CLASSIFICATION OF LOW-RANK ODD-DIMENSIONAL MODULAR CATEGORIES

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ABSTRACT. We prove that any odd-dimensional modular category of rank at most 23 is pointed. We also show that an odd-dimensional modular category of rank 25 is either pointed, perfect, or equivalent to $\text{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$. Finally, we give partial classification results for modular categories of rank up to 73.

CONTENTS

1. INTRODUCTION

Modular tensor categories (MTCs) are fusion categories with additional braiding and ribbon structures satisfying a non-degeneracy condition for the braiding, see [ENO1, T]. They are of interest for a variety of mathematical subjects, such as topological quantum field theory [T], representation theory of quantum groups [BK], von Neumann algebras [EK], conformal field theory [MS] and vertex operator algebras [H]. Furthermore, MTCs appear in the study of topological phases of matter and topological quantum computation, as anyon systems are modeled by unitary MTCs [R].

The problem of classifying MTCs is an active area of research. In [BNRW1], it was shown that there are finitely many MTCs of a fixed rank, up to equivalence. This result makes the classification of MTCs by rank a more feasible endeavor and many efforts have been made in this direction. A classification of unitary MTCs of rank at most 4 was presented in [RSW]. In [HR], all MTCs of rank at most 5 such that some object is not isomorphic to its dual (non-self-dual) were classified. Later on, all possible fusion rules for MTCs of rank 5 were determined in [BNRW2], and were used to describe their classification up to monoidal equivalence. Recently, a classification of rank 6 MTCs up to modular data was given in [NRWW].

Integral MTCs, that is, MTCs such that the Frobenius-Perron dimension of every simple object is an integer, are of particular relevance since they are in correspondence with the categories of representations of modular finite-dimensional semisimple quasi-Hopf algebras. Classification of integral MTCs has been approached from many directions, see for example [BR, B+, DLD, DN, DT]. Recently, [ABPP] reported the classification of all integral MTCs of rank at most 12.

An important class of integral MTCs are those of odd Frobenius-Perron dimension. As a consequence of [NS, Corollary 8.2] and [HR, Theorem 2.2], this condition is equivalent to asking that no non-trivial object is self-dual (i.e., the category is maximally non-self-dual, or MNSD). It was proven in [BR] that odd-dimensional MTCs of rank at most 11 are pointed, that is, all simple objects are invertible. At the same time, they exhibited an example of an odd-dimensional MTC of rank 25 that is not pointed, given by the representation category $\mathrm{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$, and asked whether this is the smallest (in rank) example of a non-pointed odd-dimensional MTC. A partial answer to this question is given in [CP], where it was shown that all odd-dimensional MTCs of rank 13 or 15 must be pointed, and all odd-dimensional MTCs of rank between 17 and 23 are either pointed or perfect, the latter meaning that the unit is the only invertible object.

In this work, we continue the study of low-rank MNSD MTCs, or equivalently, low-rank odd-dimensional MTCs. We show that odd-dimensional MTCs of ranks 17 to 23 are all pointed, which gives a positive answer to the question in [BR]. A proof of this statement can be found in Section 4. Pointed MTCs are classified by pairs (G, q), where G is a finite abelian group, and $q : G \to \mathbf{k}^\times$ is a non-degenerate quadratic form on G, see [EGNO, Example 8.13.5]. Thus our result completes the classification of odd-dimensional MTCs up to rank 23.

We also show that an odd-dimensional MTC of rank 25 is either pointed, perfect, or equivalent to the representation category $\mathrm{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$, which is the non-pointed example exhibited in [BR]. For higher ranks, we show partial results for classification of odd-dimensional MTCs of rank at most 73.

The following Theorem summarizes our main results.

Theorem 1.1. *Let $\mathcal{C}$ be an odd-dimensional MTC.*

- *If $17 \leq \mathrm{rank}(\mathcal{C}) \leq 23$, then $\mathcal{C}$ is pointed.*
- *If $\mathrm{rank}(\mathcal{C}) = 25$, then $\mathcal{C}$ is pointed, perfect, or equivalent to $\mathrm{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$.*
- *If $\mathrm{rank}(\mathcal{C}) \in \{27, 29, 31, 37, 39, 45, 47, 53, 55, 63\}$, then $\mathcal{C}$ is either pointed or perfect.*
- *If $\mathrm{rank}(\mathcal{C}) \in \{33, 61, 69, 71\}$, then $\mathcal{C}$ is either pointed, perfect, or has 3 invertible objects.*
- *If $\mathrm{rank}(\mathcal{C}) = 35$, then $\mathcal{C}$ is pointed, perfect, or the modular subcategory of $\mathcal{Z}(\mathrm{Vec}^\omega_{H_3})$ with 9 invertible objects and 26 simple objects of dimension 3, where H_3 denotes the Heisenberg group of order 3^3.*

- *If* $\text{rank}(\mathcal{C}) = 41$*, then* $\mathcal{C}$ *is pointed, perfect, or has 3 or 5 invertible objects. Moreover, if the latter case exists, then* $\mathcal{C}$ *must be equivalent to* $\mathcal{D}^{\mathbb{Z}_5}$*, where* $\mathcal{D}$ *is a categorification of the ring* $R_{5,H}$ *as defined in* [JL, Definition 1.3]*, and* H *is a finite abelian group of order* 3^4*.*
- *If* $\text{rank}(\mathcal{C}) = 43$*, then* $\mathcal{C}$ *is pointed, perfect, or has* 9 *invertible objects.*
- *If* $\text{rank}(\mathcal{C}) = 49$*, then* $\mathcal{C}$ *is pointed, perfect, or has* 3 *or* 5 *invertible objects. Additionally, if* $\mathcal{C}$ *has* 5 *invertible objects and* $7 \nmid \text{FPdim}(\mathcal{C})$*, then* $\mathcal{C} \cong \text{Rep}(D^\omega(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5))$ *with* $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ *non-abelian and for some* 3*-cocycle* ω*.*
- *If* $\text{rank}(\mathcal{C}) \in \{51, 59, 67\}$*, then* $\mathcal{C}$ *is either pointed, perfect or has 3 or 9 invertible objects.*
- *If* $\text{rank}(\mathcal{C}) \in \{57, 65\}$*, then* $\mathcal{C}$ *is either pointed, perfect or has 3 or 5 invertible objects. In particular, the category* $\text{Rep}(D^\omega(\mathbb{Z}_{13} \rtimes \mathbb{Z}_3))$ *is an example of an odd-dimensional MTC of rank* 65 *with* 3 *invertible objects.*
- *If* $\text{rank}(\mathcal{C}) = 73$*, then* $\mathcal{C}$ *is either pointed, perfect or has 3, 5 or 7 invertible objects.*

Remark 1.2. We note that if $\mathcal{C}$ is an odd-dimensional MTC that can be obtained as the Drinfeld center of another fusion category, then its dimension is a square and thus $\text{FPdim}(\mathcal{C}) \equiv 1 \mod 8$. In particular, this is the case for MTCs of the form $\text{Rep}(D^\omega G)$ for some finite group G.

We have included in Theorem 1.1 all examples of the form $\text{Rep}(D^\omega G)$ (that we know of) that appear up to rank 73. There could be more examples in ranks 33, 41, 49, 57, 65 or 73, as it is difficult to know in general the rank of $\text{Rep}(D^\omega G)$ in relation to $|G|$, see for example [E]. Possibilities for such ranks have been computed up to $|G| = 47$ in [GS, Figure 3].

Regarding the methods used in this paper, we started by proving several general results useful for classification of low rank MTCs, which can be found in Section 3.3. We also developed an algorithm based on techniques from [CP], which we improved with results from Section 3.3, that allowed us to inspect potential arrays for the Frobenius-Perron dimensions of simple objects in an odd-dimensional MTC of fixed rank. Another important tool used in this work was the de-equivariantization of the adjoint subcategory by a Tannakian subcategory, which permitted in certain cases to work with an odd-dimensional MTC of lower dimension (and sometimes lower rank) than the original one.

There is a major conjecture in fusion categories that states that every weakly integral fusion category is weakly group-theoretical [ENO2, Question 2]. As stated in [CP], a perfect odd-dimensional MTC would yield a counter-example for said conjecture, and is thus conjectured that such a category cannot exist [CP, Conjecture 1.1]. Consequently, if weakly integral MTCs are proven to be weakly group-theoretical, our results above would give a complete classification of odd-dimensional MTCs $\mathcal{C}$ such that

- $\text{rank}(\mathcal{C}) \leq 47$ and $\text{rank}(\mathcal{C}) \neq 33, 41, 43$, or
- $\text{rank}(\mathcal{C}) = 53, 55$ or 63.

The rest of this paper is organized as follows. A brief introduction to modular tensor categories and related constructions is given in Section 2. The algorithm used for computing potential arrays of Frobenius-Perron dimensions is explained in Section 3, as well as general results useful for the classification of odd-dimensional MTCs by rank. Section 4 contains the classification of odd-dimensional MTCs of rank 17 to 23. Lastly, Sections 5, 6, 7 and 8 are dedicated to advancing the classification of MTCs of ranks 25, 27 to 31, 33 to 49, and 51 to 73, respectively.

Acknowledgements

We thank the MIT PRIMES program and its organizers Prof. Pavel Etingof, Dr. Slava Gerovitch, and Dr. Tanya Khovanova for making this possible. The research of J.P. was partially supported by NSF grant DMS-2146392 and by Simons Foundation Award 889000 as part of the Simons Collaboration on Global Categorical Symmetries. J.P. would like to thank the hospitality and excellent working conditions at the Department of Mathematics at Universität Hamburg, where she has carried out part of this research as an Experienced Fellow of the Alexander von Humboldt Foundation.

2. Preliminaries

We work over an algebraically closed field $\mathbf{k}$ of characteristic zero. We refer the reader to [ENO1, ENO2, EGNO] for the basic theory of fusion categories and braided fusion categories, and for terminology used throughout this paper.

We denote by Vec the category of finite dimensional vector spaces over $\mathbf{k}$, and by $\mathrm{Rep}(G)$ the category of finite dimensional representations of a finite group G.

2.1. Fusion categories.

A *fusion category* over $\mathbf{k}$ is a semisimple rigid tensor category over $\mathbf{k}$ with finitely many isomorphism classes of simple objects.

For the rest of this section, let $\mathcal{C}$ be a fusion category. We denote by $\mathbf{1}$ its identity object, and assume $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \cong \mathbf{k}$.

For an object $X \in \mathcal{C}$, we denote its dual by $X^* \in \mathcal{C}$. We say an object is *self-dual* if $X \cong X^*$, and *non-self-dual* if $X \ncong X^*$. The unit object $\mathbf{1}$ is always self-dual, and so we say that $\mathcal{C}$ is *maximally-non-self-dual* (MNSD) if the only self-dual simple object in $\mathcal{C}$ is $\mathbf{1}$.

We shall denote by $\mathcal{O}(\mathcal{C})$ the set of isomorphism classes of simple objects of $\mathcal{C}$. For X, Y and Z in $\mathcal{O}(\mathcal{C})$, we denote the *fusion coefficients* by

$$N_{Y,Z}^{X} := \dim \mathrm{Hom}(Y \otimes Z, X).$$

A *braiding* on $\mathcal{C}$ is a natural isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying the so-called hexagonal diagrams, see [EGNO, Definition 8.1.1]. We will say that $\mathcal{C}$ is *braided* if it is equipped with a braiding.

2.1.1. *Frobenius-Perron dimension.*

Let $\mathcal{C}$ be a fusion category. We will denote by $\mathcal{K}(\mathcal{C})$ its Grothendieck ring, see e.g. [EGNO, 4.5]. For an object X in $\mathcal{C}$ we will use the same notation for its class X in $\mathcal{K}(\mathcal{C})$. We recall that there is a unique ring homomorphism $\mathrm{FPdim} : \mathcal{K}(\mathcal{C}) \to \mathbb{R}$ called *Frobenius-Perron dimension* such that $\mathrm{FPdim}(X) \geq 1$ for any object $X \neq 0$, see [EGNO, Proposition 3.3.4]. The *Frobenius-Perron dimension* $\mathrm{FPdim}(\mathcal{C})$ of $\mathcal{C}$ is defined as

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \mathrm{FPdim}(X)^2.$$

We say that $\mathcal{C}$ is *weakly integral* if $\mathrm{FPdim}(\mathcal{C})$ is an integer, and *integral* if $\mathrm{FPdim}(X)$ is an integer for all simple objects X.

If the non-invertible simple objects of $\mathcal{C}$ are $X_1, X_1^*, \ldots, X_k, X_k^*$ and their Frobenius-Perron dimensions are $\mathrm{d}_1, \mathrm{d}_1, \ldots, \mathrm{d}_k, \mathrm{d}_k$, then we say that the *dimension array* of $\mathcal{C}$ is the array $[\mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_k]$.

2.1.2. *Pointed fusion categories.* Let $\mathcal{C}$ be a fusion category. An object X in $\mathcal{C}$ is said to be *invertible* if its evaluation $X^* \otimes X \to \mathbf{1}$ and coevaluation $\mathbf{1} \to X \otimes X^*$ maps are isomorphisms, see [EGNO, Definition 2.10.1]. Equivalently, an object X is invertible if $\mathrm{FPdim}(X) = 1$.

The unit object $\mathbf{1}$ is invertible. Generalizing the notion of a perfect group, which is a finite group with only one representation of dimension 1 (the trivial one), a fusion category $\mathcal{C}$ is said to be *perfect* if the only invertible object is the unit.

A fusion category $\mathcal{C}$ is *pointed* if all simple objects are invertible. Pointed fusion categories are classified by finite groups in the following way. Any pointed fusion category $\mathcal{C}$ is equivalent to the category of finite dimensional G-graded vector spaces $\mathbf{Vec}_{\mathrm{G}}^{\omega}$, where G is a finite group and ω is a 3-cocycle on G with coefficients in $\mathbf{k}^{\times}$ codifying the associativity constraint.

We denote the group of isomorphism classes of invertible objects of $\mathcal{C}$ by $\mathcal{G}(\mathcal{C})$. The largest pointed subcategory of $\mathcal{C}$ will be denoted by $\mathcal{C}_{\mathrm{pt}}$, that is the fusion subcategory of $\mathcal{C}$ generated by $\mathcal{G}(\mathcal{C})$. We will often identify the elements in $\mathcal{G}(\mathcal{C})$ with the invertible objects in $\mathcal{C}_{\mathrm{pt}}$.

A fusion category is said to be *group-theoretical* if it is Morita equivalent to a pointed fusion category, see [EGNO, Section 9.7].

2.1.3. *The universal grading.* Let G be a finite group. A *G-grading* on a fusion category $\mathcal{C}$ is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g,$$

where $\mathcal{C}_g$ is an abelian subcategory of $\mathcal{C}$ for all $g \in G$, such that the tensor product satifies $\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$, the unit object $\mathbf{1}$ is in $\mathcal{C}_e$, and the dualizing functor maps $\mathcal{C}_g$ to $\mathcal{C}_{g^{-1}}$, see [EGNO, Section 4.14].

Such grading is said to be *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in G$. It was shown in [ENO2, Proposition 8.20] that for a faithful grading all the components $\mathcal{C}_g$ have the same Frobenius-Perron dimension. Hence,

$$\mathrm{FPdim}(\mathcal{C}) = |G| \, \mathrm{FPdim}(\mathcal{C}_e).$$

By [GN], any fusion category $\mathcal{C}$ admits a canonical faithful grading

$$\mathcal{C} = \bigoplus_{g \in U(\mathcal{C})} \mathcal{C}_g,$$

called the *universal grading.* Its trivial component coincides with the *adjoint subcategory* $\mathcal{C}_{\mathrm{ad}}$ of $\mathcal{C}$, defined as the fusion subcategory generated by $X \otimes X^*$ for all $X \in \mathcal{O}(\mathcal{C})$.

If $\mathcal{C}$ is equipped with a braiding, then $\mathcal{U}(\mathcal{C})$ is abelian. Moreover, if $\mathcal{C}$ is modular then $\mathcal{U}(\mathcal{C})$ is isomorphic to the group of (isomorphism classes of) invertibles $\mathcal{G}(\mathcal{C})$ [GN, Theorem 6.3]. We will use this fact repeatedly throughout this work.

We denote the ranks of the components of the universal grading of a fusion category $\mathcal{C}$ using an array of ordered pairs (n, r), indicating that there are n components of rank r each.

2.1.4. *Nilpotent, solvable, and weakly group-theoretical fusion categories.* The *upper central series* of a fusion category $\mathcal{C}$ is the sequence defined recursively by

$$\mathcal{C}^{(0)} = \mathcal{C} \text{ and } \mathcal{C}^{(n)} = \left(\mathcal{C}^{(n-1)}\right)_{ad} \quad \text{for all } n \geq 1.$$

When the upper central series converges to Vec, the category $\mathcal{C}$ is said to be *nilpotent*, see [GN, ENO1]. Alternatively, a fusion category is nilpotent if there is a sequence of fusion subcategories $\mathcal{C}_0 = \mathrm{Vec} \subset \cdots \subset \mathcal{C}_n = \mathcal{C}$ and finite groups $G_1, \ldots, G_n$ such that $\mathcal{C}_i$ is a G_i-extension of $\mathcal{C}_{i-1}$ for all i. If, moreover, the groups G_i are cyclic the category is said to be *cyclically nilpotent*.

As is the case for finite groups, fusion categories of Frobenius-Perron dimension a prime power are known to be nilpotent, see [ENO1, Theorem 8.28]. Also, when $\mathcal{C}$ is a nilpotent fusion category then $\mathrm{FPdim}(X)^2$ divides $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$ for all $X \in \mathcal{O}(\mathcal{C})$ [GN].

A fusion category $\mathcal{C}$ is *weakly group-theoretical*, respectively *solvable*, if it is Morita equivalent to a nilpotent fusion category, respectively to a cyclically nilpotent fusion category [ENO1]. Both of these Morita classes are closed under Deligne tensor product, Drinfeld centers, and fusion subcategories, see [ENO1, Propositions 4.1 and 4.5]. The Morita class of weakly group-theoretical fusion categories is also closed under group extensions and equivariantizations. On the other hand, the class of solvable categories is closed under extensions and equivariantizations by solvable groups, and by taking component categories of quotient categories.

It was shown in [ENO2, Proposition 4.5] that if $\mathcal{C}$ is a solvable braided fusion category, then $\mathcal{C} \cong \mathrm{Vec}$ or $\mathcal{C}$ has a non-trivial invertible object. It is also known that braided nilpotent fusion categories are solvable [ENO2, Proposition 4.5].

2.2. **Modular tensor categories.** Let $\mathcal{C}$ be a braided fusion category. A *pivotal structure* on $\mathcal{C}$ is a natural isomorphism $\psi : \mathrm{Id} \xrightarrow{\sim} (-)^{**}$, i.e., an isomorphism between the double dual and identity functors, see [BW, EGNO]. Associated to a pivotal structure we can define the left and right trace of a morphism $X \to X$, see e.g. [EGNO, 4.7]. The pivotal structure is called *spherical* if for any such morphism its right trace equals its left trace.

Associated to a pivotal structure ψ we get the notion of *quantum dimension* $\dim(X)$ of an object $X \in \mathcal{C}$, as the scalar given by

$$\dim(X) := \mathrm{Tr}_X(\psi_X) \in \mathrm{End}(1) \cong \mathbf{k}, \tag{1}$$

where $\mathrm{Tr}_X(\psi_X)$ denotes the left trace of ψ_X. The *quantum dimension* of $\mathcal{C}$ is given by $\dim(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \mathrm{d}_X \, \mathrm{d}_{X^*}$. When $\mathcal{C}$ is spherical, it satisfies that $\dim(X) = \dim(X^*)$ for all simple objects $X \in \mathcal{C}$. Note that in this case, $\dim(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \mathrm{d}_X^2$.

A *pre-modular* tensor category is a braided fusion category equipped with a spherical structure.

Equivalently, a pre-modular category is a braided fusion category endowed with a compatible ribbon structure. A *ribbon structure* on $\mathcal{C}$ is a natural isomorphism $\theta_X : X \xrightarrow{\sim} X$ for all $X \in \mathcal{C}$, satisfying

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}, \tag{2}$$

and $(\theta_X)^* = \theta_{X^*}$ for all $X, Y \in \mathcal{C}$.

Let $\mathcal{C}$ be a premodular tensor category, with braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$. The *S-matrix* S of $\mathcal{C}$ is defined by $S := (s_{X,Y})_{X,Y \in \mathcal{O}(\mathcal{C})}$, where $s_{X,Y}$ is the trace of $c_{Y,X} c_{X,Y} : X \otimes Y \to X \otimes Y$. A premodular tensor category $\mathcal{C}$ is said to be *modular* if its S-matrix is non-degenerate.

We can obtain the entries of the S-matrix in terms of the twists, fusion rules, and quantum dimensions via the so-called *balancing equation*

$$s_{X,Y} = \theta_X^{-1}\theta_Y^{-1} \sum_{Z\in\mathcal{O}(\mathcal{C})} N_{XY}^Z \theta_Z \,\mathrm{d}_Z, \tag{3}$$

for all $X, Y \in \mathcal{O}(\mathcal{C})$ [EGNO, Proposition 8.13.8].

Note that, in this work, the terms "premodular tensor category" and "modular tensor category" imply semisimplicity, as we require them to be fusion categories. This is a slight abuse of terminology, which we adopt to be consistent with prior papers.

2.2.1. *Centralizers.* Let $\mathcal{C}$ be a braided fusion category and let $\mathcal{K}$ be a fusion subcategory of $\mathcal{C}$. The *Müger centralizer* of $\mathcal{K}$ is the fusion subcategory $\mathcal{K}'$ of $\mathcal{C}$ consisting of all objects Y in $\mathcal{C}$ such that

$$c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}, \quad \text{for all } X \in \mathcal{K}, \tag{4}$$

see [M].

We say $\mathcal{C}$ is *symmetric* if $\mathcal{C}' = \mathcal{C}$. A symmetric fusion category is called *Tannakian* if it is equivalent as a braided fusion category to the category $\mathrm{Rep}(G)$ for some finite group G, with braiding given by the usual flip of vector spaces. It is known that a symmetric fusion category of odd Frobenius-Perron dimension is Tannakian [DGNO2, Corollary 2.50]. We will use this fact repeatedly throughout this work.

A necessary and sufficient condition for $\mathcal{K}$ to be modular is $\mathcal{K} \cap \mathcal{K}' = \mathrm{Vec}$. This tells us that $\mathcal{C}$ is modular if and only if $\mathcal{C}' = \mathrm{Vec}$.

For a modular tensor category $\mathcal{C}$, any fusion subcategory $\mathcal{K}$ satisfies $\mathcal{K} = \mathcal{K}''$ and $\dim(\mathcal{K})\dim(\mathcal{K}') = \dim(\mathcal{C})$ [M, Theorem 3.2]; moreover, $\mathcal{C}_{\mathrm{pt}} = (\mathcal{C}_{\mathrm{ad}})'$ and $(\mathcal{C}_{\mathrm{pt}})' = \mathcal{C}_{\mathrm{ad}}$ [GN, Corollary 6.9].

Remark 2.1. [CP, Remark 2.2] If $\mathcal{C}$ is a braided fusion category, then $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is symmetric.

2.2.2. *Equivariantization and de-equivariantization.* Let $\mathcal{C}$ be a fusion category with an action of a finite group G, see [EGNO, Definition 4.15.1]. Following [EGNO, Section 2.7] one can construct a fusion category $\mathcal{C}^G$ of G-equivariant objects in $\mathcal{C}$. We call $\mathcal{C}^G$ the *equivariantization* of $\mathcal{C}$ by G. If the action of G on $\mathcal{C}$ is braided, see [EGNO, Definition 8.23.8], then $\mathcal{C}^G$ is also braided, with braiding induced from $\mathcal{C}$. Moreover, $\mathrm{Rep}(G)$ is identified with a Tannakian subcategory of $\mathcal{C}^G$ via the canonical embedding $\mathrm{Rep}(G) \hookrightarrow \mathcal{C}^G$.

Conversely, suppose that $\mathcal{C}$ is a braided fusion category with a Tannkian subcategory $\mathrm{Rep}(G)$. Then we can consider the *de-equivariantization* $\mathcal{C}_G$ of $\mathcal{C}$ with respect to $\mathrm{Rep}(G)$, see [EGNO, Theorem 8.23.3] for the construction. The category $\mathcal{C}_G$ is a braided fusion category, and there is an equivalence of braided fusion categories $\mathcal{C} \cong (\mathcal{C}_G)^G$.

Note that dimensions are well-behaved under equivariantization and de-equivariantization, see [DGNO2, Proposition 4.26]. In fact, we have

$$\mathrm{FPdim}(\mathcal{C}^G) = |G| \cdot \mathrm{FPdim}(\mathcal{C}) \qquad \text{and} \qquad \mathrm{FPdim}(\mathcal{C}_G) = \frac{1}{|G|} \cdot \mathrm{FPdim}(\mathcal{C}).$$

Remark 2.2. [ENO2, Remark 2.3] Let $\mathcal{C}$ be a braided fusion category and let $\mathcal{E} \subseteq \mathcal{C}'$ be a Tannakian subcategory. Then the de-equivariantization of $\mathcal{C}$ by $\mathcal{E}$ is non-degenerate if and only if $\mathcal{E} = \mathcal{C}'$.

Let $\mathcal{C}$ be an odd-dimensional modular tensor category. By [CP, Remark 2.2], we have that $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is symmetric and thus Tannakian [DGNO2, Corollary 2.50]. Let G be a finite group such that $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}} \cong \mathrm{Rep}(G)$, and recall that the Müger center of $\mathcal{C}_{\mathrm{ad}}$ is $\mathcal{C}_{\mathrm{pt}} \cap \mathcal{C}_{\mathrm{ad}} = (\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$. Then by the remark above, the de-equivariantization $(\mathcal{C}_{\mathrm{ad}})_G$ is also modular. We will use these facts repeatedly throughout this work.

3. Algorithm and general results

3.1. **Basic algorithm.** We begin by presenting a recursive algorithm to generate potential lists of dimensions of the simple objects of an odd-dimensional integral MTC, given the rank and number of invertible simple objects of that MTC. The code for the implementation of this algorithm can be found in Appendix A in the function `basic_algorithm`.

Consider an odd-dimensional MTC $\mathcal{C}$ of odd rank $n = 2k + s$ with s invertible objects. Then let the non-invertible objects of $\mathcal{C}$ be $X_1, X_1^*, \ldots, X_k, X_k^*$ and their Frobenius-Perron dimensions be $\mathrm{d}_1, \mathrm{d}_1, \ldots, \mathrm{d}_k, \mathrm{d}_k$. Without loss of generality, suppose that $\mathrm{d}_1 \geq \mathrm{d}_2 \geq \cdots \geq \mathrm{d}_k > 1$.

Note that by [ENO2, Theorem 2.11], for each i, there exists a positive integer m_i such that $\mathrm{FPdim}(\mathcal{C}) = m_i \, \mathrm{d}_i^2$. Also, note that each m_i must be odd since $\mathrm{FPdim}(\mathcal{C})$ is odd.

We prove the following lemma, which allows us to initiate the recursion of the algorithm.

Lemma 3.1. *The value m_1 must satisfy the following two conditions:*

- $m_1 \equiv \mathrm{rank}(\mathcal{C}) \pmod 8$*, and*
- $s \leq m_1 \leq 2k + \frac{s}{9}$.

Proof. By [CP, Lemma 5.4], $\mathrm{FPdim}(\mathcal{C}) \equiv \mathrm{rank}(\mathcal{C}) \pmod 8$, so $m_1 \, \mathrm{d}_1^2 \equiv \mathrm{rank}(\mathcal{C}) \pmod 8$. Hence, as d_1 is odd, $m_1 \equiv \mathrm{rank}(\mathcal{C}) \pmod 8$, showing the first condition.

Additionally, let $\mathcal{C}_g$ be the component in the universal grading of $\mathcal{C}$ that contains the simple object of dimension d_1. We have $m_1 = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{d}_1^2} \geq \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(\mathcal{C}_g)} = |\mathcal{G}(\mathcal{C})| = s$.

Finally, we know that

$$m_1 \, \mathrm{d}_1^2 = \mathrm{FPdim}(\mathcal{C}) = |\mathcal{G}(\mathcal{C})| + 2\,\mathrm{d}_1^2 + \cdots + 2\,\mathrm{d}_k^2 \leq s + 2k\,\mathrm{d}_1^2,$$

so $m_1 \leq \frac{s}{\mathrm{d}_1^2} + 2k$. As $\mathrm{d}_1 \geq 3$, this implies that $m_1 \leq 2k + \frac{s}{9}$. □

The next lemma will allow us to recursively generate lists of dimensions.

