Near-Optimal Streaming Ellipsoidal Rounding for General Convex Polytopes

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ABSTRACT

We give near-optimal algorithms for computing an ellipsoidal rounding of a convex polytope whose vertices are given in a stream. The approximation factor is linear in the dimension (as in John's theorem) and only loses an excess logarithmic factor in the aspect ratio of the polytope. Our algorithms are nearly optimal in two senses: first, their runtimes nearly match those of the most efficient known algorithms for the offline version of the problem. Second, their approximation factors nearly match a lower bound we show against a natural class of geometric streaming algorithms. In contrast to existing works in the streaming setting that compute ellipsoidal roundings only for centrally symmetric convex polytopes, our algorithms apply to general convex polytopes.

We also show how to use our algorithms to construct coresets from a stream of points that approximately preserve both the ellipsoidal rounding and the convex hull of the original set of points.

CCS CONCEPTS

• **Theory of computation → Streaming, sublinear and near linear time algorithms**; **Computational geometry**; *Sketching and sampling*; *Online algorithms.*

KEYWORDS

Ellipsoidal rounding, John ellipsoid, Convex polytopes, Convex geometry, Discrete geometry, Streaming algorithms, Coresets

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1 INTRODUCTION

We consider the problem of approximating convex polytopes in $\mathbb{R}^d$ with "simpler" convex bodies. Consider a convex polytope $Z \subset \mathbb{R}^d$. Our goal is to find a convex body $\widehat{Z} \subset \mathbb{R}^d$ from a given family of convex bodies, a translation vector $\boldsymbol{c} \in \mathbb{R}^d$, and a scaling factor $\alpha \in (0, 1]$ such that

$$\boldsymbol{c} + \alpha \cdot \widehat{Z} \subseteq Z \subseteq \boldsymbol{c} + \widehat{Z}. \tag{1}$$

We say that $\widehat{Z}$ is a $1/\alpha$-approximation to Z; an algorithm that computes $\widehat{Z}$ is a $1/\alpha$-approximation algorithm. In this paper, we will be interested in approximating Z with (a) ellipsoids and (b) polytopes defined by small number of vertices.

This problem has many applications in computational geometry, graphics, robotics, data analysis, and other fields (see [1] for an overview of some applications). It is particularly relevant when we are in the big-data regime and storing polytope Z requires too much memory. In this case, instead of storing Z, we find a reasonable approximation $\widehat{Z}$ with a succinct representation and then use it as a proxy for Z. In this setting, it is crucial that we use a *low-memory* approximation algorithm to find $\widehat{Z}$.

In this paper, we study the problem of approximating convex polytopes in the streaming model. The streaming model is a canonical big-data setting that conveniently lends itself to the study of low-memory algorithms. We assume that Z is the convex hull of points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$: $Z = \operatorname{conv}(\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\})$; the stream of points $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$ contains all the vertices of Z and additionally may contain other points from polytope Z. In our streaming model, points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$ arrive one at a time. At every timestep t, we must maintain an approximating body $\widehat{Z}_t$ and translate $\boldsymbol{c}_t$ such that

$$\operatorname{conv}(\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_t\}) \subseteq \boldsymbol{c}_t + \widehat{Z}_t. \tag{2}$$

Once a new point $\boldsymbol{z}_{t+1}$ arrives, the algorithm must compute a new approximating body $\widehat{Z}_{t+1}$ and translation $\boldsymbol{c}_{t+1}$ such that the guarantee (2) holds for timestep $t + 1$. Finally, after the algorithm has seen all n points, we must have

$$\boldsymbol{c}_n + \alpha \cdot \widehat{Z}_n \subseteq \underbrace{\operatorname{conv}(\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\})}_{Z} \subseteq \boldsymbol{c}_n + \widehat{Z}_n \tag{3}$$

for some $0 < \alpha \leq 1$ (where $1/\alpha$ is the approximation factor). Note that the algorithm may not know the value of n beforehand. We consider two types of approximation.

Ellipsoidal roundings. In one thrust, we aim to calculate an *ellipsoidal rounding* of Z – we are looking for ellipsoidal approximation $\widehat{Z} = \mathcal{E}$. Formally, we would like to output an origin-centered ellipsoid $\mathcal{E}$, a center/translate $\boldsymbol{c} \in \mathbb{R}^d$, and a scaling parameter $0 < \alpha \leq 1$ such that

$$\boldsymbol{c} + \alpha \cdot \mathcal{E} \subseteq Z \subseteq \boldsymbol{c} + \mathcal{E}.$$

Ellipsoidal roundings are convenient representations of convex sets. They have applications to preconditioning convex sets for efficient sampling and volume estimation [7], algorithms for convex programming [15], robotics [17], and other areas. They also require

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the storage of at most $\sim d^2$ floating point numbers, as every ellipsoid can be represented with a center $\boldsymbol{c}$ and semiaxes $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d'}$ for $d' \leq d$.

We note that by John's theorem [8], the minimum-volume outer ellipsoid for Z achieves approximation $1/\alpha \leq d$. Moreover, the upper bound of d is tight, which is witnessed when Z is a d-dimensional simplex (that is, the convex hull of $d+1$ points in general position).

We now formally state the streaming ellipsoidal rounding problem.

Problem 1.1 (Streaming ellipsoidal rounding). For $z_1, \ldots, z_n \in \mathbb{R}^d$, let $Z = \mathrm{conv}(\{z_1, \ldots, z_n\})$. A streaming algorithm $\mathcal{A}$ receives points $z_1, \ldots, z_n$ one at a time and produces a sequence of ellipsoids $\boldsymbol{c}_t + \mathcal{E}_t$ and scalings α_t. The algorithm must satisfy the following guarantee at the end of the stream.

$$\boldsymbol{c}_n + \alpha_n \cdot \mathcal{E}_n \subseteq Z \subseteq c_n + \mathcal{E}_n$$

We say that $\boldsymbol{c}_n + \mathcal{E}_n$ is an ellipsoidal rounding of Z with approximation factor $1/\alpha_n$.

We note that in the special case where Z is centrally symmetric (i.e., $Z = -Z$), there are algorithms with nearly optimal approximation factors $O(\sqrt{d \log (n\kappa^{\mathsf{OL}})})$ and $O(\sqrt{d \log \kappa})$ due to Woodruff and Yasuda [20] and Makarychev, Manoj, and Ovsiankin [11], respectively (here, κ^{OL} is the online condition number and κ is the aspect ratio of the dataset). The running times of these algorithms nearly match those of the best-known offline solutions. However, these algorithms do not work with non-symmetric polytopes and we are not aware of any way to adapt them so that they do. We defer a more detailed discussion of the algorithms for the symmetric case to Section 1.2.

Convex hull approximation. In another thrust, we want to find a translate $\boldsymbol{c} \in \mathbb{R}^d$, subset $S \subseteq [n]$, and scale α such that

$$\begin{aligned}\mathrm{conv}(\{z_i : i \in S\}) &\subseteq \mathrm{conv}(\{z_1, \ldots, z_n\}) \\ &\subseteq \boldsymbol{c} + \frac{1}{\alpha} \cdot \mathrm{conv}(\{z_i - \boldsymbol{c} : i \in S\}).\end{aligned}$$

Note that $\boldsymbol{c} + 1/\alpha \cdot \mathrm{conv}(\{z_i - \boldsymbol{c} : i \in S\})$ is a $1/\alpha$-scaled copy of $\mathrm{conv}(\{z_i : i \in S\})$. In other words, we desire to find a *coreset* $\{z_i : i \in S\}$ that approximates Z. This approach has the advantage of yielding an interpretable solution – one can think of a coreset as consisting of the most "important" datapoints of the input dataset.

We formally state the streaming convex hull approximation problem we study in Problem 1.2.

Problem 1.2 (Streaming convex hull approximation). Let $Z = \mathrm{conv}(z_1, \ldots, z_n) \subseteq \mathbb{R}^d$. A streaming algorithm $\mathcal{A}$ receives points $z_1, \ldots, z_n$ one at a time and produces a sequence of scalings α_t, centers $\boldsymbol{c}_t$, subsets $S_t \subseteq [n]$ such that $S_t \subseteq S_{t+1}$. The algorithm must satisfy the following guarantee at the end of the stream.

$$\begin{aligned}\mathrm{conv}(\{z_i : i \in S_n\}) &\subseteq \mathrm{conv}(\{z_1, \ldots, z_n\}) \\ &\subseteq \boldsymbol{c}_n + \frac{1}{\alpha} \cdot \mathrm{conv}(\{z_i - \boldsymbol{c}_n : i \in S_n\})\end{aligned}$$

We say that $\{z_i : i \in S_n\}$ is a coreset of Z with approximation factor $1/\alpha_n$. We will also call S_n a coreset.

Note that the model considered in Problem 1.2 is essentially the same as the *online coreset model* studied by Woodruff and Yasuda [20]. Similar to Problem 1.1, Problem 1.2 has been studied in the case where Z is centrally symmetric. In particular, Woodruff and Yasuda [20] obtain approximation factor $O(\sqrt{d \log (n\kappa^{\mathsf{OL}})})$ (where κ^{OL} is the same online condition number mentioned earlier). However, whether analogous results for asymmetric polytopes hold was an important unresolved question.

1.1 Our Contributions

In this section, we present our results for Problems 1.1 and 1.2.

1.1.1 Algorithmic Results. We start with defining several quantities that we need to state the results and describe their proofs.

Notation. We will denote the linear span of a set of points A by $\mathsf{span}(A)$. That is, $\mathsf{span}(A)$ is the minimal linear subspace that contains A. We denote the affine span of A by $\mathsf{aff}(A)$. That is, $\mathsf{aff}(A)$ is the minimal affine subspace that contains A. Note that $\mathsf{aff}(A) = \boldsymbol{a} + \mathsf{span}(A - \boldsymbol{a})$ if $\boldsymbol{a} \in A$. Finally, we denote the unit ball centered at the origin by B_2^d.

Definition 1.3 (Inradius). Let $K \subset \mathbb{R}^d$ be a convex body. The *inradius* $r(K)$ of K is the largest r such that there exists a point $\boldsymbol{c}_I$ (called the *incenter*) for which $\boldsymbol{c}_I + r \cdot \left(B_2^d \cap \mathsf{span}(K - \boldsymbol{c}_I)\right) \subseteq K$.

Definition 1.4 (Circumradius). Let $K \subset \mathbb{R}^d$ be a convex body. The *circumradius* $R(K)$ of K is the smallest R such that there exists a point $\boldsymbol{c}_C$ (called the *circumcenter*) for which $K \subseteq \boldsymbol{c}_C + R \cdot B_2^d$.

Definition 1.5 (Aspect Ratio). Let $K \subset \mathbb{R}^d$ be a convex body. We say that $\kappa(K) := R(K)/r(K)$ is the *aspect ratio* of K.

We now state Theorem 1.6, which provides an algorithm for Problem 1.1. In addition to the data stream of $z_1, \ldots, z_n$, this algorithm needs a suitable initialization: a ball $\boldsymbol{c}_0 + r_0 \cdot B_2^d$ inside Z.

Theorem 1.6. *Consider the setting of Problem 1.1. Suppose the algorithm is given an initial center $\boldsymbol{c}_0$ and radius r_0 for which it is guaranteed that $\boldsymbol{c}_0 + r_0 \cdot B_2^d \subseteq \mathrm{conv}(\{z_1, \ldots, z_n\})$. There exists an algorithm that, for every timestep t, maintains an origin-centered ellipsoid $\mathcal{E}_t$, center $\boldsymbol{c}_t$, and scaling factor α_t such that at every timestep t: $\mathrm{conv}(\{z_1, \ldots, z_t\}) \subseteq \boldsymbol{c}_t + \mathcal{E}_t$, and at timestep n: $\boldsymbol{c}_n + \alpha_n \cdot \mathcal{E}_n \subseteq Z \subseteq c_n + \mathcal{E}_n$, where*

$$1/\alpha_n = O\left(\min\left(R(Z)/r_0, d \log\left(R(Z)/r_0\right)\right)\right)$$

The algorithm has runtime $\widetilde{O}(nd^2)$ and stores $O(d^2)$ floating point numbers.

