\mathrm{SL}}_2(k)$.

4.3.2 *Two double covers of* GL_2. We now define two double covers of the group $\mathrm{GL}_2(k)$ and consider extensions of the genuine representation ω_ψ^+ to these groups. Our motivation is to relate a cover arising in our analysis of modular forms on $\widetilde{F}_4(k)$ with one considered in [GP80].

The first construction is given via generators and relations as follows. Consider the group $\widetilde{\mathrm{GL}}_2^{(1)}(k)$ generated by $\widetilde{\mathrm{SL}}_2(k)$ and $\widetilde{h}_{\alpha_2}(t)$ for $t \in k^\times$, subject to the relations that if we let α_1 denote the simple root of SL_2, then:

(i) ζ is still central;
(ii) $\widetilde{h}_{\alpha_2}(t)x_{\pm\alpha_1}(u)\widetilde{h}_{\alpha_2}(t)^{-1} = x_{\pm\alpha_1}(t^{\langle\alpha_2^\vee, \pm\alpha_1\rangle}u)$, where $\langle\alpha_2^\vee, \pm\alpha_1\rangle = \mp 1$;
(iii) $\widetilde{h}_{\alpha_2}(s)\widetilde{h}_{\alpha_2}(t) = \widetilde{h}_{\alpha_2}(st)(s, t)_2$.

One can prove from these relations the following additional relations:

(iv) the commutator $\{\widetilde{h}_{\alpha_1}(s), \widetilde{h}_{\alpha_2}(t)\} = (s, t)_2$;
(v) $\widetilde{w_{\alpha_1}}(t)\widetilde{h}_{\alpha_2}(u)\widetilde{w_{\alpha_1}}(-t) = (u^{-1}, u^{-1}t)_2\widetilde{h}_{\alpha_1}(u)\widetilde{h}_{\alpha_2}(u)$

Sending $\widetilde{h}_{\alpha_2}(t)$ to $\mathrm{diag}(1, t)$, we obtain a surjective homomorphism $\pi^{(1)} : \widetilde{\mathrm{GL}}_2^{(1)}(k) \longrightarrow \mathrm{GL}_2(k)$, which we claim is a double covering map extending the cover $\pi : \widetilde{\mathrm{SL}}_2(k) \longrightarrow \mathrm{SL}_2(k)$. It is immediately checked that this map is well-defined. Moreover, by a Bruhat decomposition

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argument, one sees that the kernel is exactly the image of $\mu_2(k)$ in $\widetilde{\mathrm{GL}}_2^{(1)}(k)$. To see that this image is non-trivial, so that $\widetilde{\mathrm{GL}}_2^{(1)}(k)$ is really a double cover of $\mathrm{GL}_2(k)$, we note that $\widetilde{\mathrm{GL}}_2^{(1)}(k)$ so defined is precisely the full inverse image of the subgroup $\mathrm{GL}_2(k) \subset F_4(k)$ in the double cover $\widetilde{F}_4(k)$ described in § 2.4 where $\mathrm{GL}_2(k) \subset F_4(k)$ denotes the subgroup generated by the subgroup isomorphic to $\mathrm{SL}_2(k)$ associated to the simple root α_1 and the coroot associated to the simple root α_2.

Remark 4.3.2. In the literature (for example, [KP84]), one often finds this cover described in terms of the inverse image in $\widetilde{\mathrm{SL}}_3(k)$ of the $(2,1)$-Levi subgroup. We opt for the inclusion into F_4 as this better illustrates our interest in this covering group. In any case, we have

$$\widetilde{\mathrm{GL}}_2^{(1)}(k) \subset \widetilde{\mathrm{SL}}_3(k) \subset \widetilde{F}_4(k),$$

where the inclusion $\mathrm{SL}_3 \subset F_4$ is that discussed in § 2.8.

Let

$$G^* := \{g \in \widetilde{\mathrm{GL}}_2^{(1)}(k) : \pi^{(1)}(g) \in \mathrm{GL}_2(k) \text{ has determinant a square in } k^\times\}. \tag{16}$$

As is easily seen, this is the subgroup of $\widetilde{\mathrm{GL}}_2^{(1)}(k)$ generated by $\widetilde{\mathrm{SL}}_2(k)$ and $\widetilde{h}_{\alpha_2}(t^2)$, $t \in k^\times$.

Lemma 4.3.3. *The group G^* is generated by $\widetilde{\mathrm{SL}}_2(k)$ and $\widetilde{h}_{\alpha_2}(t^2)$ subject only to the relations defining $\widetilde{\mathrm{GL}}_2(k)$, restricted to the $\widetilde{h}_{\alpha_2}(t^2)$.*

Proof. Temporarily, let G_1^* be the group described in the statement of the lemma. Then one has a tautological surjection $G_1^* \to G^*$. Now G_1^* maps to $\mathrm{GL}_2(k)$, with kernel at most $\mu_2(k)$. Now suppose ϵ is in the kernel of $G_1^* \to G^*$. Then $\epsilon \in \mu_2(k)$. But the image of $\mu_2(k)$ in G^* has size two, so $\epsilon = 1$. □