Lemma 3.2. *Suppose that*

$$c_i \, \mathrm{d}_i^2 = s + 2\,\mathrm{d}_{i+1}^2 + 2\,\mathrm{d}_{i+2}^2 + \cdots + 2\,\mathrm{d}_k^2 \tag{5}$$

for some i with c_i a positive rational number. Then m_{i+1} satisfies

- $m_{i+1} \in \left[m_i, \frac{sm_i}{tc_i} + \frac{2(k-i)m_i}{c_i}\right]$,
- $\left(\frac{c_i m_{i+1}}{m_i} - 2\right) \mathrm{d}_{i+1}^2 = s + 2\,\mathrm{d}_{i+2}^2 + \cdots + 2\,\mathrm{d}_k^2$*, and*
- $\frac{c_i m_{i+1}}{m_i} - 2 > 0$,

where $t = 225$ if $s = 1$ and $t = 9$ otherwise.

Proof. We know that the right-hand side of equation (5) is at most $s + 2(k-i)\,\mathrm{d}_{i+1}^2$, which implies that

$$c_i \, \mathrm{d}_i^2 \leq s + 2(k-i)\,\mathrm{d}_{i+1}^2 \,. \tag{6}$$

Since $m_{i+1}\,\mathrm{d}_{i+1}^2 = \mathrm{FPdim}(\mathcal{C}) = m_i\,\mathrm{d}_i^2$, we can substitute $\mathrm{d}_i^2 = \frac{m_{i+1}}{m_i}\,\mathrm{d}_{i+1}^2$ into inequality (6), giving us

$$\frac{c_i m_{i+1}}{m_i}\,\mathrm{d}_{i+1}^2 \leq s + 2(k-i)\,\mathrm{d}_{i+1}^2$$
$$m_{i+1} \leq \frac{s m_i}{c_i\,\mathrm{d}_{i+1}^2} + \frac{2(k-i)m_i}{c_i}.$$

Additionally, if $s = 1$, by [ENO2, Corollary 7.2], as $\mathcal{C}$ is perfect, $\mathcal{C}$ cannot have a simple object whose dimension is a power of a prime. As a result, we must have $\mathrm{d}_{i+1} \geq 15$, so $\mathrm{d}_{i+1}^2 \geq 225$. If instead $s \neq 1$, we must have $\mathrm{d}_{i+1} \geq 3$, so $\mathrm{d}_{i+1}^2 \geq 9$. Clearly, $m_{i+1} \geq m_i$ (as $\mathrm{d}_{i+1} \leq \mathrm{d}_i$). We therefore can bound the value of m_{i+1}:

$$m_{i+1} \in \left[m_i, \frac{s m_i}{t c_i} + \frac{2(k-i)m_i}{c_i}\right], \tag{7}$$

where $t = 225$ if $s = 1$ and $t = 9$ otherwise.

Also, by substituting $\mathrm{d}_i^2 = \frac{m_{i+1}}{m_i}\,\mathrm{d}_{i+1}^2$ into equation (5), we know

$$\frac{c_i m_{i+1}}{m_i}\,\mathrm{d}_{i+1}^2 = s + 2\,\mathrm{d}_{i+1}^2 + 2\,\mathrm{d}_{i+2}^2 + \cdots + 2\,\mathrm{d}_k^2$$
$$\left(\frac{c_i m_{i+1}}{m_i} - 2\right)\mathrm{d}_{i+1}^2 = s + 2\,\mathrm{d}_{i+2}^2 + \cdots + 2\,\mathrm{d}_k^2\,. \tag{8}$$

Clearly, the right-hand side is positive, so we must have $\frac{c_i m_{i+1}}{m_i} - 2 > 0$. □

Now, for each i, let u_i^2 be the largest perfect square that divides m_i. The following corollary reduces the number of cases that we must check each time we recurse in our algorithm.

Corollary 3.3. *Suppose that*

$$c_i\,\mathrm{d}_i^2 = s + 2\,\mathrm{d}_{i+1}^2 + 2\,\mathrm{d}_{i+2}^2 + \cdots + 2\,\mathrm{d}_k^2 \tag{9}$$

for some i with c_i a positive rational number. Then u_{i+1} satisfies

- $u_{i+1}^2 \in \left[u_i^2, \frac{s u_i^2}{t c_i} + \frac{2(k-i)u_i^2}{c_i}\right]$,
- $\left(\frac{c_i u_{i+1}^2}{u_i^2} - 2\right)\mathrm{d}_{i+1}^2 = s + 2\,\mathrm{d}_{i+2}^2 + \cdots + 2\,\mathrm{d}_k^2$, *and*
- $\frac{c_i u_{i+1}^2}{u_i^2} - 2 > 0$,

where $t = 225$ if $s = 1$ and $t = 9$ otherwise.

Proof. Let $w = \frac{m_i}{u_i^2}$ (hence, $m_i = u_i^2 w$). Since w is squarefree, the equation $m_{i+1}\,\mathrm{d}_{i+1}^2 = m_i\,\mathrm{d}_i^2$ implies that $m_{i+1} = u_{i+1}^2 w$ for an odd integer u_{i+1} (u_{i+1} must be odd as $\mathrm{FPdim}(\mathcal{C}) = u_{i+1}^2 w\,\mathrm{d}_{i+1}^2$ is odd). As a result, checking all the values of m_{i+1} in the range (7) is equivalent to checking odd values of u_{i+1} such that u_{i+1}^2 is in the range

$$u_{i+1}^2 \in \left[u_i^2, \frac{s u_i^2}{t c_i} + \frac{2(k-i)u_i^2}{c_i}\right]$$

for whether they satisfy the equation

$$\left(\frac{c_i u_{i+1}^2}{u_i^2} - 2\right)\mathrm{d}_{i+1}^2 = s + 2\,\mathrm{d}_{i+2}^2 + \cdots + 2\,\mathrm{d}_k^2\,.$$

□

Algorithm 3.4. *Given odd integers n and s, in order to determine whether an odd-dimensional MTC $\mathcal{C}$ can exist with* $\operatorname{rank}(\mathcal{C}) = n$ *and* $|\mathcal{G}(\mathcal{C})| = s$, *perform the following steps:*

(1) *Initialize the variable k to $\frac{n-s}{2}$, and t to 225 if $s = 1$ and to 9 otherwise.*
(2) *For every integer value m_1 satisfying the conditions in Lemma 3.1, set $i = 1$ and $c_1 = m_1 - 2$. Then recursively perform the following steps.*
(3) *If $i < k$, for every odd integer value u_{i+1} that satisfies the conditions of Corollary 3.3, set c_{i+1} to $\frac{c_i u_{i+1}^2}{u_i^2} - 2$, increment i and repeat this step.*
(4) *If $i = k$ (i.e., the recursion is complete), check whether $\frac{s}{c_k}$ is a perfect square.*
    - *If yes, we know $\mathrm{d}_k^2 = \frac{s}{c_k}$. Calculate the rest of the dimensions of the simple objects of $\mathcal{C}$, using the previously computed values of $u_1, \ldots, u_k$, and verify whether this is a valid solution.*
    - *If no, backtrack to the prior value of i.*

Proof. We know that

$$\operatorname{FPdim}(\mathcal{C}) = m_1 \mathrm{d}_1^2 = s + 2\,\mathrm{d}_1^2 + \cdots + 2\,\mathrm{d}_k^2 .$$

Suppose that we consider a specific m_1 satisfying the conditions of Lemma 3.1. The value of m_1 must also satisfy the equation

$$(m_1 - 2)\,\mathrm{d}_1^2 = s + 2\,\mathrm{d}_2^2 + \cdots + 2\,\mathrm{d}_k^2 . \tag{10}$$

Now, we can recursively apply Corollary 3.3 in order to determine whether there exist valid simple object dimensions for $\mathcal{C}$: for each value of u_{i+1} satisfying the first and third conditions in the corollary, we recursively check whether the second condition has a solution. We initiate this recursion with equation (10).

Note that each time we recurse, we reduce the number of d_js on the right-hand side by 1. As a result, after we recurse a sufficient number of times, we will reach an equation of the form

$$c_k\,\mathrm{d}_k^2 = s. \tag{11}$$

Clearly, in order for there to exist a solution, $\frac{s}{c_k}$ must be a perfect square. Hence, if a case results in either:

- Equation (9) with $c_i \leq 0$, or
- Equation (11) with a c_k such that $\frac{s}{c_k}$ is not a perfect square,

that case cannot represent an odd-dimensional MTC of rank n with s invertible objects. □

3.2. **Adjoint algorithm.** The algorithm from Section 3.1 allows us to generate candidate Frobenius-Perron dimensions for all simple objects in an odd-dimensional MTC. However, this algorithm becomes slow with a large number of simple objects. Therefore, in this section, we present a second algorithm that generates the candidate dimensions of only simple objects in the adjoint subcategory $\mathcal{C}_{\mathrm{ad}}$, rather than all simple objects. The code for the implementation of this algorithm can be found in Appendix A in the function `adjoint_algorithm`.

Let the non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$ be $X_1, X_1^*, \ldots, X_k, X_k^*$, and let the dimensions of $X_1, X_2, \ldots, X_k$ be $\mathrm{d}_1 \geq \mathrm{d}_2 \geq \cdots \geq \mathrm{d}_k$. Also let $m_i = \frac{\operatorname{FPdim}(\mathcal{C})}{\mathrm{d}_i^2}$ for every i. The algorithm is based on the fact that $|\mathcal{G}(\mathcal{C})|\operatorname{FPdim}(\mathcal{C}_{\mathrm{ad}}) = \operatorname{FPdim}(\mathcal{C})$, by [ENO1, Proposition 8.20]. Specifically, if we let $s = |\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$, we know that

$$\operatorname{FPdim}(\mathcal{C}_{\mathrm{ad}}) = \frac{\operatorname{FPdim}(\mathcal{C})}{|\mathcal{G}(\mathcal{C})|} = s + 2\,\mathrm{d}_1^2 + \cdots + 2\,\mathrm{d}_k^2\,.$$

Thus, as $\operatorname{FPdim}(\mathcal{C}) = m_1\,\mathrm{d}_1^2$, in place of equation (10), we have

$$\left(\frac{m_1}{|\mathcal{G}(\mathcal{C})|} - 2\right)\mathrm{d}_1^2 = s + 2\,\mathrm{d}_2^2 + \cdots + 2\,\mathrm{d}_k^2\,.$$

Note that Lemma 3.2 and Corollary 3.3 still hold. Using these, we can recurse as we did in Algorithm 3.4 to generate potential solutions.

However, the initiation from Lemma 3.1 needs the following adjustment.

Lemma 3.5. *The value m_1 must satisfy the following two conditions:*

- $m_1 \equiv \operatorname{rank}(\mathcal{C}) \pmod 8$*, and*
- $|\mathcal{G}(\mathcal{C})| \le m_1 \le 2k\cdot|\mathcal{G}(\mathcal{C})| + \frac{s\cdot|\mathcal{G}(\mathcal{C})|}{9}$.

Proof. By [CP, Lemma 5.4], $\operatorname{FPdim}(\mathcal{C}) \equiv \operatorname{rank}(\mathcal{C}) \pmod 8$, so $m_1\,\mathrm{d}_1^2 \equiv \operatorname{rank}(\mathcal{C}) \pmod 8$. Hence, as d_1 is odd, $m_1 \equiv \operatorname{rank}(\mathcal{C}) \pmod 8$, showing the first condition.

Additionally, we have $m_1 = \frac{\operatorname{FPdim}(\mathcal{C})}{\mathrm{d}_1^2} \ge \frac{\operatorname{FPdim}(\mathcal{C})}{\operatorname{FPdim}(\mathcal{C}_{\mathrm{ad}})} = |\mathcal{G}(\mathcal{C})|$.

Finally, we know that

$$m_1\,\mathrm{d}_1^2 = \operatorname{FPdim}(\mathcal{C}) = |\mathcal{G}(\mathcal{C})|\cdot(s + 2\,\mathrm{d}_1^2 + \cdots + 2\,\mathrm{d}_k^2) \le |\mathcal{G}(\mathcal{C})|\cdot(s + 2k\,\mathrm{d}_1^2),$$

so $m_1 \le \frac{s\cdot|\mathcal{G}(\mathcal{C})|}{\mathrm{d}_1^2} + 2k\cdot|\mathcal{G}(\mathcal{C})|$. As $\mathrm{d}_1 \ge 3$, this implies that $m_1 \le 2k\cdot|\mathcal{G}(\mathcal{C})| + \frac{s\cdot|\mathcal{G}(\mathcal{C})|}{9}$. □

As a result, we have the following algorithm.

Algorithm 3.6. *In order to determine whether an odd-dimensional MTC $\mathcal{C}$ can exist with* $\operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) = n$ *and* $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = s$ *for odd integers n and s, perform the following steps, given the values of* $\operatorname{rank}(\mathcal{C})$ *and* $|\mathcal{G}(\mathcal{C})|$*:*

1. *Initialize the variable k to $\frac{n-s}{2}$, and t to 225 if $|\mathcal{G}(\mathcal{C})| = 1$ and to 9 otherwise.*
2. *For every integer value m_1 satisfying the conditions in Lemma 3.5, set $i = 1$ and $c_1 = \frac{m_1}{|\mathcal{G}(\mathcal{C})|} - 2$. Then recursively perform the following steps.*
3. *If $i < k$, for every odd integer value u_{i+1} that satisfies the conditions of Corollary 3.3, set c_{i+1} to $\frac{c_i u_{i+1}^2}{u_i^2} - 2$, increment i and repeat this step.*
4. *If $i = k$ (i.e., the recursion is complete), check whether $\frac{s}{c_k}$ is a perfect square.*
   - *If yes, we know $\mathrm{d}_k^2 = \frac{s}{c_k}$. Calculate the rest of the dimensions of the simple objects of $\mathcal{C}$, using the previously computed values of $u_1, \ldots, u_k$, and verify whether this is a valid solution.*
   - *If no, backtrack to the prior value of i.*

3.3. **General results for classification.** In this subsection, we include results that are true for general ranks, which we will apply for specific cases in later sections to classify odd-dimensional MTCs of given ranks.

We begin by proving the following Proposition, which is a generalization of [CP, Proposition 5.6].

Proposition 3.7. *Let $\mathcal{C}$ be an MTC such that $\mathcal{C}_{\text{pt}} \subseteq \mathcal{C}_{\text{ad}}$ and $\mathcal{C}_{\text{pt}}$ is odd-dimensional. Consider the action of $\mathcal{G}(\mathcal{C})$ on the set of simple objects of $\mathcal{C}$. If a simple object X is fixed by this action, then $X \in \mathcal{C}_{\text{ad}}$.*

Proof. Suppose there exists a simple object X in $\mathcal{C}$ that is fixed by the action. Since $\mathcal{C}_{\text{pt}} \subseteq \mathcal{C}_{\text{ad}}$ and $(\mathcal{C}_{\text{pt}})' = \mathcal{C}_{\text{ad}}$, it follows that $\mathcal{C}_{\text{pt}}$ is an odd-dimensional symmetric fusion category. Then by [DGNO1, Corollary 2.7] we know that $\theta_h = 1$ for all $h \in \mathcal{C}_{\text{pt}}$. Using the balancing equation (3), we obtain

$$s_{h,X} = \theta_h^{-1}\theta_X^{-1}\theta_X \dim(X) = \dim(X) = \dim(h)\dim(X), \text{ for all } h \in \mathcal{C}_{\text{pt}}.$$

Hence by [M, Proposition 2.5] we conclude that $X \in (\mathcal{C}_{\text{pt}})' = \mathcal{C}_{\text{ad}}$, as desired. □

Corollary 3.8. *Let $\mathcal{C}$ be an MTC such that $|\mathcal{G}(\mathcal{C})|$ is an odd prime p, $(\mathcal{C}_{\text{ad}})_{\text{pt}}$ is non-trivial, and the ranks of all non-adjoint components of the universal grading are p. Then, for each non-adjoint component of the universal grading, the simple objects in that component form an orbit under the action of $\mathcal{G}(\mathcal{C})$ on the simple objects of $\mathcal{C}$. Additionally, all simple objects outside of $\mathcal{C}_{\text{ad}}$ have the same Frobenius-Perron dimension.*

Proof. As $\mathcal{G}(\mathcal{C}_{\text{ad}})$ is a non-trivial subgroup of $\mathcal{G}(\mathcal{C})$ and $|\mathcal{G}(\mathcal{C})|$ is prime, $\mathcal{C}_{\text{pt}} \subseteq \mathcal{C}_{\text{ad}}$. Thus, by Proposition 3.7, a simple object $X \notin \mathcal{C}_{\text{ad}}$ is not fixed by the action of $\mathcal{G}(\mathcal{C})$ on the set of simple objects of $\mathcal{C}$. Therefore, the size of the orbit of X is p. Since $\mathcal{C}_{\text{pt}} \subseteq \mathcal{C}_{\text{ad}}$, the action preserves the grading, and hence all objects of the orbit are in the same component, so the size of the orbit of X is equal to the rank of the component of X in the universal grading. Hence, all simple objects in that component have the same Frobenius-Perron dimension.

Additionally, by [ENO1, Proposition 8.20], all components of the universal grading have the same Frobenius-Perron dimension. Hence, as all non-adjoint components have the same rank, all simple objects in the non-adjoint components must have the same Frobenius-Perron dimension. □

In the subsequent proofs, we will frequently use the following remark, as we did in the proof of Corollary 3.8.

Remark 3.9. Let $\mathcal{C}$ be an MTC such that $|\mathcal{G}(\mathcal{C})|$ is prime and $(\mathcal{C}_{\text{ad}})_{\text{pt}}$ is non-trivial. Then as $\mathcal{G}(\mathcal{C}_{\text{ad}})$ is a non-trivial subgroup of $\mathcal{G}(\mathcal{C})$, we know that $\mathcal{C}_{\text{pt}} \subseteq \mathcal{C}_{\text{ad}}$.

Lemma 3.10. *Let $\mathcal{C}$ be an odd-dimensional MTC such that $|\mathcal{G}(\mathcal{C})|$ is a prime p, $(\mathcal{C}_{\text{ad}})_{\text{pt}}$ is non-trivial, and the ranks of all non-adjoint components of the universal grading are p. Then in the multiset of the Frobenius-Perron dimensions of the non-invertible simple objects of $\mathcal{C}$, each element must either appear exactly $2pk$ times for an integer k or its Frobenius-Perron dimension must be divisible by p.*

Proof. First, by Corollary 3.8, the Frobenius-Perron dimensions of the simple objects outside of $\mathcal{C}_{\text{ad}}$ are equal. Specifically, this dimension appears $p(p-1)$ times in the multiset, which is of the form $2pk$ as $p-1$ is even. Therefore, we exclusively focus on the simple objects inside of $\mathcal{C}_{\text{ad}}$ from now on.

By Remark 3.9, consider the action of $\mathcal{G}(\mathcal{C}) \cong \mathcal{G}(\mathcal{C}_{\text{ad}})$ on the simple objects of $\mathcal{C}_{\text{ad}}$. The sizes of its orbits divide $|\mathcal{G}(\mathcal{C})| = p$, i.e., they are either 1 or p. Let $X \in \mathcal{C}_{\text{ad}}$ be a non-invertible simple object that is not fixed by this action, so the size of its orbit must be p.

Letting $\mathcal{G}(\mathcal{C})$ be generated by g, denote the elements of this orbit by $g^i \otimes X$ for integers $0 \leq i \leq p-1$. Then $\text{FPdim}(g^i \otimes X) = \text{FPdim}(X)$ for all i. Since the size of the orbit of X

is odd, the orbits of X and X^* are disjoint. We also know that $\mathrm{FPdim}(X) = \mathrm{FPdim}(X^*)$. Hence, we have exactly $2p$ simple objects of the same Frobenius-Perron dimension for every X that is not fixed by the action.

Now, suppose that X is fixed by the action. Let $Z_1, \ldots, Z_p$ be the simple objects of $\mathcal{C}_g$. By Proposition 3.7, we know that none of these elements are fixed by the action of $\mathcal{G}(\mathcal{C})$. Without loss of generality, suppose that $Z_i = g^{i-1} \otimes Z_1$.

Since $X \in \mathcal{C}_{\mathrm{ad}}$, $X \otimes Z_1 \in \mathcal{C}_g$. Therefore,

$$X \otimes Z_1 = N_1 Z_1 \oplus N_2 Z_2 \oplus \cdots \oplus N_p Z_p$$

for some integers $N_1, \ldots, N_p$. We also have

$$g \otimes X \otimes Z_1 = N_1 Z_2 \oplus N_2 Z_3 \oplus \ldots N_p Z_1.$$

But $X \otimes Z_1 = g \otimes X \otimes Z_1$, so $N_1 = N_2 = \cdots = N_p$. Consequently,

$$X \otimes Z_1 = N_1(Z_1 \oplus Z_2 \oplus \cdots \oplus Z_p).$$

Taking the Frobenius-Perron dimension of both sides and applying Corollary 3.8 gives us

$$\mathrm{FPdim}(X)\,\mathrm{FPdim}(Z_1) = N_1 \cdot p \cdot \mathrm{FPdim}(Z_1)$$
$$\mathrm{FPdim}(X) = N_1 \cdot p.$$

Hence, $\mathrm{FPdim}(X)$ is divisible by p. □

Remark 3.11. From the proof of Lemma 3.10, we can see that if a simple object in $\mathcal{C}$ is fixed by the action of $\mathcal{G}(\mathcal{C})$, then its Frobenius-Perron dimension must be divisible by p. Also, the number of non-fixed objects in $\mathcal{C}$ must be a multiple of $2p$.

Remark 3.12. Considering the action of a group of odd order on the simple objects of $\mathcal{C}$, the proof of Lemma 3.10 also shows that the orbits of a simple object $X \in \mathcal{C}$ and its dual X^* are disjoint.

We also prove a result that allows us to easily discard certain possible values of $|\mathcal{G}(\mathcal{C})|$ for a given rank.

Lemma 3.13. *If $\mathcal{C}$ is an odd-dimensional MTC, then* $\mathrm{rank}(\mathcal{C})$ *is expressible as a non-negative integral linear combination of $|\mathcal{G}(\mathcal{C})|$ and 8.*

Proof. Consider the universal grading $\mathcal{C} = \bigoplus_{g \in \mathcal{G}(\mathcal{C})} \mathcal{C}_g$ of $\mathcal{C}$. By [ENO1, Proposition 8.20], all such $\mathcal{C}_g$'s have the same Frobenius-Perron dimension. But $\mathrm{rank}(\mathcal{C}_g) \equiv \mathrm{FPdim}(\mathcal{C}_g) \pmod 8$ for all $g \in \mathcal{G}(\mathcal{C})$ [CP, Lemma 5.4], so all components of the universal grading must have ranks that are congruent mod 8. Letting the minimum value of $\mathrm{rank}(\mathcal{C}_g)$ across all g's be m, we therefore have $\mathrm{rank}(\mathcal{C}) = |\mathcal{G}(\mathcal{C})| \cdot m + 8k$ for some non-negative integer k, as desired. □

We proceed with three lemmas with necessary conditions for odd-dimensional MTCs with non-trivial $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$, which we will use in future sections. Note that non-trivial $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ implies that $\mathcal{C}$ is neither pointed (as $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) \geq \mathrm{rank}((\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}) > 1$) nor perfect (as $\mathrm{rank}(\mathcal{C}_{\mathrm{pt}}) \geq \mathrm{rank}((\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}) > 1$).

Lemma 3.14. *Let $\mathcal{C}$ be an odd-dimensional MTC with non-trivial $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ and p be a prime divisor of $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$. Then the universal grading of $\mathcal{C}$ must have at least three components of rank at least p.*

Proof. Note that p must be odd because p is a divisor of $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$, which divides $|\mathcal{G}(\mathcal{C})|$ and $\mathrm{FPdim}(\mathcal{C})$. By [CP, Lemma 5.1 (a)], there must exist a non-trivial component $\mathcal{C}_g$ of the universal grading that has at least p non-invertible simple objects. Because of the duality of universal grading components, the dual of $\mathcal{C}_g$ must also have at least p simple objects. $\mathcal{C}_{\mathrm{ad}}$ must clearly have at least p simple objects, completing the proof. □

Lemma 3.15. *Let $\mathcal{C}$ be an odd-dimensional MTC with non-trivial $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$. If $|\mathcal{G}(\mathcal{C})|$ is a prime number p, then the universal grading of $\mathcal{C}$ has at most one component with rank not divisible by p. Additionally, if such a component exists, it must be $\mathcal{C}_{\mathrm{ad}}$.*

Proof. By Remark 3.9, $\mathcal{C}_{\mathrm{pt}} \subseteq \mathcal{C}_{\mathrm{ad}}$. We also have $\mathrm{FPdim}(\mathcal{C}_{\mathrm{pt}}) = |\mathcal{G}(\mathcal{C})| = p$. As $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is non-trivial, $\mathcal{C}$ is not pointed, so by [CP, Proposition 5.6], all non-trivial components in the universal grading of $\mathcal{C}$ must have rank divisible by p. The only component whose rank may not be divisible by p is $\mathcal{C}_{\mathrm{ad}}$. □

Lemma 3.16. *Let $\mathcal{C}$ be an odd-dimensional MTC with non-trivial $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$. In the universal grading of $\mathcal{C}$, there is exactly one value of rank such that an odd number of components of the universal grading of $\mathcal{C}$ have this rank. Additionally, the value of such rank is equal to* $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}})$ *and is greater than* 1.

Proof. Given that the number of components in the universal grading must be odd (since $|\mathcal{G}(\mathcal{C})|$ is odd) and dual components of the universal grading other than $\mathcal{C}_{\mathrm{ad}}$ have the same rank, there must be an even number of non-adjoint components whose rank matches $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}})$. Including $\mathcal{C}_{\mathrm{ad}}$ gives us an odd total number of components of that rank. Since $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is non-trivial, this rank must be greater than 1. All other ranks must occur an even number of times in the universal grading of $\mathcal{C}$ because of the duality of universal grading components. □

The next two results will be used to discard multiple cases in later sections. Note that the following proposition holds for all MTCs (not just odd-dimensional ones).

Proposition 3.17. *Let $\mathcal{C}$ be an MTC. Then exactly $\frac{|\mathcal{G}(\mathcal{C})|}{|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|}$ components of the universal grading of $\mathcal{C}$ contain an invertible object. Additionally, for each component that contains an invertible object, consider the multiset of the Frobenius-Perron dimensions of all its simple objects. Then, all of those multisets are identical.*

Proof. Suppose that a component $\mathcal{C}_g$ of the universal grading contains an invertible object h. Also, let the simple objects in $\mathcal{C}_g$ be $X_1, \ldots, X_k$. We prove that all simple objects in $\mathcal{C}_{\mathrm{ad}}$ are $h^* \otimes X_1, \ldots, h^* \otimes X_k$. As $h^* \in \mathcal{C}_{g^{-1}}$ is invertible, all of those objects are non-isomorphic simple objects in $\mathcal{C}_{\mathrm{ad}}$. Hence, as $\mathrm{FPdim}(h^* \otimes X_i) = \mathrm{FPdim}(X_i)$,

$$\begin{aligned} \mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) &\geq \mathrm{FPdim}(h^* \otimes X_1)^2 + \cdots + \mathrm{FPdim}(h^* \otimes X_k)^2 \\ &= \mathrm{FPdim}(X_1)^2 + \cdots + \mathrm{FPdim}(X_k)^2, \end{aligned} \tag{12}$$

where the inequality is strictly greater if there are simple objects other than $h^* \otimes X_1, \ldots, h^* \otimes X_k$ in $\mathcal{C}_{\mathrm{ad}}$ and equal otherwise. Also, we have

$$\mathrm{FPdim}(\mathcal{C}_g) = \mathrm{FPdim}(X_1)^2 + \cdots + \mathrm{FPdim}(X_k)^2.$$

By [ENO1, Proposition 8.20], $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = \mathrm{FPdim}(\mathcal{C}_g)$, so the equality case in Inequality (12) holds. As a result, the only simple objects in $\mathcal{C}_{\mathrm{ad}}$ are $h^* \otimes X_1, \ldots, h^* \otimes X_k$. As h^* is invertible, these have the same Frobenius-Perron dimensions as $X_1, \ldots, X_k$, so the multiset of dimensions of the simple objects in $\mathcal{C}_g$ is the same as that for $\mathcal{C}_{\mathrm{ad}}$. This shows the second statement of the proposition. This also implies that every component that contains an invertible object actually contains $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$ invertible objects. Hence exactly $\frac{|\mathcal{G}(\mathcal{C})|}{|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|}$ components contain an invertible object, as desired. □

Corollary 3.18. *Let $\mathcal{C}$ be an odd-dimensional MTC with $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \lneq \mathcal{G}(\mathcal{C})$. Then there must exist at least three components of the universal grading of $\mathcal{C}$ that have the same rank as $\mathcal{C}_{\mathrm{ad}}$.*

Proof. Clearly, one such component is $\mathcal{C}_{\mathrm{ad}}$ itself. As $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \lneq \mathcal{G}(\mathcal{C})$, there must exist a non-trivial component $\mathcal{C}_g$ of the universal grading that contains an invertible object. By Proposition 3.17, $\mathrm{rank}(\mathcal{C}_g) = \mathrm{rank}(\mathcal{C}_{\mathrm{ad}})$. Since $\mathrm{rank}(\mathcal{C}_{g^{-1}}) = \mathrm{rank}(\mathcal{C}_g)$, we have the desired three components: $\mathcal{C}_{\mathrm{ad}}$, $\mathcal{C}_g$, and $\mathcal{C}_{g^{-1}}$. □

Next, we will show the following proposition, whose proof follows the proof of [DKP, Proposition 6].