Note that the final approximation factor depends on the quality of the initialization $(\boldsymbol{c}_0, r_0)$. If the radius r_0 of this ball is reasonably close to the inradius $r(Z)$ of Z, the algorithm gives an $O(\min(\kappa(Z), d \log \kappa(Z)))$ approximation. In Theorem 1.7, we adapt the algorithm form Theorem 1.6 to the setting where the algorithm does not have the initialization information. Note that the approximation guarantee of $O(\min(\kappa(Z), d \log \kappa(Z)))$ is a natural analogue of the bounds by [11] and [20] for the symmetric case (see Section 1.2).

Theorem 1.7. *Consider the setting of Problem 1.1. There exists an algorithm that, for every timestep t, maintains an ellipsoid $\mathcal{E}_t$, center $\boldsymbol{c}_t$, and approximation factor α_t such that*

$$\boldsymbol{c}_t + \alpha_t \cdot \mathcal{E}_t \subseteq \mathrm{conv}(\{z_1, \dots, z_t\}) \subseteq \boldsymbol{c}_t + \mathcal{E}_t.$$

Additionally, let r_t and R_t be the largest and smallest parameters, respectively, for which there exists $\boldsymbol{c}_t^\star$ such that

$$\boldsymbol{c}_t^\star + r_t \cdot \left(B_2^d \cap \mathrm{span}(z_1 - \boldsymbol{c}_t^\star, \dots, z_t - \boldsymbol{c}_t^\star)\right) \subseteq \mathrm{conv}(\{z_1, \dots, z_t\}) \subseteq \boldsymbol{c}_t^\star + R_t \cdot B_2^d$$

and $d_t \coloneqq \dim(\mathrm{aff}(z_1, \dots, z_t))$. Then, for all timesteps t, we have

$$1/\alpha_t = O\left(d_t \log\left(d_t \cdot \max_{t' \le t} \frac{R_t}{r_{t'}}\right)\right).$$

The algorithm runs in time $\widetilde{O}(nd^2)$ and stores $O(d^2)$ floating point numbers.

Let us now quickly compare the guarantees of Theorem 1.6 and 1.7. Notice that the algorithm in Theorem 1.7 does not require an initialization pair $(\boldsymbol{c}_0, r_0)$. Additionally, the algorithm in Theorem 1.7 outputs a per-timestep approximation as opposed to just an approximation at the end of the stream. However, these advantages come at a cost – it is easy to check that the aspect ratio term seen in Theorem 1.7 can be larger than that in Theorem 1.6, e.g., it is possible to have $R(Z)/r_0 \le \max_{t' \le n} R_n/r_{t'}$.

However, when we impose the additional constraint that the points z_t have coordinates that are integers in the range $[-N, N]$, we can improve over the guarantee in Theorem 1.7 and obtain results that are independent of the aspect ratio. This is similar in spirit to the condition number-independent bound that Woodruff and Yasuda [20] obtain for the sums of online leverage scores. However, a key difference is that our results still remain independent of the length of the stream. See Theorem 1.8.

Theorem 1.8. *Consider the setting of Problem 1.1, where in addition, the points $z_1, \dots, z_n$ are such that their coordinates are integers in $\{-N, -N+1, \dots, N-1, N\}$. There exists an algorithm that, for every timestep t, maintains an ellipsoid $\mathcal{E}_t$, center $\boldsymbol{c}_t$, and approximation factor α_t such that*

$$\boldsymbol{c}_t + \alpha_t \cdot \mathcal{E}_t \subseteq \mathrm{conv}(\{z_1, \dots, z_t\}) \subseteq \boldsymbol{c}_t + \mathcal{E}_t.$$

Let $d_t \coloneqq \dim(\mathrm{aff}(z_1, \dots, z_t))$. Then, for all timesteps t, we have

$$1/\alpha_t = O(d_t \log(dN)).$$

The algorithm runs in time $\widetilde{O}(nd^2)$ and stores $O(d^2)$ floating point numbers.

We present Theorems 1.6, 1.7, and 1.8 in the full version of the paper [12], although we discuss some of the main components for these algorithms in Section 4. With Theorems 1.7 and 1.8 in hand, obtaining results for Problem 1.2 becomes straightforward. We use the algorithm guaranteed by Theorem 1.7 along with a simple subset selection criterion to arrive at our result for Problem 1.2.

Theorem 1.9. *Consider $Z = \mathrm{conv}(\{z_1, \dots, z_n\})$. For a subset $S \subseteq [n]$, let $Z|_S = \mathrm{conv}(\{z_i : i \in S\})$. Consider the setting of Problem 1.2. There exists a streaming algorithm that, for every timestep t, maintains a subset S_t, center $\boldsymbol{c}_t$, and scaling factor α_t such that*

$$Z|_{S_t} \subseteq \mathrm{conv}(\{z_1, \dots, z_t\}) \subseteq \boldsymbol{c}_t + \frac{1}{\alpha_t} \cdot \left(Z|_{S_t} - \boldsymbol{c}_t\right).$$

Additionally, for d_t, r_t and R_t as defined in Theorem 1.7, we have for all t that

$$\frac{1}{\alpha_t} = O\left(d_t \log\left(d_t \cdot \max_{t' \le t} \frac{R_t}{r_{t'}}\right)\right) \text{ and } |S_t| = O\left(d_t \log\left(\max_{t' \le t} \frac{R_t}{r_{t'}}\right)\right)$$

and, if the z_t have integer coordinates ranging in $[-N, N]$, then

$$\frac{1}{\alpha_t} = O(d_t \log(dN)) \qquad \text{and} \qquad |S_t| = O(d_t \log(dN)).$$

Each S_t is either S_{t-1} or $S_{t-1} \cup \{t\}$ (where $t \ge 1$ and $S_0 = \emptyset$). The algorithm runs in time $\widetilde{O}(nd^2)$ and stores at most $O(d^2)$ floating point numbers.

We prove Theorem 1.9 in the full version of the paper.

1.1.2 Approximability Lower Bound.

Observe that the approximation factors obtained in Theorems 1.6, 1.7, and 1.9 all incur a mild dependence on (variants of) the aspect ratio of the dataset. A natural question is whether this dependence is necessary. In Theorem 1.10, we conclude that the approximation factor from Theorem 1.6 is in fact nearly optimal for a wide class of *monotone* algorithms. We defer the discussion of the notion of a monotone algorithm to Section 2.1. Loosely speaking, a monotone algorithm commits to the choices it makes; namely, the outer ellipsoid may only increase over time $\boldsymbol{c}_t + \mathcal{E}_t \supseteq \boldsymbol{c}_{t-1} + \mathcal{E}_{t-1}$ and the inner ellipsoid $\boldsymbol{c}_t + \alpha_t \mathcal{E}_t$ satisfies a related but more technical condition $\boldsymbol{c}_t + \alpha_t \mathcal{E}_t \subseteq \mathrm{conv}((\boldsymbol{c}_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}) \cup \{z_t\})$.

Theorem 1.10. *Consider the setting of Problem 1.1. Let $\mathcal{A}$ be any monotone algorithm (see Definition 2.1 in Section 2.1) that solves Problem 1.1 with approximation factor $1/\alpha_n$. For every $d \ge 2$, there exists a sequence of points $\{z_1, \dots, z_n\} \subset \mathbb{R}^d$ such that algorithm $\mathcal{A}$ gets an approximation factor of $1/\alpha_n \ge \Omega\left(\frac{d \log(\kappa(Z))}{\log d}\right)$ on $Z = \mathrm{conv}(\{z_1, \dots, z_n\})$.*

We prove Theorem 1.10 in the full version of the paper.

1.2 Related Work and Open Questions

Streaming asymmetric ellipsoidal roundings. To our knowledge, the first paper to study ellipsoidal roundings in the streaming model is that of Mukhopadhyay, Sarker, and Switzer [13]. The authors consider the case where $d = 2$ and prove that the approximation factor of the greedy algorithm (that which updates the ellipsoid to be the minimum volume ellipsoid containing the new point and the previous iterate) can be unbounded. Subsequent work by Mukhopadhyay, Greene, Sarker, and Switzer [14] generalizes this result to all $d \ge 2$.

Nearly-optimal streaming symmetric ellipsoidal roundings. Recently, Makarychev, Manoj, and Ovsiankin [11] and Woodruff and Yasuda [20] gave the first positive results for streaming ellipsoidal roundings. Both [11] and [20] considered the problem only in the *symmetric setting* – when the goal is to approximate the polytope $\mathrm{conv}(\{\pm z_1, \dots, \pm z_n\})$. [11] and [20] obtained $O(\sqrt{d \log \kappa(Z)})$ and

$O(\sqrt{d \log n\kappa^{\mathrm{OL}}})$ approximations, respectively (here, κ^{OL} is the online condition number; see [20] for details). Their algorithms use only $\widetilde{O}(\mathrm{poly}(d))$ space, where the $\widetilde{O}$ suppresses $\log d$, $\log n$, and aspect ratio-like terms. Note that by John's theorem, the $\Omega(\sqrt{d})$ dependence is required in the symmetric setting even for offline algorithms.

A natural question is whether the techniques of [11] or [20] extend to Problems 1.1 and 1.2. The update rule used in [11] essentially updates $\mathcal{E}_{t+1}$ to be the minimum volume ellipsoid covering both $\mathcal{E}_t$ and points $\pm \boldsymbol{z}_{t+1}$. In the non-symmetric case, it would be natural to consider the minimum volume ellipsoid covering $\mathcal{E}_t$ and point $\boldsymbol{z}_{t+1}$. However, this approach does not give an $\tilde{O}(d)$ approximation. The algorithm in [20] maintains a quadratic form that consists of sums of outer products of "important points" (technically speaking, those with a constant online leverage score). Unfortunately, this approach does not suggest how to move the previous center $\boldsymbol{c}_{t-1}$ to a new center $\boldsymbol{c}_t$ in a way that allows the algorithm to maintain a good approximation factor. It is not hard to see that there exist example streams for which the center $\boldsymbol{c}_{t-1}$ must be shifted in each iteration to maintain even a bounded approximation factor. This means that any nontrivial solution to Problems 1.1 and 1.2 must overcome this difficulty.

Offline ellipsoidal roundings for general convex polytopes. Nesterov [15] gives an efficient *offline* $O(d)$-approximation algorithm for the ellipsoidal rounding problem, with a runtime of $\widetilde{O}(nd^2)$. Observe that this is essentially the same runtime as those achieved by the algorithms we give (see Theorems 1.6 and 1.7).

Streaming convex hull approximations. Agarwal and Sharathkumar [2] studied related problems of computing *extent measures* of a convex hull in the streaming model, in particular finding coresets for the minimum enclosing ball, and obtained both positive and negative results. Blum et al. [5] showed that one cannot maintain an ε-*hull* in space proportional to the number of vertices belonging to the offline optimal solution (where a body $\widehat{Z}$ is an ε-hull for Z if every point in $\widehat{Z}$ is distance at most ε away from Z).