Fix a character χ of $k^\times$, with $\chi(-1) = 1$. Let $S^+(k)$ be the Schwartz space of even functions. We then have the genuine representation ω_ψ^+ of $\widetilde{\mathrm{SL}}_2(k)$ on $S^+(k)$. Following [GP80], one can extend the action to an action of G^* on $S^+(k)$ by letting

$$\widetilde{h}_{\alpha_2}(a^2)\phi(x) = \chi(a)|a|^{-1/2}\phi(a^{-1}x).$$

Proposition 4.3.4. *The above action gives a well-defined representation of G^* on $S^+(k)$. We denote the resulting representation as $\omega_{\psi,\chi}$.*

Proof. This is a direct check which we omit. □

In [Gel76] and [GP80], a different double cover of $\mathrm{GL}_2(k)$ is defined, which we now recall. For $y \in k^\times$, define

$$v(y,s) = \begin{cases} 1 & \text{if } c \neq 0, \\ (y,d)_2 & \text{otherwise,} \end{cases}$$

where $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Define $s^y = \mathrm{diag}(1,y)^{-1} s\, \mathrm{diag}(1,y)$. Now, for $\overline{s} = (s,\zeta) \in \widetilde{\mathrm{SL}}_2'(k)$ (defined as in (15)), let $\overline{s}^y = (s^y, v(y,s)\zeta)$. It is then proved that this gives an action of $k^\times$ on $\widetilde{\mathrm{SL}}_2'(k)$ and one defines $\widetilde{\mathrm{GL}}_2^{(0)}(k)$ to be the semidirect product $\widetilde{\mathrm{SL}}_2'(k) \rtimes k^\times$.

We now compare the double cover $\widetilde{\mathrm{GL}}_2^{(0)}(k)$ and our $\widetilde{\mathrm{GL}}_2^{(1)}(k)$. To do this, let $G^{(0)}$ be a group defined as follows. As a set, it is $\widetilde{\mathrm{GL}}_2^{(1)}(k)$. The multiplication in $G^{(0)}$ is defined as

$$g * h = g \cdot h(\det(g), \det(h))_2,$$

where $g \cdot h$ is the product in $\widetilde{\mathrm{GL}}_2^{(1)}(k)$.

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Proposition 4.3.5. *The group $G^{(0)}$ is isomorphic to $\widetilde{\mathrm{GL}}_2^{(0)}(k)$.*

To prove the proposition, we require a few lemmas. Temporarily, let $G_1^{(0)}$ be the group generated by $\widetilde{\mathrm{SL}}_2(k)$ and $\widetilde{h}_{\alpha_2}(t)$ for $t \in k^\times$, subject to the relations (i), (ii) and:

(iii) $\widetilde{h}_{\alpha_2}(s)\widetilde{h}_{\alpha_2}(t) = \widetilde{h}_{\alpha_2}(st)$.

Lemma 4.3.6. *The map $G_1^{(0)} \to G^{(0)}$ that is the identity on generators is a well-defined isomorphism.*

Proof. It is clear that the map is a well-defined homomorphism, because the relations satisfied in $G_1^{(0)}$ are again satisfied in $G^{(0)}$. Moreover, it is clear that the map is surjective, and covers the identity map on the linear group $\mathrm{GL}_2(k)$. By another Bruhat decomposition argument, the kernel of $G_1^{(0)} \to \mathrm{GL}_2(k)$ is at most $\mu_2(k)$. It follows that the kernel is exactly $\mu_2(k)$ and $G_1^{(0)} \to G^{(0)}$ is an isomorphism. □

Lemma 4.3.7. *Fix $t \in k^\times$. Define a map $\phi_t : \widetilde{\mathrm{SL}}_2(k) \to \widetilde{\mathrm{SL}}_2(k)$ on generators as $\zeta \mapsto \zeta$, $x_{\alpha_1}(u) \mapsto x_{\alpha_1}(t^{-1}u)$ and $x_{-\alpha_1}(u) \mapsto x_{-\alpha_1}(tu)$. Then this map is a well-defined isomorphism.*

Proof. One checks that the relations in the first copy of $\widetilde{\mathrm{SL}}_2(k)$ are satisfied in the second copy. Thus, the map is a well-defined surjection. Replacing t by t^{-1} gives a well-defined inverse. Thus, ϕ_t is an isomorphism. □

Lemma 4.3.8. *The map $\widetilde{\mathrm{SL}}_2(k) \rtimes_{\phi_t} \langle \widetilde{h}_{\alpha_2}(t) \rangle \to G_1^{(0)}$ defined for $h \in \widetilde{\mathrm{SL}}_2(k)$ as*

$$(h, \widetilde{h}_{\alpha_2}(t)) \longmapsto h\widetilde{h}_{\alpha_2}(t)$$

is a well-defined isomorphism.

Proof. Checking that it is well-defined amounts to the relation that $\widetilde{h}_{\alpha_2}(t_1)h_2\widetilde{h}_{\alpha_2}(t_1)^{-1} = \phi_{t_1}(h_2)$ in $\widetilde{\mathrm{SL}}_2(k)$, which is clear.