Proposition 3.19. *Let $\mathcal{C}$ be an odd-dimensional MTC of Frobenius-Perron dimension $p^r q^2$, where p and q are distinct odd prime numbers such that $p \nmid (q+1)$ and $|\mathcal{G}(\mathcal{C})| = p^{r-1}$. Then*
- *(a) Either $p \mid q-1$ or $q \mid p-1$.*
- *(b) Additionally, if $r = 2$, then $\mathcal{C} \cong \mathrm{Rep}(D^\omega(H))$ with $H \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ non-abelian for some 3-cocycle ω.*

Proof. (a) Since $|\mathcal{G}(\mathcal{C})| = p^{r-1}$, we know that $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = pq^2$. Therefore, by [ENO2, Theorem 1.6], $\mathcal{C}_{\mathrm{ad}}$ is solvable, so by [ENO2, Proposition 4.5 (iv)], it must contain a non-trivial invertible object. Since p is a prime, $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ is a subgroup of $\mathcal{G}(\mathcal{C})$, and $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$ divides $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})$, we know that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = p$, and $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \cong \mathbb{Z}_p$.

Consider the de-equivariantization $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_p}$ of $\mathcal{C}_{\mathrm{ad}}$ by $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \cong \mathbb{Z}_p$ (see Section 2.2.2). We know that $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_p}) = \frac{\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}})}{p} = q^2$. Since $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_p}$ is modular, we know that $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_p} \cong \mathrm{Vec}_G^\chi$, where G is an abelian group of order q^2 and χ is a nondegenerate quadratic form on G, see [DN, Lemma 4.11].

Note that $G \cong \mathbb{Z}_{q^2}$ or $\mathbb{Z}_q \times \mathbb{Z}_q$, and so it has either 1 or $q+1$ subgroups of order q, respectively. Since by assumption $p \nmid q+1$, then there exists at least one subgroup of order q that is invariant under the action of $\mathbb{Z}_p$. Now, take the subcategory $\mathcal{D}$ of $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_p}$ associated with said subgroup, which has Frobenius-Perron dimension q, and consider the equivariantization $\mathcal{D}^{\mathbb{Z}_p}$ of $\mathcal{D}$ by $\mathbb{Z}_p$. We know that $\mathcal{D}^{\mathbb{Z}_p} \subseteq \mathcal{C}_{\mathrm{ad}}$, $\mathrm{FPdim}(\mathcal{D}^{\mathbb{Z}_p}) = pq$, and $\mathcal{D}^{\mathbb{Z}_p}$ is integral. Since $|\mathcal{G}(\mathcal{D}^{\mathbb{Z}_p})| \leq |\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = p < pq = \mathrm{FPdim}(\mathcal{D}^{\mathbb{Z}_p})$, we also know that $\mathcal{D}^{\mathbb{Z}_p}$ is not pointed. As a result, since both p and q are odd primes, by [EGO, Theorem 6.3], we know that $\mathcal{D}^{\mathbb{Z}_p} \cong \mathrm{Rep}(K)$ where either $K \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ or $K \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is non-abelian. Therefore, either $p \mid q-1$ or $q \mid p-1$, as desired.

(b) From the proof of part (a), we know that $\mathcal{D}^{\mathbb{Z}_p} \cong \mathrm{Rep}(K) \subseteq \mathcal{C}$ with either $K \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ or $K \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$, so there exists the canonical étale algebra $k^{\mathbb{Z}_q \rtimes \mathbb{Z}_p}$ or $k^{\mathbb{Z}_p \rtimes \mathbb{Z}_q}$ of Frobenius-Perron dimension pq in $\mathcal{C}$.

Hence, as $\mathrm{FPdim}(\mathcal{C}) = p^2q^2$, by [DMNO, Corollary 4.1 (i)], $\mathcal{C} \cong \mathcal{Z}(\mathcal{B})$ as braided tensor categories, where $\mathcal{B}$ is a fusion category of Frobenius-Perron dimension pq and $\mathcal{Z}(B)$ denotes its Drinfeld center (see [EGNO, Definition 7.13.1]). By [EGO], either $\mathcal{B} \cong \mathrm{Vec}_H^\omega$ or $\mathcal{B} \cong \mathrm{Rep}(H)$ for some group H of order pq and some 3-cocycle ω of H. Then $\mathcal{C} \cong \mathcal{Z}(\mathrm{Vec}_H^\omega)$.

Given that p and q are primes, we know that one of the following is true: (1) $H \cong \mathbb{Z}_{pq}$, (2) $p \mid q-1$ and $H \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ is non-abelian, or (3) $q \mid p-1$ and $H \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is non-abelian. By casework, we will show that only option (2) is possible.

First, suppose for contradiction that $H \cong \mathbb{Z}_{pq}$. As all Sylow subgroups of $\mathbb{Z}_{pq}$ are cyclic, its Schur multiplier is trivial, and hence $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_{pq}}^{\omega})$ is pointed. But $\mathrm{FPdim}(\mathcal{C}) = p^2q^2$, while $|\mathcal{G}(\mathcal{C})| = p$, so $\mathcal{C}$ cannot be pointed, a contradiction.

Next, assume that $p \mid q-1$ and $H \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ is non-abelian. As a result, $\mathcal{C} \cong \mathcal{Z}(\mathrm{Vec}_H^{\omega}) \cong \mathrm{Rep}(D^{\omega}(H))$ for some 3-cocycle ω, as desired.

Finally, assume that $q \mid p-1$ and $H \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is non-abelian. Then $\mathcal{C} \cong \mathcal{Z}(\mathrm{Vec}_H^{\omega})$ for some 3-cocycle ω. However, $|\mathcal{G}(\mathcal{C})| = |\mathcal{G}(\mathcal{Z}(\mathrm{Vec}_H^{\omega}))| = q$ by [DKP, Proposition 4]. This is a contradiction with the assumption $|\mathcal{G}(\mathcal{C})| = p$. □

4. Odd-dimensional MTCs of rank 17–23

In this section, we complete the classification of odd-dimensional MTCs of rank 17 to 23, and we use that classification to show two general results about odd-dimensional MTCs that we use in future sections.

The classification of odd-dimensional MTCs of rank 17 to 23 was begun in [CP], which showed that such MTCs must be either pointed or perfect. The following result follows directly from an application of Algorithm 3.4.

Theorem 4.1. *All odd-dimensional MTCs of rank* 17*,* 19*,* 21*, and* 23 *must be pointed.*

Proof. Based on [CP, Theorem 6.3 (b)], all odd-dimensional MTCs of rank 17, 19, 21, and 23 must be either pointed or perfect. Running Algorithm 3.4 for each of $\mathrm{rank}(\mathcal{C}) = 17, 19, 21, 23$ and $|\mathcal{G}(\mathcal{C})| = 1$ produces no potential solutions, and therefore the perfect case is impossible. The desired result follows. □

Next, we use the above result to show two general results that are used in future sections to classify higher-rank odd-dimensional MTCs. We first extend [CP, Lemma 6.1] to odd-dimensional MTCs of higher rank.

Lemma 4.2. *Let $\mathcal{C}$ be a non-pointed odd-dimensional MTC of rank at most* 73*. Then $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is trivial if and only if $\mathcal{C}_{\mathrm{pt}}$ is trivial.*

Proof. Clearly, if $\mathcal{C}_{\mathrm{pt}}$ is trivial, then $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is trivial. We now show the opposite direction.

Suppose that $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is trivial. We know $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}})\,\mathrm{rank}(\mathcal{C}_{\mathrm{pt}}) = \mathrm{rank}(\mathcal{C})$ [CP, Lemma 5.2]. Therefore, since $\mathcal{C}$ is non-pointed, $\mathrm{rank}(\mathcal{C}_{\mathrm{pt}}) < \mathrm{rank}(\mathcal{C})$, so $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) > 1$ and $\mathcal{C}_{\mathrm{ad}}$ is not pointed. As a result, by [BR, Theorem 4.5], [CP, Theorem 6.3], and Theorem 4.1, $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}})$ must be at least 25. We also know that $\mathrm{rank}(\mathcal{C}) \leq 73$ and is odd, so $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}})$ cannot be a proper divisor of $\mathrm{rank}(\mathcal{C})$. Therefore, $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) = \mathrm{rank}(\mathcal{C})$, so $\mathcal{C}$ is perfect and $\mathcal{C}_{\mathrm{pt}}$ is trivial, as desired. □

We also use the following lemma to discard many cases where $\mathcal{C}_{\mathrm{ad}}$ has 9 simple objects in later sections.

Lemma 4.3. *Let $\mathcal{C}$ be an odd-dimensional MTC such that $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) = 9$. Then the ranks of all components of the universal grading of $\mathcal{C}$ are greater than* 1.

Proof. We know that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})|$ is odd as it divides $\mathrm{FPdim}(\mathcal{C})$. We perform casework on its value.

If $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 1$, by Proposition 3.17, all components of the universal grading of $\mathcal{C}$ have rank equal to $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) = 9$.

If $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3$, consider the de-equivariantization $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ of $\mathcal{C}_{\mathrm{ad}}$ by $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \cong \mathbb{Z}_3$. The three invertible objects of $\mathcal{C}_{\mathrm{ad}}$ create one simple object in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ (see Section 2.2.2). The six non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ are either all fixed or all non-fixed under the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$ on the simple objects of $\mathcal{C}_{\text{ad}}$.

In the first case, each non-invertible simple object creates three simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, so $\operatorname{rank}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 1 + 3 \cdot 6 = 19$, and $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is pointed by Theorem 4.1. This means that $\operatorname{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 19$ and $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 3 \cdot 19 = 57$. Since 57 is not a perfect square, no component of the universal grading of $\mathcal{C}$ can have rank 1.

In the second case, each set of 3 simple objects in $\mathcal{C}_{\text{ad}}$ creates one simple object in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, so $\operatorname{rank}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 1 + \frac{6}{3} = 3$, and $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is again pointed by [HR, Theorem 2.6]. Then $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 3 \cdot 3 = 9$. However, $\mathcal{C}_{\text{ad}}$ contains 3 invertible objects and 6 non-invertible objects, so $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) \geq 3 + 6 \cdot 3^2 > 9$, which is a contradiction.

If $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$, consider the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ of $\mathcal{C}_{\text{ad}}$ by $\mathcal{G}(\mathcal{C}_{\text{ad}}) \cong \mathbb{Z}_5$. The five invertible objects of $\mathcal{C}_{\text{ad}}$ create one simple object in the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$. The four non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ must be all fixed under the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$ on the simple objects of $\mathcal{C}_{\text{ad}}$, and each of them creates five simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$. As a result, $\operatorname{rank}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}) = 1 + 5 \cdot 4 = 21$, and by Theorem 4.1, $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ is pointed. This means that $\operatorname{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}) = 21$ and $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 5 \cdot 21 = 105$, which is not a perfect square, and no component of the universal grading of $\mathcal{C}$ can have rank 1.

If $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 7$, a similar argument holds. In the the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$ of $\mathcal{C}_{\text{ad}}$ by $\mathcal{G}(\mathcal{C}_{\text{ad}}) \cong \mathbb{Z}_7$, the seven invertible objects of $\mathcal{C}_{\text{ad}}$ create one simple object in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$, and the two non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ each create seven simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$. Hence, $\operatorname{rank}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}) = 1 + 7 \cdot 2 = 15$, so $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$ is pointed [CP, Theorem 6.3 (a)]. This means that $\operatorname{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}) = 15$ and $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 7 \cdot 15 = 105$, which is not a perfect square, so no component of the universal grading of $\mathcal{C}$ can have rank 1.

Finally, consider $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$. In this case $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 9$, so the Frobenius-Perron dimension of every component of the universal grading must be 9. As a result, each component either contains exactly 9 invertible objects or exactly 1 object of dimension 3, and hence no component contains at least 3 non-invertible objects. But this contradicts [CP, Lemma 5.1 (a)], as 3 divides $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$, so this case is impossible. □

5. Odd-dimensional MTCs of rank 25

In this section, we prove that all odd-dimensional MTCs of rank 25 must be pointed, perfect, or equivalent to $\operatorname{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$. Let $\mathcal{C}$ be an odd-dimensional MTC of rank 25. We begin by restricting the possible values of $|\mathcal{G}(\mathcal{C})|$, as a corollary of Lemma 3.13.

Corollary 5.1. *If $\mathcal{C}$ is an odd-dimensional MTC of rank 25, then we must have $|\mathcal{G}(\mathcal{C})| = 1, 3, 5, 9, 17, 25$.*

We continue by further reducing the number of possible values of $|\mathcal{G}(\mathcal{C})|$.

Proposition 5.2. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 25. Then $\mathcal{C}$ is either pointed, perfect, or has 3 invertible objects. If $\mathcal{C}$ has 3 invertible objects, then the ranks of the components of the universal grading must be 19, 3, and 3.*

Proof. We perform casework based on the possible values of $|\mathcal{G}(\mathcal{C})|$, from Corollary 5.1. Then, in each subcase, by [ENO1, Proposition 8.20] and [CP, Lemma 5.4], we know that $\operatorname{rank}(\mathcal{C}_g) \equiv \operatorname{rank}(\mathcal{C}_h) \pmod 8$ for all $g, h \in \mathcal{G}(\mathcal{C})$. This combined with $\sum_{g \in \mathcal{G}(\mathcal{C})} \operatorname{rank}(\mathcal{C}_g) = 25$ allows us to enumerate all possible multisets of the ranks of the components of the universal grading for each $|\mathcal{G}(\mathcal{C})|$.

For every $|\mathcal{G}(\mathcal{C})| > 1$, we know that $\mathcal{C}_{\mathrm{pt}}$ is non-trivial, and therefore, by Lemma 4.2, $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is also non-trivial. As a result, we can use Lemmas 3.14, 3.15, and 3.16 to discard cases.

Case 1: $|\mathcal{G}(\mathcal{C})| = 17$**.** The ranks of the components of the universal grading must be $[(1,9),(16,1)]$ (see Section 2.1.3 for a description of the notation. In this case, there is 1 component of rank 9, and 16 components of rank 1). This case is therefore discarded by Lemma 3.14.

Case 2: $|\mathcal{G}(\mathcal{C})| = 9$**.** The ranks of the components of the universal grading must be either $[(1,17),(8,1)]$ or $[(2,9),(7,1)]$. We discard both possibilities by Lemma 3.14.

Case 3: $|\mathcal{G}(\mathcal{C})| = 5$**.** The ranks of the components of the universal grading must be $[(5,5)]$. We know that $\mathcal{G}(\mathcal{C}) \cong \mathbb{Z}_5$. As $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ is a subgroup of $\mathcal{G}(\mathcal{C})$ and $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| \neq 1$ by Lemma 4.2, we know that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 5$. But $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) = 5$, so all simple objects of $\mathcal{C}_{\mathrm{ad}}$ must be invertible, i.e., their Frobenius-Perron dimensions are all 1. As a result, $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = 5$, so by [ENO1, Proposition 8.20], the Frobenius-Perron dimension of all the components of the universal grading must be 5. This is a contradiction, as the square of the Frobenius-Perron dimension of a non-invertible object of $\mathcal{C}$ must be at least 9.

Case 4: $|\mathcal{G}(\mathcal{C})| = 3$**.** The ranks of the components of the universal grading must be either $[(1,19),(2,3)]$ or $[(2,11),(1,3)]$. The latter case is discarded by Lemma 3.15.

This leaves $|\mathcal{G}(\mathcal{C})| = 1, 3, 25$ as possible, with the desired conditions for the case with $|\mathcal{G}(\mathcal{C})| = 3$, completing the proof. □

Next, we show that an MTC of rank 25 with $|\mathcal{G}(\mathcal{C})| = 3$ must be equivalent to the category $\mathrm{Rep}(D^{\omega}(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ for some 3-cocycle ω. Let $\mathcal{C}$ be an MTC of rank 25 with $|\mathcal{G}(\mathcal{C})| = 3$. Let the elements of $\mathcal{G}(\mathcal{C})$ be 1, g, and g^2. Then, we know that $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) = 19$, $\mathrm{rank}(\mathcal{C}_g) = 3$, and $\mathrm{rank}(\mathcal{C}_{g^2}) = 3$.

Lemma 5.3. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 25 with 3 invertible objects. Suppose that there are t simple objects in $\mathcal{C}$ of Frobenius-Perron dimension 3, and that r of them are not fixed by $\mathcal{G}(\mathcal{C})$. Then t is a multiple of 6, and $3t - 3r + 1$ divides $\frac{\mathrm{FPdim}(\mathcal{C})}{9}$.*

Proof. Clearly, all the simple objects of Frobenius-Perron dimension 3 are in $\mathcal{C}_{\mathrm{ad}}$ (otherwise, $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = \mathrm{FPdim}(\mathcal{C}_g) = 27$, while $\mathcal{C}_{\mathrm{ad}}$ has 16 non-invertible simple objects, which is impossible). Consider the de-equivariantization $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ of $\mathcal{C}_{\mathrm{ad}}$ by $\mathbb{Z}_3$, see Section 2.2.2. We know $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = \frac{\mathrm{FPdim}(\mathcal{C})}{9}$. We count the number of invertible objects in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$, which we know must be a divisor of $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})$.

Consider the simple objects in $\mathcal{C}_{\mathrm{ad}}$. In order for an invertible object in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ to be created by a simple object $X \in \mathcal{C}_{\mathrm{ad}}$, either X is not fixed and $\mathrm{FPdim}(X) = 1$, or X is fixed and $\mathrm{FPdim}(X) = 3$. For the former case, the only invertible object is created by $1, g, g^2 \in \mathcal{C}_{\mathrm{ad}}$, resulting in one invertible object. For the latter case, we know there are t simple objects of Frobenius-Perron dimension 3 in $\mathcal{C}_{\mathrm{ad}}$. As a result, as the non-fixed objects come in sets of 6 (an orbit of size 3 of the action by $\mathcal{G}(\mathcal{C})$ and the dual of the orbit), r is a multiple of 6, and there are $t - r$ fixed simple objects of Frobenius-Perron dimension 3 in $\mathcal{C}_{\mathrm{ad}}$. But each of these objects gives rise to 3 simple objects of Frobenius-Perron dimension 1 in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$, resulting in an additional $3t - 3r$ invertible objects.

Hence, there are a total of $3t - 3r + 1$ invertible objects in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$. As $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is modular, this must divide $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = \frac{\mathrm{FPdim}(\mathcal{C})}{9}$, as desired. □

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8	9	10	11
1	18275625	1425	1425	1425	1425	855	475	225	75	45	19	5
2	95355225	3255	3255	3255	3255	1953	1085	465	279	105	31	5
3	300155625	5775	5775	5775	5775	3465	1925	825	385	275	231	75
4	99225	105	105	105	105	63	35	15	7	5	3	3
5	300155625	5775	5775	5775	5775	3465	1925	825	385	231	225	175
6	52490025	2415	2415	2415	2415	1449	805	315	207	161	21	7
7	6125625	825	825	825	825	495	275	99	99	5	5	3
8	6125625	825	825	825	825	495	275	99	75	55	33	11
9	164025	135	135	135	135	81	45	15	15	5	5	5
10	108056025	3465	3465	3465	3465	2079	1155	385	315	315	11	7
11	99225	105	105	105	105	63	35	9	9	9	7	5
12	108056025	3465	3465	3465	3465	2079	945	693	385	385	105	11
13	1334025	385	385	385	385	231	105	77	35	35	35	11
14	7868025	935	935	935	935	561	255	165	165	51	17	5
15	1334025	385	385	385	385	231	105	55	55	35	35	35
16	38025	65	65	65	65	39	15	13	13	3	3	3
17	27225	55	55	55	55	33	11	11	11	5	5	3
18	1625625	425	425	425	425	255	85	85	75	51	51	3
19	5625	25	25	25	25	15	5	5	3	3	3	3
20	31585464729	59241	59241	59241	59241	25389	25389	19747	8463	1953	403	49
21	95355225	3255	3255	3255	3255	1395	1395	1085	465	105	31	5
22	23532201	1617	1617	1617	1617	693	693	539	231	49	21	11
23	1432809	399	399	399	399	171	171	133	57	9	7	7
24	52490025	2415	2415	2415	2415	1035	1035	805	315	161	21	7
25	99225	105	105	105	105	45	45	35	9	9	7	5
26	9801	33	33	33	33	11	11	11	9	9	3	3
27	13689	39	39	39	39	13	13	13	9	9	9	3
28	2025	15	15	15	15	5	5	5	3	3	3	3
29	38025	65	65	65	39	39	39	39	15	3	3	3
30	27225	55	55	55	33	33	33	33	11	5	5	3
31	1625625	425	425	425	255	255	255	255	75	51	51	3
32	5625	25	25	25	15	15	15	15	3	3	3	3
33	2025	15	15	15	9	9	9	5	5	5	3	3
34	441	7	7	7	3	3	3	3	3	3	3	3
35	2025	15	15	9	9	9	9	9	9	5	5	5

TABLE 1. Potential dimension arrays for MTC $\mathcal{C}$ of rank 25 with 3 invertible objects. All discarded except row 34.

Let $\mathcal{C}$ be a fusion category, and let its non-invertible simple objects be $X_1, X_1^*, \ldots, X_k, X_k^*$ and their Frobenius-Perron dimensions be $\mathrm{d}_1, \mathrm{d}_1, \ldots, \mathrm{d}_k, \mathrm{d}_k$. Recall that we call the array $[\mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_k]$ the *dimension array* of $\mathcal{C}$.

We now generate all potential dimension arrays of $\mathcal{C}$ using Algorithm 3.4 for $\mathrm{rank}(\mathcal{C}) = 25$ and $\mathcal{G}(\mathcal{C}) = 3$. This produces the cases listed in Table 1, where the columns numbered 1 through 11 represent the elements of the array, i.e., the dimensions of the non-invertible simple objects, excluding duplicates due to duals.

We discard all of these potential arrays on a case-by-case basis, except for the one in row 34, which is realizable as $\mathrm{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$.

Lemma 5.4. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 25 with 3 invertible objects. Then the dimension array of $\mathcal{C}$ is $[7, 7, 7, 3, 3, 3, 3, 3, 3, 3, 3]$, with the Frobenius-Perron dimensions of all the non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$ equal to 3 and the Frobenius-Perron dimensions of all simple objects outside of $\mathcal{C}_{\mathrm{ad}}$ equal to 7.*

Proof. Consider the potential dimension arrays listed by Table 1. By Corollary 3.8, we can eliminate case 35, as we know that $\operatorname{FPdim}(\mathcal{C}_g) = \frac{1}{3}\operatorname{FPdim}(\mathcal{C}) = 675$, and hence the Frobenius-Perron dimension of all the simple objects outside of $\mathcal{C}_{\mathrm{ad}}$ must be $\sqrt{\frac{675}{3}} = 15$, but there are only 4 simple objects with Frobenius-Perron dimension 15.

Then, using Lemma 3.10, we can eliminate all remaining cases except for cases 9, 26, 27, 28, 29, 31, 32, 33, and 34. Using Lemma 5.3, we can eliminate cases 26, 27, 31, and 33, leaving cases 9, 28, 29, 32, and 34.

We now individually rule out cases 9, 28, 29, and 32. For these cases, we can easily determine the Frobenius-Perron dimensions of any simple object Z outside of $\mathcal{C}_{\mathrm{ad}}$, again using $\operatorname{FPdim}(Z)^2 = \frac{1}{3}\operatorname{FPdim}(\mathcal{C}_g) = \frac{1}{9}\operatorname{FPdim}(\mathcal{C}_{\mathrm{ad}})$ and Corollary 3.8.

Case 9: $[135, 135, 135, 135, 81, 45, 15, 15, 5, 5, 5]$. Using the same reasoning as for case 35, the Frobenius-Perron dimensions of all of the simple objects outside of $\mathcal{C}_{\mathrm{ad}}$ must be 135. As a result, the Frobenius-Perron dimensions from the above multiset that belong to $\mathcal{C}_{\mathrm{ad}}$ are $135, 81, 45, 15, 15, 5, 5, 5$. Notice that the objects with Frobenius-Perron dimension 5 cannot be fixed by g per Remark 3.11, as they are not divisible by 3. Label the simple objects $X_1, X_2, \ldots, X_8$ from right-to-left in the array (hence, $\operatorname{FPdim}(X_1) = 5, \operatorname{FPdim}(X_2) = 5, \ldots, \operatorname{FPdim}(X_7) = 81, \operatorname{FPdim}(X_8) = 135$). Consider the decomposition of $X_1 \otimes X_1^*$, where we know that the coefficients of X_i and X_i^* are equal as $\mathcal{C}$ is a braided category:

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{8} N_i(X_i \oplus X_i^*).$$

Taking the Frobenius-Perron dimension of both sides gives

$$25 = 1 + \sum_{i=1}^{8} 2N_i \operatorname{FPdim}(X_i).$$

Clearly, N_6, N_7, and N_8 will be 0, as $\operatorname{FPdim}(X_8) > \operatorname{FPdim}(X_7) > \operatorname{FPdim}(X_6) > 25$. But as all of $X_1, \ldots, X_5$ are multiples of 5, taking the above equation mod 5 gives us $0 \equiv 1 \pmod 5$, a contradiction. As a result, we rule out case 9.

Case 28: $[15, 15, 15, 15, 5, 5, 5, 3, 3, 3, 3]$. The Frobenius-Perron dimensions of all of the simple objects outside of $\mathcal{C}_{\mathrm{ad}}$ must be 15. Label the simple objects of $\mathcal{C}_{\mathrm{ad}}$ as $X_1, X_2, \ldots, X_8$ from right-to-left. By Lemma 5.3, all of X_1, X_2, X_3, and X_4 must be fixed, creating 12 invertible objects in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$. Then, as 5 is not divisible by 3, X_5, X_6, and X_7 are not fixed, resulting in one simple object of Frobenius-Perron dimension 5 in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$. Finally, X_8 clearly has to be fixed, resulting in three simple objects of Frobenius-Perron dimension 5 in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$.

As a result, we have $\operatorname{rank}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = 33$ and $|\mathcal{G}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})| = 25$ (note that we must account for the objects created by the duals of the simple objects in $\mathcal{C}$). Consider the universal grading of $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$. By Lemma 4.2, $\operatorname{rank}((((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})_{\mathrm{ad}})_{\mathrm{pt}}) > 1$, so we must have $\operatorname{rank}(((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})_{\mathrm{ad}}) = 9$, forcing the rank of all of the remaining components to be 1. But this is impossible by Lemma 3.14, so we discard this case.

Case 29: $[65, 65, 65, 39, 39, 39, 39, 15, 3, 3, 3]$. The Frobenius-Perron dimensions of the simple objects outside of $\mathcal{C}_{\mathrm{ad}}$ must be 65. Label the simple objects of $\mathcal{C}_{\mathrm{ad}}$ as $X_1, X_2, \ldots, X_8$ from right-to-left. By Lemma 5.3, all of X_1, X_2, and X_3 are not fixed, resulting in a simple object of Frobenius-Perron dimension 3 in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$. But 3 does not divide $\operatorname{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = \frac{\operatorname{FPdim}(\mathcal{C})}{9} = 4225$, a contradiction by [ENO2, Corollary 7.2]. Therefore, we discard case 29.