Offline convex hull approximations. The problem of approximating a convex body with the convex hull of a small number of points belonging to the body has been well-studied. Existentially, Barvinok [3] shows that if the input convex set is sufficiently symmetric, then one can choose $(d/\varepsilon)^{d/2}$ points to obtain a $1+\varepsilon$ approximation. Moreover, Lu [10] shows that one can obtain a $d+2$ approximation with $d+1$ points, which is witnessed by choosing the $d+1$ points to be the maximum volume simplex contained within the convex body (for this reason, this construction is called "John's Theorem for simplices"; see [16] for more details). However, none of these works study a streaming or online setting, as we do here.

Coresets for the minimum volume enclosing ellipsoid problem (MVEE).. Let $\mathsf{MVEE}(K)$ denote the minimum volume enclosing ellipsoid for a convex body $K \subset \mathbb{R}^d$. We say that a subset $S \subseteq [n]$ is an ε-coreset for the MVEE problem if we have

$$\mathrm{vol}\left(\mathsf{MVEE}(Z)\right) \leq (1+\varepsilon)^d \, \mathrm{vol}\left(\mathsf{MVEE}(Z|_S)\right). \tag{4}$$

There is extensive literature on coresets for the MVEE problem, and we refer the reader to papers by Kumar and Yildirim [9], Todd and Yildirim [19], Clarkson [6], Bhaskara, Mahabadi, and Vakilian [4], and the book by Todd [18].

Importantly, $\mathsf{MVEE}(Z|_S)$ may not be a good approximation for $\mathsf{MVEE}(Z)$ (for that reason, some authors refer to coresets satisfying (4) as weak coresets for MVEE). Therefore, even though $\mathsf{MVEE}(Z)$ provides a good ellipsoidal rounding for Z, $\mathsf{MVEE}(Z|_S)$ generally speaking does not. See [19, page 2] and [4, Section 2.1] for an extended discussion.

2 SUMMARY OF TECHNIQUES

In this section, we give an overview of the technical methods behind our results.

2.1 Monotone Algorithms

The algorithm we give in Theorem 1.6 belongs to a class we term *monotone algorithms*, which we now define.

Definition 2.1 (Monotone algorithm). Consider the setting of Problem 1.1. Note the following invariants for every timestep t.

$$\boldsymbol{c}_t + \mathcal{E}_t \supseteq \mathrm{conv}((\boldsymbol{c}_{t-1} + \mathcal{E}_{t-1}) \cup \{\boldsymbol{z}_t\}) \tag{5}$$

$$\boldsymbol{c}_t + \alpha_t \mathcal{E}_t \subseteq \mathrm{conv}((\boldsymbol{c}_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}) \cup \{\boldsymbol{z}_t\}) \tag{6}$$

We say that an algorithm $\mathcal{A}$ is monotone if for any initial $(\boldsymbol{c}_0 + \mathcal{E}_0, \alpha_0)$ and sequence of data points $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$, the resulting sequence $\{(\boldsymbol{c}_0 + \mathcal{E}_0, \alpha_0), (\boldsymbol{c}_1 + \mathcal{E}_1, \alpha_1), \ldots, (\boldsymbol{c}_n + \mathcal{E}_n, \alpha_n)\}$ arising from applying $\mathcal{A}$ to the stream satisfies the two invariants (5) and (6). Refer to Figure 1.

We will sometimes consider how a monotone algorithm $\mathcal{A}$ makes a single update upon seeing a new point $\boldsymbol{x}$. In this setting, we will call $\mathcal{A}$ a *monotone update rule*.

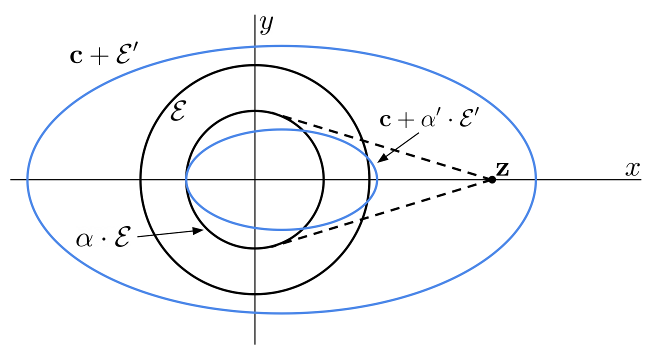

Figure 1: A monotone update step. For brevity, we refer to $\mathcal{E}$ and $\alpha \cdot \mathcal{E}$ as the previous ellipsoids $\mathcal{E}_{t-1}, \alpha\mathcal{E}_{t-1}$, and $\mathcal{E}'$ and $\alpha' \cdot \mathcal{E}'$ as the next ellipsoids $\mathcal{E}_t, \alpha_t \cdot \mathcal{E}_t$. $\mathcal{E}$ and $\alpha\mathcal{E}$ are, respectively, the larger and smaller black circles. $c + \mathcal{E}'$ and $c + \alpha'\mathcal{E}'$ are the larger and smaller blue ellipses. The dotted lines show $\partial(\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})) \setminus \partial(\alpha\mathcal{E})$, i.e. the the boundary of $\mathrm{conv}(\alpha \cdot \mathcal{E} \cup \{z\})$ minus the boundary of $\alpha\mathcal{E}$.

Here we will refer to $\boldsymbol{c}_t + \mathcal{E}_t, \boldsymbol{c} + \alpha_t \mathcal{E}_t$ as the "next" ellipsoids and to $\boldsymbol{c}_{t-1} + \mathcal{E}_{t-1}, \boldsymbol{c} + \alpha_{t-1}\mathcal{E}_{t-1}$ as the "previous" ellipsoids. The first condition we require is that

$$\boldsymbol{c}_t + \mathcal{E}_t \supseteq \boldsymbol{c}_{t-1} + \mathcal{E}_{t-1}. \tag{5a}$$

It ensures that each successive outer ellipsoid contains the previous outer ellipsoid. Thus once the algorithm decides that some $z \in c_t + \mathcal{E}_t$, it makes a commitment that $z \in c_{t'} + \mathcal{E}_{t'}$ for all $t' \geq t$. Note that (5a) implies (5), since z_t must be in $c_t + \mathcal{E}_t$ and $c_t + \mathcal{E}_t$ is convex. The second condition (6) looks more complex but is also very natural. Assume that the algorithm only knows that (a) $c_{t-1} + \alpha_{t-1}\mathcal{E}_{t-1} \subseteq Z$ (this is true from induction) and (b) $z_t \in Z$ (this is true by the definition of Z). Then, we must have that $A = \operatorname{conv}((c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}) \cup \{z_t\})$ lies in Z; as far as the algorithm is concerned, any point outside of A may also be outside of Z. Since the algorithm must ensure that $c_t + \alpha_t\mathcal{E}_t \subseteq Z$, it will also ensure that $c_t + \alpha_t\mathcal{E}_t \subseteq A$ and thus satisfy (6).

2.2 Streaming Ellipsoidal Rounding (Theorems 1.6, 1.7, and 1.8)

Now we describe the algorithm from Theorem 1.6 in more detail. Our algorithm keeps track of the current ellipsoid $\mathcal{E}_t$, center c_t, and scaling parameter α_t. Initially, $c_0 + \mathcal{E}_0$ is the ball of radius r_0 around c_0 (r_0 and c_0 are given to the algorithm), and $\alpha_0 = 1$. Each time the algorithm gets a new point z_t, it updates $\mathcal{E}_{t-1}$, c_{t-1}, α_{t-1} using a *monotone* update rule (as defined in Definition 2.1) and obtains $\mathcal{E}_t$, c_t, α_t. The monotonicity condition is sufficient to guarantee that the algorithm gets a $1/\alpha_n$ approximation to Z. Indeed, first using condition (5), we get

$$\begin{aligned} c_n + \mathcal{E}_n &\supseteq (c_{n-1} + \mathcal{E}_{n-1}) \cup \{z_n\} \\ &\supseteq (c_{n-2} + \mathcal{E}_{n-2}) \cup \{z_{n-1}, z_n\} \\ &\supseteq \cdots \supseteq \{z_1, \ldots, z_n\} \end{aligned}$$

Thus, $c_n + \mathcal{E}_n \supseteq Z$. Then, using condition (6), we get

$$\begin{aligned} c_n + \alpha_n\mathcal{E}_n &\subseteq \operatorname{conv}((c_{n-1} + \alpha_{n-1}\mathcal{E}_{n-1}) \cup \{z_n\}) \\ &\subseteq \operatorname{conv}((c_{n-2} + \alpha_{n-2}\mathcal{E}_{n-2}) \cup \{z_{n-1}, z_n\}) \\ &\subseteq \cdots \subseteq \operatorname{conv}((c_0 + \alpha_0\mathcal{E}_0) \cup \{z_1, \ldots, z_n\}). \end{aligned}$$

The initial ellipsoid $c_0 + \alpha_0\mathcal{E}_0 = c_0 + r_0 B_2^d$ is in Z and therefore $c_n + \alpha_n\mathcal{E}_n \subseteq \operatorname{conv}(z_1, \ldots, z_n) = Z$. We verified that the algorithm finds a $^1/_{\alpha_n}$ approximation for Z.

Now, the main challenge is to design an update rule that ensures that $1/\alpha_n$ is small (as in the statement Theorem 1.6) and prove that the rule satisfies the monotonicity conditions/invariants from Definition 2.1. We proceed as follows.

First, we design a monotone update rule that satisfies a particular evolution condition. This condition upper bounds the increase of the approximation factor $^1/_{\alpha_t} - {^1/_{\alpha_{t-1}}}$. Second, we prove that any monotone update rule satisfying the evolution condition yields the approximation we desire. These two parts imply Theorem 1.6. Finally, we remove the initialization requirement from Theorem 1.6 and obtain Theorem 1.7.

Designing a monotone update rule. Suppose that at the end of timestep $t-1$ our solution consists of a center c_{t-1}, ellipsoid $\mathcal{E}_{t-1}$, and scaling parameter α_{t-1} for which the invariants in Definition 2.1 hold. We give a procedure that, given the next point z_t, computes $c_t, \mathcal{E}_t, \alpha_t$ that still satisfy the invariants of Definition 2.1. Further, we prove that the resulting update satisfies an evolution condition (7)

$$\frac{^1/_{\alpha_t} - {^1/_{\alpha_{t-1}}}}{\log \operatorname{vol}(\mathcal{E}_t) - \log \operatorname{vol}(\mathcal{E}_{t-1})} \leq C, \tag{7}$$

where C is an absolute constant and $\operatorname{vol}(\mathcal{E})$ denotes the volume of the ellipsoid $\mathcal{E}$. While it is possible to find the optimal update using convex optimization (the update that satisfies the invariants and minimizes the ratio on the left of (7)), we instead provide an explicit formula for an update that readily satisfies (7) and as we show is monotone.

We now describe how we get the formula for the update rule. By applying an affine transformation, we may assume that $\mathcal{E}_{t-1}$ is a unit ball and $c_{t-1} = 0$. Further, we may assume that z_t is colinear with e_1 (the first basis vector): $z_t = \|z_t\| e_1$. Importantly, affine transformations preserve (a) the invariants in Definition 2.1 (if they hold for the original ellipsoids and points, then they also do for the transformed ones and vice versa) and (b) the value of the ratio in (7), since they preserve the value of $\operatorname{vol}(\mathcal{E}_t)/\operatorname{vol}(\mathcal{E}_{t-1})$.