The inverse map is $G_1^{(0)} \to \widetilde{\mathrm{SL}}_2(k) \rtimes_{\phi_t} \langle \widetilde{h}_{\alpha_2}(t) \rangle$ given by the obvious map on generators. The relations defining $G_1^{(0)}$ are again satisfied in the semi-direct product, so the map is well-defined. It is clear that these maps are inverses to each other, giving the lemma. □

Proof of Proposition 4.3.5. Given the previous lemmas, we simply must check that the semi-direct product defining $\widetilde{\mathrm{GL}}_2^{(0)}(k)$ is the same as that given by ϕ_t, and one must map our $\widetilde{\mathrm{SL}}_2(k)$ to $\widetilde{\mathrm{SL}}_2'(k)$. For this latter task, one checks that $\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right) \mapsto \left(\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right), 1\right)$ is a splitting to $\widetilde{\mathrm{SL}}_2'(k)$. (Use the identity on Hilbert symbols $(a,b)_2(-ab, a+b)_2 = 1$.) This splitting pins down the isomorphism $\widetilde{\mathrm{SL}}_2(k) \to \widetilde{\mathrm{SL}}_2'(k)$. One finds that $\widetilde{w}_\alpha(t) \mapsto \left(\left(\begin{smallmatrix} & t \\ -t^{-1} & \end{smallmatrix}\right), 1\right)$ and that $\widetilde{h}_{\alpha_1}(t) \mapsto (\mathrm{diag}(t, t^{-1}), (t,t)_2)$. We omit the rest of the proof. □

Note that this shows that the subgroup $G^* \subset \widetilde{\mathrm{GL}}_2^{(1)}(k)$ naturally occurs as a subgroup of $\widetilde{\mathrm{GL}}_2^{(0)}(k)$, at least once we fix the above isomorphism $G^{(0)} \cong \widetilde{\mathrm{GL}}_2^{(0)}(k)$.

4.3.3 *The Weil representation for* GL_2. The Weil representation of $\widetilde{\mathrm{GL}}_2^{(1)}(k)$ is defined as

$$\Omega^{(1)}_{\psi,\chi} := \mathrm{Ind}_{G^*}^{\widetilde{\mathrm{GL}}_2^{(1)}(k)}(\omega_{\psi,\chi}). \tag{17}$$

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In order to use results of [GP80], we will need to compare $\Omega^{(1)}_{\psi,\chi}$ with the Weil representation studied in [GP80], which is defined as

$$\Omega^{(0)}_{\psi,\chi} := \mathrm{Ind}_{G^*}^{\widetilde{\mathrm{GL}}_2^{(0)}(k)}(\omega_{\psi,\chi}) \simeq \mathrm{Ind}_{G^*}^{G^{(0)}}(\omega_{\psi,\chi}).$$

To compare these representations, suppose $V^{(1)}$ is a representation of $\widetilde{\mathrm{GL}}_2^{(1)}(k)$. Define a representation $V^{(0)}$ of $G^{(0)}$ by letting $V^{(0)} = V^{(1)}$ as vector spaces, with action

$$g * v = \frac{\gamma(\det(g)q)}{\gamma(q)} gv.$$

Here $\gamma(q)$ is as in (14).

PROPOSITION 4.3.9. *Suppose S is a representation of G^*, $V^{(1)} = \mathrm{Ind}_{G^*}^{\widetilde{\mathrm{GL}}_2^{(1)}(k)}(S)$, $V^{(0)}$ is as above, and let $V' = \mathrm{Ind}_{G^*}^{G^{(0)}}(S)$. As representations of $\widetilde{\mathrm{GL}}_2^{(0)}(k)$, $V^{(0)}$ is isomorphic to V' via the map*

$$f(g) \mapsto \frac{\gamma(\det(g)q)}{\gamma(q)} f(g).$$

In particular, the map

$$\left(\Omega^{(1)}_{\psi,\chi}\right)^{(0)} \longrightarrow \Omega^{(0)}_{\psi,\chi}$$

given by $f(g) \mapsto (\gamma(\det(g)q)/\gamma(q))f(g)$ is an isomorphism.

Proof. This is a simple check. □

Remark 4.3.10. As remarked in [GP80], the representation $\Omega^{(1)}_{\psi,\chi}$ is independent of ψ. This implies the same for $\Omega^{(0)}_{\psi,\chi}$. In any case, this fact could have been derived in the same way as [GP80]. We retain the notation above simply to keep track of our (fixed) choice of ψ, such as our analysis of various twisted Jacquet functors related to these representations.

We may now derive certain properties of $\Omega^{(1)}_{\psi,\chi}$ from the corresponding results of Gelbart and Piatetski-Shapiro [GP80]. Temporarily, let $U(k) = \{x_{\alpha_1}(t) : t \in k\}$ denote the unipotent radical of the upper triangular Borel subgroup of $\mathrm{GL}_2(k)$. This subgroup splits uniquely into both $\widetilde{\mathrm{GL}}_2^{(1)}(k)$ and $\widetilde{\mathrm{GL}}_2^{(0)}(k)$, so let $U(k)$ also denote the image under the splitting. If V is a representation of either double cover, and $t \in k^\times$, a linear functional $L : V \to \mathbf{C}$ is said to be a (U, ψ_t)-functional if $L(x_\alpha(u)v) = \psi(tu)L(v)$ for all $u \in k$ and $v \in V$.