Case 32: $[25, 25, 25, 15, 15, 15, 15, 3, 3, 3, 3]$. The Frobenius-Perron dimensions of the simple objects outside of $\mathcal{C}_{\text{ad}}$ must be 25. Label the simple objcets of $\mathcal{C}_{\text{ad}}$ as $X_1, X_2, \dots, X_8$. By Lemma 5.3, all of X_1, X_2, X_3, and X_4 must be fixed by g. We also know that either all of X_5, X_6, X_7, X_8 are fixed by g, or one of them is fixed by g and three of them are not. First suppose that three of them are not. This results in a $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ of rank 33 and with 25 invertible objects (once again recalling that we must also account for the duals of $X_1, \dots, X_8$). As a result, just as we had for case 28, we must have $\text{rank}(\mathcal{C}_{\text{ad}}) = 9$. But this contradicts [CP, Lemma 5.1 (c)]: $33 \geq 9 + 25 + 2 \cdot p - 3 \geq 37$ (as $p \geq 3$).

Thus, all of X_5, X_6, X_7, X_8 must be fixed. As a result, in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, we have 25 simple objects of Frobenius-Perron dimension 1 and 24 simple objects of Frobenius-Perron dimension 5, resulting in 25 invertibles, an overall rank of 49, and an overall Frobenius-Perron dimension of 625. Consider the universal grading of $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, with $|\mathcal{G}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})| = 25$ components and a total of 49 simple objects in those components. As a result, as the Frobenius-Perron dimension of all the components are congruent mod 8, the number of simple objects in each component must be one of the following three cases:

- [(1, 25), (24, 1)],
- [(1, 17), (1, 9), (23, 1)],
- [(3, 9), (22, 1)].

In the first two cases, by Lemma 4.2, $\text{rank}((((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})_{\text{ad}})_{\text{pt}}) > 1$. Therefore, we can discard these cases by Lemma 3.14.

Finally, consider the third case. By [ENO1, Proposition 8.20], all of the components of the universal grading must have the same Frobenius-Perron dimension. Specifically, as $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 625$, the Frobenius-Perron dimension of each component must be 25. This means that the simple object in every component that has exactly 1 simple object must have Frobenius-Perron dimension 5, resulting in 22 simple objects with Frobenius-Perron dimension 5 being used and $24 - 22 = 2$ remaining. These 2 remaining simple objects must be in the components with 9 simple objects, resulting in at least one component having Frobenius-Perron dimension greater than 25. But the Frobenius-Perron dimension of this components must also be 25, a contradiction.

Hence, we have discarded case 32. This discards all cases except for case 34, completing the proof. □

Finally, using the information from Lemma 5.4, we prove that $\mathcal{C}$ must be equivalent to $\text{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$.

Proposition 5.5. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 25 with 3 invertible objects. Then $\mathcal{C} \cong \text{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ for some 3-cocycle ω.*

Proof. Following Lemma 5.4, we know that the only possible dimension array for $\mathcal{C}$ is given in row 34 in Table 1. Then it follows from Proposition 3.19 that $\mathcal{C} \cong \text{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$. □

Therefore, in summary, we have shown the following theorem.

Theorem 5.6. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 25. Then $\mathcal{C}$ is either pointed, perfect, or equivalent to $\text{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ for some 3-cocycle ω.*

6. Odd-dimensional MTCs of rank 27–31

In this section, we prove that all odd-dimensional MTCs of rank between 27 and 31 must be either pointed or perfect. We start with a corollary to Lemma 3.13, applied to ranks 27, 29, and 31.

Corollary 6.1. *If $\mathcal{C}$ is an odd-dimensional MTC, then the possible values of $|\mathcal{G}(\mathcal{C})|$ by rank are the following:*

- $\operatorname{rank}(\mathcal{C}) = 27$*:* $|\mathcal{G}(\mathcal{C})| = 1, 3, 9, 11, 19, 27$*,*
- $\operatorname{rank}(\mathcal{C}) = 29$*:* $|\mathcal{G}(\mathcal{C})| = 1, 3, 5, 7, 13, 21, 29$*,*
- $\operatorname{rank}(\mathcal{C}) = 31$*:* $|\mathcal{G}(\mathcal{C})| = 1, 3, 5, 7, 15, 23, 31$*.*

We use this corollary to show the following result using casework.

Theorem 6.2. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 27, 29, or 31. Then $\mathcal{C}$ must be either pointed or perfect.*

Proof. We mirror the proof of Proposition 5.2 for this proof. Specifically, we again perform casework based on the possible values of $|\mathcal{G}(\mathcal{C})|$, listed in Corollary 6.1. We also can again find the possible multisets of the ranks of the components of the universal grading for each $|\mathcal{G}(\mathcal{C})|$ using the facts that $\operatorname{rank}(\mathcal{C}_g) \equiv \operatorname{rank}(\mathcal{C}_h) \pmod 8$ for all $g, h \in \mathcal{G}(\mathcal{C})$ and $\sum_{g\in\mathcal{G}(\mathcal{C})} \operatorname{rank}(\mathcal{C}_g)$ is 27, 29, or 31.

We can still use Lemmas 3.14, 3.15, and 3.16 to discard cases, as for every $|\mathcal{G}(\mathcal{C})| > 1$, we know that $\mathcal{C}_{\mathrm{pt}}$ is non-trivial, and therefore, by Lemma 4.2, $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is also non-trivial.

First, consider $\operatorname{rank}(\mathcal{C}) = 27$. Using Lemma 3.14, we can discard the following cases, where we list each case as a multiset of the ranks of the components of the universal grading:

- $|\mathcal{G}(\mathcal{C})| = 19$: [(1, 9), (18, 1)],
- $|\mathcal{G}(\mathcal{C})| = 11$: [(1, 17), (10, 1)], [(2, 9), (9, 1)],
- $|\mathcal{G}(\mathcal{C})| = 3$: [(1, 25), (2, 1)], [(1, 17), (1, 9), (1, 1)].

This leaves the cases [(9, 3)] with $|\mathcal{G}(\mathcal{C})| = 9$, and [(3, 9)] with $|\mathcal{G}(\mathcal{C})| = 3$.

For the former remaining case, [(9, 3)], we know that $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ is a subgroup of $\mathcal{G}(\mathcal{C})$. Hence, as $|\mathcal{G}(\mathcal{C})| = 9$, $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| \neq 1$ by Lemma 4.2, and $\operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) = 3$, we must have $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3$. But this means that $\operatorname{FPdim}(\mathcal{C}_g) = \operatorname{FPdim}(\mathcal{C}_{\mathrm{ad}}) = 1^2 + 1^2 + 1^2 = 3$ for all $g \in \mathcal{G}(\mathcal{C})$, i.e., $\mathcal{C}$ is pointed, a contradiction.

For the latter remaining case, [(3, 9)], as in Section 3, let the non-invertible simple objects of such a category $\mathcal{C}$ be $X_1, X_1^*, \ldots, X_{12}, X_{12}^*$ and their dimensions be $\mathrm{d}_1, \mathrm{d}_1, \ldots, \mathrm{d}_{12}, \mathrm{d}_{12}$. Define $m_1 = \frac{\operatorname{FPdim}(\mathcal{C})}{\mathrm{d}_1^2}$ (the same definition as we had in Section 3), which must be an odd positive integer by [ENO2, Theorem 2.11]. If $m_1 = 3$, then $\operatorname{FPdim}(\mathcal{C}) = 3\,\mathrm{d}_1^2$, so $\operatorname{FPdim}(\mathcal{C}_g) = \mathrm{d}_1^2$ by [ENO1, Proposition 8.20]. But the component containing the simple object X_1 clearly has Frobenius-Perron dimension greater than d_1^2 (as there are 9 simple objects in that component), a contradiction. As a result, we assume $m_1 \neq 3$.

In order to discard the cases for $m_1 \neq 3$, we run Algorithm 3.4, considering only the cases where $m_1 \geq 5$. Doing so produces one solution set of Frobenius-Perron dimensions of the simple objects. Specifically, this solution has $\operatorname{FPdim}(\mathcal{C}) = 2475$ and dimensions of non-invertible simple objects are

$$15, 15, 15, 15, 15, 5, 5, 5, 5, 3, 3, 3, 3.$$

As a result, there are 10 simple objects in $\mathcal{C}$ with Frobenius-Perron dimension 15 (the five listed above and their duals). Hence, as there are three components in the universal grading, there must exist one such component with at least 4 simple objects with Frobenius-Perron dimension 15. This component therefore must have dimension at least $4 \cdot 15^2 = 900$. But the dimension of each component is $\frac{2475}{3} = 825$, a contradiction, so we cannot have [(3, 9)].

Next, consider $\operatorname{rank}(\mathcal{C}) = 29$. By Lemma 3.14, we discard the cases listed below:

- $|\mathcal{G}(\mathcal{C})| = 21$: [(1, 9), (20, 1)],
- $|\mathcal{G}(\mathcal{C})| = 13$: [(1, 17), (12, 1)], [(2, 9), (11, 1)].

Additionally, by Lemma 3.15, we discard the cases:

- $|\mathcal{G}(\mathcal{C})| = 7$: [(1, 11), (6, 3)],
- $|\mathcal{G}(\mathcal{C})| = 5$: [(1, 25), (4, 1)], [(1, 17), (1, 9), (3, 1)], [(3, 9), (2, 1)],
- $|\mathcal{G}(\mathcal{C})| = 3$: [(1, 15), (2, 7)].

As this discards all non-pointed and non-perfect cases, if $\text{rank}(\mathcal{C}) = 29$, $\mathcal{C}$ must be pointed or perfect.

Finally, consider $\text{rank}(\mathcal{C}) = 31$. We discard the following cases by Lemma 3.14:

- $|\mathcal{G}(\mathcal{C})| = 23$: [(1, 9), (22, 1)],
- $|\mathcal{G}(\mathcal{C})| = 15$: [(1, 17), (14, 1)], [(2, 9), (13, 1)].

We also can discard the following cases by Lemma 3.15:

- $|\mathcal{G}(\mathcal{C})| = 7$: [(1, 25), (6, 1)], [(1, 17), (1, 9), (5, 1)], [(3, 9), (4, 1)],
- $|\mathcal{G}(\mathcal{C})| = 5$: [(1, 19), (4, 3)], [(2, 11), (3, 3)],
- $|\mathcal{G}(\mathcal{C})| = 3$: [(1, 21), (2, 5)], [(2, 13), (1, 5)].

This leaves just the pointed and perfect cases, as desired. □

7. Odd-dimensional MTCs of rank 33–49

In this section, we reduce the possible cases of odd-dimensional MTCs of rank between 33 and 49. We begin each case by determining all possible values of $|\mathcal{G}(\mathcal{C})|$, which we find using Lemma 3.13. Then, for each of those values, we enumerate all possible multisets that contain the ranks of the components of the universal grading, and use this information to discard possibilities.

Additionally, we can often directly apply lemmas previously shown to quickly discard many cases. Specifically, the following are the methods, which are also listed in Appendix B.

(a) Lemma 3.14,
(b) Lemma 3.16,
(c) Lemma 4.3 in conjunction with Lemma 3.16,
(d) [CP, Proposition 5.6], Proposition 3.7, and Proposition 3.17, all of which are in conjunction with Lemma 3.16.

Note that we can use options (a) and (b) whenever $|\mathcal{G}(\mathcal{C})|$ is larger than 1 (i.e., whenever the multiset representing $\mathcal{C}$ has length greater than 1), since then $\mathcal{C}_{\text{pt}}$ is non-trivial and $(\mathcal{C}_{\text{ad}})_{\text{pt}}$ is also non-trivial by Lemma 4.2.

All of the cases for each rank that can be discarded using the above options can be found in Appendix B, and we present methods to manually discard some of the remaining cases in the following propositions.

Proposition 7.1. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank* 33*. Then $\mathcal{C}$ must be pointed, perfect, or have* 3 *invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be* $[(1, 27), (2, 3)]$.

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 33, 25, 17, 11, 9, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(1, 13), (4, 5)],
- [(1, 27), (2, 3)].

Case [(1, 13), (4, 5)]: by Lemma 3.16, we know that $\operatorname{rank}(\mathcal{C}_{\mathrm{ad}}) = 13$. Since $|\mathcal{G}(\mathcal{C})| = 5$, by Corollary 3.8, we also know that the simple objects outside of $\mathcal{C}_{\mathrm{ad}}$ have Frobenius-Perron dimension d satisfying $\mathrm{d}^2 = \frac{\mathrm{FPdim}(\mathcal{C})}{25}$. Let d_1 be the largest Frobenius-Perron dimension of any simple object in $\mathcal{C}$, corresponding to simple object X_1, and let $m_1 = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{d}_1^2}$, as in Algorithm 3.4. If X_1 is not is $\mathcal{C}_{\mathrm{ad}}$, then $m_1 = 25$. If X_1 is in $\mathcal{C}_{\mathrm{ad}}$, then so is its dual, and $2\,\mathrm{d}_1^2 < \mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = \frac{\mathrm{FPdim}(\mathcal{C})}{5}$, so $m_1 = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{d}_1^2} > 10$. By [CP, Lemma 5.4], we know that $m_1 \equiv \operatorname{rank}(\mathcal{C}) \equiv 1 \pmod 8$, so $m_1 \geq 17$, and by $\mathrm{FPdim}(\mathcal{C}) = 25\,\mathrm{d}^2$, m_1 must be a perfect square, so $m_1 \geq 25$. Running the Algorithm 3.4 for rank 33 with 5 invertible objects only for cases with $m_1 \geq 25$ produces no solutions, which discards this case.

This means that $\mathcal{C}$ must be pointed, perfect, or have 3 invertible objects, with the last case represented by [(1, 27), (2, 3)], as desired. □

Proposition 7.2. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 35. Then $\mathcal{C}$ must be*

(1) pointed,
(2) perfect, or
(3) the modular subcategory of $\mathcal{Z}(\mathrm{Vec}_{H_3}^{\omega})$ with 9 invertible objects and 26 simple objects of dimension 3, where H_3 denotes the Heisenberg group of order 3^3.

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 35, 27, 19, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(1, 11), (8, 3)],
- [(1, 17), (2, 9)].

Case [(1, 11), (8, 3)]:
since $\mathcal{C}_{\mathrm{ad}}$ has rank 11 and we know that $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 9$ by Corollary 3.18, then there are only two noninvertible simple objects X and X^* in $\mathcal{C}_{\mathrm{ad}}$. Moreover, both X and X^* are fixed by the action of $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$. Consider the decomposition

$$X \otimes X^* = \bigoplus_{g \in \mathcal{G}(\mathcal{C}_{\mathrm{ad}})} g \oplus m(X \oplus X^*),$$

for some $m \geq 0$. Taking Frobenius-Perron dimension on the equation above, we get $\mathrm{FPdim}(X)^2 = 9 + 2m\,\mathrm{FPdim}(X)$. Solving the quadratic gives two potential dimensions for X: $\mathrm{FPdim}(X) = 3$ or $\mathrm{FPdim}(X) = 9$. But if $\mathrm{FPdim}(X) = 9$ then $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = 2 \cdot 9^2 + 9 = 3^2 \cdot 19$, which is not possible. In fact, let $\mathcal{C}_g$ be any other component. Then $\mathcal{C}_g$ has 3 simple objects with the same dimension d, see [CP, Lemma 5.1], and so $3^2 \cdot 19 = \mathrm{FPdim}(\mathcal{C}_g) = 3d^2$, a contradiction. Now, if $\mathrm{FPdim}(X) = 3$ then $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = 2 \cdot 3^2 + 9 = 3^2 \cdot 3$ and $\mathrm{FPdim}(\mathcal{C}) = 3^5$. Hence $\mathcal{C}$ is the modular subcategory of $\mathcal{Z}(\mathrm{Vec}_{H_3}^{\omega})$ with 9 invertible objects and 26 simple objects of dimension 3, where H_3 denotes the Heisenberg group of order 3^3, see [CPS, Example 4] and [CP, Theorem 3.5].

Case [(1, 17), (2, 9)]: note that by Lemma 4.2, $|\mathcal{G}(\mathcal{C})| = |\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3$.

We use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$. This produces the cases listed in Table 2, which exhibits these dimensions in columns 1 through 7, excluding duplicates due to duals.

We will discard all potential arrays on a case-by-case basis.

Recall that MTCs of Frobenius-Perron dimension mp^2, for m a square free integer, are pointed. In case 1, we would have that $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is an MTC of Frobenius-Perron dimension $5^2 \cdot 19$, see Section 2.2.2. It follows that $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is pointed, which is not possible. In fact, the

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7
1	$3^2 \cdot 5^2 \cdot 19$	15	15	15	3	3	3	3
2	$3^3 \cdot 5^2$	5	5	5	3	3	3	3
3	$3^2 \cdot 43$	3	3	3	3	3	3	3

TABLE 2. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$ for $\mathcal{C}$ an MTC of rank 35 with 3 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 17$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

simple objects of Frobenius-Perron dimension 15 in $\mathcal{C}_{\text{ad}}$ would generate simple objects of dimensions either 15 or 5 in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. Case 2 is discarded in the same way, since $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is an MTC of Frobenius-Perron dimension $5^2 \cdot 3$, hence pointed, but simple objects of dimension 5 in $\mathcal{C}_{\text{ad}}$ generate objects of dimension 5 in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. Lastly, in case 3, $\text{FPdim}(\mathcal{C}) = 3^2 \cdot 43$ and so $\mathcal{C}$ should be pointed, a contradiction. □

Proposition 7.3. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank* 37*. Then $\mathcal{C}$ must be either pointed or perfect.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 37, 29, 21, 13, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect case remains:

- [(2, 15), (1, 7)].

Case [(2, 15), (1, 7)]: assume for contradiction that an MTC corresponding to this case exists. By Lemma 3.15, we know that $\text{rank}(\mathcal{C}_{\text{ad}}) = 7$. We also know that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. Consider the de-equivariantization of $\mathcal{C}_{\text{ad}}$ by $\mathbb{Z}_3$. Since there are only 4 non-invertible objects in $\mathcal{C}_{\text{ad}}$ and the number of non-fixed objects must be a multiple of 6 by Remark 3.11, all must be fixed by the action of $\mathcal{G}(\mathcal{C})$ on the simple objects of $\mathcal{C}_{\text{ad}}$. Therefore, each non-invertible simple object in $\mathcal{C}_{\text{ad}}$ creates three simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, so the rank of $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ must be $1 + 4 \cdot 3 = 13$.

Since $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is modular (see Section 2.2.2), it must be pointed by [CP, Theorem 6.3 (a)]. This means that $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 13$ and $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 39$. This must be equal to the Frobenius-Perron dimension of any of the two non-adjoint components $\mathcal{C}_g$ of the universal grading of $\mathcal{C}$. The Frobenius-Perron dimension of all simple objects outside of $\mathcal{C}_{\text{ad}}$ must be at least 3 since all invertible objects are in $\mathcal{C}_{\text{ad}}$. Since the rank of the non-adjoint components is 15, the Frobenius-Perron dimension of a non-adjoint component $\mathcal{C}_g$ is at least $15 \cdot 9 = 135 > 39$, a contradiction.

This means that $\mathcal{C}$ must be pointed or perfect, as desired. □

Proposition 7.4. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank* 39*. Then $\mathcal{C}$ must be either pointed or perfect.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 39, 31, 23, 15, 13, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard all cases directly using previously shown lemmas, which are displayed in Appendix B. Hence, as all non-pointed and non-perfect cases are discarded, $\mathcal{C}$ must be pointed or perfect, as desired.

□

Proposition 7.5. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank* 41*. Then $\mathcal{C}$ is either pointed, perfect, or*

- *$\mathcal{C}$ has 5 invertible objects, and Frobenius-Perron dimension array as in row 15 of Table 3, or*
- *$\mathcal{C}$ has 3 invertible objects, and the ranks of the components of the universal grading of $\mathcal{C}$ are* $[(1,35),(2,3)]$.

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 41, 33, 25, 17, 11, 9, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(1, 21), (4, 5)],
- [(1, 35), (2, 3)].

Case [(1, 21), (4, 5)]: note that by Lemma 4.2, $|\mathcal{G}(\mathcal{C})| = |\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$.

Using Algorithm 3.4, we compute all potential cases for the Frobenius-Perron dimensions of simple objects in the category. Similarly to the case [(1, 13), (4, 5)] for rank 33 in Proposition 7.1, m_1 must be a perfect square that is at least 25. Running Algorithm 3.4 for rank 41 with 5 invertible objects only for cases with $m_1 \geq 25$ produces the arrays listed in Table 3. The first column holds the Frobenius-Perron dimension of the 20 objects outside $\mathcal{C}_{\text{ad}}$ for each case. The rest of the columns show the Frobenius-Perron dimensions of the remaining non-invertible simple objects, excluding duplicates due to duals.

We will discard all potential arrays on a case-by-case basis.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8	9
1	$3^6 \cdot 5^2 \cdot 13^2$	351	351	351	195	135	65	27	15	13
2	$3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$	9009	9009	9009	5005	3465	1365	1287	63	35
3	$3^4 \cdot 5^4 \cdot 7^2 \cdot 13^2$	4095	4095	4095	2275	1365	819	819	63	13
4	$3^6 \cdot 5^4$	135	135	135	75	45	27	25	9	5
5	$3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2$	231	231	231	105	105	35	35	33	33
6	$3^6 \cdot 5^2 \cdot 11^2$	297	297	297	135	99	99	55	55	15
7	$3^2 \cdot 5^2 \cdot 11^2$	33	33	33	15	11	11	5	5	5
8	$3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2$	273	273	273	105	91	91	91	35	13
9	$3^2 \cdot 5^2 \cdot 7^2 \cdot 13^2$	273	273	273	105	91	91	65	65	35
10	$3^2 \cdot 5^2 \cdot 17^2 \cdot 37^2$	1887	1887	1887	629	629	555	555	555	255
11	$3^2 \cdot 5^2 \cdot 19^2$	57	57	51	19	19	15	15	15	15
12	$3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$	9009	9009	6435	6435	5005	3465	1365	63	35
13	$3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2$	231	231	165	165	105	105	35	35	33
14	$3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2$	231	165	165	165	165	105	105	35	35
15	$3^4 \cdot 5^2$	9	5	5	5	5	5	5	5	5

TABLE 3. Potential dimension arrays for MTC $\mathcal{C}$ of rank 41 with 5 invertible objects, such that $m_1 \geq 25$. All but line 15 are discarded.

Note that dimensions in columns 2–9 that appear less than five times in a row must correspond with simple objects that are fixed by the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$. Hence cases 1–13 are discarded following Remark 3.11. Case 15 is treated in Proposition 7.6.

It remains to look at case 14. All simple objects in $\mathcal{C}_{\text{ad}}$ must be fixed, as no dimension in columns 2–9 is repeated 5 times. Consider the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$, which is again an MTC, see Section 2.2.2. Non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ will have Frobenius-Perron dimensions 33, 21 and 7. Moreover, there is a unique invertible object in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$.

Let X_1 be one of the simple objects of Frobenius-Perron dimension 7, and consider the decomposition

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{k} N_{X_1, X_1^*}^{X_i} (X_i \oplus X_{i^*}),$$

where $1, X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ denote all simple objects in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_5}$, and we are using that $N_{X_1,X_1^*}^{X_i} = N_{X_1,X_1^*}^{X_i^*}$ for all $i = 1, \ldots, k$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$49 = 1 + 2 \sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \operatorname{FPdim}(X_i).$$

Hence $N_{X_1,X_1^*}^{X_i} = 0$ for all X_i with $\operatorname{FPdim}(X_i) = 33$ or 21. But then

$$49 = 1 + 2m \cdot 7$$

for some $m \geq 1$, a contradiction. Hence case 14 is also discarded.

This means that $\mathcal{C}$ must be pointed, perfect, have 5 invertible objects with Frobenius-Perron dimensions as in row 15 of Table 3, or have 3 invertible objects, with the last case represented by [(1, 35), (2, 3)], as desired. □

Proposition 7.6. *Suppose there exists an MTC $\mathcal{C}$ of rank 41 with 5 invertible objects and Frobenius-Perron dimension array as in row 15 in Table 3. Then $\mathcal{C} \cong \mathcal{D}^{\mathbb{Z}_5}$, where $\mathcal{D}$ is a categorification of the ring $R_{5,H}$ as defined in* [JL, Definition 1.3]*, and H is a finite abelian group of order 3^4.*

Proof. Let $\mathcal{C}$ be as in the statement, and consider the Tannakian subcategory $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}} \cong \operatorname{Rep}(\mathbb{Z}_5)$.

Note that all non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$ are fixed by the action of $\mathcal{G}(\mathcal{C}) \cong \mathbb{Z}_5$: otherwise, the rank of the de-equivariantization $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_5}$ (which is modular by Section 2.2.2) would be of rank 33 with 31 invertible objects, which is impossible by Proposition 7.1. Hence, $\mathcal{C}_{\mathbb{Z}_5}$ has 81 invertible objects. On the other hand, each non-trivial component $\mathcal{C}_g$ contains 5 simple objects which are not fixed by the action, see Proposition 3.7. Hence the simple objects of each non-trivial component induce an object of dimension 9 in $\mathcal{C}_{\mathbb{Z}_5}$. That is, $\mathcal{C}_{\mathbb{Z}_5}$ is $\mathbb{Z}_5$-graded of rank 85, with 81 invertible objects in the trivial component, and 4 simple objects of dimension 9. Let $(\mathcal{C}_{\mathbb{Z}_5})_{\mathrm{pt}} \cong \operatorname{Vec}_H^{\omega}$ for some abelian group H of order 81 and 3-cocycle ω. Then computing the fusion rules for $\mathcal{C}_{\mathbb{Z}_5}$ shows that $\mathcal{K}(\mathcal{C}_{\mathbb{Z}_5}) \cong R_{5,H}$, where $\mathcal{K}(\mathcal{C}_{\mathbb{Z}_5})$ denotes the Grothendieck ring of $\mathcal{C}_{\mathbb{Z}_5}$ and $R_{5,H}$ is the fusion ring defined on [JL, Definition 1.3].

We have shown that, if such a category $\mathcal{C}$ exists, its de-equivariantization $\mathcal{C}_{\mathbb{Z}_5}$ must be equivalent to some categorification of $R_{5,H}$, and the statement follows. □

Remark 7.7. Categorifications of $R_{5,H}$ were parametrized in [JL, Proposition 3.1]. Let $\mathcal{D}$ be such a categorification. To obtain a category $\mathcal{C}$ as in row 15 in Table 3, by the lemma above we would need to have a (non-trivial) action of $\mathbb{Z}_5$ on $\mathcal{D}$. Since this induces an action of $\mathbb{Z}_5$ on $\mathcal{D}_{\mathrm{pt}} \cong \operatorname{Vec}_H$, we must have that $5 \mid |\operatorname{Aut}(H)|$. Since H is a finite abelian group of order 81, it follows that we should have $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Proposition 7.8. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank* 43*. Then $\mathcal{C}$ must be pointed, perfect, or have* 9 *invertible objects and Frobenius-Perron dimension array of $\mathcal{C}_{\mathrm{ad}}$ as in one of the rows 5–8 or 11 of Table 4.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 43, 35, 27, 19, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(1, 19), (8, 3)],
- [(2, 11), (7, 3)],
- [(1, 25), (2, 9)].

Case [(1, 19), (8, 3)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ or 9.

For the case $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$, since $|\mathcal{G}(\mathcal{C})| = 9 = 3 \cdot |\mathcal{G}(\mathcal{C}_{\text{ad}})|$ and $\text{rank}(\mathcal{C}_{\text{ad}}) = 19$, this case is discarded by Corollary 3.18.

For the case $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. This produces the cases listed in Table 4, which exhibits these dimensions in columns 1 through 5, excluding duplicates due to duals. We will discard potential arrays on a case-by-case basis.

#	FPdim($\mathcal{C}$)	1	2	3	4	5
1	$3^6 \cdot 5^4 \cdot 19$	675	135	75	27	27
2	$3^4 \cdot 5^2 \cdot 19$	45	9	3	3	3
3	$3^6 \cdot 7^2 \cdot 19$	189	27	27	21	9
4	$3^6 \cdot 19$	27	3	3	3	3
5	$3^7 \cdot 5^4$	225	135	75	27	27
6	$3^5 \cdot 5^2$	15	9	3	3	3
7	$3^7 \cdot 7^2$	63	27	27	21	9
8	3^7	9	3	3	3	3
9	$3^4 \cdot 43$	9	9	3	3	3
10	$3^4 \cdot 59$	9	9	9	3	3
11	$3^5 \cdot 5^2$	9	9	9	9	3
12	$3^4 \cdot 7 \cdot 13$	9	9	9	9	9
13	$3^4 \cdot 11$	3	3	3	3	3

TABLE 4. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$ for $\mathcal{C}$ an MTC of rank 43 with 9 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 19$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$. All but lines 5–8 and 11 are discarded.