Now consider the group $G = \mathbb{O}(d)_{e_1} \cong \mathbb{O}(d-1)$ of orthogonal transformations that map e_1 to itself: all of them map the unit ball $\mathcal{E}_{t-1}$ to itself and z_t to itself. Thus, it is natural to search for an update $(c_t, \mathcal{E}_t)$ that is symmetric with respect to all these transformations. It is easy to see that in this case $\mathcal{E}_t$ is defined by equation $(x_1/a)^2 + \sum_{i=2}^d (x_i/b)^2 = 1$ where a and b are some parameters (equal to the semiaxes of $\mathcal{E}_t$) and $c_t = ce_1$ for some c. Since all ellipsoids and points appearing in the invariant conditions are symmetric with respect to G, it is sufficient now to restrict our attention to their sections in the 2d-plane $\operatorname{span}(e_1, e_2)$ and prove that the invariants hold in this plane. Hence, the problem reduces to a statement in two-dimensional Euclidean geometry (however, when we analyze (7), we still use that the volume of $\mathcal{E}_t$ is proportional to ab^{d-1} and not ab).

Let us denote the coordinates corresponding to basis vectors e_1 and e_2 by x and y. For brevity, let $\mathcal{E} = \mathcal{E}_{t-1}$, $z = z_t$, $\mathcal{E}' = \mathcal{E}_t$, $c = c_t = ce_1$, $\alpha = \alpha_{t-1}$, and $\alpha' = \alpha_t$. We now need to choose parameters a, b, and c so that invariants from Definition 2.1 and (7) hold. See Figure 1. As shown in that figure, the new outer ellipse $c + \mathcal{E}'$ must contain the previous outer ellipse $\mathcal{E}$ and the newly received point z. The new inner ellipse $c + \alpha'\mathcal{E}'$ must be contained within the convex hull of the previous inner ellipse $\alpha\mathcal{E}$ and z.

It is instructive to consider what happens when point z is at infinitesimal distance Δ from $\mathcal{E}$: $\|z\| = 1 + \Delta$. We consider a minimal axis-parallel outer ellipse $\mathcal{E}'$ that contains $\mathcal{E}$ and z. It must go through $z = (1+\Delta, 0)$ and touch $\mathcal{E}$ at two points symmetric w.r.t. the x-axis, say, $(-\sin\varphi, \pm\cos\varphi)$. Angle φ uniquely determines $\mathcal{E}'$. Now we want to find the largest value of the scaling parameter α' so that $\alpha'\mathcal{E}'$ fits inside the convex hull of $\mathcal{E}$ and z. When Δ is infinitesimal, this condition splits into two lower bounds on α' – loosely speaking, they say that $\mathcal{E}$ does not extend out beyond the convex hull in the horizontal (one bound) and vertical directions (the other). The former bound becomes stronger (gives a smaller upper bound on α') when φ increases, and the latter becomes stronger when φ decreases. When $\varphi = \alpha/2 \pm O(\alpha^2)$, all terms linear in α vanish in both bounds and then $\alpha' = \alpha - \Theta(\alpha^2\Delta)$ satisfies both of them; for other choices of φ, we have $\alpha' \leq \alpha - \Omega(\alpha\Delta)$. So we let $\varphi = \alpha/2$ and from the formula for α' get $1/\alpha' = 1/\alpha + O(\Delta)$. On the other hand, $\operatorname{vol}(\mathcal{E}') \geq (1 + \Delta/2)\operatorname{vol}(\mathcal{E})$, since $\mathcal{E}'$ covers $z = (1+\Delta, 0)$. It is easy to see now that the evolution condition (7) holds: the numerator is $O(\Delta)$ and the denominator is $\Omega(\Delta)$ in (7).

We remark that letting $\boldsymbol{c} + \mathcal{E}'$ be the minimum volume ellipsoid that contains $\mathcal{E}$ and $\boldsymbol{z}$ is a highly suboptimal choice (it corresponds to setting $\varphi = \Theta(1/d)$). To derive our specific update formulas for arbitrary $\boldsymbol{z}$, we, loosely speaking, represent an arbitrary update as a series of infinitesimal updates, get a differential equation on a, b, c, and α', solve it, and then simplify the solution (remove non-essential terms, etc). We get the following.

Our updates come from a family parameterized by $\gamma \geq 0$. Define α' by ${}^1/_{\alpha'} = {}^1/_{\alpha} + 2\gamma$. With this choice of α', define the new ellipses to be

$$\underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 = 1,}_{\boldsymbol{c}+\mathcal{E}'} \qquad \underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 = \alpha'^2}_{\boldsymbol{c}+\alpha'\mathcal{E}'}$$

where we use parameters

$$\left.\begin{aligned} a &= \exp(\gamma) \\ b &= 1 + \frac{\alpha - \alpha'}{2} \\ c &= -\alpha + \alpha' \cdot a \end{aligned}\right\}.$$

Choose $\gamma \approx \ln\|\boldsymbol{z}\|$ so that $\boldsymbol{c} + \mathcal{E}'$ covers point $\boldsymbol{z}$. We use two-dimensional geometry to prove that $\mathcal{E}'$, $\boldsymbol{c}$, and α' satisfy the invariants (see Figure 1). Now to prove the evolution condition, we observe two key properties: (1) the increase in the approximation factor is given by $\frac{1}{\alpha'} - \frac{1}{\alpha} = 2\gamma$ and (2) the length of the horizontal semiaxis of the new outer ellipse is $\exp(\gamma)$. The length of the vertical semiaxis is at least 1, so by the second property we have $\log \mathrm{vol}(\mathcal{E}') - \log \mathrm{vol}(\mathcal{E}) \geq \gamma$. We combine this with the first property to prove that this update satisfies the evolution condition (7).

Finally, we obtain an upper bound on $1/\alpha_n$ from the evolution equation. We have

$$\begin{aligned} {}^1/_{\alpha_n} &= {}^1/_{\alpha_0} + \sum_{t=1}^{n} \left({}^1/_{\alpha_t} - {}^1/_{\alpha_{t-1}}\right) \\ &\overset{(\text{by } 7)}{\leq} 1 + C\sum_{t=1}^{n}(\log \mathrm{vol}(\mathcal{E}_t) - \log \mathrm{vol}(\mathcal{E}_{t-1})) \\ &= 1 + C \log \frac{\mathrm{vol}\,\mathcal{E}_n}{\mathrm{vol}\,\mathcal{E}_0}. \end{aligned}$$

It remains to get an upper bound on $\mathrm{vol}(\mathcal{E}_n)$. We know that $\mathcal{E}_n$ approximates Z, and Z, in turn, is contained in the ball of radius $R(Z)$. Loosely speaking, we get $\mathrm{vol}(\mathcal{E}_n) \approx \mathrm{vol}(Z) \leq R(Z)^d\, \mathrm{vol}(B_2^d)$. Since $\mathcal{E}_0$ is the ball of radius r, $\mathrm{vol}\,\mathcal{E}_0 = r^d\, \mathrm{vol}(B_2^d)$. We conclude that the approximation factor is at most ${}^1/_{\alpha_n} \lessapprox 1 + C\log \frac{R(Z)^d}{r^d} = 1 + O(d \log \frac{R(Z)}{r})$, as desired.

Removing the initialization assumption. Once we have a monotone update rule and guarantee on its approximation factor, we have to convert this to a guarantee where the algorithm does not have access to the initialization.

One natural approach is as follows. Let $d' \leq d$ be the largest timestep for which points $z_1, \ldots, z_{d'+1}$ are in general position. We can compute the John ellipsoid for $\mathrm{conv}(\{z_1, \ldots, z_{d'+1}\})$ and after that apply the monotone update rule guaranteed by Theorem 1.6 to obtain the rounding for every $t \geq d' + 2$, so long as for every such timestep we have $z_t \in \mathrm{aff}(z_1, \ldots, z_{t-1})$.

The principal difficulty in this approach is designing an *irregular update step* that will handle points z_t outside of $\mathrm{aff}(z_1, \ldots, z_{t-1})$; when we add these points the dimensionality of the affine hull increases by 1. We consider the special case where the new point z_t is conveniently located with respect to our previous ellipsoid $\mathcal{E}_{t-1}$ (see Figure 2 for a 2d-picture). Specifically, $\mathcal{E}_{t-1}$ is the unit ball in $\mathrm{span}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d'})$, and the new point $z_t = (0, \ldots, 0, \sqrt{1+2\alpha}), 0, \ldots)$. In z_t, only coordinate $d'+1$ is nonzero. We show that we can design an irregular update step for this special case that makes the new approximation factor ${}^1/_{\alpha_t}$ satisfy ${}^1/_{\alpha_t} = {}^1/_{\alpha_{t-1}} + 1$.

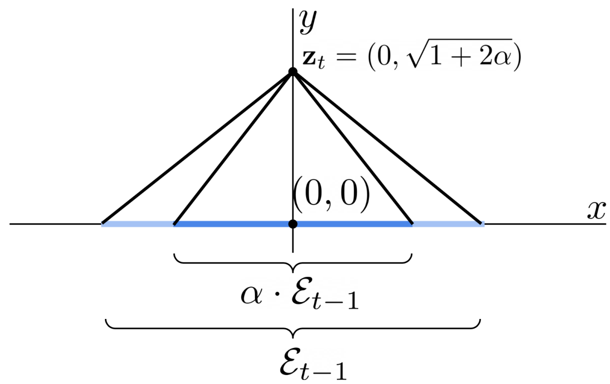

Figure 2: Irregular update step. $\mathcal{E}_{t-1}$ and $\alpha \cdot \mathcal{E}_{t-1}$ are, respectively, the light blue strip on the x-axis and the dark blue strip on the x-axis. $z_t = (0, \sqrt{1+2\alpha})$ is the newly received point.

It turns out that it is sufficient to consider only this special case. To see this, note that we can choose an affine transformation that maps any new point z_t and previous ellipsoid $\mathcal{E}_{t-1}$ to the setting shown in Figure 2. Next, observe that there are at most $d-1$ irregular update steps. This means that the irregular update steps contribute at most an additive $d-1$ to the final approximation factor.

Finally, observe that the inradius of $\mathrm{conv}(\{z_1, \ldots, z_t\})$ is not monotone in t. In particular, it can decrease after each irregular update step. Nonetheless, we can still give a bound on the radius of a ball that our convex body $\mathrm{conv}(z_1, \ldots, z_t)$ contains for all t. This will give us everything we need to apply Theorem 1.6 to this setting, and Theorem 1.7 follows.

Improved bounds on lattices. Finally, we briefly discuss how to remove the aspect ratio dependence in the setting where the input points z_t have coordinates in $[-N, N]$. At a high level, this improvement follows from carefully tracking how the approximation factors of our solutions change after an irregular update step. Following (7), recall that our goal is to analyze (where we write $\alpha_0 = 1$)

$$\sum_{t \geq 1} \frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}}.$$

By (7), we see that for all "regular" updates, we have

$$\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} \lesssim \log\left(\frac{\mathrm{vol}_{d_t}(\mathcal{E}_t)}{\mathrm{vol}_{d_t}(\mathcal{E}_{t-1})}\right),$$

where $d_t = \dim(\mathrm{aff}(z_1, \ldots, z_t))$. Furthermore, as previously mentioned, in our irregular update step, we get

$$\frac{1}{\alpha_t} - \frac{1}{\alpha_{t-1}} = 1.$$

In order to control the sum of the ${}^{1}/_{\alpha_t} - {}^{1}/_{\alpha_{t-1}}$, it remains to bound ${}^{\mathrm{vol}_{d_t}(\mathcal{E}_t)}/_{\mathrm{vol}_{d_{t-1}}(\mathcal{E}_{t-1})}$ for an irregular update step t. We will then get a telescoping upper bound whose last term is the ratio of the volume of the final ellipsoid to the Euclidean ball in the same affine span.

Similarly to the improvements of Woodruff and Yasuda [20] in the integer-valued case, it will turn out that we will be interested in the total product of these volume changes. By carefully tracking these, we will get that this product can be expressed as the determinant of a particular integer-valued matrix. Then, since this matrix has integer entries, the magnitude of its determinant must be at least 1. We then observe that the volume of $\mathcal{E}_n$ after normalizing by the volume of $\mathrm{vol}(B_2^{d_n})$ must be at most $(N\sqrt{d})^{d_n}$, since the length of any vector in this lattice is at most $N\sqrt{d}$. The desired result then follows.