PROPOSITION 4.3.11. *The space of (U, ψ_t)-functionals on $\Omega^{(1)}_{\psi,\chi}$ is one-dimensional. A basis of this space of functionals is given by*

$$f \in \Omega^{(1)}_{\psi,\chi} \mapsto f(h_{\alpha_2}(t^{-1}))(1).$$

Proof. It is immediately checked that the map $f \mapsto f(h_{\alpha_2}(t^{-1}))(1)$ is a non-zero (U, ψ_t)-functional. Thus, the key statement is the multiplicity-one claim. For the representation $\Omega^{(0)}_{\psi,\chi}$, this is due to Gelbart and Piatetski-Shapiro [GP80]. Comparing $\Omega^{(1)}_{\psi,\chi}$ with $\Omega^{(0)}_{\psi,\chi}$ using Proposition 4.3.9, we see that

$$\mathrm{Hom}_U(\Omega^{(1)}_{\psi,\chi}, \psi_t) = \mathrm{Hom}_U(\Omega^{(0)}_{\psi,\chi}, \psi_t);$$

the multiplicity one for $\Omega^{(1)}_{\psi,\chi}$ follows. □

We will also require some results on invariant vectors of $\Omega^{(1)}_{\psi,\chi}$. To state the first result, let $k = \mathbf{Q}_2$ and let $\Gamma_{1,\mathrm{GL}_2}(4)$ be the subgroup of $\mathrm{GL}_2(k)$ generated by $x_\alpha(u)$, $x_{-\alpha}(4u)$, $h_{\alpha_1}(t), h_{\alpha_2}(t)$

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with $u \in \mathbf{Z}_2$ and $t \in 1 + 4\mathbf{Z}_2$. Using the generators and relations, an easy analogue of Theorem 2.5.5 implies that $\Gamma_{1,\mathrm{GL}_2}(4)$ splits the cover $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_2)$; we set $\Gamma^*_{1,\mathrm{GL}_2}(4)$ for the image of the splitting. Similarly, we denote by $\Gamma^*_{1,\mathrm{SL}_2}(4)$ the subgroup of $\widetilde{\mathrm{SL}}_2(\mathbf{Q}_2)$ generated by $x_\alpha(u)$, $x_{-\alpha}(4u)$, $h_{\alpha_1}(t)$ with $u \in \mathbf{Z}_2$ and $t \in 1 + 4\mathbf{Z}_2$.

Corollary 4.3.12. *Let L_t denote the non-zero (U, ψ_t)-functional given in the statement of Proposition 4.3.11. If $t = 1$ or $t = -1$, there is a $\Gamma^*_{1,\mathrm{GL}_2}(4)$-invariant vector $f_t \in \Omega^{(1)}_{\psi,\chi}$ so that $L_t(f_t) = 1$. In particular, if $t = 1$ or $t = -1$ and a (U, ψ_t)-functional L on $\Omega^{(1)}_{\psi,\chi}$ vanishes on the $\Gamma^*_{1,\mathrm{GL}_2}(4)$-invariant vectors, then $L = 0$.*

Proof. Let $\phi_0 \in S^+(\mathbf{Q}_2)$ be the characteristic function of $\mathbf{Z}_2$. Define $f_1 \in \Omega^{(1)}_{\psi,\chi}$ via $f_1(1) = \phi_0$, $f_1(h_{\alpha_2}(5)) = \phi_0$ and if $g \notin G^* \cup G^* h_{\alpha_2}(5)$, then $f_1(g) = 0$. Define $f_{-1} \in \Omega^{(1)}_{\psi,\chi}$ via $f_{-1}(h_{\alpha_2}(-1)) = \phi_0$, $f_{-1}(h_{\alpha_2}(-5)) = \phi_0$ and if $g \notin G^* h_{\alpha_2}(-1) \cup G^* h_{\alpha_2}(-5)$, then $f_{-1}(g) = 0$.

By construction, $L_t(f_t) = 1$ for $t = 1, -1$. One readily verifies that f_1 and f_{-1} are $\Gamma^*_{1,\mathrm{GL}_2}(4)$-invariant: for this, one uses that ϕ_0 is $\Gamma^*_{1,\mathrm{SL}_2}(4)$ invariant under the action of ω_ψ, and that $h_{\alpha_2}(5)$, $h_{\alpha_2}(-1)$ normalize $\Gamma^*_{1,\mathrm{SL}_2}(4)$. The corollary follows. □

We have an analogous statement at the odd primes. Let $k = \mathbf{Q}_p$ with p odd and let $\mathrm{GL}_2^*(\mathbf{Z}_p)$ be the subgroup of $\widetilde{\mathrm{GL}}_2^{(1)}(k)$ generated by $x_{\pm\alpha}(u)$, $\widetilde{h}_{\alpha_2}(t)$ with $u \in \mathbf{Z}_p$ and $t \in \mathbf{Z}_p^\times$; this is the image of a splitting of $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p)$ over $\mathrm{GL}_2(\mathbf{Z}_p)$.