In each case, by Proposition 3.7, the Frobenius-Perron dimensions of all simple objects in the components of rank 3 are equal. As a result, we know that FPdim($\mathcal{C}$) is of the form $27\,\mathrm{d}^2$, so $\frac{\text{FPdim}(\mathcal{C})}{27}$ must be a perfect square. Hence, we can discard cases 1–4, 9, 10, 12, and 13. This leaves rows 5–8 and 11, as desired.

Case [(2, 11), (7, 3)]: in this case $|\mathcal{G}(\mathcal{C})| = 9$. By Lemma 3.16, we know that $\text{rank}(\mathcal{C}_{\text{ad}}) = 3$. Since $(\mathcal{C}_{\text{ad}})_{\text{pt}}$ is non-trivial by Lemma 4.2 and $|\mathcal{G}(\mathcal{C})| = 9$, we also have $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$, so $\mathcal{C}_{\text{ad}}$ is pointed and $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 3$. This is a contradiction since the Frobenius-Perron dimension of the components of rank 11 must be greater than 3.

Case [(1, 25), (2, 9)]: note that by Lemma 4.2, $|\mathcal{G}(\mathcal{C})| = |\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$.

We use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 5, which exhibits these dimensions in columns 1 through

11, excluding duplicates due to duals. We will discard all potential arrays on a case-by-case basis.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8	9	10	11
1	$3^2 \cdot 5^2 \cdot 11$	15	5	5	5	5	5	5	3	3	3	3
2	$3^8 \cdot 5^2 \cdot 19$	405	405	405	135	81	45	15	15	5	5	5
3	$3^4 \cdot 11^2 \cdot 19$	99	99	99	33	11	11	11	9	9	3	3
4	$3^4 \cdot 13^2 \cdot 19$	117	117	117	39	13	13	13	9	9	9	3
5	$3^4 \cdot 5^2 \cdot 19$	45	45	45	15	5	5	5	3	3	3	3
6	$3^2 \cdot 5^2 \cdot 13^2 \cdot 19$	195	195	195	39	39	39	39	15	3	3	3
7	$3^2 \cdot 5^4 \cdot 17^2 \cdot 19$	1275	1275	1275	255	255	255	255	75	51	51	3
8	$3^2 \cdot 5^4 \cdot 19$	75	75	75	15	15	15	15	3	3	3	3
9	$3^4 \cdot 5^2 \cdot 19$	45	45	45	9	9	9	5	5	5	3	3
10	$3^2 \cdot 7^2 \cdot 19$	21	21	21	3	3	3	3	3	3	3	3
11	$3^9 \cdot 5^2$	135	135	135	135	81	45	15	15	5	5	5
12	$3^5 \cdot 11^2$	33	33	33	33	11	11	11	9	9	3	3
13	$3^5 \cdot 13^2$	39	39	39	39	13	13	13	9	9	9	3
14	$3^5 \cdot 5^2$	15	15	15	15	5	5	5	3	3	3	3
15	$3^3 \cdot 5^2 \cdot 13^2$	65	65	65	39	39	39	39	15	3	3	3
16	$3^3 \cdot 5^4 \cdot 17^2$	425	425	425	255	255	255	255	75	51	51	3
17	$3^3 \cdot 5^4$	25	25	25	15	15	15	15	3	3	3	3
18	$3^5 \cdot 5^2$	15	15	15	9	9	9	5	5	5	3	3
19	$3^3 \cdot 7^2$	7	7	7	3	3	3	3	3	3	3	3
20	$3^5 \cdot 5^2$	15	15	9	9	9	9	9	9	5	5	5
21	$3^2 \cdot 5^2 \cdot 43$	15	15	15	15	15	15	15	3	3	3	3
22	$3^2 \cdot 67$	3	3	3	3	3	3	3	3	3	3	3

TABLE 5. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$ for $\mathcal{C}$ an MTC of rank 43 with 3 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 25$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

Dimensions in columns 1–11 that appear a number of times not divisible by 3 must correspond with simple objects that are fixed by the action of $\mathcal{G}(\mathcal{C}_{\text{ad}}) \cong \mathbb{Z}_3$. In case 3, this implies the simple objects of Frobenius-Perron dimension 3 must be fixed by the action. Hence for each one of these, we get 3 invertible objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, and so there are a total of 13 invertible objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. This is a contradiction, since 13 does not divide $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 3^2 \cdot 11^2 \cdot 19$. Cases 4, 7, 9, 12–13, 16 and 18 are discarded in the same way.

In cases 1, 8, 10, 17, 19 and 21, $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = mp^k$, where p is an odd prime, $k \leq 4$ and m is an odd square-free integer. Since $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is modular, this implies it should be pointed, see [DN]. Hence the Frobenius-Perron dimension of all simple objects in $\mathcal{C}_{\text{ad}}$ should be 3, which is not true in any of these cases.

In cases 2, 11 and 20, $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = mp^kq^l$, where p and q are odd primes, $k, l \geq 1$, and m is an odd square-free integer. It follows that $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is weakly-group-theoretical and thus solvable, see [N1, NP]. Hence by [ENO2, Proposition 4.5] $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ should have a non-invertible simple object, which contradicts the fact that in all these cases, $\mathcal{C}_{\text{ad}}$ has no simple object of Frobenius-Perron dimension 3.

In case 15, $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = 3 \cdot 5^2 \cdot 13^2$, and so again $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ should be solvable. There are 6 simple objects of Frobenius-Perron dimension 3 in $\mathcal{C}_{\mathrm{ad}}$. By Remark 3.11, either all of them are fixed, or none of them are. So $\mathcal{G}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})$ has either size 1 or 19 , respectively. The former is not possible by [ENO2, Proposition 4.5]. The latter is also not possible, since 19 does not divide $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = 3 \cdot 5^2 \cdot 13^2$.

In case 5, $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = 3^2 \cdot 5^2 \cdot 19$. Note that there are 8 simple objects of Frobenius-Perron dimension 3 in $\mathcal{C}_{\mathrm{ad}}$. By Remark 3.11, either exactly 2 of them are fixed, or all of them are. So $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ has either 25 or 7 invertible objects, respectively. The latter is not possible, since 7 does not divide $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = 3^2 \cdot 5^2 \cdot 19$. If the former is true, then because 25 does not divide $\mathrm{FPdim}(((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})_{\mathrm{ad}}) = 3^2 \cdot 19$, it must be the case that $((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})_{\mathrm{ad}}$ is perfect. But from [ENO2, Theorem 1.6] we get that $((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3})_{\mathrm{ad}}$ is solvable and so it should contain a non-trivial invertible object by [ENO2, Proposition 4.5], which is a contradiction. Cases 6 and 14 are discarded in the same way.

Lastly, in case 22 we have that $\mathrm{FPdim}(\mathcal{C}) = 3^2 \cdot 67$, and so it should be pointed, a contradiction. □

Proposition 7.9. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 45. Then $\mathcal{C}$ must be either pointed or perfect.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 45, 37, 29, 21, 15, 13, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(15, 3)],
- [(9, 5)],
- [(3, 15)].

Case [(15, 3)]: since $(\mathcal{C}_{\mathrm{ad}})_{\mathrm{pt}}$ is non-trivial, we know $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3 = \mathrm{rank}(\mathcal{C}_{\mathrm{ad}})$, and $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = 3$. That means that $\mathrm{FPdim}(\mathcal{C}_g) = 3$ for all components $\mathcal{C}_g$ of the universal grading of $\mathcal{C}$. But that means that all simple objects of $\mathcal{C}$ are invertible, and $\mathcal{C}$ is pointed, a contradiction.

Case [(9, 5)]: in this case $|\mathcal{G}(\mathcal{C})| = 9$, $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3$, and $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \cong \mathbb{Z}_3$. Consider the de-equivariantization $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ of $\mathcal{C}_{\mathrm{ad}}$ by $\mathcal{G}(\mathcal{C}_{\mathrm{ad}}) \cong \mathbb{Z}_3$. The three invertible objects of $\mathcal{C}_{\mathrm{ad}}$ create one simple object in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$. Each of the two non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$ must be fixed under the action of $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ on the simple objects of $\mathcal{C}_{\mathrm{ad}}$, and as such, each of them creates three simple objects in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$. Therefore, $\mathrm{rank}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = 1 + 3 \cdot 2 = 7$, and $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is pointed by [BR, Theorem 4.5]. This means that $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}) = 7$ and $\mathrm{FPdim}(\mathcal{C}_{\mathrm{ad}}) = 3 \cdot 7 = 21$, which implies that $\mathrm{FPdim}(\mathcal{C}_g) = 21$ for every component $\mathcal{C}_g$ of the universal grading of $\mathcal{C}$. However, there are only 6 invertible objects in the 8 non-adjoint components. That means that at least one of the components must contain no invertible objects, and the Frobenius-Perron dimensions of that component must be at least $5 \cdot 9 > 21$, leading to a contradiction, and thus discarding the case.

Case [(3, 15)]: note that by Lemma 4.2, $|\mathcal{G}(\mathcal{C})| = |\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3$.

We use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$. This produces the cases listed in Table 6, which exhibits these dimensions in columns 1 through 6, excluding duplicates due to duals.

We will discard all potential arrays on a case-by-case basis.

In case 1, we would have that $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is an MTC of Frobenius-Perron dimension $5^2 \cdot 13$, see Section 2.2.2. It follows that $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is pointed, which is not possible. In fact, the

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6
1	$3^2 \cdot 5^2 \cdot 13$	15	15	3	3	3	3
2	$3^2 \cdot 37$	3	3	3	3	3	3

TABLE 6. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$ for $\mathcal{C}$ an MTC of rank 45 with 3 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 15$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

simple objects of Frobenius-Perron dimension 15 in $\mathcal{C}_{\text{ad}}$ would generate simple objects of dimensions either 15 or 5 in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. Lastly, in case 2, $\text{FPdim}(\mathcal{C}) = 3^2 \cdot 37$ and so $\mathcal{C}$ should be pointed, a contradiction.

This means that $\mathcal{C}$ must be pointed or perfect, as desired. □

Proposition 7.10. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 47. Then $\mathcal{C}$ must be either pointed or perfect.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 47, 39, 31, 23, 15, 13, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect case remains:

- [(2, 21), (1, 5)].

Case [(2, 21), (1, 5)]: similarly to the case [(9, 5)] in the proof of Proposition 7.9 for rank 45, considering the de-equivariantization of $\mathcal{C}_{\text{ad}}$ by $\mathcal{G}(\mathcal{C}_{\text{ad}}) \cong \mathbb{Z}_3$, we know that $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 21$. However, since all invertible objects are in $\mathcal{C}_{\text{ad}}$, the non-adjoint components of the universal grading of $\mathcal{C}$ must have Frobenius-Perron dimension of at least $21 \cdot 9 > 21$, leading to a contradiction, and discarding this case.

This means that $\mathcal{C}$ must be pointed or perfect, as desired. □

Proposition 7.11. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 49. Then $\mathcal{C}$ must be pointed, perfect, or have 3 or 5 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be [(1, 43), (2, 3)], and if $|\mathcal{G}(\mathcal{C})| = 5$, then the ranks must be [(1, 29), (4, 5)].*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 49, 41, 33, 25, 17, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(7, 7)],
- [(1, 29), (4, 5)],
- [(1, 43),(2, 3)].

Case [(7, 7)]: since $(\mathcal{C}_{\text{ad}})_{\text{pt}}$ is non-trivial and $|\mathcal{G}(\mathcal{C})| = 7$, we must have $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 7 = \text{rank}(\mathcal{C}_{\text{ad}})$, which means $\mathcal{C}_{\text{ad}}$ is pointed. Then the Frobenius-Perron dimension of all components must be 7, and $\mathcal{C}$ is pointed, which is a contradiction.

This means that $\mathcal{C}$ must be pointed, perfect, or have 3 or 5 invertible objects, with the last cases represented by [(1, 43), (2, 3)] and [(1, 29), (4, 5)], respectively, as desired. □

The next proposition provides restrictions on an odd-dimensional MTC $\mathcal{C}$ of rank 49 with $|\mathcal{G}(\mathcal{C})| = 5$.

Proposition 7.12. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 49 with 5 invertible objects. If $7 \nmid \text{FPdim}(\mathcal{C})$, then $\mathcal{C} \cong \text{Rep}(D^\omega(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5))$, with $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ non-abelian and ω a 3-cocycle.*

Proof. By Proposition 7.11, the multiset of the ranks of the universal grading components of $\mathcal{C}$ is represented by [(1, 29), (4, 5)]. We perform casework based on whether there exists a simple object in $\mathcal{C}_{\text{ad}}$ that is not fixed by the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$ on the simple objects of $\mathcal{C}_{\text{ad}}$.

First, consider the case in which all non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ are fixed by that action. The Frobenius-Perron dimensions of all non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ must be divisible by 5. Define d_i and $m_i = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{d}_i^2}$ for each i, as in Section 3.2. We show that none of the values of m_i can be divisible by 5.

Suppose for the sake of contradiction that $5 \mid m_i$ for some i. As $5 \mid \mathrm{d}_i$, we know that $\mathrm{FPdim}(\mathcal{C}) = m_i \, \mathrm{d}_i^2$ is a multiple of 125, so $\mathrm{FPdim}(\mathcal{C}_{\text{ad}}) = \frac{\mathrm{FPdim}(\mathcal{C})}{5}$ is a multiple of 25. But we also know that $\mathrm{FPdim}(\mathcal{C}_{\text{ad}}) = 5 + 2\,\mathrm{d}_1^2 + \cdots + 2\,\mathrm{d}_{12}^2 \equiv 5 \pmod{25}$, as each d_i is divisible by 5, a contradiction.

Additionally, as $\mathrm{FPdim}(\mathcal{C})$ is not divisible by 7, we know that $m_1 \neq 49$. Similarly to the case [(1, 13), (4, 5)] for rank 33 in Proposition 7.1, m_1 must be a perfect square that is at least 25. We use Algorithm 3.6 modified to consider only the cases where all values of m_i are not divisible by 5, $m_1 \neq 49$, and $m_1 > 25$ to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ (note that we skip $m_1 = 25$ as m_1 cannot be divisible by 5). This produces the cases listed in Table 7, which exhibits these dimensions in columns 1 through 12, excluding duplicates due to duals.

#	$\mathrm{FPdim}(\mathcal{C})$	1	2	3	4	5	6	7	8	9	10	11	12
1	$3^4 \cdot 5^2 \cdot 29^2$	145	145	145	145	145	145	145	145	45	5	5	5
2	$3^8 \cdot 5^2 \cdot 11^2$	495	495	495	495	495	495	495	495	135	55	55	15
3	$3^4 \cdot 5^2 \cdot 11^2$	55	55	55	55	55	55	55	55	15	5	5	5
4	$3^4 \cdot 5^2 \cdot 7^2 \cdot 37^2$	1295	1295	1295	1295	1295	1295	1295	1295	315	185	185	5
5	$3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2$	455	455	455	455	455	455	455	455	105	65	65	35
6	$3^4 \cdot 5^2 \cdot 17^2 \cdot 37^2$	3145	3145	3145	3145	3145	3145	3145	3145	555	555	555	255
7	$3^4 \cdot 5^2 \cdot 19^2$	95	95	95	95	95	95	95	95	15	15	15	15
8	$3^4 \cdot 5^2 \cdot 11^2$	55	55	55	55	55	55	45	45	45	15	5	5
9	$3^4 \cdot 5^2 \cdot 11^2$	55	55	55	55	45	45	45	45	45	45	15	5
10	$3^4 \cdot 5^2 \cdot 11^2$	55	55	45	45	45	45	45	45	45	45	45	15
11	$5^2 \cdot 11^2$	5	5	5	5	5	5	5	5	5	5	5	5

TABLE 7. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$ for $\mathcal{C}$ an MTC of rank 49 with 5 invertible objects, $\mathrm{rank}(\mathcal{C}_{\text{ad}}) = 29$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$, where all values of m_i are not divisible by 5, $m_1 \neq 49$, and $m_1 > 25$. All but line 11 discarded.

First, note that we do not consider cases 4 and 5, as their Frobenius-Perron dimensions are divisible by 7.

Now, consider cases 1, 3, and 8, and consider the de-equivariantizations $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$, which are modular (see Section 2.2.2). In these cases, there are 31, 31, and 21 invertible objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$, respectively (as each non-invertible object of Frobenius-Perron dimension 5 creates 5 invertible objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$). However, $\mathrm{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}) = \frac{\mathrm{FPdim}(\mathcal{C})}{25}$ is not divisible by this count in any of the cases, a contradiction. Thus, these cases are discarded.

Next, we discard cases 2, 7, and 10. In each case, $\mathrm{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}) = \frac{\mathrm{FPdim}(\mathcal{C})}{25}$ is of the form $p^a q^b$ for primes p and q, so $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ is solvable [ENO2, Theorem 1.6]. Thus, by [ENO2, Proposition 4.5 (iv)], $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ must contain a non-trivial invertible object. But this is a contradiction, as there are no simple objects of Frobenius-Perron dimension 5 in $\mathcal{C}_{\text{ad}}$ for each case, discarding these cases.

We proceed to case 6. Non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ will have Frobenius-Perron dimensions 629, 111, 51, and possibly 3145. Moreover, there is 1 invertible object in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$. Let X_1 be one of the simple objects of Frobenius-Perron dimension 51. Then consider the decomposition

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{k} N_{X_1,X_1^*}^{X_i}(X_i \oplus X_{i^*}),$$

where $X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ denote all non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$, and we are using that $N_{X_1,X_1^*}^{X_i} = N_{X_1,X_1^*}^{X_i^*}$ for all $i = 1, \ldots, k$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$51^2 = 1 + 2\sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \operatorname{FPdim}(X_i).$$

But then

$$2601 = 1 + 2n_1 \cdot 629 + 2n_2 \cdot 111 + 2n_3 \cdot 51$$

for some $n_1, n_2, n_3 \geq 0$. As there are no solutions to this equation, we have a contradiction.

The last case we discard is case 9. Non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ will have Frobenius-Perron dimensions 11, 9, 3, and possibly 45. Moreover, there are 11 invertible objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$. Let X_1 be one of the simple objects of Frobenius-Perron dimension 3, and suppose that it is fixed by only 1 of the invertible objects. Then consider the decomposition

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{k} N_{X_1,X_1^*}^{X_i}(X_i + X_{i^*}),$$

where $X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ denote all non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$, and we are using that $N_{X_1,X_1^*}^{X_i} = N_{X_1,X_1^*}^{X_i^*}$ for all $i = 1, \ldots, k$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$9 = 1 + 2\sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \operatorname{FPdim}(X_i).$$

Hence $N_{X_1,X_1^*}^{X_i} = 0$ for all X_i with $\operatorname{FPdim}(X_i) = 11$ or 9. But then

$$9 = 1 + 2m \cdot 3$$

for some $m \geq 0$, a contradiction.

Next, suppose that X_1 is fixed by all 11 of the invertible objects. Consider the decomposition

$$X_1 \otimes X_1^* = \bigoplus_{g \in G[X]} g \oplus \bigoplus_{i=1}^{k} N_{X_1,X_1^*}^{X_i}(X_i \oplus X_{i^*}),$$

where $X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ denote all non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$9 = 11 + 2\sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \operatorname{FPdim}(X_i),$$

which is clearly a contradiction.

Finally, consider case 11. It follows from Proposition 3.19 that $\mathcal{C} \cong \mathrm{Rep}(D^\omega(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5))$.

Now, we move to the case in which there exists a non-invertible simple object in $\mathcal{C}_{\mathrm{ad}}$ that is not fixed by the action of $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ on the simple objects of $\mathcal{C}_{\mathrm{ad}}$. Define d_i and m_i similarly to the previous case for each i. Then, we know that at least 5 of the values of d_i are equal.

Similarly to the case [(1, 13), (4, 5)] for rank 33 in Proposition 7.1, m_1 must be a perfect square that is at least 25. We use Algorithm 3.6 modified to consider only the cases where at least five consecutive values $\mathrm{d}_i, \mathrm{d}_{i+1}, \ldots, \mathrm{d}_{i+4}$ are equal and m_1 is a perfect square that is at least 25 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$. This produces the cases listed in Table 8, which exhibits these dimensions in columns 1 through 12, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8	9	10	11	12
1	$3^4 \cdot 5^2 \cdot 29^2$	145	145	145	145	145	145	145	145	45	5	5	5
2	$3^8 \cdot 5^2 \cdot 11^2$	495	495	495	495	495	495	495	495	135	55	55	15
3	$3^4 \cdot 5^2 \cdot 11^2$	55	55	55	55	55	55	55	55	15	5	5	5
4	$3^4 \cdot 5^2 \cdot 7^2 \cdot 37^2$	1295	1295	1295	1295	1295	1295	1295	1295	315	185	185	5
5	$3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2$	455	455	455	455	455	455	455	455	105	65	65	35
6	$3^4 \cdot 5^2 \cdot 17^2 \cdot 37^2$	3145	3145	3145	3145	3145	3145	3145	3145	555	555	555	255
7	$3^4 \cdot 5^2 \cdot 19^2$	95	95	95	95	95	95	95	95	15	15	15	15
8	$3^4 \cdot 5^2 \cdot 11^2$	55	55	55	55	55	55	45	45	45	15	5	5
9	$5^2 \cdot 11^2$	5	5	5	5	5	5	5	5	5	5	5	5
10	$3^4 \cdot 5^2 \cdot 7^2 \cdot 37^2$	1665	1295	1295	1295	1295	1295	1295	555	555	315	185	5
11	$3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2$	585	455	455	455	455	455	455	195	195	105	65	35
12	$3^6 \cdot 5^2 \cdot 7^2$	135	105	105	105	105	105	105	45	35	35	15	15
13	$3^6 \cdot 5^2 \cdot 7^2$	135	105	105	105	105	105	105	35	35	35	35	5
14	$3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 17^2$	8415	6545	6545	6545	6545	6545	5355	5355	1785	1155	315	105
15	$5^2 \cdot 7^2 \cdot 13^2$	65	65	35	35	35	35	35	35	35	35	35	35
16	$3^4 \cdot 5^2 \cdot 11^2$	55	55	45	45	45	45	45	45	45	45	45	15
17	$3^2 \cdot 5^2 \cdot 7^2$	15	15	15	15	5	5	5	5	5	5	5	5
18	$3^4 \cdot 5^2 \cdot 11^2$	55	55	55	55	45	45	45	45	45	45	15	5
19	$3^6 \cdot 5^2 \cdot 7^2$	135	135	135	105	105	45	45	45	45	45	35	35
20	$5^2 \cdot 7^2 \cdot 11^2$	55	55	55	55	35	35	7	7	7	7	7	5
21	$5^2 \cdot 7^2 \cdot 13^2$	65	65	65	65	35	35	35	5	5	5	5	5

TABLE 8. Potential dimension arrays of $\mathcal{C}_{\mathrm{ad}}$ for $\mathcal{C}$ an MTC of rank 49 with 5 invertible objects, $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) = 29$, and $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 5$, where at least 5 values of d_i are equal and $m_1 \geq 25$ is a perfect square. All discarded.

We disregard cases 4, 5, 10–15, 17, and 19–21, as in those cases FPdim($\mathcal{C}$) is divisible by 7.

In cases 1–3, 6–9, 16, and 18, the Frobenius-Perron dimensions of all non-invertible objects in $\mathcal{C}_{\mathrm{ad}}$ are divisible by 5. We show that all these objects must be fixed under the action of $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ on the simple objects of $\mathcal{C}_{\mathrm{ad}}$. In all these cases, $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_5}) = \frac{\mathrm{FPdim}(\mathcal{C})}{25}$ is not divisible by 5. If a simple object X in $\mathcal{C}_{\mathrm{ad}}$ were not fixed by the action, then there would be a simple object in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_5}$ of dimension $\mathrm{FPdim}(X)$, which is divisible by 5 and is hence not a divisor of $\mathrm{FPdim}((\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_5})$, resulting in a contradiction [ENO2, Theorem 2.11]. Thus, all simple objects in $\mathcal{C}_{\mathrm{ad}}$ are fixed by the action.

Now, note that all of the cases we are considering now (1–3, 6–9, 16, and 18) already appeared in Table 7, and we handled them when we assumed that all simple objects in $\mathcal{C}_{\mathrm{ad}}$ were fixed.

Hence, as we discarded all cases other than the desired existing one, this completes the proof. ☐

We summarize the results from this section in the following theorem.

Theorem 7.13. *Let $\mathcal{C}$ be an odd-dimensional MTC such that $33 \leq \text{rank}(\mathcal{C}) \leq 49$.*

(a) If $\text{rank}(\mathcal{C}) \in \{37, 39, 45, 47\}$, then $\mathcal{C}$ is either pointed or perfect.
(b) If $\text{rank}(\mathcal{C}) = 33$, then $\mathcal{C}$ is pointed, perfect, or has 3 invertible objects.
(c) If $\text{rank}(\mathcal{C}) = 35$, then $\mathcal{C}$ is pointed, perfect, or the modular subcategory of $\mathcal{Z}(\text{Vec}^{\omega}_{H_3})$ with 9 invertible objects and 26 simple objects of dimension 3, where H_3 denotes the Heisenberg group of order 3^3.
(d) If $\text{rank}(\mathcal{C}) = 41$, then $\mathcal{C}$ is pointed, perfect, or has 3 or 5 invertible objects. Moreover, if the latter case exists, then $\mathcal{C}$ should be equivalent to $\mathcal{D}^{\mathbb{Z}_5}$, where $\mathcal{D}$ is a categorification of the ring $R_{5,H}$ as defined in [JL, Definition 1.3]*, and H is a finite abelian group of order 3^4.*
(e) If $\text{rank}(\mathcal{C}) = 43$, then $\mathcal{C}$ is pointed, perfect, or has 9 invertible objects.
(f) If $\text{rank}(\mathcal{C}) = 49$, then $\mathcal{C}$ is pointed, perfect, or has 3 or 5 invertible objects. Additionally, if $\mathcal{C}$ has 5 invertible objects and $7 \nmid \text{FPdim}(\mathcal{C})$, then $\mathcal{C} \cong \text{Rep}(D^{\omega}(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5))$ with $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ non-abelian and for some 3-cocycle ω.

8. Odd-dimensional MTCs of rank 51–73

We perform the same procedures throughout this section as we did in Section 7. Specifically, we begin by finding the possible values of $|\mathcal{G}(\mathcal{C})|$ using Lemma 3.13, and then find all possible multisets that contain the ranks of the components of the universal grading. We use this information to discard possibilities. Once again, many of the cases can be directly discarded using previous lemmas, which are also listed in Appendix B.

(a) Lemma 3.14,
(b) Lemma 3.16,
(c) Lemma 4.3 in conjunction with Lemma 3.16,
(d) [CP, Proposition 5.6], Proposition 3.7, and Proposition 3.17, all of which are in conjunction with Lemma 3.16.

We can use options (a) and (b) whenever $|\mathcal{G}(\mathcal{C})|$ is larger than 1 (i.e., whenever the multiset representing $\mathcal{C}$ has length greater than 1), since then $\mathcal{C}_{\text{pt}}$ is non-trivial and $(\mathcal{C}_{\text{ad}})_{\text{pt}}$ is also non-trivial by Lemma 4.2.

All cases that we discard using the above lemmas are listed in Appendix B. In the rest of this section, we present manual methods to discard many of the remaining cases for each rank.

Proposition 8.1. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 51. Then $\mathcal{C}$ must be pointed, perfect, or have 3 or 9 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be $[(1, 33), (2, 9)]$, and if $|\mathcal{G}(\mathcal{C})| = 9$, then those ranks must be $[(1, 27), (8, 3)]$ or $[(3, 11), (6, 3)]$.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 51, 43, 35, 27, 19, 17, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The non-pointed and non-perfect cases that remain are $[(1, 27), (8, 3)]$, $[(3, 11), (6, 3)]$, and $[(1, 33), (2, 9)]$.

☐

Proposition 8.2. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 53. Then $\mathcal{C}$ must be pointed or perfect.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 53, 45, 37, 29, 21, 15, 13, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(1, 23), (2, 15)].