2.3 Coresets for Convex Hull (Theorem 1.9)

We now outline our proof strategy for Theorem 1.9. Our main task is to design an appropriate selection criterion for every new point – in other words, we must check whether a new point z_t is "important enough" to be added to our previous set of points S_{t-1}. We then have to show that this selection criterion yields the approximation guarantee promised by Theorem 1.9.

To design the selection criterion, we run an instance of the algorithm in Theorem 1.7 on the stream. For every new point z_t, we ask two questions – "Does z_t result in an irregular update step? Does it cause $\mathrm{vol}(\mathcal{E}_t)$ to be much larger than $\mathrm{vol}(\mathcal{E}_{t-1})$?" If the answer to any of these questions is affirmative, we add z_t to the coreset. The first question is necessary to obtain even a bounded approximation factor (for example, imagine that the final point z_n results in an irregular update step – then, we must add it). The second question is quite natural, as it ensures that the algorithm adds "important points" – those that necessitate a significant update.

We now observe that at every irregular update step $t_{d'}$ for $d' \le d$ and subsequent timestep $t \ge t_{d'}$ for which there are no irregular update steps in between $t_{d'}$ and t, there exists a translation $\boldsymbol{c}_{d'}$ (which is the center for $\mathcal{E}_{d'}$ that the algorithm maintains) and a value $r_{d'}$ for which we know

$$\begin{aligned}\boldsymbol{c}_{d'} + r_{d'} \cdot \left(B_2^d \cap \mathrm{span}(z_1 - \boldsymbol{c}_{d'}, \ldots, z_{d'} - \boldsymbol{c}_{d'})\right) &\subseteq \mathrm{conv}(z_1, \ldots, z_t) \\ &\subseteq \boldsymbol{c}_C + R_t \cdot B_2^d,\end{aligned}$$

where $\boldsymbol{c}_C$ is the circumcenter of $\mathrm{conv}(\{z_1, \ldots, z_t\})$. The resulting bound on $|S_t|$ follows easily from the above observation and a simple volume argument.

Finally, we obtain the approximation guarantee from noting that for all t, the output of the algorithm from Theorem 1.7 given the first t points is the same as running it only on the points selected by S_t.

2.4 Lower Bound (Theorem 1.10)

Whereas in the upper bound we demonstrated a particular algorithm that satisfies the evolution condition (7), for the lower bound it suffices to show that for any monotone algorithm, there exists an instance of the problem (a sequence of $z_1,\ldots, z_n$) where the algorithm must satisfy the "reverse evolution condition", i.e.

$$\frac{{}^{1}/_{\alpha_t} - {}^{1}/_{\alpha_{t-1}}}{\log \mathrm{vol}(\mathcal{E}_t) - \log \mathrm{vol}(\mathcal{E}_{t-1})} \ge C \tag{8}$$

for some $C > 0$. In analogy to the argument of the upper bound, showing this reverse evolution condition yields a lower bound of the form $\frac{1}{\alpha_n} \ge \widetilde{\Omega}(d \log(\kappa))$. Given any monotone algorithm $\mathcal{A}$, the instance we use is produced by an adversary that repeatedly feeds $\mathcal{A}$ a point that is a constant factor away from the previous ellipsoid.

In order to simplify showing this reverse evolution condition, we use a symmetrization argument. Specifically, by a particular sequence of Steiner symmetrizations, we see that the optimal response of $\mathcal{A}$ can be completely described in two dimensions. Thus, it is sufficient to only show this reverse evolution condition in the two-dimensional case where the previous outer ellipsoid is the unit ball.

This transformed two-dimensional setting is significantly simpler to analyze. Specifically, we can assume that the point given by the adversary is always $2\boldsymbol{e}_1$. The rest of the argument proceeds by cases, again using two-dimensional Euclidean geometry. On a high level, the constraints placed on the new outer and inner ellipsoid by the monotonicity condition force the update of $\mathcal{A}$ to satisfy the reverse evolution condition.

3 PRELIMINARIES

3.1 Notation

We denote the standard Euclidean norm of a vector $\boldsymbol{v}$ by $\|\boldsymbol{v}\|$. We write $\mathbf{Diag}(a_1, \ldots, a_d)$ to mean the $d \times d$ diagonal matrix whose diagonal entries are $a_1, \ldots, a_d$.

Denote the ℓ_2-unit ball by $B_2^d = \left\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \le 1\right\}$. We use ∂S for the boundary of an arbitrary set S. We use natural logarithms unless otherwise specified.

In this paper, we work extensively with ellipsoids. We will always assume that all ellipsoids and balls we consider are centered at the origin. We use the following representation of ellipsoids: for a non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, let $\mathcal{E}_{\boldsymbol{A}} := \{\boldsymbol{x} : \|\boldsymbol{A}\boldsymbol{x}\| \le 1\}$. In other words, the matrix $\boldsymbol{A}$ defines an bijective linear map satisfying $\boldsymbol{A}\mathcal{E}_{\boldsymbol{A}} = B_2^d$. Every full-dimensional ellipsoid (centered at the origin) has such a representation. We note that this representation is not unique as matrices $\boldsymbol{A}$ and $\boldsymbol{M}\boldsymbol{A}$ define the same ellipsoid if matrix $\boldsymbol{M}$ is orthogonal (since $\|\boldsymbol{A}\boldsymbol{v}\| = \|\boldsymbol{M}\boldsymbol{A}\boldsymbol{v}\|$ for every vector $\boldsymbol{v}$).

3.2 Geometry

We restate the well-known result that five points determine an ellipse. This is usually phrased for conics, but for nondegenerate ellipses the usual condition that no three of the five points are colinear is vacuously true.

Claim 3.1 (Five points determine an ellipse). *Let $c_1+\partial\mathcal{E}_1, c_2+\partial\mathcal{E}_2$ be two ellipses in $\mathbb{R}^2$. If they intersect at five distinct points, then $c_1+\partial\mathcal{E}_1$ and $c_2+\partial\mathcal{E}_2$ are the same.*

The following claim, that every full-rank ellipsoid (i.e. an ellipsoid whose span has full dimension) can be represented by a positive definite matrix, follows from looking at the singular value decomposition of A.

Claim 3.2. *Let $\mathcal{E} \subseteq \mathbb{R}^d$ be a full-rank ellipsoid. Then there exists $A \succ 0$ such that $\mathcal{E} = \mathcal{E}_A$.*

We also have the standard result relating volume and determinants, which follows from observing $A\mathcal{E}_A = B_2^d$.

Claim 3.3. *Let $A \succ 0$. Then $\mathrm{vol}(\mathcal{E}_A) = \det(A^{-1})\,\mathrm{vol}(B_2^d)$.*

4 STREAMING ELLIPSOIDAL ROUNDING

In this section, we describe the key components of the algorithms used to prove 1.6 and 1.7. The full details are given in the full version of the paper.

4.1 Monotone Algorithms Solve Problem 1.1

To design algorithms to solve the streaming ellipsoidal rounding problem, we first show that any monotone algorithm gives a valid solution. We let $c_0 \in \mathbb{R}^d$ and $r_0 \geq 0$ be given so that $c_0 + r_0 \cdot B_2^d \subseteq Z$, and denote the initial ellipsoid as $\mathcal{E}_0 = r_0 \cdot B_2^d$. Note that r_0 need not be the inradius, although it is upper bounded by the inradius.

If we had for each intermediate step t that

$$c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \mathrm{conv}(z_1, \dots z_t) \subseteq c_t + \mathcal{E}_t$$

then clearly any algorithm that satisfies this would give a valid final solution as well. However, in intermediate steps it is not clear that $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \mathrm{conv}(z_1, \dots z_t)$, due to the initialization of $c_0 + \mathcal{E}_0$ in our monotone algorithm framework. Instead, we relax this invariant to $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \mathrm{conv}(\{z_1, \dots z_t\} \cup (c_0 + \mathcal{E}_0))$, which still suffices to produce a valid final solution.

Claim 4.1. *To solve Problem 1.1, it suffices for the sequence of ellipsoids $c_i + \mathcal{E}_i$ and scalings α_i to satisfy the invariants of Definition 2.1.*

Proof. First, we argue that $\mathrm{conv}(z_1, \dots, z_n) \subseteq c_n + \mathcal{E}_n$. As $\mathcal{E}_n$ is an ellipsoid and therefore a convex set, it suffices to show $\{z_1, \dots, z_n\} \subseteq c_n + \mathcal{E}_n$. We actually argue by induction that for all $0 \leq t \leq n$. $\{z_1, \dots, z_t\} \subseteq c_t + \mathcal{E}_t$. This is vacuously true for $t = 0$. At each step $t > 0$ the inductive hypothesis gives $\{z_1, \dots, z_{t-1}\} \subseteq c_{t-1} + \mathcal{E}_{t-1}$, and thus by (5) we have $\{z_1, \dots, z_t\} \subseteq c_t + \mathcal{E}_t$.

Now, we argue that $c_n + \alpha_n \cdot \mathcal{E}_n \subseteq \mathrm{conv}(z_1, \dots, z_n)$. We show by induction that $c_t + \alpha_t \cdot \mathcal{E}_t \subseteq \mathrm{conv}(\{z_1, \dots, z_t\} \cup (c_0 + \mathcal{E}_0))$ for all $0 \leq t \leq n$. This is sufficient as $\mathrm{conv}(\{z_1, \dots, z_n\} \cup (c_0 + \mathcal{E}_0)) = Z$. The case for $t = 0$ is trivial. For $t > 0$, the inductive hypothesis gives $c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1} \subseteq \mathrm{conv}(\{z_1, \dots, z_{t-1}\} \cup (c_0 + \mathcal{E}_0))$, and by (6) we have

$$\begin{aligned} c_t + \alpha_t \cdot \mathcal{E}_t &\subseteq \mathrm{conv}((c_{t-1} + \alpha_{t-1} \cdot \mathcal{E}_{t-1}) \cup \{z_i\}) \\ &\subseteq \mathrm{conv}(\{z_1, \dots, z_t\} \cup (c_0 + \mathcal{E}_0)) \end{aligned}$$

as desired. □

4.2 Special Case

In light of Claim 4.1, our strategy is to design an algorithm that preserves the invariants given in Definition 2.1. This algorithm can be thought of as an *update rule* that, given the previous outer and inner ellipsoids $c_{t-1} + \mathcal{E}_{t-1}, c_{t-1} + \alpha_{t-1}\mathcal{E}_{t-1}$ and next point z_t, produces the next outer and inner ellipsoids $c_t + \mathcal{E}_t, c_t + \alpha_t\mathcal{E}_t$.

It is in fact sufficient to consider the simplified case where the previous outer ellipsoid is the unit ball, and the previous inner ellipsoid is some scaling of the unit ball; we will show this in Section 4.3. We can further specialize by considering only the two-dimensional case $d = 2$. We will later show that the high-dimensional case is not much different, as all the relevant sets $c_{t_1} + \mathcal{E}_{t-1}, c_t + \mathcal{E}_t$ and $\mathrm{conv}(\alpha \cdot \mathcal{E}_{t-1} \cup \{z_t\})$ form bodies of revolution about the axis through c_{t-1} and z_t.