Lemma 4.3.13. *Suppose p is odd. Let $\phi_0 \in S^+(\mathbf{Q}_p)$ be the characteristic function of $\mathbf{Z}_p$. Let $\{1, \mu, p, \mu p\}$ with $\mu \in \mathbf{Z}_p^\times$ be representatives for $\mathbf{Q}_p^\times/(\mathbf{Q}_p^\times)^2$. Define $f_0 \in \mathrm{Ind}_{G^*}^{\widetilde{\mathrm{GL}}_2^{(1)}(k)}(S^+(\mathbf{Q}_p))$ by $f_0(1) = \phi_0$, $f_0(\widetilde{h}_{\alpha_2}(\mu)) = \phi_0$, $f_0(\widetilde{h}_{\alpha_2}(p)) = 0$ and $f_0(\widetilde{h}_{\alpha_2}(p\mu)) = 0$. Then f_0 is $\mathrm{GL}_2^*(\mathbf{Z}_p)$-invariant.*

Proof. This is a relatively direct check, which we omit. □

It is proved in [GP80] that $\Omega^{(0)}_{\psi,\chi}$ and, thus, $\Omega^{(1)}_{\psi,\chi}$ is irreducible. We will see in § 4.4 that $\Omega^{(1)}_{\psi,\chi}$ embeds in a certain principal series representation, from which it follows that the space of $\mathrm{GL}_2^*(\mathbf{Z}_p)$-invariant vectors of $\Omega^{(1)}_{\psi,\chi}$ is at most one-dimensional [GG18, § 9.2], and thus exactly one-dimensional, spanned by the f_0 of Lemma 4.3.13. We obtain the following corollary.

Corollary 4.3.14. *Suppose $t = 1$ or $t = -1$, $k = \mathbf{Q}_p$ with p odd, and L is (U, ψ_t)-functional that is 0 on the unique line of $\mathrm{GL}_2^*(\mathbf{Z}_p)$-invariant vectors of $\Omega^{(1)}_{\psi,\chi}$. Then $L = 0$.*

Proof. This follows from a similar argument to the $p = 2$ case. □

4.4 Jacquet functors

For any finite prime p, let $V_{\min} = \Pi_{\min,p}$ denote the local component of $\Pi_{\min}$ at p. Recall that $Q = LU_Q$ denotes the standard maximal parabolic of F_4 associated to the simple root α_2. In this subsection, we identify the Jacquet module $V_{\min,U_Q}$ of $V_{\min}$ with respect to U_Q with the representation $\Omega^{(1)}_{\psi,\chi}$ of $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p)$ considered in § 4.3.3. For this to make sense, we first explicate a map $\widetilde{L}(\mathbf{Q}_p) \to \widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p)$.

Recall the subgroup $\mathrm{SL}_3(\mathbf{Q}_p)$ of $F_4(\mathbf{Q}_p)$ as described before Lemma 2.5.7.

Proposition 4.4.1. *The group $\mathrm{SL}_3(\mathbf{Q}_p)$ splits into $\widetilde{F}_4(\mathbf{Q}_p)$, is normal in $\widetilde{L}(\mathbf{Q}_p)$, and one has*

$$\widetilde{L}(\mathbf{Q}_p)/\mathrm{SL}_3(\mathbf{Q}_p) \simeq \widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p).$$

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Proof. We first note that $\mathrm{SL}_3(\mathbf{Q}_p)$ is a normal subgroup of $L(\mathbf{Q}_p)$ such that

$$L(\mathbf{Q}_p)/\mathrm{SL}_3(\mathbf{Q}_p) \simeq \mathrm{GL}_2(\mathbf{Q}_p).$$

That $\mathrm{SL}_3(\mathbf{Q}_p)$ splits into $\widetilde{F}_4(\mathbf{Q}_p)$ is Lemma 2.5.4.

To see that this $\mathrm{SL}_3(\mathbf{Q}_p)$ is normal, let s denote the splitting of $\mathrm{SL}_3(\mathbf{Q}_p)$ into $\widetilde{F}_4(\mathbf{Q}_p)$. Because $\mathrm{SL}_3(\mathbf{Q}_p)$ is its own derived group, the splitting s is unique. Now, let $g' \in \widetilde{L}(\mathbf{Q}_p)$ with image $g \in L(\mathbf{Q}_p)$. Define $s_g : \mathrm{SL}_3(\mathbf{Q}_p) \to \widetilde{F}_4(\mathbf{Q}_p)$ as $s_g(h) = g's(g^{-1}hg)(g')^{-1}$. Since $\mathrm{SL}_3(\mathbf{Q}_p)$ is normal in $L(\mathbf{Q}_p)$, s_g is another splitting; thus, $s_g = s$ by uniqueness. This implies $(g')^{-1}s(h)g' = s(g^{-1}hg)$, proving $s\big(\mathrm{SL}_3(\mathbf{Q}_p)\big)$ is normal.

Finally, we have a map $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p) \to \widetilde{L}(\mathbf{Q}_p)$, because we know that the relations defining $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p)$ are satisfied in $\widetilde{L}(\mathbf{Q}_p)$. This induces $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p) \to \widetilde{L}(\mathbf{Q}_p)/\mathrm{SL}_3(\mathbf{Q}_p)$. The latter group is a non-split double cover of $\mathrm{GL}_2(\mathbf{Q}_p)$, as is $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p)$. Since the map $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p) \to \widetilde{L}(\mathbf{Q}_p)/\mathrm{SL}_3(\mathbf{Q}_p)$ is defined in terms of generators and relations, it fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p) & \longrightarrow & \mathrm{GL}_2(\mathbf{Q}_p) & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow & & \downarrow{=} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{L}(\mathbf{Q}_p)/\mathrm{SL}_3(\mathbf{Q}_p) & \longrightarrow & \mathrm{GL}_2(\mathbf{Q}_p) & \longrightarrow & 1 \end{array}$$

and is, thus, an isomorphism. □

Let χ_{exc} denote the unique exceptional character of $Z(\widetilde{T}(\mathbf{Q}_p))$; by an abuse of notation, we use the same symbol for the extension to $T_*(\mathbf{Q}_p)$ defined by setting

$$\chi_{\mathrm{exc}}(h_{\alpha_1}(t)) = |t|^{1/2}\frac{\gamma(q)}{\gamma(tq)} \tag{18}$$

for $t \in \mathbf{Q}_p$; here $\gamma(q)$ is defined in (14). We set $B_L = L \cap B = TU_{B_L}$ the associated Borel subgroup of the Levi subgroup L and set $B_{L,*}(\mathbf{Q}_p) = T_*(\mathbf{Q}_p)U_{B_L}(\mathbf{Q}_p)$.