Case [(1, 23), (2, 15)]: By Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. We use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 9, which exhibits these dimensions in columns 1 through 10, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8	9	10
1	$3^8 \cdot 5^2 \cdot 13$	405	405	135	81	45	15	15	5	5	5
2	$3^4 \cdot 11^2 \cdot 13$	99	99	33	11	11	11	9	9	3	3
3	$3^4 \cdot 13^3$	117	117	39	13	13	13	9	9	9	3
4	$3^4 \cdot 5^2 \cdot 13$	45	45	15	5	5	5	3	3	3	3
5	$3^2 \cdot 5^2 \cdot 13^3$	195	195	39	39	39	39	15	3	3	3
6	$3^2 \cdot 5^4 \cdot 13 \cdot 17^2$	1275	1275	255	255	255	255	75	51	51	3
7	$3^2 \cdot 5^4 \cdot 13$	75	75	15	15	15	15	3	3	3	3
8	$3^4 \cdot 5^2 \cdot 13$	45	45	9	9	9	5	5	5	3	3
9	$3^2 \cdot 7^2 \cdot 13$	21	21	3	3	3	3	3	3	3	3
10	$3^3 \cdot 5^2 \cdot 7$	15	15	15	5	5	5	3	3	3	3
11	$3^2 \cdot 5^2 \cdot 37$	15	15	15	15	15	15	3	3	3	3
12	$3^2 \cdot 5^3$	5	5	5	5	5	5	3	3	3	3
13	$3^2 \cdot 61$	3	3	3	3	3	3	3	3	3	3

TABLE 9. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$ for $\mathcal{C}$ an MTC of rank 53 with 3 invertible objects, rank($\mathcal{C}_{\text{ad}}$) = 23, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

We will discard all potential arrays on a case-by-case basis. In case 1, we know that the Frobenius-Perron dimension of the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is $3^6 \cdot 5^2 \cdot 13$, so $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is solvable by [N2, Corollary 5.4]. Additionally, there does not exist a simple object of dimension 3 in $\mathcal{C}_{\text{ad}}$, so $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is perfect. But this contradicts [ENO2, Proposition 4.5 (iv)]. In case 2, the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ has rank 45 and 13 invertible objects, a contradiction by Proposition 7.9.

Cases 3 and 6 are impossible as in each case, $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ has 7 invertible objects, which does not divide its dimension.

In case 4, we know that there must be either 7 or 25 invertible objects in the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. Since 7 does not divide its Frobenius-Perron dimension of $3^2 \cdot 5^2 \cdot 13$, there must be 25 invertible objects. Additionally, in this case, $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ must have rank 45. But this contradicts Proposition 7.9, discarding this case.

In case 5, as the Frobenius-Perron dimension of the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is $5^2 \cdot 13^3$, so $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ must be solvable by [ENO2, Theorem 1.6]. As a result, by [ENO2, Proposition

4.5 (iv)], $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ cannot be perfect, so it must have 19 invertible objects. However, it has rank either 29 or 45, a contradiction by Theorem 6.2 and Proposition 7.9.

In cases 7, 9, and 12, the Frobenius-Perron dimensions of the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ are $5^4 \cdot 13$, $7^2 \cdot 13$, and 5^3, respectively. Thus, in all of these cases $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is pointed, by [DN, Corollary 4.13], [NR, Proposition 4.11], and a generalization of [DN, Lemma 4.11], respectively. This means that each of their ranks must be equal to their dimensions. Additionally, in each case, the rank of $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ must be at most $3 \cdot \text{rank}(\mathcal{C}_{\text{ad}}) = 69$, i.e., when all simple objects in $\mathcal{C}_{\text{ad}}$ are fixed under the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$ on them (note that this exact value is not even possible, as all invertible objects and simple objects with dimension not divisible by 3 cannot be fixed). But this maximum possible rank is less than the actual rank in each case, hence discarding these cases.

In case 8, the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ has rank either 29 or 45 and 13 invertible objects, a contradiction by Theorem 6.2 and Proposition 7.9.

In case 10, the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ has rank 13, 29, or 45 (as the simple objects of dimension 5 in $\mathcal{C}_{\text{ad}}$ cannot be fixed under the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$ on $\mathcal{G}(\mathcal{C})$) and at least 7 invertible objects. Additionally, $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is not pointed. But this is a contradiction, as there does not exist a non-pointed modular category of rank 13, 29, or 45 with at least 7 invertible objects by [CP, Theorem 6.3 (a)], Theorem 6.2, and Proposition 7.9, respectively.

In case 11, we know that the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ has Frobenius-Perron dimension $5^2 \cdot 37$, so it must be pointed by [NR, Theorem 4.11]. But $\mathcal{C}_{\text{ad}}$ has simple objects of dimension 15, so $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ cannot be pointed, a contradiction.

Finally, in case 13, $\mathcal{C}$ is pointed by [NR, Proposition 4.11], a contradiction (as there are simple objects of Frobenius-Perron dimension 3).

As a result, we have discarded all cases in Table 9. □

Proposition 8.3. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank* 55*. Then $\mathcal{C}$ must be pointed or perfect.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 55, 47, 39, 31, 23, 15, 13, 11, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(2, 21), (1, 13)].

Case [(2, 21), (1, 13)]: we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 10, which exhibits these dimensions in columns 1 through 5, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5
1	$3^2 \cdot 5^2 \cdot 7$	15	3	3	3	3
2	$3^2 \cdot 31$	3	3	3	3	3

TABLE 10. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ MTC of rank 55 with 3 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 13$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

We will discard all potential arrays on a case-by-case basis. In case 1, the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ obtained by de-equivariantization has Frobenius-Perron dimension $5^2 \cdot 7$, hence it is pointed. This contradicts that there are simple objects of dimension 15 in $\mathcal{C}_{\text{ad}}$. In case 2, $\mathcal{C}$ should be pointed, see [NR, Proposition 4.11], which contradicts that the simple objects in $\mathcal{C}_{\text{ad}}$ have dimension 3. Thus all cases in Table 10 are discarded. □

Proposition 8.4. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 57. Then $\mathcal{C}$ must be pointed, perfect, or have 3 or 5 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be $[(1,51),(2,3)]$, and if $|\mathcal{G}(\mathcal{C})| = 5$, then those ranks must be $[(1,37),(4,5)]$.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 57, 49, 41, 33, 25, 19, 17, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(3, 17), (6, 1)],
- [(1, 15), (6, 7)],
- [(2, 27), (1, 3)],
- [(1, 37), (4, 5)],
- [(1, 51), (2, 3)].

Case [(3, 17), (6, 1)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ or $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$. The latter can be discarded by [CP, Proposition 5.6]. For the former, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 11, which exhibits these dimensions in columns 1 through 7, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7
1	$3^8 \cdot 5^2$	81	45	15	15	5	5	5
2	$3^3 \cdot 5^2 \cdot 19$	15	15	15	3	3	3	3
3	$3^4 \cdot 11^2$	11	11	11	9	9	3	3
4	$3^4 \cdot 13^2$	13	13	13	9	9	9	3
5	$3^4 \cdot 5^2$	5	5	5	3	3	3	3
6	$3^3 \cdot 43$	3	3	3	3	3	3	3

TABLE 11. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 57 with 9 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 17$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

Recall that there are components of the universal grading with a unique simple object. Since $|\mathcal{G}(\mathcal{C})| = 9$, this implies that FPdim($\mathcal{C}$) should be a perfect square. This discards rows 2 and 6. On the other hand, for the arrays in rows 3-5, we get that $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 3 \cdot p^2$, for $p = 11, 13$ or 5, respectively, and thus $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ should be pointed, see [NR, Proposition 4.11]. This contradicts that there are simple objects of dimension 11, 13 and 5 in $\mathcal{C}_{\text{ad}}$, respectively.

Lastly, to discard row 1, note that non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ have Frobenius-Perron dimensions 27, 15 or 5. Let X_1 be one of the simple objects of Frobenius-Perron dimension 5. Then consider the decomposition

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{k} N_{X_1,X_1^*}^{X_i}(X_i + X_{i^*}),$$

where $X_1, \dots, X_k, X_1^*, \dots, X_k^*$ denote all non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, and we are using that $N_{X_1,X_1^*}^{X_i} = N_{X_1,X_1^*}^{X_i^*}$ for all $i = 1, \dots, k$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$25 = 1 + 2\sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \operatorname{FPdim}(X_i).$$

But then we must have

$$25 = 1 + 2n_1 \cdot 5,$$

for some $n_1 \geq 0$, which is not possible. Thus all cases in Table 11 are discarded.

Case [(1, 15), (6, 7)]: we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the unique case listed in Table 12, which exhibits these dimensions in columns 1 through 4, excluding duplicates due to duals. Since $\operatorname{FPdim}(C) =$

#	FPdim($\mathcal{C}$)	1	2	3	4
1	$3 \cdot 7^2 \cdot 19$	7	7	7	7

TABLE 12. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 57 with 7 invertible objects, $\operatorname{rank}(\mathcal{C}_{\text{ad}}) = 15$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 7$. All discarded.

$7^2 \cdot d$, for d square free, then $\mathcal{C}$ should be pointed, see [DN, Theorem 4.7, Corollary 4.13]. Hence this case is discarded.

Case [(2, 27), (1, 3)]: since $\operatorname{rank}(\mathcal{C}_{\text{ad}}) = 3$ by Lemma 3.16, we know $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. But this means that $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 3$, which must be equal to the Frobenius-Perron dimension of the remaining two components of the universal grading of $\mathcal{C}$. This is not possible since each of those components contains 27 simple objects, which discards this case. □

Proposition 8.5. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 59. Then $\mathcal{C}$ must be pointed, perfect, or have 3 or 9 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be $[(1, 41), (2, 9)]$, and if $|\mathcal{G}(\mathcal{C})| = 9$, then those ranks must be $[(1, 35), (8, 3)]$.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 59, 51, 43, 35, 27, 19, 17, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(2, 19), (7, 3)],
- [(4, 11), (5, 3)],
- [(1, 35), (8, 3)],
- [(1, 41), (2, 9)].

Case [(2, 19), (7, 3)]: since $\operatorname{rank}(\mathcal{C}_{\text{ad}}) = 3$ by Lemma 3.16, we know $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. But this means that $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 3$, which must be equal to the Frobenius-Perron dimension of the remaining components of the universal grading of $\mathcal{C}$. This is not possible for the components of rank 19, which discards this case.

Case [(4, 11), (5, 3)]: similarly to the previous case, $\operatorname{rank}(\mathcal{C}_{\text{ad}}) = 3$ by Lemma 3.16, and we know $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. This means that $\operatorname{FPdim}(\mathcal{C}_{\text{ad}}) = 3$, which must be equal to the Frobenius-Perron dimension of the remaining components of the universal grading of $\mathcal{C}$. This is not possible for the components of rank 11, which discards this case. ☐

Proposition 8.6. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 61. Then $\mathcal{C}$ must be pointed, perfect, or have 3 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be $[(1, 31), (2, 15)]$.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 61, 53, 45, 37, 29, 21, 15, 13, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(1, 19), (14, 3)],
- [(2, 11), (13, 3)],
- [(2, 13), (7, 5)],
- [(1, 31), (2, 15)].

Case [(1, 19), (14, 3)]: in this case we know $\text{rank}(\mathcal{C}_{\text{ad}}) = 19$ by Lemma 3.16, and, given $|\mathcal{G}(\mathcal{C})| = 15$, we also know that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$, or $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 15$.

The cases $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$ can be discarded by Proposition 3.17 since there is only one component of rank 19 in the universal grading of $\mathcal{C}$, which is less than $\frac{|\mathcal{G}(\mathcal{C})|}{|\mathcal{G}(\mathcal{C}_{\text{ad}})|}$.

The case $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 15$ can be discarded by Lemma 3.14, as there is only one component of rank at least 5 (which divides $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 15$).

Case [(2, 11), (13, 3)]: since $\text{rank}(\mathcal{C}_{\text{ad}}) = 3$ by Lemma 3.16, we know $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. But this means that $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 3$, which must be equal to the Frobenius-Perron dimension of the remaining components of the universal grading of $\mathcal{C}$. This is not possible for the components of rank 11, which discards this case.

Case [(2, 13), (7, 5)]: by Lemma 3.16 we know that $\text{rank}(\mathcal{C}_{\text{ad}}) = 5$. Since $|\mathcal{G}(\mathcal{C})| = 9$, we also know that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$.

Consider the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ of $\mathcal{C}_{\text{ad}}$ by $\mathcal{G}(\mathcal{C}_{\text{ad}}) \cong \mathbb{Z}_3$. The three invertible objects of $\mathcal{C}_{\text{ad}}$ create one simple object in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. The two non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ must be fixed under the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$ on the simple objects of $\mathcal{C}_{\text{ad}}$, and each of them creates three simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. As a result, $\text{rank}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 1 + 3 \cdot 2 = 7$, and by [BR, Theorem 4.5], $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is pointed. This means that $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 7$ and $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 3 \cdot 7 = 21$. This must be equal to the Frobenius-Perron dimension of any component of the universal grading of $\mathcal{C}$. By Proposition 3.17, we know that the components with rank 13 do not contain any invertible objects, and as such their Frobenius-Perron dimension must be at least $13 \cdot 3^2 = 117$, which is larger than 21, discarding this case. ☐

Proposition 8.7. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 63. Then $\mathcal{C}$ must be pointed or perfect.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 63, 55, 47, 39, 31, 23, 21, 15, 13, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(21, 3)],
- [(3, 17), (12, 1)],
- [(9, 7)],
- [(3, 21)].

Case [(21, 3)]: in this case we must have $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ and $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 3$. This must be equal to the Frobenius-Perron dimension of the remaining components of the universal grading of $\mathcal{C}$, and since each of them has rank 3, they must each contain 3 invertible objects. This implies that $\mathcal{C}$ is pointed, a contradiction, which discards this case.

Case [(3, 17), (12, 1)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3, 5$ or 15. Note that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 15$ can be discarded by Proposition 3.7, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ can be discarded by Proposition 3.17. For $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 13, which exhibits these dimensions in columns 1 through 6, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6
1	$3^2 \cdot 5^2 \cdot 47$	15	5	5	5	5	5
2	$3^6 \cdot 5^2 \cdot 7 \cdot 11^2$	495	495	135	55	55	15
3	$3^2 \cdot 5^2 \cdot 7 \cdot 11^2$	55	55	15	5	5	5
4	$3^2 \cdot 5^2 \cdot 7^3 \cdot 13^2$	455	455	105	65	65	35
5	$3^2 \cdot 5^2 \cdot 7 \cdot 17^2 \cdot 37^2$	3145	3145	555	555	555	255
6	$3^2 \cdot 5^2 \cdot 7 \cdot 19^2$	95	95	15	15	15	15
7	$3^2 \cdot 5^2 \cdot 127$	15	15	15	15	5	5
8	$3 \cdot 5^2 \cdot 61$	5	5	5	5	5	5

Table 13. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 63 with 15 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 17$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$. All discarded.

All the generated cases are easily discarded, since the existence of components of the universal grading with a unique simple object implies that $\text{FPdim}(\mathcal{C}_{\text{ad}}) = \text{FPdim}(\mathcal{C})/15$ should be a perfect square.

Case [(9, 7)]: we know that $|\mathcal{G}(\mathcal{C})| = 9$ and $\text{rank}(\mathcal{C}_{\text{ad}}) = 7$.

Consider the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ of $\mathcal{C}_{\text{ad}}$ by $\mathcal{G}(\mathcal{C}_{\text{ad}}) \cong \mathbb{Z}_3$. The three invertible objects of $\mathcal{C}_{\text{ad}}$ create one simple object in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. The four non-invertible simple objects in $\mathcal{C}_{\text{ad}}$ must be fixed under the action of $\mathcal{G}(\mathcal{C}_{\text{ad}})$ on the simple objects of $\mathcal{C}_{\text{ad}}$, and each of them creates three simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$. As a result, $\text{rank}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 1 + 3 \cdot 4 = 13$, and by [CP, Theorem 6.3 (a)], $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is pointed. This means that $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 13$ and $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 3 \cdot 13 = 39$. This must be equal to the Frobenius-Perron dimension of any component of the universal grading of $\mathcal{C}$. By Proposition 3.17, we know that six of the components do not contain any invertible objects, and as such their Frobenius-Perron dimension must be at least $7 \cdot 3^2 = 63$, which is larger than 39, discarding this case.

Case [(3, 21)]: we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 14, which exhibits these dimensions in columns 1 through 9, excluding duplicates due to duals.

In case 12, $\text{FPdim}(\mathcal{C}) = 3^2 \cdot d$, for d square-free, and so it should be pointed, see [DN, Theorem 4.7, Corollary 4.13]. In case 7, $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}) = 5^4 \cdot 7$, and thus by [DN, Corollary 4.13] $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ should be pointed, which contradicts that there are simple objects in $\mathcal{C}_{\text{ad}}$ of Frobenius-Perron dimension greater than 3. Cases 9–11 are discarded in the same way.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8	9
1	$3^8 \cdot 5^2 \cdot 7$	405	135	81	45	15	15	5	5	5
2	$3^4 \cdot 7 \cdot 11^2$	99	33	11	11	11	9	9	3	3
3	$3^4 \cdot 7 \cdot 13^2$	117	39	13	13	13	9	9	9	3
4	$3^4 \cdot 5^2 \cdot 7$	45	15	5	5	5	3	3	3	3
5	$3^2 \cdot 5^2 \cdot 7 \cdot 13^2$	195	39	39	39	39	15	3	3	3
6	$3^2 \cdot 5^4 \cdot 7 \cdot 17^2$	1275	255	255	255	255	75	51	51	3
7	$3^2 \cdot 5^4 \cdot 7$	75	15	15	15	15	3	3	3	3
8	$3^4 \cdot 5^2 \cdot 7$	45	9	9	9	5	5	5	3	3
9	$3^2 \cdot 7^3$	21	3	3	3	3	3	3	3	3
10	$3^3 \cdot 5^3$	15	15	5	5	5	3	3	3	3
11	$3^2 \cdot 5^2 \cdot 31$	15	15	15	15	15	3	3	3	3
12	$3^2 \cdot 5 \cdot 11$	3	3	3	3	3	3	3	3	3

TABLE 14. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 63 with 3 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 21$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

In cases 2–4 and 8, $\text{rank}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})$ is at most 39, $|\mathcal{G}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})| \geq 7$, and $|\mathcal{G}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})| \neq 9$. Thus by Theorem 1.1, we should have that $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is pointed, which contradicts that there are simple objects in $\mathcal{C}_{\text{ad}}$ of Frobenius-Perron dimension > 3.

In case 5, note that 3 does not divide $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})$, so the simple objects of dimension 3 in $\mathcal{C}_{\text{ad}}$ must be fixed by the action of $\mathbb{Z}_3$. But then $|\mathcal{G}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})| = 19$, which does not divide $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})$.

In case 6, $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ has 7 invertibles, and so $((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})_{\text{ad}}$ has Frobenius-Perron dimension $5^4 \cdot 17^2$. Since 7 does not divide $5^4 \cdot 17^2$, this implies that $(((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3})_{\text{ad}})_{\text{pt}}$ is trivial, which contradicts Lemma 4.2.

Lastly, in case 1, the simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ have dimensions 135, 45, 27, 15, or 5. Let X_1 be one of the simple objects of Frobenius-Perron dimension 5. Then consider the decomposition

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{k} N_{X_1,X_1^*}^{X_i}(X_i + X_{i^*}),$$

where $X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ denote all non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$, and we are using that $N_{X_1,X_1^*}^{X_i} = N_{X_1,X_1^*}^{X_i^*}$ for all $i = 1, \ldots, k$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$25 = 1 + 2\sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \text{FPdim}(X_i).$$

But then we must have

$$25 = 1 + 2n_1 \cdot 5,$$

for some $n_1 \geq 0$, which is not possible. Thus all cases in Table 14 are discarded. □

Proposition 8.8. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 65. Then $\mathcal{C}$ must be pointed, perfect, or have 3 or 5 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the* *components of the universal grading of* $\mathcal{C}$ *must be* $[(1,59),(2,3)]$*, and if* $|\mathcal{G}(\mathcal{C})| = 5$*, then those ranks must be* $[(1,45),(4,5)]$.

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 65, 57, 49, 41, 33, 25, 19, 17, 13, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(1, 23), (6, 7)],
- [(2, 27), (1, 11)],
- [(1, 45), (4, 5)],
- [(1, 59), (2, 3)],

Case [(1, 23), (6, 7)]: we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 15, which exhibits these dimensions in columns 1 through 8, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8
1	$3^4 \cdot 5^4 \cdot 7^2$	315	175	175	105	63	35	21	7
2	$3^6 \cdot 5^2 \cdot 7^2 \cdot 17^2$	3213	1785	1785	1071	595	357	315	105
3	$3^2 \cdot 5^2 \cdot 7^2$	21	7	7	7	7	7	7	7
4	$7^2 \cdot 113$	7	7	7	7	7	7	7	7

TABLE 15. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 65 with 7 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 23$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 7$. All discarded.

Case 4 is discarded since $\mathcal{C}$ should be pointed by [NR, Proposition 4.11]. In case 3, note that 7 does not divide $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7})$ and so all simple objects in $\mathcal{C}_{\text{ad}}$ of Frobenius-Perron dimension 7 must be fixed by the action of $\mathbb{Z}_7$. Hence $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$ has rank 113 and 99 invertibles. This implies that at least one component of its universal grading will have a unique (invertible) simple, and thus all components would have Frobenius-Perron dimension 1, which is a contradiction since $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$ is not pointed.

In case 1, $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$ has rank 113 and 15 invertibles, so the ranks of the components of its universal grading should be [(1, 15), (14, 7)]. By Corollary 3.18, $|\mathcal{G}(((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7})_{\text{ad}})| = 15$. But then $\text{FPdim}((\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7})_{\text{ad}})) = 15$, which contradicts that the remaining components have 7 simples, since the Frobenius-Perron dimension of a non-invertible simple is at least 3.

It remains to discard case 2. Simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$ have dimensions 15, 45, 51, 85, 153, 255, or 459. Let X_1 be one of the simple objects of Frobenius-Perron dimension 15. Then consider the decomposition

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{k} N_{X_1,X_1^*}^{X_i}(X_i + X_{i^*}),$$

where $X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ denote all non-invertible simple objects in $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_7}$, and we are using that $N_{X_1,X_1^*}^{X_i} = N_{X_1,X_1^*}^{X_i^*}$ for all $i = 1, \ldots, k$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$225 = 1 + 2\sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \operatorname{FPdim}(X_i).$$

But then, noting that $153 = 3 \cdot 51$, we must have

$$225 = 1 + 2n_1 \cdot 15 + 2n_2 \cdot 45 + 2n_3 \cdot 51 + 2n_4 \cdot 85,$$

for some $n_1, \dots, n_4 \geq 0$. Taking equivalence mod 5, we obtain that $n_3 = 2$. But then

$$20 = 2n_1 \cdot 15 + 2n_2 \cdot 45 + 2n_4 \cdot 85,$$

which is not possible. Thus all cases in Table 15 are discarded.

Case [(2, 27), (1, 11)]: we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the unique case listed in Table 16, which exhibits these dimensions in columns 1 through 4, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4
1	$3^2 \cdot 5^2$	3	3	3	3

Table 16. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 65 with 3 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 11$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. Discarded.

Suppose that $\mathcal{C}$ is a modular category with the dimension array of $\mathcal{C}_{\text{ad}}$ of Table 16. Then each component of the universal grading of $\mathcal{C}$ that has 27 simple objects does not have any invertible objects, so their Frobenius-Perron dimensions are at least $3^2 \cdot 27 = 243$. But $\operatorname{FPdim}(\mathcal{C}) = 225$, a contradiction, so $\mathcal{C}$ cannot exist. □

Remark 8.9. It is known that the category $\operatorname{Rep}(D^\omega(\mathbb{Z}_{13} \rtimes \mathbb{Z}_3))$ is an example of an odd-dimensional MTC of rank 65 (see [DKP, Section 4.2]) with 3 invertible objects.

Proposition 8.10. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 67. Then $\mathcal{C}$ must be pointed, perfect, or have 3 or 9 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be $[(1, 49), (2, 9)]$, and if $|\mathcal{G}(\mathcal{C})| = 9$, then those ranks must be $[(1, 43), (8, 3)]$.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 67, 59, 51, 43, 35, 27, 19, 17, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(5, 11), (4, 3)],
- [(4, 15), (1, 7)],
- [(1, 43), (8, 3)],
- [(1, 49), (2, 9)].

Case [(5, 11), (4, 3)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ or 9. Note that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$ can be discarded by [CP, Proposition 5.6]. For $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 17, which exhibits these dimensions in columns 1 through 6, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4
1	$3^3 \cdot 5^2$	3	3	3	3

Table 17. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 67 with 9 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 11$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. Discarded.

To discard this case, note that the Frobenius-Perron dimension of each component of the universal grading is $3 \cdot 5^2 = 75$. By Proposition 3.17, two of the components of rank 11 do not have any invertibles, and so their Frobenius-Perron dimension is at least $11 \cdot 3^2 = 99$, a contradiction.

Case [(4, 15), (1, 7)]: we know that $\text{rank}(\mathcal{C}_{\text{ad}}) = 7$ by Lemma 3.16 and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$. Then the de-equivariantization of $\mathcal{C}_{\text{ad}}$ by $\mathcal{G}(\mathcal{C}_{\text{ad}}) \cong \mathbb{Z}_5$ must have rank $1 + 2 \cdot 5 = 11$, so it is pointed by [BR, Theorem 4.5] and has Frobenius-Perron dimension 11. As a result, the dimensions of all components of the universal grading of $\mathcal{C}$ must be $\text{FPdim}(\mathcal{C}_{\text{ad}}) = 55$. But the dimension of one of the components of rank 15 must be at least $15 \cdot 3^2 > 55$, a contradiction. □

Proposition 8.11. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 69. Then $\mathcal{C}$ must be pointed, perfect, or have 3 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be $[(1, 39), (2, 15)]$.*

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 69, 61, 53, 45, 37, 29, 23, 21, 15, 13, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(3, 17), (18, 1)],
- [(1, 27), (14, 3)],
- [(3, 11), (12, 3)],
- [(3, 13), (6, 5)],
- [(1, 39), (2, 15)].

Case [(3, 17), (18, 1)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ or 7. Note that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ can be discarded by Proposition 3.17. For $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 7$, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 18, which exhibits these dimensions in columns 1 through 5, excluding duplicates due to duals. In this case, there are components of the universal grading with a unique simple object.

#	FPdim($\mathcal{C}$)	1	2	3	4	5
1	$3^2 \cdot 5 \cdot 7^2 \cdot 13^2$	91	21	7	7	7
2	$3^2 \cdot 5^3 \cdot 7^2 \cdot 17^2$	595	119	105	7	7
3	$3^2 \cdot 7^2 \cdot 61$	21	7	7	7	7
4	$3^2 \cdot 7^2 \cdot 173$	21	21	21	21	7
5	$3 \cdot 7^2 \cdot 71$	7	7	7	7	7

Table 18. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 69 with 21 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 17$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 7$. All discarded.

This implies that $\text{FPdim}(\mathcal{C}_{\text{ad}}) = \text{FPdim}(\mathcal{C})/21$ should be a perfect square, which discards all potential arrays in Table 18.

Case [(1, 27), (14, 3)]: in this case we know $\text{rank}(\mathcal{C}_{\text{ad}}) = 27$ by Lemma 3.16, and, given $|\mathcal{G}(\mathcal{C})| = 15$, we also know that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$, or $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 15$.

The cases $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$ can be discarded by Proposition 3.17 since there is only one component of rank 27 in the universal grading of $\mathcal{C}$, which is less than $\frac{|\mathcal{G}(\mathcal{C})|}{|\mathcal{G}(\mathcal{C}_{\text{ad}})|}$.

The case $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 15$ can be discarded by Lemma 3.14, since there is only one component of rank at least 5 (which is a divisor of $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 15$).

Case [(3, 11), (12, 3)]: in this case we know that $\text{rank}(\mathcal{C}_{\text{ad}}) = 11$ by Lemma 3.16, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ or $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$.

The case $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ can be discarded by Proposition 3.17 since there are only three components of rank 11 in the universal grading of $\mathcal{C}$, which is less than $\frac{|\mathcal{G}(\mathcal{C})|}{|\mathcal{G}(\mathcal{C}_{\text{ad}})|} = 5$.

For the case $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 19, which exhibits these dimensions in columns 1 through 3, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3
1	$3^2 \cdot 5^2 \cdot 37$	15	5	5
2	$3 \cdot 5^2 \cdot 31$	5	5	5

TABLE 19. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 69 with 15 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 11$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 5$. All discarded.

In case 1, the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_5}$ has rank 31 but is neither pointed nor perfect, a contradiction by Theorem 6.2. In case 2, the Frobenius-Perron dimension of $\mathcal{C}$ is of the form $p^2 d$ with d squarefree, so $\mathcal{C}$ should be pointed by a generalization of [DN, Corollary 4.13], a contradiction.