We now describe our two-dimensional update rule. In order to simplify notation, we will let α be the previous scaling α_{t-1}, and α' be the next scaling α_t. We will assume that $\alpha \leq 1/2$ to simplify the analysis of our update rule; this will not affect the quality of our final approximation as this update rule will only be used in the "large approximation factor" regime. We will also overload notation by writing $c + \mathcal{E}$ even when c is a scalar to mean $(c, 0) + \mathcal{E}$. We can describe the previous outer ellipsoid $\mathcal{E}$ with the equation $x^2 + y^2 \leq 1$, and the previous inner ellipsoid $\alpha\mathcal{E}$ with $x^2 + y^2 \leq \alpha^2$. We define the next outer and inner ellipsoids $c + \mathcal{E}'$, $c + \alpha'\mathcal{E}'$ as

$$\underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 \leq 1,}_{c+\mathcal{E}'} \qquad \underbrace{\frac{1}{a^2}(x-c)^2 + \frac{1}{b^2}y^2 \leq \alpha'^2}_{c+\alpha'\mathcal{E}'}$$

where we use parameters

$$\left.\begin{aligned} a &= \exp(\gamma) \\ b &= 1 + \frac{\alpha - \alpha'}{2} \\ c &= -\alpha + \alpha' \cdot a \\ \alpha' &= \frac{1}{\frac{1}{\alpha} + 2\gamma} \end{aligned}\right\} \tag{9}$$

We will let z be the rightmost point of $c + \mathcal{E}'$, so that $z = (c + a, 0)$. Eventually, we will choose γ so that z coincides with z_t, the point received in the next iteration. In Section 4.4, these parameters $a(\gamma), b(\gamma), c(\gamma), \alpha'(\gamma)$ will be used as functions of the parameter $\gamma \geq 0$. However, we will not yet explicitly specify γ, so in this section these parameters can be thought of as constants for some fixed γ. This update rule is pictured in Figure 1.

We first collect a few straightforward properties of this update rule.

Claim 4.2. *The parameters in the setup (9) satisfy the following.*

(1) $\frac{1}{\alpha'} = \frac{1}{\alpha} + 2\gamma$
(2) $b \geq 1$
(3) $c \geq 0$
(4) $c + \alpha' \cdot a \geq \alpha$

Before proving these properties, we provide geometric interpretations. Intuitively, (1) means that γ is proportional to the increase in the approximation factor at this step, a fact that we use when analyzing the general-case algorithm. (2) means that the outer ellipsoid grows on every axis; and (3) means that the centers of the next ellipsoids are to the right of the y-axis, i.e. the centers of the next ellipsoids are further towards v than those of the previous ellipsoids. The rightmost point of $c + \alpha'\mathcal{E}'$ is $c + \alpha' \cdot a$, so (4) shows that this point is to the right of the rightmost point of $\alpha \cdot \mathcal{E}$.

We now prove Claim 4.2.

Proof of Claim 4.2. (1) is clear from rearranging the definition of α'. From (1) we also have $\alpha' \leq \alpha$, so that (2) follows immediately.

For (3), observe that $\frac{\alpha}{\alpha'} = 1 + 2\gamma\alpha$. When $\alpha \leq 1/2$, this means

$$\frac{\alpha}{\alpha'} \leq 1 + \gamma \leq \exp(\gamma) = a \tag{10}$$

using $1 + x \leq e^x$. By definition of c, $\alpha/\alpha' \leq a$ is equivalent to $c \geq 0$.

To show (4), by definition we have that $c + \alpha' \cdot a = -\alpha + 2\alpha' a$. Thus showing $c + \alpha' \cdot a \geq \alpha$ is equivalent to showing that $\alpha' a \geq \alpha$, which is equivalent to the inequality in (10). □

As Figure 1 depicts, the update step we defined satisfies the invariants in Definition 2.1 and so is monotone; in the rest of this section we make this picture formal. To start, we consider the invariant concerning outer ellipsoids; we will show that $\mathcal{E} \subseteq c + \mathcal{E}'$. For now we can think of z as replacing z_t , and clearly $z \in c + \mathcal{E}'$, so if we show that $\mathcal{E} \subseteq c + \mathcal{E}'$, then $\mathrm{conv}(\mathcal{E} \cup \{z\}) \subseteq c + \mathcal{E}'$ as well since $c + \mathcal{E}'$ is convex.

Claim 4.3. *We have $\mathcal{E} \subseteq c + \mathcal{E}'$.*

Proof. First, observe that $\mathcal{E} \subseteq \mathcal{E}'$ because both axes of $\mathcal{E}'$ have greater length than those of $\mathcal{E}$: $a \geq 1$ by definition, and $b \geq 1$ from Claim 4.2-(2). Now, we translate $\mathcal{E}'$ to the right until it touches $\mathcal{E}$ at two points. We call this translated ellipse $c_r + \mathcal{E}'$, as shown in Figure 3. Observe that as long as $c \leq c_r$, we have $\mathcal{E} \subseteq c + \mathcal{E}'$. We now determine c_r.

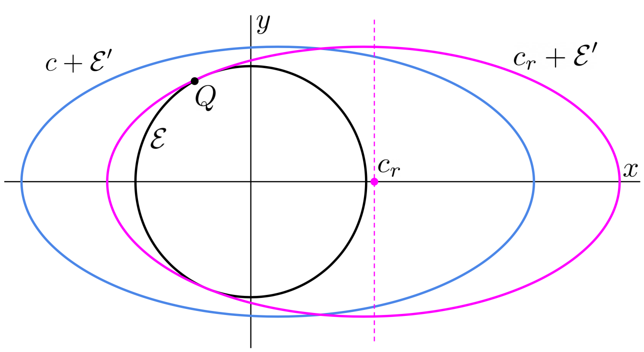

Figure 3: Outer ellipses of the update step. As before, $\mathcal{E}$ is the black circle and $c + \mathcal{E}'$ is the blue ellipse. $c_r + \mathcal{E}'$ is the magenta ellipse, with its center at c_r and the dotted magenta line showing the position of c_r along the x-axis. c_r is defined so $c_r + \mathcal{E}'$ and $\mathcal{E}$ are tangent at two points. Q is one of these two tangent points.

First, note points on the boundary of $c_r + \mathcal{E}'$ are described by the equation

$$\frac{(x - c_r)^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{11}$$

Let $Q = (x', y')$ be the point of intersection between $\mathcal{E}$ and $c_r + \mathcal{E}'$ where $y' > 0$. Since Q is on the boundary of both ellipses, the vectors $\left(\frac{2(x'-c_r)}{a^2}, \frac{2y'}{b^2}\right)$ and $(2x', 2y')$, which are the normal vectors at Q of $c_r + \mathcal{E}'$ and $\mathcal{E}$ respectively, must be parallel. Thus $\frac{4(x'-c_r)}{a^2} \cdot y' = \frac{4y'x'}{b^2}$, which simplifies to

$$x' = \frac{c_r}{1 - \frac{a^2}{b^2}}. \tag{12}$$

At this point we have a system of three equations relating (x', y') and c_r: (12), Q lying on $\mathcal{E}$, and Q satisfying (11). We now solve this system to find c_r. To start, we expand (11) into $x'^2 - 2x'c_r + c_r^2 + y'^2\frac{a^2}{b^2} = a^2$, which we rewrite into $x'^2\frac{a^2}{b^2} + x'^2\left(1 - \frac{a^2}{b^2}\right) - 2x'c_r + c_r^2 + y'^2\frac{a^2}{b^2} = a^2$. As Q lies on $\mathcal{E}$, this becomes $x'^2\left(1 - \frac{a^2}{b^2}\right) - 2x'c_r + c_r^2 + \frac{a^2}{b^2} = a^2$. Substituting in (12), we get $\frac{c_r^2}{1-\frac{a^2}{b^2}} - 2\frac{c_r^2}{1-\frac{a^2}{b^2}} + c_r^2 + \frac{a^2}{b^2} = a^2$ Simplifying, we have $c_r^2\left(1 - \frac{b^2}{b^2-a^2}\right) = a^2\left(1 - \frac{1}{b^2}\right)$, i.e. $c_r^2 = \frac{b^2-1}{b^2}(a^2 - b^2)$. To complete the proof of Claim 4.3, it suffices to show

$$c^2 \leq \frac{b^2 - 1}{b^2}(a^2 - b^2).$$

This is shown in Claim 7.5 in the full version of the paper. □

Now, we move on to the inner ellipsoid invariant of Definition 2.1. In particular, we will argue that $c + \alpha'\mathcal{E}' \subseteq \mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$. On a high level, we show this by arguing that the boundary of $c + \alpha'\mathcal{E}'$ does not intersect the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$, except at points of tangency.

We can split the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$ into two pieces: the part that intersects with the boundary of $\alpha\mathcal{E}$, which is an arc of the boundary of $\alpha\mathcal{E}$; and the remainder, which can described as two line segments connecting z to that arc. In particular, there are two lines that go through z and are tangent to $\alpha\mathcal{E}$, one of which we call line L, and the other line is the reflection of L across the x-axis. We define P_1 and P_2 as the tangent points of these lines to $\alpha\mathcal{E}$. Then, the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$ consists of an arc P_1P_2 and the line segments $\overline{P_1z}, \overline{P_2z}$. This is illustrated in Figure 4. Note that at this point it is possible a priori for the arc P_1P_2 that coincides with the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$ to be either the major or minor arc; we will later show it must be the major arc. We will take L to be the line whose tangent point to $\alpha\mathcal{E}$, P_1, is above the x-axis, though this choice is arbitrary due to symmetry across the x-axis.

We first show that $c+\alpha'\mathcal{E}'$ does not intersect $\overline{P_1z}$ and $\overline{P_2z}$, except possibly at points of tangency. In fact, we show a slightly stronger statement, in similar fashion to Claim 4.3.

Claim 4.4. *$c + \alpha'\mathcal{E}'$ lies inside the angle $\angle P_1zP_2$.*

Proof. We translate $c + \alpha'\mathcal{E}'$ to the right until it touches L (and, by symmetry, $\overline{P_2z}$). We call this translated ellipse $c_+ + \alpha'\mathcal{E}'$, as shown in Figure 5. (Formally, the center c_+ can be described not as a translation from some other ellipse, but as c_+ such that $c_+ + \alpha'\mathcal{E}'$ intersects L at one point). Observe that if $c \leq c_+$, then $c + \alpha'\mathcal{E}'$ lies inside the angle $\angle P_1zP_2$. We now determine c_+.

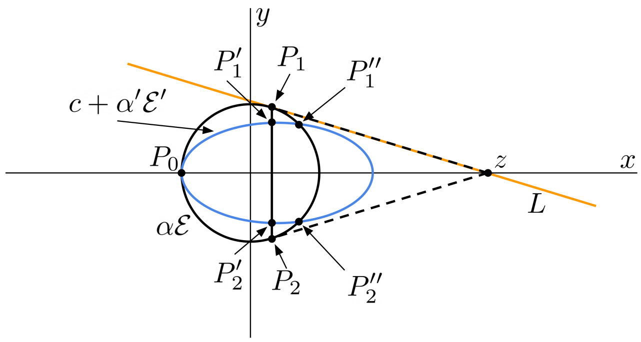

Figure 4: Inner ellipses of the update step. As before, $\alpha\mathcal{E}$ is the black circle and $c+\alpha'\mathcal{E}'$ is the blue ellipse. P_0 is the shared leftmost point of $\alpha\mathcal{E}$ and $c+\alpha'\mathcal{E}'$. There are two lines through z that are tangent to $\alpha\mathcal{E}$, one of which we call L and pictured in orange. We call the tangent points P_1 and P_2. The line segments $\overline{P_1 z}, \overline{P_2 z}$ are the dotted black lines. P_1' and P_2' are the two points of intersection between $\partial(c+\alpha'\mathcal{E}')$ and the line segment $\overline{P_1 P_2}$. P_1'' and P_2'' are the two points of intersection between $\partial(c+\alpha'\mathcal{E}')$ and $\partial\alpha\mathcal{E}$ to the right of the y-axis. Note that P_2, P_2', P_2'' are the reflections of P_1, P_1', P_1'' across the x-axis.