It follows from [LS10, § 6] that there is an embedding $V_{\min} \hookrightarrow \mathrm{Ind}_{B_*(\mathbf{Q}_p)}^{\widetilde{F}_4(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1})$ and, thus,

$$V_{\min,U_Q} \longrightarrow \ Ind_{B_*(\mathbf{Q}_p)}^{\widetilde{Q}(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1}) \cong \mathrm{Ind}_{B_{L,*}(\mathbf{Q}_p)}^{\widetilde{L}(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1}). \tag{19}$$

This latter map sends a function $f \in \mathrm{Ind}_{B_*(\mathbf{Q}_p)}^{\widetilde{F}_4(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1})$ to its restriction $f|_{\widetilde{Q}}$. It is clear that this factors through the Jacquet functor $V_{\min,U_Q}$. It is also clear that the map is non-zero.

Proposition 4.4.2. *The Jacquet functor $V_{\min,U_Q}$ is irreducible as a representation of $\widetilde{L}(\mathbf{Q}_p)$. Moreover, the representation $\mathrm{Ind}_{B_{L,*}(\mathbf{Q}_p)}^{\widetilde{L}(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1})$ has a unique irreducible subrepresentation, which is, thus, identified with $V_{\min,U_Q}$ under the above morphism.*

Proof. To prove the irreducibility of $V_{\min,U_Q}$, we follow the argument of [BFG03, Theorems 2.2 and 2.3]. This relies on the fact that the Jacquet functor of $\mathrm{Ind}_{B_*(\mathbf{Q}_p)}^{\widetilde{F}_4(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}})$ associated to any standard non-minimal parabolic subgroup has no supercuspidal subquotients [BZ77, Corollary 2.13(b)].

Suppose $V_1 \subseteq V_{\min,U_Q}$ is an $\widetilde{L}(\mathbf{Q}_p)$-invariant subspace, and V_2 the quotient of $V_{\min,U_Q}$ by V_1, giving the short exact sequence of $\widetilde{L}(\mathbf{Q}_p)$-representations

$$0 \longrightarrow V_1 \longrightarrow V_{\min,U_Q} \longrightarrow V_2 \longrightarrow 0.$$

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By exactness of the Jacquet functor down to the unipotent radical U_{B_L} of the Borel subgroup of L, we obtain

$$0 \longrightarrow V_{1,U_{B_L}} \longrightarrow \left(V_{\min,U_Q}\right)_{U_{B_L}} \cong V_{\min,U_B} \longrightarrow V_{2,U_{B_L}} \longrightarrow 0.$$

The Jacquet functor $V_{\min,U_B}$ associated to the Borel subgroup of F_4 is irreducible [LS10, Proposition 6.4]. In particular, either $V_{1,U_{B_L}} = 0$ or $V_{2,U_{B_L}} = 0$; suppose it is $V_{1,U_{B_L}} = 0$.

If V_1 has no non-zero Jacquet modules, we must have $V_1 = 0$ by [BZ77, Corollary 2.13(b)]. Otherwise, let $P_L = M_L N_L \subset L$ be the standard parabolic subgroup that is minimal among those such that $V_{1,N_L} \neq 0$. By assumption, $P_L \neq B_L$, so that V_{1,N_L} is a non-zero supercuspidal representation of $\widetilde{M}_L(\mathbf{Q}_p)$ and also a subquotient of the Jacquet module $\mathrm{Ind}_{B_*(\mathbf{Q}_p)}^{\widetilde{F}_4(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}})_{N_L}$, which is a contradiction. An argument is identical if we assume $V_{2,U_{B_L}} = 0$, completing the proof of the irreducibility of $V_{\min,U_Q}$.

The proof that $\mathrm{Ind}_{B_*(\mathbf{Q}_p)}^{\widetilde{Q}(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1})$ has a unique irreducible subrepresentation is exactly the same as the semisimple case treated in [LS10]. Now recall that one has a non-zero map (19), giving the final claim. □

Pulling back along the quotient map from Proposition 4.4.1, we now analyze the representation $\Omega_{\psi,\chi}^{(1)}$ as a representation of $\widetilde{L}(\mathbf{Q}_p)$. Define the multiplicative character $\chi(v) = |v|^{3/2}$, and recall that χ determines an extension of the representation on $S^+(\mathbf{Q}_p)$ from $\widetilde{\mathrm{SL}}_2(\mathbf{Q}_p)$ to the group G^*; see Proposition 4.3.4. Consider the corresponding Weil representation $\Omega_{\psi,\chi}^{(1)} = Ind_{G^*}^{\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p)}(S^+(\mathbf{Q}_p))$ of $\widetilde{\mathrm{GL}}_2^{(1)}(\mathbf{Q}_p)$.