Case [(3, 13), (6, 5)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ or 9. Note that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$ can be discarded by [CP, Proposition 5.6]. For $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 20, which exhibits these dimensions in columns 1 through 5, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5
1	$3^3 \cdot 5^2 \cdot 7$	15	3	3	3	3
2	$3^3 \cdot 31$	3	3	3	3	3

TABLE 20. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 69 with 9 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 13$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

Case 2 is not possible since $\mathcal{C}$ should be pointed, see [NR, Proposition 4.12]. In case 1, the de-equivariantization $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ has Frobenius-Perron dimension $5^2 \cdot 7$, so it should be pointed by [NR, Proposition 4.11]. This contradicts the fact that there are simple objects of Frobenius-Perron dimension 15 in $\mathcal{C}_{\text{ad}}$. ◻

Proposition 8.12. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 71. Then $\mathcal{C}$ must be pointed, perfect, or have 3 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be* $[(1, 29), (2, 21)]$.

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 71, 63, 55, 47, 39, 31, 23, 21, 15, 13, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect case remains:

- [(1, 29), (2, 21)].

□

Proposition 8.13. *Let $\mathcal{C}$ be an odd-dimensional MTC of rank 73. Then $\mathcal{C}$ must be pointed, perfect, or have 3, 5, or 7 invertible objects. Additionally, if $|\mathcal{G}(\mathcal{C})| = 3$, then the ranks of the components of the universal grading of $\mathcal{C}$ must be* $[(1, 67), (2, 3)]$*, if* $|\mathcal{G}(\mathcal{C})| = 5$*, then those ranks must be* $[(1, 53), (4, 5)]$*, and if* $|\mathcal{G}(\mathcal{C})| = 7$*, then those ranks must be* $[(1, 31), (6, 7)]$.

Proof. We can discard all values of $|\mathcal{G}(\mathcal{C})|$ except for 73, 65, 57, 49, 41, 33, 25, 19, 17, 13, 11, 9, 7, 5, 3, and 1 using Lemma 3.13. Then, we discard most cases directly using previously shown lemmas, which are displayed in Appendix B. The following non-pointed and non-perfect cases remain:

- [(3, 17), (2, 9), (4, 1)],
- [(2, 27), (1, 19)],
- [(1, 31), (6, 7)],
- [(1, 53), (4, 5)],
- [(1, 67), (2, 3)].

Case [(3, 17), (2, 9), (4, 1)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$ or 9. Note that $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 9$ can be discarded by [CP, Proposition 5.6]. For $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$, we use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\text{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 21, which exhibits these dimensions in columns 1 through 7, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7
1	$3^8 \cdot 5^2$	81	45	15	15	5	5	5
2	$3^3 \cdot 5^2 \cdot 19$	15	15	15	3	3	3	3
3	$3^4 \cdot 11^2$	11	11	11	9	9	3	3
4	$3^4 \cdot 13^2$	13	13	13	9	9	9	3
5	$3^4 \cdot 5^2$	5	5	5	3	3	3	3
6	$3^3 \cdot 43$	3	3	3	3	3	3	3

TABLE 21. Potential dimension arrays of $\mathcal{C}_{\text{ad}}$, for $\mathcal{C}$ an MTC of rank 73 with 9 invertible objects, $\text{rank}(\mathcal{C}_{\text{ad}}) = 17$, and $|\mathcal{G}(\mathcal{C}_{\text{ad}})| = 3$. All discarded.

In cases 2–5, the modular category $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ obtained by de-equivariantization has Frobenius-Perron dimension $\text{FPdim}(\mathcal{C})/3^3 = p^2 \cdot q$, for p and q odd primes. Hence by [NR, Proposition 4.11] $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ should be pointed, which contradicts that $\mathcal{C}_{\text{ad}}$ has simple objects of dimension greater than 3. Case 6 is discarded by [NR, Proposition 4.12], since $\mathcal{C}$ should be pointed. It remains to discard case 1. Note that $(\mathcal{C}_{\text{ad}})_{\mathbb{Z}_3}$ is solvable by [ENO2, Theorem 1.6]. It follows from [ENO2, Proposition 4.5] that $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ has a non-trivial invertible object. On the other hand, $\mathcal{C}_{\mathrm{ad}}$ has no simple objects of dimension 3, thus $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is perfect, so we get a contradiction.

Case [(2, 27), (1, 19)]: by Lemma 4.2, $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3$. We use Algorithm 3.6 to compute the potential Frobenius-Perron dimensions of non-invertible simple objects in $\mathcal{C}_{\mathrm{ad}}$. After removing the cases that violate Remark 3.11, this produces the cases listed in Table 22, which exhibits these dimensions in columns 1 through 8, excluding duplicates due to duals.

#	FPdim($\mathcal{C}$)	1	2	3	4	5	6	7	8
1	$3^8 \cdot 5^2$	135	81	45	15	15	5	5	5
2	$3^4 \cdot 11^2$	33	11	11	11	9	9	3	3
3	$3^4 \cdot 13^2$	39	13	13	13	9	9	9	3
4	$3^4 \cdot 5^2$	15	5	5	5	3	3	3	3
5	$3^2 \cdot 5^2 \cdot 13^2$	39	39	39	39	15	3	3	3
6	$3^2 \cdot 5^4 \cdot 17^2$	255	255	255	255	75	51	51	3
7	$3^2 \cdot 5^4$	15	15	15	15	3	3	3	3
8	$3^4 \cdot 5^2$	9	9	9	5	5	5	3	3
9	$3^2 \cdot 7^2$	3	3	3	3	3	3	3	3

TABLE 22. Potential dimension arrays of $\mathcal{C}_{\mathrm{ad}}$, for $\mathcal{C}$ an MTC of rank 73 with 3 invertible objects, $\mathrm{rank}(\mathcal{C}_{\mathrm{ad}}) = 19$, and $|\mathcal{G}(\mathcal{C}_{\mathrm{ad}})| = 3$. All discarded.

For case 1, consider the modular category $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$, which has non-invertible simple objects of dimensions 5, 15, 27, and 45, and let X_1 be a simple object of dimension 5. Then, as $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ has one invertible object, consider the decomposition

$$X_1 \otimes X_1^* = 1 \oplus \bigoplus_{i=1}^{k} N_{X_1, X_1^*}^{X_i}(X_i + X_{i^*}),$$

where $X_1, \ldots, X_k, X_1^*, \ldots, X_k^*$ denote all non-invertible simple objects in $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$, and we are using that $N_{X_1,X_1^*}^{X_i} = N_{X_1,X_1^*}^{X_i^*}$ for all $i = 1, \ldots, k$. Taking Frobenius-Perron dimension on both sides of the equation above, we get

$$25 = 1 + 2\sum_{i=1}^{k} N_{X_1,X_1^*}^{X_i} \,\mathrm{FPdim}(X_i).$$

But then we must have

$$25 = 1 + 2n_1 \cdot 5 + 2n_2 \cdot 15 + 2n_3 \cdot 27 + 2n_4 \cdot 45,$$

for some $n_1, \ldots, n_4 \geq 0$. Clearly, $n_3 = n_4 = 0$. But then taking equivalence mod 5 gives a contradiction, discarding this case.

Now, for each of cases 2, 3, 4, 6, 7, and 8, we must have that $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ has 7, 13, or 25 invertible objects. But we also know that the rank of the modular category $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ must be 17, 33, or 49, which are impossible by Theorem 4.1, Proposition 7.1, and Proposition 7.11, respectively, discarding these cases.

Next, consider case 5. We know that the Frobenius-Perron dimension of the deq-equivariantization $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ is $5^2 \cdot 13^2$, which is solvable by [ENO2, Theorem 1.6] and hence not perfect by [ENO2, Proposition 4.5]. As a result, the 3-dimensional simple objects in

$\mathcal{C}_{\mathrm{ad}}$ must be fixed by the action of $\mathcal{G}(\mathcal{C}_{\mathrm{ad}})$ on $\mathcal{C}_{\mathrm{ad}}$, so $(\mathcal{C}_{\mathrm{ad}})_{\mathbb{Z}_3}$ has 19 invertible objects. But 19 does not divide its dimension $5^2 \cdot 13^2$, discarding this case.

The final remaining case is case 9. We know that the Frobenius-Perron dimension of each component of the universal grading of $\mathcal{C}$ must be $3 \cdot 7^2 = 147$. But the components with 27 simple objects cannot have any invertible objects, so their dimensions are each at least $27 \cdot 3^2 = 243$, a contradiction.

Hence, all cases in Table 22 are discarded. □

We summarize the results from this section in the following theorem.

Theorem 8.14. *Let $\mathcal{C}$ be an odd-dimensional MTC such that $51 \leq \mathrm{rank}(\mathcal{C}) \leq 73$.*

- *(a) If $\mathrm{rank}(\mathcal{C}) \in \{53, 55, 63\}$, then $\mathcal{C}$ is either pointed or perfect.*
- *(b) If $\mathrm{rank}(\mathcal{C}) \in \{51, 59, 67\}$, then $\mathcal{C}$ is either pointed, perfect or has 3 or 9 invertible objects.*
- *(c) If $\mathrm{rank}(\mathcal{C}) \in \{57, 65\}$, then $\mathcal{C}$ is either pointed, perfect or has 3 or 5 invertible objects.*
- *(d) If $\mathrm{rank}(\mathcal{C}) \in \{61, 69, 71\}$, then $\mathcal{C}$ is either pointed, perfect or has 3 invertible objects.*
- *(e) If $\mathrm{rank}(\mathcal{C}) = 73$, then $\mathcal{C}$ is either pointed, perfect or has 3, 5 or 7 invertible objects.*

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Appendix A. Code for basic and adjoint algorithms

The following is the code for Algorithms 3.4 and 3.6, written in the Julia programming language. It utilizes the built-in rational data type in order to avoid arithmetic precision errors. Algorithm 3.4 is implemented in the function `basic_algorithm` and Algorithm 3.6 is implemented in the function `adjoint_algorithm`.

```
using Primes

function validate_result(dk_squared)
    global GC, s, k, w, u_squared, solutions
    FPdimC_over_w = dk_squared * u_squared[k]
    d_squared = Vector{Rational{Int128}}(undef, k)
    d = Vector{Int128}(undef, k)
    ds_are_integers = true
    power_of_prime = false
    for i in eachindex(u_squared)
        d_squared[i] = FPdimC_over_w // u_squared[i]
        d[i] = round(Int128, sqrt(d_squared[i]))
        if !isinteger(d_squared[i]) || d[i] * d[i] != d_squared[i]
            ds_are_integers = false
            break
        end
        if length(factor(d[i])) == 1
            power_of_prime = true
        end
    end
    # for perfect MTCs, exclude cases with a simple object
    # whose Frobenius-Perron dimension is a power of a prime
    if ds_are_integers && (GC > 1 || !power_of_prime)
        FPdimC = FPdimC_over_w * w
        solution = Vector{Int128}(undef, k + 2)
        solution[1] = FPdimC
        solution[2] = s
        for i in eachindex(d)
            solution[i + 2] = d[i]
        end
        push!(solutions, solution)
    end
end

function solve(i, last_c, last_u)
    global s, k, t, u_squared
    last_u_squared = last_u * last_u
    max_new_u_squared =
        s * last_u_squared // (t * last_c) +
        (2 * (k - i) * last_u_squared) // last_c
    min_new_u = round(Int128, sqrt(2 * last_u_squared // last_c))
```

```
    # defensive coding
    if min_new_u >= 1 &&
        (min_new_u - 1) * (min_new_u - 1) >
        (2 * last_u_squared // last_c)
        throw(ErrorException("min_new_u rounded incorrectly,
            which can lead to missing solutions."))
    end
    new_u = max(min_new_u, last_u)
    if new_u % 2 == 0
        new_u += 1
    end
    while new_u * new_u <= max_new_u_squared
        new_u_squared = new_u * new_u
        new_c = last_c * new_u_squared // last_u_squared - 2
        u_squared[i + 1] = new_u_squared
        if k == i + 1
            if new_c > 0 && isinteger(s // new_c)
                dk_squared = s // new_c
                dk = round(Int128, sqrt(dk_squared))
                if dk * dk == dk_squared
                    validate_result(Int128(dk_squared))
                end
            end
        elseif new_c > 0
            solve(i + 1, new_c, new_u)
        end
        new_u += 2
    end
end

function basic_algorithm(arg_rank, arg_GC, arg_min_m1)
    n = Int128(arg_rank)
    global GC = Int128(arg_GC)
    global s = GC
    global k = (n - s) ÷ 2
    global t
    global w
    global u_squared = Vector{Int128}(undef, k)
    global solutions = Vector{Vector{Int128}}()
    min_m1 = Int128(arg_min_m1)
    if s == 1
        t = 225
    else
        t = 9
    end
    m1 = n % 8
    while m1 <= 2 * k + s ÷ 9
```

```
        w = 1
        if m1 >= s && m1 >= min_m1
            c1 = m1 - 2
            if (c1 > 0)
                u1_squared = m1
                factorization = factor(m1)
                for i in factorization
                    if i.second % 2 == 1
                        u1_squared ÷= i.first
                        w *= i.first
                    end
                end
                u_squared[1] = u1_squared
                u1 = floor(Int128, sqrt(u1_squared))
                i = 1
                solve(i, c1, u1)
            end
        end
        m1 += 8
    end
    return solutions
end

function adjoint_algorithm(arg_rank, arg_n, arg_GC, arg_s, arg_min_m1)
    rank = Int128(arg_rank)
    n = Int128(arg_n)
    global GC = Int128(arg_GC)
    global s = Int128(arg_s)
    global k = (n - s) ÷ 2
    global t
    global w
    global u_squared = Vector{Int128}(undef, k)
    global solutions = Vector{Vector{Int128}}()
    min_m1 = Int128(arg_min_m1)
    if GC == 1
        t = 225
    else
        t = 9
    end
    m1 = rank % 8
    while m1 <= 2 * k * GC + (s * GC) ÷ 9
        w = 1
        if m1 >= GC && m1 >= min_m1
            c1 = m1 // GC - 2
            if (c1 > 0)
                u1_squared = m1
                factorization = factor(m1)
```

```
                for i in factorization
                    if i.second % 2 == 1
                        u1_squared ÷= i.first
                        w *= i.first
                    end
                end
                u_squared[1] = u1_squared
                u1 = floor(Int128, sqrt(u1_squared))
                i = 1
                if k > 1
                    solve(i, c1, u1)
                elseif isinteger(s // c1)
                    dk_squared = s // c1
                    dk = round(Int128, sqrt(dk_squared))
                    if dk * dk == dk_squared
                        validate_result(Int128(dk_squared))
                    end
                end
            end
        end
        m1 += 8
    end
    return solutions
end
```

Appendix B. All non-pointed and non-perfect cases for ranks 33–73

Let $\mathcal{C}$ be an odd-dimensional MTC. For each value of rank($\mathcal{C}$) between 33 and 73, we present all possible multisets of ranks of the universal grading components of $\mathcal{C}$ that satisfy Lemma 3.13.

In each rank, we split the cases based on whether they are handled or not, and, if they are handled, the method we use to handle them. These methods are:

(a) Discarded by Lemma 3.14,
(b) Discarded by Lemma 3.16,
(c) Discarded by Lemma 4.3 in conjunction with Lemma 3.16,
(d) Discarded by [CP, Proposition 5.6], Proposition 3.7, and Proposition 3.17, all of which are in conjunction with Lemma 3.16,
(e) Specific methods listed in the respective proposition for each rank, found in Sections 7 and 8.

The cases that remain after using these methods are listed in the last row of each table, which is labelled "Open."

Rule	Cases
(a)	$[(1,9),(24,1)]$, $[(1,17),(16,1)]$, $[(2,9),(15,1)]$, $[(1,25),(8,1)]$, $[(1,17),(1,9),(7,1)]$
(b)	$[(1,19),(1,11),(1,3)]$
(c)	$[(3,9),(6,1)]$
(d)	$[(11,3)]$, $[(3,11)]$
(e)	$[(1,13),(4,5)]$
Open	$[(1,27),(2,3)]$

Table 23. Non-pointed and non-perfect cases for rank 33. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(26,1)]$, $[(1,17),(18,1)]$, $[(2,9),(17,1)]$, $[(1,25),(10,1)]$, $[(1,17),(1,9),(9,1)]$, $[(1,33),(2,1)]$, $[(1,25),(1,9),(1,1)]$, $[(2,17),(1,1)]$
(b)	None
(c)	$[(3,9),(8,1)]$
(d)	$[(7,5)]$, $[(5,7)]$
(e)	$[(1,11),(8,3)]$, $[(1,17),(2,9)]$
Open	None

Table 24. Non-pointed and non-perfect cases for rank 35. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(28,1)]$, $[(1,17),(20,1)]$, $[(2,9),(19,1)]$, $[(1,25),(12,1)]$, $[(1,17),(1,9),(11,1)]$, $[(1,33),(4,1)]$, $[(1,25),(1,9),(3,1)]$, $[(2,17),(3,1)]$

(b)	$[(4,9),(1,1)]$
(c)	$[(3,9),(10,1)]$
(d)	$[(1,19),(6,3)]$, $[(2,11),(5,3)]$, $[(1,17),(2,9),(2,1)]$, $[(1,23),(2,7)]$
(e)	$[(2,15),(1,7)]$
Open	None

Table 25. Non-pointed and non-perfect cases for rank 37. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(30,1)]$, $[(1,17),(22,1)]$, $[(2,9),(21,1)]$, $[(1,25),(14,1)]$, $[(1,17),(1,9),(13,1)]$, $[(1,33),(6,1)]$, $[(1,25),(1,9),(5,1)]$, $[(2,17),(5,1)]$
(b)	$[(4,9),(3,1)]$, $[(1,19),(1,11),(3,3)]$, $[(1,21),(1,13),(1,5)]$
(c)	$[(3,9),(12,1)]$
(d)	$[(13,3)]$, $[(1,17),(2,9),(4,1)]$, $[(1,27),(4,3)]$, $[(3,11),(2,3)]$, $[(1,29),(2,5)]$, $[(3,13)]$
(e)	None
Open	None

Table 26. Non-pointed and non-perfect cases for rank 39. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(32,1)]$, $[(1,17),(24,1)]$, $[(2,9),(23,1)]$, $[(1,25),(16,1)]$, $[(1,17),(1,9),(15,1)]$, $[(1,33),(8,1)]$, $[(1,25),(1,9),(7,1)]$, $[(2,17),(7,1)]$
(b)	$[(4,9),(5,1)]$, $[(1,27),(1,11),(1,3)]$
(c)	$[(3,9),(14,1)]$
(d)	$[(1,11),(10,3)]$, $[(1,17),(2,9),(6,1)]$, $[(2,13),(3,5)]$, $[(2,19),(1,3)]$, $[(1,19),(2,11)]$
(e)	None
Open	$[(1,21),(4,5)]$, $[(1,35),(2,3)]$

Table 27. Non-pointed and non-perfect cases for rank 41. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(34,1)]$, $[(1,17),(26,1)]$, $[(2,9),(25,1)]$, $[(1,25),(18,1)]$, $[(1,17),(1,9),(17,1)]$, $[(1,33),(10,1)]$, $[(1,25),(1,9),(9,1)]$, $[(2,17),(9,1)]$, $[(1,41),(2,1)]$, $[(1,33),(1,9),(1,1)]$, $[(1,25),(1,17),(1,1)]$
(b)	$[(4,9),(7,1)]$
(c)	$[(3,9),(16,1)]$
(d)	$[(1,17),(2,9),(8,1)]$, $[(1,13),(6,5)]$, $[(1,15),(4,7)]$, $[(2,17),(1,9)]$

(e)	$[(2,11),(7,3)]$, $[(1,25),(2,9)]$
Open	$[(1,19),(8,3)]$

TABLE 28. Non-pointed and non-perfect cases for rank 43. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(36,1)]$, $[(1,17),(28,1)]$, $[(2,9),(27,1)]$, $[(1,25),(20,1)]$, $[(1,17),(1,9),(19,1)]$, $[(1,33),(12,1)]$, $[(1,25),(1,9),(11,1)]$, $[(2,17),(11,1)]$, $[(1,41),(4,1)]$, $[(1,33),(1,9),(3,1)]$, $[(1,25),(1,17),(3,1)]$
(b)	$[(4,9),(9,1)]$, $[(1,19),(1,11),(5,3)]$, $[(1,17),(3,9),(1,1)]$, $[(1,23),(1,15),(1,7)]$
(c)	$[(3,9),(18,1)]$, $[(2,17),(1,9),(2,1)]$
(d)	$[(1,17),(2,9),(10,1)]$, $[(1,27),(6,3)]$, $[(3,11),(4,3)]$, $[(1,25),(2,9),(2,1)]$, $[(5,9)]$, $[(1,31),(2,7)]$
(e)	$[(15,3)]$, $[(9,5)]$, $[(3,15)]$
Open	None

TABLE 29. Non-pointed and non-perfect cases for rank 45. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(38,1)]$, $[(1,17),(30,1)]$, $[(2,9),(29,1)]$, $[(1,25),(22,1)]$, $[(1,17),(1,9),(21,1)]$, $[(1,33),(14,1)]$, $[(1,25),(1,9),(13,1)]$, $[(2,17),(13,1)]$, $[(1,41),(6,1)]$, $[(1,33),(1,9),(5,1)]$, $[(1,25),(1,17),(5,1)]$
(b)	$[(4,9),(11,1)]$, $[(1,17),(3,9),(3,1)]$, $[(1,27),(1,11),(3,3)]$, $[(1,29),(1,13),(1,5)]$
(c)	$[(3,9),(20,1)]$, $[(2,17),(1,9),(4,1)]$, $[(5,9),(2,1)]$
(d)	$[(1,17),(2,9),(12,1)]$, $[(1,11),(12,3)]$, $[(1,25),(2,9),(4,1)]$, $[(1,35),(4,3)]$, $[(2,19),(3,3)]$, $[(1,19),(2,11),(2,3)]$, $[(4,11),(1,3)]$, $[(1,37),(2,5)]$, $[(1,21),(2,13)]$
(e)	$[(2,21),(1,5)]$
Open	None

TABLE 30. Non-pointed and non-perfect cases for rank 47. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(40,1)]$, $[(1,17),(32,1)]$, $[(2,9),(31,1)]$, $[(1,25),(24,1)]$, $[(1,17),(1,9),(23,1)]$, $[(1,33),(16,1)]$, $[(1,25),(1,9),(15,1)]$, $[(2,17),(15,1)]$, $[(1,41),(8,1)]$, $[(1,33),(1,9),(7,1)]$, $[(1,25),(1,17),(7,1)]$

(b)	$[(4,9),(13,1)]$, $[(1,17),(3,9),(5,1)]$, $[(1,21),(1,13),(3,5)]$, $[(1,35),(1,11),(1,3)]$, $[(1,27),(1,19),(1,3)]$
(c)	$[(3,9),(22,1)]$, $[(2,17),(1,9),(6,1)]$, $[(5,9),(4,1)]$
(d)	$[(1,17),(2,9),(14,1)]$, $[(1,19),(10,3)]$, $[(2,11),(9,3)]$, $[(1,25),(2,9),(6,1)]$, $[(3,13),(2,5)]$, $[(1,27),(2,11)]$, $[(2,19),(1,11)]$
(e)	$[(7,7)]$
Open	$[(1,29),(4,5)]$, $[(1,43),(2,3)]$

Table 31. Non-pointed and non-perfect cases for rank 49. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(42,1)]$, $[(1,17),(34,1)]$, $[(2,9),(33,1)]$, $[(1,25),(26,1)]$, $[(1,17),(1,9),(25,1)]$, $[(1,33),(18,1)]$, $[(1,25),(1,9),(17,1)]$, $[(2,17),(17,1)]$, $[(1,41),(10,1)]$, $[(1,33),(1,9),(9,1)]$, $[(1,25),(1,17),(9,1)]$, $[(1,49),(2,1)]$, $[(1,41),(1,9),(1,1)]$, $[(1,33),(1,17),(1,1)]$, $[(2,25),(1,1)]$
(b)	$[(4,9),(15,1)]$, $[(1,17),(3,9),(7,1)]$, $[(1,19),(1,11),(7,3)]$, $[(1,25),(1,17),(1,9)]$
(c)	$[(3,9),(24,1)]$, $[(2,17),(1,9),(8,1)]$, $[(5,9),(6,1)]$
(d)	$[(1,17),(2,9),(16,1)]$, $[(17,3)]$, $[(1,25),(2,9),(8,1)]$, $[(1,21),(6,5)]$, $[(2,13),(5,5)]$, $[(1,23),(4,7)]$, $[(2,15),(3,7)]$, $[(3,17)]$
(e)	$[(1,27),(8,3)]$
Open	$[(3,11),(6,3)]$, $[(1,33),(2,9)]$

Table 32. Non-pointed and non-perfect cases for rank 51. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(44,1)]$, $[(1,17),(36,1)]$, $[(2,9),(35,1)]$, $[(1,25),(28,1)]$, $[(1,17),(1,9),(27,1)]$, $[(1,33),(20,1)]$, $[(1,25),(1,9),(19,1)]$, $[(2,17),(19,1)]$, $[(1,41),(12,1)]$, $[(1,33),(1,9),(11,1)]$, $[(1,25),(1,17),(11,1)]$, $[(1,49),(4,1)]$, $[(1,41),(1,9),(3,1)]$, $[(1,33),(1,17),(3,1)]$, $[(2,25),(3,1)]$
(b)	$[(4,9),(17,1)]$, $[(1,17),(3,9),(9,1)]$, $[(1,27),(1,11),(5,3)]$, $[(1,25),(1,17),(1,9),(2,1)]$, $[(1,25),(3,9),(1,1)]$, $[(2,17),(2,9),(1,1)]$, $[(1,31),(1,15),(1,7)]$
(c)	$[(3,9),(26,1)]$, $[(2,17),(1,9),(10,1)]$, $[(5,9),(8,1)]$
(d)	$[(1,17),(2,9),(18,1)]$, $[(1,11),(14,3)]$, $[(1,25),(2,9),(10,1)]$, $[(1,13),(8,5)]$, $[(1,35),(6,3)]$, $[(2,19),(5,3)]$, $[(1,19),(2,11),(4,3)]$, $[(4,11),(3,3)]$, $[(1,33),(2,9),(2,1)]$, $[(3,17),(2,1)]$, $[(1,17),(4,9)]$, $[(1,39),(2,7)]$, $[(2,23),(1,7)]$
(e)	$[(1,23),(2,15)]$
Open	None

Table 33. Non-pointed and non-perfect cases for rank 53. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(46,1)]$, $[(1,17),(38,1)]$, $[(2,9),(37,1)]$, $[(1,25),(30,1)]$, $[(1,17),(1,9),(29,1)]$, $[(1,33),(22,1)]$, $[(1,25),(1,9),(21,1)]$, $[(2,17),(21,1)]$, $[(1,41),(14,1)]$, $[(1,33),(1,9),(13,1)]$, $[(1,25),(1,17),(13,1)]$, $[(1,49),(6,1)]$, $[(1,41),(1,9),(5,1)]$, $[(1,33),(1,17),(5,1)]$, $[(2,25),(5,1)]$
(b)	$[(4,9),(19,1)]$, $[(1,17),(3,9),(11,1)]$, $[(1,25),(1,17),(1,9),(4,1)]$, $[(1,25),(3,9),(3,1)]$, $[(2,17),(2,9),(3,1)]$, $[(6,9),(1,1)]$, $[(1,35),(1,11),(3,3)]$, $[(1,27),(1,19),(3,3)]$, $[(1,19),(3,11),(1,3)]$, $[(1,37),(1,13),(1,5)]$, $[(1,29),(1,21),(1,5)]$
(c)	$[(3,9),(28,1)]$, $[(2,17),(1,9),(12,1)]$, $[(5,9),(10,1)]$
(d)	$[(1,17),(2,9),(20,1)]$, $[(1,25),(2,9),(12,1)]$, $[(1,19),(12,3)]$, $[(2,11),(11,3)]$, $[(11,5)]$, $[(1,33),(2,9),(4,1)]$, $[(3,17),(4,1)]$, $[(1,17),(4,9),(2,1)]$, $[(1,43),(4,3)]$, $[(1,27),(2,11),(2,3)]$, $[(2,19),(1,11),(2,3)]$, $[(5,11)]$, $[(1,45),(2,5)]$, $[(1,29),(2,13)]$
(e)	$[(2,21),(1,13)]$
Open	None