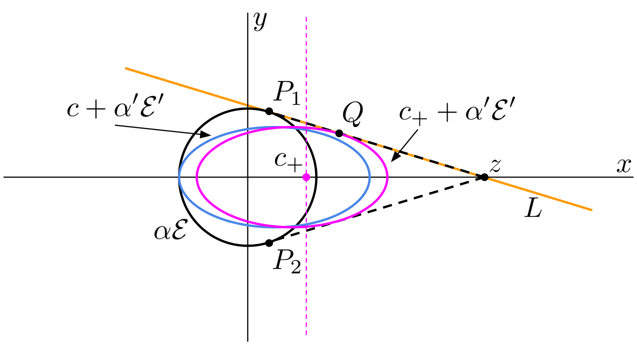

Figure 5: Inner ellipses of the update step. As before, $\alpha\mathcal{E}$ is the black circle, $c + \alpha'\mathcal{E}$ is the blue ellipse, L is the orange line through z and tangent to $\alpha\mathcal{E}$, P_1 and P_2 are the tangent points on the lines through z tangent to $\alpha\mathcal{E}$, and $\overline{P_1 z}, \overline{P_2 z}$ are the dotted black lines. $c_+ + \alpha'\mathcal{E}'$ is the magenta ellipse, with its center at c_+ and magenta dotted line showing its position on the x-axis. c_+ is defined so that $c_+ + \alpha'\mathcal{E}'$ is tangent to $\overline{P_1 z}$ and $\overline{P_2 z}$, with Q as the tangent point of $c_+ + \alpha'\mathcal{E}'$ and $\overline{P_1 z}$.

The equation defining L is

$$\underbrace{\frac{1}{c+a}}_{\ell_1} \cdot x + \underbrace{\sqrt{\frac{1}{\alpha^2} - \frac{1}{(c+a)^2}}}_{\ell_2} \cdot y = 1,$$

where we define ℓ_1, ℓ_2 as the coefficients for x and y. Observe that z is on L, and L is tangent to $\alpha\mathcal{E}$ at P_1, which has coordinates

$$P_1 = \left(\frac{\alpha^2}{c+a}, \alpha^2\sqrt{\frac{1}{\alpha^2} - \frac{1}{(c+a)^2}}\right). \tag{13}$$

Tangency can be confirmed by checking that P_1 is parallel to (ℓ_1, ℓ_2), the normal vector defining L.

Let $Q = (x', y')$ be the point of intersection of L and $c_+ + \alpha'\mathcal{E}$, there are three properties that define Q. First it lies on the boundary of $c_+ + \alpha'\mathcal{E}$, so it satisfies

$$\frac{(x' - c_+)^2}{a^2} + \frac{y'^2}{b^2} = \alpha'^2. \tag{14}$$

Second, at Q the normal vectors for the equations defining $c_+ + \alpha'\mathcal{E}$ and L are parallel, i.e. $(\frac{2(x-c_+)}{a^2}, \frac{2y}{b^2})$ is parallel to (ℓ_1, ℓ_2). So

$$\frac{(x' - c_+)}{a^2}\ell_2 = \frac{y'}{b^2}\ell_1. \tag{15}$$

Finally, Q lies on L, so we have $\ell_1 x' + \ell_2 y' = 1$. Solving this for y', we get

$$y' = \frac{1 - \ell_1 x'}{\ell_2}. \tag{16}$$

These three equations form a system for x', y' and c_+, which we now solve to find c_+. Taking the square of (15) and rearranging gives $\frac{y'^2}{b^2} = \frac{b^2(x'-c_+)^2\ell_2^2}{a^4\ell_1^2}$. Substituting this into (14), we get $\frac{(x'-c_+)^2}{a^2} + \frac{b^2(x'-c_+)^2\ell_2^2}{a^4\ell_1^2} = \alpha'^2$. Now, defining $r := \frac{a^2\ell_1^2}{b^2\ell_2^2}$, we group the terms of this equation into the form

$$(x' - c_+)^2 \cdot \frac{1}{a^2}\left(1 + \frac{1}{r}\right) = \alpha'^2. \tag{17}$$

We substitute (16) into (15) to get $\frac{x'-c_+}{a^2}\ell_2 = \frac{\ell_1}{b^2}\frac{1-x'\ell_1}{\ell_2}$. Grouping for x' and rearranging yields

$$x' - c_+ = \frac{r}{1+r}\left(\frac{1}{\ell_1} - c_+\right). \tag{18}$$

Next, we substitute (18) into (17), and get after some cancellation

$$\left(\frac{1}{\ell_1} - c_+\right)^2 = \alpha'^2 a^2 \cdot \frac{1+r}{r}.$$

Observe on the left hand side that $\frac{1}{\ell_1} - c_+ = c + a - c_+$. Clearly the center c_+ must be to the left of z, so this must be non-negative. Hence after taking the positive square root, we obtain $c_+ = c + a - \alpha' \cdot a\sqrt{\frac{1+r}{r}}$. It remains to show that $c \le c_+$, or equivalently that

$$a - \alpha' \cdot a\sqrt{\frac{1+r}{r}} \ge 0.$$

We verify this in Claim 7.6 in the full version of the paper. □

Now, we build on the previous claim to show the inner ellipsoid invariant.

Claim 4.5. *We have $c + \alpha' \cdot \mathcal{E}' \subseteq \operatorname{conv}(\alpha \cdot \mathcal{E} \cup \{z\})$.*

Proof. We will argue that the boundary of $c + \alpha'\mathcal{E}'$ does not intersect the boundary of $\operatorname{conv}(\alpha\mathcal{E} \cup \{z\})$, except at points of tangency. This is sufficient to establish the claim, as Claim 4.4 shows that $c + \alpha'\mathcal{E}'$ is internal to $\angle P_1 z P_2$, and so if $c + \alpha'\mathcal{E}'$ does not intersect the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$, $c + \alpha'\mathcal{E}'$ must lie inside of, or be disjoint from $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$. Since the leftmost points of $\alpha\mathcal{E}$ and $c + \alpha'\mathcal{E}'$ coincide, $c + \alpha'\mathcal{E}'$ must then lie inside of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$. Recall that the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$ consists of the arc P_1P_2 and the line segments $\overline{P_1 z}, \overline{P_2 z}$. Claim 4.4 already shows that the boundary of $c + \alpha'\mathcal{E}'$ does not intersect $\overline{P_1 z}$ and $\overline{P_2 z}$, so we only need to show that the boundary of $c + \alpha'\mathcal{E}'$ does not intersect the arc P_1P_2.

To do this, we start by enumerating the points of intersection of $\partial\alpha\mathcal{E}$ and $\partial(c + \alpha'\mathcal{E}')$, recalling that P_1P_2 is an arc of $\partial\alpha\mathcal{E}$. Observe that the leftmost points of $\alpha\mathcal{E}$ and $c+\alpha'\mathcal{E}'$ coincide, as the leftmost point of $c + \alpha'\mathcal{E}'$ is $c - \alpha' \cdot a = -\alpha$ by definition; we call this point P_0. P_0 is a point of tangency and hence has intersection multiplicity 2, because the centers of $\alpha\mathcal{E}$ and $c + \alpha' \cdot \mathcal{E}'$ both lie on the x-axis.

Next, we argue for the existence of two more distinct intersection points P_1'', P_2'' as depicted in Figure 4. The leftmost point of $c+\alpha'\mathcal{E}'$ is $(-\alpha, 0)$, and the rightmost point is $c + \alpha'$, which by Claim 4.2-(4) is to the right of $(\alpha, 0)$, the rightmost point of $\alpha\mathcal{E}$. Thus, by lying on $\partial\alpha\mathcal{E}$, P_1, P_2 lie between the leftmost and rightmost points of $c + \alpha'\mathcal{E}'$, and so $c + \alpha'\mathcal{E}'$ intersects the line through P_1 and P_2. Further, by Claim 4.4, as $c + \alpha'\mathcal{E}'$ lies in the angle $\angle P_1 \boldsymbol{v} P_2$, $c+\alpha'\mathcal{E}'$ actually intersects the line segment $\overline{P_1P_2}$. Observe that this intersection happens at two distinct points, which we call P_1' and P_2'. Both points are inside of $\alpha\mathcal{E}$, yet $\partial(c + \alpha'\mathcal{E}')$ is a continuous path that connects both to the rightmost point of $c+\alpha'\mathcal{E}'$, which is outside of $\alpha\mathcal{E}$. Thus $\partial(c+\alpha'\mathcal{E}')$ intersects $\partial\alpha\mathcal{E}$ at two more distinct points, which we call P_1'' and P_2''.

Now, we argue that P_1'' and P_2'' lie on the minor arc P_1P_2. First, observe that the arc P_1P_2 containing P_0 is the major arc. This is because P_1 lies to the right of the y-axis, as determined in (13); and by symmetry so does P_2. This also implies that major arc P_1P_2 is the arc with which the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$ coincides. P_1' and P_2' are colinear with P_1 and P_2, and as P_1'' and P_2'' are to the right of P_1' and P_2', this implies that they must lie on the minor arc P_1P_2.

Counting all the intersection points of $\partial\alpha\mathcal{E}$ and $\partial(c + \alpha'\mathcal{E}')$, we have P_0 (with multiplicity 2) and P_1'' and P_2'' (both with multiplicity 1); with total multiplicity 4. Using Claim 3.1, it is impossible for them to have another intersection point without both ellipses being the same. Thus $\partial(c + \alpha'\mathcal{E}')$ cannot intersect the major arc P_1P_2 except at P_0, and so except at points of tangency the boundary of $c + \alpha'\mathcal{E}'$ does not intersect the boundary of $\mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$. ◻

4.3 Generalizing to High Dimension and Arbitrary Previous Ellipsoids

Now that we have demonstrated the invariants of Definition 2.1 for the special two-dimensional case where the previous ellipsoid is the unit ball, we generalize slightly to higher dimensions. However, we first still assume the previous ellipsoid is the unit ball.

Using the parameters as defined in (9), we will let $\mathcal{E} = B_2^d$, and define the boundary of $\mathcal{E}'$ as $\frac{1}{a^2}(\boldsymbol{x}_1 - c)^2 + \frac{1}{b^2}\boldsymbol{x}_2^2 + \ldots + \frac{1}{b^2}\boldsymbol{x}_d^2 = 1$. Similarly to before, we let $\boldsymbol{z} = (c + a, 0, 0, \ldots, 0) \in \mathbb{R}^d$, the furthest point of $c + \mathcal{E}'$ in the positive direction of the x_1-axis.

Now, we argue that the invariants of Definition 2.1 still hold in this setting.