Lemma 4.4.3. *Consider the functional*

$$B : \Omega_{\psi,\chi}^{(1)} \longrightarrow \mathbf{C}$$
$$B(f) = f(1)(0).$$

Then $B(t \cdot f) = (\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1})(t)B(f)$ *for all* $t \in T_*(\mathbf{Q}_p)$, *where* χ_{exc} *is the exceptional character* χ_{exc} *of* $T_*(\mathbf{Q}_p)$ *given by* (18).

Proof. Using the formulas in § 4.3.1, one has

$$B(h_{\alpha_1}(t) \cdot f) = |t|^{1/2}\frac{\gamma(tq)}{\gamma(q)}B(f)$$

and

$$B(h_{\alpha_2}(v^2) \cdot f) = \chi(v)|v|^{-1/2}B(f) = |v|B(f).$$

Moreover, $B(h_{\alpha_3}(v) \cdot f) = B(h_{\alpha_4}(v) \cdot f) = B(f)$. Now observe that for each simple root $\delta_B^{1/2}(h_\alpha(t)) = |t|$. The lemma now follows from the definition of χ_{exc}. □

Because $\Omega_{\psi,\chi}^{(1)}$ is irreducible [GP80], Frobenius reciprocity provides an embedding of $\widetilde{L}(\mathbf{Q}_p)$-representations

$$\Omega_{\psi,\chi}^{(1)} \longrightarrow \mathrm{Ind}_{B_{L,*}(\mathbf{Q}_p)}^{\widetilde{L}(\mathbf{Q}_p)}(\delta_B^{1/2}\chi_{\mathrm{exc}}^{-1}).$$

Corollary 4.4.4. *The Jacquet module* $V_{\min,U_Q}$ *is isomorphic to* $\Omega_{\psi,\chi}^{(1)}$.

We recall from Remark 4.3.10 that the latter representation is independent of ψ, as we should expect for $V_{\min,U_Q}$.

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4.5 The minimal modular form

We return now to the global setting. Let $J = H_3(\mathbf{Q})$ be the symmetric 3×3 matrices with $\mathbf{Q}$ coefficients. Fourier coefficients of modular forms on F_4 are parameterized by elements $\omega = (a, b, c, d) \in W_J(\mathbf{Q})$ where

$$W_J(\mathbf{Q}) = \mathbf{Q} \oplus J \oplus J^\vee \oplus \mathbf{Q} = \mathbf{Q} \oplus J \oplus J \oplus \mathbf{Q}$$

as $J^\vee$ is identified with J via the trace pairing. In this subsection, we show that we may choose $v_f \in \Pi_{\min,f}$ such that the modular form $\Theta_{F_4} := \theta(v_f)$ satisfies that it has:

(i) $U_{F_4}(4)$ level; and
(ii) non-zero $(0, 0, 0, 1)$-Fourier coefficient.

This will rely on the following purely local result. Let p be a finite prime. Denote by K_p^* the compact open subgroup of $\widetilde{F}_4(\mathbf{Q}_p)$ at p introduced in § 2.7, so that $K_2^* = K_R'(4)$ and $K_p^* = F_4^*(\mathbf{Z}_p)$ for odd p. Let $U_R = U_{\alpha_1} U_Q$ be the unipotent radical of the parabolic subgroup $R \subset F_4$ associated to the simple roots α_1 and α_2; it splits canonically into $\widetilde{F}_4(\mathbf{Q}_p)$. For $t = 1$ or $t = -1$, define a character $\psi_{1,t}$ on $U_R(\mathbf{Q}_p)$ by using the fixed additive character ψ_t on the root space U_{α_1}.

Theorem 4.5.1. *Let V_p denote the vector space underlying $\Pi_{\min,p}$. Suppose L is $(U_R, \psi_{1,t})$-functional such that L is 0 on the K_p^*-fixed vectors of V_p. Then $L = 0$. In particular, the twisted Jacquet functor associated to $(U_R, \psi_{1,t})$ induces a surjection*

$$V_p^{K_p^*} \longrightarrow V_{p,(U,\psi_{1,t})},$$

which is an isomorphism for $p \neq 2$.

Proof. There are two cases: $p = 2$ and $p > 2$.

Let us first handle the case where p is odd. First observe that $V_p^{K_p^*} \to V_{U_Q}^{\widetilde{L} \cap K_p^*}$ is well-defined and non-zero. Indeed, it is clear that the map is well-defined. To see that it is non-zero, consider the further map to V_{U_B} (recall U_B denotes the unipotent radical of the Borel). Recalling the embedding of V_p into $\mathrm{Ind}_{B_*(\mathbf{Q}_p)}^{\widetilde{F}_4(\mathbf{Q}_p)}(\delta_B^{1/2} \chi_{\mathrm{exc}}^{-1})$, we may consider the linear functional on V_p given by composing this map can with the evaluation-at-1 map: this gives a non-zero functional

$$V_p \longrightarrow V_{U_Q} \longrightarrow V_{U_B} \longrightarrow \mathbf{C}.$$

The spherical vector in this induced representation is non-zero at $t = 1$, so that this functional is non-vanishing on $V_p^{K_p^*}$. In particular, the composition

$$V_p^{K_p^*} \longrightarrow V_{U_Q}^{\widetilde{L} \cap K_p^*} \longrightarrow V_{U_B}^{\widetilde{T} \cap K_p^*} \tag{20}$$

is non-zero.