Table 34. Non-pointed and non-perfect cases for rank 55. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(48,1)]$, $[(1,17),(40,1)]$, $[(2,9),(39,1)]$, $[(1,25),(32,1)]$, $[(1,17),(1,9),(31,1)]$, $[(1,33),(24,1)]$, $[(1,25),(1,9),(23,1)]$, $[(2,17),(23,1)]$, $[(1,41),(16,1)]$, $[(1,33),(1,9),(15,1)]$, $[(1,25),(1,17),(15,1)]$, $[(1,49),(8,1)]$, $[(1,41),(1,9),(7,1)]$, $[(1,33),(1,17),(7,1)]$, $[(2,25),(7,1)]$
(b)	$[(4,9),(21,1)]$, $[(1,17),(3,9),(13,1)]$, $[(1,19),(1,11),(9,3)]$, $[(1,25),(1,17),(1,9),(6,1)]$, $[(1,25),(3,9),(5,1)]$, $[(2,17),(2,9),(5,1)]$, $[(6,9),(3,1)]$, $[(1,29),(1,13),(3,5)]$, $[(1,43),(1,11),(1,3)]$, $[(1,35),(1,19),(1,3)]$, $[(1,27),(1,19),(1,11)]$
(c)	$[(3,9),(30,1)]$, $[(2,17),(1,9),(14,1)]$, $[(5,9),(12,1)]$
(d)	$[(1,17),(2,9),(22,1)]$, $[(19,3)]$, $[(1,25),(2,9),(14,1)]$, $[(1,27),(10,3)]$, $[(3,11),(8,3)]$, $[(1,33),(2,9),(6,1)]$, $[(1,17),(4,9),(4,1)]$, $[(2,21),(3,5)]$, $[(1,21),(2,13),(2,5)]$, $[(4,13),(1,5)]$, $[(1,35),(2,11)]$, $[(3,19)]$
(e)	$[(3,17),(6,1)]$, $[(1,15),(6,7)]$, $[(2,27),(1,3)]$
Open	$[(1,37),(4,5)]$, $[(1,51),(2,3)]$

Table 35. Non-pointed and non-perfect cases for rank 57. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	[(1, 9), (50, 1)], [(1, 17), (42, 1)], [(2, 9), (41, 1)], [(1, 25), (34, 1)], [(1, 17), (1, 9), (33, 1)], [(1, 33), (26, 1)], [(1, 25), (1, 9), (25, 1)], [(2, 17), (25, 1)], [(1, 41), (18, 1)], [(1, 33), (1, 9), (17, 1)], [(1, 25), (1, 17), (17, 1)], [(1, 49), (10, 1)], [(1, 41), (1, 9), (9, 1)], [(1, 33), (1, 17), (9, 1)], [(2, 25), (9, 1)], [(1, 57), (2, 1)], [(1, 49), (1, 9), (1, 1)], [(1, 41), (1, 17), (1, 1)], [(1, 33), (1, 25), (1, 1)]
(b)	[(4, 9), (23, 1)], [(1, 17), (3, 9), (15, 1)], [(1, 25), (1, 17), (1, 9), (8, 1)], [(1, 25), (3, 9), (7, 1)], [(2, 17), (2, 9), (7, 1)], [(6, 9), (5, 1)], [(1, 27), (1, 11), (7, 3)], [(1, 21), (1, 13), (5, 5)], [(1, 23), (1, 15), (3, 7)], [(1, 33), (1, 17), (1, 9)]
(c)	[(3, 9), (32, 1)], [(2, 17), (1, 9), (16, 1)], [(5, 9), (14, 1)]
(d)	[(1, 17), (2, 9), (24, 1)], [(1, 25), (2, 9), (16, 1)], [(1, 11), (16, 3)], [(1, 33), (2, 9), (8, 1)], [(3, 17), (8, 1)], [(1, 17), (4, 9), (6, 1)], [(1, 19), (2, 11), (6, 3)], [(1, 29), (6, 5)], [(3, 13), (4, 5)], [(1, 31), (4, 7)], [(3, 15), (2, 7)], [(2, 25), (1, 9)], [(1, 25), (2, 17)]
(e)	[(2, 19), (7, 3)], [(4, 11), (5, 3)]
Open	[(1, 35), (8, 3)], [(1, 41), (2, 9)]

TABLE 36. Non-pointed and non-perfect cases for rank 59. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	[(1, 9), (52, 1)], [(1, 17), (44, 1)], [(2, 9), (43, 1)], [(1, 25), (36, 1)], [(1, 17), (1, 9), (35, 1)], [(1, 33), (28, 1)], [(1, 25), (1, 9), (27, 1)], [(2, 17), (27, 1)], [(1, 41), (20, 1)], [(1, 33), (1, 9), (19, 1)], [(1, 25), (1, 17), (19, 1)], [(1, 49), (12, 1)], [(1, 41), (1, 9), (11, 1)], [(1, 33), (1, 17), (11, 1)], [(2, 25), (11, 1)], [(1, 57), (4, 1)], [(1, 49), (1, 9), (3, 1)], [(1, 41), (1, 17), (3, 1)], [(1, 33), (1, 25), (3, 1)]
(b)	[(4, 9), (25, 1)], [(1, 17), (3, 9), (17, 1)], [(1, 25), (1, 17), (1, 9), (10, 1)], [(1, 25), (3, 9), (9, 1)], [(2, 17), (2, 9), (9, 1)], [(6, 9), (7, 1)], [(1, 35), (1, 11), (5, 3)], [(1, 27), (1, 19), (5, 3)], [(1, 19), (3, 11), (3, 3)], [(1, 33), (1, 17), (1, 9), (2, 1)], [(1, 33), (3, 9), (1, 1)], [(1, 25), (1, 17), (2, 9), (1, 1)], [(3, 17), (1, 9), (1, 1)], [(1, 39), (1, 15), (1, 7)], [(1, 31), (1, 23), (1, 7)]
(c)	[(3, 9), (34, 1)], [(2, 17), (1, 9), (18, 1)], [(5, 9), (16, 1)], [(2, 25), (1, 9), (2, 1)]
(d)	[(1, 17), (2, 9), (26, 1)], [(1, 25), (2, 9), (18, 1)], [(1, 33), (2, 9), (10, 1)], [(3, 17), (10, 1)], [(1, 17), (4, 9), (8, 1)], [(1, 21), (8, 5)], [(1, 43), (6, 3)], [(1, 27), (2, 11), (4, 3)], [(2, 19), (1, 11), (4, 3)], [(5, 11), (2, 3)], [(1, 41), (2, 9), (2, 1)], [(1, 25), (2, 17), (2, 1)], [(1, 25), (4, 9)], [(2, 17), (3, 9)], [(1, 47), (2, 7)], [(2, 23), (1, 15)]
(e)	[(1, 19), (14, 3)], [(2, 11), (13, 3)], [(2, 13), (7, 5)]
Open	[(1, 31), (2, 15)]

TABLE 37. Non-pointed and non-perfect cases for rank 61. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(54,1)]$, $[(1,17),(46,1)]$, $[(2,9),(45,1)]$, $[(1,25),(38,1)]$, $[(1,17),(1,9),(37,1)]$, $[(1,33),(30,1)]$, $[(1,25),(1,9),(29,1)]$, $[(2,17),(29,1)]$, $[(1,41),(22,1)]$, $[(1,33),(1,9),(21,1)]$, $[(1,25),(1,17),(21,1)]$, $[(1,49),(14,1)]$, $[(1,41),(1,9),(13,1)]$, $[(1,33),(1,17),(13,1)]$, $[(2,25),(13,1)]$, $[(1,57),(6,1)]$, $[(1,49),(1,9),(5,1)]$, $[(1,41),(1,17),(5,1)]$, $[(1,33),(1,25),(5,1)]$
(b)	$[(4,9),(27,1)]$, $[(1,17),(3,9),(19,1)]$, $[(1,25),(1,17),(1,9),(12,1)]$, $[(1,25),(3,9),(11,1)]$, $[(2,17),(2,9),(11,1)]$, $[(6,9),(9,1)]$, $[(1,19),(1,11),(11,3)]$, $[(1,33),(1,17),(1,9),(4,1)]$, $[(1,33),(3,9),(3,1)]$, $[(1,25),(1,17),(2,9),(3,1)]$, $[(3,17),(1,9),(3,1)]$, $[(1,17),(5,9),(1,1)]$, $[(1,43),(1,11),(3,3)]$, $[(1,35),(1,19),(3,3)]$, $[(1,27),(1,19),(1,11),(2,3)]$, $[(1,27),(3,11),(1,3)]$, $[(1,45),(1,13),(1,5)]$, $[(1,37),(1,21),(1,5)]$, $[(1,29),(1,21),(1,13)]$
(c)	$[(3,9),(36,1)]$, $[(2,17),(1,9),(20,1)]$, $[(5,9),(18,1)]$, $[(2,25),(1,9),(4,1)]$, $[(2,17),(3,9),(2,1)]$
(d)	$[(1,17),(2,9),(28,1)]$, $[(1,25),(2,9),(20,1)]$, $[(1,33),(2,9),(12,1)]$, $[(1,17),(4,9),(10,1)]$, $[(1,27),(12,3)]$, $[(3,11),(10,3)]$, $[(1,13),(10,5)]$, $[(1,41),(2,9),(4,1)]$, $[(1,25),(2,17),(4,1)]$, $[(1,25),(4,9),(2,1)]$, $[(7,9)]$, $[(1,51),(4,3)]$, $[(2,27),(3,3)]$, $[(1,35),(2,11),(2,3)]$, $[(3,19),(2,3)]$, $[(2,19),(2,11),(1,3)]$, $[(1,19),(4,11)]$, $[(1,53),(2,5)]$, $[(2,29),(1,5)]$, $[(1,37),(2,13)]$
(e)	$[(21,3)]$, $[(3,17),(12,1)]$, $[(9,7)]$, $[(3,21)]$
Open	None

TABLE 38. Non-pointed and non-perfect cases for rank 63. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(56,1)]$, $[(1,17),(48,1)]$, $[(2,9),(47,1)]$, $[(1,25),(40,1)]$, $[(1,17),(1,9),(39,1)]$, $[(1,33),(32,1)]$, $[(1,25),(1,9),(31,1)]$, $[(2,17),(31,1)]$, $[(1,41),(24,1)]$, $[(1,33),(1,9),(23,1)]$, $[(1,25),(1,17),(23,1)]$, $[(1,49),(16,1)]$, $[(1,41),(1,9),(15,1)]$, $[(1,33),(1,17),(15,1)]$, $[(2,25),(15,1)]$, $[(1,57),(8,1)]$, $[(1,49),(1,9),(7,1)]$, $[(1,41),(1,17),(7,1)]$, $[(1,33),(1,25),(7,1)]$

(b)	[(4,9),(29,1)], [(1,17),(3,9),(21,1)], [(1,25),(1,17),(1,9),(14,1)], [(1,25),(3,9),(13,1)], [(2,17),(2,9),(13,1)], [(6,9),(11,1)], [(1,27),(1,11),(9,3)], [(1,33),(1,17),(1,9),(6,1)], [(1,33),(3,9),(5,1)], [(1,25),(1,17),(2,9),(5,1)], [(3,17),(1,9),(5,1)], [(1,17),(5,9),(3,1)], [(1,37),(1,13),(3,5)], [(1,29),(1,21),(3,5)], [(1,21),(3,13),(1,5)], [(1,51),(1,11),(1,3)], [(1,43),(1,19),(1,3)], [(1,35),(1,27),(1,3)], [(1,35),(1,19),(1,11)]
(c)	[(3,9),(38,1)], [(2,17),(1,9),(22,1)], [(5,9),(20,1)], [(2,25),(1,9),(6,1)], [(2,17),(3,9),(4,1)], [(7,9),(2,1)]
(d)	[(1,17),(2,9),(30,1)], [(1,25),(2,9),(22,1)], [(1,11),(18,3)], [(1,33),(2,9),(14,1)], [(3,17),(14,1)], [(1,17),(4,9),(12,1)], [(13,5)], [(1,35),(10,3)], [(2,19),(9,3)], [(1,19),(2,11),(8,3)], [(4,11),(7,3)], [(1,41),(2,9),(6,1)], [(1,25),(2,17),(6,1)], [(1,25),(4,9),(4,1)], [(2,15),(5,7)], [(1,29),(2,13),(2,5)], [(2,21),(1,13),(2,5)], [(5,13)], [(1,43),(2,11)], [(1,27),(2,19)]
(e)	[(1,23),(6,7)], [(2,27),(1,11)]
Open	[(1,45),(4,5)], [(1,59),(2,3)]

Table 39. Non-pointed and non-perfect cases for rank 65. Each case is denoted as an array of ordered pairs (n, r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	[(1,9),(58,1)], [(1,17),(50,1)], [(2,9),(49,1)], [(1,25),(42,1)], [(1,17),(1,9),(41,1)], [(1,33),(34,1)], [(1,25),(1,9),(33,1)], [(2,17),(33,1)], [(1,41),(26,1)], [(1,33),(1,9),(25,1)], [(1,25),(1,17),(25,1)], [(1,49),(18,1)], [(1,41),(1,9),(17,1)], [(1,33),(1,17),(17,1)], [(2,25),(17,1)], [(1,57),(10,1)], [(1,49),(1,9),(9,1)], [(1,41),(1,17),(9,1)], [(1,33),(1,25),(9,1)], [(1,65),(2,1)], [(1,57),(1,9),(1,1)], [(1,49),(1,17),(1,1)], [(1,41),(1,25),(1,1)], [(2,33),(1,1)]
(b)	[(4,9),(31,1)], [(1,17),(3,9),(23,1)], [(1,25),(1,17),(1,9),(16,1)], [(1,25),(3,9),(15,1)], [(2,17),(2,9),(15,1)], [(6,9),(13,1)], [(1,33),(1,17),(1,9),(8,1)], [(1,33),(3,9),(7,1)], [(1,25),(1,17),(2,9),(7,1)], [(3,17),(1,9),(7,1)], [(1,17),(5,9),(5,1)], [(1,35),(1,11),(7,3)], [(1,27),(1,19),(7,3)], [(1,19),(3,11),(5,3)], [(1,29),(1,13),(5,5)], [(1,31),(1,15),(3,7)], [(1,41),(1,17),(1,9)], [(1,33),(1,25),(1,9)]
(c)	[(3,9),(40,1)], [(2,17),(1,9),(24,1)], [(5,9),(22,1)], [(2,25),(1,9),(8,1)], [(2,17),(3,9),(6,1)], [(7,9),(4,1)]
(d)	[(1,17),(2,9),(32,1)], [(1,25),(2,9),(24,1)], [(1,33),(2,9),(16,1)], [(3,17),(16,1)], [(1,17),(4,9),(14,1)], [(1,19),(16,3)], [(2,11),(15,3)], [(1,41),(2,9),(8,1)], [(1,25),(2,17),(8,1)], [(1,25),(4,9),(6,1)], [(1,27),(2,11),(6,3)], [(2,19),(1,11),(6,3)], [(1,37),(6,5)], [(2,21),(5,5)], [(1,21),(2,13),(4,5)], [(4,13),(3,5)], [(1,39),(4,7)], [(2,23),(3,7)], [(1,23),(2,15),(2,7)], [(1,33),(2,17)], [(2,25),(1,17)]

(e)	$[(5,11),(4,3)]$, $[(4,15),(1,7)]$
Open	$[(1,43),(8,3)]$, $[(1,49),(2,9)]$

Table 40. Non-pointed and non-perfect cases for rank 67. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases
(a)	$[(1,9),(60,1)]$, $[(1,17),(52,1)]$, $[(2,9),(51,1)]$, $[(1,25),(44,1)]$, $[(1,17),(1,9),(43,1)]$, $[(1,33),(36,1)]$, $[(1,25),(1,9),(35,1)]$, $[(2,17),(35,1)]$, $[(1,41),(28,1)]$, $[(1,33),(1,9),(27,1)]$, $[(1,25),(1,17),(27,1)]$, $[(1,49),(20,1)]$, $[(1,41),(1,9),(19,1)]$, $[(1,33),(1,17),(19,1)]$, $[(2,25),(19,1)]$, $[(1,57),(12,1)]$, $[(1,49),(1,9),(11,1)]$, $[(1,41),(1,17),(11,1)]$, $[(1,33),(1,25),(11,1)]$, $[(1,65),(4,1)]$, $[(1,57),(1,9),(3,1)]$, $[(1,49),(1,17),(3,1)]$, $[(1,41),(1,25),(3,1)]$, $[(2,33),(3,1)]$
(b)	$[(4,9),(33,1)]$, $[(1,17),(3,9),(25,1)]$, $[(1,25),(1,17),(1,9),(18,1)]$, $[(1,25),(3,9),(17,1)]$, $[(2,17),(2,9),(17,1)]$, $[(6,9),(15,1)]$, $[(1,19),(1,11),(13,3)]$, $[(1,33),(1,17),(1,9),(10,1)]$, $[(1,33),(3,9),(9,1)]$, $[(1,25),(1,17),(2,9),(9,1)]$, $[(3,17),(1,9),(9,1)]$, $[(1,17),(5,9),(7,1)]$, $[(1,21),(1,13),(7,5)]$, $[(1,43),(1,11),(5,3)]$, $[(1,35),(1,19),(5,3)]$, $[(1,27),(1,19),(1,11),(4,3)]$, $[(1,27),(3,11),(3,3)]$, $[(1,41),(1,17),(1,9),(2,1)]$, $[(1,33),(1,25),(1,9),(2,1)]$, $[(1,41),(3,9),(1,1)]$, $[(1,33),(1,17),(2,9),(1,1)]$, $[(2,25),(2,9),(1,1)]$, $[(1,25),(2,17),(1,9),(1,1)]$, $[(4,17),(1,1)]$, $[(1,25),(1,17),(3,9)]$, $[(1,47),(1,15),(1,7)]$, $[(1,39),(1,23),(1,7)]$, $[(1,31),(1,23),(1,15)]$
(c)	$[(3,9),(42,1)]$, $[(2,17),(1,9),(26,1)]$, $[(5,9),(24,1)]$, $[(2,25),(1,9),(10,1)]$, $[(2,17),(3,9),(8,1)]$, $[(7,9),(6,1)]$
(d)	$[(1,17),(2,9),(34,1)]$, $[(1,25),(2,9),(26,1)]$, $[(23,3)]$, $[(1,33),(2,9),(18,1)]$, $[(1,17),(4,9),(16,1)]$, $[(1,41),(2,9),(10,1)]$, $[(1,25),(2,17),(10,1)]$, $[(1,25),(4,9),(8,1)]$, $[(1,29),(8,5)]$, $[(1,51),(6,3)]$, $[(2,27),(5,3)]$, $[(1,35),(2,11),(4,3)]$, $[(3,19),(4,3)]$, $[(2,19),(2,11),(3,3)]$, $[(1,19),(4,11),(2,3)]$, $[(6,11),(1,3)]$, $[(1,49),(2,9),(2,1)]$, $[(1,33),(2,17),(2,1)]$, $[(2,25),(1,17),(2,1)]$, $[(1,33),(4,9)]$, $[(3,17),(2,9)]$, $[(1,55),(2,7)]$, $[(2,31),(1,7)]$, $[(3,23)]$
(e)	$[(3,17),(18,1)]$, $[(1,27),(14,3)]$, $[(3,11),(12,3)]$, $[(3,13),(6,5)]$
Open	$[(1,39),(2,15)]$

Table 41. Non-pointed and non-perfect cases for rank 69. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases

(a)	$[(1,9),(62,1)]$, $[(1,17),(54,1)]$, $[(2,9),(53,1)]$, $[(1,25),(46,1)]$, $[(1,17),(1,9),(45,1)]$, $[(1,33),(38,1)]$, $[(1,25),(1,9),(37,1)]$, $[(2,17),(37,1)]$, $[(1,41),(30,1)]$, $[(1,33),(1,9),(29,1)]$, $[(1,25),(1,17),(29,1)]$, $[(1,49),(22,1)]$, $[(1,41),(1,9),(21,1)]$, $[(1,33),(1,17),(21,1)]$, $[(2,25),(21,1)]$, $[(1,57),(14,1)]$, $[(1,49),(1,9),(13,1)]$, $[(1,41),(1,17),(13,1)]$, $[(1,33),(1,25),(13,1)]$, $[(1,65),(6,1)]$, $[(1,57),(1,9),(5,1)]$, $[(1,49),(1,17),(5,1)]$, $[(1,41),(1,25),(5,1)]$, $[(2,33),(5,1)]$
(b)	$[(4,9),(35,1)]$, $[(1,17),(3,9),(27,1)]$, $[(1,25),(1,17),(1,9),(20,1)]$, $[(1,25),(3,9),(19,1)]$, $[(2,17),(2,9),(19,1)]$, $[(6,9),(17,1)]$, $[(1,33),(1,17),(1,9),(12,1)]$, $[(1,33),(3,9),(11,1)]$, $[(1,25),(1,17),(2,9),(11,1)]$, $[(3,17),(1,9),(11,1)]$, $[(1,17),(5,9),(9,1)]$, $[(1,27),(1,11),(11,3)]$, $[(1,41),(1,17),(1,9),(4,1)]$, $[(1,33),(1,25),(1,9),(4,1)]$, $[(1,41),(3,9),(3,1)]$, $[(1,33),(1,17),(2,9),(3,1)]$, $[(2,25),(2,9),(3,1)]$, $[(1,25),(2,17),(1,9),(3,1)]$, $[(4,17),(3,1)]$, $[(1,25),(1,17),(3,9),(2,1)]$, $[(1,25),(5,9),(1,1)]$, $[(2,17),(4,9),(1,1)]$, $[(1,51),(1,11),(3,3)]$, $[(1,43),(1,19),(3,3)]$, $[(1,35),(1,27),(3,3)]$, $[(1,35),(1,19),(1,11),(2,3)]$, $[(1,35),(3,11),(1,3)]$, $[(1,27),(1,19),(2,11),(1,3)]$, $[(3,19),(1,11),(1,3)]$, $[(1,53),(1,13),(1,5)]$, $[(1,45),(1,21),(1,5)]$, $[(1,37),(1,29),(1,5)]$, $[(1,37),(1,21),(1,13)]$
(c)	$[(3,9),(44,1)]$, $[(2,17),(1,9),(28,1)]$, $[(5,9),(26,1)]$, $[(2,25),(1,9),(12,1)]$, $[(2,17),(3,9),(10,1)]$, $[(7,9),(8,1)]$
(d)	$[(1,17),(2,9),(36,1)]$, $[(1,25),(2,9),(28,1)]$, $[(1,33),(2,9),(20,1)]$, $[(3,17),(20,1)]$, $[(1,17),(4,9),(18,1)]$, $[(1,11),(20,3)]$, $[(1,41),(2,9),(12,1)]$, $[(1,25),(2,17),(12,1)]$, $[(1,25),(4,9),(10,1)]$, $[(1,35),(12,3)]$, $[(2,19),(11,3)]$, $[(1,19),(2,11),(10,3)]$, $[(4,11),(9,3)]$, $[(1,21),(10,5)]$, $[(2,13),(9,5)]$, $[(1,15),(8,7)]$, $[(1,49),(2,9),(4,1)]$, $[(1,33),(2,17),(4,1)]$, $[(2,25),(1,17),(4,1)]$, $[(1,33),(4,9),(2,1)]$, $[(3,17),(2,9),(2,1)]$, $[(1,17),(6,9)]$, $[(1,59),(4,3)]$, $[(1,43),(2,11),(2,3)]$, $[(2,27),(1,11),(2,3)]$, $[(1,27),(2,19),(2,3)]$, $[(1,27),(4,11)]$, $[(2,19),(3,11)]$, $[(1,61),(2,5)]$, $[(1,45),(2,13)]$, $[(2,29),(1,13)]$
(e)	None
Open	$[(1,29),(2,21)]$

TABLE 42. Non-pointed and non-perfect cases for rank 71. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

Rule	Cases

(a)	$[(1,9),(64,1)]$, $[(1,17),(56,1)]$, $[(2,9),(55,1)]$, $[(1,25),(48,1)]$, $[(1,17),(1,9),(47,1)]$, $[(1,33),(40,1)]$, $[(1,25),(1,9),(39,1)]$, $[(2,17),(39,1)]$, $[(1,41),(32,1)]$, $[(1,33),(1,9),(31,1)]$, $[(1,25),(1,17),(31,1)]$, $[(1,49),(24,1)]$, $[(1,41),(1,9),(23,1)]$, $[(1,33),(1,17),(23,1)]$, $[(2,25),(23,1)]$, $[(1,57),(16,1)]$, $[(1,49),(1,9),(15,1)]$, $[(1,41),(1,17),(15,1)]$, $[(1,33),(1,25),(15,1)]$, $[(1,65),(8,1)]$, $[(1,57),(1,9),(7,1)]$, $[(1,49),(1,17),(7,1)]$, $[(1,41),(1,25),(7,1)]$, $[(2,33),(7,1)]$
(b)	$[(4,9),(37,1)]$, $[(1,17),(3,9),(29,1)]$, $[(1,25),(1,17),(1,9),(22,1)]$, $[(1,25),(3,9),(21,1)]$, $[(2,17),(2,9),(21,1)]$, $[(6,9),(19,1)]$, $[(1,33),(1,17),(1,9),(14,1)]$, $[(1,33),(3,9),(13,1)]$, $[(1,25),(1,17),(2,9),(13,1)]$, $[(3,17),(1,9),(13,1)]$, $[(1,17),(5,9),(11,1)]$, $[(1,35),(1,11),(9,3)]$, $[(1,27),(1,19),(9,3)]$, $[(1,19),(3,11),(7,3)]$, $[(1,41),(1,17),(1,9),(6,1)]$, $[(1,33),(1,25),(1,9),(6,1)]$, $[(1,41),(3,9),(5,1)]$, $[(1,33),(1,17),(2,9),(5,1)]$, $[(2,25),(2,9),(5,1)]$, $[(1,25),(2,17),(1,9),(5,1)]$, $[(4,17),(5,1)]$, $[(1,25),(1,17),(3,9),(4,1)]$, $[(1,25),(5,9),(3,1)]$, $[(2,17),(4,9),(3,1)]$, $[(8,9),(1,1)]$, $[(1,23),(1,15),(5,7)]$, $[(1,45),(1,13),(3,5)]$, $[(1,37),(1,21),(3,5)]$, $[(1,29),(1,21),(1,13),(2,5)]$, $[(1,29),(3,13),(1,5)]$, $[(1,59),(1,11),(1,3)]$, $[(1,51),(1,19),(1,3)]$, $[(1,43),(1,27),(1,3)]$, $[(1,43),(1,19),(1,11)]$, $[(1,35),(1,27),(1,11)]$
(c)	$[(3,9),(46,1)]$, $[(2,17),(1,9),(30,1)]$, $[(5,9),(28,1)]$, $[(2,25),(1,9),(14,1)]$, $[(2,17),(3,9),(12,1)]$, $[(7,9),(10,1)]$
(d)	$[(1,17),(2,9),(38,1)]$, $[(1,25),(2,9),(30,1)]$, $[(1,33),(2,9),(22,1)]$, $[(3,17),(22,1)]$, $[(1,17),(4,9),(20,1)]$, $[(1,19),(18,3)]$, $[(2,11),(17,3)]$, $[(1,41),(2,9),(14,1)]$, $[(1,25),(2,17),(14,1)]$, $[(1,25),(4,9),(12,1)]$, $[(1,13),(12,5)]$, $[(1,43),(10,3)]$, $[(1,27),(2,11),(8,3)]$, $[(2,19),(1,11),(8,3)]$, $[(5,11),(6,3)]$, $[(1,49),(2,9),(6,1)]$, $[(1,33),(2,17),(6,1)]$, $[(2,25),(1,17),(6,1)]$, $[(1,33),(4,9),(4,1)]$, $[(1,17),(6,9),(2,1)]$, $[(3,15),(4,7)]$, $[(2,29),(3,5)]$, $[(1,37),(2,13),(2,5)]$, $[(3,21),(2,5)]$, $[(2,21),(2,13),(1,5)]$, $[(1,21),(4,13)]$, $[(2,35),(1,3)]$, $[(1,51),(2,11)]$, $[(1,35),(2,19)]$
(e)	$[(3,17),(2,9),(4,1)]$, $[(2,27),(1,19)]$
Open	$[(1,31),(6,7)]$, $[(1,53),(4,5)]$, $[(1,67),(2,3)]$

TABLE 43. Non-pointed and non-perfect cases for rank 73. Each case is denoted as an array of ordered pairs (n,r), indicating that there are n components of rank r in the universal grading of $\mathcal{C}$.

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