Claim 4.6. *The inner and outer ellipsoid invariants hold in this setting:*

(1) $\mathcal{E} \subseteq c \cdot \boldsymbol{e}_1 + \mathcal{E}'$
(2) $c \cdot \boldsymbol{e}_1 + \alpha'\mathcal{E}' \subseteq \mathrm{conv}(\alpha\mathcal{E} \cup \{z\})$

Proof. Observe that $\mathcal{E}$, $c\cdot\boldsymbol{e}_1+\mathcal{E}'$, $c\cdot\boldsymbol{e}_1+\alpha'\mathcal{E}'$, and $\mathrm{conv}(\alpha\mathcal{E}\cup\{z\})$ are all bodies of revolution about the x_1-axis, with their cross-sections given by their counterparts in Section 4.2. As Claim 4.3 and Claim 4.5 hold for these cross sections, the set containments hold for the bodies of revolution as well. ◻

We further generalize to the case where the previous ellipsoid is arbitrary. In particular, let $\boldsymbol{c}^\circ + \mathcal{E}$ be the previous ellipsoid, with a vector $\boldsymbol{c}^\circ \in \mathbb{R}^d$ and $\mathcal{E} = \mathcal{E}_{\boldsymbol{A}}$ for non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$. Let $\boldsymbol{z}^\circ \in \mathbb{R}^d$ be an arbitrary vector, representing the next point received. We let $\boldsymbol{u} = \boldsymbol{A}(\boldsymbol{z}^\circ - \boldsymbol{c}^\circ)$, and $\boldsymbol{W} \in \mathbb{R}^{d\times d}$ be an orthogonal matrix with $\boldsymbol{w} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$ as its first column (e.g. by using as its columns an orthonormal basis containing $\boldsymbol{w}$). We define the next outer ellipsoid as $\boldsymbol{c}^\circ + c\boldsymbol{A}^{-1}\boldsymbol{w} + \mathcal{E}'$ for $\mathcal{E}' = \mathcal{E}_{\boldsymbol{W}\boldsymbol{D}\boldsymbol{W}^\top\boldsymbol{A}}$, with $\boldsymbol{D} = \mathbf{Diag}\left(\frac{1}{a^2}, \frac{1}{b^2}, \ldots, \frac{1}{b^2}\right)$. Observe that $\boldsymbol{z} = \boldsymbol{c}^\circ + (c + a)\boldsymbol{A}^{-1}\boldsymbol{w}$ is the furthest point of $\boldsymbol{c}^\circ + c\boldsymbol{A}^{-1}\boldsymbol{w} + \mathcal{E}'$ from the previous center $\boldsymbol{c}^\circ$ towards $\boldsymbol{z}^\circ$.

This setup works to preserve the key invariants, as we see in the next claim.

Claim 4.7. *The inner and outer ellipsoid invariants hold in this setting:*

(1) $\boldsymbol{c}^\circ + \mathcal{E} \subseteq \boldsymbol{c}^\circ + c\boldsymbol{A}^{-1}\boldsymbol{w} + \mathcal{E}'$
(2) $\boldsymbol{c}^\circ + c\boldsymbol{A}^{-1}\boldsymbol{w} + \alpha'\mathcal{E}' \subseteq \mathrm{conv}((\boldsymbol{c}^\circ + \alpha\mathcal{E}) \cup \{z\})$

Proof. We translate both set inclusions by $-\boldsymbol{c}^\circ$, then apply the nonsingular linear transformation $\boldsymbol{W}^\top\boldsymbol{A}$. Observe that the set inclusions we wish to prove hold if and only if the transformed ones do. Noting that $\boldsymbol{W}^\top\boldsymbol{A}\mathcal{E}' = \mathcal{E}_{\boldsymbol{W}\boldsymbol{D}}$, the transformed set inclusions are $\mathcal{E}_{\boldsymbol{W}} \subseteq c\cdot\boldsymbol{e}_1+\mathcal{E}_{\boldsymbol{W}\boldsymbol{D}}$ and $c\cdot\boldsymbol{e}_1+\alpha'\mathcal{E}_{\boldsymbol{W}\boldsymbol{D}} \subseteq \mathrm{conv}(\alpha\mathcal{E}_{\boldsymbol{W}}\cup\{(c+a)\cdot\boldsymbol{e}_1\})$. However, since $\boldsymbol{W}$ is an orthogonal matrix, $\mathcal{E}_{\boldsymbol{W}} = B_2^d$ and $\mathcal{E}_{\boldsymbol{W}\boldsymbol{D}} = \mathcal{E}_{\boldsymbol{D}}$, and so the inclusions are exactly those shown in Claim 4.6. ◻

Choosing γ correctly in (9) ensures that $\boldsymbol{z} \in \boldsymbol{c}^\circ + c\boldsymbol{A}^{-1}\boldsymbol{w} + \mathcal{E}'$ coincides with $\boldsymbol{z}^\circ$, as stated in the upcoming claim. This can be seen by looking at the definition of $\boldsymbol{z}$.

Claim 4.8. *If γ is chosen so that $c + a = \|\boldsymbol{u}\|$, then $\boldsymbol{z} = \boldsymbol{z}^\circ$.*

4.4 General Update Step

In this section, in Algorithm 1 we give the general update step, which is the primary primitive for the algorithm that solves Problem 1.1. The analysis of this step builds on that of the previous sections.

In Lines 3, 4 and 5, we use the definition of $a(\gamma), b(\gamma), c(\gamma), \alpha'(\gamma)$ from (9), substituting α_{t-1} for α. Although the update step does not explicitly mention ellipsoids, we use $\mathcal{E}_t = \mathcal{E}_{\boldsymbol{A}_t}$ so that at iteration t the next outer and inner ellipsoids are $\boldsymbol{c}_t + \mathcal{E}_{\boldsymbol{A}_t}$ and $\boldsymbol{c}_t + \alpha_t\mathcal{E}_{\boldsymbol{A}_t}$, respectively.

Observe also that if in iteration t we let $\boldsymbol{W} \in \mathbb{R}^{d\times d}$ be an orthogonal matrix with $\boldsymbol{w}$ as its first column, we can write

$$\hat{\boldsymbol{A}} = \boldsymbol{W} \cdot \mathbf{Diag}\left(\frac{1}{a(\gamma_t^\star)}, \frac{1}{b(\gamma_t^\star)}, \cdots, \frac{1}{b(\gamma_t^\star)}\right) \cdot \boldsymbol{W}^\top. \tag{19}$$

Algorithm 1 Full update step $\mathcal{A}^{\mathrm{full}}$

Input: $\boldsymbol{A}_{t-1} \in \mathbb{R}^{d\times d}, \boldsymbol{c}_{t-1} \in \mathbb{R}^d, \alpha_{t-1} \in [0, \frac{1}{2}], \boldsymbol{z}_t \in \mathbb{R}^d$

Output: $\boldsymbol{A}_t \in \mathbb{R}^{d\times d}, \boldsymbol{c}_t \in \mathbb{R}^d, \alpha_t \in [0, \alpha_{t-1}]$

1: Let $\boldsymbol{u} = \boldsymbol{A}_{t-1}(\boldsymbol{z}_t - \boldsymbol{c}_{t-1}), \boldsymbol{w} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$

2: **if** $\|\boldsymbol{u}\| > 1$ **then**

3: Let $\gamma_t^\star$ be such that $a(\gamma_t^\star) + c(\gamma_t^\star) = \|\boldsymbol{u}\|$

4: $\hat{\boldsymbol{A}} = \frac{1}{b(\gamma_t^\star)} \boldsymbol{I}_d + \left(\frac{1}{a(\gamma_t^\star)} - \frac{1}{b(\gamma_t^\star)}\right) \boldsymbol{w}\boldsymbol{w}^\top$

5: **return** $\boldsymbol{A}_t = \hat{\boldsymbol{A}} \cdot \boldsymbol{A}_{t-1}, \boldsymbol{c}_t = \boldsymbol{c}_{t-1} + c(\gamma_t^\star)\boldsymbol{A}_{t-1}^{-1}\boldsymbol{w}, \alpha_t = \alpha'(\gamma_t^\star)$

6: **else**

7: **return** $\boldsymbol{A}_t = \boldsymbol{A}_{t-1}, \boldsymbol{c}_t = \boldsymbol{c}_{t-1}, \alpha_i = \alpha_{t-1}$

Now, we argue that this algorithm satisfies the invariants defined in Definition 2.1. This argument is essentially the observation that the update step in the algorithm is the one analyzed in Claim 4.7.

Claim 4.9. *Algorithm 1 is a monotone update; i.e., it satisfies the invariants in Definition 2.1.*

Proof. If $\|\boldsymbol{u}\| \leq 1$, then $\boldsymbol{z}_i \in \boldsymbol{c}_n + \mathcal{E}_n$ and the inner and outer ellipsoids are not updated, so the invariants clearly hold. Otherwise, we apply Claim 4.7 and Claim 4.8 setting $\boldsymbol{A} = \boldsymbol{A}_{t-1}, \boldsymbol{c}^\circ = \boldsymbol{c}_{t-1}, \boldsymbol{z}^\circ = \boldsymbol{z}_t, \alpha = \alpha_{t-1}$. Using (19), $\mathcal{E}_{\boldsymbol{A}_t}$ is the same as $\mathcal{E}'$ in Claim 4.7; and clearly $\alpha_t = \alpha'$. This establishes the inner ellipsoid invariant $\boldsymbol{c}_t + \alpha_t\mathcal{E}_t \subseteq \mathrm{conv}((\boldsymbol{c}_{t-1} + \alpha_{t-1}\mathcal{E}_{t-1}) \cup \{\boldsymbol{z}_t\})$ directly. To show $\mathrm{conv}((\boldsymbol{c}_{t-1} + \mathcal{E}_{t-1}) \cup \{\boldsymbol{z}_t\}) \subseteq \boldsymbol{c}_t + \mathcal{E}_t$, observe that we have $\boldsymbol{c}_{t-1} + \mathcal{E}_{t-1} \subseteq \boldsymbol{c}_t + \mathcal{E}_t$ from Claim 4.7, and $\boldsymbol{z}_t \in \boldsymbol{c}_t + \mathcal{E}_t$ from Claim 4.8. Then the outer ellipsoid invariant follows as $\boldsymbol{c}_t + \mathcal{E}_t$ is a convex set. □

Finally, we bound the relevant quantities that will be used in the analysis of the full algorithm's approximation factor. In particular, we show that $\exp(\gamma_t^\star)$ gives a lower bound on the increase in volume at each iteration t, showing that the evolution condition holds. If $\|\boldsymbol{u}\| \leq 1$, and the ellipsoids are not updated, in that iteration we think of $\gamma_t^\star = 0$.

Claim 4.10. *For any input given to Algorithm 1, we have* $\mathrm{vol}(\mathcal{E}_i) \geq \exp(\gamma_t^\star)\,\mathrm{vol}(\mathcal{E}_{t-1})$.

Proof. This formula is clearly true when the ellipsoids are not updated because $\gamma_t^\star = 0$, so we consider the nontrivial case. Recall the formula $\mathrm{vol}(\mathcal{E}_{\boldsymbol{A}}) = \det(\boldsymbol{A}^{-1})\,\mathrm{vol}(B_2^d)$ from Claim 3.3. Then we have

$$\begin{aligned}\mathrm{vol}(\mathcal{E}_{\boldsymbol{A}_i}) &= \det(\boldsymbol{A}_i^{-1})\,\mathrm{vol}(B_2^d) = \det(\hat{\boldsymbol{A}}^{-1}) \cdot \det(\boldsymbol{A}_{t-1}^{-1}) \cdot \mathrm{vol}(B_2^d)\\ &= \det(\hat{\boldsymbol{A}}^{-1})\,\mathrm{vol}(\mathcal{E}_{\boldsymbol{A}_{t-1}})\end{aligned}$$

where we use the definition of $\hat{\boldsymbol{A}}$ from Line 4 on the t-th iteration. Then

$$\begin{aligned}\det(\hat{\boldsymbol{A}}^{-1}) &= a(\gamma_t^\star) \cdot b(\gamma_t^\star)^{d-1} && \text{using (19)}\\ &\geq a(\gamma_t^\star) && \text{by Claim 4.2-(2)}\\ &= \exp(\gamma_t^\star) && \text{by definition of } a \text{ in (9)}\end{aligned}$$

and using $\mathrm{vol}(\mathcal{E}_{\boldsymbol{A}_i}) = \det(\hat{\boldsymbol{A}}^{-1}) \cdot \mathrm{vol}(\mathcal{E}_{\boldsymbol{A}_{t-1}})$ completes the proof. □

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