Now observe that both $V_p^{K_p^*}$ and $V_{p,U_Q}^{\widetilde{L} \cap K_p^*}$ are at most one-dimensional [GG18, § 9.2]. In fact, each is exactly one-dimensional: in the case of V_p, this follows from the intertwining operator calculations of [LS10]. In the case of V_{p,U_Q}, it now follows from the non-vanishing of the map (20) and, in any case, we constructed a spherical vector in Lemma 4.3.13. The claim of the theorem now follows by Corollary 4.3.14 and the isomorphism

$$V_{p,(U_R,\psi_{1,t})} \cong \left(V_{p,U_Q}\right)_{U_{\alpha_1},\psi_t} \cong \left(\Omega_{\psi,\chi}^{(1)}\right)_{U_{\alpha_1},\psi_t}.$$

We now discuss the case of $p = 2$. First observe that $K_2^* = K_R'(4)$ has an Iwahori factorization with respect to $Q = LU_Q$, as proved in Corollary 2.5.14. Now, it follows by [Cas95, Theorem 3.3.3]

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that $V^{K_R^*(4)} \to V_{U_Q}^{\widetilde{L}\cap K_R^*(4)}$ is surjective. In light of Corollary 4.4.4, the claim of the theorem thus follows as above by Corollary 4.3.12. □

Remark 4.5.2. The p odd case may also be handled in a similar fashion to the $p = 2$ case by instead considering the subgroup $I_p^* \subset K_p^*$ associated to the Iwahori subgroup. The only non-trivial step is noting that

$$V_p^{I_p^*} \cong V_p^{K_p^*}$$

as both are one-dimensional. This follows for K_p^* as noted above and follows for I_p^* as $V_p = \Pi_{\min,p}$ corresponds to the trivial representation of the Iwahori–Hecke algebra under the Shimura correspondence proved in [LS10, § 9]. We thank Gordan Savin for pointing this out to us.

Using Theorem 4.5.1, we obtain the following corollary, completing the proof of Theorem 3.3.1.

Corollary 4.5.3. *There is a quaternionic modular form Θ_{F_4} of weight $\frac{1}{2}$ on $\widetilde{F}_4(\mathbf{A})$ with $U_{F_4}(4)$ level and non-zero $(0,0,0,1)$-Fourier coefficient.*

Proof. Let $\omega_1 := (0,0,0,1) \in W_J(\mathbf{Q})$ and consider the ω_1-Fourier coefficient

$$\theta \longmapsto \int_{[N_J]} \theta(n)\psi^{-1}(\langle \omega_1, \overline{n}\rangle)\, dn,$$

where θ is a vector in the space of automorphic forms $\Pi_{\min}$. By [Gin19, Proposition 3], this gives a non-zero linear functional L_{ω_1} on $\Pi_{\min}$; that is, there are vectors in $\Pi_{\min}$ with non-zero ω_1-Fourier coefficient. Moreover, such a vector can be chosen to be a quaternionic modular form (in other words, to lie in the minimal $\widetilde{K}_\infty$-type at the archimedean place) by the explicit formula for the generalized Whittaker function proved in Theorem 3.2.2. Indeed, a corollary of the proof of the explicit formula is that there is a unique moderate growth $(N_J(\mathbf{R}), \psi(\langle \omega_1, -\rangle))$-equivariant functional on $\Pi_{\min,\infty}$ up to scalar multiple, and these functionals are nonvanishing on the minimal $\widetilde{K}_\infty$-type in $\Pi_{\min,\infty}$.

Now consider the linear map on $\Pi_{\min,f}$ given by $v_f \mapsto L_{\omega_1}(\theta(v_f))$; see (13) for the notation. By what was just said, this map is non-zero on $\Pi_{\min,f}$. Moreover, [Gin19, Proposition 4] implies that for any θ, we have

$$\int_{[N_J]} \theta(n)\psi^{-1}\big(\langle \omega_1, \overline{n}\rangle\big)\, dn = \int_{[N_S]} \left(\int_{[N_J]} \theta(nn')\psi^{-1}\big(\langle \omega_1, \overline{n}\rangle\big)\, dn \right) dn',$$

where N_S denote the unipotent radical of the Siegel parabolic subgroup of $H_J = \mathrm{GSp}_6$. But

$$\int_{[N_S]} \left(\int_{[N_J]} \theta(nn')\psi^{-1}\big(\langle \omega_1, \overline{n}\rangle\big)\, dn \right) dn' = \int_{[U_R]} \theta(u)\psi_{1,-1}^{-1}(u)\, du,$$

where U_R is the unipotent radical of the parabolic R from Theorem 4.5.1 and $\psi_{1,-1} = \prod_v \psi_{1,-1,v}$ is the global analogue of the character considered locally. By that result, the non-zero linear map on $\Pi_{\min,f}$ given by $v_f \mapsto L_{\omega_1}(\theta(v_f))$ does not vanish on the $\prod_p K_p^*$-invariant vectors. The corollary follows. □

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Conflicts of Interest

None.

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